Checking the Adequacy of Quantile Regression Models

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December 24, 2024

Abstract

We propose a new class of tests for the correct specification of parametric quantile regression models over a continuum of quantile levels. The test is based on a cumulative sum process of the gradient vector with respect to uni-dimensional linear projections of the covariates. Besides, we show that the use of an orthogonal projection on the tangent space of nuisance parameters eliminates the estimation effect and facilitates the computation of critical values via a multiplier bootstrap procedure. We investigate the asymptotic behavior of the proposed test statistics under the null hypothesis, fixed alternatives and a sequence of local alternatives with the rate of $n^{-1/2}$ converging to the null. Simulation studies show good performance under finite samples. A real data example is also included to illustrate the application of the proposed method.

Keywords: Dimension Reduction, Lack-of-Fit, Multiplier Bootstrap, Projection, Vector-Weighted Cusum Process

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Financial support from the National Natural Science Foundation of China (Grant Numbers 72373007 and 72333001) is gratefully acknowledged.

1 Introduction

Since the seminal work of Koenker and Bassett (1978), quantile regression (QR) has been widely applied in empirical studies due to its ability to document the heterogeneous effect of covariates and robustness to outliers in the response measurements. The general quantile regression model is given by

$$Y = m_\tau(X) + \epsilon_\tau, \text{ for all } \tau \in \mathcal{T}, \tag{1}$$

where Y and $X = (X_1, \ldots, X_d)^\top$ denote the scalar response variable and a vector of d-dimensional covariates, respectively, and $\mathcal{T}$ is compact subset of $(0, 1)$. In addition, $m_\tau(X)$ represents the conditional quantile regression function at the τ quantile and ϵ_τ is the quantile error satisfying $\mathbb{P}(\epsilon_\tau \leq 0|X) = \tau$ almost surely (a.s.) for all $\tau \in \mathcal{T}$.

This paper considers testing whether some parametric specification for $m_\tau(X)$ is adequate in (1). The null hypothesis is

$$H_0 : \mathbb{E}\left[\mathbb{1}\left(Y \leq m(X, \theta_0(\tau))\right) - \tau|X\right] = 0 \text{ a.s. for some } \theta_0 \in \mathcal{B} \text{ and some } \tau \in \mathcal{T}, \tag{2}$$

where $m(\cdot, \theta(\cdot))$ represents a pre-specified parametric quantile regression function, and $\mathcal{B}$ is a family of uniformly bounded functions from $\mathcal{T}$ to $\Theta \in \mathbb{R}^p$. The alternative hypothesis is the negation of the null. If the null hypothesis cannot be rejected, we can rely on some parametric quantile regression models to fit the data. Compared with the nonparametric QR models, parametric models can have better estimation efficiency and interpretation. However, if this is not true, the conclusions based on the assumed parametric regression models can be misleading. In sum, it is important to conduct testing procedures to ascertain whether the hypothetical parametric models are adequate.

Testing for the validity of parametric quantile regression models against unspecified alternatives have been developed in lots of literature. However, the analysis is mostly limited to a fixed quantile, see, e.g., Zheng (1998), Bierens and Ginther (2001), Horowitz and Spokoiny (2002), He and Zhu (2003), Whang (2006a,b), Otsu (2008), Horowitz and Lee (2009), Conde-Amboage et al. (2015), and Horvath et al. (2022).

In the classical setup, Stute (1997) proposed a model checking framework based on the cusum process of the residuals, which converges to a centered Gaussian process. This framework, in the context of a fixed quantile, is that

$$R_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n)\right)\right) \mathbb{1}\left(X_i \leq x\right), \tag{3}$$

where $x \in \Pi$ with Π a suitable subset in $\mathbb{R}^d$, $\hat{\theta}_n$ is a $\sqrt{n}$-consistent estimator of θ_0 derived via quantile "check function". This approach transfers the conditional moment restriction of (2) to equivalently infinite number of unconditional restrictions for the construction of empirical process, which is also proposed or applied by Bierens (1982); Bierens and Ploberger (1997); Bierens and Wang (2012), Stute et al. (1998); Stute and Zhu (2002), Escanciano (2006) and so on beyond the literature mentioned before.

However, as noted in Eubank and Hart (1992) and He and Zhu (2003), the uni-dimensional cusum process (3) is sensitive to certain alternatives but tends to have low power against departures in some forms of oscillation around the null. Therefore, He and Zhu (2003) updated the cusum process by using all components in the gradient equation, that is,

$$\tilde{R}_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{m}(X_i, \hat{\theta}_n) \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n)\right)\right) \mathbb{1}\left(X_i \leq x\right), \tag{4}$$

where $\dot{m}(X, \theta) := \partial m(X, \theta)/\partial \theta$ denotes the gradient of $m(X, \theta)$. Note that process (4) is a $(p \times 1)$-dimensional vector. More details are discussed in Section 2.1.

In addition, another drawback of process (3) as well as (4) is that, the weight function $\mathbb{1}(X \leq x)$ suffers from "curse of dimensionality". As also mentioned in Bierens (1990), Bierens and Wang (2012), Escanciano (2006), etc., the power of test statistics built on the original indicator function increases relatively slower than those with the characteristic of dimension reduction as it may meet information loss when the dimensionality of covariates is large or even moderate. To alleviate this issue, we adopt a class of weight functions with the property of dimension-reduction, denoted as $w(X, x)$. Then, we have a further modified vector of process based on (4), that is,

$$\breve{R}_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{m}(X_i, \hat{\theta}_n) \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n)\right)\right) w(X_i, x). \tag{5}$$

However, the discussion above is the case of fixed quantile. So far, little attention has been paid towards a more general case of continuum of quantiles, and there are only Escanciano and Velasco (2010) for the specification of dynamic conditional quantiles, Rothe and Wied (2013) for a class of conditionally distributional models and Escanciano and Goh (2014) for checking the linearity of quantile regression models. We follow the setup of Escanciano and Goh (2014) and rewrite the process of (5) in the context of continuum quantiles as:

$$\breve{R}_n(x, \tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(X_i, Y_i, \hat{\theta}_n(\tau)) w(X_i, x), \tag{6}$$

where we denote $\psi(X, Y, \theta(\tau)) := \dot{m}(X, \theta(\tau)) \left(\tau - \mathbb{1}\left(Y \leq m(X, \theta(\tau))\right)\right)$ for notational simplicity. In order to elicit another contribution of this paper, we provide the following decomposition of process (6) under the null hypothesis and some mild regularity conditions including the $\sqrt{n}$ consistency of $\hat{\theta}_n(\tau)$, that is,

$$
\begin{aligned}
\breve{R}_n(x,\tau) =& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(X_i,Y_i,\theta_0(\tau))w(X_i,x) \\
&- \mathbb{E}\left[f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}(X,\theta_0(\tau))^\top w(X,x)\right]\sqrt{n}\left(\hat{\theta}_n-\theta_0\right)+o_p(1),
\end{aligned}
\tag{7}
$$

uniformly in $(x,\tau)\in\Pi\times\mathcal{T}$, where $f(y|X)$ denotes the conditional density function of Y given X, evaluated at $Y=y$. The second component at the right-hand side of (7) is called "parameter estimation effect" due to the non-ignorable term, $\sqrt{n}||\hat{\theta}_n-\theta_0||=\mathcal{O}_p(1)$. The presence of preliminary parameter estimation effect complicates the limiting null distribution of process $\breve{R}_n(x,\tau)$ in (6) and the approximation of the corresponding critical values.

To overcome this problem, we consider a transformation on the weight function $w(X,x)$ such that the transferred second component at the right-hand side of (7) can be equal to zero, and thus eliminated. This idea was firstly proposed by Neyman (1959) in the context of fully parametric models, that is, we acknowledge our lack of knowledge about parameter, which however, can be accounted for by an orthogonal projection of a certain weight function into the so-called tangent space of nuisance parameter. For this body of literature on specification testing equipped with such projection approach, see Escanciano (2009b), Zhu et al. (2017), Sun et al. (2017), Sun et al. (2018), Sant'Anna and Song (2019), and Sant'Anna and Song (2020), among others. Besides, Bickel et al. (2006) extended this idea to the semiparametric context, and Escanciano and Goh (2014) proposed a nonparametric projection operator.

As a result, we establish a "vector-weighted dimension-reduced projection-based" cusum process in the context of continuum of quantiles. In addition, we also introduce a Cramér-von Mises (CvM) type test statistic with two different norms. To the best of our knowledge, there is not such literature concerning about all the issues above with generalized continuum of quantiles. Besides, as we illustrate subsequently, several commonly used resampling techniques for the inference of critical values under the fixed quantile is not valid anymore under a continuum of quantiles, while the multiplier bootstrap procedure is feasible and even can be facilitated due to our proposed projection. More specifically, the idea of multiplier bootstrap is to obtain the first order (uniform) expansion of an empirical process in order to mimic its asymptotic behavior by multiplying the estimated expansion with mean zero, unit variance and path independent random vectors. Since this procedure does not generate new samples and require estimations in each bootstrap repetition, it is fast and easy-to-implement. However, one premise is that we need to obtain the expansion of process which is not complicated either. And our projection equipped on process (6) exactly satisfies this condition due to the elimination of parameter estimation effect.

The remainder of this paper is organized as follows. Section 2 presents our testing framework and constructs corresponding CvM-type test statistics. Section 3 illustrates the asymptotic null distributions and the power properties of our tests under the fixed and sequence of local alternatives, respectively. Section 4 introduces a convenient multiplier bootstrap procedure for the computation of critical values. Section 5 contains a series of simulations under finite samples and an empirical application for checking the validity of quantile regression models. Section 6 concludes. All proofs are gathered into the Appendices.

The following notation is adopted throughout this paper. For a generic matrix A, let $A^\top$, $||A||_1$, $||A||_E$ and $||A||_F$ be the transpose, L_1 norm, Euclidean norm and Frobenius norm of A, respectively. Denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{1}(X \in \mathcal{A})$ be the indicator function which is equal to 1 if $X \in \mathcal{A}$, and 0 otherwise. For a generic set $\mathcal{G}$, let $l^\infty(\mathcal{G})$ be the Banach space of all uniformly bounded real functions on $\mathcal{G}$. Let "$\Rightarrow$" be the weak convergence on $(l^\infty(\mathcal{G}), \mathcal{B}_\infty)$, where $\mathcal{B}_\infty$ represents the corresponding Borel σ-algebra. Denote "$\xrightarrow{\mathcal{P}}$" and "$\xrightarrow{\mathcal{D}}$" as convergence in probability and convergence in distribution, respectively.

2 Testing Setup

2.1 A Modified Empirical Process

Let $\rho_\tau(\epsilon) = (\tau - \mathbb{1}(\epsilon \leq 0))\epsilon$. Under H_0, the QR vector $\theta_0(\tau)$ solves the population minimization problem $\theta_0(\tau) := \arg\min_{\theta\in\Theta} \mathbb{E}[\rho_\tau(Y - m(X, \theta))]$, assuming integrability and uniqueness of the solution. Let $\psi(W_i, \theta(\tau)) = \dot{m}(X_i, \theta(\tau))(\tau - \mathbb{1}(Y_i \leq m(X_i, \theta(\tau))))$, which is one of the subgradient of $\rho_\tau(Y_i - m(X_i, \theta))$ with respect to θ. He and Zhu (2003) propose a test based on the cusum process of the gradient vector

$$\breve{R}_n(x, \tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{m}(X_i, \hat{\theta}_n(\tau)) \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n(\tau))\right)\right) w(X_i, x). \tag{8}$$

Then, we obtain the partial subgradient (derivative) of ρ_τ with the residuals of $Y_i - m(X_i, \hat{\theta}_n(\tau))$, that is, a p-dimensional vector $\psi(X_i, Y_i, \hat{\theta}_n(\tau))$. As a result, the sample analog of empirical process can be built as (6) with the uniform decomposition of (7). One advantage of (6) over the traditional cusum process of

$$\bar{R}_n(x, \tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n(\tau))\right)\right) w(X_i, x),$$

is that the partial subgradient of quantile check function of (**??**) under the null hypothesis is equal to 0 only if $\theta = \theta_0$, and non-zero otherwise. Hence, it is able to be more sensitive to detect the departures from null hypothesis. As a matter of fact, this idea has been studied in the literature, for example, Qu (2008) constructed the subgradient test statistic in quantile regression for testing the structural changes. Zhang et al. (2014) proposed a score function-based test for checking the jumping threshold effect in the threshold models.

Next, we turn to solve the obstacle from parameter estimation effect. Consider the following infeasible projection operator

$$\mathcal{P}_\tau w(X,x) = w(X,x)I_p - G^\top(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))g(X,\theta_0(\tau)), \tag{9}$$

where I_p is the identity matrix of dimension p,

$$g(X,\theta_0(\tau)) = f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}^\top(X,\theta_0(\tau)), \tag{10}$$

and

$$\Delta(\theta_0(\tau)) = \mathbb{E}[g(X,\theta_0(\tau))g^\top(X,\theta_0(\tau))],\ G(x,\theta_0(\tau)) = \mathbb{E}[g(X,\theta_0(\tau))w(X,x)].$$

We can see that $G^\top(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))g(X,\theta_0(\tau))$ is the best linear predictor of $w(X,x)$ given $g(X,\theta_0(\tau))^\top$, and thus, for all x and τ,

$$\mathbb{E}\left[\mathcal{P}_\tau w(X,x)g^\top(X,\theta_0(\tau))\right] = G^\top(x,\theta_0(\tau)) - G^\top(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))\Delta(\theta_0(\tau)) = 0.$$

Therefore, we turn to a further modified cusum process for the continuum of quantiles,

$$\hat{R}_n(x,\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{P}_{n,\tau}w(X_i,x)\psi(X_i,Y_i,\hat{\theta}_n(\tau)), \tag{11}$$

where $\mathcal{P}_{n,\tau}w(X,x)$, the sample analog of $\mathcal{P}_\tau w(X,x)$ in (9), is defined as:

$$\mathcal{P}_{n,\tau}w(X_i,x) = w(X_i,x)I_p - \hat{G}^\top(x,\hat{\theta}_n(\tau))\hat{\Delta}^{-1}(\hat{\theta}_n(\tau))\hat{g}(X_i,\hat{\theta}_n(\tau)), \tag{12}$$

where

$$\hat{g}(X_i,\hat{\theta}_n(\tau)) = \hat{f}(m(X_i,\hat{\theta}_n(\tau))|X_i)\dot{m}(X_i,\hat{\theta}_n(\tau))\dot{m}^\top(X_i,\hat{\theta}_n(\tau)),$$

and

$$\hat{\Delta}(\hat{\theta}_n(\tau)) = \frac{1}{n}\sum_{i=1}^n \hat{g}(X_i,\hat{\theta}_n(\tau))\hat{g}^\top(X_i,\hat{\theta}_n(\tau)),\ \hat{G}(x,\hat{\theta}_n(\tau)) = \frac{1}{n}\sum_{i=1}^n \hat{g}(X_i,\hat{\theta}_n(\tau))w(X_i,x).$$

In addition, as we illustrate in Theorem 3.1, we can obtain the decomposition of process (11) as:

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\hat{R}_n(x,\tau) - R_n(x,\tau)\right| = o_p(1), \tag{13}$$

where

$$R_n(x,\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathcal{P}_\tau w(X_i,x)\psi(X_i,Y_i,\theta_0(\tau)). \tag{14}$$

As we can see, the process $\hat{R}_n(x,\tau)$ is asymptotically invariant to $\hat{\theta}_n(\tau)$. This also enables a convenient multiplier bootstrap procedure for the computation of critical values. More details are shown in Section 4.

Remark 2.1 *In the above discussion, we have used the full vectors of gradient $\dot{m}(X,\theta_0(\tau))$ to construct test statistics. It is also possible to only use a subvector of $\dot{m}(X,\theta_0(\tau))$. Denote $\dot{m}_s(X,\theta_0(\tau))$ the subvector, whose dimension is p_s with $1 \le p_s \le p$. In this case, similar to (9), the infeasible projection operator is given by*

$$\mathcal{P}_{\tau,s} w(X,x) = w(X,x)I_{p_s} - G_s^\top(x,\theta_0(\tau))\Delta_s^{-1}(\theta_0(\tau))g_s(X,\theta_0(\tau)),$$

where I_{p_s} is the identity matrix of dimension p_s,

$$g_s(X,\theta_0(\tau)) = f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}_s^\top(X,\theta_0(\tau)),$$

and

$$\Delta_s(\theta_0(\tau)) = \mathbb{E}[g_s(X,\theta_0(\tau))g_s^\top(X,\theta_0(\tau))],\ G_s(x,\theta_0(\tau)) = \mathbb{E}[g_s(X,\theta_0(\tau))w(X,x)].$$

Similar to (12), $\mathcal{P}_{n,\tau,s}w(X,x)$, the feasible version of $\mathcal{P}_{\tau,s}w(X,x)$, can be obtained readily. Then it is sufficient to study the following process

$$\hat{R}_{n,s}(x,\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathcal{P}_{n,\tau,s} w(X_i,x)\dot{m}_s(X_i,\hat{\theta}_n(\tau)) \left(\tau - \mathbb{1}\left(Y_i \le m(X_i,\hat{\theta}_n(\tau))\right)\right),$$

which may direct powers to specific alternatives. For simplicity, we do not focus on $\hat{R}_{n,s}(x,\tau)$.

2.2 Conditional Density Estimation

We consider a consistent estimator for $f(m(X,\theta(\tau))|X)$ in (10) with characteristic of dimension reduction (Escanciano and Goh (2019)). Instead of the direct estimation on X_i, Escanciano and Goh (2019) turned to exploit the behavior of fitted conditional τ_r for $r = 1,\dots,m$ over a range of discretized quantiles from a uniform distribution in the subset of $(0,1)$. Explicitly speaking, we have the following identity of distribution function

$$F(y|X_i) \equiv \tau_0 + \int_{\mathcal{T}} \mathbb{1}\left(F^{-1}(\tau|X_i) \le y\right) d\tau,\ \text{for } F(y|X_i) \in \mathcal{T} \subset (0,1),$$

which suggests a smooth approximation of the indicator function to obtain the new type of estimator for conditional density function, that is,

$$f(y|X_i) = \frac{|\mathcal{T}|}{h}\mathbb{E}\left[K\left(\frac{y - F^{-1}(\tau|X_i)}{h}\right)\Big|X_i\right] + o_p(1),$$

where $|\mathcal{T}| = \tau_1 - \tau_0$ denotes the length of quantile interval. $K(\cdot)$ is a smoothing kernel satisfying some mild regularity conditions, h is bandwidth that can approximate $f(y|X_i)$ as $h \to 0$, and $\tau|X_i$ is uniformly distributed on $\mathcal{T}$.

Note that for the simplification of computation, we uniformly discretize τ with m_n-grid. Then, under the null hypothesis, we have the estimator of $f(m(X,\theta(\tau))|X)$ as:

$$\hat{f}(m(X_i, \hat{\theta}_n(\tau))|X_i) = \frac{|\mathcal{T}|}{m_n h_m}\sum_{r=1}^{m} K\left(\frac{m(X_i, \hat{\theta}_n(\tau_r)) - m(X_i, \hat{\theta}_n(\tau))}{h_m}\right), \tag{15}$$

where m_n and h_m satisfies $m_n \to \infty$ and $h_m \to 0$ as $n \to \infty$, respectively.

Compared with classical conditional density estimators, the convergence rate of estimator in the perspective of quantiles is free of dimensionality. On the other hand, it does not have random denominators which leads to the robustness of bandwidth choice as well.

2.3 Test Statistics

Based on the process of (11), we proceed to investigate our tests based on CvM-type functionals with two different norms: Euclidean and Frobenius norm. Denote test statistics as:

$$CvM_n = \int_{\Pi\times\mathcal{T}} ||\hat{R}_n(x,\tau_r)||^2\Psi(dx, d\tau), \tag{16}$$

where $||\cdot||$ can be $||\cdot||_E$ or $||\cdot||_F$, $\Psi(dx)$ is some feasible integrating measures with respect to Π and $w(X,x)$, that is, from an implementational point of view, the choice of $\Psi(dx)$ is to obtain the closed form expression of test statistics and ease the computation. For integration of quantiles, we generate $\{\tau_r\}_{r=1}^{m}$ in the estimation of (15), where m is increasing with n. Then, we obtain the practical version of test statistics,

$$CvM_n^{\mathrm{E}} = \lambda_{\max}\left(\frac{1}{m}\sum_{r=1}^{m}\int_{\Pi}\hat{R}_n(x,\tau_r)\hat{R}_n^{\top}(x,\tau_r)\Psi(dx)\right), \tag{17}$$

and

$$CvM_n^{\mathrm{F}} = \mathrm{trace}\left(\frac{1}{m}\sum_{r=1}^{m}\int_{\Pi}\hat{R}_n(x,\tau_r)\hat{R}_n^{\top}(x,\tau_r)\Psi(dx)\right), \tag{18}$$

where λ_{max} and trace denote the largest eigenvalue and trace of matrix, respectively. Note that these two test statistics can be asymptotically equivalent to (16), see, e.g., Chang (1990), Escanciano and Goh (2014).

We first consider the linear indicator weight function, $w(X,x) = \mathbb{1}\left(\beta^\top X \le x\right)$, where $\beta \in \mathbb{S}^d := \left\{\beta \in \mathbb{R}^d : ||\beta||_E = 1\right\}$, with the integrating measures as $\Psi^{\text{lin}}(dx) = F_{n,\beta}(dx)d\beta$, where $F_{n,\beta}(x) = n^{-1}\sum_{i=1}^n \mathbb{1}\left(\beta^\top X_i \le x\right)$ is the estimation of $P(\beta^\top X \le x)$, and $d\beta$ is the rescaled uniform density of $\mathbb{S}^d$. The integral is over $\Pi \times \mathbb{S}^d$.

Then, we have the following closed form expression of CvM-type test statistics as:

$$CvM_n^{\text{lin,E}} = \lambda_{\text{max}}(\mathcal{B}_n) \text{ and } CvM_n^{\text{lin,F}} = \text{trace}(\mathcal{B}_n), \tag{19}$$

where $\mathcal{B}_n$ is a p-dimensional square matrix with the expression as:

$$\mathcal{B}_n = \frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n \xi_i(\tau_r)B_{ij}\xi_j^\top(\tau_r), \tag{20}$$

with

$$\xi_i(\tau_r) = \psi(X_i, Y_i, \hat{\theta}_n(\tau_r)) - \hat{g}^\top(X_i, \hat{\theta}_n(\tau_r))\hat{\Delta}^{-1}(\hat{\theta}_n(\tau_r))\frac{1}{n}\sum_{s=1}^n \hat{g}(X_s, \hat{\theta}_n(\tau_r))\psi(X_s, Y_s, \hat{\theta}_n(\tau_r)), \tag{21}$$

a p-dimensional vector and

$$B_{ij} = \sum_{v=1}^n \left(\pi - \frac{(X_i - X_v)^\top(X_j - X_v)}{||X_i - X_v||_E||X_j - X_v||_E}\right)\frac{\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}, \tag{22}$$

respectively, where the first component on the left-hand side is the complementary angle between vectors $X_i - X_v$ and $X_j - X_v$ measured in radians.

Next, we turn to the exponential weighting function $w(X,x) = \exp\{\mathrm{i}x^\top X\}$ with $\mathrm{i} = \sqrt{-1}$ the imaginary unit and Gaussian-type and Laplace-type integrating measures, which are $\Psi^{\text{gau}}(dx) = \exp\{-a||x||_E^2\}dx$ and $\Psi^{\text{lap}}(dx) = \exp\{-b||x||_1\}dx$, respectively, where a and b are two positive constants directing the power of test statistics. In the following we simply focus on $a = 0.5$ and $b = 1$.

For CvM-type test statistics with Gaussian-type integrating measures, we have that

$$CvM_n^{\text{gau,E}} = \lambda_{\text{max}}(\mathcal{U}_n) \text{ and } CvM_n^{\text{gau,F}} = \text{trace}(\mathcal{U}_n), \tag{23}$$

respectively, where $\mathcal{U}_n$ is a p-dimensional square matrix with the expression as:

$$\mathcal{U}_n = \frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n \xi_i(\tau_r)U_{ij}\xi_j^\top(\tau_r), \tag{24}$$

where ξ_i and ξ_j are defined as in (21), and

$$U_{ij} = \exp\left\{-\frac{||X_i - X_j||_E^2}{2}\right\}. \tag{25}$$

Likewise, for the CvM-type test statistics with Laplace-type integrating measure, we can similarly obtain that

$$CvM_n^{\text{lap,E}} = \lambda_{\max}(\mathcal{L}_n) \text{ and } CvM_n^{\text{lap,F}} = \text{trace}(\mathcal{L}_n), \tag{26}$$

respectively, where $\mathcal{L}_n$ is also a p-dimensional squre matrix with the expression as:

$$\mathcal{L}_n = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i(\tau_r)L_{ij}\xi_j^\top(\tau_r), \tag{27}$$

where ξ_i and ξ_j are defined as in (21), and

$$L_{ij} = \frac{1}{1 + ||X_i - X_j||^2}. \tag{28}$$

3 Large Sample Properties

In this section, we investigate the limiting behaviors of process (11) under the null hypothesis, fixed alternative and a sequence of local alternatives converging to the null at a parametric rate of $n^{-1/2}$. The asymptotic distributions of our test statistics can be yielded with the combination of weak convergence of process (11) and continuous mapping theorem (CMT, see, e.g., Van Der Vaart and Wellner (1996)).

