6 Monte Carlo simulation of Hull-White model and sensitivities computation

The objective of this project is to implement Monte Carlo simulation for pricing and sensitivity analysis of fixed income instruments under the Hull-White one-factor short-rate model.

The Hull-White one-factor short-rate model is defined by:

$$dr(t) = \big(\theta(t) - ar(t)\big)\,dt + \sigma\,dW_t, \text{ and we assume } r(0) = 0.012,$$

where a is the mean reversion speed assumed equal to 1, $\sigma = 0.1$. In this project we consider $\theta(t)$ defined on the interval $[0, 10]$ by the following expression

$$\theta(t) = [0.012 + 0.0014 * t]1_{0 \le t < 5} + [0.019 + 0.001 * (t-5)]1_{5 \le t \le 10}. \tag{7}$$

The distribution of $r(t)$ at t, knowing its value r_s at time $s < t$, is Gaussian and can be simulated by

$$r(t) = m_{s,t} + \Sigma_{s,t} G, \tag{8}$$

where G has a standard normal distribution and

$$m_{s,t} = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du, \quad \Sigma_{s,t} = \sqrt{\frac{\sigma^2(1-e^{-2a(t-s)})}{2a}}.$$

The zero coupon bond price $P(t,T)$ represents the amount of money to pay at time t to obtain unity at T. Its value is given by

$$P(t,T) = E_t\left(\exp(-\int_t^T r_s ds),\right) \tag{9}$$

when $t=0$, $E_0 = E$ is a straight expectation. The forward rate $f(0,T)$ is expressed by

$$f(0,T) = \frac{\partial \ln(P(0,T))}{\partial T}.$$

1. Write a cuda program that, for a range of maturities of $T \in [0,10]$, computes Monte Carlo estimates of $P(0,T)$ and of $f(0,T)$. The integral $\int_0^T r_s ds$ should be approximated using the trapezoidal rule on a uniform time grid. (8 points)

In practice, the values of $\{P(0,T)\}_{T\in[0,10]}$ are quoted in the market and it is rather θ that is obtained using the following expression

$$\theta(T) = \frac{\partial f(0,T)}{\partial T} + af(0,T) + \frac{\sigma^2(1-e^{-2aT})}{2a}. \tag{10}$$

Also, under Hull-white, one can prove that the zero coupon defined in (9) has the following analytical expression (cf. [3])

$$P(t,T) = A(t,T)\exp\big(-B(t,T)r(t)\big), \text{ with } B(t,T) = \frac{1-e^{-a(T-t)}}{a} \text{ and}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[B(t,T)f(0,t) - \frac{\sigma^2(1-e^{-2aT})}{4a}B(t,T)^2\right].$$

Consequently, with this analytical expression, one can simulate the trajectories of $\{P(t,T)\}_{0\le t\le T}$ through the simulation of $\{r_t\}_{0\le t\le T}$ and thus with a Monte Carlo simulation compute the value of the European call option

$$\mathbf{ZBC}(S_1,S_2,K) = E\left(e^{-\int_0^{S_1} r_s ds}(P(S_1,S_2)-K)_+\right), \text{ with } (x)_+ = \max(x,0) \text{ and } 0 < S_1 < S_2 \le 10.$$

2. Use the values of $\{P(0,T)\}_{T\in\mathcal{T}_{10}}$ and of $\{f(0,T)\}_{T\in\mathcal{T}_{10}}$, with $\mathcal{T}_{10}$ being a uniform discretization of the interval $[0,10]$, computed in question 1
   a) to recover, from (10), the piecewise linear expression of θ given in (7) using a cuda kernel. (4 points)
   b) to perform a Monte Carlo simulation on GPU of $\mathbf{ZBC}(5,10,e^{-0.1})$. (4 points)

We call sensitivity of $\mathbf{ZBC}(S_1,S_2,K)$ with respect to the parameter σ its derivative expressed by

$$\partial_\sigma \mathbf{ZBC}(S_1,S_2,K) = E\left(\partial_\sigma P(S_1,S_2)e^{-\int_0^{S_1} r_s ds}1_{P(S_1,S_2)>K} - \left[\int_0^{S_1}\partial_\sigma r_s ds\right]e^{-\int_0^{S_1} r_s ds}(P(S_1,S_2)-K)_+\right),$$

and using the same G involved in (8), $\partial_\sigma r_t$ can be generated with the induction,

$$\partial_\sigma r(t) = M_{s,t} + \frac{\Sigma_{s,t}}{\sigma}G \text{ with } M_{s,t} = \partial_\sigma r(s)e^{-a(t-s)} + \frac{2\sigma e^{-at}[\cosh(at)-\cosh(as)]}{a^2} \text{ and } \partial_\sigma r(0) = 0.$$

3. We sssume the values of $\{P(0,T)\}_{T\in\mathcal{T}_{10}}$ and of $\{f(0,T)\}_{T\in\mathcal{T}_{10}}$, computed in question 1, as market data (fixed). Compute with Monte Carlo simulation on GPU the value of $\partial_\sigma\mathbf{ZBC}(5,10,e^{-0.1})$ and compare it to a finite difference approximation of the derivative at $\sigma = 0.1$. (3 points + 1 point)