Part 1 - Finite Elements in 1D

1.1. The 1D Poisson problem with zero boundary condition

(a simple hands-on example)

Model Problem - 1D Poisson equation

Given a function $f(x)$,

find $u(x)$ solving the differential equation

$$-u''(x) = f(x), \qquad x \in I := (0,1),$$

with the boundary condition

$$u(0) = u(1) = 0.$$

Model Problem - 1D Poisson equation

Find $u(x)$ with $u(0) = u(1) = 0$ and

$$-u''(x) = f(x), \qquad x \in I := (0, 1).$$

<u>Comments:</u>

- Problem can be solved by hand by integrating twice.
- We will solve it numerically using the finite element method.

Step 1: Variational problem formulation

Find $u(x)$ with $u(0) = u(1) = 0$ and for $x \in I$:

$$-u''(x) = f(x).$$

Idea: multiply equation with a differentiable function $v(x)$ with $v(0) = v(1) = 0$ and integrate by parts:

$$\begin{aligned}
\int_I f \cdot v \, \mathrm{d}x &\overset{\text{diff. eqn.}}{=} \int_I -u'' \cdot v \, \mathrm{d}x \\
&\overset{\text{part. int.}}{=} \int_I u' \cdot v' \, \mathrm{d}x - u'(1)v(1) + u'(0)v(0) \\
&\overset{v(0)=v(1)=0}{=} \int_I u' \cdot v' \, \mathrm{d}x
\end{aligned}$$

Step 1: Variational problem formulation

Find $u(x)$ with $u(0) = u(1) = 0$ and for $x \in I$:

$$-u''(x) = f(x).$$

Equivalent Variational Formulation:

For all differentiable $v(x)$ with $v(0) = v(1) = 0$:

$$\int_I u' \cdot v' \, \mathrm{d}x = \int_I f \cdot v \, \mathrm{d}x.$$

Next step:

Replace u, v by piecewise linear functions u_h, v_h:

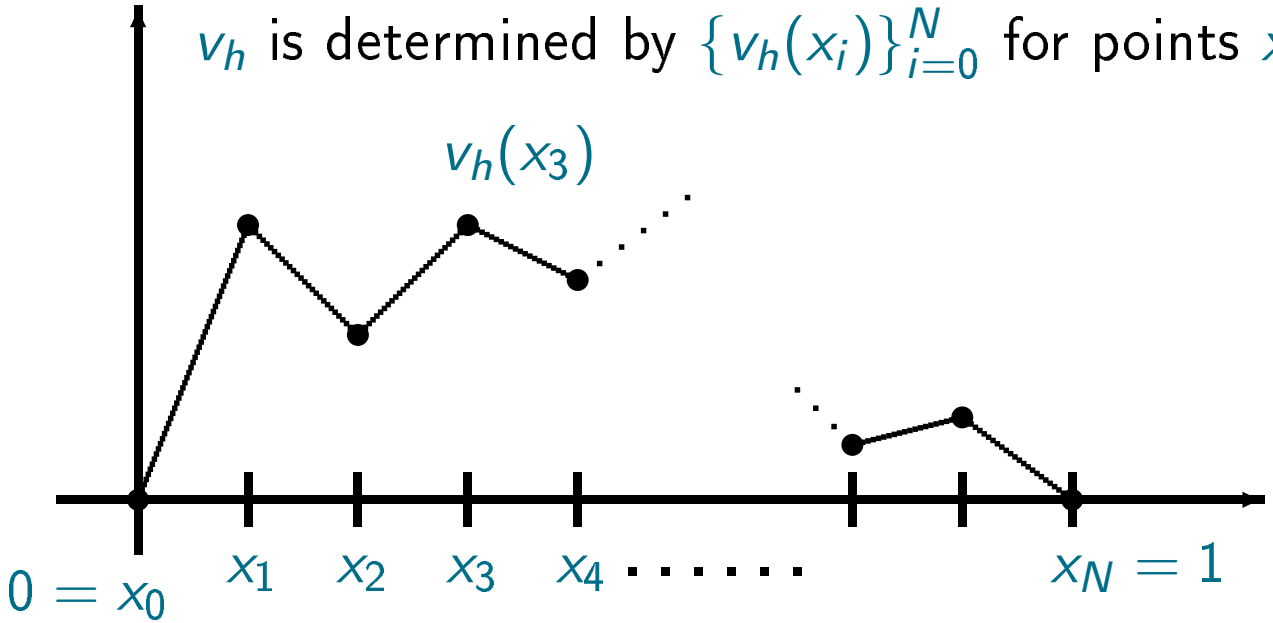
$$\int_I u_h' \cdot v_h' \, \mathrm{d}x = \int_I f \cdot v_h \, \mathrm{d}x$$