3.1 Regularity Conditions

We first list the regularity conditions for the derivation of asymptotic distribution of our test statistics.

Assumption 3.1 *(i) $\{(Y_i, X_i^\top)^\top\}_{i=1}^n$ is a sequence of independently and identically distributed (i.i.d.) random variables; (ii) $\mathbb{E}[||X||_E^2] < \infty$; (iii) $|\mathcal{T}| > 0$.*

Assumption 3.2 *(i) $f(y|x)$ is uniformly bounded from above and below on $y \in \mathcal{Y}$ and $x \in \Pi$, where $\mathcal{Y} = \{m(x, \theta_0(\tau)) : x \in \Pi$ and $\tau \in \mathcal{T}\}$; (ii) $\partial f(y|x)/\partial y$ is uniformly bounded with respect to $y \in \mathcal{Y}$.*

Assumption 3.3 *(i) $\sup_{\theta\in\Theta_0} ||\dot{m}(x, \theta(\tau))||_E \le M(x)$ and $||\dot{m}(x, \theta_1(\tau)) - \dot{m}(x, \theta_2(\tau))|| \le N(x)||\theta_1 - \theta_2||_E$, where $M(x)$ and $N(x)$ are some integrable functions, Θ_0 is the neighborhood of $\theta_0(\tau)$ defined as $\Theta_0 := \{\theta(\tau) : ||\theta(\tau) - \theta_0(\tau)||_E \le C\}$ for some constant C and for all $\tau \in \mathcal{T}$; (ii) $\Delta(\theta(\tau))$ and $\mathbb{E}[g(X, \theta(\tau))]$ are nonsingular in Θ_0.*

Assumption 3.4 *(i) Θ is a compact subset of $\mathbb{R}^p$; (ii) $\theta_0(\tau)$ is at the interior subset of Θ; (iii) $\sqrt{n}||\hat{\theta}_n(\cdot) - \theta_0(\cdot)||_E^2 = o_p(1)$.*

Assumption 3.5 *(i) $K(\cdot)$ is of bounded variation and $|K(u)| \leq \infty$ for all $u \in \mathbb{R}$; (ii) $K(\cdot)$ satisfies a Lipschitz condition on $\mathbb{R}$; (iii) $\int_{-\infty}^{\infty} K(u)du = 1$, $\int_{-\infty}^{\infty} uK(u)du = 0$, $\int_{-\infty}^{\infty} u^2 K(u)du = \mu_2$ and $\int_{-\infty}^{\infty}(K(u))^2 du = \nu_2$ for some $\mu_2, \nu_2 \in (0, \infty)$.*

Assumption 3.6 *The bandwidth h_m satisfies $\lim_{m\to\infty} P(a_m \leq h_m \leq b_m) = 1$ for deterministic sequences of positive numbers a_m and b_m such that $ma_m/\log(m) \to \infty$ and $b_m \to 0$ as $n \to \infty$.*

Assumption 3.7 *$\Psi(dx, d\tau)$ is absolutely continuous with respect to the Lebesgue measure on $\Pi \times \mathcal{T}$.*

Assumption 3.1(i) is standard for IID observations, (ii) imposes a regular moment condition for the limiting property, and (iii) excludes the case of singleton quanitle. Assumption 3.2 is commonly used for the regulation of conditional density function. Assumption 3.3(i) concerns the smoothness of parametric model $m(x, \theta(\tau))$ with only requiring the finite first moment and a Lipschitz condition; (ii) requires the existence of matrix $\Delta(\theta(\tau))$ and $\mathbb{E}[g(X, \theta(\tau))]$ in a neighborhood of $\theta_0(\tau)$ for projection and non-projection cases, respectively. Assumption 3.4 includes regular conditions in the context of quantile regression estimator, see, e.g., Koenker and Bassett (1978). Note that we relax the assumption of $\sqrt{n}||\hat{\theta}_n(\cdot) - \theta_0(\cdot)|| = \mathcal{O}_p(1)$ to (ii) due to our imposed projection. Assumptions 3.5 and 3.6 allow for the widely used smoothing kernels with data-driven bandwidths. Assumption 3.7 is imposed for the consistency of test statistics.

3.2 Asymptotic Null Distribution

We first illustrate the asymptotic distribution of process (11) under the null hypothesis.

Theorem 3.1 *Suppose Assumptions 3.1-3.6 hold. Then, under the null hypothesis,*

$$\hat{R}_n(x, \tau) \Rightarrow R_\infty(x, \tau), \tag{29}$$

where R_∞ is a centered Gaussian process with the covariance function

$$cov\left((x_1, \tau_1), (x_2, \tau_2)\right) = (\tau_1 \wedge \tau_2 - \tau_1 \cdot \tau_2)\, \mathbb{E}\left[\dot{m}(X, \theta_0(\tau_1))\dot{m}^\top(X, \theta_0(\tau_2))\mathcal{P}_{\tau_1} w(X, x_1)\mathcal{P}_{\tau_2} w(X, x_2)\right]. \tag{30}$$

From Theorem 3.1, we see that the limiting behavior of process $\hat{R}_n(x, \tau)$ does not rely on $\hat{\theta}_n$. To prove this theorem, we first need to show the uniform decomposition of $\hat{R}_n(x, \tau)$ in (14), and then its weak convergence of under the null.

Next, Theorem 3.1 and CMT yield the asymptotic null distribution of continuous functionals of process $\hat{R}_n(x, \tau)$.

Corollary 3.1 *Suppose Assumptions 3.1-3.7 hold. Then, under the null hypothesis,*

$$CvM_n \xrightarrow{\mathcal{D}} \int_{\Pi\times\mathcal{T}} ||R_\infty||^2 \Psi(dx, d\tau), \tag{31}$$

where CvM_n are the test statistics proposed in Section 2.3, R_∞ is defined the same as in Theorem 3.1. The limiting distribution of discretized $\{\tau_r\}_{r=1}^m$ is guaranteed by the arguments in Chang (1990).

3.3 Asymptotic Power

In this section, we investigate the power properties of our tests under two types of alternatives. The first theorem illustrates the asymptotic properties of our tests under the fixed alternative which is exactly the negation of null hypothesis.

Theorem 3.2 *Suppose Assumptions 3.1-3.6 hold. Then, under the fixed alternative, we have that*

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}} \left| \frac{\hat{R}_n(x,\tau)}{\sqrt{n}} - \mathbb{E}\left[\mathcal{P}_\tau w(X,x)\psi(X,Y,\theta_0(\tau))\right] \right| = o_p(1). \tag{32}$$

Note that from Theorem 3.2, our test statistics are not consistent against all fixed alternative hypotheses. Explicitly speaking, under the fixed alternatives, we have that

$$\mathbb{E}\left[w(X,x)I_p\psi(X,Y,\theta_0(\tau))\right] - G^\top(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))\mathbb{E}\left[g(X,\theta_0(\tau))\psi(X,Y,\theta_0(\tau))\right] = 0, \tag{33}$$

when $\psi(X,Y,\theta_0(\tau))$ and $g(X,\theta_0(\tau))$ are collinear.

Next, we derive the asymptotic distribution of process $\hat{R}_n(x,\tau)$ as well as the test statistics under a certain sequence of local alternatives converging to the null at parametric rate of $n^{-1/2}$. Specifically, the sequence of local alternatives is given by

$$H_{1n} : \mathbb{E}\left[\mathbb{1}\left(Y \le m(X,\theta_0(\tau))\right) - \tau \middle| X\right] = \frac{d(X,\tau)}{\sqrt{n}} \text{ a.s., for all } \tau\in\mathcal{T} \text{ and some } \theta_0(\cdot)\in\Theta, \tag{34}$$

where $d(X,\tau)$ determines the direction of departure from the null hypothesis.

Assumption 3.8 *$d(X,\cdot)$ is continuous a.s., and $d(X,\tau) < \infty$ for $\tau\in\mathcal{T}$.*

Theorem 3.3 *Suppose Assumptions 3.1-3.6 and Assumption 3.8 hold. Then, under the sequence of local alternatives (34), we have that*

$$\hat{R}_n(x,\tau) \Rightarrow R_\infty + D(x,\tau), \tag{35}$$

where R_∞ is defined the same as in Theorem 3.1, and $D(x,\tau) := \mathbb{E}\left[\mathcal{P}_\tau w(X,x)\dot{m}(X,\theta_0(\tau))d(X,\tau)\right]$ is a deterministic shift term.

Similarly, with CMT at hand, we have the following corollary.

Corollary 3.2 *Under Assumptions 3.1-3.8 and the sequence of local alternatives (34), we have that*

$$CvM_n \xrightarrow{\mathcal{D}} \int_{\Pi\times\mathcal{T}} ||R_\infty + D(x,\tau)||^2\Psi(dx,d\tau), \tag{36}$$

where CvM_n can be any test statistics proposed in Section 2.3, R_∞ is defined the same as in Theorem 3.1.

The proof of Theorem 3.3 is straightforward. Besides, we remark that $D(x,\tau) \neq 0$ for at least some $(x,\tau) \in \Pi \times \mathcal{T}$, which means our proposed test statistics possess non-trivial power against local alternatives of (34). The situation where our tests cannot detect such local alternatives is when direction $\dot{m}(X,\theta_0(\tau))d(X,\tau)$ is collinear with $g(X,\theta_0(\tau))$[1], under which, (33) is equal to the null hypothesis so that it cannot be detected.

Remark 3.1 *Note that for the fixed and sequence of lcoal alternatives, the lack of power under collinearity is not a limitation. Since all of the cusum process-based type of tests have trivial local power against several alternatives, the power in the direction of $g(X,\theta_0(\tau))$ may also be low, even without this projection transformation.*

On the other hand, once we do not waste power in such difficultly detectable directions, we may obtain more power against other alternatives, see more detailed discussions in Strasser (1990), Escanciano (2009a) and Sant'Anna and Song (2019). As a result, our tests are consistent against a broad class of alternatives. And practically, our tests are shown to be easy-to-implement in the bootstrap procedure due to this projection.

4 Bootstrap Procedure

As shown in Theorem 3.1, the limiting processes of our test statistics rely on the underlying data generating process (DGP). Hence, we propose a multiplier bootstrap to approximate the null distribution for the computation of critical values. The advantage of this procedure over other bootstrap methods is that it does not require the estimation of nuisance parameter at each bootstrap replication, especially we need to optimize the check function in the quantile context. Besides, it is fully data-driven, and does not involve any need to select tuning parameters.

[1]See (33), just replacing $(\tau - \mathbb{1}(Y \leq m(X,\theta_0(\tau))))$ with $d(X,\tau)$.

As a matter of fact, in the context of a continuum of quantiles, some other commonly used bootstrap approaches such as wild bootstrap and parametric bootstrap, etc., are invalid due to the quantile-dependent newly generated samples. We will also elaborate it subsequently, which is another superior and even necessity of multiplier bootstrap.

4.1 Validity of Multiplier Bootstrap

Multiplier bootstrap is to mimic the asymptotic behavior of stochastic process by first order expansion with multiplying a sequence of mean zero, unit variance and path independent random variables on it. Due to the projection for eliminating the parameter estimation effect, we can onbtain a simple fisrt order expansion of process and thus facilitate this procedure.

The multiplier bootstrap-based empirical process for (11) is

$$\hat{R}_n^*(x,\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i \left\{ \mathcal{P}_{n,\tau} w(X_i, x) \psi(X_i, Y_i, \hat{\theta}_n(\tau)) \right\}, \tag{37}$$

where $\{V_i\}_{i=1}^n = \left\{ \{V_{i1}, \ldots, V_{ip}\}^\top \right\}_{i=1}^n$ is a sequence of p-dimensional IID random vectors with mean zero, unit variance and path independent of process $\hat{R}_n(x,\tau)$. An distribution rule of $\{V_i\}_{i=1}^n$ proposed in Mammen (1993) is

$$P(V = 1-\kappa) = \frac{\kappa}{\sqrt{5}},\ \ P(V=\kappa) = 1 - \frac{\kappa}{\sqrt{5}},$$

where $\kappa = (\sqrt{5}+1)/2$.

The multiplier bootstrap-based test statistics based on (37) are

$$CvM_n^* = \int_{\Pi \times \mathcal{T}} \left|\left| \hat{R}_n^*(x,\tau) \right|\right|^2 \Psi(dx, d\tau), \tag{38}$$

with the following closed form expressions as:

$$\begin{aligned} CvM_n^{*,\text{lin,E}} &= \lambda_{\max}(\mathcal{B}_n^*),\ \ CvM_n^{*,\text{lin,F}} = \operatorname{trace}(\mathcal{B}_n^*), \\ CvM_n^{*,\text{gau,E}} &= \lambda_{\max}(\mathcal{U}_n^*),\ \ CvM_n^{*,\text{gau,F}} = \operatorname{trace}(\mathcal{U}_n^*), \\ CvM_n^{*,\text{lap,E}} &= \lambda_{\max}(\mathcal{L}_n^*),\ \ CvM_n^{*,\text{lap,F}} = \operatorname{trace}(\mathcal{L}_n^*), \end{aligned} \tag{39}$$

where $\mathcal{B}_n^*$, $\mathcal{U}_n^*$ and $\mathcal{L}_n^*$ are p-dimensional square matrices with the expression as:

$$(\mathcal{B}_n^*)_{(k,l)} = \frac{1}{mn^2} \sum_{r=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i^*(\tau_r) B_{ij} \xi_j^{*\top}(\tau_r), \tag{40}$$

$$(\mathcal{U}_n^*)_{(k,l)} = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i^*(\tau_r)U_{ij}\xi_j^{*\top}(\tau_r), \tag{41}$$

$$(\mathcal{L}_n^*)_{(k,l)} = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i^*(\tau_r)L_{ij}\xi_j^{*\top}(\tau_r), \tag{42}$$

respectively, and

$$\xi_i^*(\tau_r) = V_i\psi(X_i, Y_i, \hat{\theta}_n(\tau_r)) - \hat{g}^\top(X_i, \hat{\theta}_n(\tau_r))\hat{\Delta}^{-1}(\hat{\theta}_n(\tau_r))\frac{1}{n}\sum_{s=1}^{n}\hat{g}(X_s, \hat{\theta}_n(\tau_r))V_s\psi(X_s, Y_s, \hat{\theta}_n(\tau_r)), \tag{43}$$

B_{ij}, U_{ij} and L_{ij} are defined the same as before.

Next, we establish the validity of multiplier bootstrap procedure to the asymptotic null distribution of test statistics.

Theorem 4.1 *Suppose Assumptions 3.1-3.7 hold. Then, we have that the multiplier bootstrap-based tests*

$$CvM_n^* \xrightarrow{\mathcal{D}} \int_{\Pi\times\mathcal{T}} ||R_\infty||^2\Psi(dx, d\tau) \text{ a.s.}, \tag{44}$$

conditional on the original samples.

Then, the bootstrapped critical values is given by

$$\text{cv}_{n,\alpha} = \inf\left\{c_\alpha \middle| P\left(CvM_n^* \geq c_\alpha \middle| \left\{\left(Y_i, X_i^\top\right)^\top\right\}_{i=1}^{n}\right) \leq \alpha\right\}, \tag{45}$$

where α is the nominal significance level. In the multiplier bootstrap procedure, $\text{cv}_{n,\alpha}$ can be approximatedaccurately by the $B\cdot(1-\alpha)$-th order statistic from B-time replications of $\left\{(CvM_n^*)^b\right\}_{b=1}^{B}$ to the original test statistics.

4.2 Invalidity of Resampling Schemes

In this section, we briefly illustrate the reason of infeasibility for other bootstrap methods to simulate the critical values in the context of continuum of quantiles.

When we investigate the property of process over a continuum of quantiles, we generally need to simulation a sequence of "uniform error term". Otherwise, the newly generated samples are τ-dependent and thus cannot curve the global behavior of process. Denote $\{Y_i^\star(\tau)\}_{i=1}^n$ as the newly generated outcomes. Then, we have the empirical process based on resampling, that is,

$$\hat{R}_n^\star(x,\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathcal{P}_{n,\tau}w(X_i, x)\psi(X_i, Y_i^\star(\tau), \hat{\theta}_n(\tau)), \tag{46}$$

and the bootstrapped test statistics are

$$CvM_n^\star = \int_{\Pi \times \mathcal{T}} \left|\left|\hat{R}_n^\star(x,\tau)\right|\right|^2 \Psi(dx, d\tau), \tag{47}$$

where $Y_i^\star(\tau)$ in (46) is not random but supposed to be non-decreasing with respect to τ. As a result, the discretized version of test statistic $CvM_n^\star$ is

$$CvM_n^\star = \frac{1}{m}\sum_{r=1}^{m} \int_{\Pi} \left|\left|\hat{R}_n^\star(x,\tau_r)\right|\right|^2 \Psi(dx), \tag{48}$$

which is obviously not asymptotically equivalent to (47), and we have no idea on the closed form expression of bootstrapped test statistics based on (46).

In practice, when we use (48) to mimic the process over a range of quantiles, the rejection rates under the null hypothesis is over inflated beyond nominal significance level due to the distortion between (47) and (48). While in the multiplier bootstrap procedure, this simulation-based method just incorporate a vector of path-independent random variables without changing the originally global properties of process over whole quantiles by generating newly, quantile-dependent samples. Therefore, the multiplier bootstrap-based test statistics can be expressed as

$$CvM_n^* = \frac{1}{m}\sum_{r=1}^{m} \int_{\Pi} \left|\left|\hat{R}_n^*(x,\tau_r)\right|\right|^2 \Psi(dx), \tag{49}$$

which is asymptotically equivalent to (38) and can compute the critical values as accurately as desired.

5 Simulation and Empirical Study

5.1 Comparison to the Tests without Projection

We first give the closed form expression of the CvM-type test statistics without projection based on process (6), that is

$$CvM_n^{\text{wp}} = \int_{\Pi \times \mathcal{T}} ||\breve{R}_n(x,\tau)||^2 \Psi(dx, d\tau), \tag{50}$$

where $||\cdot||$ is also denoted by Euclidean norm $||\cdot||_E$ and Frobenius norm $||\cdot||_F$, respectively.

For brevity, we directly give the closed form expression of several test statistics which possess the same derivation as in Section 2.3.

- (Linear indicator weight function)

$$CvM_n^{\text{lin,E,wp}} = \lambda_{\max}(\mathcal{B}_n^{\text{wp}}) \text{ and } CvM_n^{\text{lin,F,wp}} = \text{trace}(\mathcal{B}_n^{\text{wp}}), \tag{51}$$

where $\mathcal{B}_n^{\text{wp}}$ is a p-dimensional squre matrix with the expression as

$$\mathcal{B}_n^{\text{wp}} = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(X_i, Y_i, \hat{\theta}_n(\tau_r))B_{ij}\psi^\top(X_j, Y_j, \hat{\theta}_n(\tau_r)),$$

and B_{ij} defined in (22).

- (Exponential weight function with Gaussian integrating measure)

$$CvM_n^{\text{gau,E,wp}} = \lambda_{\max}(\mathcal{U}_n^{\text{wp}}) \text{ and } CvM_n^{\text{gau,F,wp}} = \text{trace}(\mathcal{U}_n^{\text{wp}}), \tag{52}$$

where $\mathcal{U}_n^{\text{wp}}$ is a p-dimensional squre matrix with the expression as:

$$(\mathcal{U}_n^{\text{wp}})_{(k,l)} = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(X_i, Y_i, \hat{\theta}_n(\tau_r))U_{ij}\psi^\top(X_j, Y_j, \hat{\theta}_n(\tau_r)),$$

and U_{ij} defined in (25).

- (Exponential weight function with Laplace integrating measure)

$$CvM_n^{\text{lap,E,wp}} = \lambda_{\max}(\mathcal{L}_n^{\text{wp}}) \text{ and } CvM_n^{\text{lap,F,wp}} = \text{trace}(\mathcal{L}_n^{\text{wp}}), \tag{53}$$

where $\mathcal{L}_n^{\text{wp}}$ is a p-dimensional squre matrix with the (k,l)-th element as:

$$(\mathcal{L}_n^{\text{wp}})_{(k,l)} = \frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(X_i, Y_i, \hat{\theta}_n(\tau_r))L_{ij}\psi^\top(X_j, Y_j, \hat{\theta}_n(\tau_r)),$$

and L_{ij} defined in (28).

The multiplier bootstrap procedure in this case is kind of complicates as the existence of parameter estimation effect. By combining the expansion of (7) as well as the Bahadur-type (linear) representation of $\hat{\theta}_n$ derived from check funtion (**??**), we have the following multiplier bootstrap-based process as:

$$\breve{R}_n^*(x,\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i \left(\tau - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n(\tau))\right)\right)\left(w(X_i, x)I_p - \overline{\mathbb{W}}\right)\dot{m}(X_i, \hat{\theta}_n(\tau)), \tag{54}$$

where

$$\overline{\mathbb{W}} = \frac{1}{n}\sum_{i=1}^{n}\hat{g}(X_i,\hat{\theta}_n(\tau))\left(\frac{1}{n}\sum_{i=1}^{n}\hat{g}(X_i,\hat{\theta}_n(\tau))\right)^{-1} w(X_i,x). \tag{55}$$

After simple algebra, we have that

$$CvM_n^{*,\text{lin,E,wp}} = \lambda_{\max}(\mathcal{B}_n^{*,\text{wp}}) \text{ and } CvM_n^{*,\text{lin,F,wp}} = \text{trace}(\mathcal{B}_n^{*,\text{wp}}), \tag{56}$$

$$CvM_n^{*,\text{gau,E,wp}} = \lambda_{\max}(\mathcal{U}_n^{*,\text{wp}}) \text{ and } CvM_n^{*,\text{gau,F,wp}} = \text{trace}(\mathcal{U}_n^{*,\text{wp}}), \tag{57}$$

$$CvM_n^{*,\text{lap,E,wp}} = \lambda_{\max}(\mathcal{L}_n^{*,\text{wp}}) \text{ and } CvM_n^{*,\text{lap,F,wp}} = \text{trace}(\mathcal{L}_n^{*,\text{wp}}), \tag{58}$$

respectively, where $\mathcal{B}_n^{*,\text{wp}}$, $\mathcal{U}_n^{*,\text{wp}}$ and $\mathcal{L}_n^{*,\text{wp}}$ can be expressed in unification as:

$$\frac{1}{mn^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i^{*,\text{wp}}(\tau_r)W_{ij}\xi_j^{*,\text{wp}\top}(\tau_r), \tag{59}$$

where W_{ij} can be (22), (25) and (28), respectively, and

$$\xi_i^{*,\text{wp}}(\tau_r) = V_i\psi(X_i,Y_i,\hat{\theta}_n(\tau_r)) - \hat{g}^\top(X_i,\hat{\theta}_n(\tau_r))\left(\frac{1}{n}\sum_{i=1}^{n}\hat{g}(X_i,\hat{\theta}_n(\tau_r))\right)^{-1}\frac{1}{n}\sum_{s=1}^{n}V_s\psi(X_s,Y_s,\hat{\theta}_n(\tau_r)). \tag{60}$$

Next, we show the closed form expression of tests without gradient. Note that the uni-dimensional empirical process is

$$\bar{R}_n^{\text{pro}}(x,\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\tau - \mathbb{1}\left(Y_i \le m(X_i,\hat{\theta}_n(\tau))\right))\hat{\mathcal{P}}_{n,\tau}w(X_i,x), \tag{61}$$

where

$$\hat{\mathcal{P}}_{n,\tau}w(X_i,x) = w(X_i,x) - s_n(X_i,\hat{\theta}_n(\tau))\Xi_n^{-1}(\hat{\theta}_n(\tau))S_n(x,\hat{\theta}_n(\tau)), \tag{62}$$

with $s_n(X_i,\hat{\theta}_n(\tau)) := f(m(X_i,\hat{\theta}_n(\tau))|X_i)\dot{m}(X_i,\hat{\theta}_n(\tau))$ and

$$\Xi_n(\hat{\theta}_n(\tau)) = \frac{1}{n}\sum_{i=1}^{n}s_n(X_i,\hat{\theta}_n(\tau))s_n^\top(X_i,\hat{\theta}_n(\tau)),\ S_n(x,\hat{\theta}_n(\tau)) = \frac{1}{n}\sum_{i=1}^{n}s_n(X_i,\hat{\theta}_n(\tau))w(X_i,x).$$

Then, the test statistics established on (61) are

$$
\begin{aligned}
CvM_n^{\text{wg}} =& \int_{\Pi\times\mathcal{T}} ||\bar{R}_n^{\text{pro}}(x,\tau)||^2 \Psi(dx, d\tau) \\
=& \frac{1}{nm^2}\sum_{r=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{\text{wg}}(Y_i, X_i, \tau_r) W_{ij} \xi_{\text{wg}}(Y_j, X_j, \tau_r),
\end{aligned} \tag{63}
$$

where W_{ij} can be (22), (25) and (28), respectively, and

$$
\begin{aligned}
\xi_{\text{wg}}(Y_i, X_i, \tau_r) =& (\tau_r - \mathbb{1}\left(Y_i \leq m(X_i, \hat{\theta}_n(\tau_r))\right)) - s_n^\top(X_i, \hat{\theta}_n(\tau_r))\Xi_n^{-1}(\hat{\theta}_n(\tau)) \\
& \times \frac{1}{n}\sum_{z=1}^{n} s_n^\top(X_z, \hat{\theta}_n(\tau_r))(\tau_r - \mathbb{1}\left(Y_z \leq m(X_z, \hat{\theta}_n(\tau_r))\right)),
\end{aligned} \tag{64}
$$

the closed form expression of multiplier bootstrap-based test statistics are similar, so we omit here for brevity.

