Consensus of Double-Integrator Multi-Agent Systems with Directed Networks and Nonuniform Communication Time Delays

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***Abstract*— In this paper, the consensus problem for a class of double-integrator multi-agent systems with directed communication networks and nonuniform communication time delays is considered. A distributed control algorithm is adopted to drive all agents to reach consensus. And the upper bound of the nonuniform communication time delays is derived by the frequency domain approach. Then, the simulation examples are provided to verify the correctness of the theoretical results.**

***Index Terms*— Consensus, double-integrator multi-agent systems, directed network, nonuniform communication time delays.**

I. INTRODUCTION

As a significant cooperative control problem of multi-agent systems, the consensus problem requires a group of agents agree upon a common value of interest via local communication. The researches on the consensus problem lay a foundation for some more complicated cooperative control problems, such as distributed formation control problem [1, 2], flocking problem [3, 4], containment problem [5, 6] and so on.

It is noted that, as an unavoidable issue in practical applications, communication time delays have been frequently considered in the study of distributed consensus problems [7–16]. For example, in [7], the authors addressed the average-consensus problem in the presence of constant uniform communication time delays for single-integrator dynamic systems. More specifically, they derived the maximal delay which do not spoil the average-consensus by means of frequency domain approach. Subsequently, in [9], the authors studied the influence of nonuniform communication time delays for discrete-time multi-agent systems, in which all agents reach a fixed point with zero velocity. In [10], the authors studied the consensus problem with nonuniform communication time delays by the Lyapunov-Razumikhin functional method. Based on a reduced-order Lyapunov-Krasovskii functional and linear matrix inequalities (LMIs), the authors in [11] studied the average consensus problem with switching topology and uniform time delays. In [13], Lyapunov-Krasovskii functional and linear matrix inequalities were also used to solve the consensus problem of multi-agent systems. In [14], the authors applied algebra method to study the influence of the nonuniform communication time delays and the systems was convergent with a exponent rate. Consensus problems with communication time delays were also reported in [17–20], to name a few.

However, it is worth noting that the authors in [16] only considered the nonuniform communication time delays for fixed undirected networks by frequency method, which could not be applied for the case with directed networks and nonuniform communication time delays directly. Additionally, in [10, 11, 13], the consensus problems were solved by Lyapunov-Razumikhin or Lyapunov-Krasovskii methods, in which the upper bound of communication time delays could not be derived. Thus, it is meaningful and challenging to study the case with directed networks and nonuniform communication time delays. The main contribution in this paper is to design a suitable distributed control protocol for double-integrator multi-agent systems with the directed topology and nonuniform communication time delays such that the consensus is achieved. More specially, we derived the upper bound of tolerable communication time delays by frequency domain approach, which includes the results of [16] as a special case.

Notations: R^n denotes the set of n dimensional real column vectors; C^n denotes the set of n dimensional complex vectors; I_m represents the m dimensional identity matrix; $\mathbf{1}_n$ denotes the n dimensional column vector with all elements being 1; 0 represents a zero vector or matrix with appropriate dimension; x^T denotes the transposition of x; x^H denotes the conjugate transposition of x; j denotes the imaginary unit; $\otimes$ denotes the kronecker product. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary part of a complex number. $\arg(a) = \arctan \frac{\mathrm{Im}(a)}{\mathrm{Re}(a)} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ represents the argument of the complex number a.

II. GRAPH THEORY

A graph is denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \cdots, v_n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathcal{A} = (a_{ij})_{n\times n}$ is the weighted matrix of a graph $\mathcal{G}$. The element $a_{ij} \neq 0$ $(i \neq j)$ if and only if the edge $(v_i, v_j) \in \mathcal{E}$ and $a_{ii} = 0$, $i \in \{1, \cdots, n\}$, otherwise. For a directed graph, we will always define that $a_{ij} \geq 0$. The Laplacian matrix L is defined as $L = (l_{ik})_{n\times n}$, where $l_{ii} = \sum_{k=1}^{n} a_{ik}$ and $l_{ik} = -a_{ik}$, $i \neq k$. A (directed) path from v_1 to v_l is a sequence of edges of the form $(v_1, v_2), (v_2, v_3), ..., (v_{l-1}, v_l)$ with distinct nodes in a (directed) graph. The neighbor set of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$.

