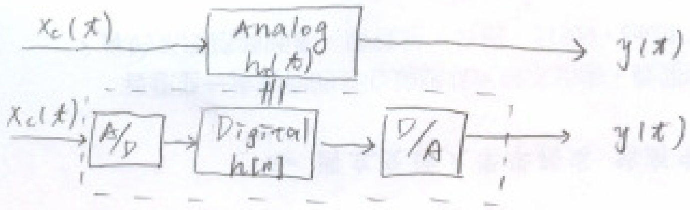
DSP 2009 Spring Midtern 1 Answer

1.

(a)



有說明沒式子2分
有畫圖說明沒式子3分

Impulse invariance:

the impulse response of the discreate-time system is a sampled and scaled version of its continuous-time impulse response.

$\boxed{h[n] = T \cdot h_c(nT)}$, T is the sampling period.

(b) A white noise signal is $\phi_{xx}[m] = \sigma_x^2 \delta[m]$, and the power spectrum of a white noise is $\Phi_{xx}(e^{j\omega}) = \sigma_x^2$ for all ω.

cross-correlation $\phi_{xy}[m] \triangleq E\{x[n] \cdot y[n+m]\}$

$$= E\left\{x[n] \cdot \sum_{k=-\infty}^{\infty} h[k]\, x[n+m-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, \phi_{xx}[m-k]$$

$\Updownarrow$ F.T

$$\Phi_{xy}(e^{j\omega}) = H(e^{j\omega}) \cdot \Phi_{xx}(e^{j\omega})$$

$\Rightarrow$ for white noise signal input: $\Phi_{xy}(e^{j\omega}) = \sigma_{xx}^2 \cdot H(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \frac{\Phi_{xy}(e^{j\omega})}{\sigma_{xx}^2}}$$

$\therefore$ we can obtain frequency response of a system using white noise input.

文字說明觀念正確沒式子2分或3分
只寫式子，寫錯，沒說明2分

(c) For random input signals, we only concern about their autocorrelation functions and power spectra density functions

$$\phi_{yy}[m] = \phi_{yy}[n, n+m] = E\{y(n)\cdot y[n+m]\}$$

$$= E\left\{\sum_k\sum_r h[k]\cdot x[n-k]h[r]\cdot x[n+m-r]\right\}$$

$$= \sum_k\sum_r h[k]h[r]\cdot \phi_{xx}[m+k-r]$$

$\ell = -k+r$

$$= \sum_\ell \phi_{xx}[m-\ell]\cdot \underbrace{\sum_k h[k]h[k+\ell]}_{C_{hh}[\ell] = h[\ell] * h[-\ell]}$$

$$\Rightarrow \boxed{\phi_{yy}[m] = \phi_{xx}[m] * C_{hh}[m]}$$

, $\phi_{xx}[m]$ and $\phi_{yy}[m]$ are autocorrelation sequences of input and output.

F.T

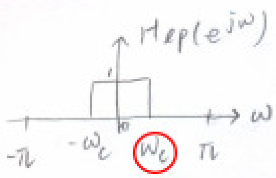
$$\boxed{\Phi_{yy}(e^{jw}) = \Phi_{xx}(e^{jw})\cdot \mathbb{C}(e^{jw})}$$

, $\Phi_{xx}(e^{jw})$ and $\Phi_{yy}(e^{jw})$ are power spectra density functions of input and output

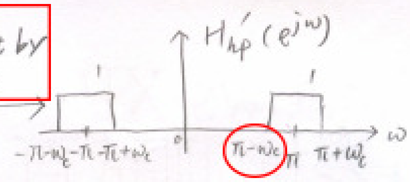
and $\mathbb{C}(e^{jw}) = F\{h[\ell] * h[-\ell]\} = H(e^{jw})\cdot H^*(e^{jw}) = |H(e^{jw})|^2$

(2分)兩式寫其中一個就算對
(5分)證明要從頭開始導才會給5分

(d)



$\boxed{\text{shift by } \pi}$ $\longrightarrow$



$\boxed{\text{multiply by } e^{jwM}}$ $\longrightarrow$ $H_{hp}(e^{jw})$

H'hp並不是=1-Hlp!!