In all, we have listed 9 test statistics for the comparison with our proposed tests in this paper.

5.2 Numerical Evidence

In this part, we examine the finite sample performance of our proposed tests compared with other tests illustrated in Section 5.1 under the null and alternative hypotheses via Monte Carlo simulations. We consider 1,000 replications with 500 sequences of bootstrap generated for each replication in all experiments. The quantile intervals are considered as $\mathcal{T}_1 = [0.1, 0.4]$, $\mathcal{T}_2 = [0.4, 0.6]$ and $\mathcal{T}_3 = [0.6, 0.9]$, respectively, with evenly spaced point of 0.01. For bandwidth, we consider $h = n^{-1/5}$ for all cases below. The significance levels are shown as 1%, 5% and 10%, respectively.

We first focus on the following DGPs under the null hypothesis, which are

- DGP1: $Y_i = 1 + \theta^\top X_i + \varepsilon_i$,
- DGP2: $Y_i = 1 + \theta^\top X_i + e_i$,
- DGP3: $Y_i = 1 + \theta^\top X_i + \varepsilon_i(X_i)$,
- DGP4: $Y_i = 1 + \theta^\top X_i + e_i(X_i)$,

where we have the τ_r-th quantile $F^{-1}_{Y_i|X_i}(\tau_r)$ for $r = \{1, 2, \ldots, m\}$, and X_i are d-dimensional covariates generated from uniform distribution $\mathcal{U}[0, \pi]$, which are mutually independent for each dimension, ε_i and e_i are error terms generated from standard normal and centered exponential distributions, respectively. $\varepsilon_i(X_i) = (\theta^\top X_i + 0.5)\varepsilon_i$, $e_i(X_i) = (\theta^\top X_i + 0.5)e_i$. DGP1 and DGP2 represent the homoscedastic quantile regression models, while DGP3 and DGP4 are proposed for checking the robustness of our tests under heteroscedastic model. Let $d = 2$ and 5. $\theta^\top = \{1, \ldots, 1\}$ for each dimension. The sample sizes are considered as 100 and 200.

Next, we consider the following fixed alternatives with different deviations against DGP1,

- DGP5: $Y_i = 1 + \theta^\top X_i + 0.1p \times \left(\sum_{k=1}^{p} X_{ki}^2 + \sum_{k\neq l} X_{ki}X_{li}\right) + \varepsilon_i,$
- DGP6: $Y = 1 + \theta^\top X_i + 2p \times \exp\{-\frac{1+\theta^\top X_i}{2}\} + \varepsilon_i,$
- DGP7: $Y = 1 + \theta^\top X_i + 3p \times \log\left(1 + \theta^\top X_i\right) + \varepsilon_i,$

where X_i and ε_i are drawn independently from the same distributions as previous. The dimensions of covariates are also considered the same as before. DGP5 represents a quadratic drift away from the null hypothesis; DGP6 and DGP7 adopt nonlinear deviations with different forms. The sample sizes are also considered as 100 and 200.

In the last, we consider a quadratic deviation of local alternatives against the null hypothesis of DGP1,

- DGP8: $Y_i = 1 + \theta^\top X_i + \delta\left(\theta^\top X_i\right)^2 + \varepsilon_i,$

where τ_r, X_i and ε_i are same as previous. $\delta = \{0.01, 0.02, \ldots, 0.09, 0.10\}$. The dimensions of covariates are also equal to $d = 2$ and 5, respectively. The sample sizes are considered as 100 and 200.

The simulation results are shown in Appendix B. Tables 2-3 report the empirical size with respect to DGP1-DGP4. Regardless of the four different forms of residuals, all types of test statistics but those without projection are well controlled near the significance levels, especially when the sample size increases to 200. Although the empirical size of the test statistics without projection gradually moves towards significance levels at $n = 200$, it still has somewhat distortion against the significance levels, especially for the test statistics with linear indicator weight function under $d = 5$. This result shows that our projection can effectively correct the type-I error, and thus is necessary.

The empirical powers corresponding to DGP5-DGP7 are displayed in Tables 4-5. Note that due to the over large type-I error of tests statistics without projection, the corresponding type-II error can be smaller, that is, the power is relatively larger. However, these results are not effective under such circumstance. In addition, the powers under three DGPs of alternatives are significantly larger than the significance levels under three quantile intervals, and as the sample size increases, their powers are all increasing as expected.

The trend of empirical local powers in DGP8 at different significance levesl are shown in Table 6-11 with respect to $n = 100$ and 200 under three quantile intervals. Similar as previous, we do not refer the local power of the test statistics without projection as they fail to control the empirical size. For other test statistics, the local powers are all going larger as δ as well as sample size increases.

In summary, according to the finite sample performance, our proposed tests can be robust to size level, and on the other hand, they can display significant power against the null hypothesis. Besides, our tests are also robust under large-dimensional covariates.

Remark 5.1 *Note that the empirical size and power results of the tests without gradient are similar to our proposed tests in that the gradients under linear quantile regression models are constant vectors, whose effect is not obvious. However, as illustrated in several literature, such gradient-based weight is necessary as it can improve the power performance in nolinear quantile regression models under finite sample, see, e.g., Eubank and Hart (1992) and He and Zhu (2003).*

5.3 Real Data Application

We apply our proposed tests to check of model validity in the context of quantile regression models. The empirical data is from Angrist et al. (2009), which studied an evaluation of strategies designed to improve academic performance among college freshmen. The data was collected from several randomized controlled trials (RCTs) with three different treatments (strategies): academic support services, financial incentives and both interventions. The model is:

$$y_{it} = X_i^\top \delta_t + \alpha \cdot ssp_i^* + \beta \cdot sfp_i^* + \gamma \cdot sfsp_i^* + \xi_i + \varepsilon_{it}, \tag{65}$$

where $t = 0, 1$ represents the baseline investigation and post-experiment, respectively. y_{it} is academic performance (Grade Point Average, GPA), X_i are some covariates with respect to individuals. ssp_i^*, sfp_i^* and $sfsp_i^*$ are three strategies stated above, which are implemented at $t = 1$. ξ_i captures the unobservable individual-effect and ε_{it} is error term. After first-order difference, we have

$$\Delta y_i = X_i^\top \delta_0 + \alpha \cdot ssp_i^* + \beta \cdot sfp_i^* + \gamma \cdot sfsp_i^* + \epsilon_i, \tag{66}$$

where $\Delta y_i := y_{i1} - y_{i0}$.

Quantile regression adopted herein is to describe the distribution feature of GPA. In this section, we examine the specification issue of quantile regression model (66) under lower, median, upper and whole range of quantiles via our proprosed tests, respectively. The testing issue is proposed in three different setups:

- $\mathcal{H}_0^1 : \Delta y_i = \alpha \cdot ssp_i^* + \beta \cdot sfp_i^* + \gamma \cdot sfsp_i^* + \epsilon_i,$
- $\mathcal{H}_0^2 : \Delta y_i = \theta_0 \cdot hs_i + \alpha \cdot ssp_i^* + \beta \cdot sfp_i^* + \gamma \cdot sfsp_i^* + \epsilon_i,$
- $\mathcal{H}_0^3 : \Delta y_i = X_i^\top \delta_0 + \alpha \cdot ssp_i^* + \beta \cdot sfp_i^* + \gamma \cdot sfsp_i^* + \epsilon_i,$

where hs_i denotes the high school GPA for individual i, X_i represents the covariates mentioned in Angrist et al. (2009), such as gender, high school GPA, expected hours of work, and parents' educational attainment, etc. The corresponding alternatives are the negation of null hypotheses listed above.

Table 1 reports the critical values and p-value obtained from the application of three testing issues. We see that only $\mathcal{H}_0^1$ is accepted among all cases of quantile intervals. For $\mathcal{H}_0^2$, the p-values are around the significance levels of 1% and 5% in the lower, middle and whole range of quantiles, which result in the rejection of null hypothesis, however, the p-values increase over 10% in upper interval of quantiles which indicates that our tests do not find any evidence of model misspecification. The situation of $\mathcal{H}_0^3$ is similar to $\mathcal{H}_0^2$. Besides, we see that in most cases, the corresponding critical values of the each group of test statistics are the same, which results from the asymptotically uniform equivalence of test statistics when sample size increases to large[2].

One may double the testing results for intuitively, more covariates included could decrease the likelihood of misspecification in the regression models. However, the characteristic of RCT data is that outcome variable is always determined by treatments whatever the cases of quantile are, while it is difficult to guarantee that the characteristics of covariates used for the analysis of treatment effect heterogeneity are uniform at each quantile. As a result, there will be the possibility of misspecification in quantile regression models when the covariates are incorporated at some quantiles.

Remark 5.2 *Note that the proposed tests in our paper is about checking the validity of quantile regression models but not selection, once a parametric form is established, our tests can provide evidence of its reliability but cannot tell what the "true" regression model is supposed to be.*

Table 1: Critical values and p-values associated with proposed tests

	$[0.1, 0.3]$				$[0.4, 0.6]$				$[0.7, 0.9]$				$[0.1, 0.9]$			
	cv(1%)	cv(5%)	cv(10%)	p-value	cv(1%)	cv(5%)	cv(10%)	p-value	cv(1%)	cv(5%)	cv(10%)	p-value	cv(1%)	cv(5%)	cv(10%)	p-value
								$\mathcal{H}_0^1$ (10^{-33})								
$CvM_n^{\text{lin,E}}$	1.42	0.92	0.70	0.99	1.89	1.33	1.01	0.93	1.53	0.87	0.62	0.96	0.72	0.87	0.23	0.99
$CvM_n^{\text{lin,F}}$	1.42	0.92	0.70	0.99	1.89	1.33	1.01	0.97	1.53	0.87	0.62	0.99	0.72	0.87	0.23	1.00
$CvM_n^{\text{gau,E}}$	1.26	0.79	0.62	0.97	1.67	1.19	0.89	0.86	1.31	0.72	0.55	0.97	1.04	0.76	0.64	1.00
$CvM_n^{\text{gau,F}}$	1.26	0.80	0.62	0.98	1.67	1.17	0.89	0.94	1.31	0.72	0.55	1.00	1.04	0.76	0.64	1.00
$CvM_n^{\text{lap,E}}$	1.26	0.80	0.61	0.97	1.67	1.17	0.89	0.86	1.32	0.72	0.55	0.97	1.04	0.76	0.64	1.00
$CvM_n^{\text{lap,F}}$	0.84	1.02	1.55	0.98	2.00	1.43	1.14	0.94	1.55	0.95	0.71	1.00	1.39	1.05	0.90	1.00
								$\mathcal{H}_0^2$								
$CvM_n^{\text{lin,E}}$	0.93	0.64	0.50	0.02	1.56	0.99	0.81	0.01	0.87	0.59	0.47	0.28	1.03	0.74	0.61	0.00
$CvM_n^{\text{lin,F}}$	0.93	0.64	0.50	0.02	1.56	0.99	0.81	0.01	0.87	0.59	0.47	0.28	1.03	0.74	0.61	0.00
$CvM_n^{\text{gau,E}}$	1.48	1.14	0.96	0.06	2.21	1.85	1.61	0.03	1.43	1.10	0.97	0.37	1.46	1.21	1.15	0.02
$CvM_n^{\text{gau,F}}$	1.48	1.14	0.96	0.06	2.21	1.85	1.61	0.03	1.43	1.10	0.97	0.37	1.46	1.21	1.15	0.02
$CvM_n^{\text{lap,E}}$	1.33	1.04	0.94	0.04	2.01	1.71	1.50	0.01	1.36	1.02	0.91	0.21	1.37	1.15	1.09	0.01
$CvM_n^{\text{lap,F}}$	1.33	1.04	0.94	0.04	2.01	1.71	1.50	0.01	1.36	1.02	0.91	0.21	1.37	1.15	1.09	0.01
								$\mathcal{H}_0^3$								
$CvM_n^{\text{lin,E}}$	0.87	0.48	0.37	0.02	1.31	0.78	0.62	0.01	0.86	0.63	0.56	0.13	0.69	0.50	0.42	0.00
$CvM_n^{\text{lin,F}}$	0.87	0.48	0.37	0.02	1.31	0.78	0.62	0.01	0.86	0.63	0.56	0.13	0.69	0.50	0.42	0.00
$CvM_n^{\text{gau,E}}$	1.01	0.92	0.89	0.01	1.48	1.37	1.29	0.09	0.97	0.91	0.87	0.52	1.12	1.03	0.96	0.08
$CvM_n^{\text{gau,F}}$	1.01	0.92	0.89	0.01	1.48	1.37	1.29	0.09	0.97	0.91	0.87	0.52	1.12	1.03	0.96	0.08
$CvM_n^{\text{lap,E}}$	1.00	0.89	0.85	0.01	1.57	1.41	1.34	0.06	0.94	0.88	0.83	0.27	1.11	1.06	1.00	0.02
$CvM_n^{\text{lap,F}}$	1.00	0.89	0.85	0.01	1.57	1.41	1.34	0.06	0.94	0.88	0.83	0.27	1.11	1.06	1.00	0.02

Note: This table reports the critical values and p-values of three specification problems via our proposed tests in interval quantiles. Multiplier bootstrap procudere is based on 500 computations. All entries are under the 1%, 5% and 10% significance levels.

[2]The sample size is 1173 after cleaning. See Bierens (1982) for more details.

6 Conclusion

We proposed a new class of tests for the correct specification of quantile regression models. For the construction of test statistics, we constructed a "dimension-reduced projection-based vector-weighted" cusum process to eliminate the parameter estimation effect. Then, we derived the asymptotic behavior of our proposed tests. We have also shown that our projection method can facilitate a simple-to-implement multiplier bootstrap procedure for the simulation of critical values that are asymptotically equal to the nominal significance level. In addition, we noted that our testing procedure is feasible and convenient for checking the adequacy of quantile regression models over a continuum of interval quantiles since multiplier bootstrap procedure can avoid the newly generated samples related to quantile. We finally conduct a series of Monte Carlo simulations and an empirical application to illustrate the excellent performance of our proposed tests under finite samples.

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Appendices

A.1 Main Proofs

A.1.1 Preliminary Lemmas

Lemma A.1 *Define a generic function class*

$$\mathcal{F} := \{(x, y) \to f(x, y, \theta, \epsilon) : \theta \in \Theta,\, \epsilon \in \mathcal{E}\},$$

where θ and ϵ are generic Banach spaces with associated norms of $||\cdot||_\Theta$ and $||\cdot||_\mathcal{E}$, respectively.

Next, we assume that the functions in $\mathcal{F}$ satisfy

$$\mathbb{E}\left[\sup_{||\theta_1-\theta_2||_\Theta \le \delta}\ \sup_{||\epsilon_1-\epsilon_2||_\mathcal{E} \le \delta} \left| f(X, Y, \theta_1, \epsilon_1) - f(X, Y, \theta_2, \epsilon_2)\right|^2\right] \le C\delta^s,$$

where C and $s \in (0, 2]$ are some constants. Then, we have that

- *$N_{[\cdot]}(e, \mathcal{F}, ||\cdot||_2) \le N\left([\frac{e}{2C}]^{\frac{2}{s}}, \Theta, ||\cdot||_\Theta\right) \times N\left([\frac{e}{2C}]^{\frac{2}{s}}, \mathcal{E}, ||\cdot||_\mathcal{E}\right)$ for any $e > 0$, where $||\cdot||_2$ is L_2 norm, $N_{[\cdot]}(e, \Theta, ||\cdot||_\Theta)$ denotes the minimal number of e-brackets to cover Θ and $N(e, \Theta, ||\cdot||_\Theta)$ denotes the minimal number of e-neighborhoods to cover Θ. More details about definitions are shown in Van Der Vaart and Wellner (1996);*
- *If Θ is a compact subset of $\mathbb{R}^p$, $\int_0^{+\infty} \sqrt{\log N(e^{\frac{2}{s}}, \mathcal{E}, ||\cdot||_\mathcal{E})}de < \infty$;*
- *For any sequence of positive constants $\delta_n = o(1)$ and $\delta_n \to \delta$,*

$$\sup_{||\theta_1-\theta_2||_\Theta \le \delta_n}\ \sup_{||\epsilon_1-\epsilon_2||_\mathcal{E} \le \delta_n} ||\Delta\mathbb{E}_n[f(Y_i, X_i, \theta_1, \epsilon_1)] - \Delta\mathbb{E}_n[f(Y_i, X_i, \theta_2, \epsilon_2)]|| = o_p(1),$$

where

$$\Delta\mathbb{E}_n[f(Y_i, X_i, \theta_1, \epsilon_1)] := \frac{1}{\sqrt{n}}\sum_{i=1}^n f(Y_i, X_i, \theta, \epsilon) - \sqrt{n}\mathbb{E}[f(Y_i, X_i, \theta_1, \epsilon_1)],$$

It shows the asymptotically stochastically equicontinuous of empirical process $\mathbb{E}_n[f(Y_i, X_i, \theta_1, \epsilon_1)]$.

Proof of Lemma A.1. The results of Lemma A.1 follows from Theorem 3 of Chen et al. (2003). $\mathcal{Q.E.D.}$

Lemma A.2 *Define the envelope function for the class $\mathcal{F}$ as F, which is measurable, squared-integrable and satisfies $\sup_{f\in\mathcal{F}} |f(x, y, \theta, \epsilon)| \le F(x, y, \theta, \epsilon)$, $\mathbb{E}[|F(\cdot)|^2] = O(1)$ and $\mathbb{E}[|F(\cdot)|^2 \mathbb{1}\left(\frac{F(\cdot)}{\sqrt{n}} > \varepsilon\right)] = o(1)$ for each $\varepsilon > 0$.*

Define the folllowing cusum process

$$
\begin{aligned}
S_n(x,\tau,b) =&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\psi(Y_i,X_i,b)-\mathbb{E}\left[\psi(Y_i,X_i,b)\Big|X_i\right]\right)w(X_i,x)\\
=:&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Delta\psi(Y_i,X_i,b)w(X_i,x),
\end{aligned}
$$

where $b\in\mathcal{B}$ is a class of bounded, Lipschitz $\mathbb{R}^p$-valued functions on $\mathcal{T}$. $w(X_i,x)\in\mathcal{W}:=\{w(X_i,x):x\in\mathbb{R}^d\}$ with a envelope function $W(x)$, and $\sup_{|F_X(x_1)-F_X(x_2)|<\delta}||w(\cdot,x_1)-w(\cdot,x_2)||_2\le C\delta$ for a sufficiently smal $\delta>0$.

Then, under Assumptions 3.1-3.7 and $\lim_{\delta_n\to 0}\int_0^{\delta_n}\sqrt{\log N_{[\cdot]}\left(e,\mathcal{W},||\cdot||_2\right)}de=0$, we have that process $S_n(x,\tau,b)$ is ρ-stochastically equicontinuous.

Proof of Lemma A.2. Define $\mathcal{D}:=\{(x,y)\to\Delta\psi(x,y,b):(b,\tau)\in\mathcal{B}\times\mathcal{T}\}$. Then, for any fixed (b_1,τ_1), we have that

$$
\max_{1\le i\le n}\left|\mathbb{E}\left[\sup_{||b_1-b||\le d^2}\ \sup_{||\tau_1-\tau||_{\mathcal{T}}\le d^2}|\Delta\psi(Y_i,X_i,b_1)-\Delta\psi(Y_i,X_i,b)|^2\right]\right|=Cd^2,
$$

which follows from the triangle inequality illustrated in Escanciano and Goh (2014), where C and $d\in(0,1)$ are some constants.

By Lemma A.1, the rest of proof follows from

$$
\begin{aligned}
&N(2e||WD||_2,\mathcal{W}\cdot\mathcal{D},||\cdot||_2)\le N(e||W||_2,\mathcal{W},||\cdot||_2)\times N(e||D||_2,\mathcal{D},||\cdot||_2),\\
&N_{[\cdot]}(e,\mathcal{W}\cdot\mathcal{D},||\cdot||_2)\le N_{[\cdot]}(e,\mathcal{W},||\cdot||_2)\times N_{[\cdot]}(e,\mathcal{D},||\cdot||_2),
\end{aligned}
$$

Theorem 2.7.11 and Theorem 19.28 of Van Der Vaart and Wellner (1996). $\mathcal{Q.E.D}$.

Lemma A.3 *Define*

$$
\hat f_{h_m}(x,\beta):=\frac{1}{mh_m}\sum_{r=1}^{m}K\left(\frac{[m(x,\theta_0(\tau))-m(x,\theta_0(\tau_r))]+\frac{m(x,\beta(\tau))-m(x,\beta(\tau_r))}{\sqrt{n}}}{h_m}\right).
$$

Then, under Assumptions 3.1-3.6, we have that

$$
\begin{aligned}
&\sup_{a_m\le h_m\le b_m}\sup_{\tau\in\mathcal{T}}\sup_{\beta\in\mathcal{B}}\max_{1\le i\le n}\left|\hat f_{h_m}(X_i,\beta)-f\left(m(X_i,\theta_0(\tau))|X_i\right)\right|\\
=&\mathcal{O}_p\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{\log(a_m^{-1})\vee\log\log(m)}{ma_m}}+b_m^2\right).
\end{aligned}
$$

Proof of Lemma A.3. This proof follows from Lemma B.5 of Escanciano et al. (2014), just by replacing w with parametric model. Thus we omit here for brevity. $\mathcal{Q.E.D}$.

Lemma A.4 *Suppose Assumptions 3.1-3.4 hold. Then, for estimator θ_n derived from check function, we have that*

$$\sqrt{n}(\hat{\theta}_n(\tau) - \theta_0(\tau)) = \frac{\mathbb{E}^{-1}[f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}^\top(X,\theta_0(\tau))]}{\sqrt{n}} \sum_{i=1}^{n} \psi(Y_i, X_i, \theta_0(\tau)) + o_p(1).$$

Proof of Lemma A.4. Lemma A.4 derives from the Bahadur-type representation for M-estimator $\hat{\theta}_n(\cdot)$, see, e.g., Koenker and Portnoy (1987) and He and Shao (1996) for more details. $\mathcal{Q.E.D}$.

Lemma A.5 *Suppose Assumptions 3.1-3.6 hold. Then, under the null hypothesis, we have that*

$$\begin{aligned}\breve{R}_n(x,\tau) =& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(Y_i, X_i, \theta_0(\tau)) w(X_i, x) \\ &+ \mathbb{E}[f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}^\top(X,\theta_0(\tau)) w(X,x)]\sqrt{n}(\hat{\theta}_n(\tau) - \theta_0(\tau)) + o_p(1),\end{aligned}$$

uniformly in $(x,\tau) \in \Pi \times \mathcal{T}$.

Proof of Lemma A.5. The class of weight functions $\{X \to w(X,x) : x \in \Pi\}$ satisfies the conditions of Theorem 2.7.1 in Van Der Vaart and Wellner (1996). Then, by the uniform consistency of kernel estimator in Lemma A.3 and the definition of $S_n(x,\tau,b)$ in Lemma A.2, we have that

$$\begin{aligned}\breve{R}_n(x,\tau) =& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(Y_i, X_i, \theta_0(\tau)) w(X_i, x) \\ &+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\mathbb{E}[\psi(Y_i, X_i, \theta_0(\tau))|X_i] - \mathbb{E}[\psi(Y_i, X_i, \hat{\theta}_n(\tau))|X_i]\Big) w(X_i, x) + o_p(1),\end{aligned}$$

uniformly in $(x,\tau) \in \Pi \times \mathcal{T}$, where θ_0, $\hat{\theta}_n \in \mathcal{B}$ almost surely.

Then, by the mean value theorem as well as Lemma A.4, we have that

$$\begin{aligned}&\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\mathbb{E}[\psi(Y_i, X_i, \theta_0(\tau))|X_i] - \mathbb{E}[\psi(Y_i, X_i, \hat{\theta}_n(\tau))|X_i]\Big) w(X_i, x) \\ =& \left(\frac{1}{n} \sum_{i=1}^{n} f(m(X_i, \tilde{\theta}_n(\tau))|X_i)\dot{m}(X_i, \tilde{\theta}_n(\tau))\dot{m}^\top(X_i, \tilde{\theta}_n(\tau)) w(X_i, x)\right) \sqrt{n}(\hat{\theta}_n - \theta_0) + o_p(1),\end{aligned}$$

uniformly in $(x,\tau) \in \Pi \times \mathcal{T}$, where $\tilde{\theta}_n(\tau)$ lies between $\theta_0(\tau)$ and $\hat{\theta}_n(\tau)$ for each $\tau \in \mathcal{T}$ almost surely.

