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Bipartite Consensus for a Class of Double-Integrator Multi-Agent Systems with Antagonistic Interactions

Yanhua Yang, Rui Ding and Wenfeng Hu*

***Abstract*— In this paper, the bipartite consensus problem is addressed for a class of double-integrator multi-agent systems with antagonistic interactions. The cases with and without communication time delay are considered. In particular, if the communication time delay is not taken into account, the bipartite consensus of the multi-agent systems with directed signed graph can be achieved by the proposed distributed controller. If the communication time delay is considered, the bipartite consensus of the multi-agent systems with undirected signed graph can be achieved and the upper bound of the time delay is further derived. Finally, some numerical examples are provided to illustrate the correctness of the results.**

***Index Terms*— Multi-agent systems, antagonistic network, double-integrator systems, time delay.**

I. Introduction

Cooperative control problems of multi-agent systems have been widely investigated in many domains in the past several years. In many multi-agent systems such as biological systems, robotic systems and so on, the information exchange and interaction among agents are usually local, which leads to more concerns on distributed control protocols. As a fundamental cooperative control problem, the so-called consensus problem requires the states of all agents to converge to a common value of interest. Some typical results on distributed consensus protocols can be found in [1–6]. More specifically, in [1], the consensus problem was solved for multi-agent systems with fixed and switching topologies, respectively. Furthermore, in [3], the authors showed that the consensus problem can be solved if the interaction network is jointly connected.

As for the information exchange among agents, the communication time delay is unavoidable due to the unexpected disturbances. The time delay affects the control performance and even stability of the controlled systems severely. Thus, numerical works have been done to study the effect of the time delay in [7–10]. In [7] and [9], the authors analysed the influence of the time-varying delay for fixed and switching networks and got the upper bound of the time delay under which the consensus can be achieved.

It is noted that all the above-mentioned works are concerned with cooperative control protocols. However, in many scenarios, some agents in multi-agent systems not only collaborate, but also compete, see [11, 12]. For example, two groups of mobile cars or robots may keep their velocities in two opposite directions in order to complete some tasks. In this case, the edges in a graph representing the interaction between two agents may not always have positive weights. Such a graph is called a signed graph or an antagonistic network.

The consensus problem of multi-agent systems with antagonistic networks was investigated in the literature [13] by C. Altafini firstly and the multi-agent systems with the first order dynamics are considered. It is shown that the bipartite consensus was reached if the connected signed graph is structurally balanced. Moreover, the continuous-time Altafini model has been studied in [14], where the authors relaxed the strongly connected assumption to the less conservative spanning tree condition. What is more, the finite-time bipartite consensus problem was studied in literature [15], which estimated the settling time effectively. More related results on this topic can be found in [16, 17]. However, in the cases with antagonistic interactions, communication time delays are usually neglected. To the best of our knowledge, there are few results on the bipartite consensus problem of double-integrator multi-agent systems with the communication time delay taken into consideration. It is further desired that the bipartite consensus is achieved with a nonzero velocity, which may be useful in future engineering work. In this paper, the bipartite consensus problem of double-integrator multi-agent systems is studied with and without time delay considered, respectively. Main tools used in this paper are matrix theory, algebraic theory, frequency analysis and control theory.

The rest of the paper is organized as follows: basic definitions about algebraic graph theory are introduced in section II; problem formulation is introduced in section III; in section IV, we give the main results and detailed proofs; some simulation results are shown in section V to illustrate our results; and the conclusion is given in section VI.

II. Graph Theory

Notations: R^n denotes the set of n dimensional real column vectors; C^n denotes the set of n dimensional complex vectors; I_m represents the m dimensional identity matrix; $\mathbf{1}_n$ denotes the n dimensional column vector with all elements being 1; 0 represents zero vectors or zero matrixes with appropriate dimensions; x^T denotes the transposition of x; x^* denotes the conjugate transposition of x; j denotes the imaginary unit; $\otimes$ denotes the kronecker product.

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$\mathrm{diag}(a_1, a_2, \cdots, a_n)$ denotes a diagonal matrix with the diagonal elements are $a_1, a_2, ...,$ and a_n.