$\Updownarrow$ F.T $\qquad\qquad$ $\Updownarrow$ F.T $\qquad\qquad$ $\Updownarrow$ F.T

$h_{lp}[n] = \dfrac{\sin w_c n}{\pi n}$ $\longrightarrow$ frequency shift $\equiv$ time $\times\, e^{j\pi n}$ $\longrightarrow$ frequency $\times\, e^{jwM}$ $\equiv$ time shift M

$$h'_{hp}[n] = \frac{\sin w_c n}{\pi n}\cdot \underbrace{e^{j\pi n}}_{=(-1)^n}$$

$$\boxed{h_{hp}[n] = \frac{\sin w_c(n-M)}{\pi(n-M)}\cdot e^{j\pi(n-M)}}$$

$e^{j\pi(n-M)} = (-1)^{(n-M)}$

做法觀念對答案錯3分
積分，但積錯2分

(e)

input : $x[n] \rightarrow$ output : $x[Mn] = y[n]$

input : $x[n-k] \rightarrow$ output : $x[M(n-k)] = x[Mn-Mk] \neq y[n-k]$

$= x[Mn-k]$

$\therefore$ It's not a time-invariant system.

2、

(a) when input $\boxed{x[n] = e^{j\omega_0 n}}$, called "eigenfunction"

proof: $y[n] = h[n] * e^{j\omega_0 n}$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j\omega_0(n-k)}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} h[k]} \cdot e^{j\omega_0 n} \cdot \underbrace{e^{-j\omega_0 k}}$$

$$= \boxed{H(e^{j\omega_0})} \cdot e^{j\omega_0 n}$$

只寫答案沒proof2分

the output is simply the input scale a constant $H(e^{j\omega_0})$, called the corresponding "eigenvalue".

(b) $x[n] = \sum_{k=1}^{N} A_k \sin(\omega_k n + \phi_k) = \sum_{k=1}^{N} \frac{A_k}{2j}\left[e^{j(\omega_k n + \phi_k)} - e^{-j(\omega_k n + \phi_k)}\right]$

$$= \sum_{k=1}^{N} \frac{A_k}{2j}\left(e^{j\omega_k n} \cdot e^{j\phi_k} - e^{-j\omega_k n} \cdot e^{-j\phi_k}\right)$$

導到這個形式就給分

$$y[n] = \sum_{k=1}^{N} \frac{A_k}{2j}\left[|H(e^{j\omega_k})| e^{j\angle H(e^{j\omega_k})} \cdot e^{j\omega_k n} \cdot e^{j\phi_k} - |H(e^{-j\omega_k})| e^{j\angle H(e^{-j\omega_k})} \cdot e^{-j\omega_k n} \cdot e^{-j\phi_k}\right]$$

for $h[n]$ real value, $|H(e^{j\omega_k})| = |H(e^{-j\omega_k})|$ and $\angle H(e^{j\omega_k}) = -\angle H(e^{-j\omega_k})$

$$\therefore y[n] = \sum_{k=1}^{N} \frac{A_k}{2j} |H(e^{j\omega_k})| \left[e^{j\angle H(e^{j\omega_k})} \cdot e^{j\omega_k n} \cdot e^{j\phi_k} - e^{-j\angle H(e^{j\omega_k})} \cdot e^{-j\omega_k n} \cdot e^{-j\phi_k}\right]$$

$$= \underline{\sum_{k=1}^{N} A_k |H(e^{j\omega_k})| \sin[\omega_k n + \phi_k + \angle H(e^{j\omega_k})]}$$

只寫答案沒推導2分
計算小錯4分
錯很大2分

(3.)