Next, since $(x,\tau) \to f(mX, \theta(\tau)|X)\dot{m}(X,\theta(\tau))\dot{m}^\top(X,\theta(\tau))w(X,x)$ is Glivenko–Cantelli, we further have that

$$\breve{R}_n(x,\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(Y_i, X_i, \theta_0(\tau)) w(X_i, x)$$

$$+\mathbb{E}\left[f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}^\top(X,\theta_0(\tau))w(X,x)\right]\sqrt{n}(\hat{\theta}_n-\theta_0)+o_p(1),$$

uniformly in $(x,\tau)\in\Pi\times\mathcal{T}$. $\mathcal{Q.E.D.}$

Lemma A.6 *Define $\breve{S}_n(x,\tau) = n^{-1/2}\sum_{i=1}^n g(X_i,\hat{\theta}_n(\tau))\psi(Y_i,X_i,\hat{\theta}_n(\tau))$. Then, under Assumptions 3.1-3.6, we have that,*

$$\breve{S}_n(x,\tau)=\frac{1}{\sqrt{n}}\sum_{i=1}^n g(X_i,\theta_0(\tau))\psi(Y_i,X_i,\theta_0(\tau))+\Delta(\theta_0(\tau))\sqrt{n}(\hat{\theta}_n(\tau)-\theta_0(\tau))+o_p(1),$$

uniformly in $(x,\tau)\in\Pi\times\mathcal{T}$,

Proof of Lemma A.6.

We first decompose process $\breve{S}_n(x,\tau)$ into:

$$\begin{aligned}
&\breve{S}_n(x,\tau)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^n g(X_i,\theta_0(\tau))\psi(Y_i,X_i,\theta_0(\tau))+\frac{1}{\sqrt{n}}\sum_{i=1}^n(g(X_i,\hat{\theta}_n(\tau))-g(X_i,\theta_0(\tau)))\psi(Y_i,X_i,\theta_0(\tau))\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^n g(X_i,\hat{\theta}_n(\tau))\left(\mathbb{E}[\psi(Y_i,X_i,\theta_0(\tau))|X_i]-\mathbb{E}[\psi(Y_i,X_i,\hat{\theta}_n(\tau))|X_i]\right)+o_p(1)\\
=:&\frac{1}{\sqrt{n}}\sum_{i=1}^n g(X_i,\theta_0(\tau))\psi(Y_i,X_i,\theta_0(\tau))+S_n^1(x,\tau)+S_n^2(x,\tau)+o_p(1),
\end{aligned}$$

while for $S_n^1(x,\tau)$, we have that $\sup_{(x,\tau)\in\Pi\times\mathcal{T}}|S_n^1(x,\tau)|=o_p(1)$ via the stochastic equicontinuity. Lemma A.3 and Lemma A.5 yield that

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}}|S_n^2(x,\tau)-\Delta(\theta_0(\tau))\sqrt{n}(\hat{\theta}_n(\tau)-\theta_0(\tau))|=o_p(1).$$

$\mathcal{Q.E.D.}$

Lemma A.7 *Suppose Assumptions 3.1-3.4 hold. Then, we have that*

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\hat{G}(x,\hat{\theta}_n(\tau))-G(x,\theta_0(\tau))\right|=o_p(1).$$

Proof of Lemma A.7. The proof follows from Lemma A.3, and uniformly law of large number (ULLN). $\mathcal{Q.E.D.}$

Lemma A.8 *Suppose Assumptions 3.1-3.4 hold. Then, we have that*

$$\sup_{\tau\in\mathcal{T}}\left|\hat{\Delta}^{-1}(\hat{\theta}_n(\tau))-\Delta^{-1}(\theta_0(\tau))\right|=o_p(1).$$

Proof of Lemma A.8. The proof follows from Lemma A.3, ULLN and the CMT. $\mathcal{Q.E.D.}$

Lemma A.9 *For the weight function of $w(X,x) = \mathbb{1}\left(\beta^\top X \leq x\right)$, the closed form expression of test statistics with Euclidean and Frobenius norm are:*

$$CvM_n^{\textit{lin,E}} = \lambda_{\textit{max}}(\mathcal{B}_n), \;\; CvM_n^{\textit{lin,F}} = \textit{trace}(\mathcal{B}_n),$$

respectively, where $\mathcal{B}_n$ is a p-dimensional square matrix with the expression as:

$$\mathcal{B}_n = \frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n \xi_i(\tau_r)B_{ij}\xi_j^\top(\tau_r),$$

where

$$\xi_i(\tau_r) = \psi(Y_i, X_i, \hat\theta_n(\tau_r)) - \hat g^\top(X_i,\hat\theta_n(\tau_r))\hat\Delta^{-1}(\hat\theta_n(\tau_r))\frac{1}{n}\sum_{s=1}^n \hat g(X_s,\hat\theta_n(\tau_r))\psi(X_s,Y_s,\hat\theta_n(\tau_r)),$$

and

$$B_{ij} = \sum_{v=1}^n\left(\pi - \frac{(X_i-X_v)^\top(X_j-X_v)}{||X_i-X_v||_E||X_j-X_v||_E}\right)\frac{\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}.$$

Proof of Lemma A.9. Recall the definition of projection operator with $\mathcal{P}_{n,\tau}\mathbb{1}\left(\beta^\top X_i \leq x\right)$, we have that

$$\begin{aligned}
&CvM_n^{\text{lin}}\\
=&\frac{1}{mn}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\int_\Pi \mathcal{P}_{n,\tau}\mathbb{1}\left(\beta^\top X_i \leq x\right)\psi(Y_i,X_i,\hat\theta_n(\tau_r))\psi_{\tau_r}^\top(Y_i,X_i,\hat\theta_n(\tau_r))\mathcal{P}_{n,\tau}^\top\mathbb{1}\left(\beta^\top X_j\leq x\right)F_{n,\beta}(dx)d\beta\\
=&\frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\int_\Pi \mathcal{P}_{n,\tau}\mathbb{1}\left(\beta^\top X_i\leq\beta^\top X_v\right)\psi(Y_i,X_i,\hat\theta_n(\tau_r))\\
&\times\psi_{\tau_r}^\top(Y_i,X_i,\hat\theta_n(\tau_r))\mathcal{P}_{n,\tau}^\top\mathbb{1}\left(\beta^\top X_j\leq\beta^\top X_v\right)d\beta.
\end{aligned}$$

Let $\psi_k(X,Y,\theta) := \dot m_k(X,\theta(\tau))\left(\tau - \mathbb{1}\left(Y\leq m(X,\theta(\tau))\right)\right)$ the k-th element of $\psi(X,Y,\theta)$. Then, for each (k,l)-th element, we have that

$$\begin{aligned}
(\mathcal{B}_n)_{(k,l)} =&\frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\int_\Pi\mathcal{P}_{n,\tau,k}\mathbb{1}\left(\beta^\top X_i\leq\beta^\top X_v\right)\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))\\
&\times\psi_{\tau_r l}^\top(Y_i,X_i,\hat\theta_n(\tau_r))\mathcal{P}_{n,\tau,l}^\top\mathbb{1}\left(\beta^\top X_j\leq\beta^\top X_v\right)d\beta\\
=&\frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))\psi_l(Y_j,X_j,\hat\theta_n(\tau_r))B_{ijv}\\
&-\frac{1}{mn^3}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\sum_{w=1}^n\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))\psi_l(Y_j,X_j,\hat\theta_n(\tau_r))
\end{aligned}$$

$$
\begin{aligned}
&\times g_k^\top(X_i,\hat\theta_n(\tau_r))\hat\Delta_k^{-1}(\hat\theta_n(\tau_r))g_l^\top(X_w,\hat\theta_n(\tau_r))B_{iwv}\\
&-\frac{1}{mn^3}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\sum_{u=1}^n\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))\psi_l(Y_j,X_j,\hat\theta_n(\tau_r))\\
&\times g_l^\top(X_j,\hat\theta_n(\tau_r))\hat\Delta_l^{-1}(\hat\theta_n(\tau_r))g_k^\top(X_u,\hat\theta_n(\tau_r))B_{juv}\\
&+\frac{1}{mn^4}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\sum_{v=1}^n\sum_{u=1}^n\sum_{w=1}^n\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))\psi_l(Y_j,X_j,\hat\theta_n(\tau_r))g_k^\top(X_i,\hat\theta_n(\tau_r))\\
&\times \hat\Delta_k^{-1}(\hat\theta_n(\tau_r))g_l^\top(X_w,\hat\theta_n(\tau_r))g_k(X_u,\hat\theta_n(\tau_r))\hat\Delta_k^{-1}(\hat\theta_n(\tau_r))g_l(X_j,\hat\theta_n(\tau_r))B_{uwv}\\
&=:\frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\xi_k(Y_i,X_i,\tau_r)B_{ij}\xi_l(Y_j,X_j,\tau_r),
\end{aligned}
$$

where B_{ij} is the summation of proportion of spherical wedge volume between vectors X_i-X_v and X_j-X_v, that is,

$$
\begin{aligned}
B_{ij}=&\sum_{v=1}^n\int_{\mathbb{S}^d}\mathbb{1}\left(\beta^\top X_i\le\beta^\top X_v\right)\mathbb{1}\left(\beta^\top X_j\le\beta^\top X_v\right)d\beta\\
=&\sum_{v=1}^n\left(\pi-\arccos\left(\frac{(X_i-X_v)^\top(X_j-X_v)}{||X_i-X_v||_E||X_j-X_v||_E}\right)\right)\frac{\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})},
\end{aligned}
$$

and

$$
\xi_k(Y_i,X_i,\tau_r)=\psi_k(Y_i,X_i,\hat\theta_n(\tau_r))-g_k^\top(X_i,\hat\theta_n(\tau_r))\hat\Delta_k^{-1}(\hat\theta_n(\tau_r))\frac{1}{n}\sum_{s=1}^n g_k(X_s,\hat\theta_n(\tau_r))\psi_k(X_s,Y_s,\hat\theta_n(\tau_r)),
$$

where the subscript k denotes the k-th element of vector.

Then, we can readily obtain the matrix expression of $\mathcal{B}_n$ as:

$$
\mathcal{B}_n=\frac{1}{mn^2}\sum_{r=1}^m\sum_{i=1}^n\sum_{j=1}^n\xi_i(\tau_r)B_{ij}\xi_j^\top(\tau_r),
$$

where

$$
\xi_i(\tau_r)=\psi(Y_i,X_i,\hat\theta_n(\tau_r))-\hat g^\top(X_i,\hat\theta_n(\tau_r))\hat\Delta^{-1}(\hat\theta_n(\tau_r))\frac{1}{n}\sum_{s=1}^n\hat g(X_s,\hat\theta_n(\tau_r))\psi(X_s,Y_s,\hat\theta_n(\tau_r)).
$$

At last, for the test statistics with Euclidean and Frobenius norm, we just take the largest eigenvalue and the trace, respectively. $\mathcal{Q.E.D}$.

Lemma A.10 *For the weight function of $w(X,x)=\exp\{ix^\top X\}$, the closed form expression of test statistics with Gaussian integrating meausre under Euclidean and Frobenius norm are:*

$$
CvM_n^{gau,E}=\lambda_{max}(\mathcal{U}_n),\ \textit{and}\ CvM_n^{gau,F}=trace(\mathcal{U}_n),
$$

respectively, where $\mathcal{U}_n$ is p-dimensional square matrix with the expression as:

$$\mathcal{U}_n = \frac{1}{mn^2} \sum_{r=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i(\tau_r) U_{ij} \xi_j^\top(\tau_r),$$

and

$$U_{ij} = \exp\{-\frac{||X_i - X_j||_E^2}{2}\}.$$

Likewise, the closed form expression of test statistics with Laplace integrating meausre under Euclidean and Frobenius norm are:

$$CvM_n^{lap,E} = \lambda_{max}(\mathcal{L}_n), \text{ and } CvM_n^{lap,F} = trace(\mathcal{L}_n),$$

respectively, where $\mathcal{L}_n$ is p-dimensional square matrix with the expression as:

$$\mathcal{L}_n = \frac{1}{mn^2} \sum_{r=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i(\tau_r) L_{ij} \xi_j^\top(\tau_r),$$

and

$$L_{ij} = \frac{1}{1 + ||X_i - X_j||_E^2},$$

$\xi_i(\tau_r)$ is defined the same as in Lemma A.9.

Proof of Lemma A.10. The difference between Lemma A.9 and Lemma A.10 is weight functions.

For the exponential weight function equipped with Gaussian integrating measure, we have that

$$U_{ij} = \int_{\mathbb{R}} \exp\{ix^\top(X_i - X_j) - 0.5 \cdot ||x||_E^2\}dx = \sqrt{\frac{\pi}{2}} \exp\{-\frac{||X_i - X_j||_E^2}{2}\}.$$

Likewise, for the exponential weight function equipped with Laplace integrating measure, we have that

$$L_{ij} = \int_{\mathbb{R}} \exp\{ix^\top(X_i - X_j) - ||x|_1|\}dx = \frac{2}{1 + ||X_i - X_j||_E^2}.$$

We omit other derivation here for brevity. $\mathcal{Q.E.D}$.

A.1.2 Theorem Proofs

Proof of Theorem 3.1. We first show that process $\hat{R}_n(x,\tau)$ is asymptotically equivalent to $R_n(x,\tau)$ in (14), that is,

$$
\begin{aligned}
&\hat{R}_n(x,\tau)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w(X_i,x)I_p\psi(Y_i,X_i,\hat{\theta}_n(\tau)) - G^{\top}(x,\hat{\theta}_n(\tau))\Delta^{-1}(\hat{\theta}_n(\tau))\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(X_i,\hat{\theta}_n(\tau))\psi(Y_i,X_i,\hat{\theta}_n(\tau))\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w(X_i,x)I_p\psi(Y_i,X_i,\theta_0(\tau)) + \mathbb{E}[f(m(X,\theta_0(\tau))|X)\dot{m}(X,\theta_0(\tau))\dot{m}^{\top}(X,\theta_0(\tau))w(X,x)]\\
&\times \sqrt{n}(\hat{\theta}_n(\tau)-\theta_0(\tau)) - G^{\top}(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(X_i,\theta_0(\tau))\psi(Y_i,X_i,\theta_0(\tau))\\
&-\Delta(\theta_0(\tau))\sqrt{n}(\hat{\theta}_n(\tau)-\theta_0(\tau)) + o_p(1)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w(X_i,x)I_p\psi(Y_i,X_i,\theta_0(\tau)) - G^{\top}(x,\theta_0(\tau))\Delta^{-1}(\theta_0(\tau))\\
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(X_i,\theta_0(\tau))\psi(Y_i,X_i,\theta_0(\tau)) + o_p(1)\\
=&R_n(x,\tau)+o_p(1),
\end{aligned}
$$

where the second equality follows from Lemmas A.5-A.8.

Then, the weak convergence of process $R_n(x,\tau)$ follows from the joint weak convergence of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w(X_i,x)I_p\psi(X_i,\theta_0(\tau)) \text{ and } \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(X_i,\theta_0(\tau))\psi(X_i,\theta_0(\tau)),$$

note that this joint asymptotic equicontinuity follows from that of the marginals, and a standard multivariate CLT implies the convergence of finite-dimensional distributions, where the first process follows from Lemma A.2 by applying to $\mathcal{B}=\theta_0$ and $\mathcal{W}$, while the second process also follows from Lemma A.2 by applying to $\mathcal{B}=\theta_0$ and $\{x\to g(x,\theta_0(\tau)):\tau\in\mathcal{T}\}$. $\mathcal{Q.E.D.}$

Proof of Corollary 3.1. For CvM-type test statistic, we have the following inequality that

$$
\begin{aligned}
&\left|\int_{\Pi\times\mathcal{T}} ||\hat{R}_n(x,\tau)||^2\Psi_n(dx,d\tau) - \int_{\Pi\times\mathcal{T}} ||R_\infty||^2\Psi(dx,d\tau)\right|\\
\leq&\left|\int_{\Pi\times\mathcal{T}}\left(||\hat{R}_n(x,\tau)||^2-||R_\infty||^2\right)\Psi_n(dx,d\tau)\right| + \left|\int_{\Pi\times\mathcal{T}} ||\hat{R}_n(x,\tau)||^2\left(\Psi_n(dx,d\tau)-\Psi(dx,d\tau)\right)\right|,
\end{aligned}
$$

where $\Psi_n(dx,d\tau)$ is equivalently iscretized to $\Psi_n(dx)/m$.

For the first term of the right-hand side of the above inequality is $o(1)$ a.s., due to Theorem 3.1 and Skorohod construction, and the trajectories of R_∞ are bounded and continuous almost surely. The second term of the right-hand side of the above inequality is also $o(1)$ almost surely due to

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}} |\Psi_n(dx,d\tau)-\Psi(dx,d\tau)|\to 0,\ \text{as } n\to\infty \text{ and } m\to\infty,$$

and Helly-Bray Theorem, see, e.g., Rao (1965). $\mathcal{Q.E.D.}$

Proof of Theorem 3.2. Under Assumptions 3.1-3.6, we have that

$$\begin{aligned}
&\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathcal{P}_\tau w(X_i,x)\psi(Y_i,X_i,\theta_0(\tau))-\mathbb{E}[\mathcal{P}_\tau w(X,x)\psi(X,Y,\theta_0(\tau))]\right)\right|\\
=&\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\frac{\hat{R}_n(x,\tau)}{\sqrt{n}}-\mathbb{E}[\mathcal{P}_\tau w(X,x)\psi(X,Y,\theta_0(\tau))]\right|\\
=&o_p(1),
\end{aligned}$$

where the second equality follows from Lemmas A.5-A.8 and ULLN. $\mathcal{Q.E.D.}$

Proof of Theorem 3.3. This proof is similar to Theorem 3.1 and Corollary 3.1. Under Assumptions 3.1-3.8 and a sequence of local alternatives with the form of (34), we have that

$$\begin{aligned}
\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\Bigg|\hat{R}_n(x,\tau)-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\psi(Y_i,X_i,\theta_0(\tau))-\frac{d(X_i,\tau)}{\sqrt{n}}\dot{m}(X_i,\theta_0(\tau))\right)\mathcal{P}_\tau w(X_i,x)\\
-\mathbb{E}\left[d(X,\tau)\dot{m}(X,\theta_0(\tau))\mathcal{P}_\tau w(X,x)\right]\Bigg|=o_p(1),
\end{aligned}$$

which follows from ULLN. Since $\psi(Y_i,X_i,\theta_0(\tau))-\frac{d(X_i,\tau)}{\sqrt{n}}\dot{m}(X_i,\theta_0(\tau))$ follows a mean zero IID summand, we can apply the functional central limit theorem (CLT) via its asymptotically stochastic equicontinuity and the functional central limit theorem, which lead to the distribution of $\hat{R}_n(x,\tau)$ convergence to $R_\infty+\mathbb{E}\left[d(X,\tau)\dot{m}(X,\theta_0(\tau))\mathcal{P}_\tau w(X,x)\right]$. $\mathcal{Q.E.D.}$

Proof of Corollary 3.2. This proof is as same as in Corollary 3.1, so we omit here for brevity. $\mathcal{Q.E.D.}$

Proof of Theorem 4.1. Define $\mathcal{M}:=\{(X,Y,V)\to\psi(X,Y,\theta(\tau))w(X,x)V:(x,\tau)\in\Pi\times\mathcal{T}\text{ and }\theta(\cdot)\in\mathcal{B}\}$ and $\mathcal{N}:=\{(X,Y,V)\to\psi(X,Y,\theta(\tau))\hat{f}_h\dot{m}(X,\theta(\tau))V:(\tau\in\mathcal{T}\text{ and }\theta(\cdot)\in\mathcal{B}\}$. Then, we have that $\mathcal{M}$ and $\mathcal{N}$ are P-Donsker class (Lemma A.1). Following from the consistency of $\hat{\theta}_n(\tau)$, $\dot{m}(X,\hat{\theta}_n(\tau))$ and the stochastic equicontinuity, we have that

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(Y_i,X_i,\hat{\theta}_n(\tau))w(X_i,x)V_i-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(Y_i,X_i,\theta_0(\tau))w(X_i,x)V_i\right|=o_p(1),$$

and

$$\sup_{(x,\tau)\in\Pi\times\mathcal{T}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(Y_i,X_i,\hat{\theta}_n(\tau))\hat{f}_h\dot{m}(X_i,\hat{\theta}_n(\tau))V_i\right.$$

$$-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(Y_i,X_i,\theta_0(\tau))f(m(X_i,\theta_0(\tau))|X_i)\dot{m}(X_i,\theta_0(\tau))w(X_i,x)V_i\Big| = o_p(1).$$

Recall the properties of multipliers $\{V_i\}_{i=1}^n$, and the rest of this proof follows from CMT and multiplier CLT (Theorem 2.9.2 of Van Der Vaart and Wellner (1996)) applied to the process $\hat{R}_n^*(x,\tau)$, regardless of whether the null hypothesis is holding. $\mathcal{Q.E.D}$.

A.2 Simulation Tables

A.2.1 Empirical Sizes

Table 2: Rejection rates associated with DGP1-DGP4 (n=100).

	DGP1									DGP2								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.8	4.8	9.7	0.8	6.2	11.7	0.9	4.7	4.7	0.2	2.9	9.0	0.8	5.1	11.8	0.8	4.6	4.6
$CvM_n^{\text{lin,F}}$	0.9	4.9	9.5	0.6	6.8	11.6	0.7	4.6	4.6	0.3	3.5	8.1	1.0	5.5	12.5	0.5	4.5	4.5
$CvM_n^{\text{gau,E}}$	0.9	4.6	10.3	1.0	5.7	11.5	0.7	5.0	5.0	0.3	3.1	9.4	0.8	5.6	10.9	1.0	5.4	5.4
$CvM_n^{\text{gau,F}}$	1.0	5.2	9.9	1.1	6.1	12.1	0.8	5.2	5.2	0.4	3.2	9.3	0.9	5.4	11.4	0.9	5.6	5.6
$CvM_n^{\text{lap,E}}$	0.5	4.6	10.3	1.1	5.6	11.6	0.7	4.2	4.2	0.2	3.8	8.9	1.1	5.3	10.5	0.7	5.1	5.1
$CvM_n^{\text{lap,F}}$	0.7	4.8	10.3	1.0	5.9	11.6	0.7	4.5	4.5	0.2	3.5	8.3	1.0	5.7	11.2	0.7	5.2	5.2
without projection																		
$CvM_n^{\text{lin,E,wp}}$	4.1	22.7	41.7	3.5	17.7	31.6	4.1	22.1	22.1	3.8	22.4	43.1	3.9	16.8	31.4	4.2	21.9	21.9
$CvM_n^{\text{lin,F,wp}}$	3.7	18.0	35.5	3.1	15.3	27.6	3.0	18.4	18.4	2.2	18.9	36.2	3.1	14.1	26.1	3.0	18.3	18.3
$CvM_n^{\text{gau,E,wp}}$	1.7	7.3	14.0	1.6	8.4	13.5	1.0	7.7	7.7	0.6	5.8	13.4	1.3	7.0	14.0	1.1	6.4	6.4
$CvM_n^{\text{gau,F,wp}}$	1.7	7.5	14.1	1.5	8.5	14.0	1.1	7.6	7.6	0.6	5.7	13.2	1.0	7.2	13.9	1.1	6.9	6.9
$CvM_n^{\text{lap,E,wp}}$	1.3	6.8	14.2	1.4	7.4	14.6	1.0	6.5	6.5	0.6	5.9	12.9	0.8	7.9	13.4	1.0	6.3	6.3
$CvM_n^{\text{lap,F,wp}}$	1.5	6.4	14.0	1.3	7.3	14.3	0.9	6.2	6.2	0.5	6.1	13.0	1.2	7.3	13.3	0.9	6.4	6.4
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.9	4.3	10.2	0.7	6.3	12.0	1.1	5.1	5.1	0.2	3.7	8.0	0.8	5.7	11.9	0.4	4.3	4.3
$CvM_n^{\text{gau,wg}}$	1.2	5.0	10.9	1.1	6.5	11.7	1.5	6.1	6.1	0.6	4.3	9.1	0.9	6.1	11.8	0.6	4.9	4.9
$CvM_n^{\text{lap,wg}}$	1.0	4.9	10.1	1.0	6.0	12.4	1.1	4.8	4.8	0.4	4.1	8.9	0.9	6.4	11.9	0.6	4.4	4.4
$d=5$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.4	3.3	7.5	0.4	3.6	9.1	0.2	2.7	5.7	0.6	3.1	7.2	0.6	5.3	12.2	0.3	2.1	5.5
$CvM_n^{\text{lin,F}}$	0.3	3.1	7.9	0.2	4.1	9.8	0.2	2.6	7.6	0.1	3.8	8.6	1.2	6.7	12.8	0.1	2.0	6.4
$CvM_n^{\text{gau,E}}$	0.1	2.7	7.1	0.1	4.1	9.5	0.1	2.3	6.7	0.3	3.2	6.7	0.7	5.7	12.4	0.0	1.2	6.3
$CvM_n^{\text{gau,F}}$	0.4	2.8	7.5	0.2	4.0	9.6	0.3	2.5	6.3	0.1	3.4	6.9	0.7	5.6	13.0	0.0	1.4	5.9
$CvM_n^{\text{lap,E}}$	0.2	3.2	6.1	0.1	3.4	8.6	0.1	1.3	4.3	0.1	2.6	5.2	0.3	3.6	10.2	0.0	0.4	2.5
$CvM_n^{\text{lap,F}}$	0.1	2.4	6.6	0.2	3.9	8.4	0.2	1.5	4.8	0.1	2.4	5.8	0.3	3.4	10.4	0.0	0.3	3.0
without projection																		
$CvM_n^{\text{lin,E,wp}}$	99.6	100.0	100.0	96.4	99.5	100.0	99.8	100.0	100.0	99.6	100.0	100.0	96.2	99.7	100.0	99.4	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	97.6	100.0	100.0	82.5	98.3	99.4	98.8	100.0	100.0	97.2	99.8	100.0	84.2	97.6	99.8	97.3	99.9	100.0
$CvM_n^{\text{gau,E,wp}}$	0.2	3.4	12.1	0.5	6.7	14.6	0.1	3.1	11.1	0.1	4.1	11.6	1.3	9.1	17.8	0.1	3.9	11.5
$CvM_n^{\text{gau,F,wp}}$	0.2	3.8	12.7	0.6	6.2	15.0	0.1	2.8	11.2	0.1	4.2	11.9	1.5	9.0	18.5	0.1	3.6	11.9
$CvM_n^{\text{lap,E,wp}}$	0.1	2.4	11.7	0.6	7.0	16.6	0.0	1.9	11.2	0.0	3.2	12.0	1.3	9.4	21.2	0.0	2.9	10.4
$CvM_n^{\text{lap,F,wp}}$	0.1	2.7	12.3	0.6	6.2	17.1	0.0	1.8	11.3	0.0	3.2	12.0	1.2	9.3	22.0	0.0	2.7	10.6
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.4	2.3	8.5	0.2	3.4	9.2	0.0	0.7	5.0	0.1	2.9	7.5	0.6	5.7	12.3	0.0	1.0	5.0
$CvM_n^{\text{gau,wg}}$	0.1	2.9	8.9	0.7	4.4	10.2	0.0	1.2	6.2	0.2	3.4	9.2	1.0	6.3	15.6	0.1	2.1	6.0
$CvM_n^{\text{lap,wg}}$	0.1	2.2	6.3	0.2	2.4	8.1	0.0	0.2	2.3	0.2	2.1	5.9	0.7	4.5	13.7	0.0	0.5	3.1