Lemma 1. *If the graph $\mathcal{G}$ is directed with a spanning tree, then its Laplacian matrix L has a simple eigenvalue at 0 with associated eigenvector $\mathbf{1}_n$ and all its other $n-1$ eigenvalues have positive real parts. Denote all eigenvalues as $\mu_i, i = 1, 2, \cdots, n$, where $\mu_1 = 0$, $\mathrm{Re}(\mu_i) = a_i > 0, i = 2, \cdots, n$.*

III. Problem Formulation

Consider a group of n agents, and ith agent has the following dynamics:

$$\begin{array}{l}\dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), i \in \{1, \cdots, n\},\end{array} \tag{1}$$

where $x_i(t)$, $v_i(t)$, $u_i(t) \in R$ represent the position, velocity and the control input of ith agent, respectively. The initial conditions are given as $x_i(s) = x_i(0),\ s \in (-\infty, 0]$, $v_i(s) = v_i(0),\ s \in (-\infty, 0]$.

If the communication time delays are considered, and suppose that the time delay between agents i and k is τ_{ik} (without loss of generality, it is assumed $\tau_{ik} = \tau_{ki}$), the protocol is given as

$$\begin{array}{l}u_i(t) = \\ f_i(x_i(t-\tau_{ik}), x_k(t-\tau_{ik}), v_i(t-\tau_{ik}), v_k(t-\tau_{ik}), k \in N_i).\end{array} \tag{2}$$

Definition 1. *Consider the multi-agent system (1) with a directed graph $\mathcal{G}$. Develop a distributed control scheme of form* (2)*, such that for any finite $x_i(0)$, $v_i(0)$, $i = 1, \cdots, n$, the following conditions are satisfied*

$$\begin{array}{l}\lim\limits_{t\to+\infty} [x_i(t) - x_j(t)] = 0, \\ \lim\limits_{t\to+\infty} [v_i(t) - v_j(t)] = 0,\end{array} \tag{3}$$

then, it is said that the consensus problem is solved.

The objective of the paper is to design a distributed control protocol of form (2), such that the consensus problem as defined in Definition 1 can be solved. Meanwhile, for the case with nonuniform time delays, the upper bound is derived by a series of analysis. To obtain the main results, we further need the following assumption and lemmas.

Assumption 1. The directed graph has a spanning tree.

In this paper, we consider the consensus problem in presence of nonuniform communication time delays, the control input $u_i(t)$ of the form (2) is given as

$$\begin{aligned}u_i(t) = &-k_1 \sum_{k\in N_i} a_{ik}(x_i(t-\tau_{ik}) - x_k(t-\tau_{ik})) \\ &-k_2 \sum_{k\in N_i} a_{ik}(v_i(t-\tau_{ik}) - v_k(t-\tau_{ik})).\end{aligned} \tag{4}$$

where $k_1, k_2 > 0$ are control parameters to be designed, and τ_{ik} is communication time delay between agent i and k. Suppose that there are M different delays, denoted by $\tau_m \in \{\tau_{ik}, i, k \in \mathbf{V}\} (m = 1, 2, \cdots, M)$.

If all the communication time delays $\tau_{ik} = 0$, the distributed control protocol (4) is degenerated as follows:

$$\begin{aligned}u_i(t) = &-k_1 \sum_{j\in N_i} a_{ij}(x_i(t) - x_j(t)) \\ &-k_2 \sum_{j\in N_i} a_{ij}(v_i(t) - v_j(t)),\end{aligned} \tag{5}$$

where $k_1, k_2 > 0$ are control gains to be determined. Under Assumption 1, the following lemma presents the consensus conditions of system (1) with control protocol (5), which is derived by Yu et al in [21].

Lemma 2. *[21] Consensus of the double-integrator multi-agent system* (1) *can be achieved if and only if the network contains a directed spanning tree and $k_1 > 0$, $k_2 > 0$*

$$\tfrac{k_2^2}{k_1} > \max_{2\le i\le n} \frac{\mathrm{Im}(\mu_i)^2}{\|\mu_i\|^2 \mathrm{Re}(\mu_i)}, \tag{6}$$

where μ_i are the nonzero eigenvalues of the Laplacian matrix L, $i = 2, \cdots, n$.

IV. Main Results

In this section, we focus on the consensus analysis of the closed-loop systems with uniform or nonuniform communication time delays. Under Assumption 1, section IV-A is concerned with the consensus problem with uniform communication time delays and section IV-B focuses on consensus problem with the nonuniform communication time delays.