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

with $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$

(a) $x[n] = 0$

$$1 - 5\lambda^{-1} + 6\lambda^{-2} = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \quad (1)$$

$$\lambda = 3 \text{ or } 2$$

$$y_{ZIR}[n] = k_1 (3)^n + k_2 (2)^n \quad (2)$$

$$\Rightarrow \begin{cases} \frac{11}{6} = \frac{k_1}{3} + \frac{k_2}{2} \\ \frac{37}{36} = \frac{k_1}{9} + \frac{k_2}{4} \end{cases} \Rightarrow \begin{cases} 11 = 2k_1 + 3k_2 \\ 37 = 4k_1 + 9k_2 \end{cases} \Rightarrow \begin{cases} k_2 = 5 \\ k_1 = -2 \end{cases}$$

$$\therefore y_{ZIR}[n] = -2(3)^n + 5(2)^n \quad (2)$$

(b)

$$x[n] = 2^{-n} u[n] = \left(\tfrac{1}{2}\right)^n u[n] \qquad X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H(z) = \frac{3z^{-1} + 5z^{-2}}{1 - 5z^{-1} + 6z^{-2}}$$

$$Y(z) = \frac{\frac{28}{5}}{1 - 3z^{-1}} + \frac{-\frac{22}{3}}{1 - 2z^{-1}} + \frac{\frac{26}{15}}{1 - \frac{1}{2}z^{-1}} \quad (2)$$

$z = 3, 2, \frac{1}{2}$, since causal ROC: $|z| > 3$

$$y_{ZSR}[n] = \frac{28}{5}(3)^n u[n] - \frac{22}{3} \times 2^n u[n] + \frac{26}{15}\left[\tfrac{1}{2}\right]^n u[n] \quad (2)$$

$$y[n] = y_{ZIR}[n] + y_{ZSR}[n] = \frac{26}{15}\left(\tfrac{1}{2}\right)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n, \quad n \geq 0$$

(1)

④

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

$$y[n] = 6\left(\tfrac{1}{2}\right)^n u[n] - 6\left(\tfrac{3}{4}\right)^n u[n]$$

(a)

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \quad , \ \tfrac{1}{2} < |z| < 2$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \quad (0.5)$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \quad (0.5) \quad , \ |z| > \tfrac{3}{4}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \ (1) \Rightarrow h[n] = \left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{3}{4}\right)^{n-1} u[n-1] \ (2) \quad , \ |z| > \tfrac{3}{4}$$

(b)

$\because$ X(Z) 的 ROC : $\frac{1}{2} < |z| < 2$ $\qquad Y(z) = X(z) \cdot H(z)$

Y(Z) 的 ROC : $|z| > \frac{3}{4}$ $\qquad$, $ROC_{Y(z)} = ROC_{X(z)} \cap ROC_{H(z)}$ (1)

$\therefore$ Y(z) 要有 $|z| > \frac{3}{4}$ ，故 H(z) 之 ROC 選擇 $|z| > \frac{3}{4}$ (1)

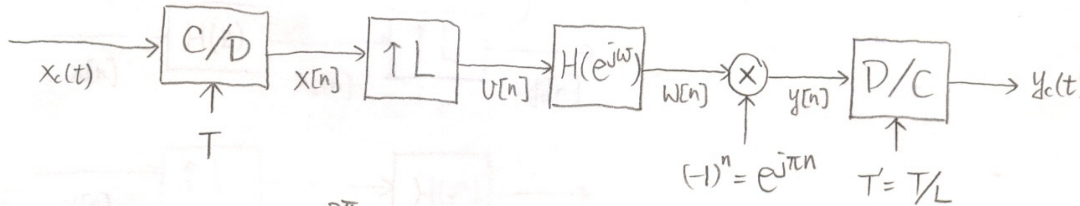
而 X(z) 在 $z=2$ 的 pole 被 H(z) 消去了 (1) ，

$\because |z| > \alpha \overset{\text{right-sided}}{\Rightarrow}$ causal

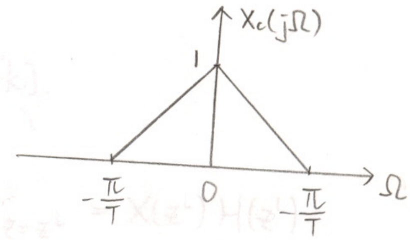
$\because$ ROC 包含 unit circle $\Rightarrow$ stable