	DGP3									DGP4								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.9	4.4	9.4	0.7	6.1	11.7	0.8	4.0	4.0	0.7	4.3	9.9	0.7	4.9	11.1	0.4	3.9	3.9
$CvM_n^{\text{lin,F}}$	0.9	4.3	8.9	0.5	5.9	12.8	0.4	4.3	4.3	0.7	4.5	10.0	0.9	5.3	11.5	0.5	4.1	4.1
$CvM_n^{\text{gau,E}}$	1.0	4.7	9.0	0.6	6.4	12.5	0.5	4.1	4.1	1.0	4.4	9.9	0.9	5.0	10.0	0.3	4.7	4.7
$CvM_n^{\text{gau,F}}$	1.1	4.6	9.1	0.6	6.6	12.8	0.4	4.2	4.2	1.0	4.7	10.4	1.0	5.0	10.6	0.3	4.5	4.5
$CvM_n^{\text{lap,E}}$	0.7	4.4	8.3	0.9	6.6	12.1	0.5	3.8	3.8	1.1	4.5	9.5	0.7	5.0	9.1	0.4	3.9	3.9
$CvM_n^{\text{lap,F}}$	0.7	4.7	8.4	0.9	6.6	12.5	0.5	3.8	3.8	1.0	4.3	8.9	0.8	4.7	9.9	0.4	4.1	4.1
without projection																		
$CvM_n^{\text{lin,E,wp}}$	3.3	15.5	33.2	3.4	15.2	26.0	3.2	19.3	19.3	4.7	19.1	37.0	2.7	14.6	26.3	2.3	16.9	16.9
$CvM_n^{\text{lin,F,wp}}$	2.5	13.7	29.7	2.2	13.3	24.3	2.7	16.1	16.1	3.7	17.7	31.5	2.2	12.8	24.3	1.8	15.6	15.6
$CvM_n^{\text{gau,E,wp}}$	0.9	6.4	12.6	1.6	7.7	15.5	1.2	6.7	6.7	1.3	7.3	13.5	1.0	6.9	13.5	0.6	6.0	6.0
$CvM_n^{\text{gau,F,wp}}$	1.0	6.1	13.3	1.4	8.0	15.1	1.2	6.6	6.6	1.8	7.3	13.3	0.8	6.9	13.4	0.6	6.1	6.1
$CvM_n^{\text{lap,E,wp}}$	0.7	5.5	11.0	1.4	8.6	15.3	0.8	6.5	6.5	1.6	6.6	13.4	0.8	6.7	13.0	0.5	5.4	5.4
$CvM_n^{\text{lap,F,wp}}$	0.8	5.6	11.2	1.1	8.4	14.6	0.9	6.3	6.3	1.6	6.9	13.0	0.9	6.4	12.9	0.6	5.8	5.8
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.7	4.3	9.1	1.0	5.6	12.5	1.1	4.6	4.6	0.9	5.4	10.0	0.5	5.8	12.1	0.4	2.8	2.8
$CvM_n^{\text{gau,wg}}$	0.5	5.6	10.2	1.3	6.8	13.1	1.4	5.2	5.2	1.1	5.7	11.3	0.6	6.0	12.0	0.8	3.4	3.4
$CvM_n^{\text{lap,wg}}$	0.6	5.3	9.7	1.2	6.6	12.8	1.0	5.0	5.0	1.0	5.3	10.1	0.4	6.0	11.3	0.6	3.4	3.4
$d=5$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.4	3.1	6.7	0.2	1.8	5.7	0.1	0.9	3.7	0.5	3.3	6.9	0.2	3.4	7.4	0.2	1.7	4.2
$CvM_n^{\text{lin,F}}$	0.1	3.3	6.5	0.1	2.3	5.6	0.0	1.0	2.8	0.1	3.1	7.5	0.3	2.9	8.1	0.0	1.2	3.5
$CvM_n^{\text{gau,E}}$	0.4	2.6	6.4	0.1	2.7	7.1	0.1	1.1	4.4	0.1	2.4	6.7	0.3	3.9	9.1	0.0	0.8	3.2
$CvM_n^{\text{gau,F}}$	0.2	2.7	6.2	0.1	2.6	7.5	0.0	1.0	4.3	0.1	2.1	7.2	0.5	4.1	9.3	0.0	0.5	3.2
$CvM_n^{\text{lap,E}}$	0.1	2.1	5.5	0.1	2.5	6.6	0.0	0.2	1.7	0.1	2.3	6.1	0.1	3.2	7.9	0.0	0.0	1.2
$CvM_n^{\text{lap,F}}$	0.2	2.4	5.5	0.3	2.8	7.1	0.0	0.3	1.8	0.1	2.1	4.8	0.2	3.3	8.6	0.0	0.0	1.4
without projection																		
$CvM_n^{\text{lin,E,wp}}$	98.4	100.0	100.0	87.4	98.6	99.6	98.7	100.0	100.0	98.4	100.0	100.0	88.7	98.8	99.7	97.2	99.9	100.0
$CvM_n^{\text{lin,F,wp}}$	90.9	99.5	99.9	67.2	93.2	98.5	91.8	99.5	99.9	91.2	99.8	99.9	68.1	93.8	98.3	88.3	99.2	100.0
$CvM_n^{\text{gau,E,wp}}$	0.4	3.5	10.6	0.3	4.2	11.7	0.0	3.1	8.0	0.1	3.0	9.0	0.7	5.9	13.9	0.0	1.6	7.2
$CvM_n^{\text{gau,F,wp}}$	0.5	3.9	10.7	0.3	4.7	11.8	0.0	3.2	8.2	0.0	2.7	9.0	0.6	5.8	14.2	0.0	1.6	7.5
$CvM_n^{\text{lap,E,wp}}$	0.1	2.7	9.5	0.3	3.6	14.3	0.0	1.2	7.2	0.0	2.2	8.0	0.5	5.7	14.3	0.0	1.0	6.0
$CvM_n^{\text{lap,F,wp}}$	0.1	2.4	9.1	0.3	4.3	13.9	0.0	1.3	7.3	0.0	1.7	7.5	0.6	5.2	15.0	0.0	1.0	5.9
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.7	3.8	8.5	0.1	1.6	4.9	0.0	0.9	3.4	0.3	3.4	7.2	0.4	2.8	7.0	0.1	1.4	3.5
$CvM_n^{\text{gau,wg}}$	0.5	2.3	7.6	0.1	2.8	7.4	0.0	1.3	4.4	0.1	3.3	7.6	0.5	3.7	8.8	0.0	0.8	3.6
$CvM_n^{\text{lap,wg}}$	0.2	2.4	5.8	0.1	2.2	7.5	0.0	0.3	1.7	0.2	2.9	4.9	0.1	2.4	6.4	0.0	0.2	1.7

Note: This table reports the rejection rates under the null hypothese. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 3: Rejection rates associated with DGP1-DGP4 (n=200).

	DGP1									DGP2								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	1.0	4.8	9.3	0.4	3.3	8.5	1.1	4.5	10.3	0.4	4.1	10.1	1.1	4.6	9.9	0.8	4.6	9.3
$CvM_n^{\text{lin,F}}$	1.2	4.9	9.4	0.4	3.5	8.6	0.9	4.9	10.6	0.5	4.5	10.1	1.0	4.8	10.1	1.1	4.8	9.0
$CvM_n^{\text{gau,E}}$	1.0	4.9	10.5	0.3	3.9	8.9	0.9	5.0	10.0	0.5	4.0	10.7	0.8	5.2	10.2	1.0	4.4	8.9
$CvM_n^{\text{gau,F}}$	1.0	4.8	10.5	0.3	4.4	8.9	0.9	5.4	10.0	0.3	3.8	11.1	0.5	5.1	10.0	0.9	4.4	8.9
$CvM_n^{\text{lap,E}}$	1.2	4.6	10.3	0.4	4.3	8.9	0.8	4.9	9.1	0.4	4.2	9.7	0.8	5.0	10.8	1.2	4.5	9.3
$CvM_n^{\text{lap,F}}$	1.3	4.6	10.7	0.4	4.5	9.4	0.7	4.8	9.4	0.4	4.0	10.0	0.5	4.7	10.0	0.9	4.7	8.7
without projection																		
$CvM_n^{\text{lin,E,wp}}$	2.1	13.3	23.8	1.4	8.6	15.4	2.5	12.2	21.6	1.9	11.1	22.0	1.5	8.2	18.1	2.4	11.2	21.2
$CvM_n^{\text{lin,F,wp}}$	2.1	10.2	20.7	1.3	7.8	14.6	2.2	10.7	20.2	1.3	10.4	21.3	1.6	8.2	16.2	2.3	10.3	19.3
$CvM_n^{\text{gau,E,wp}}$	1.2	6.0	13.6	0.7	5.4	10.8	1.5	7.1	12.4	0.5	5.5	13.0	0.8	6.2	10.8	1.5	5.7	11.6
$CvM_n^{\text{gau,F,wp}}$	1.2	6.0	13.3	0.6	5.4	10.7	1.1	6.4	12.7	0.5	5.3	13.0	0.9	6.2	10.6	1.4	5.9	11.6
$CvM_n^{\text{lap,E,wp}}$	0.8	5.4	13.9	0.7	5.5	11.2	1.4	6.0	11.7	0.7	5.1	12.4	1.0	6.2	11.5	1.4	6.2	11.5
$CvM_n^{\text{lap,F,wp}}$	1.1	5.3	13.1	0.6	5.5	11.0	1.2	5.7	12.1	0.6	5.4	12.4	1.0	5.6	10.7	1.4	5.8	11.1
without gradient																		
$CvM_n^{\text{lin,wg}}$	1.0	4.4	9.0	0.7	4.2	9.7	1.0	5.0	9.6	0.5	4.2	10.3	0.9	4.1	9.5	0.8	4.3	8.4
$CvM_n^{\text{gau,wg}}$	1.5	5.4	9.9	0.6	4.4	9.5	1.1	5.2	10.2	0.7	4.4	10.8	1.0	4.7	8.9	0.9	4.4	9.2
$CvM_n^{\text{lap,wg}}$	1.2	4.5	9.3	0.5	4.6	10.2	1.0	4.9	10.1	0.5	4.4	10.1	1.0	3.8	10.4	0.9	4.3	8.4
$d=5$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.9	4.3	9.2	1.2	6.1	12.4	10.1	4.3	10.2	0.9	4.9	9.7	0.7	5.8	11.6	10.4	4.6	8.5
$CvM_n^{\text{lin,F}}$	0.8	4.3	8.8	1.1	6.0	12.1	0.8	4.6	9.4	0.5	5.4	11.4	0.7	5.5	11.3	0.9	4.0	10.2
$CvM_n^{\text{gau,E}}$	0.8	3.9	8.4	0.9	4.9	11.7	0.6	4.3	8.9	0.4	3.9	8.9	1.1	6.1	11.2	0.6	3.3	8.1
$CvM_n^{\text{gau,F}}$	0.7	3.9	8.7	0.8	5.2	11.9	0.5	4.2	9.1	0.3	3.7	9.2	0.9	6.3	10.9	0.8	3.1	8.5
$CvM_n^{\text{lap,E}}$	0.2	3.0	7.4	0.6	4.5	11.0	0.5	3.8	8.2	0.2	3.3	8.5	0.3	4.8	10.8	0.2	3.6	9.1
$CvM_n^{\text{lap,F}}$	0.3	3.1	8.1	0.6	5.3	11.4	0.3	3.3	5.1	0.2	3.3	8.6	0.4	4.7	9.9	0.4	4.1	4.2
without projection																		
$CvM_n^{\text{lin,E,wp}}$	82.6	97.9	99.8	50.2	86.8	95.6	79.8	97.6	99.6	81.5	97.7	99.8	50.7	83.8	94.8	79.4	97.9	99.4
$CvM_n^{\text{lin,F,wp}}$	48.6	87.7	96.3	24.5	61.2	80.7	45.2	85.8	95.4	50.7	87.9	96.7	26.2	59.1	77.7	46.0	84.8	94.8
$CvM_n^{\text{gau,E,wp}}$	0.7	3.6	9.5	1.4	6.7	14.6	0.4	4.3	10.6	0.4	4.2	10.9	1.6	7.3	13.6	0.4	3.4	9.2
$CvM_n^{\text{gau,F,wp}}$	0.7	3.9	9.9	1.3	7.3	14.3	0.4	4.6	10.3	0.3	4.2	11.0	1.5	7.7	13.5	0.5	3.3	9.1
$CvM_n^{\text{lap,E,wp}}$	0.5	3.0	8.5	1.0	7.0	15.4	0.3	3.1	9.0	0.1	2.9	9.9	0.8	6.5	14.4	0.3	3.2	8.5
$CvM_n^{\text{lap,F,wp}}$	0.6	3.0	9.2	0.8	7.3	15.9	0.3	3.4	9.5	0.0	2.4	10.4	1.1	6.7	14.7	0.4	3.3	7.8
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.3	3.5	8.9	1.0	5.8	12.6	0.4	3.8	7.6	0.2	3.9	9.2	0.8	5.5	11.3	0.2	4.4	8.2
$CvM_n^{\text{gau,wg}}$	1.0	3.9	9.1	0.9	6.2	12.1	0.5	4.1	8.3	0.5	4.2	9.3	1.2	6.1	13.0	0.5	5.1	9.4
$CvM_n^{\text{lap,wg}}$	0.3	2.9	5.7	0.5	5.4	11.4	0.3	3.3	6.3	0.1	3.5	7.8	0.6	5.1	11.4	0.2	3.6	6.7

	DGP3									DGP4								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.8	3.9	8.8	1.0	5.3	10.4	0.9	3.9	8.9	0.9	4.9	9.8	1.3	5.3	10.2	0.5	5.1	10.0
$CvM_n^{\text{lin,F}}$	0.9	4.2	9.4	1.4	6.7	9.5	0.7	3.8	10.2	0.8	5.4	10.3	1.7	6.2	10.3	0.7	5.3	9.9
$CvM_n^{\text{gau,E}}$	0.8	4.5	9.9	1.2	5.3	10.4	0.7	4.5	10.8	0.8	5.3	10.1	1.6	5.4	10.5	0.6	5.0	10.5
$CvM_n^{\text{gau,F}}$	0.7	4.7	10.2	1.5	5.5	10.8	0.6	4.5	10.4	0.9	5.2	10.3	1.5	5.6	11.3	0.7	5.2	10.7
$CvM_n^{\text{lap,E}}$	0.9	4.6	9.2	1.2	5.2	9.7	0.6	4.0	10.0	1.2	4.7	9.6	1.5	5.7	11.2	0.6	5.0	11.0
$CvM_n^{\text{lap,F}}$	0.8	4.6	9.8	1.3	5.7	9.8	0.6	3.8	10.5	0.8	5.3	10.0	1.8	5.8	11.0	0.5	4.8	10.7
without projection																		
$CvM_n^{\text{lin,E,wp}}$	1.9	9.7	18.9	2.2	8.6	16.3	1.2	8.3	17.7	1.9	10.5	20.2	2.2	9.3	17.7	2.0	9.3	17.3
$CvM_n^{\text{lin,F,wp}}$	1.9	9.3	17.2	2.7	7.8	15.0	1.2	8.2	17.4	1.5	9.8	17.9	2.4	9.0	16.7	1.8	8.9	16.9
$CvM_n^{\text{gau,E,wp}}$	1.7	6.5	12.4	1.3	6.6	11.8	0.8	4.9	11.6	1.0	6.3	11.9	1.9	6.7	12.3	1.6	6.5	11.7
$CvM_n^{\text{gau,F,wp}}$	1.6	6.2	12.6	1.7	6.3	11.5	0.7	4.8	12.1	1.0	6.1	12.1	2.0	6.5	12.8	1.6	6.5	12.2
$CvM_n^{\text{lap,E,wp}}$	1.5	5.4	12.0	1.4	6.0	11.5	0.7	5.0	10.8	1.1	5.8	11.5	2.2	6.8	13.2	1.7	5.4	12.6
$CvM_n^{\text{lap,F,wp}}$	1.5	6.0	11.9	1.6	6.7	11.2	0.6	5.3	11.9	1.4	5.6	11.9	2.1	6.8	13.4	1.5	5.8	12.5
without gradient																		
$CvM_n^{\text{lin,wg}}$	1.5	5.0	10.0	1.5	6.5	12.2	0.3	3.6	9.3	1.3	4.9	10.4	1.9	6.0	12.9	0.9	5.2	10.7
$CvM_n^{\text{gau,wg}}$	1.4	5.4	10.4	1.2	6.5	12.2	0.5	4.4	10.5	1.1	5.4	11.4	1.9	6.1	12.1	1.0	6.6	11.1
$CvM_n^{\text{lap,wg}}$	1.3	4.7	10.5	1.3	6.6	12.2	0.3	4.2	9.8	1.3	5.1	10.9	1.8	6.3	12.6	1.0	6.5	11.1
$d=5$																		
proposed tests																		
$CvM_n^{\text{lin,E}}$	0.6	3.3	8.7	0.4	3.4	8.1	0.5	4.2	9.7	0.4	4.6	8.8	0.7	4.1	8.9	0.4	4.9	9.7
$CvM_n^{\text{lin,F}}$	0.4	3.4	8.9	0.4	3.9	8.3	0.4	4.2	9.2	0.6	4.9	8.4	0.3	3.6	7.6	0.4	4.2	8.5
$CvM_n^{\text{gau,E}}$	0.5	3.3	8.3	0.6	5.0	9.7	0.2	3.8	10.2	0.7	4.9	8.9	0.7	4.5	8.7	0.3	4.4	9.9
$CvM_n^{\text{gau,F}}$	0.1	3.3	8.6	0.8	4.4	9.6	0.3	4.6	10.6	0.3	4.6	9.1	0.4	4.0	9.0	0.3	4.5	10.0
$CvM_n^{\text{lap,E}}$	0.2	3.1	7.7	0.3	4.2	8.9	0.1	3.6	8.7	0.1	4.1	8.3	0.3	3.8	7.9	0.2	3.6	8.2
$CvM_n^{\text{lap,F}}$	0.2	3.1	7.4	0.3	3.9	8.6	0.2	3.5	8.9	0.1	4.2	8.4	0.2	3.5	8.1	0.4	3.4	8.5
without projection																		
$CvM_n^{\text{lin,E,wp}}$	54.5	89.0	96.5	32.4	52.7	86.1	61.7	92.8	98.8	63.1	92.7	97.7	34.0	70.5	86.5	54.5	88.3	96.2
$CvM_n^{\text{lin,F,wp}}$	23.8	68.3	86.2	14.3	44.8	65.2	27.6	74.4	91.2	30.5	75.2	89.6	13.9	44.5	66.6	23.5	67.9	85.5
$CvM_n^{\text{gau,E,wp}}$	0.2	2.1	6.4	0.8	5.6	12.2	0.1	4.0	8.5	0.4	3.5	8.8	0.9	4.8	9.8	0.1	3.8	8.5
$CvM_n^{\text{gau,F,wp}}$	0.2	2.3	6.6	0.9	5.1	11.7	0.2	3.7	8.6	0.5	3.4	8.8	0.5	4.1	9.4	0.1	3.9	8.9
$CvM_n^{\text{lap,E,wp}}$	0.1	1.3	4.6	0.6	5.1	11.8	0.1	2.4	7.5	0.1	2.3	7.3	0.5	4.3	9.8	0.4	2.7	8.2
$CvM_n^{\text{lap,F,wp}}$	0.1	1.4	5.3	0.6	4.3	11.3	0.1	2.2	7.7	0.2	2.2	7.4	0.5	4.4	9.5	0.2	2.6	8.5
without gradient																		
$CvM_n^{\text{lin,wg}}$	0.1	3.9	7.8	0.3	3.2	8.0	0.1	2.9	6.3	0.9	4.3	8.5	0.7	3.7	7.4	0.3	2.9	7.2
$CvM_n^{\text{gau,wg}}$	0.4	4.4	7.9	0.5	4.4	8.9	0.5	3.1	8.9	0.7	4.9	9.2	0.4	3.9	9.2	0.6	3.8	9.1
$CvM_n^{\text{lap,wg}}$	0.1	3.7	7.1	0.3	3.8	8.5	0.2	2.3	4.6	0.2	3.7	7.1	0.3	4.1	8.8	0.2	2.5	6.6

Note: This table reports the rejection rates under the null hypothese. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

A.2.2 Empirical Powers (DGP5-DGP7)

Table 4: Rejection rates associated with DGP5-DGP7 (n=100).