A. Consensus with uniform communication time delays

In this part, we consider the case with uniform communication time delays. With uniform communication time delays $\tau_{ik} = \tau$, the control protocol (4) is degenerated into the following form

$$\begin{aligned}u_i(t) = &-k_1 \sum_{k\in N_i} a_{ij}(x_i(t-\tau) - x_k(t-\tau)) \\ &-k_2 \sum_{k\in N_i} a_{ij}(v_i(t-\tau) - v_k(t-\tau)),\end{aligned} \tag{7}$$

where k_1, k_2 are control parameters to be designed.

Let $\delta(t) = [x_1(t), \cdots, x_n(t), v_1(t), \cdots, v_n(t)]^T$. We can rewrite system (1) with control protocol (7) into the following compact form

$$\dot{\delta}(t) = (A \otimes I_n)\delta(t) - (B \otimes L)\delta(t-\tau), \tag{8}$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$.

Define $\hat{x}_i(t) = x_i(t) - x_1(t)$, $\hat{v}_i(t) = v_i(t) - v_1(t)$, $i = 2, \cdots, n$. Let the state error vector as $\hat{z}(t) = [\hat{x}^T(t), \hat{v}^T(t)]^T$, where $\hat{x}(t) = [\hat{x}_2(t), \cdots, \hat{x}_n]^T$, $\hat{v}(t) = [\hat{v}_2(t), \cdots, \hat{v}_n]^T$. We obtain the following error dynamics:

$$\dot{\hat{z}}(t) = (A \otimes I_{n-1})\hat{z}(t) - (B \otimes \hat{L})\hat{z}(t-\tau), \tag{9}$$

where $\hat{L} = L_{22} + \mathbf{1}_{n-1}\alpha^T$ with

$$L_{22} = \begin{bmatrix} l_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & l_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & l_{nn} \end{bmatrix}, \alpha = \begin{bmatrix} a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{bmatrix}.$$

It is noted that the eigenvalues of matrix $\hat{L}$ are $\mu_2, \cdots, \mu_n$ [22]. Apparently, system (1) achieves consensus if and only if the error system (9) is asymptotically stable.

In the proof of following theorem, we need following lemma.

Lemma 3. *([23]) Consider the exponential polynomial like* $h(s, e^{-\tau s}) = q(s)e^{-\tau s} + p(s)$ *where,* $q(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n, p(s) = s^n + a_1 s^{n-1} + \cdots + a_n$. *If* $h(s, e^{-\tau s})$ *is stable when* $\tau = 0$ *and* $f(s, e^{-\tau s})$ *is unstable if there exists some* $\tau > 0$. *Then there must exist some* τ^* *such that* $h(s, e^{-\tau s})$ *is stable when* $\tau < \tau^*$ *and* $f(s, e^{-\tau s})$ *is unstable when* $\tau > \tau^*$, *where* $h(s, e^{-\tau^* s}) = 0, s = jw$.

Theorem 1. Under Assumption 1, consider the double-integrator multi-agent system of form (1) with a directed communication network. With $k_1 > 0$, $k_2 > 0$, $\frac{k_2^2}{k_1} > \max_{1 \le i \le n} \frac{\mathrm{Im}^2(\mu_i)}{\|\mu_i\|^2 \mathrm{Re}(\mu_i)}$, the consensus problem is solved by (7) if and only if

$$\tau < \min_{2 \le i \le n} \frac{\arctan(\frac{k_1 b_i + k_2 w_i a_i}{k_1 a_i - k_2 w_i b_i})}{w_i}, \tag{10}$$

$w_i = \sqrt{\frac{\|\mu_i\|^2 k_2^2 + \sqrt{\|\mu_i\|^4 k_2^4 + 4\|\mu_i\|^2 k_1^2}}{2}}$, where $\mu_i, i = 2, \cdots, n$ are non-zero eigenvalues of matrix L.

Proof. By utilising the laplace transformation to (9), we get

$$\hat{z}(s) = [sI_{2n-2} - (A \otimes I_{n-1}) + e^{-\tau s}(B \otimes \hat{L})]^{-1}\delta(0). \tag{11}$$

Define $G(s) = sI_{2n-2} - (A \otimes I_{n-1}) + e^{-\tau s}(B \otimes \hat{L})$. Thus, one has $\det(G(s)) = \det \begin{bmatrix} sI_{n-1} & -I_{n-1} \\ k_1 \hat{L} e^{-\tau s} & sI_{n-1} + k_2 \hat{L} e^{-\tau s} \end{bmatrix} = \prod_{i=2}^n [s^2 + k_2 s \mu_i e^{-\tau s} + k_1 \mu_i e^{-\tau s}] = 0.\ i = 2, \cdots, n.$

