Cryptographic Protocols - Lecture Notes

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1 Lecture 1

1.1 Overview of what the course will cover

- Computing with encrypted data
- AuthN without giving away any secret
- E-voting / cryptographic voting protocols
- Blockchain is transparent, cannot keep secrets
- Generate a true random & unbiased value
- Sealed bid auction w/o trusted entity

1.2 Example: Generate a random bit (coin flip)

Needs cryptography to ensure that the output is not biased among two parties A and B, since the party that sends the last message can precompute/bias the output

Use cryptography: Simultaneous commitments

- hash the message, send it
- We add randomness as hashing bit
- hash functions are collision-free: No two inputs give the same output
- output gives no info about the input

Hash func H:

$$H : \{0,1\}^* \to \{0,1\}^k$$

SHA-2: $k = 256$ as a special case of a commitment scheme

Protocol

A	B
$a \xleftarrow{\mathbb{R}} \{0,1\}$	$b \xleftarrow{\mathbb{R}} \{0,1\}$
$x \xleftarrow{\mathbb{R}} \{0,1\}^k$	$y \xleftarrow{\mathbb{R}} \{0,1\}^k$
$c \leftarrow H(a\|x)$	$d \leftarrow H(b\|y)$

$$\xrightarrow{\quad c \quad}$$
$$\xleftarrow{\quad d \quad}$$
$$\xrightarrow{\quad a,x \quad}$$
$$\xleftarrow{\quad b,y \quad}$$

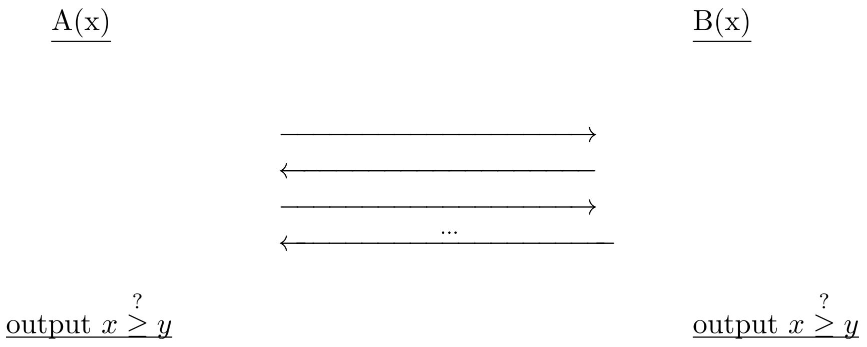
verify $d \stackrel{?}{=} H(b\|y)$	verify $c \stackrel{?}{=} H(a\|x)$
$\underline{\text{output } a \oplus b}$	$\underline{\text{output}}$ $a \oplus b$

$\tilde{b}$, $\tilde{y} \Rightarrow$ not possible as this would violate collision-freeness

- c and d hide a & b, because this hash function hides its input.
- A & B cannot change their inputs because H is collision free

Ex: Millionaires' Problem

A and B want to find out who is richer without disclosing their wealth


A(x)
B(x)
...
output $x \overset{?}{\geq} y$
output $x \overset{?}{\geq} y$

This is easy with a trusted entity T:

$$\begin{tikzcd} A \arrow[r] & \boxed{T} & B \arrow[l] \end{tikzcd}$$

$$b \leftarrow (x \overset{?}{\geq} y)$$

$$\xleftarrow{b} \qquad\qquad \xrightarrow{b}$$

We want to hide as much as we can
Important example for:

- Auctions
- Matchings
- Elections $\longrightarrow$ solution comes later

Ex: Computing with encrypted data

$A(x)$	$B(f)$
user	cloud
input x (data)	function f

Encryption scheme $(\mathsf{Enc}, \mathsf{Dec})$ with keypair (pk, sk)
$c_x \leftarrow \mathsf{Enc}(pk, x)$
Send pk, $c_x \longrightarrow$

$c_y \leftarrow \mathrm{Eval}(pk, f, c_x)$
Eval "runs" program f on data x
encrypt inside c_x

$\longleftarrow c_y$
$y \leftarrow \mathsf{Dec}(sk, c_y)$
s.t. $y = \mathsf{Dec}(sk, \mathrm{Eval}(pk, f, \mathsf{Enc}(pk, x))) = f(x)$

Why is computing on secret data difficult?