A signed graph is denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{G}$ means a graph, $\mathcal{V} = \{v_1, \cdots, v_n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathcal{A} \in R^{n \times n}$ is the weights matrix of a signed graph $\mathcal{G}$. The element $a_{ij} \neq 0$ $(i \neq j)$ if and only if the edge$(v_j, v_i) \in \mathcal{E}$ and $a_{ii} = 0$, $i \in \{1, \cdots, n\}$, otherwise. For a directed signed graph, we will always define that $a_{ij}a_{ji} \geq 0$, which implies that the edges pairs of any two nodes always have the same sign. The Laplacian matrix L_c is defined as $L_c = [l_{ik}]$, where $l_{ii} = \sum_{k=1}^{n} |a_{ik}|$ and $l_{ik} = -a_{ik}$, $i \neq k$. A (directed) path from v_1 to v_l is a sequence of edges of the form $(v_1, v_2), (v_2, v_3), ..., (v_{l-1}, v_l)$ with distinct nodes in a (directed) graph. A graph is strongly connected if there exists a directed path between any two agents. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$.

Lemma 1 ([13]): A (directed) connected signed graph $\mathcal{G}$ is structurally balanced if and only if any of the following conditions holds:

1. All nodes can be partitioned into two sets (one possibly empty) in such a way that edges joining two nodes in the same set are positive while edges joining two nodes in different sets are negative.

2. There is a Gauge transformation matrix $D \in \mathcal{D}$ satisfying DAD has all nonnegative entries where $\mathcal{D} = \{D = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n),\ \ \sigma_i \in \{+1, -1\}\}$.

III. Problem Formulation

Consider a group of n agents, and the ith agent has the following dynamics

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \end{aligned} \tag{1}$$

where $x_i(t)$, $v_i(t)$, $u_i(t) \in R$ represent the position, velocity and the control input of the ith agent at time t, respectively, with the initial condition $x_i(s) = x_i(0)$, $s \in (-\infty, 0]$, $v_i(s) = v_i(0)$, $s \in (-\infty, 0]$. Consider the control protocol as

$$u_i(t) = f_i(x_i(t), x_{j,j\in N_i}(t), v_i(t), v_{j,j\in N_i}(t)). \tag{2}$$

If the communication time delay is considered and suppose that the time delays are τ, the protocol is given as follows

$$\begin{aligned} &u_i(t) = \\ &f_i(x_i(t-\tau), x_{j,j\in N_i}(t-\tau), v_i(t-\tau), v_{j,j\in N_i}(t-\tau)). \end{aligned} \tag{3}$$

Definition 1: Consider the multi-agent system (1) with the signed graph $\mathcal{G}$. Develop a distributed control scheme such that, for any finite $x_i(0)$, $v_i(0)$, $i = 1, \cdots, n$, the following conditions are satisfied

$$\begin{aligned} \lim_{t\to+\infty} [|x_i(t)| - |x_j(t)|] = 0, \\ \lim_{t\to+\infty} [|v_i(t)| - |v_j(t)|] = 0, \end{aligned} \tag{4}$$

then, it is said that the bipartite consensus problem is solved.

To obtain the main results, we further need the following lemmas.

Lemma 2: Let $\xi_{\pm} = (k_2\mu \pm \sqrt{k_2^2\mu^2 + 4k_1\mu})/2$, where $\xi, \mu \in C$, k_1, k_2 are tuning parameters. If $\mathrm{Re}(\mu) < 0$, $k_1 > 0$ and

$$k_2 > \sqrt{\frac{2k_1}{|\mu|\cos(\frac{\pi}{2} - \arctan(\frac{-\mathrm{Re}(\mu)}{\mathrm{Im}(\mu)}))}}, \tag{5}$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real part and the imaginary part of a complex number, respectively. Then $\mathrm{Re}(\xi_{\pm}) < 0$.

Proof: Motivated by [6], we use a graph as shown in Fig. 1 to denote the notations we need.

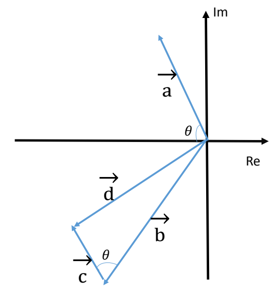

Fig. 1. Notation used in Lemma 2.

