Time Reversal Notes

student

yesterday

1 Operators

Θ is defined as the total time reversal operator. It is an **anti-unitary** operator.
An anti-unitary operator is always able to be written as

$$\Theta = UK \tag{1}$$

where U is a **unitary operator** and K is the **complex-conjugate operator**.

1.1 K, the complex-conjugate operator

K act on the ket without changing anything but like follows

$$K|\alpha\rangle = K\int \mathrm{d}r\,|r\rangle\langle r|\alpha\rangle = \int \mathrm{d}r\,|r\rangle\langle\alpha|r\rangle \tag{2}$$

K is **anti-linear** operator

$$K(c_1|\alpha\rangle + c_2|\alpha\rangle) = c_1^*K|\alpha\rangle + c_2^*K|\alpha\rangle \tag{3}$$

K is also **anti-unitary** operator

$$(K\phi, K\psi) = (\phi^*, \psi^*) = (\psi, \phi) = (\phi, \psi)^* \tag{4}$$

K is the total time-reversal operator of **time-independent** state.

1.2 Time-dependent state

Consider the time-dependent state, the total time reversal operator is defined as (1). To make distinction, we mark the unitary operator U for reversing t in the time-dependent state as T, and the total time-reversal operator of **time-dependent** state is

$$\Theta = TK \tag{5}$$

The unitary operator T acts on the time-dependent state and reverse the t like this,

$$\Theta|\alpha, t\rangle = TK|\alpha, t\rangle = K|\alpha, -t\rangle \tag{6}$$

We also know that $\Theta^{-1} = \Theta$, which is due to the fact that $\Theta\Theta = 1$.
Take momentum operator as an example,

$$\begin{aligned}
\Theta \mathbf{P} \Theta^{-1} |\Psi(t)\rangle &= TK\mathbf{P}K^{-1}T^{-1} |\Psi(t)\rangle \\
&= TK\mathbf{P}TK |\Psi(t)\rangle \\
&= TK\mathbf{P} \int \mathrm{d}r \, |r\rangle \langle \Psi(-t)|r\rangle \\
&= TK(-i\hbar\nabla) \int \mathrm{d}r \, |r\rangle \langle \Psi(-t)|r\rangle \\
&= T(+i\hbar\nabla)K \int \mathrm{d}r \, |r\rangle \langle \Psi(-t)|r\rangle \\
&= +i\hbar\nabla |\Psi(t)\rangle \\
&= -\mathbf{P} |\Psi(t)\rangle
\end{aligned}$$

1.3 Plus spin-space

If we have to deal with spins, we need to add a unitary operator, say U, who only play a role in the spin-space. Thus the total time-reversal operator is

$$\Theta = UTK \tag{7}$$

Consider the spin-$1/2$ only, we take the S_z representation, in which

$$U = -U^{-1} = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

[Sakurai] choose $U = -i\sigma_y$, because he always rotate/translate coordinates. There is no difference between those choices.

Let's check this. We know the fact that the spin is analogue to angular momentum, that's why they are able to couple. So under time reversal transformation, the spin must change its sign as **L**.

$$\begin{aligned}
\Theta \mathbf{S} \Theta^{-1} &= UTK\mathbf{S}K^{-1}T^{-1}U^{-1} \\
&= U\mathbf{S}^*TKK^{-1}T^{-1}U^{-1} \\
&= U\mathbf{S}^*U^{-1} = -\mathbf{S}
\end{aligned}$$

Since at S_z representation, only S_y is complex. For each component, we required that

$$US_xU^{-1} = -S_x \, , US_y^*U^{-1} = -S_y \, , US_zU^{-1} = -S_z \tag{9}$$

It is easy to check that (8) is really correct.

The U operator only acts on spin-space, and it is pure real. So we could use the fact that $[U, TK] = 0$

How about the raising, $S_+ = S_x + iS_y$, and lowering, $S_- = S_x - iS_y$, operators? For example,

$$\begin{aligned}
\Theta S_+ \left|-\right\rangle &= \Theta S_+ \Theta^{-1} \Theta \left|-\right\rangle \\
&= [UTK(S_x + iS_y)K^{-1}T^{-1}U^{-1}](UTK\left|-\right\rangle) \\
&= [U(S_x K - iKS_y)K^{-1}TT^{-1}U^{-1}](-i\left|-\right\rangle) \\
&= [U(S_x K - iS_y^* K)K^{-1}TT^{-1}U^{-1}](-i\left|+\right\rangle) \\
&= [U(S_x + iS_y)KK^{-1}TT^{-1}U^{-1}](-i\left|+\right\rangle) \\
&= (U(S_x + iS_y)U^{-1})(-i\left|+\right\rangle) \\
&= (US_xU^{-1} + iUS_yU^{-1})(-i\left|+\right\rangle) \\
&= (-S_x + iS_y)(-i\left|+\right\rangle) \\
&= -S_-(-i\left|+\right\rangle)
\end{aligned}$$

