# Notes

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## Chapter 2. Linear Transformations, Basis, Dimension Theorem

**Definition.** Let $U, W$ be subspaces of vector space $V$. The sum of $U$ and $W$, denoted $U+W$, is the subspace

$$U + W = \text{Span}(U \cup W) = \{u + w | u \in U \text{ and } w \in W\}.$$

In addition, if $U \cap W = \{0\}$, then the sum is called direct sum, denoted $U \oplus W$.

**Definition.** Let $V, W$ be non-zero finite-dimensional vector spaces. Let $\alpha = v_1, \ldots, v_n, \beta = w_1, \ldots, w_m$ be bases for $V$ and $W$ resepctively, and let $T \in L(V, W)$. The matrix representation wrt the bases $\alpha$ and $\beta$ is denoted $[T]_\alpha^\beta$ such that for all $x \in V$,

$$[Tx]_\beta = [T]_\alpha^\beta [x]_\alpha.$$

**Definition.** The matrix $[I]_\alpha^\beta$ is called the change of basis matrix from $\alpha$-coordinates to $\beta$-coordinates, such that for every $x \in V$,

$$[x]_\beta = [I]_\alpha^\beta [x]_\alpha.$$

**Definition.** Matrices $A, B$ are similar if there exists invertible matrix $S$ such that

$$A = SBS^{-1}.$$

**Theorem.** Let $\alpha = v_1, \ldots, v_n$ be a basis for vector space $V$.

- If $\beta = w_1, \ldots, w_n$ is a basis for $V$, the change of basis matrix $[I]_\alpha^\beta$ is invertible and its inverse is $[I]_\beta^\alpha$.
- If $S \in M_{n \times n}(\mathbb{R})$ is invertible, then there is a basis $\beta$ for $V$ such that $S = [I]_\beta^\alpha$.

**Theorem.** Let $V$ be n-dimensional vector space, and let $T \in L(V)$.

- Let $\alpha, \beta$ be two bases for $V$, and let $S = [I]_\alpha^\beta$. Then $S$ is invertible and

$$[T]_\alpha^\alpha = [I]_\beta^\alpha [T]_\beta^\beta [I]_\alpha^\beta = S[T]_\beta^\beta S^{-1}.$$

- Let $S \in M_{n \times n}(\mathbb{R})$ be invertible, and let $\alpha$ be a basis for $V$. Then there exists basis $\beta$ for $V$ such that

$$[T]_\alpha^\alpha = S[T]_\beta^\beta S^{-1}.$$

**Theorem.** If $U \cap W = \{0\}$, then each vector in $U \oplus W$ is uniquely expressible as a sum of a vector in $U$ and a vector in $W$.

**Theorem** (Dimension Theorem)**.** Let $V, W$ be vector spaces. Suppose $V$ is finite-dimensional, and let $T \in L(V, W)$. Then

$$\dim(\ker T) + \dim(\text{Im} T) = \dim(V).$$

**Theorem** (Rank-Nullity Theorem)**.** For any matrix $A$,

$$\text{nullity } A + \text{rank } A = \text{ number of columns of } A.$$

**Theorem.** Let $A, B \in M_{n\times n}(\mathbb{R})$, then $A$ and $B$ are similar iff there exists n-dimensional vector space $V$ with bases $\alpha, \beta$, and there exists $T \in L(V)$ such that

$$A = [T]_{\alpha}^{\alpha} \text{ and } B = [T]_{\beta}^{\beta}.$$

## Chapter 4. Eigen and Diagonalizability

**Definition.** $x \in V$ is an eigenvector if $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{R}$. $\lambda$ is called an eigenvalue of $T$ corresponding to x.

**Definition.** For a given eigenvalue $\lambda$, the eigenspace $E_\lambda$ is defined as

$$E_\lambda = \{x \in V, Tx = \lambda x\}. \tag{1}$$

**Definition.** A linear mapping $T$ of finite-dimensional vector space $V$ is said to be diagonalizable if there exists a basis of $V$, all of whose vectors are eigenvectors of $T$.

**Definition.** The geometric multiplicity of $\lambda$ is the dimension of associated subspace $E_\lambda(T)$.

**Definition.** The algebraic multiplicity of $\lambda$ is the number of times $\lambda$ appears as a root of $p_T(\lambda)$.

**Theorem.** Let $x_i$ be eigenvectors of $T : V \to V$ corresponding to distinct eigenvalues $\lambda_i$. Then $\{x_1, x_2, \dots, x_k\}$ is a linearly independent subset of $V$.

