Halo notes

was (Threshold behavior of the proton-proton fusion S-factor)

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Abstract

We discuss the calculation of P-wave contribution to the proton-proton fusion S-factor. We define the partial wave decomposition of the S-factor and explain how to calculate the related cross section in the pionless effective field theory approach.

I. INTRODUCTION: THE PP-FUSION S-FACTOR

Proton-proton fusion is the process in which two protons fuse to a deuteron while the system emits a positron and a neutrino. It is the first process in the so-called pp chain which is the reaction network responsible for sun's energy generation.

The cross section for this process is very small due to the strong Coulomb repulsion between the two protons. The reaction cross section $\sigma(E)$ is therefore reexpressed in terms of the so-called S-factor

$$\sigma(E) = \frac{S(E)}{E} \exp(-2\pi\eta(E)) \ , \tag{1}$$

where $\eta(E)$ denotes the system dependent Sommerfeld parameter.

It is usually assumed that it is sufficient to calculate the S-factor and it's derivatives with respect to the energy at threshold ($E = 0$). In practive, however, the process is calculated at very small energies and to assume that the first two derivatives of the S-factor at threshold are sufficient to capture its energy dependence.

Recent publications [**?** **?**] have raised the question about the absolute size of the p-wave contribution to pp-fusion. Marcucci et al. calculated the energy-dependent S-factor including also p-waves.

Following the standard partial wave decomposition of the total cross section, we write down the partial wave decomposition of the S-factor

$$\sigma(E) = \sum_l \sigma_l(E) = \frac{\exp(-2\pi\eta)}{E} \sum_l S_l(E) \ , \tag{2}$$

which implies that we can relate partial cross section for pp-fusion out of a relative p-wave to the required S-factor piece.

The pionless EFT is perfectly suited for this process. It uses the minimal degrees of freedom required in this process and parametrizes results in terms of a few measurable parameters. Kong and Ravndal [**?**] were the first to consider pp-fusion within the pionless EFT and S-wave pp-fusion was succeedingly calculated up to 5th order by others.

In this paoer, we calculate the leading p-wave contributions to proton-proton fusion in the pionless EFT in order obtain a reliable estimate for the impact of p-waves on the total reaction.

II. THEORETICAL FOUNDATIONS

We use pionless effective field theory to calculate the pp-fusion S-factor. This EFT is constructed from nucleon fields and their derivatives alone

A. Pionless effective field theory

The Lagrangian for the pionless EFT at leading order can be written as

$$\mathcal{L} = N^\dagger(i\partial_t + \frac{\nabla^2}{2m})N + C_0(N^T P_i N)^\dagger (N^T P_i N) + \dots \ . \tag{3}$$

Here m denotes the nucleon mass and N the field that includes two two-component spinors for spin and isospin. The projector $P_i = \frac{1}{\sqrt{8}}\tau_2\sigma_2\sigma_i$ projects spins and isospins of two-nucleons on the Cartesian i component of deuteron state (spin-triplet and isospin-singlet). Its trace is

$$\mathrm{Tr}[P_i^\dagger P_j] = \frac{1}{2}\delta_{ij} \ . \tag{4}$$

The wave function renormalization factor is given

$$Z_d = \frac{8\pi\gamma}{m^2} \tag{5}$$

Chen, Rupak and Savage [**?**] seem to have a negative Z-factor, odd.

We follow Butler and Chen and include the weak interaction terms by defining the weak part of the Lagrangian as

$$\mathcal{L}_W = -\frac{G_F}{\sqrt{2}} l_+^\mu J_\mu^- + \mathrm{h.c} \tag{6}$$

The hadronic current has vector and axial vector parts,

$$J_\mu^- = V_\mu^- - A_\mu^- \ . \tag{7}$$

The leading part of the axial vector current piece

$$A_k^- = \frac{g_A}{2} N^\dagger \tau^- \sigma_k N + \dots \ , \tag{8}$$

where $\tau^- = \tau_1 - i\tau_2$ and the first subleading corrections arise from two-body operators.

