6.23. Causal LTI system with system function:

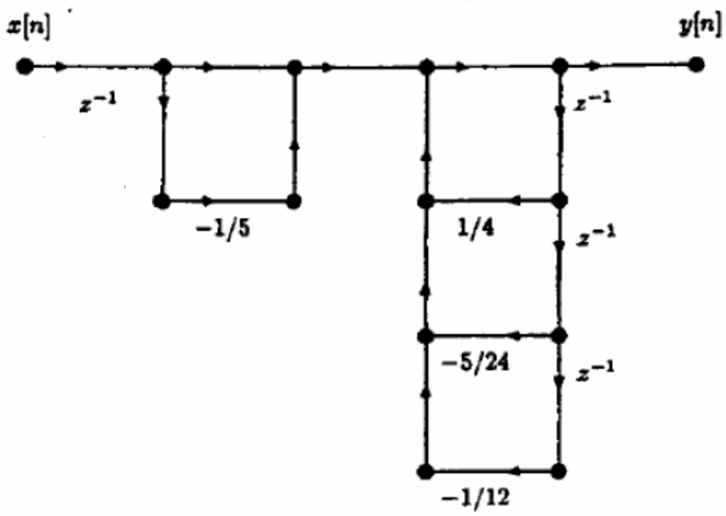
$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2})(1+\frac{1}{4}z^{-1})}.$$

(a) (i) Direct form I.

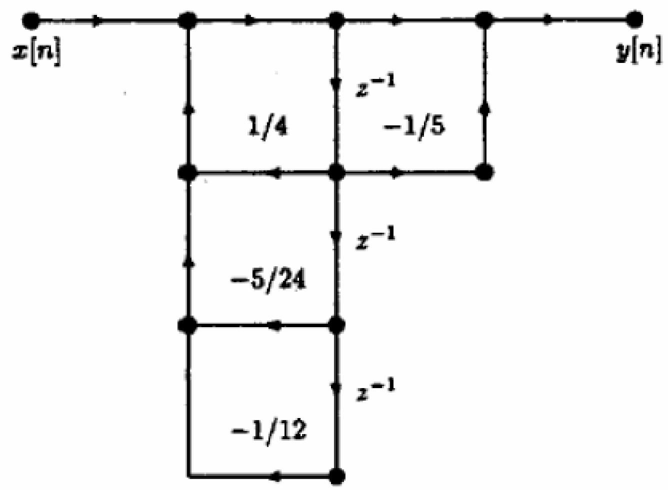
$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z(-3)}$$

so

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5}\ \text{and}\ a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$



(ii) Direct form II.

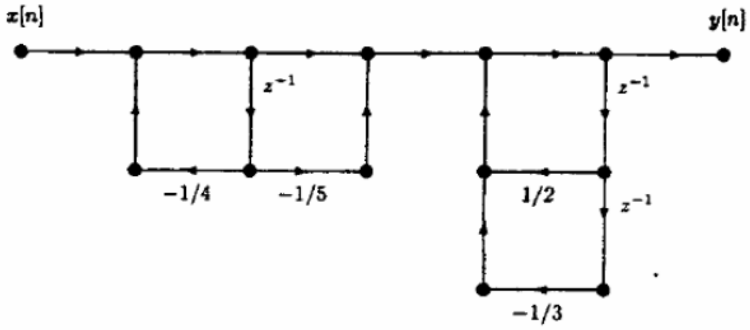


(iii) **Cascade form using first and second order direct form II sections.**

$$H(z) = \left(\frac{1-\frac{1}{5}z^{-1}}{1+\frac{1}{4}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}\right).$$

So

$$b_{01} = 1\ ,\ b_{11} = -\tfrac{1}{5}\ ,\ b_{21} = 0\ ,$$
$$b_{02} = 1\ ,\ b_{12} = 0\ ,\ b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$

