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ORIGINAL ARTICLE

Bipartite consensus of double-integrator multi-agent systems with nonuniform communication time delays

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Abstract
In this paper, the bipartite consensus problem is addressed for a class of double-integrator multi-agent systems with antagonistic interactions. The cases with and without communication time delays are considered. In particular, if the communication time delays are not taken into account, the bipartite consensus of the studied multi-agent systems with directed signed graph can be achieved by the proposed distributed controller. If the nonuniform communication time delays are considered, the bipartite consensus of the considered multi-agent systems with undirected signed graph can be achieved if the time delays are less than a derived upper bound. Moreover, we propose an algorithm to solve the so-called grouping problem. Finally, some numerical examples are provided to illustrate the correctness of the results.

Keywords Bipartite consensus · Double-integrator multi-agent systems · Communication time delays · Antagonistic network

1 Introduction

Cooperative control problems of multi-agent systems have been widely investigated in many domains in the past several years. Due to their flexibility and scalability, the distributed control protocols have been developed for many application scenarios. As a fundamental cooperative control problem, the so-called consensus problem requires the states of all agents to converge to a common value of interest. Some typical results on distributed cooperative control protocols can be found in [1–5]. In particular, the authors in [1, 2] derived the conditions on consensus problems for single-integrator multi-agent systems, and the work was further extended to the double-integrator agent dynamics in [3, 6]. The authors in [7] investigated the containment control problem with multiple leaders. In [8], the rotating consensus problem was investigated for a group of double-integrator agents with event-based communication. In [4], the authors studied flocking problems of double-integrator multi-agent systems.

It is noted that during the information exchange among agents, the communication time delays are unavoidable due to the unexpected disturbances such as the congestion of the communication channels, see [9, 10]. It is well known that the communication time delays may affect the control performance and even unstabilize the controlled systems severely. Thus, numerous works have been done to study the effect of the time delays in [11, 12]. In [11], the authors analyzed the influence of the nonuniform time delays by a frequency domain approach and got an upper bound of the time delays under which the consensus can be achieved. In [12, 13], the time-varying transmission delay was further taken into consideration, where the studied networked control systems were in the general linear or nonlinear dynamics. With taking time-varying communication delays into consideration, the authors in [14] further proved that the convergence was achieved even though the network topology is time varying.

It is worth mentioning that in the above-mentioned papers, the consensus is achieved through collaboration. However, in many scenarios, there exist the collaborative and competitive interactions simultaneously in many

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complex systems, for example activators–inhibitors in biological systems, competing cartels in economic systems, and competing robots in engineering. For example, there exists the repulsive force between some robots in a robotic swarm [15]. In a specific social dynamics, the authors in [16, 17] investigated opinion formation for trust–mistrust networks through DeGroot-type or Laplacian-type dynamics. In these cases, the edges in a graph representing the interaction between two agents may not always have positive weights. In contrast, the negative weights can be used to characterize such repulsive forces or distrust/hostile relationship between two agents. A graph with both positively weighted edges and negatively weighted edges is called a signed graph or an antagonistic network.

The consensus problem of multi-agent systems with antagonistic networks was investigated firstly in [18], where the bipartite consensus can be achieved by a distributed Laplacian-like control protocol if the strongly connected signed graph is structurally balanced. Moreover, the continuous-time Altafini model has been studied in [19–22], where the authors relaxed the strongly connected assumption to the less conservative spanning tree condition in [19] and studied internal consensus problem for the signed network which is structurally unbalanced in [22]. In [23], the authors studied the bipartite consensus problem for double-integrator multi-agent systems with the uniform communication time delays taken into consideration. Some adaptive control strategies were proposed for bipartite consensus of heterogeneous linear multi-agent systems in [24]. As for the discrete-time counterpart, the authors in [25] considered a more general case which does not require the sign-symmetry assumption, and necessary and sufficient conditions on the sequence of signed digraphs were obtained. In [26], the bipartite consensus of double-integrator multi-agent systems was reached, where the final velocity of each agent converges to zero eventually. The control performance of multi-agent systems with antagonistic interactions was further considered in [27], where fixed-time nonlinear consensus protocols were proposed. More related results on this topic can be found in [28, 29]. However, in the aforementioned literature, communication time delays are usually neglected. The consideration of the communication time delays makes the existing results fail to be applicable to our case. To the best of our knowledge, there are few results on the bipartite consensus problem of double-integrator multi-agent systems with the nonuniform communication time delays. Furthermore, it is sometimes desired that the bipartite consensus is achieved with a nonzero velocity, which may be useful in some engineering application. In the context of antagonistic networks, a so-called gauge transformation is usually applied based on the structurally balanced assumption. However, it is hardly discussed how to get the transformation matrix and how to identify the two groups each agent belongs to in the end. The above-mentioned discussions motivate our study.

In this paper, the bipartite consensus problem of double-integrator multi-agent systems is studied. The cases with and without the communication time delays are considered. With the help of frequency domain approach and some matrix inequality technologies, an upper bound of the communication time delays is obtained. In addition, we propose an algorithm to indicate how to get the transformation matrix for a given structurally balanced topology.

The rest of the paper is organized as follows: Basic facts about algebraic graph theory are introduced in Sect. 2; problem formulation is introduced in Sect. 3; in Sect. 4, we give the main results and detailed proofs; the algorithm to solve the grouping problem is proposed in Sect. 5; some simulation results are shown in Sect. 6 to illustrate our results; and the conclusion is given in Sect. 7.

2 Graph theory

Notations: $\mathbb{R}^n$ and $\mathbb{C}^n$ are used to represent the set of n-dimensional real column vectors and the set of n-dimensional complex vectors, respectively; I_m represents the m-dimensional identity matrix; $\mathbf{1}_n$ is the n-dimensional column vector with all elements being 1; 0 represents a zero vector or matrix with appropriate dimension; x^{T} and x^* denote the transposition and the conjugate transposition of x, respectively; j is the imaginary unit; $\otimes$ denotes the Kronecker product; $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ denotes diagonal matrix with the diagonal elements $a_1, a_2, \ldots, a_n$. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary part of a complex number, respectively.