B. Decomposition of the amplitude

We write the amplitude as the contraction of leptonic tensor l_+^ν and hadronic amplitude $\mathcal{A}_\nu$

$$i\mathcal{M} = i l_+^\nu \mathcal{A}_\nu \ , \tag{9}$$

and since we need to calculate the absolute square of the amplitude, we write

$$|\mathcal{M}|^2 = l_+^\mu \mathcal{A}_\mu \left(l_+^\nu \mathcal{A}_\nu\right)^* \ . \tag{10}$$

However, we want to calculate the partial cross section $\sigma_l^{\lambda\lambda'}$. Here, λ and λ' denote the spin projections of the two incoming protons. We do not have to worry about isospin since the isospin incoming and outgoing states are the same for all diagrams. The amplitude has of course the same structure and we can write is a

$$\left(\mathcal{A}_{lm,i}^{\lambda\lambda'}(E)\right)^\nu = \chi_\lambda^T[\mathrm{GT}](P_i^\dagger)\chi_{\lambda'}\mathcal{A}_{lm}(E) = (iV^\nu)^{\lambda\kappa}(P_i^\dagger)^{\kappa\lambda'}\mathcal{A}_{lm}(E) \ , \tag{11}$$

and we factorized the amplitude in a part that contains the spin algebra and another one that contains the loop integral. We wrote in the above equation a general vertex factor iV that has either the spin structure of the Gamow-Teller or the Fermi vertex. Now we can *square* the amplitude to obtain

$$\left(\mathcal{A}_{lm,i}^{\lambda\lambda'}(E)\right)^\nu (\mathcal{A}) \tag{12}$$

$$i\mathcal{A}_{k,1m'}^{\lambda\lambda'} = \langle d,\, \mathbf{k},\, i|J_k^-|p,\, 1m,\, \lambda\lambda'\rangle, \tag{13}$$

where k denotes the Cartesian index of the deuteron spin projection. We are interested in the p-wave contribution to the S-factor and we set therefore $l = 1$.

$$i\mathcal{A}_{1m,1m'}^{\lambda\lambda'} = \langle d,\, \mathbf{k},\, i|J_k^-|p,\, lm,\, \lambda\lambda'\rangle. \tag{14}$$

where it's important to note that $\mathcal{A}(E)$ is a c-number. In order to calculate the cross section, we need to calculate the absolute square of these amplitudes.

$$(\mathcal{A}^*)_{lm,i}^{\lambda'\lambda}\,\mathcal{A}_{lm,i}^{\lambda\lambda'} \ . \tag{15}$$

For the above example of the Gamow-Teller current we get therefore

$$(P_i)_{\lambda'\kappa}[(\mathrm{GT})^\dagger]^{\kappa\lambda}[\mathrm{GT}]_{\lambda\kappa}(P_i^\dagger)_{\kappa\lambda'} \tag{16}$$

which with averaging over the incoming spins becomes

$$\frac{1}{4}\,\mathrm{Tr}[P_i(GT)^\dagger(GT)P_i^\dagger] = \frac{1}{4}2 \tag{17}$$

FIG. 1. Diagrams required for the calculation of momentum density n_1.

1. *Matrix elements of the Gamow-Teller operator*

There are two types of matrix elements that we need to consider

$$\langle ppS = 0|\sigma_i|d, k\rangle \longrightarrow \mathrm{Tr}[\bar{P}\sigma_i P_k^\dagger] = 2\delta_{ik} \tag{18}$$

2. *Fermi operator*

$$\langle ppS = 0|1|d, k\rangle \longrightarrow \mathrm{Tr}[\bar{P}\sigma_i P_k^\dagger] \tag{19}$$

$$\tag{20}$$

III. THE PP-FUSION AMPLITUDE

We calculate the contribution to proton-proton fusion from the channel that has incoming angular momentum $l = 1$. At leading order, we have only a small number of diagrams to consider for this process.