Let $\vec{a} = k_2\mu$, $\vec{b} = (k_2\mu)^2$, $\vec{c} = 4k_1\mu$, and $\vec{d} = \vec{b} + \vec{c}$. To satisfy $\mathrm{Re}(k_2\mu \pm \sqrt{k_2^2\mu^2 + 4k_1\mu}) < 0$, we need to show that $|\vec{d}| < |\vec{a}|^2$. It follows from the law of cosines that $|\vec{d}|^2 = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos(\theta)$. Then, it can be calculated that $k_2 > \sqrt{\frac{2k_1}{|\mu|\cos(\theta)}}$, where $\theta = \pi/2 - \arctan(-\mathrm{Re}(\mu)/\mathrm{Im}(\mu)), \mathrm{Im}(\mu) \neq 0$. ■

Lemma 3 ([7]): Define $D(w) = (\arctan w)/w$. For $w \in (0, +\infty)$, $(d/dw)D(w) < 0$.

IV. Main Results

In this section, we focus on the stability analysis for the double-integrator dynamics systems with and without time delays. First part is concerned with the directed signed graph without time delay and the other part focuses on undirected signed graph with the communication time delay.

A. Directed signed graph without time delay

With the directed signed graph, a distributed control protocol is designed as follows

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j\in N_i} |a_{ij}|(x_i(t) - \mathrm{sgn}(a_{ij})x_j(t)) \\ &-k_2 \sum_{j\in N_i} |a_{ij}|(v_i(t) - \mathrm{sgn}(a_{ij})v_j(t)), \end{aligned} \tag{6}$$

where $k_1, k_2 > 0$ are control parameters and $\mathrm{sgn}(\cdot)$ is the sign functon.

Define $x = [x_1, x_2, ..., x_n]^T \in R^n$, $v = [v_1, v_2, ..., v_n]^T \in R^n$. With protocol (6), system (1) can be rewritten in the following compact form

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -k_1 L_c x - k_2 L_c v, \end{aligned} \tag{7}$$

where L_c is the Laplacian matrix for a signed graph. Eq. (7) is written as

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} &= \begin{bmatrix}0 & I_n\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} + \begin{bmatrix}0 & 0\\ -k_1L_c & -k_2L_c\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix}\\ &= \Gamma\begin{bmatrix}x\\ v\end{bmatrix},\end{aligned} \tag{8}$$

where $\Gamma = \begin{bmatrix}0 & I_n\\ -k_1L_c & -k_2L_c\end{bmatrix}$. If the directed connected signed graph is structurally balanced, it is obvious that there exists a transformation matrix D as defined in Lemma 1. Define the extended transformation matrix $\bar{D} = \mathrm{diag}(D, D) \in R^{2n\times 2n}$. Let $\bar{x} = Dx$, $\bar{v} = Dv$, $z = \bar{D}[x^T, v^T]^T = [\bar{x}^T, \bar{v}^T]^T$. We could get the derivative of the variable z as follows

$$\dot{z} = \begin{bmatrix}D & 0\\ 0 & D\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \Gamma_D z, \tag{9}$$

where $\Gamma_D = \begin{bmatrix}0 & I_n\\ -k_1L_D & -k_2L_D\end{bmatrix}$ and $L_D = DL_cD$ is the new Laplacian matrix of a directed connected unsigned graph.

It follows from (9) that

$$z = e^{\Gamma_D t}z(0), t \geq 0. \tag{10}$$

To find the eigenvalues of Γ_D, we need to calculate the solutions of $\det(\lambda I_{2n} - \Gamma_D) = 0$, which is the characteristic polynomial of matrix Γ_D. Note that

$$\begin{aligned}\det(\lambda I_{2n} - \Gamma_D) &= \det\left(\begin{bmatrix}\lambda I_n & -I_n\\ k_1L_D & \lambda I_n + k_2L_D\end{bmatrix}\right)\\ &= \det(\lambda^2 I_n + (k_1 + k_2\lambda)L_D).\end{aligned} \tag{11}$$

Also note that

$$\det(\lambda I_n + L_D) = \prod_{i=1}^{n}(\lambda - \mu_i), \tag{12}$$

where μ_i is the ith eigenvalue of $-L_D$.