	DGP5									DGP6									DGP7								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																											
proposed tests																											
$CvM_n^{\text{lin,E}}$	3.8	13.1	23.7	6.9	21.2	33.3	4.0	19.2	31.1	5.9	20.2	32.0	10.9	30.7	43.0	11.9	33.7	46.7	19.4	44.7	57.8	20.6	44.0	58.5	7.9	26.3	42.3
$CvM_n^{\text{lin,F}}$	3.9	14.8	28.2	9.0	24.4	37.0	4.8	22.0	34.6	6.7	22.4	36.5	12.5	34.3	48.1	14.2	38.6	53.6	22.6	51.7	65.3	24.2	51.4	63.2	9.5	33.2	48.6
$CvM_n^{\text{gau,E}}$	4.9	16.5	28.5	8.9	24.9	36.3	5.6	22.1	35.4	5.6	17.4	29.0	9.3	26.8	39.2	10.0	29.2	44.7	13.5	41.6	56.7	18.3	41.1	55.6	7.4	28.4	42.6
$CvM_n^{\text{gau,F}}$	5.2	17.4	29.3	9.3	25.0	37.2	5.9	22.4	36.1	5.8	19.4	31.2	9.9	28.7	41.5	10.7	31.4	48.5	16.2	45.2	60.1	18.7	44.2	58.4	7.7	29.7	44.2
$CvM_n^{\text{lap,E}}$	4.1	14.0	24.5	6.9	22.0	32.7	4.6	18.1	31.1	3.7	14.3	25.1	6.7	22.1	34.3	6.2	21.8	37.1	8.6	31.9	49.0	13.7	35.0	50.0	5.9	22.6	36.8
$CvM_n^{\text{lap,F}}$	4.3	14.6	26.4	7.4	23.1	33.9	4.5	19.4	32.1	4.4	16.6	28.0	8.2	24.9	38.1	8.2	26.3	42.1	11.8	38.3	53.9	16.6	40.1	54.1	6.7	26.1	40.0
without projection																											
$CvM_n^{\text{lin,E,wp}}$	33.9	64.5	79.7	32.2	60.6	75.4	31.8	67.2	83.0	47.6	76.1	87.4	49.3	75.0	87.6	54.6	85.0	92.4	66.3	90.3	96.5	62.9	85.4	93.7	59.5	85.1	94.2
$CvM_n^{\text{lin,F,wp}}$	25.5	56.0	73.3	25.6	52.3	69.8	26.0	61.1	77.7	40.6	72.8	84.4	45.7	72.2	84.5	53.0	82.6	92.0	62.1	86.7	94.8	57.0	81.2	90.6	48.5	79.9	90.3
$CvM_n^{\text{gau,E,wp}}$	16.1	36.9	50.4	20.3	40.7	54.7	15.9	41.7	55.9	22.0	47.8	61.2	31.0	56.8	68.4	34.4	62.1	73.5	49.7	74.5	83.2	46.0	71.5	80.6	33.6	61.0	73.8
$CvM_n^{\text{gau,F,wp}}$	13.3	34.2	48.1	18.3	38.8	52.6	15.1	38.1	53.7	21.5	47.2	60.9	30.7	57.3	69.5	34.9	62.5	74.0	48.4	73.6	82.8	44.6	70.4	79.6	30.0	59.2	72.3
$CvM_n^{\text{lap,E,wp}}$	11.3	32.6	44.6	17.2	36.0	50.2	13.0	37.2	51.9	17.0	40.7	56.3	25.6	50.5	64.4	27.1	56.2	68.9	41.4	69.1	78.8	40.0	65.8	76.3	26.8	53.7	68.9
$CvM_n^{\text{lap,F,wp}}$	10.1	30.8	44.0	15.9	34.8	48.7	12.4	35.2	50.4	16.6	41.2	56.0	26.2	52.0	65.4	28.7	57.8	71.1	41.8	69.2	79.5	39.6	64.8	75.4	24.2	53.3	67.2
without gradient																											
$CvM_n^{\text{lin,wg}}$	8.0	24.9	37.2	13.9	32.8	45.1	10.3	31.5	45.6	14.8	37.2	50.9	27.9	53.7	65.0	31.1	58.6	71.1	47.9	72.7	82.1	45.1	68.8	79.5	23.4	51.2	65.4
$CvM_n^{\text{gau,wg}}$	10.6	28.7	41.7	16.5	36.6	48.5	13.7	35.0	49.3	18.3	42.1	54.9	31.3	56.3	67.9	35.7	62.3	73.7	53.4	75.9	84.2	50.3	72.9	82.8	28.9	57.6	70.2
$CvM_n^{\text{lap,wg}}$	9.1	24.2	37.4	13.8	32.0	44.8	11.0	31.2	45.1	13.1	36.7	49.8	27.5	51.0	64.7	30.1	57.5	70.8	47.7	72.4	81.4	43.7	67.6	78.6	22.6	48.7	64.4
$d=5$																											
proposed tests																											
$CvM_n^{\text{lin,E}}$	12.7	47.5	78.2	41.0	80.2	92.5	26.1	73.7	90.1	2.9	10.1	20.5	3.6	12.9	23.5	0.8	6.2	14.7	10.5	38.7	67.4	10.2	33.1	48.7	2.7	11.5	23.1
$CvM_n^{\text{lin,F}}$	11.8	42.3	71.2	32.7	71.8	86.6	25.6	68.5	86.2	2.1	9.3	19.8	2.1	11.3	20.9	0.8	5.7	14.5	10.0	35.9	61.2	8.6	27.7	43.1	1.1	8.3	19.1
$CvM_n^{\text{gau,E}}$	21.3	63.9	88.5	56.6	85.0	93.4	36.3	79.0	90.8	1.8	8.4	13.5	0.9	6.9	15.8	0.1	1.7	7.5	2.5	27.4	48.3	4.2	19.5	36.3	0.7	5.8	15.0
$CvM_n^{\text{gau,F}}$	18.4	60.2	87.9	54.4	85.1	92.5	37.7	80.7	92.5	1.4	9.5	11.4	0.9	6.8	15.8	0.1	2.1	7.8	2.4	29.1	53.8	4.3	20.8	37.6	0.6	5.4	15.3
$CvM_n^{\text{lap,E}}$	8.9	40.3	70.5	34.6	71.5	86.9	11.7	52.1	77.8	0.3	4.1	9.6	0.1	3.3	11.0	0.0	0.2	2.7	1.8	19.8	37.4	1.6	12.1	25.5	0.0	2.1	7.6
$CvM_n^{\text{lap,F}}$	8.3	37.7	68.3	34.3	72.1	87.4	13.3	57.8	80.3	0.4	5.2	8.4	0.2	3.5	12.4	0.0	0.3	3.0	1.3	17.5	39.1	1.6	13.3	28.9	0.0	1.9	7.3
without projection																											
$CvM_n^{\text{lin,E,wp}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.0	99.9	99.9	99.6	100.0	100.0	100.0	100.0	100.0	99.7	100.0	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.2	100.0	100.0	94.0	99.4	99.8	99.2	100.0	100.0	99.9	100.0	100.0	97.3	99.6	99.9	99.8	100.0	100.0
$CvM_n^{\text{gau,E,wp}}$	58.1	89.3	95.5	89.1	98.0	99.0	82.6	98.5	99.5	1.2	10.0	23.7	4.1	18.0	32.3	0.8	13.1	28.2	13.6	44.8	65.9	19.3	48.3	65.5	4.1	22.7	43.2
$CvM_n^{\text{gau,F,wp}}$	50.4	85.3	94.1	85.2	97.3	98.7	79.9	97.2	99.4	1.0	8.5	22.9	4.1	17.9	32.3	0.9	13.2	28.2	12.9	43.0	63.4	17.9	44.8	62.4	2.7	20.2	39.9
$CvM_n^{\text{lap,E,wp}}$	48.0	87.4	96.2	88.1	98.6	99.3	75.7	97.5	99.4	0.6	8.1	24.4	3.3	20.0	37.4	0.3	10.5	28.6	8.7	41.8	64.4	16.6	47.8	69.0	2.4	19.3	42.9
$CvM_n^{\text{lap,F,wp}}$	37.1	82.4	93.6	82.6	97.4	98.9	69.9	96.2	99.3	0.5	7.6	23.3	3.3	18.7	36.7	0.3	10.2	28.7	7.0	38.5	61.9	14.0	43.7	64.9	1.2	15.3	38.9
without gradient																											
$CvM_n^{\text{lin,wg}}$	34.2	73.3	88.1	83.9	96.8	98.6	77.7	97.3	99.0	0.3	5.2	13.3	3.2	14.9	30.0	1.1	12.6	24.9	17.1	51.4	67.9	21.4	50.5	67.3	2.7	16.3	32.8
$CvM_n^{\text{gau,wg}}$	36.4	75.8	87.8	84.3	96.3	98.2	83.6	98.1	99.4	0.5	6.1	14.0	3.5	15.6	29.3	1.4	11.0	25.3	17.4	47.3	65.8	17.8	45.1	62.8	2.1	15.4	31.8
$CvM_n^{\text{lap,wg}}$	14.3	55.3	78.4	73.8	93.4	97.6	67.1	93.5	98.4	0.2	2.4	7.2	1.3	10.6	24.1	0.4	5.5	17.0	6.8	34.1	54.8	10.7	37.6	57.5	0.5	6.2	18.5

Note: This table reports the rejection rates under the fix hypothese. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 5: Rejection rates associated with DGP5-DGP7 (n=200).

	DGP5									DGP6									DGP7								
	$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$			$\mathcal{T}_1$			$\mathcal{T}_2$			$\mathcal{T}_3$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																											
proposed tests																											
$CvM_n^{\text{lin,E}}$	8.7	30.6	45.7	16.3	44.8	57.8	17.1	42.3	56.5	18.2	47.2	62.0	35.0	63.2	74.2	38.2	63.1	73.7	60.3	82.0	91.2	53.7	77.6	86.1	28.9	59.1	73.9
$CvM_n^{\text{lin,F}}$	12.6	35.8	52.3	23.9	50.5	64.2	22.6	49.3	63.4	21.6	54.3	68.4	40.4	69.6	80.8	44.3	70.3	80.1	67.4	87.7	93.5	58.8	82.6	89.6	34.5	67.4	82.0
$CvM_n^{\text{gau,E}}$	15.8	37.7	51.9	24.4	50.4	62.3	21.8	46.4	62.1	15.5	43.8	59.7	29.7	56.8	72.1	29.6	57.5	70.1	49.2	77.9	86.6	46.5	72.2	83.1	24.6	58.7	73.4
$CvM_n^{\text{gau,F}}$	15.8	37.8	52.3	25.4	51.9	63.3	23.2	48.0	63.0	17.8	47.6	61.6	33.3	60.6	74.8	32.8	61.8	73.6	53.7	81.0	89.3	50.0	74.6	85.0	28.1	61.0	75.8
$CvM_n^{\text{lap,E}}$	12.9	32.8	46.4	20.6	44.4	58.5	17.7	42.5	55.7	12.9	37.0	52.9	22.9	49.7	64.3	22.6	49.7	64.4	38.9	69.5	81.2	38.3	66.0	78.2	20.3	51.2	66.9
$CvM_n^{\text{lap,F}}$	12.9	33.2	48.6	21.7	46.8	60.5	20.5	44.3	58.0	15.0	42.0	57.5	28.5	55.8	70.2	28.9	57.3	69.1	47.3	76.8	86.6	44.3	70.7	81.8	23.6	55.8	71.3
without projection																											
$CvM_n^{\text{lin,E,wp}}$	52.2	77.1	87.7	58.2	80.5	88.1	55.9	81.8	90.0	75.6	91.7	96.3	83.0	94.5	97.2	82.9	94.2	97.9	95.7	99.2	99.7	91.4	98.1	99.5	87.4	97.1	98.6
$CvM_n^{\text{lin,F,wp}}$	39.8	68.7	81.7	47.5	75.3	84.2	48.0	77.2	85.9	69.9	89.6	95.5	77.6	93.4	96.7	80.5	92.8	97.0	94.9	99.1	99.5	88.3	97.3	98.8	80.0	95.6	97.7
$CvM_n^{\text{gau,E,wp}}$	37.3	62.0	74.7	46.9	72.2	81.1	42.5	71.5	80.8	57.2	80.9	88.7	70.0	88.5	93.5	70.9	87.9	92.5	89.5	97.5	98.8	83.5	94.5	97.7	73.3	90.7	95.3
$CvM_n^{\text{gau,F,wp}}$	31.2	57.7	71.0	43.3	69.3	79.4	40.1	69.0	78.8	55.1	80.7	88.5	69.6	88.1	93.5	71.2	88.4	92.4	89.6	97.2	99.0	82.6	93.9	97.5	69.2	89.7	95.3
$CvM_n^{\text{lap,E,wp}}$	29.2	55.3	68.3	41.7	67.5	77.2	37.3	65.9	77.6	50.4	75.7	85.2	63.6	83.7	90.2	63.9	84.8	90.7	85.1	96.0	98.4	78.4	91.7	95.8	64.3	87.0	92.6
$CvM_n^{\text{lap,F,wp}}$	27.3	52.7	66.0	39.0	64.6	76.0	35.2	64.2	75.8	49.2	75.8	85.6	62.6	84.8	90.0	64.9	84.7	90.7	85.6	96.6	98.4	78.3	91.4	95.5	61.9	86.0	92.8
without gradient																											
$CvM_n^{\text{lin,wg}}$	28.0	53.5	66.5	41.6	66.5	76.8	38.5	66.1	76.7	50.2	76.8	86.0	69.4	87.4	93.2	71.3	88.2	92.9	90.6	97.8	98.7	84.6	94.4	97.2	68.1	89.3	94.4
$CvM_n^{\text{gau,wg}}$	32.6	58.8	70.5	45.4	70.4	79.3	42.9	68.8	78.9	56.3	80.6	88.7	72.8	90.2	94.6	75.7	89.7	93.5	92.4	98.3	99.1	87.1	95.8	98.4	73.5	91.9	95.9
$CvM_n^{\text{lap,wg}}$	26.8	52.9	65.6	40.6	65.0	75.6	39.0	63.8	75.4	49.2	75.3	86.1	68.4	87.1	92.9	70.0	88.0	92.2	90.7	97.6	98.7	83.5	94.0	96.9	66.1	88.5	94.1
$d=5$																											
proposed tests																											
$CvM_n^{\text{lin,E}}$	75.0	96.7	99.4	97.5	99.9	100.0	98.0	99.9	99.9	3.5	22.0	30.4	9.2	24.5	43.7	13.6	45.2	6.5	52.3	90.7	91.6	42.3	74.2	85.4	25.4	64.9	89.4
$CvM_n^{\text{lin,F}}$	62.8	92.0	97.3	95.6	99.8	99.9	97.1	99.8	99.9	2.6	19.1	29.8	6.4	20.5	39.7	9.9	39.6	54.3	48.2	87.1	98.1	33.6	65.2	78.8	21.7	61.2	85.2
$CvM_n^{\text{gau,E}}$	86.4	97.4	98.9	99.3	100.0	100.0	99.4	100.0	100.0	1.8	16.4	26.6	3.2	16.5	33.8	3.2	21.0	30.8	37.2	74.2	89.1	26.9	56.7	72.6	16.3	54.2	80.5
$CvM_n^{\text{gau,F}}$	84.0	96.8	98.8	99.3	100.0	100.0	99.4	100.0	100.0	1.7	16.1	27.2	3.8	16.8	32.8	4.6	22.0	34.3	40.1	75.8	92.4	26.8	57.8	73.6	14.5	52.2	81.0
$CvM_n^{\text{lap,E}}$	72.8	93.6	97.6	98.0	100.0	100.0	97.4	99.6	100.0	1.1	13.2	21.4	3.1	12.7	28.5	2.1	9.0	17.9	21.4	58.3	80.2	18.1	48.3	66.6	9.6	32.0	57.7
$CvM_n^{\text{lap,F}}$	68.3	91.8	97.0	97.4	99.9	100.0	98.2	99.6	100.0	1.2	13.1	20.0	2.6	13.1	29.3	1.2	11.0	22.2	24.3	62.9	83.4	19.2	49.4	67.0	8.2	32.6	56.1
without projection																											
$CvM_n^{\text{lin,E,wp}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	96.7	99.5	99.9	88.5	98.4	99.0	96.6	99.8	100.0	100.0	100.0	100.0	99.8	100.0	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	78.8	96.6	98.8	67.4	91.8	96.6	87.6	98.3	99.5	99.5	100.0	100.0	96.6	99.6	99.9	96.0	100.0	100.0
$CvM_n^{\text{gau,E,wp}}$	98.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	3.8	17.1	29.0	11.1	29.1	41.3	9.0	34.7	47.3	71.3	90.2	95.6	62.1	85.2	91.9	25.5	57.9	74.5
$CvM_n^{\text{gau,F,wp}}$	98.0	99.9	100.0	99.9	100.0	100.0	100.0	100.0	100.0	3.1	16.3	28.5	10.2	28.3	40.3	8.6	33.0	46.9	69.7	89.0	95.1	57.8	83.0	90.7	20.9	52.1	71.3
$CvM_n^{\text{lap,E,wp}}$	98.4	99.9	100.0	99.9	100.0	100.0	100.0	100.0	100.0	1.9	15.7	29.0	10.7	30.1	45.1	7.0	31.1	48.3	66.2	89.1	95.2	61.4	85.5	92.8	20.5	55.2	73.6
$CvM_n^{\text{lap,F,wp}}$	96.3	99.9	100.0	99.9	100.0	100.0	100.0	100.0	100.0	1.6	13.2	26.8	8.9	28.0	42.6	7.0	29.1	46.7	62.3	88.0	94.4	54.6	83.0	91.3	14.5	46.7	68.8
without gradient																											
$CvM_n^{\text{lin,wg}}$	98.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	4.1	14.6	27.8	14.9	9.4	54.9	16.8	41.4	54.6	82.7	95.1	97.8	73.2	91.3	95.5	28.4	63.1	76.7
$CvM_n^{\text{gau,wg}}$	97.8	99.8	100.0	100.0	100.0	100.0	100.0	100.0	100.0	3.5	14.6	26.9	13.9	7.2	58.9	14.0	39.3	53.2	80.4	93.3	96.8	65.2	87.1	93.0	21.3	56.2	71.1
$CvM_n^{\text{lap,wg}}$	92.1	99.2	100.0	100.0	100.0	100.0	100.0	100.0	100.0	1.1	9.2	20.4	9.3	30.1	47.4	8.6	30.4	45.8	69.9	90.0	95.1	59.5	84.1	91.8	12.5	42.0	64.2

Note: This table reports the rejection rates under the fix hypothese. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

A.2.3 Local Powers (n=100)

Table 6: Rejection rates associated with DGP8 in interval $\mathcal{T}_1$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.4	3.2	8.3	0.3	4.5	9.6	0.6	4.1	11.7	0.5	4.2	10.4	1.1	5.6	11.4	0.9	6.0	12.0	1.6	7.5	16.3	1.6	9.0	16.5	2.0	8.9	17.4	1.8	10.3	18.6
$CvM_n^{\text{lin,F}}$	0.7	3.3	8.4	0.5	4.4	9.9	0.6	5.1	10.0	0.6	4.4	11.3	1.2	6.4	13.7	0.9	6.3	13.7	1.6	9.0	17.1	1.8	9.7	17.8	2.3	10.9	18.7	2.1	11.9	20.7
$CvM_n^{\text{gau,E}}$	0.7	3.8	9.7	0.7	5.0	9.9	0.8	5.2	11.6	0.5	5.4	11.9	1.3	6.1	12.6	1.0	7.1	13.6	1.1	9.3	15.9	1.6	9.5	17.8	2.3	9.7	19.3	1.8	11.1	19.4
$CvM_n^{\text{gau,F}}$	0.6	3.3	9.8	0.7	4.7	10.3	0.7	5.2	11.3	0.5	5.5	12.1	1.4	6.2	13.2	1.1	7.6	13.8	1.2	9.5	16.5	1.5	10.1	18.2	2.7	9.5	19.5	1.7	11.5	20.1
$CvM_n^{\text{lap,E}}$	0.6	3.7	9.5	0.5	4.3	10.2	0.4	5.0	11.0	0.6	4.5	10.5	1.2	4.9	12.0	0.5	6.4	11.9	1.1	8.3	13.7	1.3	8.2	16.5	1.9	9.4	17.2	1.1	8.7	16.0
$CvM_n^{\text{lap,F}}$	0.6	3.6	9.2	0.5	4.3	10.1	0.3	5.2	10.8	0.5	5.4	10.9	0.9	5.5	12.4	0.7	6.9	13.3	1.1	8.7	14.3	1.4	8.5	16.6	2.0	9.2	18.7	1.1	9.3	18.3
without projection																														
$CvM_n^{\text{lin,E,wp}}$	4.3	23.1	43.9	5.7	25.4	44.0	5.7	28.0	47.9	6.9	27.4	47.2	10.3	33.9	53.3	12.2	36.0	55.8	12.8	38.2	56.3	14.7	43.4	63.2	17.5	46.6	66.1	22.1	50.6	67.4
$CvM_n^{\text{lin,F,wp}}$	3.4	18.0	39.4	3.7	21.3	39.5	4.0	23.0	41.0	5.1	22.0	41.5	8.5	29.0	47.5	8.2	28.6	49.8	8.9	34.9	52.4	11.3	35.9	56.5	13.4	40.7	61.1	16.1	44.9	62.8
$CvM_n^{\text{gau,E,wp}}$	0.8	5.9	13.3	1.6	7.6	15.7	1.6	8.6	18.2	2.4	9.1	17.1	3.6	13.0	21.7	3.9	14.5	23.3	4.0	16.6	28.5	5.5	18.7	30.5	7.2	20.6	33.4	8.2	25.0	37.5
$CvM_n^{\text{gau,F,wp}}$	0.8	5.5	13.7	1.1	7.2	16.3	1.7	8.3	17.3	2.1	9.0	16.6	3.2	13.0	21.2	3.2	13.7	22.6	3.9	15.9	27.4	5.0	18.0	29.7	6.6	19.4	31.7	7.5	24.0	36.7
$CvM_n^{\text{lap,E,wp}}$	0.8	5.6	13.7	1.3	6.9	15.6	1.0	8.5	16.8	1.9	9.1	15.5	3.1	10.8	20.2	2.3	12.8	21.0	3.5	13.8	24.8	3.8	15.9	27.8	5.6	17.7	29.9	5.9	22.5	32.9
$CvM_n^{\text{lap,F,wp}}$	0.9	5.1	14.2	1.1	7.0	15.8	1.2	8.5	16.6	1.9	8.5	15.2	2.9	10.8	20.0	2.0	12.5	21.0	2.7	14.1	24.1	4.0	16.0	27.9	5.2	17.2	29.3	5.2	21.3	32.5
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.2	3.6	8.7	0.3	4.6	9.7	0.9	3.9	10.3	1.1	5.7	11.2	1.3	7.1	15.0	1.2	9.0	14.4	1.9	8.6	17.2	2.5	11.2	19.6	4.9	13.2	22.7	3.3	15.5	26.0
$CvM_n^{\text{gau,wg}}$	0.5	3.8	9.5	0.4	4.8	11.2	1.0	5.3	11.9	1.1	6.5	12.7	1.7	9.7	16.2	1.9	9.4	16.3	2.3	10.8	19.6	3.6	13.1	21.6	5.8	15.8	25.3	5.0	19.1	28.9
$CvM_n^{\text{lap,wg}}$	0.5	3.9	10.0	0.3	4.5	10.3	0.9	4.3	11.5	1.1	6.5	11.5	1.4	8.3	15.6	1.5	8.9	15.1	1.8	8.7	17.9	2.5	11.6	20.5	4.2	13.8	22.2	2.9	16.1	25.5
$d=5$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.2	2.6	8.3	0.5	3.5	9.0	0.5	4.0	9.0	0.7	5.5	12.3	1.0	6.8	12.1	1.5	7.8	17.6	2.1	10.8	20.8	2.9	13.1	22.3	2.4	12.7	23.8	3.0	14.4	25.5
$CvM_n^{\text{lin,F}}$	0.2	2.6	7.5	0.1	3.6	8.5	0.1	2.8	8.1	0.4	4.2	11.8	0.4	4.2	8.6	0.5	6.7	14.3	1.5	7.5	17.6	1.1	9.4	19.5	0.7	8.7	19.5	1.8	9.8	20.4
$CvM_n^{\text{gau,E}}$	0.2	1.3	5.1	0.1	1.5	5.5	0.0	2.4	7.6	0.2	2.3	7.7	0.2	2.8	8.9	0.1	3.6	10.6	0.1	4.7	14.3	0.2	6.1	14.6	0.2	5.6	14.9	0.8	5.2	16.7
$CvM_n^{\text{gau,F}}$	0.2	1.3	5.3	0.0	1.7	5.1	0.0	2.2	7.4	0.1	2.3	6.9	0.1	2.5	3.6	0.2	3.5	10.6	0.0	4.7	14.1	0.2	5.8	13.6	0.2	5.7	15.0	1.0	4.8	16.8
$CvM_n^{\text{lap,E}}$	0.0	0.7	2.2	0.0	0.7	2.4	0.0	0.5	3.8	0.0	0.9	3.6	0.0	0.9	4.2	0.0	1.1	5.6	0.0	1.7	6.7	0.0	1.5	7.6	0.0	1.4	7.5	0.0	2.3	7.7
$CvM_n^{\text{lap,F}}$	0.0	0.5	2.9	0.0	0.7	2.3	0.0	0.7	3.9	0.0	1.0	3.5	0.0	0.5	100.0	0.0	1.5	5.4	0.0	2.0	6.3	0.0	1.5	7.7	0.0	1.5	8.1	0.1	2.2	8.4
without projection																														
$CvM_n^{\text{lin,E,wp}}$	99.8	100.0	100.0	99.8	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	97.6	99.9	100.0	97.8	100.0	100.0	99.1	100.0	100.0	98.4	100.0	100.0	98.5	100.0	22.4	98.8	100.0	100.0	99.2	99.9	100.0	99.1	100.0	100.0	99.6	100.0	100.0	99.8	100.0	100.0
$CvM_n^{\text{gau,E,wp}}$	0.3	3.6	11.3	0.4	3.5	13.3	0.6	7.2	16.4	1.0	6.9	19.7	1.1	10.8	21.9	1.5	11.9	27.4	1.8	16.5	33.7	3.2	20.4	36.3	3.5	22.6	40.6	5.2	25.8	46.2
$CvM_n^{\text{gau,F,wp}}$	0.3	3.7	11.1	0.2	3.7	13.4	0.6	6.6	16.2	0.9	6.5	19.4	0.8	10.1	23.8	1.1	10.8	26.2	1.4	15.3	31.9	2.2	17.9	33.4	2.7	20.3	39.9	3.6	22.9	43.5
$CvM_n^{\text{lap,E,wp}}$	0.3	2.7	11.1	0.1	3.2	13.3	0.4	5.5	18.5	0.3	6.0	20.0	0.6	9.5	22.3	0.7	10.8	29.4	0.5	15.1	35.2	1.3	17.6	38.3	2.2	20.7	44.2	3.1	23.2	48.6
$CvM_n^{\text{lap,F,wp}}$	0.2	2.7	10.8	0.1	3.1	12.5	0.3	4.9	16.6	0.4	4.8	19.4	0.2	8.4	13.8	0.2	9.9	27.2	0.3	13.6	31.9	0.9	14.6	34.4	1.3	18.0	40.6	2.1	19.4	43.2
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.0	1.6	5.1	0.0	1.7	5.6	0.3	2.0	7.2	0.5	3.4	9.8	0.7	4.7	13.6	0.7	6.9	15.9	1.0	9.9	20.7	1.4	11.2	25.6	2.3	14.1	27.9	3.1	15.7	31.7
$CvM_n^{\text{gau,wg}}$	0.2	1.8	6.6	0.1	2.1	6.8	0.3	3.2	9.3	0.4	3.8	10.9	0.4	5.6	7.6	0.4	7.3	16.8	0.7	9.5	19.9	1.1	10.3	23.6	0.6	13.2	25.3	1.9	13.4	30.0
$CvM_n^{\text{lap,wg}}$	0.0	0.5	2.9	0.0	1.1	3.0	0.1	1.1	4.4	0.0	1.3	5.5	0.1	2.0	32.5	0.2	2.5	9.8	0.2	3.9	12.4	0.2	4.1	13.4	0.1	5.2	16.7	0.5	5.9	17.8