**Theorem.** $x$ is an eigenvector of $T$ with eigenvalue $\lambda \iff x \in \ker(T - \lambda I)$.

**Theorem.** Let $V$ be finite-dimensional vector space, and let $T \in L(V)$. Suppose

$$p_T(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_k)^{n_k} q(\lambda)$$

where $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of $T$, and $q(\lambda)$ is some polynomial. Then the following are equivalent:

- $T$ is diagonalizable.
- For all $\lambda$ of $T$, the AM and GM of $\lambda$ are equal.
- $V = E_{\lambda_1}(T) + \dots + E_{\lambda_k}(T)$.
- $[T]_\alpha^\alpha$ is similar to a diagonal matrix for any basis $\alpha$ of $V$.

## Chapter 5. $\mathbb{C}, F$

**Definition.** The set of complex numbers $\mathbb{C}$ is the real vector space $\mathbb{R}^2$ together with multiplication operator $\odot$ defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$

The standard basis for $\mathbb{C}$ is commonly written as the list $\{1, i\}$.

**Definition.** A field is a set $F$ together with two operations called addition $(+)$ and multiplication $(\odot)$ such that the following axioms hold

1. (AA) Additive Associativity. $\forall a, b, c \in F, (a + b) + c = a + (b + c)$
2. (AC) Additive Commutativity. $\forall a, b \in F, a + b = b + a$
3. (Z) Zero (or additive identity). $\exists! 0 \in F$ such that $\forall a \in F, a + 0 = a$
4. (AI) Additive Inverse. $\forall a \in F, \exists! (-a) \in F$ such that $a + (-a) = 0$
5. (MA) Multiplicative Associativity. $\forall a, b, c \in F, a + (b + c) = (a + b) + c$
6. (MC) Multiplicative Commutativity. $\forall a, b \in F, a \odot b = b \odot a$
7. (D) Distributivity. $\forall a, b, c \in F, a \odot (b + c) = a \odot b + a \odot c$
8. (O) One. $\exists! 1 \in F$ such that $\forall a \in F, 1 \odot a = a$
9. (MI) Multiplicative Inverse. $\forall a \neq 0 \in F, \exists! a^{-1} \in F$ such that $a \odot a^{-1} = 0$

**Definition.** A field $F$ is called algebraically closed if every non-constant polynomial with coefficients in $F$ has a root in $F$.

**Theorem** (Fundamental Theorem of Algebra)**.** Every polynomial $p(x)$ of degree $n$ with complex coefficients has a unique factorization of the form

$$p(x) = a(x - a_1)^{n_1}(x - a_2)^{n_2} \ldots (x - a_k)^{n_k}$$

where $a$ and $a_i$'s are complex numbers and $n_i$'s are natural numbers such that

$$n = n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i.$$

# Chapter 6

## Triangular Form

**Definition.** Let $T : V \to V$ be a linear mapping. A subspace $W \subset V$ is said to be invariant (or stable) under $T$ if $T(W) \subset W$.

**Definition.** A linear mapping $T : V \to V$ on a finite-dimensional vector space $V$ is said to be triangularizable if there exists a basis $\beta$ such that $[T]_\beta^\beta$ is upper-triangular.

**Theorem.** Let $V$ be a vector space, let $T \in L(V)$, and let $\beta = \{x_1, \dots, x_n\}$ be a basis for $V$. Then $[T]_\beta^\beta$ is upper-triangular $\iff$ each of the subspaces $W_i = \text{Span}\{x_1, \dots, x_i\}$ [1] is invariant under $T$.

**Theorem.** Let $T : V \to V$, let $W \subset V$ be invariant subspace. Then the characteristic polynomial of $T|_W$ divides the polynomial of $T$. This implies that every eigenvalues of $T|_W$ is also an eigenvalue of $T$.

**Theorem.** Let $V$ be a finite-dimensional vector space over a field $F$, and let $T : V \to V$ be a linear mapping. Then $T$ is triangularizable $\iff$ the characteristic polynomial equation of $T$ has $\dim(V)$ roots in the field $F$.

**Theorem.** Assume the characteristic polynomial of $T$ has $n = \dim(V)$ roots in $F$. If $W \subset V, W \neq V$ is an invariant subspace under $T$. Then there exists a vector $x \neq 0$ in $V$ such that $x \notin W$, and $W + \text{Span}(\{x\})$ is invariant under $T$.

**Theorem.** If $T$ is triangularizable, with eigenvalues $\lambda_i$ and respective multiplicities $m_i$, then there exists basis $\beta$ for $V$ such that $[T]_\beta^\beta$ is upper-triangular, and the diagonal entries of $[T]_\beta^\beta$ are $m_1$ of $\lambda_1$'s, followed by $m_2$ of $\lambda_2$'s, and so on.