By comparing (11) with (12), we see that

$$\det(\lambda^2 I_n + (k_1 + k_2\lambda)L_D) = \prod_{i=1}^{n}(\lambda^2 - (k_1 + k_2\lambda)\mu_i), \tag{13}$$

which implies that the roots of (13) can be obtained by solving $\lambda^2 = (k_1 + k_2\lambda)\mu_i$. Therefore, it is straightforward to see that the eigenvalues of Γ_D are given by

$$\lambda_{i\pm} = \frac{k_2\mu_i \pm \sqrt{k_2^2\mu_i^2 + 4k_1\mu_i}}{2}, \tag{14}$$

where λ_{i+} and λ_{i-} are two conjugate eigenvalues of Γ_D associated with μ_i. Eq. (14) has $2n$ eigenvalues if matrix $-L_D$ has n eigenvalues. It is natural that $-L_D$ has at least one eigenvalue 0 with the eigenvector $\mathbf{1}_n$ for the fact that the row sum of $-L_D$ is 0. Therefore, Eq. (14) has at least two eigenvalues 0 corresponding to $u_i = 0$ with $\lambda_+ = 0$ and $\lambda_- = 0$.

From the above analysis, we have the following theorem.

Theorem 1: Consider the multi-agent systems (1) with a strongly connected directed signed graph. If the graph is structurally balanced, the bipartite consensus problem is solved by (6) with $k_1 > 0$, $k_2 > \max_{\mu_i\neq 0}\sqrt{\frac{2k_1}{|\mu_i|\cos(\frac{\pi}{2} - \arctan(\frac{-\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)}))}}$.

Proof: Since L_D of the directed graph may have complex eigenvalues, we have to prove that all eigenvalues of Γ_D are located in left-hand-plane (LHP). For a directed graph which is strongly connected, the eigenvalues of $-L_D$ have negative real parts except for $\mu_1 = 0$, which implies that $\mathrm{Re}(\mu_i) < 0, i = 2, 3, ..., n$. From Lemma 2, we could conclude that if $k_1 > 0$, $k_2 > \max_{\mu_i\neq 0}\sqrt{\frac{2k_1}{|\mu_i|\cos(\frac{\pi}{2} - \arctan(\frac{\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)}))}}$, then $\mathrm{Re}(\lambda_{i\pm}) < 0, i = 2, 3, \cdots, n$. And then, we need to prove that the consensus will be achieved since the matrix Γ_D only has two eigenvalues 0 and others are located in LHP.

First, we prove that Γ_D has only one linearly independent eigenvector associated with eigenvalue 0. Define $[q_1^T, q_2^T]^T$ as the eigenvector of Γ_D corresponding to eigenvalue 0, where $q_1, q_2 \in C^n$.

So

$$\begin{bmatrix}0 & I_n\\ -k_1L_D & -k_2L_D\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{15}$$

From (15), we could get $q_2 = \mathbf{0}$ and $k_1L_Dq_1 = \mathbf{0}$. For the fact that L_D only has one eigenvector $\mathbf{1}_n$ corresponding to eigenvalue 0, then we could only get a linearly independent eigenvector q_1, which means that the matrix Γ_D only has one linearly independent eigenvector while it has two eigenvalues 0. Thus the matrix Γ_D could be written as following form:

$$\begin{aligned}&\Gamma_D = PJP^{-1} =\\ &W\begin{bmatrix}0 & 1 & 0_{1\times(2n-2)}\\ 0 & 0 & 0_{1\times(2n-2)}\\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J'\end{bmatrix}V,\end{aligned} \tag{16}$$

where $W = \begin{bmatrix}w_1 & \ldots & w_{2n}\end{bmatrix}$ and $V = \begin{bmatrix}v_1 & \ldots & v_{2n}\end{bmatrix}^T$, in which w_i, $i \in \{1, \ldots, 2n\}$ are chosen from the right eigenvectors or generalized eigenvectors of Γ_D and v_i, $i \in \{1, \ldots, 2n\}$ are chosen from the left eigenvector or generalized left eigenvectors of Γ_D, and J' is the Jordan upper diagonal block matrix corresponding to non-zero eigenvalues λ_{i+} and λ_{i-}, $i = 2, ..., n$.

