A Method for Simultaneous Measurement of Moments Using Quantum Circuits

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The quantum circuit is shown in the following figure:

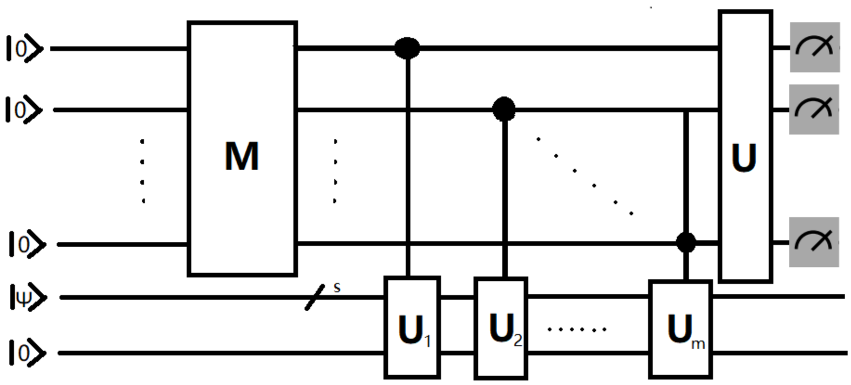

Figure 1: Quantum Circuit

For convenience of expression, let $|\psi\rangle\ |0\rangle = |\phi\rangle$, and let $\langle\phi|U_k|\phi\rangle = E_k$ (where $1 \le k \le m$), $|\psi\rangle$ contains s qubits.

Where the form of U_k is as follows:

$$U_k = \left[\begin{array}{cccc|cccc} x_{1k} & 0 & \cdots & 0 & y_{1k} & 0 & \cdots & 0 \\ 0 & x_{2k} & \cdots & 0 & 0 & y_{2k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nk} & 0 & 0 & \cdots & y_{nk} \\ \hline y_{1k} & 0 & \cdots & 0 & -x_{1k} & 0 & \cdots & 0 \\ 0 & y_{2k} & \cdots & 0 & 0 & -x_{2k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{nk} & 0 & 0 & \cdots & -x_{nk} \end{array}\right]$$

Where $x_{ik} = \left(\frac{i-1}{n-1}\right)^k$, $y_{ik} = \sqrt{1-x_{ik}^2}$ $(1 \le i \le n = 2^s, 1 \le k \le m)$, it is easy to verify that U_k is a unitary matrix. Note that $E_k = \langle\phi|U_k|\phi\rangle = \sum_{i=1}^{n} x_{ik}\psi_i^2$, which is the k-th moment (ψ_i is the component of $|\psi\rangle$ on each basis vector).

M transforms $|00\cdots0\rangle$ into $\frac{1}{\sqrt{m+1}}(|00\cdots0\rangle+|10\cdots0\rangle+|01\cdots0\rangle+\cdots+|00\cdots1\rangle)$
Then the state before U becomes $\frac{1}{\sqrt{m+1}}(|00\cdots0\rangle\,|\phi\rangle+|10\cdots0\rangle\,U_1\,|\phi\rangle+|01\cdots0\rangle\,U_2\,|\phi\rangle+\cdots+|00\cdots1\rangle\,U_m\,|\phi\rangle)$

U transforms $|00\cdots0\rangle$ into $\frac{1}{\sqrt{2m}}(|10\cdots0\rangle+|01\cdots0\rangle+\cdots+|00\cdots1\rangle)+\frac{1}{\sqrt{2m}}(|01\cdots1\rangle+|10\cdots1\rangle+\cdots+|11\cdots0\rangle)$ transforms $|10\cdots0\rangle$ into $\frac{1}{\sqrt{2}}(|10\cdots0\rangle$-$|01\cdots1\rangle)$, transforms $|01\cdots0\rangle$ into $\frac{1}{\sqrt{2}}(|01\cdots0\rangle$-$|10\cdots1\rangle)$, $\cdots$, transforms $|00\cdots1\rangle$ into $\frac{1}{\sqrt{2}}(|00\cdots1\rangle$-$|11\cdots0\rangle)$

The state after U is $\frac{1}{\sqrt{m+1}}|10\cdots0\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_1|\phi\rangle}{\sqrt{2}})+\frac{1}{\sqrt{m+1}}|01\cdots0\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_2|\phi\rangle}{\sqrt{2}})+\cdots+\frac{1}{\sqrt{m+1}}|00\cdots1\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_m|\phi\rangle}{\sqrt{2}})+\cdots$ $+\frac{1}{\sqrt{m+1}}|10\cdots0\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_1|\phi\rangle}{\sqrt{2}})+\frac{1}{\sqrt{m+1}}|01\cdots0\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_2|\phi\rangle}{\sqrt{2}})+\cdots+\frac{1}{\sqrt{m+1}}|00\cdots1\rangle(\frac{|\phi\rangle}{\sqrt{2m}}+\frac{U_m|\phi\rangle}{\sqrt{2}})$

The probability of measuring 0 in the i-th bit can be calculated as $prob(k) = \frac{1}{2} + \frac{(E_1+\cdots+E_m)-2E_k}{(m+1)\sqrt{m}}$ From $\sum_{k=1}^{m} prob(k) = \frac{m}{2} + \frac{(m-2)(E_1+\cdots+E_m)}{(m+1)\sqrt{m}}$, we can obtain $\sum_{k=1}^{m} E_k$. Substituting the result into the expression of $prob(k)$ can estimate E_k. It is also possible to calculate E_1 using the probability of $|10\cdots0\rangle$ minus the probability of $|01\cdots1\rangle$, which is $\frac{4E_1}{(m+1)\sqrt{m}}$, and similarly for E_k.