Define $g(s) = s^2 + k_2 s \mu_i e^{-\tau s} + k_1 \mu_i e^{-\tau s}$. From Lemma 3, there exists a upper bound $\bar{\tau}$ such that $g(s)$ is hurwitz stable if and only if $\tau \in [0, \bar{\tau})$. When it reaches to the critical point $s = jw_i$, one has

$$\begin{aligned} &g(jw_i) = (jw_i)^2 + jk_2 w_i \mu_i e^{-j\tau w_i} + k_1 \mu_i e^{-j\tau w_i} \\ &= -w_i^2 + e^{-j\tau w_i}(jk_2 w_i + k_1)\mu_i \\ &= -w_i^2 + (\cos(\tau w_i) - j\sin(\tau w_i))(jk_2 w_i + k_1)(a_i + jb_i) \\ &= 0, \end{aligned} \tag{12}$$

where $\mu_i = a_i + jb_i$, $a_i > 0$, $b_i \in R$.

Divide the real part and imaginary part of (12), one has

$$\begin{aligned} &-w_i^2 + \\ &\cos(\tau w_i)(a_i k_1 - k_2 w_i b_i) + \sin(\tau w_i)(a_i k_2 w_i + k_1 b_i) = 0, \\ &\cos(\tau w_i)(a_i k_2 w_i + k_1 b_i) + \sin(\tau w_i)(k_2 w_i b_i - a_i k_1) = 0, \end{aligned} \tag{13}$$

From (13), we get that

$$\begin{aligned} \cos(\tau w_i) &= \frac{w_i^2(a_i k_1 - k_2 w_i b_i)}{(a_i k_1 - k_2 w_i b_i)^2 + (b_i k_1 + k_2 w_i a_i)^2}, \\ \sin(\tau w_i) &= \frac{w_i^2(b_i k_1 + k_2 w_i a_i)}{(a_i k_1 - k_2 w_i b_i)^2 + (b_i k_1 + k_2 w_i a_i)^2}. \end{aligned} \tag{14}$$

With $\cos^2(\tau w_i) + \sin^2(\tau w_i) = 1$, one has

$$\begin{aligned} &\cos^2(\tau w_i) + \sin^2(\tau w_i) \\ &= \frac{w_i^4}{(a_i k_1 - k_2 w_i b_i)^2 + (b_i k_1 + k_2 w_i a_i)^2} \\ &= 1. \end{aligned} \tag{15}$$

Namely,

$$w_i^4 - k_2^2(a_i^2 + b_i^2)w_i^2 - k_1^2(a_i^2 + b_i^2) = 0. \tag{16}$$

From (16), one has $w_i^2 = \frac{k_2^2\|\mu_i\|^2 + \sqrt{(k_2^2\|\mu_i\|^2)^2 + 4k_1^2\|\mu_i\|^2}}{2}$. From (14), we could get

$$\tan(\tau w_i) = \frac{k_1 b_i + k_2 w_i a_i}{k_1 a_i - k_2 w_i b_i}. \tag{17}$$

From (17), one has

$$\tau = \frac{\arctan(\frac{k_1 b_i + k_2 w_i a_i}{k_1 a_i - k_2 w_i b_i})}{w_i}. \tag{18}$$

Thus, to satisfy that error system (9) is asymptotically stable, it is concluded that

$$\bar{\tau} = \min_{2 \le i \le n} \frac{\arctan(\frac{k_1 b_i + k_2 w_i a_i}{k_1 a_i - k_2 w_i b_i})}{w_i}, \tag{19}$$

where $w_i = \sqrt{\frac{k_2^2\|\mu_i\|^2 + \sqrt{(k_2^2\|\mu_i\|^2)^2 + 4k_1^2\|\mu_i\|^2}}{2}}$, $u_i = a_i + jb_i, i = 2, \cdots, n$ are non-zero eigenvalues of matrix L. Thus, from Lemma 3, the error system (9) is asymptotically stable if and only if $\tau < \bar{\tau}$. Thus, the system (1) with control protocol (7) reaches consensus. □