Without loss of generality, we choose $w_1 = [\mathbf{1}_n^T, \mathbf{0}^T]^T$ and $w_2 = [\mathbf{0}^T, \mathbf{1}_n^T]^T$, where w_1 and w_2 are an eigenvector and a generalized eigenvector of Γ_D associated with eigenvalue 0 respectively. Find a vector $p \in C^n$ satisfying $p^TL_D = 0$, i.e., p is left eigenvector of L_D associated with eigenvalue 0 and $p^T\mathbf{1}_n = 1$. we choose $v_1 = [p^T, \mathbf{0}^T]^T$ and $v_2 = [\mathbf{0}^T, p^T]^T$, where v_1 and v_2 are a generalized left eigenvector and a left eigenvector of Γ_D associated with eigenvalue 0 respectively, and $v_1^Tw_1 = 1, v_2^Tw_2 = 1$.

For matrix Γ_D, define matrix $P = [w_1, w_2, p_w]$, where $p_w \in C^{2n\times 2(n-1)}$ and $P^{-1} = [v_1, v_2, p_v]^T$, where $p_v \in C^{2n\times 2(n-1)}$. Based on the fact that $\lambda_{i+}, \lambda_{i-}, i \in \{2, \ldots, n\}$

have negative real parts, one has

$$\begin{aligned}
&e^{\Gamma_D t} = Pe^{Jt}P^{-1} \\
&= [w_1, w_2, p_w]\begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J' t} \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \\ p_v^T \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{1}_n p^T & t\mathbf{1}_n p^T \\ 0_{n\times n} & \mathbf{1}_n p^T \end{bmatrix} + p_w e^{J' t} p_v^T.
\end{aligned} \tag{17}$$

Thus, $z = e^{\Gamma_D t}\begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n p^T & t\mathbf{1}_n p^T \\ 0_{n\times n} & \mathbf{1}_n p^T \end{bmatrix}\begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} + \tilde{z}(t)$ in which $\tilde{z}(t) = p_w e^{J' t} p_v^T \begin{bmatrix} \bar{x}(0) \\ \bar{v}(0) \end{bmatrix} = \begin{bmatrix} \tilde{z}_1(t) \\ \tilde{z}_2(t) \end{bmatrix}$ with $\tilde{z}_1(t) = [z_{11}(t), \cdots, z_{1n}(t)]^T$ and $\tilde{z}_2(t) = [z_{21}(t), \cdots, z_{2n}(t)]^T$. It can be verified that $\tilde{z}_1(t)$, $\tilde{z}_2(t) \to 0$ as $t \to \infty$. Then, $\lim_{t\to+\infty}[|x_i(t)| - |x_j(t)|] = \lim_{t\to+\infty}[(p^T\bar{x}(0) + tp^T\bar{v}(0) + z_{1i}(t)) - (p^T\bar{x}(0) + tp^T\bar{v}(0) + z_{1j}(t))] = 0$, $\lim_{t\to+\infty}[|v_i(t)| - |v_j(t)|] = \lim_{t\to+\infty}[(p^T\bar{v}(0) + z_{2i}(t)) - (p^T\bar{v}(0) + z_{2j}(t))] = 0$. Thus, the proof is completed.

■

If graph $\mathcal{G}$ is an undirected connected signed graph which is structurally balanced, a transportation matrix D is easy to be found such that $L_D = DL_cD$ is a new Laplacian of the unsigned connected graph. In this case, $-L_D$ is a symmetric matrix and its eigenvalues are reals. The nonzero eigenvalues λ_i of Γ_D have negative real parts, which means that the conditions of Theorem 1 are satisfied. Thus, the multi-agent systems with undirected signed graphs could reach bipartite consensus asymptotically. In conclusion, we have the following corollary.

Corollary 1: Consider the multi-agent system (1) with a connected undirected signed graph. If the graph is structurally balanced, the bipartite consensus problem is solved by (6) with $k_1 > 0$ and $k_2 > 0$.