- The leading order Gamow-Teller contribution
- higher order contributions due to

A. Gamow-Teller contribution

The Gamow-Teller contribution arises from the leading order charge changing weak axial vector Lagrangian that is given by

$$A_k^- = \frac{g_A}{2} N^\dagger \tau^- \sigma_k N \ , \tag{21}$$

where $\tau^- = \tau_1 - i\tau_2$.

The amplitude is then usually parametrized in the form [**?**]

$$|\langle d; i|A_k^-|pp\rangle = g_A C_\eta \sqrt{\frac{32\pi}{\gamma^3}} \Lambda(p)\delta_{kj} \ , \tag{22}$$

Note that Kong & Ravndal [**?**] have as the relation between the amplitude

$$|\mathcal{T}_{fi}(p)| = \sqrt{\frac{8\pi C_\eta^2}{\gamma^3}} \Lambda(p) \tag{23}$$

On the surface this definition seems to be different by a factor 3 and also doesn't include the factor of g_A used in the Butler & Chen expression.

B. NLO one-body contributions

At NLO we get terms in the Lagrangian with the structure

$$N^\dagger i \frac{\overleftrightarrow{\nabla}}{2m} N \longrightarrow \frac{i}{2m}(\mathbf{p}_{\mathrm{in}} + \mathbf{p}_{\mathrm{out}}) \ , \tag{24}$$

where the right-hand

C. Leading order Gamow-Teller and Fermi diagrams

Let us discuss here the amplitude part $\mathcal{A}$ with spin and isospin structures factored out.

The diagram that contains the Coulomb t-matrix gives through application of the Feynman rules

$$
\begin{aligned}
i\mathcal{A}_1 = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, i\left[q_0 + E/2 - \frac{q^2}{2m} + i\epsilon\right]^{-1} \\
\times\, i\left[-q_0 + E/2 - \frac{q^2}{2m} + i\epsilon\right]^{-1} t_C(E,p,q)\, i\left[-q_0 + E - \kappa_0 - \frac{(\mathbf{q}-\mathbf{k})^2}{2m} + i\epsilon\right]^{-1} \ . \quad (25)
\end{aligned}
$$

where $E = p^2/m$ denotes the total energy of the two incoming protons. Doing the contour integration gives

$$
i\mathcal{A}_1 = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left[E - \frac{q^2}{m} + i\epsilon\right]^{-1} t_c i\left[E - \kappa_0 - \frac{q^2}{2m} - \frac{(\mathbf{q}-\mathbf{k})^2}{2m} + i\epsilon\right]^{-1} \ . \tag{26}
$$

We also have the integral without any Coulomb exchanges and that gives

$$
i\mathcal{A}_2 = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right) i\left[\frac{p^2}{2m} - \kappa_0 - \frac{(\mathbf{p}-\mathbf{k})^2}{2m} + i\epsilon\right]^{-1} \ . \tag{27}
$$

We can combine the two amplitudes such that we end up with an integral that involves the Coulomb wave function. We do this by writing the second term as an integral over some momentum and including an additional δ-function

$$
\begin{aligned}
i\mathcal{A}_{\mathrm{LO}} = \sqrt{Z_D}\left(i\frac{g_A G_V}{\sqrt{2}}\right)\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left[(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) + \frac{t_C}{E - \frac{q^2}{2m} + i\epsilon}\right] \\
\times\, i\left[E - \kappa_0 - \frac{q^2}{2m} - \frac{(\mathbf{q}-\mathbf{k})^2}{2m} + i\epsilon\right]^{-1} \ . \quad (28)
\end{aligned}
$$

We can not insert the wave function after using the identity

$$
\chi_{\mathbf{p}}^{+}(\mathbf{q}) = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) + \frac{t_C}{E - \frac{q^2}{2m} + i\epsilon} \ . \tag{29}
$$

This gives

$$
i\mathcal{A}_{\mathrm{LO}} = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\chi_{\mathbf{p}}^{(+)}(\mathbf{q})\, i\left[E - \kappa_0 - \frac{q^2}{2m} - \frac{(\mathbf{q}-\mathbf{k})^2}{2m} + i\epsilon\right]^{-1} \tag{30}
$$