Layers

App
VMs
Containers
OS
Hypervisor
Base OS
Hardware

high: Trusted Platform Module (TPM)
TCB Trusted Computer Base

lower levels can always see everythin that happens on higher levels/above them
TCB controls everything

1.3 Ex: Secret vote among three

Parties p_1, p_2, p_3
Each has a binary vote v_i as input. Goal is to compute the sum s of the votes and not disclose any more information than that.

Protocol: Additive secret sharing

Primitive: split$(b) \rightarrow (x_1, x_2, x_3)$ to distribute or "share" b
Use prime p

```
split(p): ${0, 1, 2, ..., p-1}$

$x_1 \leftarrow \mathbb{Z}_p$
$x_2 \leftarrow \mathbb{Z}_p$
$x_3 \leftarrow \mathbb{Z}_p$ s.t. $x_1 + x_2 + x_3 \equiv b \pmod{p}$
return $(x_1, x_2, x_3)$
```

secure channel: confidential & authenticated

$$\begin{tikzcd} & P_2 \arrow[dl, dash] \arrow[dr, dash] & \\ P_1 \arrow[rr, dash] & & P_3 \end{tikzcd}$$

```
Party $p_i(v_i)$:
$(x_{i1}, x_{i2}, x_{i3}) \leftarrow$ split$(v_i)$
send $x_{ij}$ to $p_j$ for $j \in \{1, 2, 3\}$
receive $x_{ji}$ from $p_j$
$y_i \leftarrow (x_{1i} + x_{2i} + x_{3i}) \bmod p$
send $y_i$ to $p_j$ for $j \neq i, ..., 3$
receive $y_j$ from $p_j$
output $(y_1 + y_2 + y_3) \bmod p$
where $(y_1 + y_2 + y_3) =$ s
```

Completeness

claim: $s \equiv v_1 + v_2 + v_3$

Proof:

$$\begin{aligned}
s &= \sum_{i=1}^{3} y_i \\
&= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} x_{ji} \right) \\
&= \sum_{j=1}^{3} \left(\sum_{i=1}^{3} x_{ji} \right) && \text{(columns sum)} \\
&= \sum_{j=1}^{3} (x_{j1} + x_{j2} + x_{j3}) && \text{(rows sum)} \\
& && v_j \text{ to split()} \\
&= \sum_{j=1}^{3} v_j \pmod p
\end{aligned}$$

Security

split$(b) \to (x_1, x_2, x_3)$

No two values of x_i, x_j give information about b

(Generation of a one time pad)

Output s reveals nothing more than it should: nothing about v_j for $j \neq i$; only s.

Goals

Privacy: No party learns more than it should (the output)
Correctness: Every party receives the output as specified by the function they are evaluating
Input Independence: Inputs of corrupted or faulty parties do not depend on input of correct parties.
Fairness: All parties either receive an output or no party receives an output
$\longrightarrow$ faulty parties receive outputs if and only if correct parties receive output

Faults

All faulty parties are modeled as controlled (or corrupted) by one adversary $\mathcal{A}$.
semi-honest behaviour

- Corrupted parties follow protocol
- Leak all data to $\mathcal{A}$
- "passive attack", "passive adversary", "read-only attack"

Malicious behaviour

- Faulty parties behave arbitrarily, controlled by $\mathcal{A}$

2 Lecture 2

Consider Programs formulated as circuits.

- Finite state automata (Turing machines)
- Circuits
- stack automata

What can be computed on one can be computed on all
$\Longrightarrow$ For this course, consider circuits.

2.1 Basic technique

2.1.1 Programs as circuits

Every program on inputs $x_1, ..., x_n$ computes a function $f(x_1, ..., x_n)$, represented by a Turing machine or by a circuit.
Deterministic vs. Non-Deterministic Turing Machine:
TODO: add this
Polynomial-time: $O(n^k) \approx$ Polynomial time, so the run time depends on the input size in a polynomial relation.
Cook-Levin Theorem: $NP \overset{?}{=} P$
Every problem decided by a non-deterministic Turing Machine (= NP-problem) in Polynomial time can be formulated as the satisfiability problem (SAT) of a polynomial-size (boolean) circuit.
$\Longrightarrow$ One time use circuits, not circuits with clocks.
In every computer the CPU represents computations as stateful circuits.
Here we will represent any computation as a circuit and evaluate the circuit in "encrypted" form.