B. Undirected signed graph with communication time delay

Suppose that there is uniform time delay τ. Consider the control protocol with fixed time delay τ as

$$\begin{aligned}
u_i(t) = &-k_1\sum_{j\in N_i}|a_{ij}|(x_i(t-\tau) - \mathrm{sgn}(a_{ij})x_j(t-\tau)) \\
&-k_2\sum_{j\in N_i}|a_{ij}|(v_i(t-\tau) - \mathrm{sgn}(a_{ij})v_j(t-\tau)),
\end{aligned} \tag{18}$$

where k_1, k_2 are positive gains.

Define $\omega(t) = [x_1(t), v_1(t), x_2(t), v_2(t), ..., x_n(t), v_n(t)]^T$. With protocol (18), the system (1) can be rewritten in the following compact form

$$\dot{\omega}(t) = (I_n \otimes A)\omega(t) - (L_c \otimes B)\omega(t-\tau), \tag{19}$$

with the initial condition $\omega(s) = \omega(0), s \in (-\infty, 0]$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$. Let $\phi(t) = (D \otimes I_2)\omega(t)$, from (19),

$$\dot{\phi}(t) = (I_n \otimes A)\phi(t) - (L_D \otimes B)\phi(t-\tau), \tag{20}$$

where $L_D = DL_cD$. Let $\delta(t) = \phi(t) - \mathbf{1}_n \otimes [\alpha(t), \beta]^T$, where $\alpha(t) = (1/n)\sum_{i=1}^n[\sigma_i x_i(0) + \beta t]$ and $\beta = (1/n)\sum_{i=1}^n \sigma_i v_i(0)$. It is easy to see that the multi-agent system (19) is equal to

$$\dot{\delta}(t) = (I_n \otimes A)\delta(t) - (L_D \otimes B)\delta(t-\tau). \tag{21}$$

For the matrix L_D, the nonzero eigenvalues denote as $\lambda_2, \cdots, \lambda_n$, where $\lambda_i > 0, i \in [2, \cdots, n]$. For a symmetrical matrix L_D, there exists an orthogonal matrix W satisfying

$$\Lambda = W^T L_D W = \mathrm{diag}\{0, \lambda_2, \cdots, \lambda_n\}. \tag{22}$$

Premultiplying both sides of (21) by $W^T \otimes I_2$, we have

$$\dot{\delta}_c(t) = (I_{n-1} \otimes A)\delta_c(t) - (\Lambda_m \otimes B)\delta_c(t-\tau), \tag{23}$$

where $(W^T \otimes I_2)\delta(t) = [0, 0, \delta_c^T(t)]^T$ and $\Lambda_m = \mathrm{diag}\{\lambda_2, \cdots, \lambda_n\}$. When the time delay $\tau = 0$, by simple calculations, the eigenvalues of $A - \lambda_i B$, $i = 2, 3, ..., n$ have negative real parts and hence the eigenvalues of the matrix $I_n \otimes A - \Lambda_m \otimes B$ have negative real parts except for eigenvalues 0.

Theorem 2: Consider the multi-agent systems (1) with a connected undirected signed graph. If the graph is structurally balanced and $\tau_{max} < (\arctan(k_2 z/k_1))/z$ where $z = \sqrt{(k_2^2\lambda_{max}^2(L_c) + \sqrt{k_2^4\lambda_{max}^4(L_c) + 4k_1^2\lambda_{max}^2(L_c)})/2}$, the bipartite consensus problem is solved by (18) with $k_1 > 0$ and $k_2 > 0$.

Proof: For the cases with uniform time delay τ, we use the frequency domain approach to analyze the stability of (23), then

$$\delta_c(s) = (sI_{2n-2} - I_{n-1} \otimes A - (\Lambda_m \otimes B)e^{-\tau s})^{-1}\delta_c(0). \tag{24}$$

Consider $G_\tau(s) = sI_{2n-2} - I_{n-1} \otimes A - (\Lambda_m \otimes B)e^{-\tau s}$. To consider the stability of closed-loop system (23), we consider the nonzero roots of $G_\tau(s)$ on imaginary axis. Suppose that $s_G = jw \neq 0$ is an imaginary root of $G_\tau(s)$ and $u \in C^{2n-2}$ is the eigenvector corresponding to $s_G = jw \neq 0$ and $\|u\| = 1$, we have:

$$(jwI_{2n-2} - I_{n-1} \otimes A - (\Lambda_m \otimes B)e^{-j\tau w})u = 0. \tag{25}$$