A signed graph is denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{G}$ denotes a graph, $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathcal{A} = (a_{ij})_{n\times n}$ is the weighted matrix of a signed graph $\mathcal{G}$. The element $a_{ji} \neq 0\,(i \neq j)$ if and only if the edge $(v_i, v_j) \in \mathcal{E}$ and $a_{ii} = 0$, $i \in \{1, \ldots, n\}$, otherwise. For a directed signed graph, we will always suppose that $a_{ij}a_{ji} \geq 0$, which implies that the edge pair of any two nodes always has the same sign (also called digon sign-symmetric in Altafini [18]). The Laplacian matrix L_c is defined as $L_c = (l_{ik})_{n\times n}$, where $l_{ii} = \sum_{k=1}^{n} |a_{ik}|$ and $l_{ik} = -a_{ik}$, $i \neq k$. A (directed) path from v_1 to v_l is a sequence of edges of the form $(v_1, v_2), (v_2, v_3), \ldots, (v_{l-1}, v_l)$ with distinct nodes in a (directed) graph. A directed graph is strongly connected if there exists a directed path between any two nodes. The neighbor set of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$.

Lemma 1 (*Altafini* [18]) *For a connected undirected graph or a strongly connected, digon sign-symmetric* *directed graph, if the signed graph $\mathcal{G}$ is structurally balanced, then the following conditions hold:*

1. *All nodes can be partitioned into two groups (one possibly empty) in such a way that edges joining two nodes in the same group are positive, while edges joining two nodes in different groups are negative.*
2. *There exists a Gauge transformation matrix $D \in \mathcal{D}$ satisfying that DAD has all nonnegative entries, where $\mathcal{D} = \{D = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)\}$, where $\sigma_i \in \{+1, -1\}$.*

3 Problem formulation

Consider a group of n agents, and ith agent has the following dynamics:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), i \in \{1, \ldots, n\}, \end{aligned} \tag{1}$$

where $x_i(t)$, $v_i(t)$, $u_i(t) \in \mathbb{R}$ represent the position, velocity, and the control input of ith agent, respectively. The initial conditions are given as $x_i(s) = x_i(0)$, $s \in (-\infty, 0]$, $v_i(s) = v_i(0)$, $s \in (-\infty, 0]$. Consider the control protocol as

$$u_i(t) = f_i(x_i(t), x_j(t), v_i(t), v_j(t), j \in N_i). \tag{2}$$

If the communication time delays are considered, and suppose that the time delays between agents i and j are τ_{ij}(without loss of generality, it is assumed $\tau_{ij} = \tau_{ji}$), the protocol is given as

$$\begin{aligned} u_i(t) = f_i(&x_i(t - \tau_{ij}), x_j(t - \tau_{ij}), \\ &v_i(t - \tau_{ij}), v_j(t - \tau_{ij}), j \in N_i). \end{aligned} \tag{3}$$

Definition 1 Consider the multi-agent system (1) with a signed graph $\mathcal{G}$. Develop a distributed control scheme of form (2) or (3), such that for any finite $x_i(0)$, $v_i(0)$, $i = 1, \ldots, n$, the following conditions are satisfied:

$$\begin{aligned} &\lim_{t \to +\infty} [|x_i(t)| - |x_j(t)|] = 0, \\ &\lim_{t \to +\infty} [|v_i(t)| - |v_j(t)|] = 0, \end{aligned} \tag{4}$$

and then, it is said that the bipartite consensus problem is solved.

The objective of the paper is to design a distributed control protocol of form (2) or (3), such that the bipartite consensus problem as defined in Definition 1 can be solved. To obtain the main results, we further need the following assumption and lemmas.

Assumption 1 The signed graph is structurally balanced.

Lemma 2 *Let* $\xi_{\pm} = \frac{k_2\mu \pm \sqrt{k_2^2\mu^2 + 4k_1\mu}}{2}$, *where* $\xi, \mu \in \mathbb{C}$, $\mathrm{Re}(\mu) < 0$, $k_1 > 0$ *and* k_2 *are tuning parameters. If and only if*

$$k_2 > \sqrt{\frac{k_1 \mathrm{Im}^2(\mu)}{|\mu|^2 \mathrm{Re}(-\mu)}}, \tag{5}$$

then $\mathrm{Re}(\xi_{\pm}) < 0$.

Proof Define $\mu = -a + bj$, $\sqrt{k_2^2\mu^2 + 4\mu} = p_1 + p_2 j$, where $a > 0$, $p_1, p_2 \in \mathbb{R}$. It is easy to see that

$$k_2^2(a^2 - b^2 - 2abj) - 4k_1(a - bj) = p_1^2 - p_2^2 + 2p_1p_2 j. \tag{6}$$

Separating the real and imaginary parts, one has

$$\begin{aligned} &k_2^2(a^2 - b^2) - 4k_1 a = p_1^2 - p_2^2, \\ &\quad -2abk_2^2 + 4k_1 b = 2p_1p_2. \end{aligned} \tag{7}$$

Thus,

$$p_2 = \frac{-abk_2^2 + 2k_1 b}{p_1}. \tag{8}$$

Substituting (8) into the upper one of (7), we have

$$p_1^4 - (k_2^2(a^2 - b^2) - 4k_1 a)p_1^2 - (2k_1 b - abk_2^2)^2 = 0. \tag{9}$$

With $-(2k_1b - abk_2^2)^2 < 0$, by calculating the solution of Eq. (9), one has that

$$\begin{aligned} 2p_1^2 = &(k_2^2(a^2 - b^2) - 4k_1 a) \\ &+ \sqrt{(k_2^2(a^2 - b^2) - 4k_1 a)^2 + 4(-abk_2^2 + 2k_1 b)^2}. \end{aligned} \tag{10}$$

With the fact that $\mathrm{Re}(k_2\mu) = -k_2 a < 0$, then $\mathrm{Re}(\xi_{\pm}) < 0$ if and only if $2p_1^2 < 2|\mathrm{Re}(k_2\mu)|^2 = 2(k_2 a)^2$. In other words, $p_1^2 - k_2^2 a^2 < 0$. Equivalently,

$$\begin{aligned} 2p_1^2 - 2k_2^2a^2 = &(k_2^2(a^2 - b^2) - 4k_1 a) - 2k_2^2 a^2 \\ &+ \sqrt{(k_2^2(a^2 - b^2) - 4k_1 a)^2 + 4(-abk_2^2 + 2k_1 b)^2} \\ &< 0, \end{aligned} \tag{11}$$

Namely,

$$\begin{aligned} &(k_2^2(a^2 - b^2) - 4k_1 a) - 2k_2^2 a^2 \\ &\quad < -\sqrt{(k_2^2(a^2 - b^2) - 4k_1 a)^2 + 4(-abk_2^2 + 2k_1 b)^2}, \end{aligned} \tag{12}$$