Step 2: discrete approximation space

Define space of piecewise linear functions $V_{h,0}$ with

- $N+1$: equidistant points x_i $\quad (0 \le i \le N)$;
- $h := 1/N$: mesh size. $\quad (h = x_{i+1} - x_i)$
- $v_h \in V_{h,0}$ is globally continuous, and it is linear on each $[x_i, x_{i+1}]$.

v_h is determined by $\{v_h(x_i)\}_{i=0}^{N}$ for points $x_i = ih$.

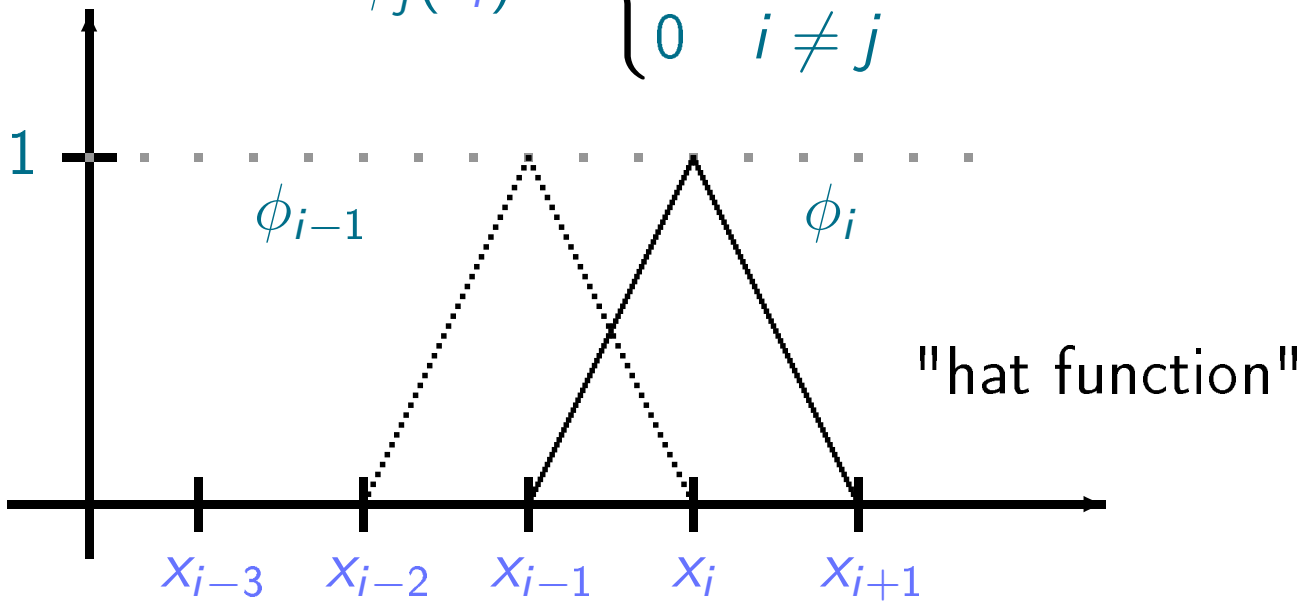


Step 2: discrete approximation space

Goal: simple representation of the space $V_{h,0}$.