2.1.2 Circuit for testing if two numbers are equal

Given two numbers in binary.
$(x)_2 = (x_{n-1}, ..., x_0)$
$(y)_2 = (y_{n-1}, ..., y_0)$
To compute if $x \overset{?}{=} y$

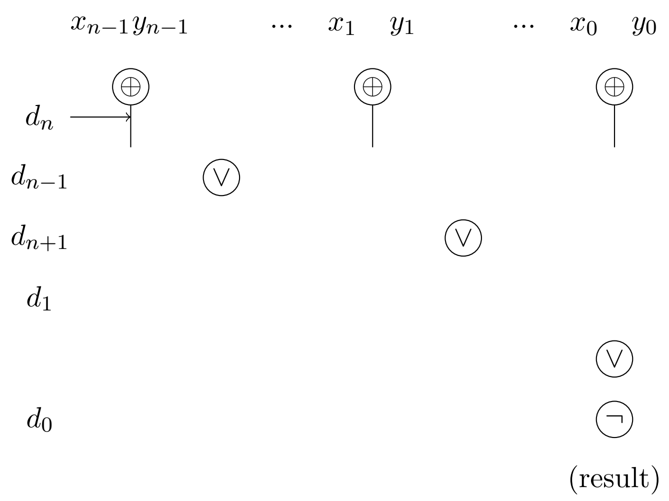
```
i ← n
while i > 0 do
    i ← i−1
    if $x_i \neq y_i$ then
        return FALSE
return TRUE
```

In the circuit one has to unroll all loops and eliminate imperative operations.
Circuit:

```
i ← n
$d_n \leftarrow 0$
while i > 0 do
```

```
        i ← i-1
        $d_i \leftarrow d_{i+1} \vee (x_i \oplus y_i)$
    return $\neg d_0$
```

Circuit: $O(n)$ complexity



2.2 Mathematics of public key Cryptography

Modular Arithmetic: "computing modulo m"

$x \leftarrow a \bmod m$ (operation)
$x \equiv a \pmod m$ (Relation, can be true/false)
$\mathbb{Z}_m = \{0, ..., m-1\}$ with addition modulo m
$\mathbb{Z}_m^* = \{1, ..., m-1\}$ with multiplication modulo m, typically m is p, prime.

Cyclic groups

A cyclic group G has a generator $g \in G$ s.t. every element of G is obtained by computing g^i for $i = 0, 1, ...$

$$g^0 = 1$$

G is finite and has $|G|$ elements: $G = \{1, g, g^2, ..., g^{|G|-1}\}$
because $g^{|G|} = g^0 = 1$
write $\langle g \rangle = G$ (generator)

- $\mathbb{Z}_m$ with addition operation is cyclic with $g = 1$
- $\mathbb{Z}_p^*$ with multiplication is cyclic with $|\mathbb{Z}_p^*| = p - 1$

Ex: $\mathbb{Z}_{11}^*$
$\langle 2 \rangle_{11} = \{1, 2, 4, 8, 5, 10, 9, 7, 3, 6\}$
$\langle 3 \rangle_{11} = \{1, 3, 9, 5, 4, ...\}$ only a subgroup

if $q \mid p-1$ is also prime then there is a cyclic subgroup with q elements, defined by multiplication modulo $p \Rightarrow$ used in cryptography

with $|p|$ about 2000 bit
and $|q|$ about 256 bit such that $p = nq + 1$
or such that $p = 2q + 1$
$(|q| = |p| - 1)$

Discrete logarithms

$G = \langle g \rangle$
the discrete log of $a \in G$ with respect to (w.r.t.) base g is the $l \in \mathbb{Z}_{|G|-1}$ s.t.

$$g^l = a$$

computing discrete logarithms is assumed to be hard (in some G) (computationally hard)
DLP: Discrete Logarithm Problem
Given y, where $\begin{cases} r \leftarrow \mathbb{Z}_q \\ y \leftarrow g^r \end{cases}$ with public info or from key
Find r.