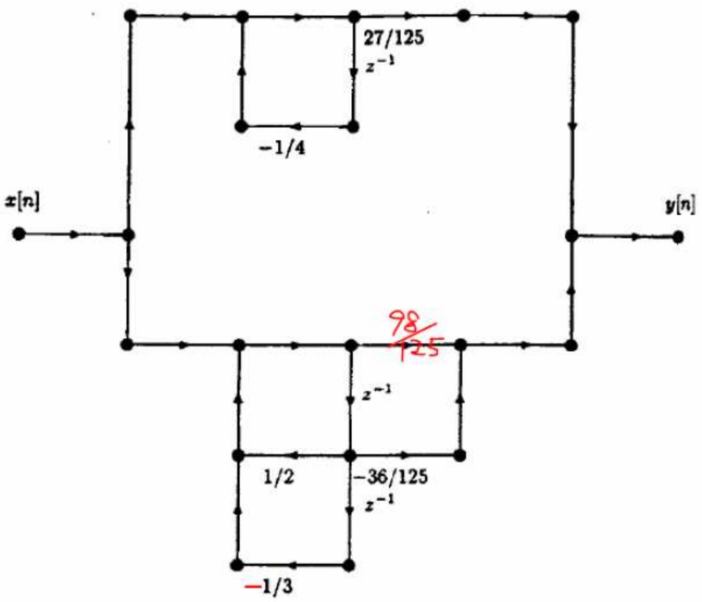


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

$$H(z) = \frac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{98}{125} - \frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}.$$

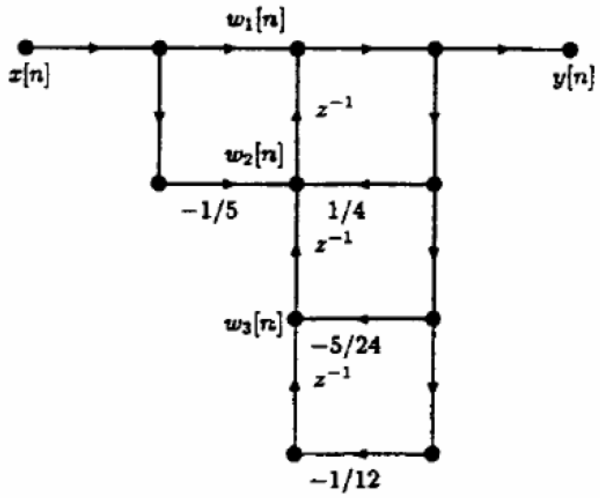
So

$$e_{01} = \tfrac{27}{125}\ ,\ e_{11} = 0\ ,$$
$$e_{02} = \tfrac{98}{125}\ ,\ e_{12} = -\tfrac{36}{125}\ ,\ \text{and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$



(v) Transposed direct form II

We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}$$

Combining the above equations, we get:

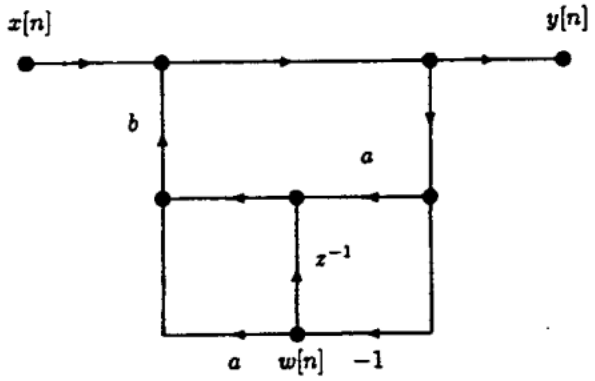
$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.28.



(a)

$$
\begin{aligned}
y[n] &= x[n] + abw[n] + bw[n-1] + aby[n] \\
w[n] &= -y[n].
\end{aligned}
$$

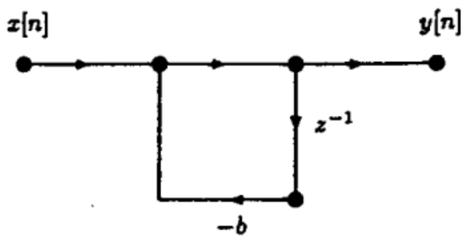
Eliminate w[n]:

$$
\begin{aligned}
y[n] &= x[n] - aby[n] - by[n-1] + aby[n] \\
y[n] &= x[n] - by[n-1]
\end{aligned}
$$