Let $\{\phi_j\}_{j=0}^{N} \subset V_{h,0}$ be functions so that for $i, j = 0, \ldots, N$:

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$



$$v_h(x) = \sum_{i=1}^{N-1} y_i \, \phi_i(x), \quad \text{where } y_i = v_h(x_i).$$

Step 2: discrete approximation space

Let $\{\phi_j\}_{j=0}^{N} \subset V_{h,0}$ be such that for $i, j = 0, \ldots, N$:

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

We can directly prescribe the values in the points x_j:

$$v_h(x) = \sum_{i=1}^{N-1} y_i \, \phi_i(x).$$

Check:

$$v_h(x_j) = \sum_{i=1}^{N-1} y_i \, \phi_i(x_j) = y_j \cdot 1 + \sum_{i \neq j} y_i \cdot 0 = y_j.$$

Step 2: discrete approximation space

The functions $\{\phi_j\}_{j=1}^{N-1}$ form a basis of $V_{h,0}$,

i.e. any $v_h \in V_{h,0}$ can be written as

$$v_h(x) = \sum_{i=1}^{N-1} y_i \, \phi_i(x), \quad \text{where } y_i = v_h(x_i).$$

We have

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

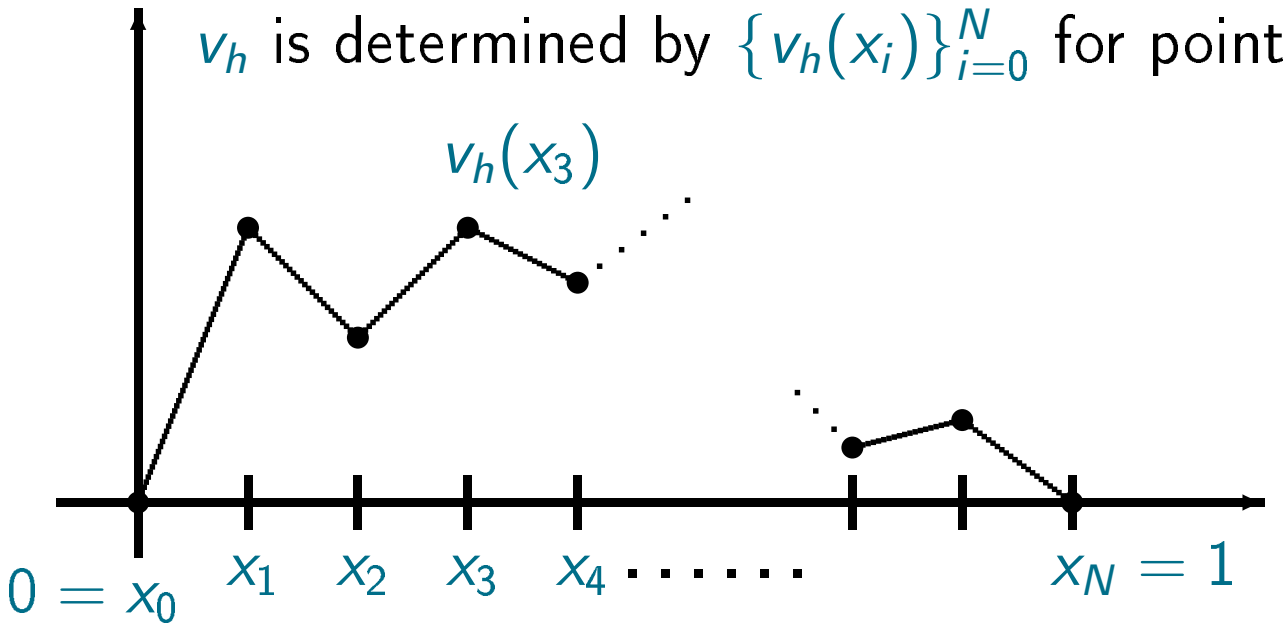
Step 2: discrete approximation space

Observe:

$$v_h = \sum_{i=1}^{N-1} y_i \, \phi_i \qquad \Rightarrow \qquad v_h(0) = 0 \text{ and } v_h(1) = 0,$$

because the ϕ_i for $i = 0$ and $i = N$ are not part of the sum!

v_h is determined by $\{v_h(x_i)\}_{i=0}^{N}$ for points $x_i = ih$.