Define two variables $m_1 = (k_2^2(a^2 - b^2) - 4k_1a) - 2k_2^2a^2$ and $m_2 = (k_2^2(a^2 - b^2) - 4k_1a)^2 + 4(-abk_2^2 + 2k_1b)^2$. For the reason that $(k_2^2(a^2 - b^2) - 4k_1a) - 2k_2^2a^2 < 0$, (12) is equal to

$$m_1 > m_2. \tag{13}$$

Thus, $m_1 - m_2 = k_1k_2^2a(a^2 + b^2) - k_1^2b^2 > 0$. Then, we could conclude that $k_2 > \sqrt{\frac{k_1b^2}{(a^2+b^2)a}}$. Namely, $k_2 > \sqrt{\frac{k_1\mathrm{Im}^2(\mu)}{|\mu|^2\mathrm{Re}(-\mu)}}$. □

Lemma 3 *(Lin et al. [11]) Define a function $f(w) = (\arctan w)/w$. For $w \in (0, +\infty)$, $(d/dw)f(w) < 0$.*

Definition 2 (The grouping problem) Consider a given network which is structurally balanced, and divide all agents into two groups V_1, V_2 such that the interactions between two agents in the same group are cooperative, while the interactions between the agents in different groups are competitive. In this case, the Gauge transformation matrix D can be found in such a way that $\sigma_i = 1$ if $i \in V_1$ and $\sigma_i = -1$ if $i \in V_2$. In other words, the Gauge transformation matrix D can be determined.

4 Main results

In this section, we focus on the stability analysis of the closed-loop systems with or without time delays. The first part is concerned with the directed signed graph without communication time delays, and the other part focuses on undirected signed graph with the communication time delays.

4.1 Directed signed graph without time delays

If the communication time delays are not considered, a distributed control protocol is designed as follows:

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j\in N_i} |a_{ij}|(x_i(t) - \mathrm{sgn}(a_{ij})x_j(t)) \\ &- k_2 \sum_{j\in N_i} |a_{ij}|(v_i(t) - \mathrm{sgn}(a_{ij})v_j(t)), \end{aligned} \tag{14}$$

where $k_1, k_2 > 0$ are control gains to be determined and $\mathrm{sgn}(\cdot)$ is the sign function.

Define $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $v = [v_1, v_2, \ldots, v_n]^{\mathrm{T}} \in \mathbb{R}^n$. With protocol (14), system (1) can be rewritten into the following compact form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -k_1L_c & -k_2L_c \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} = \Gamma\begin{bmatrix} x \\ v \end{bmatrix}, \tag{15}$$

where $\Gamma = \begin{bmatrix} 0 & I_n \\ -k_1L_c & -k_2L_c \end{bmatrix}$ and L_c is the Laplacian matrix for a signed graph. If the directed signed graph is structurally balanced, it is obvious that there exists a transformation matrix D as defined in Lemma 1. Define the extended transformation matrix $\bar{D} = \mathrm{diag}(D, D) \in \mathbb{R}^{2n\times 2n}$. Let $\bar{x} = Dx$, $\bar{v} = Dv$, $z = \bar{D}[x^{\mathrm{T}}, v^{\mathrm{T}}]^{\mathrm{T}} = [\bar{x}^{\mathrm{T}}, \bar{v}^{\mathrm{T}}]^{\mathrm{T}}$. We could get the derivative of variable z as follows:

$$\dot{z}(t) = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \Gamma_D z, \tag{16}$$

where $\Gamma_D = \begin{bmatrix} 0 & I_n \\ -k_1L_D & -k_2L_D \end{bmatrix}$ and $L_D = DL_cD$ is a new Laplacian matrix associated with an unsigned graph. It follows from (16) that

$$z = e^{\Gamma_D t}z(0), t \geq 0. \tag{17}$$

To find the eigenvalues of Γ_D, we need to calculate the solutions of $\det(\lambda I_{2n} - \Gamma_D) = 0$, which is the characteristic polynomial of matrix Γ_D. Note that

$$\begin{aligned} \det(\lambda I_{2n} - \Gamma_D) &= \det\left(\begin{bmatrix} \lambda I_n & -I_n \\ k_1L_D & \lambda I_n + k_2L_D \end{bmatrix}\right) \\ &= \det(\lambda^2 I_n + (k_1 + k_2\lambda)L_D). \end{aligned} \tag{18}$$

Also note that

$$\det(\lambda I_n + L_D) = \prod_{i=1}^{n}(\lambda - \mu_i), \tag{19}$$

where μ_i is the ith eigenvalue of $-L_D$. By comparing (18) with (19), we see that

$$\det(\lambda^2 I_n + (k_1 + k_2\lambda)L_D) = \prod_{i=1}^{n}(\lambda^2 - (k_1 + k_2\lambda)\mu_i), \tag{20}$$

which implies that the roots of (20) can be obtained by solving $\lambda^2 = (k_1 + k_2\lambda)\mu_i$. Therefore, it is straightforward to see that the eigenvalues of Γ_D are given by

$$\lambda_{i\pm} = \frac{k_2\mu_i \pm \sqrt{k_2^2\mu_i^2 + 4k_1\mu_i}}{2}, i = 1, 2, \ldots, n \tag{21}$$

where λ_{i+} and λ_{i-} are two conjugate eigenvalues of Γ_D associated with μ_i. Equation (21) has $2n$ eigenvalues if matrix $-L_D$ has n eigenvalues. If $\mathcal{G}$ is strongly connected, it is obvious that $-L_D$ has one zero eigenvalue with at least one multiplicity. Therefore, Eq. (21) has at least two zero eigenvalues corresponding to $u_1 = 0$ denoted as $\lambda_{1+} = 0$ and $\lambda_{1-} = 0$.

From the above analysis, we have the following theorem.

Theorem 1 *Consider the multi-agent systems* (1) *with a directed signed graph, which is strongly connected and digon sign-symmetric. Under Assumption* 1, *the problem is solved by* (14) *if* $k_1 > 0$, $k_2 > \max_{\mu_i \neq 0} \sqrt{\frac{k_1 \mathrm{Im}^2(\mu_i)}{|\mu_i|^2 \mathrm{Re}(-\mu_i)}}$, *where* μ_i *is the ith eigenvalue of* $-L_D$.

Proof Since L_D of a directed graph may have complex eigenvalues, we need to prove that all eigenvalues of Γ_D are located in left-hand plane (LHP). For a directed graph which is strongly connected, the eigenvalues of $-L_D$ have negative real parts except for $\mu_1 = 0$, which implies that $\mathrm{Re}(\mu_i) < 0, i = 2, 3, \ldots, n$. From Lemma 2, we get that if

$$k_1 > 0, \qquad k_2 > \max_{\mu_i \neq 0} \sqrt{\frac{k_1 \mathrm{Im}^2(\mu_i)}{|\mu_i|^2 \mathrm{Re}(-\mu_i)}}, \qquad \text{then}$$

$\mathrm{Re}(\lambda_{i\pm}) < 0, i = 2, 3, \ldots, n$.