If the communication topology is an undirected graph, the eigenvalues of matrix $\hat{L}$ are real. Namely, $\mu_i = a_i > 0, b_i = 0,\ i = 2, \cdots, n$. One has $k_1 > 0, k_2 > 0$. Furthermore, $\tau < \bar{\tau} = \min_{2 \le i \le n} \frac{\arctan(\frac{k_2 w_i}{k_1})}{w_i}$, $w_i = \sqrt{\frac{a_i^2 k_2^2 + \sqrt{a_i^4 k_2^4 + 4a_i^2 k_1^2}}{2}}$. It is noted that $f(w) = \frac{\arctan(\frac{k_2 w}{k_1})}{w}$ is a decreasing function along the variable w. And the function $w(a) = \sqrt{\frac{a^2 k_2^2 + \sqrt{a^4 k_2^4 + 4a^2 k_1^2}}{2}}$ is a increasing function along the variable a. Thus, one has $\min_{2 \le i \le n} \frac{\arctan(\frac{k_2 w_i}{k_1})}{w_i} = \frac{\arctan(\frac{k_2 w_{max}}{k_1})}{w_{max}}$, $w_{max} = \sqrt{\frac{a_n^2 k_2^2 + \sqrt{a_n^4 k_2^4 + 4a_n^2 k_1^2}}{2}}$, where a_n is maximum eigenvalue of L. Thus, we have following corollary.

Corollary 1. Consider the multi-agent systems (1) with an undirected connected graph. With $k_1 > 0$, $k_2 > 0$, the consensus problem is solved by (7) if and only if $\tau < \frac{\arctan(\frac{k_2 w_{max}}{k_1})}{w_{max}}$, $w_{max} = \sqrt{\frac{a_n^2 k_2^2 + \sqrt{a_n^4 k_2^4 + 4a_n^2 k_1^2}}{2}}$, where a_n is the maximum eigenvalue of matrix L.

B. Consensus with nonuniform communication time delays

With considering the nonuniform communication time delays, the control protocol is rewritten as follows

$$\begin{aligned} u_i(t) = &-k_1 \sum_{k \in N_i} a_{ik}(x_i(t - \tau_{ik}) - x_k(t - \tau_{ik})) \\ &-k_2 \sum_{k \in N_i} a_{ik}(v_i(t - \tau_{ik}) - v_k(t - \tau_{ik})), \end{aligned} \tag{20}$$

where τ_{ik} is the communication time delay between agent i and k.

Define variable $\delta(t) = [x_1(t), v_1(t), \cdots, x_n(t), v_n(t)]^T$, we get

$$\dot{\delta}(t) = (I_n \otimes A)\delta(t) - \sum_{m=1}^{M}(L_m \otimes B)\delta(t - \tau_m), \tag{21}$$

with the initial condition $\delta(s)=\delta(0), s\in(-\infty,0]$, where $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}$ and L_m combined with B denotes the coefficient matrix of the variable $\delta(t-\tau_m)$ for $m=1,\cdots,M$. Clearly, $L=\sum_{m=1}^{M}L_m$.

Define $\hat{x}_i(t)=x_i(t)-x_1(t)$, $\hat{v}_i(t)=v_i(t)-v_1(t)$, $i=2,\cdots,n$. Let the state error vector as $\hat{z}(t)=[\hat{x}_2(t),\hat{v}_2(t),\cdots,\hat{x}_n(t),\hat{v}_n(t)]^T$. We obtain the following error dynamics:

$$\dot{\hat{z}}(t)=(I_{n-1}\otimes A)\hat{z}(t)-\sum_{m=1}^{M}(\hat{L}_m\otimes B)\hat{z}(t-\tau_m), \tag{22}$$

where $\hat{L}_m=L_{22m}+\mathbf{1}_{n-1}\alpha_m^T$ with

$$L_{22m}=\begin{bmatrix}l_{22m}&-a_{23m}&\cdots&-a_{2nm}\\-a_{32m}&l_{33m}&\cdots&-a_{3nm}\\\vdots&\vdots&\ddots&\vdots\\-a_{n2m}&-a_{n3m}&\cdots&l_{nnm}\end{bmatrix},\alpha_m^T=\begin{bmatrix}a_{12m}\\a_{13m}\\\vdots\\a_{1nm}\end{bmatrix}.$$

Apparently, $\sum_{m=1}^{M}\hat{L}_m=\hat{L}$. Thus, system (1) achieves consensus if and only if the error system (22) is asymptotically stable.