**Theorem** (Cayley-Hamilton)**.** Let $T \in L(V)$ on finite-dimensional vector space $V$, and let $p(t) = \det(T - tI)$ be its characteristic polynomial. Assume that $p(t)$ has $\dim(V)$ roots in the field $F$ over which $V$ is defined. Then $p(T) = 0$ (the zero mapping on $V$). Note that the linear mapping $p$ is defined as

$$p(T) = a_n T^n + a_{n-1} T^{n-1} = \cdots + a_2 T^2 + a_1 I.$$

**Corollary.** Under all linear mappings $T : V \to V$,

- $\{0\}$ and $V$ are invariant subspaces under $T$.
- $\ker(T)$ and $\text{Im}(T)$ are invariant subspaces under $T$.
- If $\lambda$ is an eigenvalue of $T$, then the eigenspace $E_\lambda$ is invariant under $T$[2].

## Nilpotent, Jordan Canonical

**Definition.** A matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

**Definition.** A direct sum of matrices is a block diagonal matrix, i.e. every off-diagonal block is zero:

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

this can be genenralized from 2 to $n$.

[1] Note that the subspaces $W_i$ are related as $\{0\} \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V$. The $W_i$ form an increasing sequence of subspaces.

[2] This is true since for $v \in E_\lambda, Tv = \lambda v \in E_\lambda$.

**Definition.** A Jordan matrix J is a direct sum of Jordan blocks

$$J = J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus \cdots \oplus J_{n_r}(\lambda_r)$$

where $n_i \in \mathbb{N}, 1 \leq i \leq r$.

**Definition.** A nilpotent Jordan block is a Jordan block with eigenvalue 0. We denote the $k \times k$ nilpotent Jordan block $J_k(0)$ by $J_k$.
A nilpotent Jordan Matrix $J$ is a direct sum of Jordan blocks

$$J = J_{n_1} \oplus \cdots \oplus J_{n_r}$$

where $n_i \in \mathbb{N}, 1 \leq i \leq r$.

**Definition.** Let $V$ be a vector space, let $T \in L(V)$, and let $x \in V$ . The cyclic subspace generated by $x$, denoted $C(x)$, is defined to be

$$C(x) = \text{Span}\{x, Tx, T^2x, T^3x, \dots\}$$

A subspace $W \subseteq V$ is called cyclic with respect to $T$ if there exists $x \in V$ such that $W = C(x)$.

**Definition.** A Jordan form for an $n \times n$ matrix $A$ is a Jordan matrix $J$ to which $A$ is similar, meaning for some invertible $n \times n$ matrix $S$,

$$J = SAS^{-1}$$

**Definition.** Let $J$ and $\mathcal{J}$ be $n \times n$ nilpotent Jordan matrices. Then $J$ is similar to $\mathcal{J}$ iff rank $J^p =$ rank $\mathcal{J}^p$ for each $p = 1, \dots, n$.

**Theorem.** Let $V$ be a finite-dimensional vector space, let and $N : V \to V$ be a linear mapping. Then $N$ has one eigenvalue $\lambda = 0$ with multiplicity $n = \dim(V) \iff N$ is nilpotent.

*Proof.* Let $N : V \to V$ be a linear mapping on finite-dimensional vector space $V$, with only one eigenvalue $\lambda = 0$, with multiplicity $n = \dim(V)$. Then by the Cayley-Hamilton theorem, $N^n = 0$, hence nilpotent.
Conversely, if $N$ is nilpotent, with $N^k = 0, k \in \mathbb{N}$. Then every eigenvalue of $N$ is equal to 0. ☐

**Corollary.**
- $N^{k-1}(x)$ is an eigenvector of $N$ with eigenvalue $\lambda = 0$.
- $C(x)$ is an invariant subspace of $V$ under $N$.
- The cycle generated by $x \neq 0$ is a linearly independent set. Hence $\dim(C(x)) = k$.

**Corollary.** Each nilpotent matrix is similar to a nilpotent Jordan matrix (that is unique up to the ordering of the direct summands). Re-worded in the language of linear transformations, the claim is: If $V$ is a finite-dimensional vector space and $T \in L(V)$ is nilpotent, then there exists a basis $\alpha$ for $V$ such that $[T]_\alpha^\alpha$ is a nilpotent Jordan matrix.