This minus sign is really necessary. And the imaginary part, which means the phaseη, is essential. Be aware that $\Theta^2 = (UTK)^2 = -1$ due to the fact that $UU = (i\sigma_y)(i\sigma_y) = -1$, thus $\Theta^2 S_+ \left|-\right\rangle = -S_+ \left|-\right\rangle$.

This is because (see[Sakurai(4.4.56)] or (4.4.79))

$$\Theta \left|l, m\right\rangle = (-1)^{-m} \left|l, -m\right\rangle \tag{10}$$

Again we differ from [Sakurai] by a minus sign of m in the term, $(-1)^{-m}$. This again really depend on if you rotate coordinates or states.

So $\Theta\left|+\right\rangle = (-1)^{1/2}\left|-\right\rangle = -i\left|-\right\rangle, \Theta\left|-\right\rangle = (-1)^{-1/2}\left|+\right\rangle = i\left|+\right\rangle$, and easy to see that for spin-less states, $\Theta^2 = 1$.

If we have multiple spin-$1/2$ particles, then the total U is direct product of every particle's $i\sigma_y$.

$$U = \prod \otimes (i\sigma_y) \tag{11}$$

1.4 Plus isospin-space

We'd like also to consider isospin. We can add another unitary operator, say V, stands for volunteers, acts only on isospin-space. The total time-reversal operators is thus

$$\Theta = VUTK \tag{12}$$

Also we take $\mathcal{T}_z$ representation, and in which

$$V = I_2 \tag{13}$$

Let's prove this in two aspects.

1.4.1 Evidence I

Since we barely know the property of time-reversal operator of isospin from textbook, we would like to start with something we are really certain of. The first thing we know is that τ_z's two eigenstates, $\left|+\right\rangle$, means proton, $\left|-\right\rangle$, means neutron, are time-reversal even. We assert that

$$\Theta\left|+\right\rangle = \left|+\right\rangle, \Theta\left|-\right\rangle = \left|-\right\rangle \tag{14}$$

Start from this point, we say τ_z is also even under time-reversal. Since we know $\tau_z\left|\pm\right\rangle = +1\left|\pm\right\rangle$, we consider following identities,

$$\Theta\tau_z\left|\pm\right\rangle = \Theta\tau_z\Theta^{-1}\Theta\left|\pm\right\rangle = \Theta\tau_z\Theta^{-1}\left|\pm\right\rangle = +1\left|\pm\right\rangle$$

Thus

$$\Theta\tau_z\Theta^{-1} = +1\tau_z \tag{15}$$

Now we consider the other two components. Since we know $|\pm\rangle = \tau_\pm |\mp\rangle$, we are really sure about that the τ_+ and τ_- should not change under time reversal. Let's consider those identities,

$$\Theta|\pm\rangle = \Theta\tau_\pm|\mp\rangle = \Theta\tau_\pm\Theta^{-1}\Theta|\mp\rangle = \Theta\tau_\pm\Theta^{-1}|\mp\rangle = |\pm\rangle$$

If we have $\Theta\tau_\pm\Theta^{-1} = \tau_\mp$, this would make $\tau_\mp|\mp\rangle = 0$, which means $|\pm\rangle = 0$. This is truly wrong. So we have no choice but

$$\Theta\tau_\pm\Theta^{-1} = \tau_\pm \tag{16}$$

Now consider $\tau_\pm = \frac{1}{2}(\tau_x \pm i\tau_y)$, we can easily find out that

$$\Theta\tau_x\Theta^{-1} = \tau_x, \Theta\tau_y\Theta^{-1} = -\tau_y \tag{17}$$

To be more specific,

$$\begin{aligned}
\Theta\tau_\pm\Theta^{-1} &= \frac{1}{2}VUTK(\tau_x \pm i\tau_y)K^{-1}T^{-1}U^{-1}V^{-1} \\
&= \frac{1}{2}V(K\tau_x \pm Ki\tau_y)K^{-1}V^{-1} \\
&= \frac{1}{2}V(\tau_x K \mp iK\tau_y)K^{-1}V^{-1} \\
&= \frac{1}{2}V(\tau_x K \mp i\tau_y^* K)K^{-1}V^{-1} \\
&= \frac{1}{2}V(\tau_x \pm i\tau_y)V^{-1} \\
&= V\tau_\pm V^{-1} = \tau_\pm
\end{aligned}$$

So only the τ_y would change into $-\tau_y$ under time-reversal transformation. And this is due to K, the complex conjugate operator. V operator in the isospin space is really just I_2, which is not necessarily shown in the full expression.