Theorem 2. Under Assumption 1, consider the multi-agent system of form (1) with a directed communication network. With $k_1>0$, $k_2>0$, $\frac{k_2^2}{k_1}>\max_{2\le i\le n}\frac{\mathrm{Im}(\mu_i)^2}{||\mu_i||^2\mathrm{Re}(\mu_i)}$, the consensus problem is solved by control protocol (20) if

$$\tau_m<\bar{\tau}=\min_{2\le i\le n}\frac{\arctan\frac{k_2\bar{w}_i}{k_1}+\arg(\mu_i)}{\bar{w}_i}, \tag{23}$$

$\bar{w}_i=\sqrt{\frac{\|\mu_i\|^2k_2^2+\sqrt{\|\mu_i\|^4k_2^4+4\|\mu_i\|^2k_1^2}}{2}}$, where $\mu_i, i=2,\cdots,n$ are non-zero eigenvalues of matrix L.

Proof. By basic mathematical theorem, the original proposition and its inverse negative proposition are logically equivalent. So, if we can prove the original proposition is true, then, the inverse negative proposition is also true. So we have following proof.

First, we define the original proposition and the inverse negative proposition as follows.

(1) The Original Proposition: if the multi-agent systems (1) with control protocol (20) does not reach consensus, then there at least exists a τ_m, $m\in\{1,2,\cdots,M\}$ such that $\tau_m\ge\bar{\tau}$.

(2) The Inverse Negative Proposition: if $\tau_m<\bar{\tau}$, then the multi-agent systems (1) with control protocol (20) reach consensus.

Then, we firstly prove that The Original Proposition is true.

Proof. With a Laplace transformation to (22), one has

$$\begin{aligned}&s\hat{z}(s)-\hat{z}(0)\\&\quad=(I_{n-1}\otimes A)\delta(s)-\textstyle\sum_{m=1}^{M}(\hat{L}_m\otimes B)\hat{z}(s)e^{-\tau_m s},\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}&\hat{z}(s)\\&=[sI_{2n-2}-(I_{n-1}\otimes A)-\sum_{m=1}^{M}e^{-\tau_m s}(\hat{L}_m\otimes B)]^{-1}\hat{z}(0).\end{aligned} \tag{25}$$

Define $G(s)=sI_{2n-2}-(I_{n-1}\otimes A)-\sum_{m=1}^{M}e^{-\tau_m s}(\hat{L}_m\otimes B)$. If the error dynamics is not asymptotically stable, in other words, $\det(G(s))=0$ at least exists a non-zero root at the imaginary axis, namely, $s=jw_i\neq 0$.

We could get $G(jw_i)=jw_iI_{2n-2}-(I_{n-1}\otimes A)+\sum_{m=1}^{M}e^{-j\tau_m w_i}(\hat{L}_m\otimes B)$. Let $u=u_1\otimes\begin{bmatrix}1\\0\end{bmatrix}+u_2\otimes\begin{bmatrix}0\\1\end{bmatrix}\in C^{2n-2}$ is eigenvector corresponding to $s=jw_i$, where $u^Hu=1$. It is obvious that

$$\begin{aligned}&[jw_iI_{2n-2}-(I_{n-1}\otimes A)-\sum_{m=1}^{M}e^{-j\tau_m w_i}(\hat{L}_m\otimes B)]u\\&=0.\end{aligned} \tag{26}$$

From (26), by utilising its odd row, one has

$$jw_iu_1=u_2. \tag{27}$$

Premultiplying u^H to (26), one has

$$\begin{aligned}&u^H[jw_iI_{2n-2}-(I_{n-1}\otimes A)-\sum_{m=1}^{M}e^{-j\tau_m w_i}(\hat{L}_m\otimes B)]u\\&=0.\end{aligned} \tag{28}$$

With a series of calculations, one has

$$\textstyle\sum_{m=1}^{M}a_me^{-j\tau_m w_i}=\frac{w_i^2}{k_1+jw_ik_2}, \tag{29}$$

where $a_m=\frac{u^H(\hat{L}_m\otimes I_2)u}{u^Hu}$.

Define

$$B_a=\textstyle\sum_{m=1}^{M}a_me^{-j\tau_m w_i}=\frac{w_i^2}{k_1+jw_ik_2}. \tag{30}$$

Taking modulus of the both sides of (30), one has

$$w_i(\|B_a\|)=\sqrt{\frac{\|B_a\|^2k_2^2+\sqrt{\|B_a\|^4k_2^4+4\|B_a\|^2k_1^2}}{2}}, \tag{31}$$

which is an increasing function along the variable $\|B_a\|$.