**Theorem** (Decomposition Theorem)**.** Let $V$ be a finite dimensional vector space, and let $T \in L(V)$. Then $V$ can be decomposed into a direct sum of cyclic subspaces.

**Theorem.** Let V be a finite-dimensional vector space and let $T \in L(V)$ be nilpotent. Then there exists a basis $\alpha$ for $V$ such that $[T]_\alpha^\alpha$is a nilpotent Jordan matrix.

**Theorem.** Let $A$ be an $n \times n$ matrix, and let $J$ and $\mathcal{J}$ be two nilpotent Jordan matrices that are similar to $A$. Then $J$ can be obtained from $\mathcal{J}$ by permuting its direct summands.

## Generalized Eigenspaces & Jordan Forms

**Definition.** Let $V$ be a finite-dimensional vector space and let $T \in L(V)$. Let $\lambda$ be an eigenvalue of $T$ with algebraic multiplicity $m$. The $\lambda$-generalized eigenspace is

$$K_\lambda = \ker(T - \lambda I)^m$$

The non-zero vectors in $K_\lambda$ are called generalized eigenvectors.

**Theorem.** Let $V$ be a finite-dimensional complex vector space and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $T \in L(V)$. Then

$$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$$

**Theorem** (Jordan's Theorem)**.** Let V be a finite-dimensional complex vector space and let $T \in L(V)$. Then there exists a basis $\alpha$ for $V$ such that $[T]_\alpha^\alpha$ is in Jordan form.

**Corollary.** Let $A, B \in M_{n\times n}(\mathbb{C})$ have the same set of eigenvalues. Then $A$ and $B$ are similar iff $\operatorname{rank}(A - \lambda I)^p = \operatorname{rank}(B - \lambda I)^p$ for each eigenvalue $\lambda$ of $A$, and each $p = 1, 2, \ldots, n$.

# Writing Assignments

## Writing Assignment 1

1. Let $x_1, \dots, x_n$ be vectors in a vector space $V$ where at least one of the vectors is nonzero. Then

$$\text{span}\{x_1, \dots, x_r\} = \text{span}\{x_j, j = 1, \dots, r \text{ and } x_j \neq 0\}$$

2. Let $U$ be a subset of vector space $V$. Then span($U$) is the intersection of all the subspaces of $V$ that contains $U$. Furthermore, if $U = \emptyset$, then $\text{span}(U) = \text{span}\{\emptyset\} = \{0\}$, which is the smallest subspace possible for vector space $V$.

## Writing Assignment 2

Let $V$ be an $n$-dimensional vector space. Let $W_1$ and $W_2$ be unequal subspaces of $V$, each of dimension $n-1$. Then

$$V = W_1 + W_2 = \{w_1 + w_2 | w_1 \in W_1 \text{ and } w_2 \in w_2\}$$
$$\dim(W_1 \cap W_2) = n - 2.$$

## Writing Assignment 3

1. $T$ has a left inverse $\iff$ $T$ is injective.

2. If $\alpha$ is a basis for $V$ and $\beta$ is a basis for $W$, then $[T]_\alpha^\beta$ has a left inverse iff its columns are linearly independent.

3. $T$ has a right inverse $\iff$ $T$ is surjective. If $\alpha$ is a basis for $V$ and $\beta$ is a basis for $W$, then $[T]_\alpha^\beta$ has a left inverse iff its rows are linearly independent.

## Writing Assignment 4

Let $A, B$ be $n \times n$ matrices.

1. If $A$ is similar to $B$, then $A - \lambda I$ is similar to $B - \lambda I$, $\forall \lambda \in \mathbb{R}$.

2. If $\exists \lambda \in R$ such that $A - \lambda I$ is similar to $B - \lambda I$, then $A$ is similar to $B$.

3. Consider basis $\alpha, \beta$ of $\mathbb{R}^n$, where

$$\alpha = v_1, v_2, \dots, v_n$$
$$\beta = w_1, w_2, \dots, w_n$$

Define $n \times n$ matrices $A, B$ as

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$
$$B = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}$$

Let $C = [I]_\alpha^\beta$. Then $A = BC$ and $C = B^{-1}A$.

## Writing Assignment 5

1. Let $A, B \in M_{n \times n}(\mathbb{R})$. If $A$ and $B$ are each similar to a diagonal matrix, and have the same eigenvectors (not necessarily the same eigenvalues). Then $AB = BA$. We say $A$ and $B$ are commute.

2. Let $A \in M_{n \times n}(\mathbb{R})$. Suppose the only eigenvalues of $A$ are $\pm 1$ and $A$ is similar to a diagonal matrix. Then $A^{-1} = A$.