Inspired by Ren and Atkins [3], we prove that Γ_D has only one linearly independent eigenvector associated with eigenvalue zero. Define $[q_1^{\mathrm{T}}, q_2^{\mathrm{T}}]^{\mathrm{T}}$ as the eigenvector of Γ_D corresponding to eigenvalue 0, where $q_1, q_2 \in \mathbb{C}^n$. So

$$\begin{bmatrix} 0 & I_n \\ -k_1 L_D & -k_2 L_D \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{22}$$

From (22), we could get $q_2 = \mathbf{0}$ and $k_1 L_D q_1 = \mathbf{0}$. According to the fact that L_D only has one linearly independent eigenvector associated with eigenvalue zero, which means that the matrix Γ_D only has one linearly independent eigenvector, while it has two zero eigenvalues. Thus, matrix Γ_D could be written in the following Jordan canonical form:

$$\begin{aligned} \Gamma_D &= PJP^{-1} \\ &= W \begin{bmatrix} 0 & 1 & 0_{1\times(2n-2)} \\ 0 & 0 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix} V, \end{aligned} \tag{23}$$

where $W = [\, w_1 \quad \ldots \quad w_{2n} \,]$ and $V = [\, v_1 \quad \ldots \quad v_{2n} \,]^{\mathrm{T}}$, in which $w_i \in \mathbb{C}^{2n}$, $i \in \{1, \ldots, 2n\}$ are chosen to be the right eigenvectors or generalized eigenvectors of Γ_D and $v_i \in \mathbb{C}^{2n}$, $i \in \{1, \ldots, 2n\}$ are chosen to be the left eigenvectors or generalized left eigenvectors of Γ_D, and J' is the Jordan upper diagonal block matrix corresponding to nonzero eigenvalues $\lambda_{i\pm}$, $i = 2, \ldots, n$.

Without loss of generality, it can be verified that $w_1 = [\mathbf{1}_n^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ and $w_2 = [\mathbf{0}^{\mathrm{T}}, \mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}}$ are an eigenvector and a generalized eigenvector of Γ_D associated with eigenvalue zero, respectively. Find a vector $p \in \mathbb{C}^n$ satisfying $p^{\mathrm{T}} L_D = 0$, i.e., p is the left eigenvector of L_D associated with eigenvalue 0 and $p^{\mathrm{T}} \mathbf{1}_n = 1$. We choose $v_1 = [p^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ and $v_2 = [\mathbf{0}^{\mathrm{T}}, p^T]^{\mathrm{T}}$, where v_1 and v_2 are a generalized left eigenvector and a left eigenvector of Γ_D associated with eigenvalue zero, respectively, where $v_1^{\mathrm{T}} w_1 = 1, v_2^{\mathrm{T}} w_2 = 1$.

For matrix Γ_D, define matrix $P = [w_1, w_2, p_w]$, where $p_w \in \mathbb{C}^{2n\times 2(n-1)}$ and $P^{-1} = [v_1, v_2, p_v]^{\mathrm{T}}$ where $p_v \in \mathbb{C}^{2n\times 2(n-1)}$. Based on the fact that λ_{i+}, λ_{i-}, $i \in \{2, \ldots, n\}$ have negative real parts, one has

$$\begin{aligned} e^{\Gamma_D t} &= P e^{Jt} P^{-1} \\ &= [w_1, w_2, p_w] \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J' t} \end{bmatrix} \\ &\quad \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ p_v^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}_n p^{\mathrm{T}} & t\mathbf{1}_n p^{\mathrm{T}} \\ 0_{n\times n} & \mathbf{1}_n p^{\mathrm{T}} \end{bmatrix} + p_w e^{J' t} p_v^{\mathrm{T}}. \end{aligned} \tag{24}$$

Thus,

$$z = e^{\Gamma_D t} \begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n p^{\mathrm{T}} & t\mathbf{1}_n p^{\mathrm{T}} \\ 0_{n\times n} & \mathbf{1}_n p^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} + \tilde{z}(t)$$

in which $\tilde{z}(t) = p_w e^{J' t} p_v^{\mathrm{T}} \begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} = \begin{bmatrix} \tilde{z}_1(t) \\ \tilde{z}_2(t) \end{bmatrix}$ with $\tilde{z}_1(t) = [z_{11}(t), \ldots, z_{1n}(t)]^{\mathrm{T}}$ and $\tilde{z}_2(t) = [z_{21}(t), \cdots, z_{2n}(t)]^{\mathrm{T}}$. It can be verified that $\tilde{z}_1(t), \tilde{z}_2(t) \to 0$ as $t \to \infty$. Then,

$$\begin{aligned} &\lim_{t\to+\infty} [|x_i(t)| - |x_j(t)|] = \lim_{t\to+\infty} [|p^{\mathrm{T}} \bar{x}(0) \\ &+ tp^{\mathrm{T}} \bar{v}(0) + z_{1i}(t)| - |p^{\mathrm{T}} \bar{x}(0) + tp^{\mathrm{T}} \bar{v}(0) + z_{1j}(t)|] = 0, \end{aligned}$$

$$\begin{aligned} &\lim_{t\to+\infty} [|v_i(t)| - |v_j(t)|] = \lim_{t\to+\infty} [|p^{\mathrm{T}} \bar{v}(0) \\ &+ z_{2i}(t)| - |p^{\mathrm{T}} \bar{v}(0) + z_{2j}(t)|] = 0. \end{aligned}$$

Thus, the proof is completed. □

If graph $\mathcal{G}$ is an undirected connected signed graph, under Assumption 1, there always exists a transportation matrix D such that $L_D = DL_c D$ is a new Laplacian matrix associated with an unsigned undirected graph. In this case, $-L_D$ is a symmetric matrix and its eigenvalues are real. It is straightforward to verify that the nonzero eigenvalues λ_i of Γ_D have negative real parts, which means that the conditions of Theorem 1 are satisfied. As a result, we have the following corollary.

Corollary 1 *Consider the multi-agent system* (1) with a connected undirected signed graph. Under Assumption 1, *the bipartite consensus problem is solved by* (14) *if* $k_1 > 0$ *and* $k_2 > 0$.