Step 3: finite element method (FEM)

The finite element approximation $u_h \in V_{h,0}$ is the solution to

$$\int_I u_h' \cdot v_h' = \int_I f \cdot v_h \qquad \text{for all } v_h \in V_{h,0}.$$

Since $\{\phi_i\}_{1 \le i \le N}$ are basis of $V_{h,0}$ this is the same as demanding

$$\int_I u_h' \cdot \phi_i' = \int_I f \cdot \phi_i \qquad \text{for all } 1 \le i \le N-1.$$

Step 3: finite element method (FEM)

Finite Element Approximation $u_h \in V_{h,0}$ is solution to

$$\int_I u_h' \cdot \phi_i' = \int_I f \cdot \phi_i \qquad \text{for all } 1 \le i \le N-1.$$

Write

$$u_h = \sum_{j=1}^{N-1} \xi_j \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

then for $1 \le i \le N-1$

$$\int_I f \cdot \phi_i = \int_I u_h' \cdot \phi_i' = \int_I \left(\sum_{j=1}^{N-1} \xi_j \phi_j' \right) \cdot \phi_i' = \sum_{j=1}^{N-1} \xi_j \int_I \phi_j' \cdot \phi_i'.$$

Step 3: finite element method (FEM)

Finite Element Approximation

$$u_h = \sum_{j=1}^{N-1} \xi_j \, \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

solves

$$\sum_{j=1}^{N-1} \xi_j \int_I \phi_j' \cdot \phi_i' = \int_I f \cdot \phi_i \qquad \text{for all } 1 \le i \le N-1,$$

which can be written as system of equations

$$\begin{pmatrix} \int_I \phi_1' \cdot \phi_1' & \cdots & \int_I \phi_{N-1}' \cdot \phi_1' \\ \vdots & \ddots & \vdots \\ \int_I \phi_{N-1}' \cdot \phi_1' & \cdots & \int_I \phi_{N-1}' \cdot \phi_{N-1}' \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N-1} \end{pmatrix} = \begin{pmatrix} \int_I f \cdot \phi_1 \\ \vdots \\ \int_I f \cdot \phi_{N-1} \end{pmatrix}$$

Step 3: finite element method (FEM)

Finite element approximation

$$u_h = \sum_{j=1}^{N-1} \xi_j \, \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

solves

$$\mathbf{A}\boldsymbol{\xi} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\mathbf{b} \in \mathbb{R}^{N-1}$ are given by

$$\mathbf{A}_{ij} := \int_I \phi_j' \, \phi_i' \qquad \text{and} \qquad \mathbf{b} := \begin{pmatrix} \int_I f \cdot \phi_1 \\ \vdots \\ \int_I f \, \phi_{N-1} \end{pmatrix}$$

Goal: compute $\mathbf{A}$ and $\mathbf{b}$.

Step 3: finite element method (FEM)

Remark:

$$\mathbf{A} = \begin{pmatrix} \int_I \phi_1' \cdot \phi_1' & \cdots & \int_I \phi_{N-1}' \cdot \phi_1' \\ \vdots & \ddots & \vdots \\ \int_I \phi_{N-1}' \cdot \phi_1' & \cdots & \int_I \phi_{N-1}' \cdot \phi_{N-1}' \end{pmatrix}$$

is called the stiffness matrix and

$$\mathbf{b} := \begin{pmatrix} \int_I f \cdot \phi_1 \\ \vdots \\ \int_I f \, \phi_{N-1} \end{pmatrix}$$

is the load vector.