We convert the expression into coordinate space by using the identities

$$
\chi_{\mathbf{p}}^{(+)}(\mathbf{q}) = \int \mathrm{d}^3 r e^{i\mathbf{q}\cdot\mathbf{r}}\tilde{\chi}_{\mathbf{p}}^{(+)}(\mathbf{r}) \ , \tag{31}
$$

for the Fourier transform of the Coulomb wave function, and

$$
im\left[m(E-\kappa_0) - q^2 - \frac{k^2}{2} - \mathbf{q}\cdot\mathbf{k} + i\epsilon\right] = -m\int \mathrm{d}^3 r\, e^{-i\mathbf{r}\cdot(\mathbf{q}-\frac{\mathbf{k}}{2})}\frac{e^{-\gamma r}}{4\pi r} \ , \tag{32}
$$

This leads to

$$i\mathcal{A}_{\mathrm{LO}} = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)(-m)\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\mathrm{d}^3 r \mathrm{d}^3 r' \exp(i\mathbf{q}\cdot\mathbf{r}')\tilde{\chi}^{(+)}_{\mathbf{p}}(\mathbf{r}')$$
$$\times \exp\left(-i\mathbf{r}\cdot(\mathbf{q}-\frac{\mathbf{k}}{2})\right)\frac{\exp(-\gamma r)}{4\pi r} \ . \quad (33)$$

The momentum space integral can be removed and gives a δ-function

$$i\mathcal{A}_{\mathrm{lo}} = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)(-m)\int \mathrm{d}^3 r \mathrm{d}^3 r' \exp(-i\mathbf{r}\cdot\frac{\mathbf{k}}{2})\delta^{(3)}(\mathbf{r}-\mathbf{r}')\frac{\exp(-\gamma r)}{4\pi r}\tilde{\chi}^{(+)}_{\mathbf{p}}(\mathbf{r}') \ . \quad (34)$$

Integrating out the δ-function leaves us with

$$i\mathcal{A}_{\mathrm{lo}} = \sqrt{Z_d}\left(i\frac{g_A G_V}{\sqrt{2}}\right)(-m)\int \mathrm{d}^3 r \exp(-i\mathbf{r}\cdot\frac{\mathbf{k}}{2})\frac{\exp(-\gamma r)}{4\pi r}\tilde{\chi}^{(+)}_{\mathbf{p}}(\mathbf{r}) \ . \quad (35)$$

The above expression was derived without a partial wave projected initial state. Working with a partial wave projected initial state is equivalent to using an integral over the angle

$$a = \quad (36)$$

Working with partial wave projected states will lead to

$$i\mathcal{A}_{1,\tilde{m}} = \frac{-mC_A}{(2\pi)^3}\int \mathrm{d}\Omega_{\mathbf{p}} Y_{1\tilde{m}}(\hat{\mathbf{p}})\int \mathrm{d}^3 r e^{-i\mathbf{r}\cdot\frac{\mathbf{r}}{2}}\frac{e^{-\gamma r}}{4\pi r}\chi^{(+)}_{\mathbf{p}}(\mathbf{r}) \ . \quad (37)$$

Now we can carry out a partial wave expansion for the exponential (current operator) and for the Coulomb wave function We get

$$i\mathcal{A}_{1,\tilde{m}} = imC_A\frac{(4\pi)^2}{(2\pi)^3}e^{i\sigma_1}\int \mathrm{d}r\, r^2\frac{F_1}{pr}\frac{\exp(-\gamma r)}{4\pi r}j_1(\frac{kr}{2})Y_{1\tilde{m}}(\hat{\mathbf{k}}) \ . \quad (38)$$

IV. CONCLUSION

We have calculated the leading p-wave contribution to proton-proton fusion.[**?**]

Appendix A: Spin projectors

It's a relevant question how the spin and isospin projectors in our formalism relate to the analogous coupling of spins with Clebsch Gordans. A quantum mechanical spin-triplet state is written as (let's ignore the momentum part for a moment)

$$|S=1,m_s\rangle = \sum_{\lambda,\lambda'} C(\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda',1m_s)|\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda'\rangle \quad (A1)$$