4.2 Undirected signed graph with communication time delays

If the communication time delays τ_{ij} are considered, we consider the following control protocol:

$$u_i(t) = -k_1 \sum_{j\in N_i} |a_{ij}|(x_i(t-\tau_{ij}) - \mathrm{sgn}(a_{ij})x_j(t-\tau_{ij})) - k_2 \sum_{j\in N_i} |a_{ij}|(v_i(t-\tau_{ij}) - \mathrm{sgn}(a_{ij})v_j(t-\tau_{ij})). \tag{25}$$

With taking the communication time delays into consideration, another question is how to get the upper bound of the tolerable communication delays, defined as $\tau_{\max}$, such that the bipartite consensus can be reached as long as $\tau_{ij} \le \tau_{\max}$, for any i, j. Then, we can derive the following theorem.

Theorem 2 *Consider multi-agent system (1) with a connected undirected signed graph. Under Assumption 1, and if* $\tau_{\max} < \arctan((k_2/k_1)z)/z$, *where*

$$z = \sqrt{(k_2^2\lambda_{\max}^2(L_c) + \sqrt{k_2^4\lambda_{\max}^4(L_c) + 4k_1^2\lambda_{\max}^2(L_c)})/2},$$

the bipartite consensus problem is solved by (25) with $k_1 > 0, k_2 > 0$.

Proof Inspired by Lin et al. [11], define

$$\omega(t) = [x_1(t), v_1(t), x_2(t), v_2(t), \cdots, x_n(t), v_n(t)]^{\mathrm{T}}.$$

With protocol (25), the system (1) can be rewritten as the following compact form:

$$\dot{\omega}(t) = (I_n \otimes A)\omega(t) - \sum_{m=1}^{M} (L_m \otimes B)\omega(t-\tau_m), \tag{26}$$

with the initial condition $\omega(s) = \omega(0), s \in (-\infty, 0]$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$, $\sum_{m=1}^{M} L_m = L_C$ and $\tau_m \in \{\tau_{ij}, i,j \in \mathcal{V}\}$, $M = n(n-1)/2$, $m = 1, 2, \ldots, M$. Let $\phi(t) = (D \otimes I_2)\omega(t)$, it follows from (26) that

$$\dot{\phi}(t) = (I_n \otimes A)\phi(t) - \sum_{m=1}^{M} (L_m^D \otimes B)\phi(t-\tau_m), \tag{27}$$

where $L_m^D = DL_mD$. Let $\delta(t) = \phi(t) - \mathbf{1}_n \otimes [\alpha(t), \beta]^{\mathrm{T}}$, where $\alpha(t) = (1/n)\sum_{i=1}^{n}[\sigma_i x_i(0) + \beta t]$ and $\beta = (1/n)\sum_{i=1}^{n} \sigma_i v_i(0)$. It is easy to see that (26) is equivalent to

$$\dot{\delta}(t) = (I_n \otimes A)\delta(t) - \sum_{m=1}^{M} (L_m^D \otimes B)\delta(t-\tau_m). \tag{28}$$

Clearly, $\sum_{m=1}^{M} L_m^D = L_D$, $\sum_{m=1}^{M} L_m = L_C$ and $L_D = DL_CD$.

For the symmetrical matrix L_D, the nonzero eigenvalues are denoted as $\lambda_2, \ldots, \lambda_n$, where $\lambda_i > 0, i \in \{2, \ldots, n\}$. Meanwhile, there exists an orthogonal matrix W with the first column equal to $\frac{1}{\sqrt{n}}\mathbf{1}_n$ satisfying

$$\Lambda = W^{\mathrm{T}} L_D W = \mathrm{diag}(0, \lambda_2, \cdots, \lambda_n). \tag{29}$$

Since L_D is a Laplacian matrix corresponding to an undirected graph, it follows that $\sum_{i=1}^{n} \sigma_i \dot{v}_i(t) = 0$, $\sum_{1}^{n} \sigma_i \dot{x}_i(t) = \sum_{i=1}^{n} \sigma_i v_i(t)$. Thus, $\sum_{i=1}^{n} \sigma_i v_i(t) = n\beta$, $\sum_{i=1}^{n} \sigma_i x_i(t) = n\alpha(t)$. It follows that $(W^{\mathrm{T}} \otimes I_2)\delta(t) = [0, 0, \delta_c^{\mathrm{T}}(t)]^{\mathrm{T}}$. Premultiplying both sides of (28) by $W^{\mathrm{T}} \otimes I_2$, we have

$$\dot{\delta}_c(t) = (I_{n-1} \otimes A)\delta_c(t) - \sum_{m=1}^{M} (\tilde{L}_m^D \otimes B)\delta_c(t-\tau_m), \tag{30}$$

where $W^{\mathrm{T}} L_m^D W = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{L}_m^D \end{bmatrix}$. If the time delays $\tau_m = 0$, $\sum_{m=1}^{M} W^{\mathrm{T}} \tilde{L}_m^D W = \mathrm{diag}(\lambda_2, \lambda_3, \cdots, \lambda_n)$, by simple calculations, the eigenvalues of $A - \lambda_i B, i = 2, \cdots, n$ have negative real parts.

For the case with nonzero time delays $\tau_m \neq 0$, we use the frequency domain approach to analyze the stability of (30), and then,

$$\delta_c(s) = G_\tau^{-1}(s)(0). \tag{31}$$

where $G_\tau(s) = sI_{2n-2} - I_{n-1} \otimes A + \sum_{m=1}^{M} (\tilde{L}_m^D \otimes B)e^{-\tau_m s}$. Suppose that $s_G = jw \neq 0$ is an imaginary root of $G_\tau(s)$ and $u \in \mathbb{C}^{2n-2}$ is the eigenvector corresponding to $s_G = jw \neq 0$ and $\|u\| = 1$, we have

$$(jwI_{2n-2} - I_{n-1} \otimes A + \sum_{m=1}^{M} (\tilde{L}_m^D \otimes B)e^{-j\tau_m w})u = 0. \tag{32}$$

Let $u = u_1 \otimes [1, 0]^{\mathrm{T}} + u_2 \otimes [0, 1]^{\mathrm{T}}$, where $u_1, u_2 \in \mathbb{C}^{n-1}$. Calculate the odd rows of (32), and we have

$$jwu_1 = u_2. \tag{33}$$

Since $u^* u = u_1^* u_1 + u_2^* u_2 = 1$, we could get the following equations:

$$\begin{aligned} u_1^* u_1 &= \frac{1}{w^2+1}, \\ u_2^* u_2 &= \frac{w^2}{w^2+1}. \end{aligned} \tag{34}$$