So:

$$H(z) = \frac{1}{1 + bz^{-1}}.$$

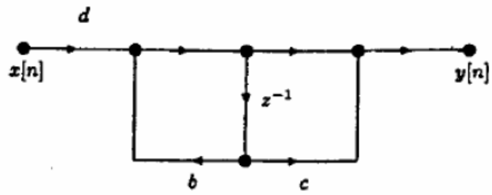
(b)



6.33.

$$H(z) = \frac{z^{-1} - 0.54}{1 - 0.54z^{-1}}.$$

(a)

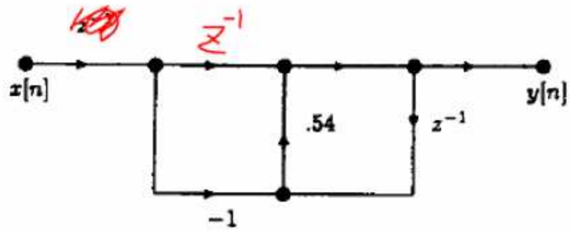


$$H(z) = \frac{cdz^{-1} + d}{1 - bz^{-1}}$$

so set $b = 0.54$, $c = -1.852$, and $d = -0.54$.

(b) With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{c}\hat{d} \neq 1$ and $\hat{d} \neq -\hat{b}$ in general, so the resulting system would not be allpass.

(c)

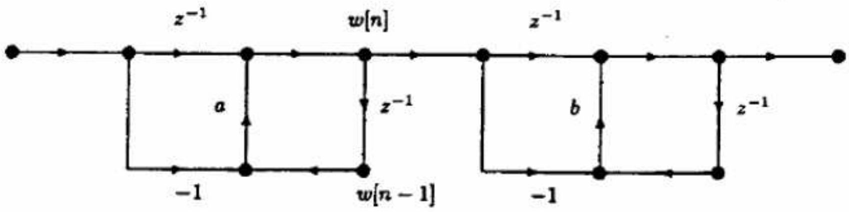


(d) Yes, since there is only one "0.54" to quantize.

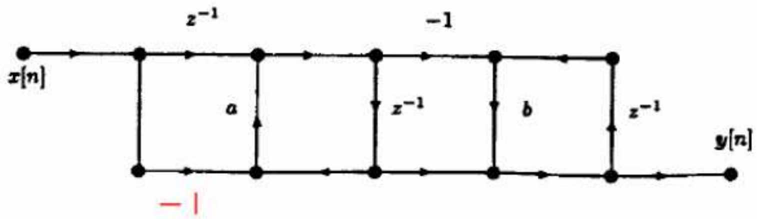
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections like (c) gives



The first delay in the second section has output $w[n-1]$ so we can combine with the second delay of the first section.

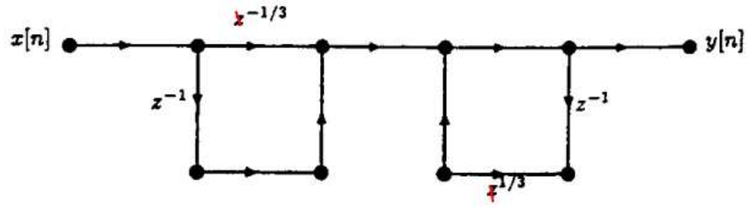


(f) Yes, same reason as part (d).

6.35.

$$H(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}.$$

(a) Direct form I:



From the graph above, it is clear that 2 delays and 2 multipliers are needed.