By calculating the argument of the both sides of (30), we gain

$$\arg(B_a)=-\arctan(\tfrac{k_2w_i}{k_1}). \tag{32}$$

Furthermore, from (30), we get

$$\arg(B_a)\ge\arg(\textstyle\sum_{m=1}^{M}a_m)-\max(w_i\tau_m). \tag{33}$$

Namely,

$$\begin{aligned}\max(w_i\tau_m)&\ge\arg(\textstyle\sum_{m=1}^{M}a_m)-\arg(B_a)\\&=\arg(\textstyle\sum_{m=1}^{M}a_m)+\arctan(\frac{k_2w_i}{k_1}).\end{aligned} \tag{34}$$

It is noted that $\sum_{m=1}^{M}a_m=u^H(\hat{L}\otimes I_2)u$ and the possible value is the eigenvalues of matrix $\hat{L}$, u_i, $i=2,\cdots,n$. Thus $\sum_{m=1}^{M}a_m=\mu_i$, namely, $\|B_a\|\le\|\mu_i\|$. Thus, $w_i(\|B_a\|)\le w_i(\|\mu_i\|)=\bar{w}_i$, where $\bar{w}_i=\sqrt{\frac{\|\mu_i\|^2k_2^2+\sqrt{\|\mu_i\|^4k_2^4+4\|\mu_i\|^2k_1^2}}{2}}$.

Meanwhile, it is noted that with different μ_i, one has different $s = jw_i$ corresponding to $\det(G(s)) = 0$.

Define τ_{max} is maximum delay of the $\tau_m, m = 1, \cdots, M$. From (34), it is convenient to get that $\tau_{max} \geq \frac{\arg(\mu_i)+\arctan(\frac{k_2 w_i}{k_1})}{w_i} \geq \frac{\arg(\mu_i)+\arctan(\frac{k_2 \bar{w}_i}{k_1})}{\bar{w}_i}$, with the fact that $f(w) = \frac{\arg(\mu_i)+\arctan(\frac{k_2 w}{k_1})}{w}$ is a decreasing function along the variable w. Then, $\tau_{max} \geq \frac{\arg(\mu_i)+\arctan(\frac{k_2 w_i}{k_1})}{w_i} \geq \frac{\arg(\mu_i)+\arctan(\frac{k_2 \bar{w}_i}{k_1})}{\bar{w}_i}$. Thus, The Original Proposition is true. $\square$

With the fact that the original proposition and its inverse negative proposition are logically equivalent, it is concluded that The Inverse Negative Proposition is also true.

Then, if $\tau_{max} < \bar{\tau} = \min\limits_{2 \leq i \leq n} \dfrac{\arg(\mu_i) + \arctan(\frac{k_2 w_i}{k_1})}{w_i}$, $\bar{w}_i = \sqrt{\frac{\|\mu_i\|^2 k_2^2 + \sqrt{\|\mu_i\|^4 k_2^4 + 4\|\mu_i\|^2 k_1^2}}{2}}$, there is no w_i on imaginary axis, which means that the error system (22) is asymptotically stable.

Thus, it is concluded that with $\tau_m < \tau_{max} < \bar{\tau}, m = 1, 2, \cdots, M$, the consensus of the system (1) is reached. $\square$

Remark 1. *When consider the nonuniform communication time delays with connected undirected graph, one has all eigenvalues* μ_i *of matrix are real except for* $\mu_1 = 0$. *Namely,* b_i *= 0,* $\arg(\mu_i) = 0$. *Inequality* (23) *is rewritten as* $\tau_m < \bar{\tau} = \min\limits_{2 \leq i \leq n} \dfrac{\arctan \frac{k_2 \bar{w}_i}{k_1}}{\bar{w}_i}$, $\bar{w}_i = \sqrt{\frac{a_i^2 k_2^2 + \sqrt{a_i^4 k_2^4 + 4a_i^2 k_1^2}}{2}}$, *where* $0 = a_1 < a_2 < \cdots < a_n$ *are eigenvalues of matrix L. Then, one has* $\bar{\tau} = \frac{\arctan \frac{k_2 \bar{w}_{max}}{k_1}}{\bar{w}_{max}}$*,where* $\bar{w}_i = \sqrt{\frac{a_n^2 k_2^2 + \sqrt{a_n^4 k_2^4 + 4a_n^2 k_1^2}}{2}}$. *Clearly, it is consistent with results in [16].*

Remark 2. *It is easy to observe that when taking all delays are uniform with* $\tau_{ik} = \tau$*, the result of Theorem 2 could reduced to the case of Theorem 1.*

V. Simulation

In this part, we would like to show several simulations to illustrate the correctness of our theoretical results. Firstly, we choose a directed graph with 4 nodes and the topology is shown in Fig 1.