Step 3: finite element method (FEM)

Algorithm:

1. Initiate a mesh $x_0 < x_1 < \cdots < x_N$ with mesh size $h = x_{i+1} - x_i$ (we have N elements $[x_i, x_{i+1}]$)
2. Compute $\mathbf{A}$ and $\mathbf{b}$.
3. Solve system $\mathbf{A}\boldsymbol{\xi} = \mathbf{b}$.
4. Obtain FEM approximation by setting

$$u_h = \sum_{j=1}^{N-1} \xi_j \phi_j.$$

Step 3: finite element method (FEM)

Recall $h = 1/N$ and

$$0 = x_0 < h = x_1 < 2h = x_2 < \cdots < ih = x_i < \cdots Nh = x_N = 1.$$

<u>Goal:</u>
compute entries of matrix $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ with

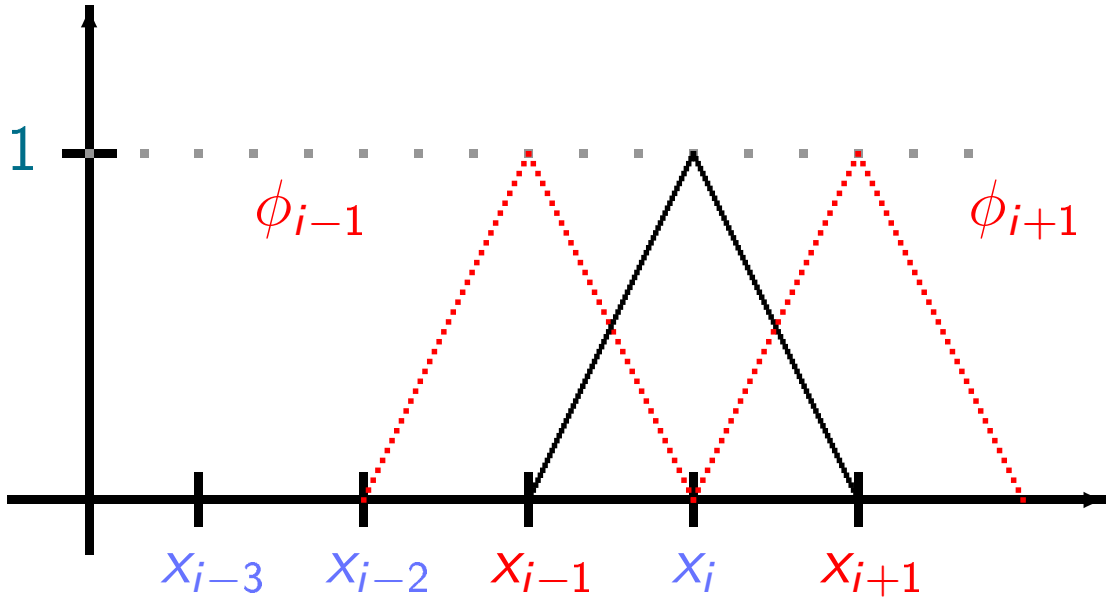
$$\mathbf{A}_{ij} := \int_0^1 \phi_j' \, \phi_i'.$$

We have 3 cases.

Step 3: finite element method (FEM)

Case 1: $|x_i - x_j| \geq 2h$ (i.e. $|i-j| \geq 2$) then $\phi_i(x) \cdot \phi_j(x) = 0$ and therefore

$$\mathbf{A}_{ij} = \int_0^1 \phi_j' \, \phi_i' = 0 \qquad \text{for } |i-j| \geq 2.$$



Step 3: finite element method (FEM)

Case 2: $|x_i - x_j| = h$, i.e. $j = i + / - 1$.

To compute

$$\mathbf{A}_{ij} := \int_0^1 \phi_j' \phi_i',$$

we have $(1 \leq i \leq N-1)$:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi_i'(x) = \begin{cases} \frac{1}{h}, & x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

We obtain:

$$\mathbf{A}_{i+1,i} = \mathbf{A}_{i,i+1} = \int_0^1 \phi_i' \, \phi_{i+1}' = \int_0^1 \phi_{i+1}' \, \phi_i' = \int_{x_i}^{x_{i+1}} \frac{-1}{h} \frac{1}{h} = -\frac{1}{h}.$$

Step 3: finite element method (FEM)

Case 3: $|x_i - x_j| = 0$, i.e. $j = i$.