(b)

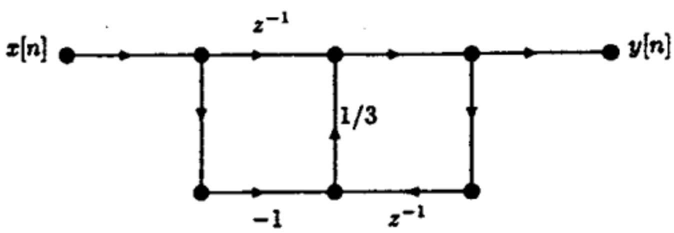
$$(1 - \frac{1}{3}z^{-1})Y(z) = (-\frac{1}{3} + z^{-1})X(z)$$

Inverse $Z-$transforming, we get:

$$y[n] - \frac{1}{3}y[n-1] = -\frac{1}{3}x[n] + x[n-1]$$

$$y[n] = \frac{1}{3}(y[n-1] - x[n]) + x[n-1]$$

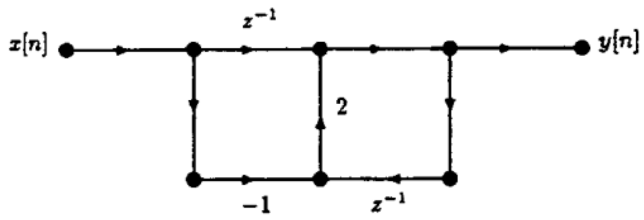
Which can be implemented with the following flow diagram:



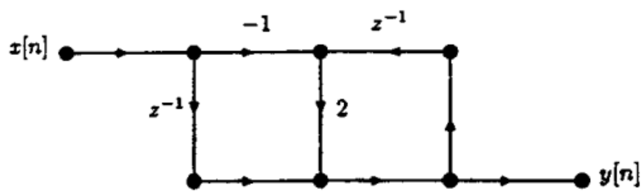
(c)

$$H(z) = \left(\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}\right)\left(\frac{z^{-1} - 2}{1 - 2z^{-1}}\right).$$

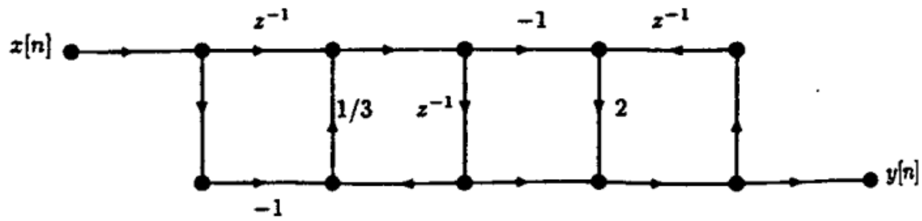
This can be implemented as the cascade of the flow graph in part (b) with the following flow graph:



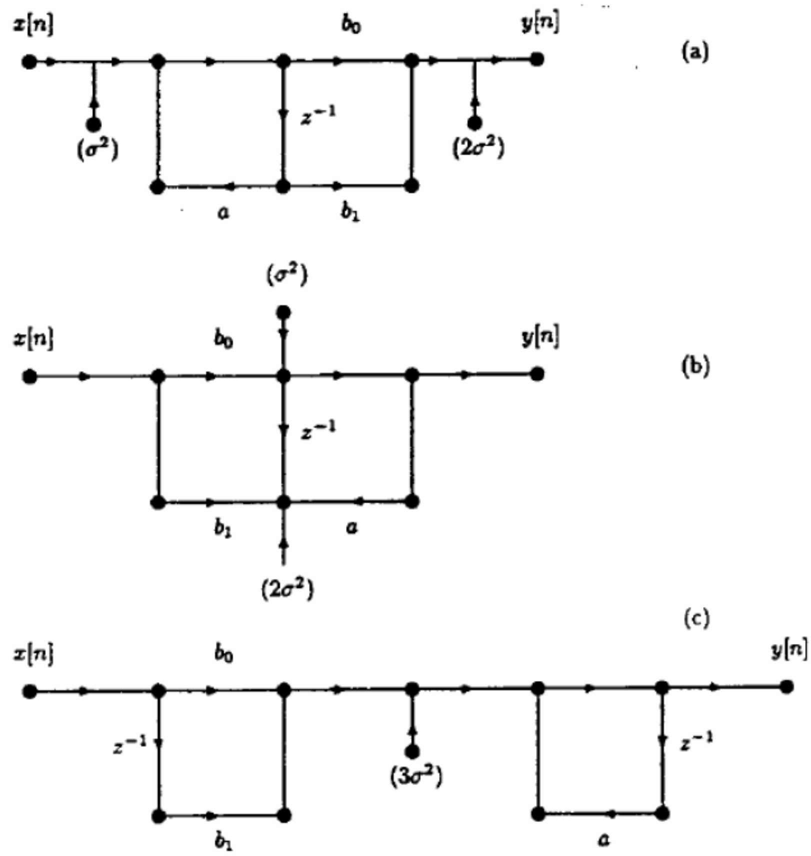
However the above flow graph can be redrawn as:



Now cascading the above flow graph with the one from part (b) and grouping the delay element we get the following system with two multipliers and three delays:



6.42. (a) The linear noise model for each system is drawn below.



(c) For network (c),

$$\sigma_f^2 = 3\sigma^2 \sum_{n=0}^{\infty} (a^n)^2 = \frac{3\sigma^2}{1-a^2},$$

or using the frequency domain formula,

$$\begin{aligned}
\sigma_f^2 &= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az} \frac{dz}{z} \\
&= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{dz}{(z-a)(1-az)} \\
&= \frac{3\sigma^2}{1-a^2}.
\end{aligned}$$

For network (a),

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}}$$

$u[n-1]$

$$h[n] = b_0\delta[n] + (b_0 + \frac{b_1}{a})a^n \cancel{u[n]}$$

- **Time domain calculation:**

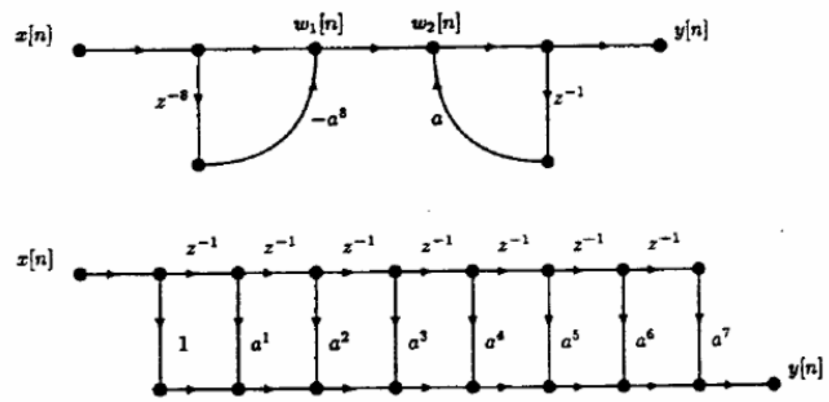
$$\sigma_f^2 = 2\sigma^2 + \sigma^2 \sum_n h^2[n]$$

$$= 2\sigma^2 + \sigma^2 \left(b_0^2 + \left(b_0 + \frac{b_1}{a} \right)^2 \underbrace{\sum_{n=1}^{\infty} a^{2n}}_{\frac{a^2}{1-a^2}} \right)$$

$$= 2\sigma^2 + \sigma^2 \left(b_0^2 + \frac{(ab_0 + b_1)^2}{1 - a^2} \right).$$

and network (b)= network (c)

6.45. The flow graphs for networks 1 and 2 respectively are:



(a) For Network 1, we have:

$$w_1[n] = x[n] - a^8 x[n-8]$$
$$w_2[n] = ay[n-1] + w_1[n]$$
$$y[n] = w_2[n]$$

Taking the Z−transform of the above equations and combining terms, we get:

$$Y(z)(1 - az^{-1}) = (1 - a^8 z^{-8})X(z)$$

That is:

$$H(z) = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

For Network 2, we have:

$$y[n] = x[n] + ax[n-1] + a^2 x[n-2] + \ldots + a^7 x[n-7].$$

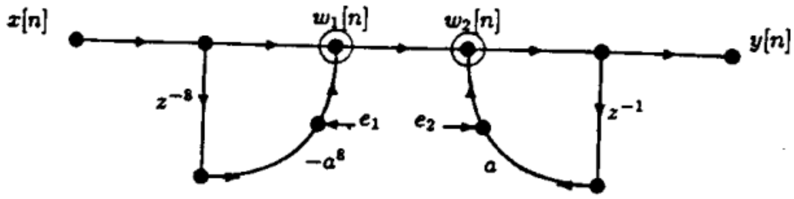
Taking the Z-transform, we get:

$$Y(z) = (1 + az^{-1} + a^2z^{-2} + ... + a^7z^{-7})X(z).$$

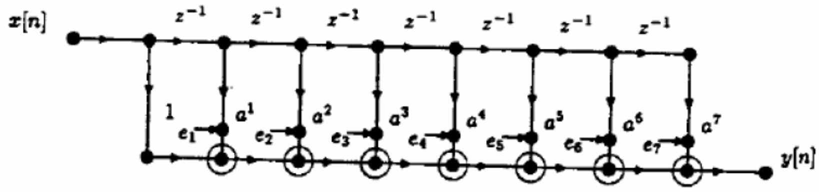
So:

$$H(z) = 1 + az^{-1} + a^2z^{-2} + ... + a^7z^{-7} = \frac{1 - a^8z^{-8}}{1 - az^{-1}}.$$

(b) Network 1:



Network 2:



(c) The nodes are circled on the figures in part (b).

(d) In order to avoid overflow in the system, each node in the network must be constrained to have a magnitude less than 1. That is if $w_k[n]$ denotes the value of the kth node variable and $h_k[n]$ denotes the impulse response from the input $x[n]$ to the node variable $w_k[n]$, a sufficient condition for $|w_k[n]| < 1$ is

$$x_{max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k[m]|}.$$

In this problem, we need to make sure overflow does not occur in each node, i.e. we need to take the tighter bound on x_{max}. For network 1, the impulse response from $w_2[n]$ to $y[n]$ is $a^n u[n]$, therefore the condition to avoid overflow from that node to the output is

$$w_{max} < 1 - |a|.$$

Where we assumed that $|a| < 1$. The transfer function from $x[n]$ to $w_1[n]$ is $1 - a^8z^{-8}$, therefore to avoid overflow at that node we need:

$$w_1[n] < x_{max}(1 - a^8) < 1 - |a|.$$

We thus conclude that to avoid overflow in network 1, we need:

$$x_{max} < \frac{1-|a|}{1-a^8}.$$

Now, for network 2, the transfer function from input to output is given by $\delta[n] + a\delta[n-1] + a^2\delta[n-2] + ... + a^7\delta[n-7]$, therefore to avoid overflow, we need:

$$x_{max} < \frac{1}{1+|a|+a^2+...+|a|^7}.$$

(e) For network 1, the total noise power is $\frac{2\sigma_e^2}{1-|a|^2}$. For network 2, the total noise power is $7\sigma_e^2$. For network 1 to have less noise power than network 2, we need

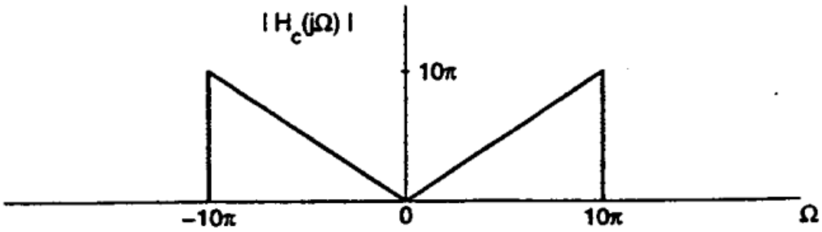
$$\frac{2\sigma_e^2}{1-|a|^2} < 7\sigma_e^2.$$

That is:

$$|a| < \sqrt{\frac{5}{7}}.$$

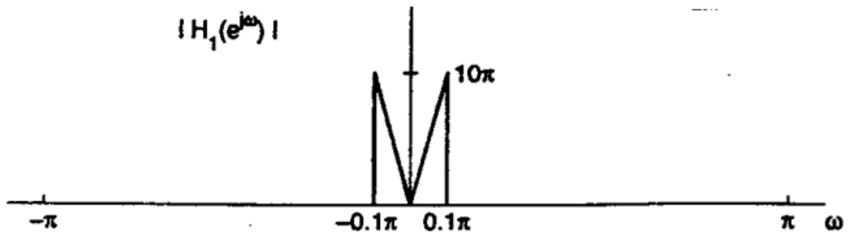
The largest value of $|a|$ such that the noise in network 1 is less than network 2 is therefore $\sqrt{\frac{5}{7}}$.