Related: Computational Diffie-Hellman Problem
Given x and y, where
$a \overset{\mathbb{R}}{\leftarrow} \mathbb{Z}_q$ $\quad b \overset{\mathbb{R}}{\leftarrow} \mathbb{Z}_q$
$x \leftarrow g^a$ $\quad y \leftarrow g^b$
Find g^{ab} $[\in G]$

DDHP: Decisional DH Problem

Given either

1. (x, y, z), where
   $a \leftarrow \mathbb{Z}_q$ $\quad y \leftarrow g^a$
   $b \leftarrow \mathbb{Z}_q$ $\quad x \leftarrow g^b$
   $c \leftarrow \mathbb{Z}_q$ $\quad z \leftarrow g^c$

or

2. (x, y, z), where
   $a \leftarrow \mathbb{Z}_q$ $\quad y \leftarrow g^a$
   $b \leftarrow \mathbb{Z}_q$ $\quad x \leftarrow g^b$
   $z \leftarrow g^{ab}$

Task is not compute z, but instead which Tripel ① or ② we have

2.3 Public Key Encryption

$\mathsf{KeyGen}() \rightarrow (pk, sk)$
$\mathsf{Enc}(pk, m) \rightarrow c$
$\mathsf{Dec}(sk, c) \rightarrow m$

Completeness

$\forall m : (pk, sk) \leftarrow \mathsf{KeyGen}()$
$\mathsf{Dec}(sk, \mathsf{Enc}(pk, m)) = m$

Security

- An encryption of some message m is indistinguishable from a random element of the ciphertext space.
- For two messages m_1 and m_2, no efficient adversary can distinguish $\mathsf{Enc}(pk, m_1)$ from $\mathsf{Enc}(pk, m_2)$ except with negligible probability.

$\Longrightarrow$ $\mathsf{Enc}(pk, m_1)$ is indistinguishable from $\mathsf{Enc}(pk, m_2)$
Security parameter: λ
$f(\cdot)$ is negligible if

$$\exists \lambda_0 : \forall c > 0 : f(\lambda) < \frac{1}{\lambda^c} \text{ for } \lambda \geq \lambda_0$$

2.4 ElGamal public-key encryption

- Textbook version
- Cyclic Group: $G = \langle g \rangle$, $|G| = q$

```
KeyGen()
$x \leftarrow \mathbb{Z}_q$
$y \leftarrow g^x$
return $(y, x)$                    $y : pk,\ x : sk$

Enc($y, m$)                  $m \in G$
        $r \overset{R}{\leftarrow} \mathbb{Z}_q$
$R \leftarrow g^r$
$c \leftarrow y^r \cdot m$
return $(R, c) \in G \times G$

Dec($x, (R, c)$)
$\hat{m} \leftarrow c / R^x$
return $\hat{m}$
```

$$\hat{m} = c/R^x = (y^r \cdot m) \cdot (g^{-r})^x = g^{xr} \cdot m \cdot g^{-rx} = m$$

Security

Decide whether for some m, (y, R, c) is $(g^x, g^r, m \cdot g^{xr})$ (DH exponent, independent) or (y, R, c) is $(g^x, g^r, m \cdot g^z)$ for some $z \overset{\$}{\leftarrow} \mathbb{Z}_p$
This is equivalent to the DDHP

Digital Signature (DS) Scheme is pub key cryptography

3 Lecture 3

3.1 Digital Signatures

Provides:

- Integrity
- Authenticity
- Non-repudiation

3.1.1 Signature Scheme

- $\mathsf{KeyGen}() \to (pk, sk)$
- $\mathsf{Sign}(sk, m) \to \sigma$
- $\mathsf{Verify}(pk, m, \sigma) \to 0/1$

Completeness:
$\forall m : (pk, sk) \leftarrow \mathsf{KeyGen}()$
$\mathsf{Verify}(pk, m, \mathsf{Sign}(sk, m)) = 1$ (TRUE)
Security:

- $(pk, sk) \leftarrow \mathsf{KeyGen}()$
- Adversary $\mathcal{A}$ obtains pk, can request repeatedly:
  - $\sigma \leftarrow \mathsf{Sign}(sk, m)$ for m chosen by $\mathcal{A}$
- Let M be the set of m asked by $\mathcal{A}$
- $\mathcal{A}$ cannot output (m^*, σ^*) such that $m^* \in M \wedge \mathsf{Verify}(pk, m^*, \sigma^*) = 1$