To compute

$$\mathbf{A}_{ij} := \int_0^1 \phi_j' \phi_i',$$

we have $(1 \leq i \leq N-1)$:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi_i'(x) = \begin{cases} \frac{1}{h}, & x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

We obtain:

$$\mathbf{A}_{i,i} = \int_0^1 \phi_i' \, \phi_i' = \int_{x_{i-1}}^{x_i} (\phi_i')^2 + \int_{x_i}^{x_{i+1}} (\phi_i')^2 = \frac{1}{h} + \frac{1}{h} = \frac{2}{h}$$

Step 3: finite element method (FEM)

We summarize:

$$\mathbf{A}_{ij} = \begin{cases} 0 & \text{if } |i-j| \geq 2 \\ -1/h & \text{if } |i-j| = 1 \\ 2/h & \text{if } i = j \end{cases}$$

and therefore

$$\mathbf{A} = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -1 & 2 & -1 \\ 0 & \vdots & 0 & -1 & 2 \end{pmatrix}$$

Step 3: finite element method (FEM)

To approximate the entries

$$\mathbf{b} := \begin{pmatrix} \int_I f \cdot \phi_1 \\ \vdots \\ \int_I f\, \phi_{N-1} \end{pmatrix}$$

we could simply use the trapezoidal rule to obtain

$$\mathbf{b}_i = \int_I f\, \phi_i = \int_{x_{i-1}}^{x_{i+1}} f\, \phi_i \approx h\, f(x_i)\, \phi_i(x_i) = h\, f(x_i).$$

Hence, we can use approximately

$$\mathbf{b} = \begin{pmatrix} h\, f(x_1) \\ \dots \\ h\, f(x_{N-1}) \end{pmatrix}$$

Step 3: finite element method (FEM)

Now we can solve $\mathbf{A}\boldsymbol{\xi} = \mathbf{b}$,

i.e.

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_{N-1} \end{pmatrix} = \mathbf{b} = \begin{pmatrix} h\,f(x_1) \\ h\,f(x_2) \\ \dots \\ h\,f(x_{N-1}) \end{pmatrix}$$

The (piecewise linear) FEM approximation is

$$u_h = \sum_{j=1}^{N-1} \xi_j\,\phi_j.$$

With $\xi_0 = \xi_N = 0$, we have $u(x_i) \approx \xi_i$ for $0 \le i \le N$.