Premultiplying both sides of (32) by u^*, we have

$$
\begin{aligned}
& jwu_1^* u_1 + k_1 \sum_{m=1}^{M} u_2^* (\tilde{L}_m^D \otimes B) e^{-j\tau_m w} u_1 - u_1^* u_2 \\
& \quad + jwu_2^* u_2 + k_2 \sum_{m=1}^{M} u_2^* (\tilde{L}_m^D \otimes B) e^{-j\tau_m w} u_2 = 0.
\end{aligned} \tag{35}
$$

By (33), (34) and (35), let $a_m = u_1^* \tilde{L}_m^D u_1 / u_1^* u_1$, we could get:

$$
\begin{aligned}
\sum_{m=1}^{M} a_m \sin(w\tau_m) &= \frac{(k_2/k_1)w^3}{(k_2^2/k_1)w^2 + k_1}, \\
\sum_{m=1}^{M} a_m \cos(w\tau_m) &= \frac{w^2}{(k_2^2/k_1)w^2 + k_1}.
\end{aligned} \tag{36}
$$

Considering the case of all equal delays, the following equations hold:

$$
\begin{aligned}
\sum_{m=1}^{M} a_m \sin(w\tau_{\max}) &= \frac{(k_2/k_1)w^3}{(k_2^2/k_1)w^2 + k_1}, \\
\sum_{m=1}^{M} a_m \cos(w\tau_{\max}) &= \frac{w^2}{(k_2^2/k_1)w^2 + k_1}.
\end{aligned} \tag{37}
$$

Let $w\tau_{\max} < \frac{\pi}{2}$, $\left(\sum_{m=1}^{M} a_m\right)^2 = w^4/(k_2^2 w^2 + k_1^2)$, then one has

$$
w^2 = \left[k_2^2 \left(\sum_{m=1}^{M} a_m \right)^2 + \sqrt{ \left(k_2 \sum_{m=1}^{M} a_m \right)^4 + 4 \left(k_1 \sum_{m=1}^{M} a_m \right)^2 } \right] /2,
$$

which is an increasing function of the variable $(\sum_{m=1}^{M} a_m)$. From Lemma 3, $D(w) = (\arctan(k_2/k_1)w)/w$ is a decreasing function with respect to w. To find maximal tolerable delay $\tau_{\max}$, and $(\sum_{m=1}^{M} a_m) = u_1^* (\sum_{m=1}^{M} \tilde{L}_m^D) u_1 / u_1^* u_1 \le \lambda_{\max}$, we get that $\tau_{\max} = \arctan(k_2/k_1)w)/w$, where $w = \sqrt{(k_2^2 \lambda_{\max}^2 + \sqrt{k_2^4 \lambda_{\max}^4 + 4k_1^2 \lambda_{\max}^2})/2}$. If $\tau_m < \tau_{\max}$, Eq. (37) is not satisfied. Meanwhile, if $\tau_m = 0$, all eigenvalues are located in LHP. Due to the continuity of function (36), the eigenvalues of function (31) have negative real parts. Under these conditions, $\delta_c(t)$ converges to 0 as $t \to +\infty$. Then, $\delta(t) = 0$, $t \to +\infty$. By premultiplying matrix D to $\phi(t)$, the bipartite consensus problem is solved. □

Remark 1 To solve the bipartite consensus problem, the protocols (14) and (25) are further developed based on conventional consensus protocols, which extends the original consensus problem to a more general case. Different from the results in [11], the final convergence state of all agents is also related to the negatively weighted edges in the graph. As a consequence, in this paper, the final convergence state is determined by $\alpha(t) = (1/n) \sum_{i=1}^{n} [\sigma_i x_i(0) + \beta t]$ and $\beta = (1/n) \sum_{i=1}^{n} \sigma_i v_i(0)$.

Remark 2 It is desirable to find the largest tolerable communication delay, defined as $\bar{\tau}_{\max}$, where $\bar{\tau}_{\max} = (\arctan(k_2/k_1)w)/w$ with $w = \sqrt{\frac{k_2^2 \lambda_{\max}^2 + \sqrt{k_2^4 \lambda_{\max}^4 + 4k_1^2 \lambda_{\max}^2}}{2}}$. Obviously, for the fixed $k_1, k_2, f(w) = (\arctan(k_2/k_1)w)/w$ is a decreasing function with respect to variable w, and w is proportional to $\lambda_{\max}$. Thus, $\bar{\tau}_{\max}$ could get the largest value when w reaches the minimum. Namely, the smaller the $\lambda_{\max}$ is, the greater the $\bar{\tau}_{\max}$ is. Thus, if we could design a suitable communication topology such that a smaller value of $\lambda_{\max}$ is obtained, one could get a greater value of $\bar{\tau}_{\max}$, which means this case is more robust to the communication time delays.

5 Solution to the grouping problem

In this section, we will solve the grouping problem as defined in Definition 2. For a given weighted matrix $\mathcal{A}$ associated with a signed graph, there exist positive and negative weights. As shown in the previous session, the solution to the grouping problem plays a key role in determining the Gauge matrix and identifying which group each agent belongs to. As a consequence, the final convergence state of each agent can also be determined.

For a signed graph which is strongly connected, all agents could be separated into two parts $\mathcal{V}_1$ and $\mathcal{V}_2$ if it is structurally balanced. Meanwhile, there exists the clear relationship between the ith agent and its neighbors N_i, which makes it easy to identify the relationship among all agents. The pseudocode to solve the grouping problem is listed in Algorithm 1.

Algorithm 1 The Algorithm to Determine D

Get the corresponding Laplacian matrix L of $\mathcal{A}$ and calculate the rank of matrix L. There exists the Gauge transformation matrix D if and only if the rank of L is equal to $n-1$. Initialize empty sets V_1, V_2, Row and the first scanned row $index = 1$, flag position $isheadv_1 = true$.
while $true$ **do**
  Store $index$ into Row. Define two empty sets V_{1t}, V_{2t}
  if $isheadv_1 == true$ **then**
    Store the $index$ into V_{1t}. Then, go to scan the $index$ row of $\mathcal{A}$. If the element is positive, put the corresponding column index into V_{1t}. If the element is negative, put the corresponding column index into V_{2t}.
  else
    Store the $index$ into V_{2t}. Then, go to scan the $index$ row of $\mathcal{A}$. If the element is positive, put the corresponding column index into V_{2t}. If the element is negative, put the corresponding column index into V_{1t}.
  end if
  $V_1 = V_1 \cup V_{1t}, V_2 = V_2 \cup V_{2t}, Row_t = V_1 - (V_1 \cap Row)$
  if Row_t is not empty **then**
    $isheadv_1 = true, index = \min(Row_t)$
  else
    $Row_t = V_2 - (V_2 \cap Row)$
    if Row_t is not empty **then**
      $isheadv_1 = false, index = \min(Row_t)$
    else
      break
    end if
  end if
end while
Get the corresponding transformation matrix D by the two groups V_1 and V_2.