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 7: Rejection rates associated with DGP8 in interval $\mathcal{T}_2$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	1.3	5.7	12.3	1.1	5.1	10.6	0.9	5.1	11.5	1.6	6.8	14.8	2.2	8.7	15.7	2.6	10.2	17.7	2.4	10.7	17.8	3.1	10.4	19.5	4.0	13.5	22.7	4.7	15.4	25.2
$CvM_n^{\text{lin,F}}$	1.3	6.1	12.6	0.9	5.3	12.0	0.8	5.4	11.2	1.5	6.3	14.6	2.0	8.9	15.8	3.2	10.9	17.9	2.6	10.9	19.0	2.7	11.7	20.8	4.2	14.1	24.5	5.6	17.0	28.2
$CvM_n^{\text{gau,E}}$	1.2	6.8	12.7	0.8	5.1	11.7	1.1	5.3	10.3	1.0	6.5	14.5	2.0	8.5	15.8	2.4	9.5	16.6	1.7	10.7	18.3	2.4	10.7	18.2	3.3	13.3	21.7	4.5	14.7	24.0
$CvM_n^{\text{gau,F}}$	1.0	6.7	12.1	0.7	4.9	11.6	1.1	5.0	10.4	1.1	6.7	13.9	1.7	9.1	15.9	2.6	10.2	16.7	1.8	10.9	18.9	2.9	11.0	18.8	4.4	13.7	21.8	5.0	15.3	25.2
$CvM_n^{\text{lap,E}}$	1.4	6.0	12.0	0.6	4.5	10.5	0.9	6.0	11.5	0.9	6.8	12.4	1.5	8.2	15.3	2.4	9.5	15.4	2.1	9.6	17.6	1.8	10.1	16.8	2.9	11.5	20.3	3.6	12.9	21.6
$CvM_n^{\text{lap,F}}$	1.2	6.0	11.8	0.7	4.4	10.9	1.0	6.4	11.8	0.9	6.9	12.6	1.6	8.2	15.7	2.5	9.5	15.9	2.0	9.9	18.1	2.4	10.5	17.9	3.6	12.4	21.5	4.1	14.5	22.6
without projection																														
$CvM_n^{\text{lin,E,wp}}$	4.1	17.0	32.9	3.9	16.4	30.9	4.0	17.1	31.7	4.6	18.8	34.4	6.6	23.5	39.2	9.7	29.0	44.1	10.3	29.3	45.8	11.7	34.3	49.8	15.6	41.0	57.0	18.4	41.1	57.4
$CvM_n^{\text{lin,F,wp}}$	3.9	15.4	29.6	2.9	14.8	27.5	2.9	15.3	27.5	3.9	17.6	31.5	5.7	21.5	36.0	8.6	25.9	40.8	8.5	25.4	41.0	10.0	31.3	46.0	14.1	36.7	53.7	16.5	39.4	54.1
$CvM_n^{\text{gau,E,wp}}$	2.4	8.5	16.1	1.1	6.7	15.5	1.4	8.4	14.8	2.4	9.1	17.1	3.0	12.9	21.0	5.2	16.4	23.8	4.1	16.9	26.1	5.7	18.2	28.8	8.9	23.9	35.6	9.7	26.5	38.6
$CvM_n^{\text{gau,F,wp}}$	2.2	7.8	16.0	1.2	7.4	14.4	1.6	8.9	14.9	2.3	9.2	17.0	3.4	12.3	21.3	5.1	15.7	24.3	4.4	16.2	26.7	5.7	17.1	29.2	8.6	22.3	35.6	9.3	26.2	37.6
$CvM_n^{\text{lap,E,wp}}$	2.3	7.5	15.6	0.9	6.7	13.3	1.6	8.8	16.0	1.9	8.6	15.5	2.9	12.0	19.5	4.9	14.4	22.7	3.4	15.4	25.7	4.5	16.8	26.2	7.9	21.5	34.0	8.5	23.4	34.4
$CvM_n^{\text{lap,F,wp}}$	2.3	7.7	15.0	1.2	6.6	13.8	1.5	8.4	16.0	2.1	8.8	15.6	3.1	11.9	20.1	4.9	14.4	22.8	3.6	16.1	24.8	5.0	16.4	26.4	7.9	21.1	34.1	8.1	24.1	35.0
without gradient																														
$CvM_n^{\text{lin,wg}}$	2.1	7.0	12.8	0.9	5.4	11.5	0.8	6.5	12.1	1.6	6.1	12.2	1.9	9.7	17.4	4.0	11.7	19.7	3.2	12.8	20.7	4.3	14.0	24.0	5.6	17.8	28.4	7.8	19.8	31.0
$CvM_n^{\text{gau,wg}}$	1.7	7.3	12.9	0.7	5.7	11.5	0.8	6.4	12.8	1.7	6.6	13.3	2.5	10.1	17.7	4.2	12.2	19.9	3.3	13.9	22.6	5.0	15.8	25.0	6.4	18.9	28.6	8.7	22.1	34.6
$CvM_n^{\text{lap,wg}}$	1.6	7.3	12.7	1.0	4.7	10.9	0.6	6.8	12.0	1.6	6.3	12.2	1.9	9.4	16.9	3.8	10.6	18.6	3.2	11.7	22.0	4.3	13.5	23.7	6.4	17.1	27.8	7.3	18.8	31.1
$d=5$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.4	4.5	10.5	1.4	7.1	15.1	1.2	7.6	15.1	3.0	11.7	20.4	3.3	14.7	22.4	4.1	19.1	32.8	5.9	21.6	36.2	7.8	27.1	43.6	10.4	32.2	49.3	14.2	38.2	55.8
$CvM_n^{\text{lin,F}}$	0.2	4.5	11.5	0.8	7.1	14.2	1.3	7.2	14.4	1.7	9.8	19.9	2.2	11.9	18.2	3.0	17.2	28.2	4.9	18.1	32.1	5.7	21.3	38.7	6.8	27.3	42.2	11.1	33.9	50.2
$CvM_n^{\text{gau,E}}$	0.3	4.3	10.5	0.5	4.4	12.7	0.5	5.9	12.6	1.7	8.3	17.1	0.9	8.9	18.2	2.5	13.0	24.0	1.9	14.1	26.0	2.5	19.2	31.3	4.8	21.5	36.7	6.2	25.9	40.9
$CvM_n^{\text{gau,F}}$	0.3	3.6	11.1	0.3	4.7	12.9	0.6	5.5	13.3	1.5	8.7	17.2	1.1	9.6	13.4	2.4	13.0	24.5	2.5	14.9	27.1	3.1	19.1	32.0	4.9	22.4	37.4	6.4	26.8	41.8
$CvM_n^{\text{lap,E}}$	0.2	2.1	7.3	0.1	2.8	8.9	0.3	3.5	9.4	0.3	5.2	11.4	0.1	6.2	14.6	1.2	7.5	18.2	0.3	7.4	19.9	1.0	11.5	23.6	1.5	12.3	28.6	2.4	15.7	31.2
$CvM_n^{\text{lap,F}}$	0.2	2.2	8.0	0.2	2.7	9.7	0.3	3.3	10.5	0.4	5.3	12.7	0.2	5.5	100.0	1.2	8.0	18.7	0.3	8.0	20.4	1.0	12.7	25.2	1.5	14.1	29.5	2.6	16.7	33.6
without projection																														
$CvM_n^{\text{lin,E,wp}}$	97.1	99.7	99.9	96.2	99.5	99.9	96.5	99.6	99.9	98.2	99.7	100.0	98.4	99.9	99.8	98.6	100.0	100.0	99.4	100.0	100.0	99.6	100.0	100.0	99.5	100.0	100.0	99.9	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	85.9	98.7	99.5	84.6	98.3	99.5	85.4	97.8	99.6	89.2	99.2	99.8	91.3	99.5	35.1	93.1	98.9	99.9	94.7	99.7	100.0	96.5	99.7	100.0	97.4	99.6	100.0	98.0	100.0	100.0
$CvM_n^{\text{gau,E,wp}}$	0.9	7.1	15.8	1.2	10.8	19.7	1.7	11.3	24.3	3.4	16.2	26.6	4.7	19.7	34.3	8.9	26.6	41.6	9.8	31.2	47.3	14.1	39.8	54.6	18.7	46.1	63.0	24.8	53.3	71.4
$CvM_n^{\text{gau,F,wp}}$	0.7	8.1	16.8	1.4	10.3	19.9	1.5	11.1	23.8	3.0	15.8	27.3	4.3	19.2	40.4	7.9	25.6	40.8	8.5	29.6	46.2	12.5	38.0	53.4	17.6	44.9	61.3	22.3	50.8	70.1
$CvM_n^{\text{lap,E,wp}}$	0.4	7.0	19.9	1.1	10.4	22.5	1.4	12.1	27.6	2.6	17.1	30.9	4.7	21.2	38.5	8.4	28.8	46.1	8.8	32.6	52.2	12.7	41.6	60.2	17.1	48.4	66.9	23.1	56.0	74.3
$CvM_n^{\text{lap,F,wp}}$	0.5	7.5	20.2	1.1	10.1	22.4	1.4	11.7	27.9	2.5	16.3	31.3	3.5	19.9	28.0	7.2	26.7	45.7	7.1	30.8	50.3	10.6	38.2	57.9	13.7	45.5	64.9	19.1	52.5	72.7
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.4	3.7	11.9	0.8	6.7	13.5	1.0	8.7	16.8	1.8	10.3	21.8	2.7	14.3	26.8	5.6	21.8	36.4	7.7	27.7	44.2	10.4	35.6	51.7	15.1	42.5	58.3	22.1	52.4	67.3
$CvM_n^{\text{gau,wg}}$	0.5	5.7	12.5	0.7	7.7	15.7	1.2	8.9	17.3	2.9	11.3	22.3	3.4	13.4	21.6	6.2	20.2	33.2	7.1	24.4	40.0	9.4	33.1	48.2	13.3	38.7	54.6	19.2	46.5	62.2
$CvM_n^{\text{lap,wg}}$	0.1	3.1	10.3	0.3	4.7	12.6	0.7	6.1	13.9	1.3	8.2	18.1	1.4	10.2	40.8	3.4	15.3	29.8	3.7	17.9	35.4	4.9	25.7	42.7	7.8	31.1	50.2	13.3	39.0	56.9

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 8: Rejection rates associated with DGP8 in interval $\mathcal{T}_3$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10	
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%
$d=2$																													
proposed tests																													
$CvM_n^{\text{lin,E}}$	1.0	4.8	8.3	1.1	6.1	10.4	0.7	6.1	11.6	0.8	5.2	12.0	1.3	6.7	11.5	1.0	5.9	12.8	2.7	8.9	15.3	2.3	9.5	16.9	3.1	12.7	21.4	4.4	13.4
$CvM_n^{\text{lin,F}}$	0.9	5.8	9.3	1.2	6.4	11.4	0.6	6.2	10.8	1.0	5.8	12.9	1.4	7.2	12.5	0.9	6.1	13.4	2.9	9.5	16.4	2.8	10.5	18.2	3.7	13.4	23.5	4.8	15.7
$CvM_n^{\text{gau,E}}$	0.6	5.1	10.0	1.0	5.2	12.1	0.5	5.9	11.3	1.1	6.3	13.5	1.4	6.9	11.9	0.8	6.3	13.7	2.2	8.3	14.5	1.9	9.0	16.7	2.6	11.3	20.5	3.1	14.3
$CvM_n^{\text{gau,F}}$	0.6	5.2	10.0	1.2	5.5	12.2	0.7	6.0	11.2	1.3	6.5	14.0	1.5	7.3	12.3	0.7	5.9	14.3	2.6	8.7	15.1	2.3	9.1	18.2	2.7	13.0	21.5	3.7	14.7
$CvM_n^{\text{lap,E}}$	0.4	4.8	9.2	0.5	4.4	10.8	0.2	5.2	10.5	0.7	5.6	12.2	1.2	6.1	10.9	0.7	5.1	12.0	1.7	7.1	13.2	1.3	7.9	14.0	1.4	9.7	18.3	2.2	11.2
$CvM_n^{\text{lap,F}}$	0.5	4.8	10.1	0.6	4.8	11.2	0.4	5.4	11.1	0.8	5.9	12.5	1.5	6.7	11.1	0.7	5.6	12.7	2.2	7.8	13.6	1.7	8.7	15.2	1.6	9.7	20.1	2.5	12.6
without projection																													
$CvM_n^{\text{lin,E,wp}}$	4.2	23.8	42.9	4.4	25.6	45.9	4.7	25.3	46.0	6.1	27.6	48.6	7.6	27.5	49.2	7.3	34.1	53.8	10.5	37.2	58.5	12.3	40.3	60.9	15.1	44.3	62.3	17.7	51.9
$CvM_n^{\text{lin,F,wp}}$	3.4	20.4	36.5	4.9	22.3	39.7	4.5	21.9	39.2	4.8	23.8	44.3	6.7	24.5	42.7	6.5	30.1	49.9	9.0	33.5	53.3	12.0	35.6	57.0	14.7	42.2	59.6	18.1	48.0
$CvM_n^{\text{gau,E,wp}}$	1.5	7.6	14.8	1.5	7.6	16.1	1.6	7.6	15.2	1.7	10.6	18.3	2.9	11.5	19.8	2.4	12.0	22.4	4.2	15.4	25.9	5.3	18.0	30.2	6.4	21.6	33.3	9.0	26.7
$CvM_n^{\text{gau,F,wp}}$	1.3	7.6	15.1	1.6	7.8	16.9	1.6	8.3	16.0	2.0	10.5	18.8	3.1	11.6	19.3	2.1	12.5	22.4	4.3	15.7	25.5	5.7	17.9	30.1	6.8	21.5	32.9	8.7	27.2
$CvM_n^{\text{lap,E,wp}}$	1.2	6.9	13.7	1.4	6.9	14.3	1.2	7.6	15.2	1.5	8.9	17.5	2.6	10.5	17.6	2.0	11.0	20.9	2.8	12.9	21.8	4.2	15.3	26.2	4.9	18.7	28.1	7.6	22.8
$CvM_n^{\text{lap,F,wp}}$	1.2	6.9	13.9	1.4	7.2	15.6	1.3	7.7	15.1	1.5	9.6	18.5	2.7	10.5	18.6	1.7	11.5	21.0	3.5	13.7	22.9	3.7	16.7	26.4	5.8	19.6	29.1	7.9	24.1
without gradient																													
$CvM_n^{\text{lin,wg}}$	0.5	3.6	9.5	0.6	5.2	10.9	0.8	4.7	10.9	1.1	6.0	14.3	1.6	7.4	13.3	1.1	8.4	14.6	3.1	9.2	18.2	3.6	13.0	22.3	4.1	15.1	24.3	6.4	18.7
$CvM_n^{\text{gau,wg}}$	0.7	5.2	10.3	0.8	6.4	12.4	1.1	5.2	11.7	1.3	8.0	15.1	1.9	8.5	14.4	1.6	9.0	16.7	3.7	11.8	21.2	4.1	15.7	23.7	5.2	17.3	27.9	7.6	22.6
$CvM_n^{\text{lap,wg}}$	0.5	4.6	10.0	0.9	5.4	11.7	0.9	5.2	10.6	1.0	7.0	14.8	1.7	8.0	13.4	1.2	8.4	14.8	3.0	9.8	18.5	3.2	13.1	22.1	3.8	15.3	24.8	6.7	19.5
$d=5$																													
proposed tests																													
$CvM_n^{\text{lin,E}}$	0.5	3.9	8.1	0.1	3.1	8.3	0.8	4.3	9.5	1.1	5.3	12.4	1.0	6.9	16.3	1.1	10.5	22.1	2.7	14.9	30.9	5.4	21.9	35.3	6.6	25.9	42.8	6.9	30.9
$CvM_n^{\text{lin,F}}$	0.4	3.8	9.2	0.1	3.0	9.8	0.4	4.2	10.9	0.5	4.8	10.9	0.8	7.0	11.8	0.8	9.8	21.9	1.9	13.8	29.4	2.7	18.7	35.0	5.4	24.5	39.7	5.7	29.5
$CvM_n^{\text{gau,E}}$	0.0	2.5	7.0	0.0	1.6	6.3	0.0	2.4	7.2	0.1	3.0	8.9	0.0	2.6	11.9	0.4	5.8	15.8	0.8	9.0	21.3	0.6	12.7	26.8	1.9	16.8	33.5	1.5	20.1
$CvM_n^{\text{gau,F}}$	0.0	2.2	7.0	0.0	1.5	6.8	0.0	2.6	7.7	0.1	3.5	10.0	0.0	3.2	4.5	0.4	6.0	16.3	1.1	9.2	22.9	0.7	13.1	28.4	2.0	17.6	34.3	2.4	21.6
$CvM_n^{\text{lap,E}}$	0.0	0.7	3.5	0.0	0.4	2.6	0.0	0.7	3.0	0.1	0.7	3.4	0.0	0.7	4.9	0.0	1.4	6.9	0.1	2.9	10.6	0.0	3.1	15.6	0.1	4.2	17.7	0.1	7.2
$CvM_n^{\text{lap,F}}$	0.0	0.7	3.9	0.0	0.5	3.2	0.0	0.7	3.9	0.1	0.6	4.8	0.0	1.2	100.0	0.0	1.6	8.1	0.2	3.5	12.1	0.0	3.7	16.7	0.3	5.1	20.2	0.2	9.5
without projection																													
$CvM_n^{\text{lin,E,wp}}$	99.7	100.0	100.0	99.6	100.0	100.0	99.8	100.0	100.0	99.6	100.0	100.0	99.8	100.0	100.0	99.7	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	99.8	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	97.5	100.0	100.0	98.4	100.0	100.0	98.8	100.0	100.0	98.5	100.0	100.0	98.9	100.0	27.2	99.4	100.0	100.0	99.6	100.0	100.0	100.0	100.0	100.0	99.7	100.0	100.0	99.8	100.0
$CvM_n^{\text{gau,E,wp}}$	0.3	5.7	13.3	0.1	5.1	16.8	0.5	7.0	15.6	1.2	8.7	21.5	1.0	13.6	27.4	3.5	20.3	37.3	5.9	30.1	47.7	9.1	38.1	54.9	12.1	45.6	65.5	16.9	56.1
$CvM_n^{\text{gau,F,wp}}$	0.2	5.8	13.9	0.1	5.2	16.3	0.5	6.9	16.8	1.2	8.4	20.9	0.7	12.7	27.7	2.7	19.8	37.3	5.4	29.6	46.8	8.8	37.7	54.9	11.9	44.4	64.6	16.3	54.6
$CvM_n^{\text{lap,E,wp}}$	0.1	4.4	13.6	0.1	3.4	15.7	0.2	6.6	15.8	0.7	6.8	21.9	0.3	11.9	27.7	1.5	18.0	38.3	3.9	26.4	48.7	4.8	34.6	55.1	8.9	41.5	66.4	11.2	50.1
$CvM_n^{\text{lap,F,wp}}$	0.1	4.4	13.2	0.1	4.1	15.6	0.0	6.1	16.8	0.6	6.6	21.5	0.2	11.0	20.9	1.4	16.8	37.3	3.0	25.9	47.8	4.2	33.4	54.9	8.3	39.9	65.7	10.9	48.8
without gradient																													
$CvM_n^{\text{lin,wg}}$	0.0	2.3	7.1	0.1	2.0	8.4	0.1	4.8	11.0	0.6	5.0	12.7	1.2	9.8	21.6	1.8	16.9	30.7	5.3	24.6	43.4	7.1	32.8	49.8	13.5	41.4	61.4	17.4	50.6
$CvM_n^{\text{gau,wg}}$	0.2	2.8	8.9	0.0	3.0	10.4	0.2	4.8	11.2	0.7	5.2	13.6	0.7	9.2	13.4	1.9	15.7	30.0	5.3	23.4	39.7	6.7	32.8	47.7	11.8	41.0	59.5	16.5	49.8
$CvM_n^{\text{lap,wg}}$	0.0	1.0	5.0	0.0	0.4	4.4	0.0	1.9	7.0	0.0	2.2	7.8	0.2	4.1	38.3	0.4	7.4	20.6	1.7	14.2	29.9	2.3	20.4	39.7	5.2	26.3	48.1	6.6	34.7

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

A.2.4 Local Powers (n=200)

Table 9: Rejection rates associated with DGP8 in interval $\mathcal{T}_1$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	1.1	4.8	9.1	0.7	5.0	10.6	1.8	6.0	12.4	1.4	8.1	13.8	2.2	9.1	18.0	2.1	11.8	20.1	2.8	12.1	20.8	4.2	15.8	24.1	5.3	19.3	28.6	6.9	22.6	34.4
$CvM_n^{\text{lin,F}}$	0.8	4.3	9.4	0.9	4.7	11.0	1.9	6.7	12.5	0.9	8.7	14.9	2.7	9.9	18.5	2.2	13.1	21.6	2.8	13.2	23.0	5.2	17.3	25.5	6.8	22.7	31.6	7.5	25.4	37.8
$CvM_n^{\text{gau,E}}$	1.1	4.8	9.4	0.7	5.5	11.2	1.2	5.6	12.2	1.6	8.0	14.9	2.5	10.7	19.6	2.3	11.6	20.4	3.1	13.2	21.4	4.1	14.6	24.6	6.2	20.1	30.1	6.1	21.7	34.8
$CvM_n^{\text{gau,F}}$	1.0	5.0	9.3	0.8	5.1	11.2	1.3	6.0	11.8	1.7	8.3	14.7	2.4	10.8	19.7	2.2	12.0	20.6	2.8	13.8	21.5	4.4	15.1	25.6	6.6	20.8	30.0	6.3	22.6	35.9
$CvM_n^{\text{lap,E}}$	0.9	4.3	9.5	0.7	5.6	11.2	1.2	5.0	11.2	1.7	7.3	14.0	1.8	9.4	17.1	1.9	10.5	19.2	3.1	11.2	18.9	2.9	12.9	22.8	4.6	17.6	27.0	4.5	18.9	29.8
$CvM_n^{\text{lap,F}}$	0.8	4.4	9.1	0.7	5.7	11.2	1.2	6.2	11.5	1.9	7.3	14.2	1.8	9.8	17.4	1.8	11.0	19.8	2.7	11.7	20.0	3.6	13.9	24.1	5.0	18.6	28.8	5.7	19.6	32.6
without projection																														
$CvM_n^{\text{lin,E,wp}}$	1.9	11.0	21.3	2.9	13.9	24.7	4.7	16.4	28.8	5.5	18.9	32.1	9.9	27.2	39.7	13.2	31.6	45.1	15.2	34.4	50.1	18.1	43.4	58.2	26.1	49.6	64.4	27.3	54.8	68.8
$CvM_n^{\text{lin,F,wp}}$	1.4	9.2	19.7	1.5	12.1	22.3	3.8	15.0	26.6	4.6	17.3	30.0	8.0	23.0	36.7	9.9	29.5	41.8	12.9	31.2	46.0	14.7	37.1	53.1	21.1	44.5	59.1	23.9	51.6	65.5
$CvM_n^{\text{gau,E,wp}}$	1.0	5.5	11.2	1.1	8.0	15.3	2.7	9.1	15.7	2.7	12.3	19.4	5.3	17.8	26.2	6.5	21.0	31.4	9.2	23.5	34.2	11.6	27.1	40.9	15.5	35.4	47.0	17.4	39.3	53.5
$CvM_n^{\text{gau,F,wp}}$	0.9	4.9	11.7	1.0	7.2	15.3	2.6	9.4	16.3	2.6	12.0	19.6	4.8	17.0	25.9	6.5	20.5	31.1	7.8	22.0	33.4	11.1	25.6	39.8	14.4	34.0	46.8	16.5	37.5	53.5
$CvM_n^{\text{lap,E,wp}}$	1.1	5.2	12.0	1.1	7.9	14.2	1.9	8.8	15.7	2.8	10.3	19.3	4.4	14.7	24.8	5.5	17.5	28.8	6.8	18.9	31.4	9.5	24.6	36.2	13.2	30.5	43.3	14.4	34.0	47.7
$CvM_n^{\text{lap,F,wp}}$	1.1	5.0	11.2	1.0	7.2	13.9	1.9	8.4	15.8	2.7	9.8	19.3	4.3	15.1	23.5	4.8	17.3	28.7	6.5	18.7	30.1	9.3	23.8	35.9	12.7	30.1	43.1	13.7	34.1	48.1
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.5	4.9	8.8	0.6	5.6	11.0	1.9	8.0	13.6	1.5	8.8	16.1	3.6	12.9	20.2	3.6	16.0	24.2	4.5	16.8	26.3	7.2	21.7	31.7	10.5	29.2	39.2	12.5	31.7	45.0
$CvM_n^{\text{gau,wg}}$	1.1	4.6	9.4	0.9	6.1	12.5	2.3	8.4	14.5	1.8	10.2	17.9	4.3	14.6	23.1	4.8	17.1	26.1	5.9	19.3	29.9	8.5	23.9	34.4	13.0	31.0	43.4	15.2	36.1	49.8
$CvM_n^{\text{lap,wg}}$	0.7	4.8	9.3	0.8	5.7	11.9	2.0	7.7	14.0	1.7	8.7	16.1	3.5	12.7	20.2	4.0	15.3	22.8	4.6	16.9	26.2	6.8	20.5	30.5	9.2	28.3	38.3	11.8	30.2	43.6
$d=5$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.9	3.6	8.7	0.7	5.2	11.4	1.8	7.7	13.8	2.7	11.9	21.3	3.7	13.7	22.9	3.6	18.1	30.5	7.1	25.1	42.0	9.7	29.6	48.3	12.9	37.6	54.9	14.8	41.3	60.5
$CvM_n^{\text{lin,F}}$	0.7	4.2	9.1	0.8	4.7	12.1	1.0	5.3	12.4	1.7	8.5	17.0	1.8	12.0	20.0	2.3	13.2	25.5	3.5	19.9	35.1	5.2	23.7	40.8	8.9	30.4	48.3	9.9	33.6	51.4
$CvM_n^{\text{gau,E}}$	0.4	3.5	7.7	0.3	3.8	9.1	0.8	4.6	10.7	0.6	6.1	14.4	0.8	9.8	19.4	1.3	10.8	22.7	2.4	16.8	31.8	4.4	20.8	36.5	7.5	26.5	45.0	8.8	31.3	48.0
$CvM_n^{\text{gau,F}}$	0.3	3.7	7.7	0.3	3.5	8.7	0.8	4.2	10.2	0.7	5.9	13.9	0.6	9.0	12.8	1.0	10.6	21.5	2.5	16.5	30.3	3.8	20.8	36.3	7.1	25.9	44.1	8.3	30.8	47.3
$CvM_n^{\text{lap,E}}$	0.1	1.6	5.7	0.2	1.9	5.8	0.4	2.6	6.6	0.2	3.2	9.6	0.2	5.4	13.0	0.4	5.3	14.8	0.8	8.8	21.7	1.6	11.8	26.4	3.4	16.9	32.9	3.0	19.8	37.2
$CvM_n^{\text{lap,F}}$	0.2	2.1	5.7	0.1	1.6	6.4	0.3	2.9	6.6	0.2	3.0	9.8	0.1	4.1	100.0	0.5	5.3	14.6	0.9	7.9	22.0	1.1	11.3	26.6	2.7	16.2	31.5	2.4	19.7	35.9
without projection																														
$CvM_n^{\text{lin,E,wp}}$	82.2	98.6	99.9	87.4	98.4	99.8	89.9	99.4	99.6	93.9	99.4	100.0	96.2	99.8	99.3	97.3	99.9	100.0	99.2	100.0	100.0	99.5	100.0	100.0	99.8	100.0	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	48.2	86.0	96.5	54.5	89.9	97.1	60.8	91.7	98.3	69.6	95.2	98.8	77.3	97.2	34.7	84.0	98.0	99.5	90.6	99.2	99.7	93.9	99.6	100.0	94.3	99.7	100.0	96.7	99.8	100.0
$CvM_n^{\text{gau,E,wp}}$	0.6	4.9	10.3	0.9	5.4	14.3	2.1	8.7	17.9	2.1	14.7	28.6	5.0	20.7	31.7	7.5	27.6	43.2	11.5	38.6	58.0	17.8	48.4	64.9	25.3	57.8	73.4	31.7	63.3	77.5
$CvM_n^{\text{gau,F,wp}}$	0.6	4.9	10.4	0.7	5.2	14.0	1.6	8.1	17.0	1.7	13.2	26.6	3.4	18.6	33.3	6.4	24.5	39.7	9.4	34.5	54.3	14.8	43.5	62.0	20.5	53.4	69.5	25.2	57.9	74.6
$CvM_n^{\text{lap,E,wp}}$	0.2	3.2	9.8	0.4	3.8	13.2	1.1	7.1	16.5	1.3	12.7	27.6	2.8	18.0	29.7	5.9	25.1	43.1	9.0	36.4	57.0	13.4	46.2	65.5	20.2	55.9	73.8	24.8	60.2	77.9
$CvM_n^{\text{lap,F,wp}}$	0.2	3.2	9.7	0.3	4.1	12.8	0.9	6.6	15.4	0.8	10.0	23.6	2.1	15.5	30.7	4.1	20.7	38.7	5.8	29.7	53.0	9.6	39.4	61.2	15.6	49.0	69.3	18.8	52.9	72.8
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.3	2.5	7.8	0.5	4.0	10.0	1.2	6.6	12.8	2.1	10.9	21.8	2.6	17.7	27.6	6.4	22.7	38.9	10.1	35.7	53.7	15.1	42.1	61.3	22.4	53.0	69.8	27.3	60.6	75.4
$CvM_n^{\text{gau,wg}}$	0.4	3.8	9.0	0.5	4.8	10.7	0.9	7.1	14.1	1.5	9.4	21.2	2.8	15.7	20.5	4.6	20.1	35.5	7.1	30.2	47.3	12.1	36.2	56.5	18.3	47.5	64.1	23.1	53.0	69.0
$CvM_n^{\text{lap,wg}}$	0.1	1.5	5.1	0.1	2.7	7.0	0.2	3.3	10.8	0.8	4.9	14.3	0.8	9.3	60.7	2.2	13.2	26.6	2.6	19.5	39.0	5.6	26.4	47.4	9.6	35.7	54.9	12.3	41.7	60.9