Verification is performed on qiskit, and the quantum circuit diagram is as follows:

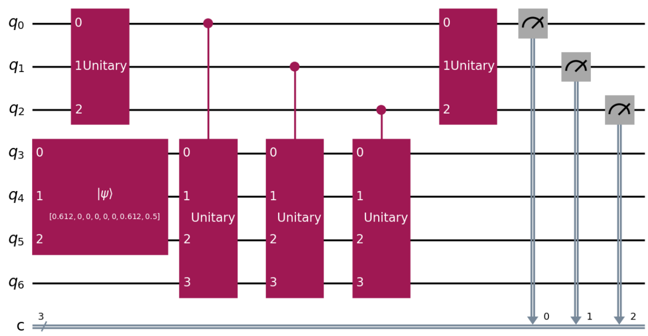

Figure 2: Experimental Circuit

Taking $m = 3$, $s = 3$, $\psi = [\sqrt{\frac{3}{8}}, 0, 0, 0, 0, 0, \sqrt{\frac{3}{8}}, \sqrt{\frac{1}{8}}]$, the values of the corresponding first, second, and third moments are 0.5714, 0.5255, 0.4862 respectively. The estimated values of the first, second, and third moments are $2\sqrt{3}(prob(2) + prob(3) - 1)$, $2\sqrt{3}(prob(1) + prob(3) - 1)$, $2\sqrt{3}(prob(1) + prob(2) - 1)$. Experiment results on qiskit are as follows:

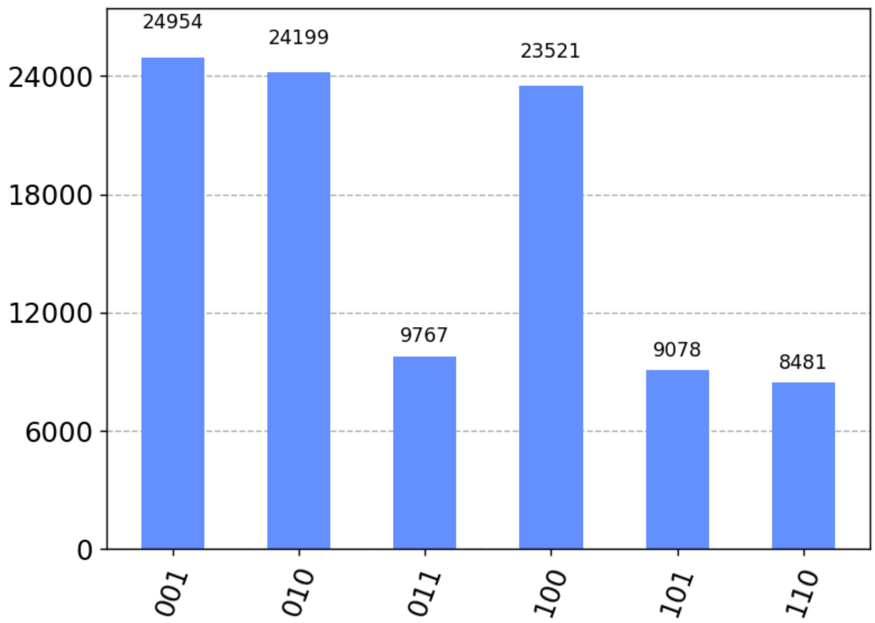

Figure 3: Experimental Results

$prob(1) = 0.5620$, $prob(2) = 0.5755$, $prob(3) = 0.5892$. From this, we estimate $E_1 = 0.5705$, $E_2 = 0.5238$, $E_3 = 0.4763$. Using the second method, we estimate $E_1 = 0.5706$, $E_2 = 0.5238$, $E_3 = 0.4765$. Except for E_3, the errors are relatively small. Another experiment is conducted, estimating $E_3 = 0.4898$, which confirms the correctness of the calculation method.

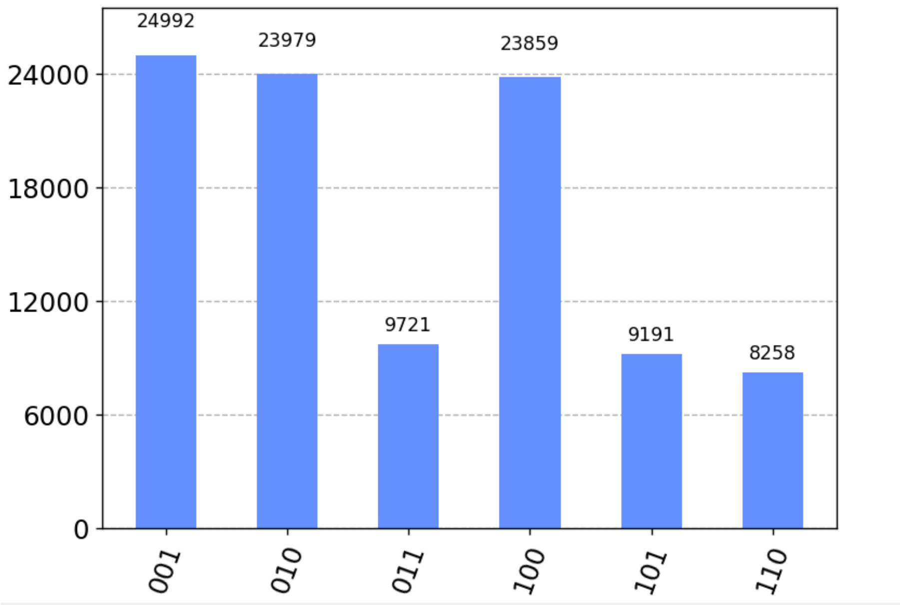

Figure 4: Experimental Results 2

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