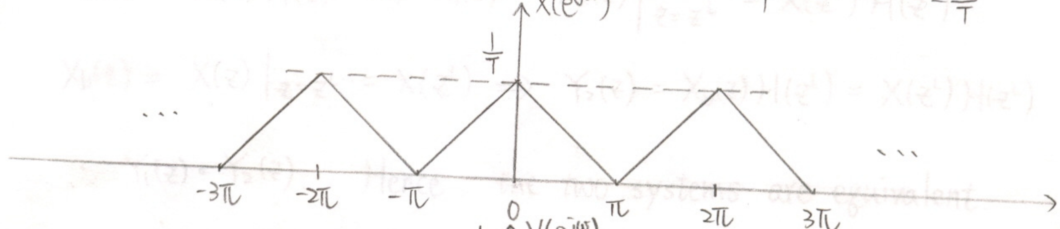
(2)

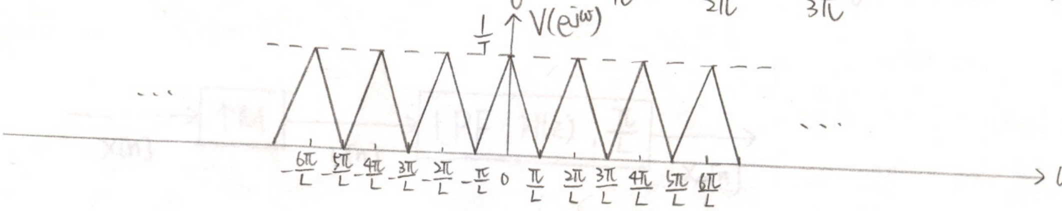
5.

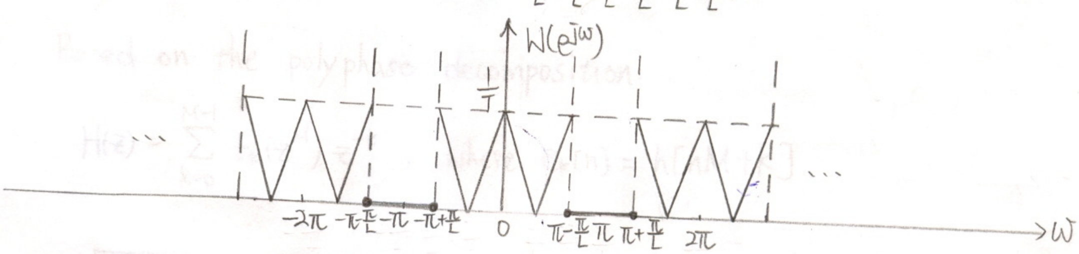


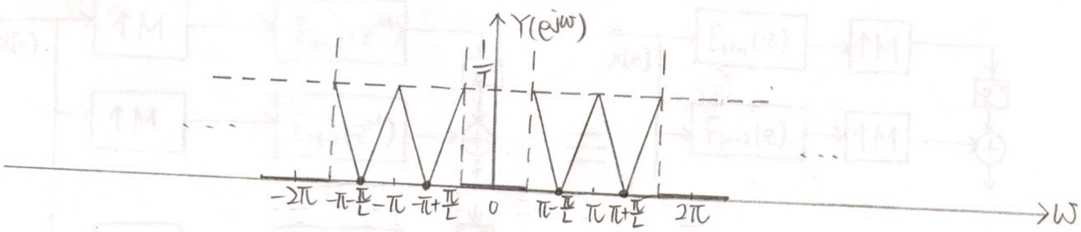
where $H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{2\pi}{L}, \\ 0, & \frac{2\pi}{L} < |\omega| \le \pi. \end{cases}$ and $L > 2$.