Let $u = u_1 \otimes [1, 0]^T + u_2 \otimes [0, 1]^T$, where $u_1, u_2 \in C^{n-1}$. Calculate the odd rows of (25), we have

$$jwu_1 = u_2. \tag{26}$$

Because $u^*u = u_1^*u_1 + u_2^*u_2 = 1$, we could get the following equations

$$\begin{aligned} u_1^*u_1 &= \tfrac{1}{w^2+1}, \\ u_2^*u_2 &= \tfrac{w^2}{w^2+1}. \end{aligned} \tag{27}$$

Premultiplying both sides of (25) by u^*, we have

$$P\mathrm{diag}\{F_1, F_2, \ldots, F_n\}Q = 0, \tag{28}$$

where

$$\begin{aligned}
P &= [u_{11}^*, u_{21}^*, \ldots, u_{1(n-1)}^*, u_{2(n-1)}^*], \\
Q &= [u_{11}, u_{21}, \ldots, u_{1(n-1)}, u_{2(n-1)}]^T, \\
F_i &= \begin{bmatrix} jw & -1 \\ \lambda_i k_1 e^{-j\tau w} & jw + \lambda_i k_2 e^{-j\tau w} \end{bmatrix}, i \in \{2, \cdots, n\}.
\end{aligned} \tag{29}$$

From (28), we have

$$
\begin{aligned}
&jwu_1^* u_1 + k_1 u_2^* \Lambda) m e^{-j\tau w} u_1 - u_1^* u_2 + jwu_2^* u_2 \\
&+ k_2 u_2^* \Lambda_m e^{-j\tau w} u_2 = 0.
\end{aligned} \tag{30}
$$

By (26), (27) and (30), let $a_m = u_1^* \Lambda_m u_1 / u_1^* u_1$, we could get:

$$
\begin{aligned}
a_m \sin(w\tau) &= \tfrac{(k_2/k_1)w^3}{(k_2^2/k_1)w^2 + k_1}, \\
a_m \cos(w\tau) &= \tfrac{w^2}{(k_2^2/k_1)w^2 + k_1}.
\end{aligned} \tag{31}
$$

So

$$
\begin{aligned}
a_m \sin(w\tau_{max}) &= \tfrac{(k_2/k_1)w^3}{(k_2^2/k_1)w^2 + k_1}, \\
a_m \cos(w\tau_{max}) &= \tfrac{w^2}{(k_2^2/k_1)w^2 + k_1}.
\end{aligned} \tag{32}
$$

Let $w\tau_{max} < \frac{\pi}{2}$, $a_m^2 = w^4/(k_2^2 w^2 + k_1^2)$, then $w^2 = (k_2^2 a_m^2 + \sqrt{(k_2 a_m)^4 + 4(k_1 a_m)^2})/2$ is an increasing function of the variable a_m. From Lemma 3, $D(w) = (\arctan(k_2 w/k_1))/w$ is a decreasing function of the variable w, $\tau_{max} = \min\{(\arctan(k_2 w/k_1))/w\}$. Our purpose is to find maximal a_m to get maximal τ_{max}, and $a_m = u_1^* \Lambda_m u_1 / u_1^* u_1 \leq \lambda_{max}$. Then, we get $\tau_{max} = \min\{(\arctan(k_2 w/k_1))/w\}$, where $w = \sqrt{(k_2^2 \lambda_{max}^2 + \sqrt{k_2^4 \lambda_{max}^4 + 4k_1^2 \lambda_{max}^2})/2}$. Meanwhile, if $\tau = 0$, all eigenvalues are located in LHP. Due to the continuity of function, the eigenvalues of function (25) have negative real parts. Under this condition, the system has no root located in right hand plane. Thus, $\delta_c(t) = 0$, $t \rightarrow +\infty$. Then, $\delta(t) = 0$, $t \rightarrow +\infty$. Then, consensus is reached for $\phi(t)$. By premultiplying matrix D to $\phi(t)$, the bipartite consensus problem is solved. ■

V. SIMULATION

In this section, numerical simulations are given to verify the obtained theoretical results. Fig. 2 shows an undirected signed graph with ten agents, where the blue real line denotes the edges with the positive weights while the red dashed line denotes the edges with the negative weights. According to Corollary 1, choose $k_1 = 2$, $k_2 = 2$. The simulated results are displayed in Fig. 3 and Fig. 4, which show that the bipartite consensus is reached as we expected. Consider the uniform time delay τ, we choose $\tau = 0.1 < \tau_{max} = 0.1256$. Fig. 5 and Fig. 6 show that the bipartite consensus is also reached with the proposed controller.