6 Simulation

In this section, numerical simulations are given to verify the obtained theoretical results.

Figure 1 shows an undirected signed graph with ten agents, where the blue solid line denotes the edges with the positive weights, while the red dashed line denotes the edges with the negative weights. It is easy to be observed that Fig. 1 is structurally balanced with $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2 = \{v_6, v_7, v_8, v_9, v_{10}\}$. We first consider the case without considering communication time delays, and according to Corollary 1, we choose $k_1 = 2$, $k_2 = 2$.

The simulated results are displayed in Figs. 2 and 3, which show that the bipartite consensus is reached as expected.

Then, we consider the case with nonuniform communication time delays τ_{ij} and choose $k_1 = 1$, $k_2 = 2$. According to Theorem 2, the maximal tolerable delay is $\tau_{\max} = 0.1190$. Suppose $\tau_{12} = \tau_{23} = \tau_{24} = \tau_{45} = \tau_{25} = 0.11 < \tau_{\max}$ and the communication delays between other pairs are $0.1 < \tau_{\max}$.

Figures 4 and 5 show that the bipartite consensus is also reached with the proposed controller, and it can be observed that all agents converge to a dynamic state.

Then, we further consider the case with communication delays beyond the upper bound. Suppose $\tau_{12} = \tau_{23} = \tau_{24} = \tau_{45} = \tau_{25} = 0.12 > \tau_{\max}$ and the communication delays between other pairs are $0.121 > \tau_{\max}$.

Figures 6 and 7 show that the bipartite consensus cannot be reached with the proposed controller, and it can be observed that the velocities and positions are divergent.

Furthermore, we consider the case that there exists only one time delay between two agents that is larger than $\tau_{\max}$, while others are less than $\tau_{\max}$. We choose $\tau_{12} = \tau_{21} = 0.7 > \tau_{\max}$, and communication delays between other pairs are 0.11 which is less than $\tau_{\max}$.

The simulated results as shown in Figs. 8 and 9 show that both the velocities and positions are divergent, which implies that in this case the consensus cannot be achieved.