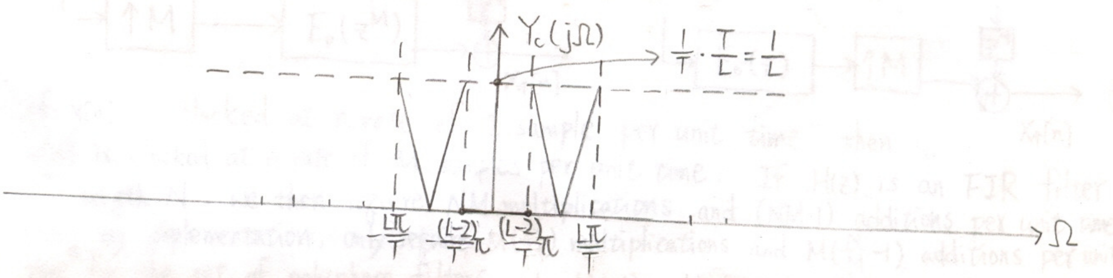




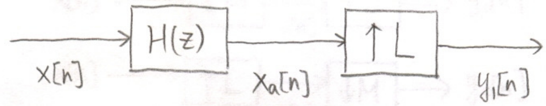


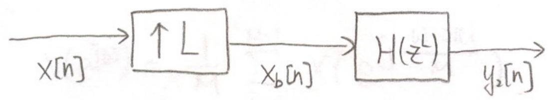






6.





$$H(z) = \sum_{k=0}^{M-1} E_k(z^M)\, z^{-k}, \text{ where } e_k[n] = h[nM+k].$$

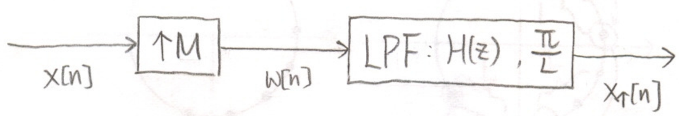
(a)

$$X_a(z) = X(z)H(z) \Longrightarrow Y_1(z) = X_a(z)\Big|_{z=z^L} = X(z^L)H(z^L)$$

$$X_b(z) = X(z)\Big|_{z=z^L} = X(z^L) \Longrightarrow Y_2(z) = X_b(z)H(z^L) = X(z^L)H(z^L)$$

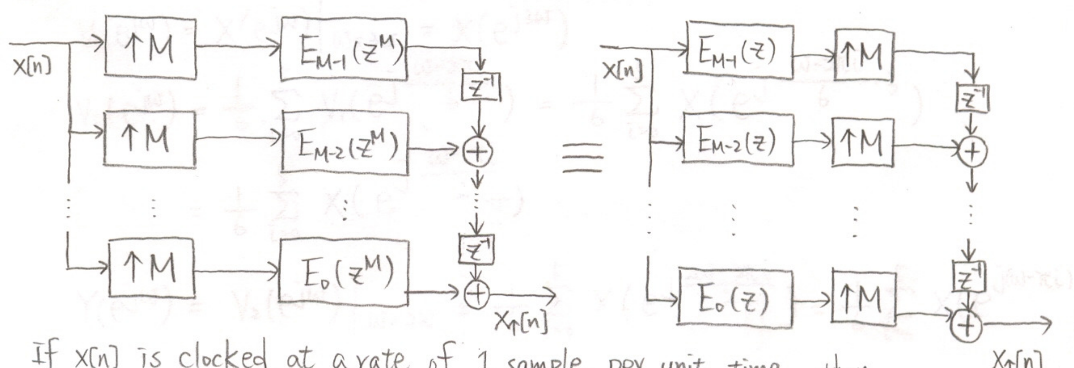
$\therefore Y_1(z) = Y_2(z)$. Hence, the two systems are equivalent.