7.23. We start with $|H_c(j\Omega)|$,



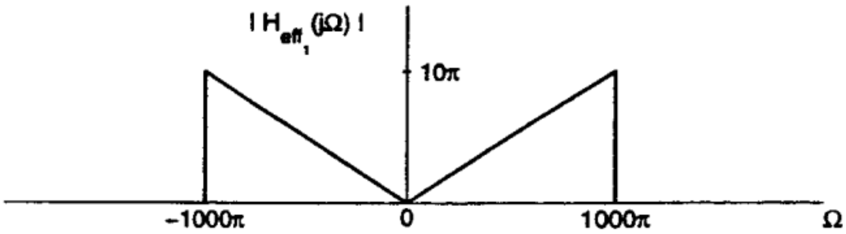
(a) By impulse invariance we scale the frequency axis by T_d to get

$$|H_1(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right)\right|$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_1}(j\Omega)| = \begin{cases} |H_1(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

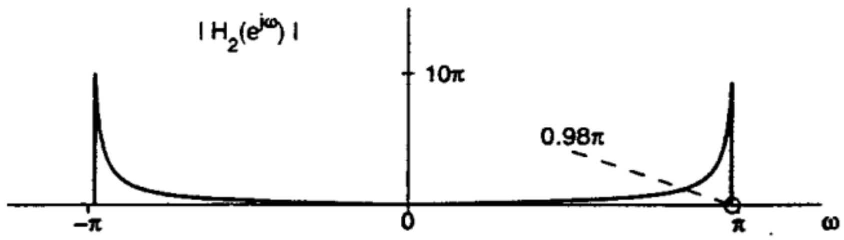


(b) Using the frequency mapping relationships of the bilinear transform,

$$\begin{aligned}\Omega &= \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right),\\ \omega &= 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right),\end{aligned}$$

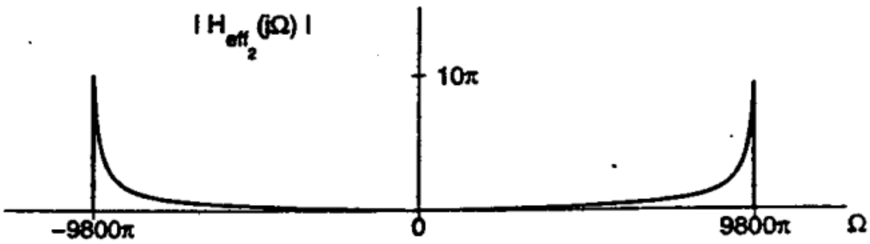
we get

$$|H_2(e^{j\omega})| = \begin{cases} |\tan\left(\frac{\omega}{2}\right)|, & |\omega| < 2\tan^{-1}(10\pi) = 0.98\pi \\ 0, & \text{otherwise} \end{cases}$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_2}(j\Omega)| = \begin{cases} |H_2(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$



7.25. (a) **Answer:** Only the bilinear transform design will guarantee that a minimum phase discrete-time filter is created from a minimum phase continuous-time filter. For the following explanations remember that a discrete-time minimum phase system has all its poles and zeros inside the unit circle.