The Clebsch-Gordan coefficients can be viewed as the entries of a matrix that carries an additional *spherical* index m_s. We extract 3 matrices

$$P_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{i}{\sqrt{2}}\sigma_2\sigma_- \ , \tag{A2}$$

$$P_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{i}{\sqrt{2}}\sigma_2\sigma_3 \ , \tag{A3}$$

$$P_{+1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{i}{\sqrt{2}}\sigma_2\sigma_+ \ , \tag{A4}$$

We define the raising and lowering operators

$$\sigma_\pm = \mp\frac{1}{\sqrt{2}}(\sigma_1 \pm i\sigma_2) \ . \tag{A5}$$

I can test these results by simply starting out from the Cartesian projection matrices

$$P_i = \frac{i}{\sqrt{2}}\sigma_2\sigma_i \ . \tag{A6}$$

where i is a Cartesian index. We can the calculate the spherical components by setting

$$-\frac{i}{\sqrt{2}}(P_1 + iP_2) = P_{+1} \ , \tag{A7}$$

$$\frac{i}{\sqrt{2}}(P_1 - iP_2) = P_{-1} \ , \tag{A8}$$

$$P_3 = -P_0 \tag{A9}$$

We find that we still have a sign difference in the z-component. It is not clear to me whether this difference matters.

An additional factor of $1/\sqrt{2}$ needs to be included since

1. Comments on spin traces

The spins of the two initial nucleons are either coupled to a spin-singlet or spin-triplet state, the leading weak operators either have a 1 or a σ_i in spin-space, and the deuteron is in spin-triplet state. Note also that we only have one specific trace to evaluate in isospin space

$$o\,\mathrm{Tr}[\tau]\sigma \tag{A10}$$

Appendix B: The normalization of states and factors for external lines

We want to calculate the partial cross section for pp-fusion into a deuteron state d with polarization i. This means that we calculate the matrix element

$$\langle d\, i|\mathbf{J}^{-}|(\mathbf{P}, k, lm)\rangle \ , \tag{B1}$$

where $\mathbf{P}$ denotes the center of mass momentum, k the relative momentum, l the partial wave and m the corresponding magnetic quantum number. The normalization of the partial wave projected states is relevant if a cross section is supposed to be calculated from such a matrix element.

For example Ref. [**?**] gives the normalization of partial wave projected states in the following form

$$|(\ P, k, lm, \lambda\lambda')\rangle = \frac{k}{(2\pi)^3}\int \mathrm{d}\Omega_k Y_{lm}(\hat{\mathbf{k}}) \int N^{\dagger}_{\lambda}(\mathbf{P}/2+k)N^{\dagger}_{\lambda'}(\mathbf{P}/2-k)|0\rangle \tag{B2}$$

These states are normalized such that the overlap of two different such states gives delta functions. However, when we try to calculate the cross section with these states, the normalization leads to unwanted side effects. Note that Gloeckle's partial wave states are normalized such that

$$\langle plm|p'l'm'\rangle = \frac{\delta(p-p')}{pp'}\delta_{ll'}\delta_{mm'} \ , \quad \langle \mathbf{p}|p'lm\rangle = Y_{lm}(\hat{\mathbf{p}})\frac{\delta(p-p')}{pp'} \ . \tag{B3}$$

For example, the cross section for elastic 2-body scattering is calculated using

$$\sigma(E) = \frac{1}{v}|\mathcal{A}|^2\Phi_2 \tag{B4}$$

In the case of scattering, we know that we can calculate the total cross section using the formula

$$\sigma(E) = \int \tag{B5}$$

Appendix C: Coulomb functions

The Coulomb wave function can be expressed in coordinate space as

$$\Psi_{\mathbf{p}}(\mathbf{r}) = \frac{1}{\rho}\sum_{l=0}^{\infty}(2l+1)i^l e^{i\sigma_l}F_l(\rho)P_l(\cos\theta) \ , \tag{C1}$$