1.4.2 Evidence II

Both spin and isospin have the same matrix representation, Pauli Matrices. The most important thing that need to be conserved under time-reversal is the commutation relations,

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

Now we assert that every operator would be either even or odd under time-reversal. We could express this by $\Theta\hat{\mathcal{O}}\Theta^{-1} = \xi\hat{\mathcal{O}}$, where $\xi = \pm 1$. For each component of Pauli Matrices we say $\Theta S_i\Theta^{-1} = \xi_i S_i$. Consider the commutation relations,

$$\begin{aligned}
&\text{LHS: } \Theta[S_i, S_j]\Theta^{-1} = \Theta(S_iS_j - S_jS_i)\Theta^{-1} = \xi_i\xi_j[S_i, S_j] \\
&\text{RHS: } \Theta(i\epsilon_{ijk}S_k)\Theta^{-1} = -i\epsilon_{ijk}\Theta S_k\Theta^{-1} = -i\epsilon_{ijk}\xi_k S_k
\end{aligned}$$

To preserve the commutation relation, we require that

$$\xi_i\xi_j = -\xi_k \tag{18}$$

We can just list all the possibilities for there are not too much of them.

a: $\xi_i = 1, \xi_j = 1, \xi_k = -1$
b: $\xi_i = 1, \xi_j = -1, \xi_k = 1$
c: $\xi_i = -1, \xi_j = 1, \xi_k = 1$
d: $\xi_i = -1, \xi_j = -1, \xi_k = -1$

d is what we have for spin-1/2. If we require third component to be time even, only b and c are possible. And we can find out that b is what we did in the last subsection. But how about c? Remember we have two ways to construct rising and lowering operators, $\tau_\pm$ and $\tau^\pm = -\tau_\mp$. So b and c are just the same.

Another important thing is that for d, $\Theta^2 = -1$, which we have checked before. This implies the fermionic nature of the particle. But for a, b and c we have $\Theta^2 = 1$, which we also checked before. This would imply the bosonic behaviors. Though we know that proton and neutron are Fermions, it won't bother us for isospin is an additional degrees of freedom. And now we know that a SU(2) structure under time-reversal could have both possibilities.

One more thing to be mentioned is that there is not way to let $\Theta\vec{\tau}\Theta^{-1} = \vec{\tau}$. This is really a common mistake in some textbooks.

2 Some quantity under time-reversal

2.1 Two isospins

$$\Theta(\vec{\tau_1} \cdot \vec{\tau_2})\Theta^{-1} = \Theta(\tau_x^2 + \tau_y^2 + \tau_z^2)\Theta^{-1} = \vec{\tau_1} \cdot \vec{\tau_2} \tag{19}$$

$$\Theta(\vec{\tau_1} \pm \vec{\tau_2})\Theta^{-1} = (\tau_{1x} \pm \tau_{2x})\hat{x} \mp (\tau_{1y} \pm \tau_{2y})\hat{y} + (\tau_{1z} \pm \tau_{2z})\hat{z} \tag{20}$$

$$\begin{aligned} \Theta(\vec{\tau_1} \times \vec{\tau_2})\Theta^{-1} &= \Theta\left((\tau_{1y}\tau_{2z} - \tau_{1z}\tau_{2y})\hat{x} + (\tau_{1x}\tau_{2z} - \tau_{1z}\tau_{2x})\hat{y} + (\tau_{1x}\tau_{2y} - \tau_{1y}\tau_{2x})\hat{z}\right)\Theta^{-1} \\ &= (-\tau_{1y}\tau_{2z} + \tau_{1z}\tau_{2y})\hat{x} + (\tau_{1x}\tau_{2z} - \tau_{1z}\tau_{2x})\hat{y} + (-\tau_{1x}\tau_{2y} + \tau_{1y}\tau_{2x})\hat{z} \end{aligned} \tag{21}$$

$$\Theta[\vec{\tau_1}\vec{\tau_2}]_2^{zz}\Theta^{-1} = \Theta(2\tau_{1z}\tau_{2z} - \frac{2}{3}\vec{\tau_1} \cdot \vec{\tau_2})\Theta^{-1} = [\vec{\tau_1}\vec{\tau_2}]_2^{zz} \tag{22}$$