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 10: Rejection rates associated with DGP8 in interval $\mathcal{T}_2$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	1.5	5.7	10.8	1.1	5.2	11.8	1.5	7.6	13.7	1.9	7.7	15.1	2.5	11.2	19.4	3.6	12.5	20.3	4.9	15.2	23.9	6.8	20.7	33.2	8.4	24.3	36.9	10.2	27.3	39.0
$CvM_n^{\text{lin,F}}$	1.5	5.4	10.0	1.1	5.6	11.9	1.6	8.5	14.7	1.9	8.0	16.3	2.8	12.2	19.9	4.1	13.4	22.9	4.7	16.1	26.2	7.8	22.8	35.1	9.3	27.3	40.2	11.4	29.9	42.6
$CvM_n^{\text{gau,E}}$	1.0	6.1	11.3	1.4	5.1	11.4	1.4	7.1	12.7	1.3	7.4	15.0	2.6	9.9	17.2	3.8	11.7	21.0	4.6	13.2	22.9	5.9	20.5	31.6	7.7	23.1	34.1	9.1	25.8	37.5
$CvM_n^{\text{gau,F}}$	1.0	5.9	10.7	1.4	5.3	11.1	1.2	7.2	12.5	1.3	7.2	14.6	2.9	10.4	17.5	3.9	11.9	21.1	4.7	13.9	23.3	6.7	20.9	32.4	8.3	24.5	35.7	9.9	27.6	39.6
$CvM_n^{\text{lap,E}}$	0.9	5.5	10.9	1.6	5.9	11.6	1.3	6.4	13.0	0.8	7.2	12.8	2.0	8.7	15.5	2.8	11.3	19.1	4.0	12.3	21.0	4.8	17.6	28.3	6.6	20.1	31.5	7.5	22.2	34.3
$CvM_n^{\text{lap,F}}$	1.1	5.2	10.9	1.6	5.9	10.9	1.3	6.6	13.1	0.9	7.4	14.4	2.8	9.3	16.9	3.1	11.5	19.5	4.3	13.0	22.1	5.7	19.9	29.7	7.4	22.1	33.6	8.9	25.0	36.4
without projection																														
$CvM_n^{\text{lin,E,wp}}$	2.1	10.2	17.8	2.4	9.4	19.1	4.4	14.5	24.8	5.1	17.4	29.3	7.4	22.0	36.3	9.9	28.1	41.1	15.1	33.9	47.1	20.8	44.1	57.1	26.4	50.3	62.2	29.9	54.0	68.8
$CvM_n^{\text{lin,F,wp}}$	2.0	8.7	16.2	2.1	9.2	17.9	4.4	13.9	23.7	4.7	16.1	27.5	7.0	20.5	34.0	8.8	26.0	37.6	12.8	31.8	44.9	17.8	40.8	54.7	22.4	47.5	59.2	26.5	51.6	65.0
$CvM_n^{\text{gau,E,wp}}$	1.7	6.6	11.8	1.4	6.1	12.2	2.7	10.3	17.3	3.3	11.5	20.4	5.1	16.0	24.1	6.9	20.2	30.7	10.5	25.5	36.1	14.5	33.6	45.5	19.0	39.6	51.2	21.1	43.7	55.7
$CvM_n^{\text{gau,F,wp}}$	1.9	6.5	11.7	1.3	6.5	12.1	2.6	9.3	17.6	3.2	11.8	20.7	5.4	15.8	24.0	6.8	19.5	30.6	9.6	24.9	35.7	13.7	32.6	45.7	18.1	39.0	51.0	20.7	43.0	54.9
$CvM_n^{\text{lap,E,wp}}$	1.7	6.6	12.1	1.6	6.8	11.8	2.9	9.5	16.5	3.2	10.3	18.8	4.7	14.2	22.0	5.8	17.3	28.3	8.8	22.8	33.2	12.0	29.8	42.4	15.8	33.8	47.3	18.0	39.2	52.7
$CvM_n^{\text{lap,F,wp}}$	1.8	6.4	12.1	1.5	6.8	11.6	2.7	9.0	16.8	3.1	10.4	18.9	4.4	14.0	22.5	6.2	17.4	27.9	8.1	21.9	32.6	11.8	29.9	42.7	16.2	34.1	47.4	18.0	39.1	52.9
without gradient																														
$CvM_n^{\text{lin,wg}}$	1.4	5.7	11.4	1.2	6.5	10.7	2.1	8.9	15.2	2.1	10.1	18.5	4.0	14.4	21.8	5.2	17.3	25.5	8.2	22.5	31.9	10.0	28.3	39.5	15.8	33.0	45.9	18.9	38.7	50.1
$CvM_n^{\text{gau,wg}}$	1.5	5.6	11.9	1.5	5.4	11.3	1.8	9.5	15.9	2.4	10.9	18.6	4.2	14.9	24.3	6.0	17.8	28.0	9.2	24.0	34.4	11.6	31.4	41.6	16.5	37.1	48.7	20.9	41.6	55.0
$CvM_n^{\text{lap,wg}}$	1.5	5.6	11.5	1.3	6.4	10.2	2.1	8.9	14.8	1.8	9.4	17.7	4.1	13.3	21.6	4.9	16.5	24.8	7.5	20.8	30.8	9.8	27.5	38.7	15.7	32.3	43.9	18.0	38.5	50.2
$d=5$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	1.2	6.3	12.4	1.3	8.3	18.1	3.7	13.1	22.1	6.3	20.2	32.1	8.7	25.7	35.0	12.9	37.0	53.4	20.1	49.6	66.6	31.4	60.5	73.9	40.8	69.5	83.1	47.3	78.5	88.9
$CvM_n^{\text{lin,F}}$	1.1	5.4	11.5	1.3	8.4	17.4	3.0	10.9	18.2	4.1	15.8	26.8	5.7	20.8	31.6	8.8	30.0	43.7	15.9	39.4	56.2	24.8	51.8	68.7	31.6	60.4	74.1	35.2	68.3	81.3
$CvM_n^{\text{gau,E}}$	0.6	5.3	11.8	0.9	7.2	16.1	2.2	7.5	15.1	3.6	12.3	22.5	4.5	17.3	31.4	8.0	25.8	41.3	12.9	36.9	52.1	20.3	46.2	63.2	26.9	54.4	68.4	32.4	64.2	77.4
$CvM_n^{\text{gau,F}}$	0.4	5.1	11.4	1.2	7.4	16.2	2.0	7.7	15.5	3.5	13.2	23.2	4.0	17.4	26.2	8.1	25.4	40.6	13.3	36.6	51.3	20.0	45.1	64.0	25.7	54.8	68.3	31.8	63.3	76.7
$CvM_n^{\text{lap,E}}$	0.5	3.9	9.4	0.7	5.1	12.5	0.8	6.5	13.5	2.9	10.4	19.1	2.1	12.8	26.8	5.0	20.0	35.4	8.5	30.8	46.1	12.6	37.9	57.3	19.3	46.7	63.7	24.0	54.6	71.6
$CvM_n^{\text{lap,F}}$	0.5	3.8	9.2	0.7	5.2	14.1	1.2	6.4	14.1	2.5	10.3	20.0	1.8	13.2	99.7	4.9	20.1	34.7	8.7	30.1	46.1	13.0	38.1	57.7	19.5	47.9	63.5	23.9	54.3	71.9
without projection																														
$CvM_n^{\text{lin,E,wp}}$	55.0	87.5	95.3	61.0	90.1	96.3	69.0	93.8	98.4	80.8	96.5	98.4	89.4	98.6	97.1	93.2	99.3	99.9	97.6	99.7	100.0	98.9	100.0	100.0	99.9	100.0	100.0	99.8	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	25.8	65.9	82.7	33.6	69.9	85.5	42.0	75.9	90.6	57.6	85.6	93.9	68.0	91.3	49.4	77.2	94.4	98.0	85.9	98.0	99.3	91.9	99.2	99.8	95.3	99.6	99.9	97.9	99.9	100.0
$CvM_n^{\text{gau,E,wp}}$	1.0	7.0	14.2	2.1	11.2	20.1	3.9	13.6	24.6	8.5	23.4	36.9	12.3	35.0	46.7	23.7	47.5	62.3	33.1	61.3	74.3	42.9	72.8	83.7	54.7	79.7	88.8	65.5	86.6	92.7
$CvM_n^{\text{gau,F,wp}}$	0.8	6.7	13.9	1.8	11.3	19.6	3.6	12.9	23.2	7.3	22.2	34.7	10.7	32.6	53.2	20.7	44.7	60.2	30.4	58.0	72.0	38.9	69.3	82.3	51.1	75.9	87.0	61.2	85.0	91.1
$CvM_n^{\text{lap,E,wp}}$	0.7	6.4	14.9	1.6	10.6	20.9	3.5	14.6	26.4	8.2	24.7	39.8	12.5	35.8	50.1	23.0	50.1	65.2	32.9	62.6	75.3	43.4	74.6	85.5	54.8	81.0	89.9	66.6	87.2	94.3
$CvM_n^{\text{lap,F,wp}}$	0.4	6.5	14.7	1.5	10.2	20.9	3.5	13.1	24.6	7.4	21.6	37.3	10.2	32.1	50.7	19.3	45.0	61.8	28.1	57.6	72.9	38.6	70.5	83.4	48.5	76.8	88.0	59.7	85.1	92.2
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.7	4.8	11.4	2.0	9.8	17.2	3.4	11.8	23.2	8.2	23.8	36.9	12.6	34.7	45.0	23.6	48.6	63.3	34.3	63.4	75.4	46.9	76.7	85.5	59.7	82.6	91.3	68.2	89.4	95.3
$CvM_n^{\text{gau,wg}}$	0.7	5.2	10.6	2.7	9.5	17.0	3.4	10.6	19.8	6.9	20.1	32.8	10.1	29.8	42.3	19.5	42.3	57.1	28.2	56.6	69.5	39.7	69.2	80.7	50.3	76.7	86.0	61.1	83.2	91.1
$CvM_n^{\text{lap,wg}}$	0.4	3.9	9.6	1.4	7.4	15.7	1.9	9.6	18.0	5.4	18.2	30.3	7.5	25.9	69.8	15.4	38.1	54.2	23.5	51.5	67.2	34.6	65.1	79.0	44.6	72.6	84.4	54.8	81.3	89.3

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.

Table 11: Rejection rates associated with DGP8 in interval $\mathcal{T}_3$.

δ	0.01			0.02			0.03			0.04			0.05			0.06			0.07			0.08			0.09			0.10		
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
$d=2$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.9	4.8	10.2	0.6	4.5	9.8	1.7	6.4	11.7	0.7	6.8	13.6	2.6	8.9	16.3	3.5	11.3	20.4	5.0	15.4	25.9	4.5	16.2	27.2	7.0	22.2	32.0	11.3	27.9	41.7
$CvM_n^{\text{lin,F}}$	0.8	4.9	10.9	0.6	4.1	9.0	1.5	6.0	12.5	1.2	8.0	14.6	2.9	9.0	17.0	3.3	11.7	21.8	6.0	17.6	28.0	5.1	18.3	29.7	8.1	24.3	36.3	13.6	31.6	45.1
$CvM_n^{\text{gau,E}}$	0.8	5.1	10.4	0.6	4.2	10.8	1.4	6.0	13.1	1.3	7.0	13.7	2.0	9.2	15.8	2.0	10.4	19.0	4.4	15.8	24.6	3.9	16.2	26.2	6.0	20.2	33.2	10.0	26.6	40.3
$CvM_n^{\text{gau,F}}$	0.7	5.2	10.9	0.7	3.7	10.6	1.6	6.0	13.0	1.4	7.5	13.8	2.1	9.7	16.5	2.6	10.6	19.6	4.6	16.4	25.0	4.5	17.7	27.8	6.5	21.3	34.2	10.9	27.6	42.3
$CvM_n^{\text{lap,E}}$	0.7	4.7	9.7	0.6	4.8	11.0	1.5	5.8	11.6	1.4	6.5	12.2	1.3	8.3	13.9	1.5	8.8	17.3	3.0	13.4	23.1	2.8	14.4	23.7	4.6	17.6	28.9	6.9	23.0	35.7
$CvM_n^{\text{lap,F}}$	0.7	5.0	10.4	0.6	4.8	10.3	1.4	5.9	12.1	1.6	6.7	12.4	1.6	8.7	14.8	1.9	8.6	18.0	4.1	14.8	23.6	3.3	15.7	25.5	5.9	19.0	30.8	9.1	25.4	38.8
without projection																														
$CvM_n^{\text{lin,E,wp}}$	1.3	11.8	23.4	2.5	11.7	23.2	2.8	15.1	27.3	5.0	18.2	30.7	7.0	24.1	36.3	8.8	29.2	45.2	14.2	36.5	51.1	16.5	41.4	55.6	24.3	49.0	62.8	29.4	60.0	74.4
$CvM_n^{\text{lin,F,wp}}$	1.6	10.5	22.0	2.3	11.5	21.7	3.0	13.2	26.7	4.3	17.5	29.1	6.7	21.0	34.5	8.0	26.6	43.4	13.9	33.9	48.7	15.0	39.4	53.1	21.7	47.6	61.5	28.4	57.4	71.5
$CvM_n^{\text{gau,E,wp}}$	1.2	5.8	12.9	1.2	6.6	13.3	1.8	8.4	16.7	2.8	12.3	19.7	4.3	14.9	23.8	5.6	18.5	29.1	9.6	25.4	36.9	9.8	30.2	41.2	14.9	36.8	49.9	22.3	45.3	60.3
$CvM_n^{\text{gau,F,wp}}$	1.1	6.0	12.9	1.1	6.3	12.8	2.0	8.2	16.6	2.9	11.8	19.2	4.5	14.7	23.3	5.8	18.2	29.4	9.9	24.9	36.2	10.5	29.5	40.7	15.5	36.4	50.3	22.5	45.2	59.9
$CvM_n^{\text{lap,E,wp}}$	1.3	6.0	11.7	1.0	6.6	13.8	1.7	6.7	16.6	2.3	10.5	17.9	3.2	11.9	21.6	4.3	15.8	26.8	7.9	22.7	35.0	8.7	26.1	38.1	11.8	33.1	45.6	19.3	39.0	55.2
$CvM_n^{\text{lap,F,wp}}$	1.2	6.0	11.7	1.0	6.8	13.1	1.7	7.2	16.3	2.4	10.4	17.7	3.3	13.1	21.5	4.4	16.4	27.2	7.5	22.4	34.2	8.4	26.2	38.1	13.2	34.2	45.9	19.9	39.9	55.5
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.8	4.5	10.3	0.6	5.0	10.3	1.4	5.6	13.0	2.1	9.2	15.8	3.2	10.6	19.1	4.4	15.0	26.1	6.8	21.1	31.4	7.7	23.7	35.5	12.8	31.0	43.6	19.3	39.7	54.2
$CvM_n^{\text{gau,wg}}$	0.6	4.5	11.1	0.6	5.6	10.2	1.7	7.0	13.8	2.0	9.9	17.3	4.3	12.2	21.6	4.7	17.3	26.1	8.1	22.5	33.8	9.3	26.7	38.3	13.6	34.4	46.9	21.3	42.9	57.7
$CvM_n^{\text{lap,wg}}$	0.7	4.6	10.8	0.5	4.9	10.5	1.4	6.0	13.9	2.1	9.0	15.8	3.2	10.8	19.7	4.8	16.3	24.9	7.2	20.1	31.6	7.6	23.5	36.0	11.8	30.5	42.9	19.6	39.2	53.1
$d=5$																														
proposed tests																														
$CvM_n^{\text{lin,E}}$	0.2	4.3	9.5	1.4	5.7	10.8	2.3	9.2	17.2	3.2	15.5	26.4	7.9	28.4	36.8	13.4	35.4	49.5	22.3	51.4	68.8	29.9	62.0	76.8	41.8	73.3	84.3	52.8	83.0	91.7
$CvM_n^{\text{lin,F}}$	0.3	5.0	10.4	0.5	4.2	11.3	1.4	8.3	16.6	2.8	13.4	25.2	5.7	23.4	31.1	9.8	30.0	45.4	19.2	45.1	62.5	24.2	54.2	70.9	34.1	65.0	80.5	45.2	76.2	87.9
$CvM_n^{\text{gau,E}}$	0.3	3.2	8.6	0.2	2.5	8.0	1.0	5.6	13.6	1.7	9.2	20.1	3.2	17.3	31.3	5.0	23.8	39.1	10.9	37.7	55.6	18.2	48.2	65.5	25.8	60.6	76.8	37.8	70.5	82.5
$CvM_n^{\text{gau,F}}$	0.2	3.4	8.3	0.1	2.7	8.9	0.7	5.7	12.9	1.9	8.7	19.7	3.4	17.7	20.7	5.3	23.9	39.0	12.1	37.8	56.3	19.4	49.2	65.7	26.9	61.3	77.9	40.4	72.2	83.4
$CvM_n^{\text{lap,E}}$	2.0	2.3	5.3	0.1	1.3	4.6	0.1	3.0	8.4	0.6	5.1	13.4	0.9	9.1	22.8	1.6	14.1	28.4	4.0	24.0	42.8	8.4	35.3	54.0	13.5	43.6	66.1	20.5	57.2	74.6
$CvM_n^{\text{lap,F}}$	0.2	2.1	5.5	0.1	1.4	5.0	0.2	3.4	8.7	0.8	5.4	13.7	1.1	8.9	100.0	1.9	14.7	29.9	4.6	25.7	44.0	8.9	36.9	55.3	15.3	46.6	67.7	24.3	58.7	76.6
without projection																														
$CvM_n^{\text{lin,E,wp}}$	78.7	98.3	99.5	84.4	99.2	99.6	87.7	99.5	99.9	92.2	99.6	99.9	96.9	100.0	100.0	97.9	100.0	100.0	98.7	100.0	100.0	99.7	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0
$CvM_n^{\text{lin,F,wp}}$	49.2	87.8	95.7	55.4	92.0	97.4	63.4	93.5	98.2	76.4	95.9	99.2	87.6	98.9	51.3	91.7	99.2	99.9	95.8	99.6	100.0	97.6	99.8	100.0	99.3	99.9	100.0	99.7	100.0	100.0
$CvM_n^{\text{gau,E,wp}}$	0.6	4.6	11.5	0.7	5.2	14.3	2.1	12.8	22.4	4.6	19.4	32.6	12.1	34.4	49.5	18.2	47.3	64.3	33.1	64.8	78.7	47.4	77.3	87.2	59.5	85.4	91.9	72.8	93.4	97.3
$CvM_n^{\text{gau,F,wp}}$	0.7	4.2	11.1	0.5	5.0	14.6	1.8	11.4	22.7	4.2	18.3	31.7	10.7	32.6	50.0	16.8	45.3	62.2	30.6	62.8	77.9	44.5	75.8	85.6	56.9	84.1	90.7	71.6	92.4	96.8
$CvM_n^{\text{lap,E,wp}}$	0.3	3.3	9.9	0.5	4.6	12.9	1.2	10.3	21.5	3.3	17.2	31.8	8.4	31.9	47.9	13.9	45.5	63.2	27.4	62.8	78.5	40.8	75.5	87.7	53.5	83.7	91.8	68.3	93.0	97.3
$CvM_n^{\text{lap,F,wp}}$	0.3	3.1	8.9	0.2	3.6	12.3	1.0	9.0	21.3	2.9	16.3	29.7	7.1	29.6	50.9	11.7	42.3	61.7	24.6	59.8	76.8	38.3	73.5	85.2	51.0	81.8	90.9	65.9	91.5	96.5
without gradient																														
$CvM_n^{\text{lin,wg}}$	0.3	3.6	8.8	0.3	4.0	11.0	1.7	10.4	20.9	5.2	19.7	31.8	13.6	35.2	46.7	19.4	49.4	64.2	41.8	69.1	80.7	52.1	78.6	88.3	65.0	88.2	94.4	81.5	95.7	97.9
$CvM_n^{\text{gau,wg}}$	0.2	4.0	9.6	0.5	4.2	11.1	2.8	10.2	20.3	4.6	18.2	30.5	11.1	32.5	39.3	18.4	44.6	59.1	35.3	64.7	77.0	47.7	73.8	84.9	61.4	85.2	91.7	77.2	92.9	96.8
$CvM_n^{\text{lap,wg}}$	0.2	2.6	5.3	0.2	1.7	7.2	1.0	6.0	13.7	2.1	12.6	23.9	5.9	22.8	71.5	9.3	34.2	52.6	24.0	55.4	71.7	38.6	67.7	79.8	49.5	79.4	88.7	65.6	90.3	94.8

Note: This table reports the rejection rates under local alternatives. Simulations are based on 1,000 Monte Carlo experiments with 500 bootstrap computations each. All entries are the proportion of rejections at 1%, 5% and 10% levels in percentage points. See the main text for more details.