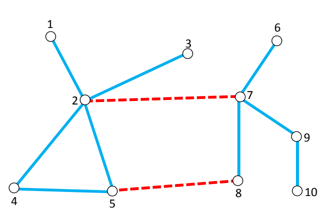

Fig. 2. Undirected graph $\mathcal{G}$.

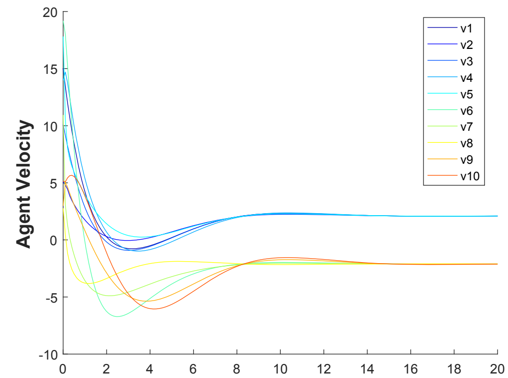

Fig. 3. Velocities of all agents.

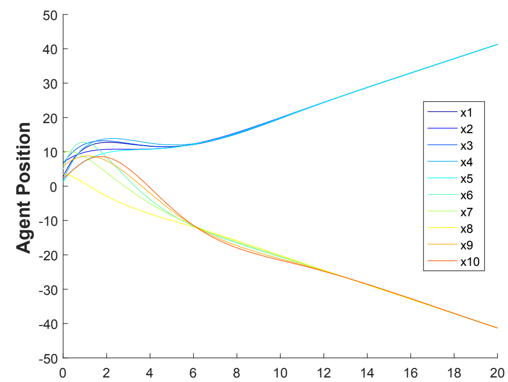

Fig. 4. Positions of all agents.

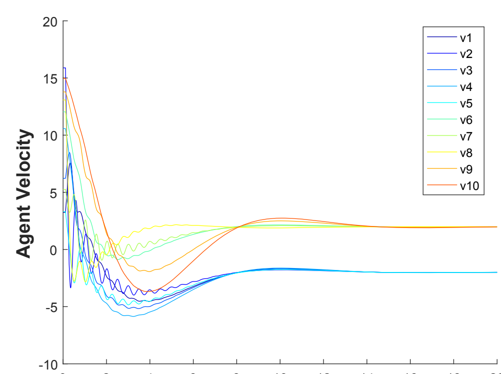

Fig. 5. Velocities of all agents with $\tau = 0.1$.

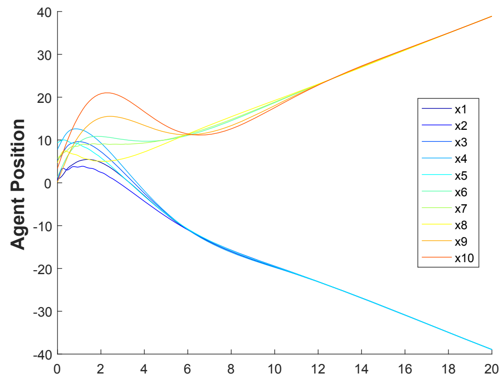

Fig. 6. Positions of all agents with $\tau = 0.1$.

VI. Conclusions

In this paper, the bipartite consensus problem for double-integrator multi-agent systems over antagonistic network with and without time delay are discussed. For the directed signed graph, sufficient conditions are given to solve the bipartite consensus problem. The bipartite consensus problem with uniform time delay for undirected graph is addressed and the maximal tolerable time delay is calculated. In the future, we will consider the non-uniform time delays, which are more practical in real world.

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