(b)



Based on the polyphase decomposition:

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M)\, z^{-k}, \quad \text{where } e_k[n] = h[nM+k].$$



(c) If $x[n]$ is clocked at a rate of 1 sample per unit time, then $w[n]$ is clocked at a rate of M samples per unit time. If $H(z)$ is an FIR filter of length N, we then require NM multiplications and $(NM-1)$ additions per unit time. Using my implementation, only require $M(\frac{N}{M})$ multiplications and $M(\frac{N}{M}-1)$ additions per unit time for the set of polyphase filters, plus $(M-1)$ additions, to obtain $x_{\uparrow}[n]$.

7.

$$x[n] \longrightarrow \boxed{\downarrow M} \xrightarrow[v[n]]{} \boxed{\uparrow L} \longrightarrow y_1[n]$$

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow[w[n]]{} \boxed{\downarrow M} \longrightarrow y_2[n]$$

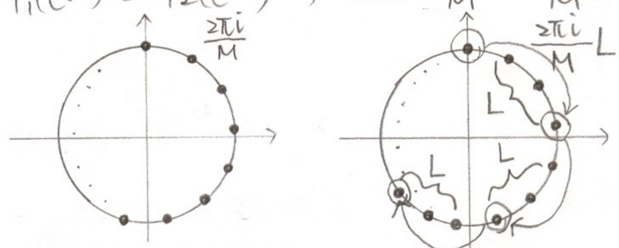
(a) $$V(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j\frac{\omega-2\pi i}{M}})$$

$$Y_1(e^{j\omega}) = V(e^{j\omega})\Big|_{\omega=\omega L} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega L}{M}-\frac{2\pi i}{M})})$$

$$W(e^{j\omega}) = X(e^{j\omega})\Big|_{\omega=\omega L} = X(e^{j\omega L})$$

$$Y_2(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} W(e^{j\frac{\omega-2\pi i}{M}}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega L}{M}-\frac{2\pi i L}{M})})$$

If $Y_1(e^{j\omega}) = Y_2(e^{j\omega})$, then $\frac{2\pi i}{M} = \frac{2\pi i}{M}L$ for $i=0,1,\cdots,M-1$.



The condition for $\frac{2\pi i}{M} = \frac{2\pi i}{M}L$ for $i=0,1,\cdots,M-1$, is that L and M are relatively prime.

(b) $$\xrightarrow[x[n]]{} \boxed{\uparrow 5} \to \boxed{\begin{matrix}\text{LPF: } \frac{\pi}{5} \\ \text{Gain}=5\end{matrix}} \to \boxed{\begin{matrix}\text{LPF: } \frac{\pi}{8} \\ \text{Gain}=1\end{matrix}} \to \boxed{\downarrow 8} \xrightarrow[y[n]]{}$$

2 points

Y= resample (X, 5, 8);

3 points

(c) $$x[n] \longrightarrow \boxed{\uparrow 3} \xrightarrow[v_1[n]]{} \boxed{\downarrow 2} \longrightarrow \boxed{\downarrow 3} \xrightarrow[v_2[n]]{} \boxed{\uparrow 2} \longrightarrow y[n]$$

$$V_1(e^{j\omega}) = X(e^{j\omega})\Big|_{\omega=3\omega} = X(e^{j3\omega})$$

$$V_2(e^{j\omega}) = \frac{1}{6}\sum_{i=0}^{5} V_1(e^{j\frac{\omega-2\pi i}{6}}) = \frac{1}{6}\sum_{i=0}^{5} X(e^{j\frac{\omega-2\pi i}{6}\cdot 3})$$

$$= \frac{1}{6}\sum_{i=0}^{5} X(e^{j\frac{\omega-2\pi i}{2}})$$

$$Y(e^{j\omega}) = V_2(e^{j\omega})\Big|_{\omega=2\omega} = \frac{1}{6}\sum_{i=0}^{5} X(e^{j\frac{2\omega-2\pi i}{2}}) = \frac{1}{6}\sum_{i=0}^{5} X(e^{j(\omega-\pi i)})$$

$$= \frac{1}{2}\left[X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right]$$

$$\therefore\ y[n] = \frac{1}{2}\left\{x[n] + (-1)^n x[n]\right\}.$$