3.1.2 Schnorr Signatures

Very Important Signature Scheme based on the hardness of the discrete logarithm problem & its associated assumptions.
Recall: $G = \langle g \rangle$
$|G| = q$
$G \in \mathbb{Z}_p^*$, where $p = nq + 1$

```
KeyGen()
$x \overset{\mathbb{R}}{\leftarrow} \mathbb{Z}_q$
$y \leftarrow g^x$
return $(y, x)$
```

```
Sign(x, m)
$r \overset{\mathbb{R}}{\leftarrow} \mathbb{Z}_q$
$t \leftarrow g^r$
$c \leftarrow H(m \| g^s \cdot y^c)$ (challenge)
$s \leftarrow r - c \cdot x$ (signature)
return $(c, s)$
```

```
Verify(y, m, (c, s))
return $c \overset{?}{=} H(m \| g^s \cdot y^c)$
```

Completeness

Ensure that argument to $H()$ is same:

$$g^s \cdot y^c = g^{r-c\cdot x} \cdot g^{xc} = g^r \cdot g^{-c} = g^r = t$$

Security

One can prove this is secure under the DL assumption, in Random Oracle Model

3.2 RSA Signatures

$N = pq$, two primes p and q
$\mathbb{Z}_N$ compute modulo N
$|\mathbb{Z}_N^*| = \varphi(N) = (p-1)(q-1)$
$\hookrightarrow$ Euler function
Euler's theorem:
For all $m \in \mathbb{Z}_N^*$: $m^{\varphi(N)} \bmod N = 1$. Without factoring N, inverting exponentiation modulo N seems difficult.

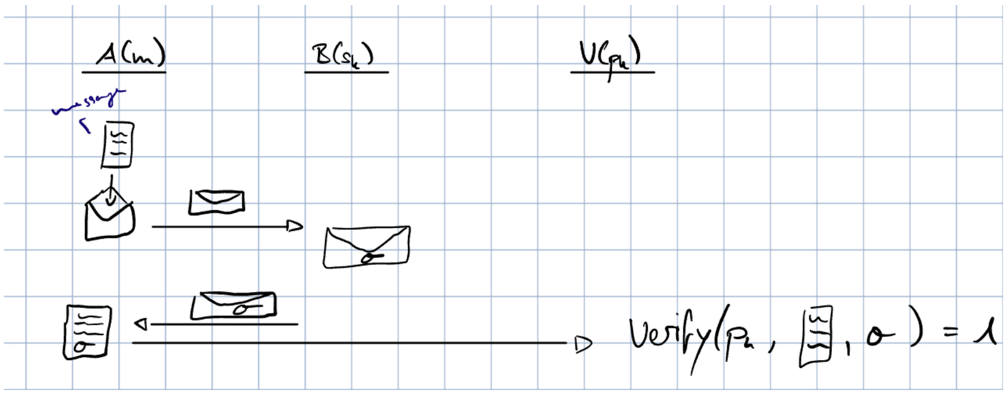
```
KeyGen()
  $p, q \leftarrow$ random primes
  $N \leftarrow pq$
  $e$ fixed prime, small, coprime to $p-1, q-1$
  $d \leftarrow e^{-1} \bmod \varphi(N)$        $\exists l : d \cdot e = 1 + l \cdot \varphi(N)$
  return $((N, e), d)$
```

```
Sign($d, m$)
  $\sigma \leftarrow H(m)^d$ where $H : \{0,1\}^* \rightarrow \mathbb{Z}_N$ (modeled as Random Oracle Model)
  return $\sigma$
```

```
Verify($(N, e), m, \sigma$)
  return $\sigma^e \overset{?}{=} H(m) \pmod N$
```

3.3 Blind Signatures

Protocol between user A and signer B to issue a digital Signature, A inputs message m, B signs m without learning m, Signature verifies as an ordinary signature.



- B must not learn any m it signs
- B learns how many times it signs
- Signature itself unforgeable

3.3.1 Blind RSA Signatures

Ordinary RSA Signatures do not hide m properly with $H(m)$ because adversary B could distinguish $H(m)$ from $H(m')$

This protocol hides m

- $\mathsf{KeyGen}()$ as before
- $\mathsf{Verify}()$ as before

Sign:

$\underline{A(N, e = m)}$ $\underline{B(d)}$ $\underline{V(N, e)}$

$r \overset{\mathbb{R}}{\leftarrow} \mathbb{Z}_p$

$\hat{h} \leftarrow r^e \cdot H(m) \bmod N$

$$\xrightarrow{\hat{h}}$$

$\bar{s} \leftarrow \hat{h}^d \bmod N$

$$\xleftarrow{\bar{s}}$$

$s \leftarrow \bar{s}/r \bmod N$

s is a RSA signature on $H(m)$

$$\xrightarrow{(m,s)} s^e \overset{?}{\equiv} H(m) \pmod N$$

Completeness

$$s^e = \bar{s}^e \cdot r^{-e} = \hat{h}^{de} \cdot r^{-e} \equiv \hat{h} \cdot r^e$$
$$\equiv r^e \cdot H(m) \cdot r^{-e} \equiv H(m) \pmod N$$

Blindness

- B signs a random value in $\mathbb{Z}_N$
- Signature allows no correlation with signing operation

Security

Same as ordinary RSA signature

3.4 Blind Schnorr Signatures

$\mathsf{Verify}(pk, m, \sigma) \to 0/1$

- $\mathsf{KeyGen}() \to (pk, sk)$ same as before

Signing Protocol:

$\underline{A(m)}$ $\underline{B(x)}$ $\underline{V(y)}$

B: $r \xleftarrow{R} \mathbb{Z}_q$

B: $t \leftarrow g^r$

$\xleftarrow{\quad t \quad}$

$\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$

$\hookrightarrow$ blinding factors

$\bar{t} \leftarrow t \cdot g^{-\alpha} \cdot y^{-\beta} \in G$

$\bar{c} \leftarrow H(m \| \bar{t})$

$c \leftarrow (\bar{c} + \beta) \bmod q$

$\xrightarrow{\quad c \quad} S \leftarrow r - c \cdot x \bmod q$

$\xleftarrow{\quad S \quad}$

$\bar{s} \leftarrow (S - \alpha) \bmod q$

<u>return</u> $(\bar{c}, \bar{s})$

$\xrightarrow{\quad (\bar{c},\bar{s},m) \quad} \mathsf{Verify}(y, m, (\bar{c}, \bar{s}))$

$\bar{c} \overset{?}{=} H(m \| g^{\bar{s}} \cdot y^{\bar{c}})$, where $g^{\bar{s}} \cdot y^{\bar{c}} = \hat{t}$

Completeness

Show that $\hat{t} = \bar{t}$:

$$\begin{aligned}
\hat{t} &= g^{\bar{s}} \cdot y^{\bar{c}} = g^{s-\alpha} \cdot g^{x\bar{c}} = g^{r-cx-\alpha} \cdot g^{x\cdot(c-\beta)} \\
&= g^{r-c\cdot x-\alpha+x\cdot c-x\cdot\beta} = g^{r-\alpha-x\cdot\beta} = g^{r-\alpha} \cdot y^{-\beta} \\
&= g^r \cdot g^{-\alpha} \cdot y^{-\beta} = t \cdot g^{-\alpha} \cdot y^{-\beta} = \bar{t} \in \mathbb{Z}_q
\end{aligned}$$

Blindness

- B sees only $H(m \| \hat{t})$ but F random
- B/V see signature $(\bar{c}, \bar{s})$ but during Protocol see only (c, s), and (c, s) are, for random α, β:

$$\bar{c} = c - \beta$$

$$\bar{s} = s - \alpha$$

- This hides c, s as in a OTP (Oblivious Transfer Protocol)

Security

Same as for Schnorr

3.5 Anonymous Digital Cash Scheme

D. Chaum, 1985

- User Alice: wallet with coins
- Shop S: receives coins
- Bank B: creates coins
  stores balance for A

Withdraw:
Denomination: u

$$\begin{tikzcd}\text{User } A \arrow[d] \\ \text{Bank } B\end{tikzcd}$$

User A: $m \overset{R}{\leftarrow} \{0,1\}^k$ coin identifier
send (m, u) to B
Bank B: - runs the blind-sig-protocol with B with A's input being m
User A obtains blind Signature σ on m

Bank B: $\text{balance}_A \leftarrow \text{balance}_A - u$
Spend:
User A sends (m, σ) to S
Shop verifies that σ is a valid signature on m
Deposit:

- Shop goes to B, sends (m, σ)
- Verify that σ is a valid signature on m
- If B has not credited m before, then:
  - Bank credits u to account of S
  - $\text{bal}_S \leftarrow \text{bal}_S + u$
- Bank tells S that coin is valid
- Shop performs service

$\longrightarrow$ spending must be online