Non-Unitary Distance Transformation

September 16, 2021

1 Résumé

Let $G = (V, E)$ be an undirected graph with n vertices and m edges with positive costs c_e, $e \in E$. Let $D \subseteq V \times V$ be a set of *demands*, each demand $(u, v) \in D$ has its *connectivity requirements*: the existence of a certain number of (u, v)-paths, possibly those paths may need to satisfy some side constraints. The class of Network Design Problems (NDPs) considered in this project consists in finding a subgraph of G with minimum cost satisfying the connectivity requirements of all demands. For examples:

- Steiner Tree Problem (STP): the input gives a set $T \subseteq V$ of terminals. D can be defined as $\{(u, v) : u, v \in T, u \neq v\}$. The connectivity requirement of a demand $(u, v) \in D$ is the existence of a path joining u and v.
- Hop-constrained Steiner Tree Problem (HSTP): the input gives a set $T \subseteq V$ of terminals, a root vertex $r \in T$ and an integer H. The demand-set D is defined as $\{(r, v) : v \in T \setminus \{r\}\}$. The connectivity requirement is the existence of a path with at most H edges (hops) joining the vertices in each demand.

Several other NDPs found in the literature also belong to this class, the connectivity requirement may ask for a number of node-disjoint paths, side constraints may include maximum delay, etc.

On all those cases, the natural formulations over the design variables x_e ($x_e = 1$ iif edge e belongs to the solution), $e \in E$, take the following format:

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e && \text{(1a)} \\ \text{S.t.} \quad & x \in P(u, v) \quad \forall (u, v) \in D && \text{(1b)} \\ & x_e \text{ binary} \quad \forall e \in E, && \text{(1c)} \end{aligned}$$

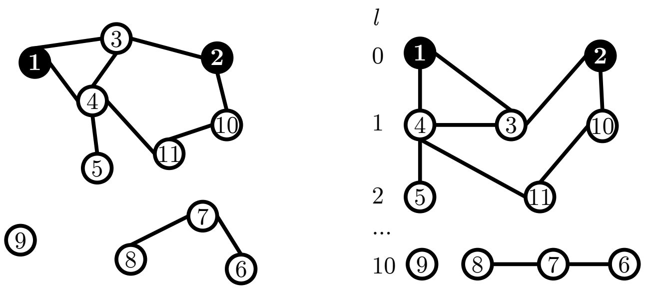
where the binary points in polyhedra $P(u, v)$ correspond to all subgraphs satisfying the connectivity requirements of demand (u, v). Even on cases when each polyhedron $P(u, v)$ is integral, i.e., when the formulation already contains the best possible inequalities that are valid for each individual demand, the overall formulation is often still weak.

The Distance Transformation (DT) is an original reformulation technique proposed to obtain stronger formulations for general NDPs, including those where the currently know reformulation techniques do not seem to work. Starting from Formulation (1), the resulting reformulation will have the following format:

$$\begin{aligned} \text{Min} \quad & \sum_{e \in E} c_e x_e && \text{(2a)} \\ \text{S.t.} \quad & Ax + Bw + Cy \geq b && \text{(2b)} \\ & (w, y) \in P'(u, v) \quad \forall (u, v) \in D && \text{(2c)} \\ & x_e \text{ binary} \quad \forall e \in E, && \text{(2d)} \end{aligned}$$

where A, B, C and b are matrices of appropriated dimensions. Constraints (2b) are problem-independent. They are devised to transform a solution over the variables x into a distance expanded solution over the new variables w and y. The original connectivity constraints in each $P(u, v)$, over the x variables, are transformed into new connectivity constraints $P'(u, v)$, over the (w, y) variables.

The purpose of DT can be informally described as follows. Let x be a *binary* vector in $\{0, 1\}^m$ and $G(x) = (V, E(x))$ be the subgraph induced by x. The distance transformation maps $G(x)$ into a subgraph of



(a) G (b) $DT_S(G)$

Figure 1: Original graph G and $DT_S(G)$ with $S = \{1, 2\}$.

the layered graph that consists of $n+1$ copies of G plus edges between adjacent layers. The mapping considers a chosen source subset $S \subseteq V$ and sends each vertex $i \in V$ to a vertex in layer l, $0 \leq l \leq n$, according to its distance from set S. Each edge $\{i, j\} \in G(x)$ is sent to an edge in the layered graph according to the layers of i and j. The transformed graph is represented by variables w and y, the mapping is encoded in Constraints (2b).

Figure 1 shows an example of distance transformation. The interest of the transformation is the following. If x is integral, the transformed graph is isomorphic to $G(x)$ (as shown on Figure 1). However, its extension to graphs $G(x)$ induced by *fractional* vectors leads to transformed graphs that are *less connected* than $G(x)$. Hence, the corresponding (w, y) fractional solution is much more likely to be cut by inequalities (2c).

The purpose of the internship is to study the strength of distance transformation based on arbitrary distances. The latter can provide stronger linear programming relaxations albeit at a higher computational cost. This trade-off will be analyzed numerically, and possibly theoretically.

2 Références

Mahjoub, A. R., Poss, M., Simonetti, L., Uchoa, E. (2019). Distance transformation for network design problems. SIAM Journal on Optimization, 29(2), 1687-1713.

3 Encadrant et lieu du stage

The internship will take place in the MAORE team from the LIRMM laboratory, under the direction of Michael Poss.