Impulse Invariance: Impulse invariance maps left-half s-plane poles to the interior of the z-plane unit circle. However, left-half s-plane zeros will *not necessarily* be mapped inside the z-plane unit circle. Consider:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} = \frac{\sum_{k=1}^{N} A_k \prod_{\substack{j=1 \\ j\neq k}}^{N} (s - s_j)}{\prod_{\ell=1}^{N} (s - s_\ell)}$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} = \frac{\sum_{k=1}^{N} T_d A_k \prod_{\substack{j=1 \\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})}{\prod_{\ell=1}^{N} (1 - e^{s_\ell T_d} z^{-1})}$$

If we define $\text{Poly}_k(z) = \prod_{\substack{j=1 \\ j\neq k}}^{N} \left(1 - e^{s_j T_d} z^{-1}\right)$, we can note that all the roots of $\text{Poly}_k(z)$ are inside the unit circle. Since the numerator of $H(z)$ is a sum of $A_k \text{Poly}_k(z)$ terms, we see that there are *no guarantees* that the roots of the numerator polynomial are inside the unit circle. In other words, the sum of minimum phase filters is not necessarily minimum phase. By considering the specific example of

$$H_c(s) = \frac{s + 10}{(s+1)(s+2)},$$

and using $T = 1$, we can show that a minimum phase filter is transformed into a non-minimum phase discrete time filter.

Bilinear Transform: The bilinear transform maps a pole or zero at $s = s_0$ to a pole or zero (respectively) at $z_0 = \frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}$. Thus,

$$|z_0| = \left|\frac{1 + \frac{T}{2}s_0}{1 - \frac{T}{2}s_0}\right|$$

Since $H_c(s)$ is minimum phase, all the poles of $H_c(s)$ are located in the left half of the s-plane. Therefore, a pole $s_0 = \sigma + j\Omega$ must have $\sigma < 0$. Using the relation for s_0, we get

$$\begin{aligned} |z_0| &= \sqrt{\frac{(1 + \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1 - \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} \\ &< 1 \end{aligned}$$

Thus, all poles and zeros will be inside the z-plane unit circle and the discrete-time filter will be minimum phase as well.

(b) **Answer:** Only the bilinear transform design will result in an allpass filter.

Impulse Invariance: In the impulse invariance design we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

The aliasing terms can destroy the allpass nature of the continuous-time filter.

Bilinear Transform: The bilinear transform only warps the frequency axis. The magnitude response is not affected. Therefore, an allpass filter will map to an allpass filter.

(c) **Answer:** Only the bilinear transform will guarantee

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Impulse Invariance: Since impulse invariance may result in aliasing, we see that

$$H(e^{j0}) = H_c(j0)$$

if and only if

$$H(e^{j0}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = H_c(j0)$$

or equivalently

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0$$

which is generally not the case.

Bilinear Transform: Since, under the bilinear transformation, $\Omega = 0$ maps to $\omega = 0$,

$$H(e^{j0}) = H_c(j0)$$

for all $H_c(s)$.

(d) **Answer:** Only the bilinear transform design is guaranteed to create a bandstop filter from a bandstop filter.

If $H_c(s)$ is a bandstop filter, the bilinear transform will preserve this because it just warps the frequency axis; however aliasing (in the impulse invariance technique) can fill in the stop band.

(e) **Answer:** The property holds under the bilinear transform, but not under impulse invariance.

Impulse Invariance: Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_{a_1}(s) = H_{a_2}(s) = e^{-sT/2}$.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &\equiv H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

(f) **Answer:** The property holds for both impulse invariance and the bilinear transform.

Impulse Invariance:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= \sum_{k=-\infty}^{\infty} H_{c1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) + \sum_{k=-\infty}^{\infty} H_{c2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= H_1(e^{j\omega}) + H_2(e^{j\omega}) \end{aligned}$$

Bilinear Transform:

$$\begin{aligned}
H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_1(z) + H_2(z)
\end{aligned}$$

(g) **Answer:** Only the bilinear transform will result in the desired relationship.

Impulse Invariance: By impulse invariance,

$$\begin{aligned}
H_1\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right) \\
H_2\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)
\end{aligned}$$

We can clearly see that due to the aliasing, the phase relationship is not guaranteed to be maintained.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned}
H_1(e^{j\omega}) &= H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \\
H_2(e^{j\omega}) &= H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)
\end{aligned}$$

therefore,

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \frac{H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right)}{H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)} = \begin{cases} e^{-j\pi/2}, & 0 < \omega < \pi \\ e^{j\pi/2}, & -\pi < \omega < 0 \end{cases}$$