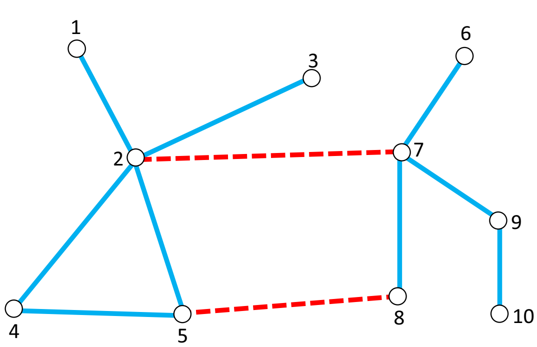

Fig. 1 Undirected graph $\mathcal{G}$

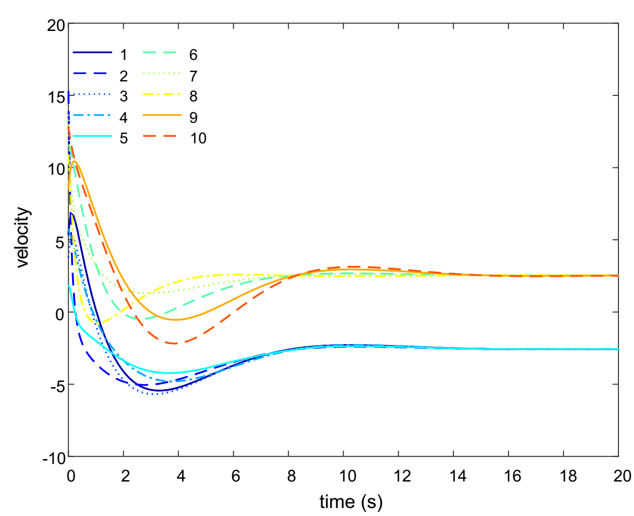

Fig. 2 Velocities of all agents without considering the time delays

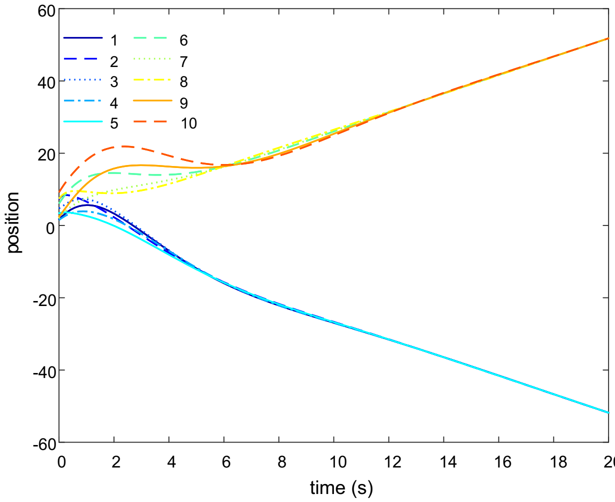

Fig. 3 Positions of all agents without considering the time delays

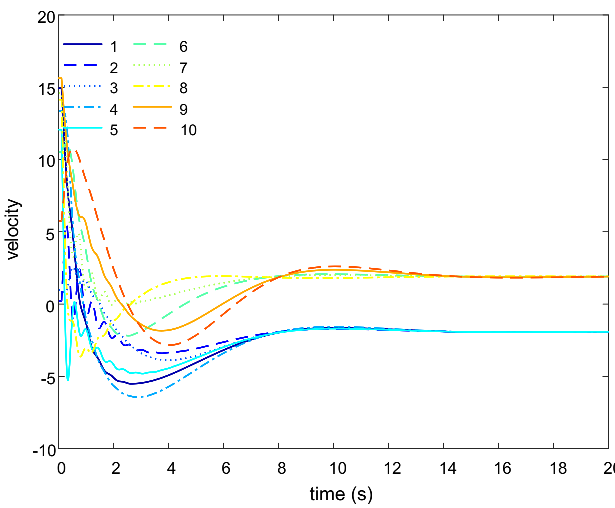

Fig. 4 Velocities of all agents with $\tau_{ij} < \tau_{\max}$ for all i, j

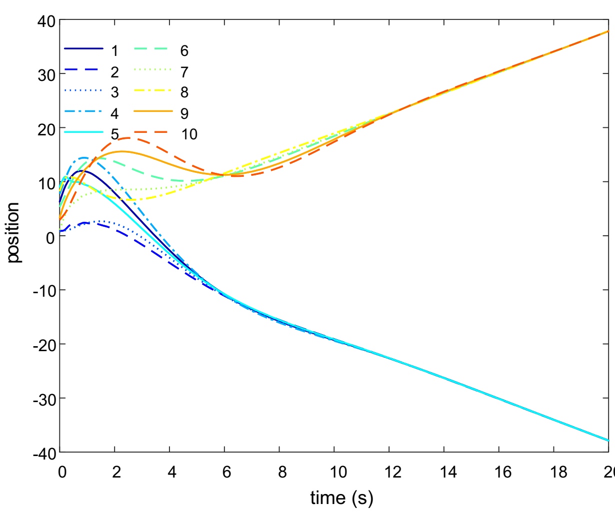

Fig. 5 Positions of all agents with $\tau_{ij} < \tau_{\max}$ for all i, j

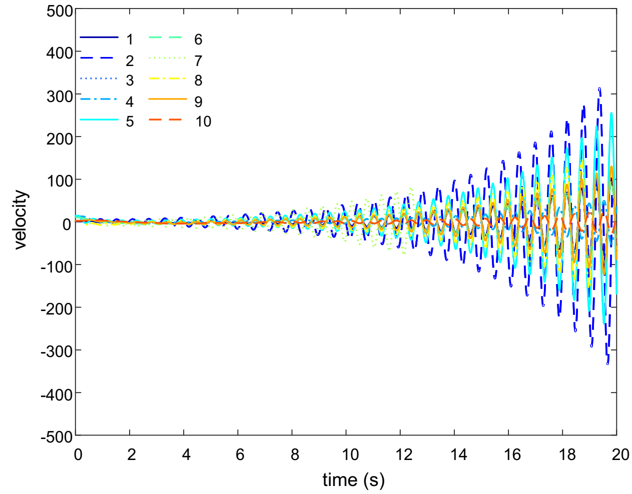

Fig. 6 Velocities of all agents with $\tau_{ij} > \tau_{\max}$ for all i, j

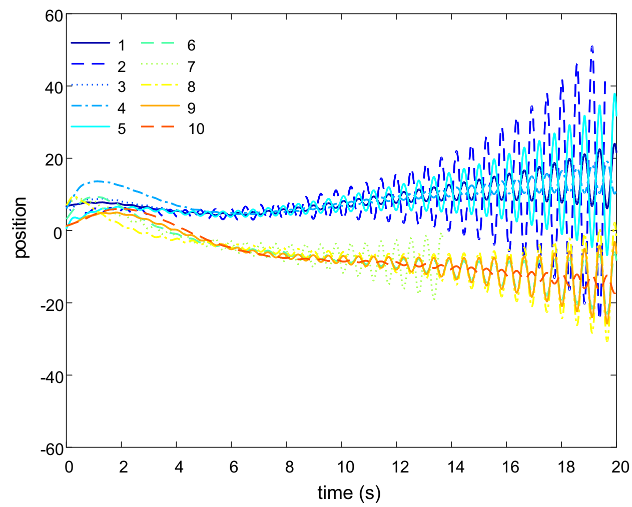

Fig. 7 Positions of all agents with $\tau_{ij} > \tau_{\max}$ for all i, j

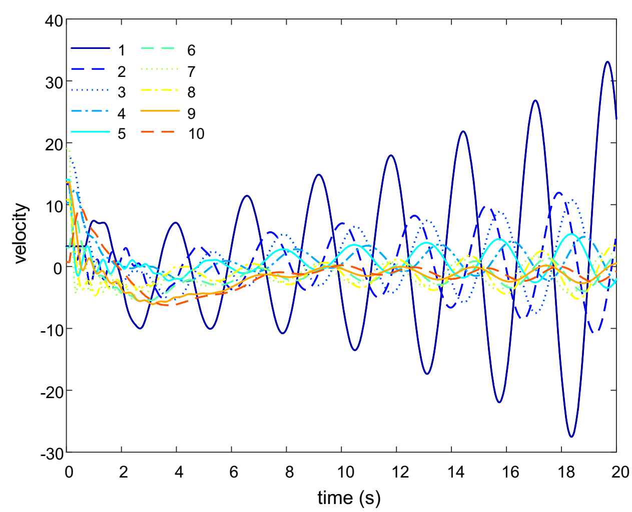

Fig. 8 Velocities of all agents with $\tau_{12} = \tau_{21} > \tau_{\max}$

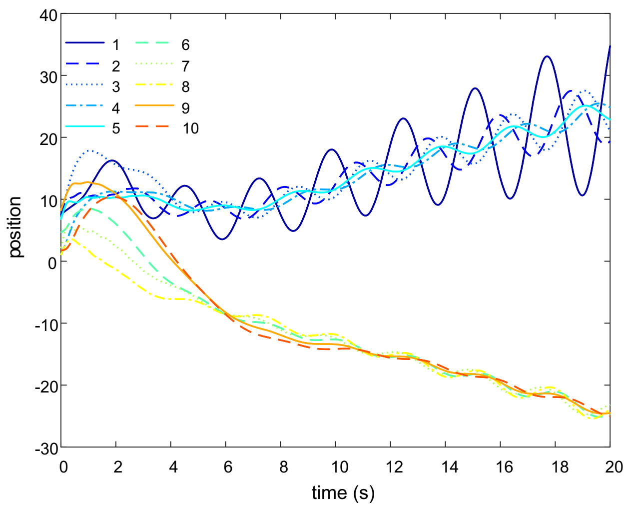

Fig. 9 Positions of all agents with $\tau_{12} = \tau_{21} > \tau_{max}$

7 Conclusions

In this paper, the bipartite consensus problems for double-integrator multi-agent systems over antagonistic networks are discussed. For the directed signed graph, sufficient conditions are given to solve the bipartite consensus problem if the communication time delays are not considered. The bipartite consensus problem with nonuniform time delays for undirected graph is addressed, and the maximal tolerable time delays are calculated by utilizing the frequency domain approach. In addition, the algorithm to solve the grouping problem for a structurally balanced graph is obtained. In the future, we will further consider the time delays in directed networks, which are more practical in real world.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest. This article does not contain any studies with animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

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