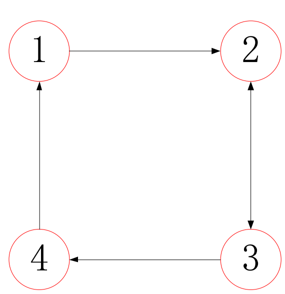

Fig. 1. Directed graph $\mathcal{G}$.

Given $k_1 = 1$, $k_2 = 2$, when consider the case with nonuniform communication time delays, from Theorem 2, we get a upper bound with $\tau_{max} < \bar{\tau} = 0.2676$. Thus, we choose nonuniform communication time delays with $\tau_{12} = 0.2176$, $\tau_{23} = \tau_{32} = 0.2276$, $\tau_{41} = 0.2376$, $\tau_{34} = 0.2476$, and all of them are less than $\bar{\tau} = 0.2676$. The simulation results are shown in Figs 2 and 3. And the consensus is achieved as we expected.

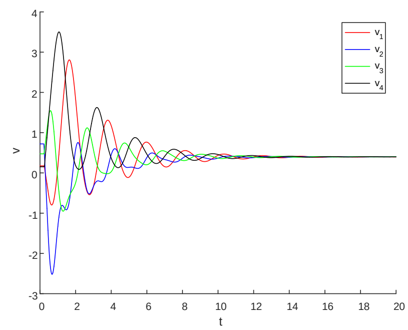

Fig. 2. Velocity of all agents with nonuniform delays.

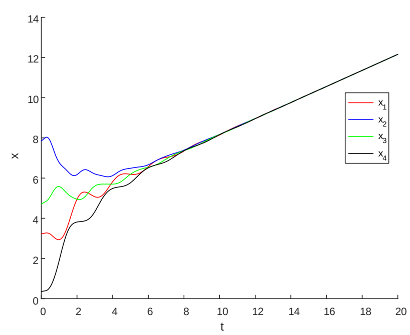

Fig. 3. Position of all agents with nonuniform delays.

An additional simulation example was included to show the divergence of the system when all the time delays exceed the bound value by a very small amount. With $\tau_{12} = 0.2680$, $\tau_{32} = \tau_{23} = 0.2720$, $\tau_{34} = 0.2760$, $\tau_{41} = 0.2800$, all of them are greater than $\bar{\tau}$, the simulation results are shown in Figs 4 and 5, which are divergent. After many trails of similar simulations, a conjecture is raised: the derived bound value for the delays might be the "delay margin" that if all the delays are greater than this value, the system will be unstable. In the future, we would like to study how to derive the sufficient and necessary conditions on the upper bound.

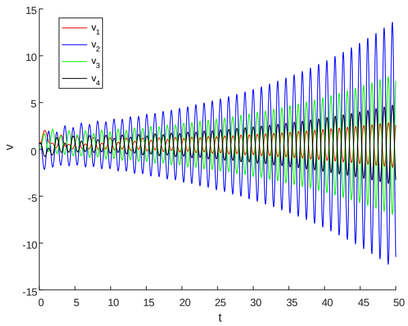

Fig. 4. Velocity of all agents with nonuniform delays.

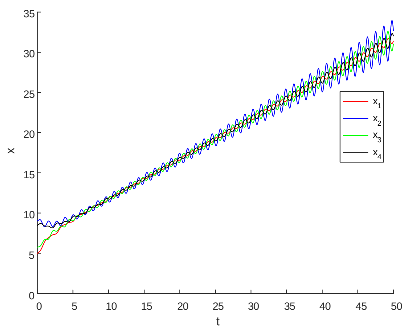

Fig. 5. Position of all agents with nonuniform delays.

VI. Conclusions

In this paper, we study the consensus problem of a class of double-integrator multi-agent systems with directed networks and nonuniform communication time delays. And the cases with uniform and nonuniform communication time delays are both considered. The suitable distributed algorithms are designed to make all agents reach consensus. By utilizing the frequency domain approach, the sufficient conditions are given to guarantee the consensus of multi-agent systems and the upper bound of communication delays is derived mathematically. Inspired by simulations, future research can focus on getting sufficient and necessary conditions for consensus problem with directed networks and nonuniform communication time delays.

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