Part 1 - Finite Elements in 1D

1.2. Motivation and Modeling

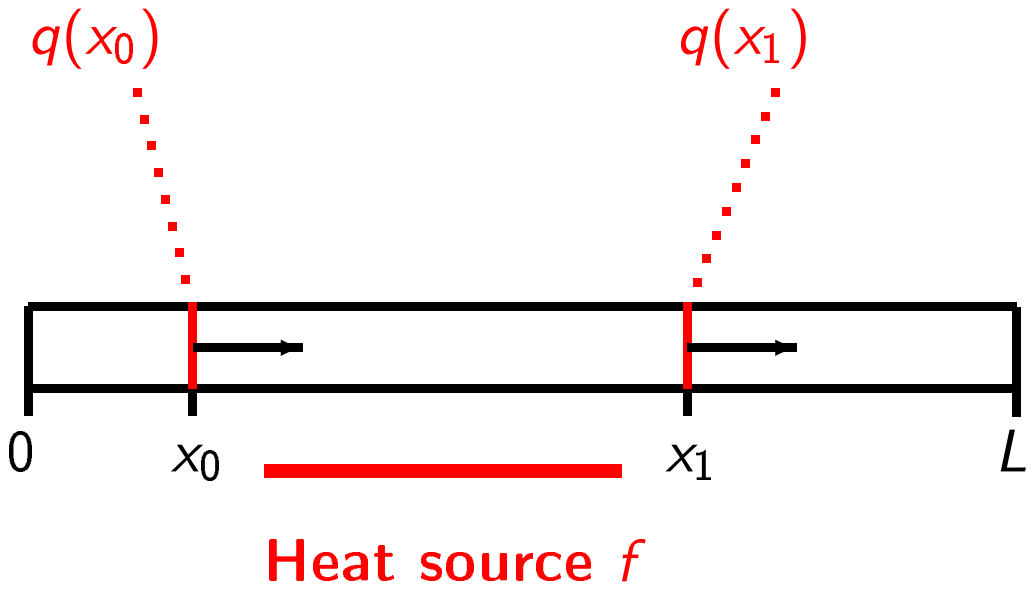
Temperature in a heated bar

Example: Thermal conduction in a $1d$-bar (conductive material).

Heat source

- $T(x)$: temperature at location x,
- $\mathbf{k}(x)$: thermal conductivity, $\mathbf{k}(x) \geq \mathbf{k_0} > 0$
- $f(x)$: heat source.

Temperature in a heated bar



q: heat flux along x-axis. Conservation of energy yields:

$$q(x_1) - q(x_0) = \int_{x_0}^{x_1} f(x)\,dx \quad \Rightarrow \quad \int_{x_0}^{x_1} q'(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx.$$

Temperature in a heated bar

We have

$$\int_{x_0}^{x_1} q'(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx.$$

Heat flows from hot to cold: heat flux q is proportional to negative temperature gradient, i.e.

$$q = -\mathbf{k}\,T', \qquad \text{with thermal conductivity } \mathbf{k}.$$

This gives us

$$-\int_{x_0}^{x_1} \left(\mathbf{k}\,T'\right)'(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx.$$

Since the interval $[x_0, x_1]$ was arbitrary, we get

$$-\left(\mathbf{k}(x)\,T'(x)\right)' = f(x) \quad \text{for all } x \in I = (0, L).$$

Temperature in a heated bar

Stationary heat equation for temperature T

$$-\left(\mathbf{k}(x)\,T'(x)\right)' = f(x) \quad \text{for all } x \in I = (0, L).$$

Boundary condition?

Temperature in a heated bar

Three important types of boundary conditions (BC):

1. Dirichlet: $T(0) = \alpha$ and $T(L) = \beta$.

   Strong BC: temperature is kept as constant value at the boundary points (temperature regulator).

2. Neumann: $T'(0) = \alpha$ and $T'(L) = \beta$.

   Natural BC: if $T'(0) = 0$ then $q(0) = 0$, which means that we do not have flow over the boundary (no flow in, no flow out).

3. Robin: $T'(0) = \alpha T(0)$ and $T'(L) = \beta T(L)$.

   Flux is proportional to the heat - the larger the heat the larger the flow.

Here, α and β are real numbers.

Part 1 - Finite Elements in 1D

1.3. 1D elliptic problems with Dirichlet boundary condition

Notation - Two important Lebesgue spaces

- $[a, b]$ interval in $\mathbb{R}$.
- Lebesgue space of square-integrable functions

$$L^2(a, b) := \{v | \int_a^b |v|^2 \, \mathrm{d}x < \infty\}$$

- Lebesgue space of pointwise bounded functions

$$L^\infty(a, b) := \{v | \text{ ess } \sup_{a \leq x \leq b} |v(x)| < \infty\}$$

Examples:

- if $v(x) := \exp(x)$ on $[0, 1]$ then $v \in L^\infty(0, 1)$.
- if $v(x) := \mathrm{sign}(x)$ on $[-1, 1]$ then $v \in L^\infty(-1, 1)$.
- if $v(x) := 1/x$ on $[0, 1]$ then $v \notin L^\infty(0, 1)$.

Notation - H^1-Sobolev spaces

- The space of continuous functions is

$$C^0(a,b) := \{v \mid v \text{ is continuous on } (a,b)\}.$$

- The Sobolev space of weakly differentiable is

$$H^1(a,b) := \{v \in C^0(a,b) \cap L^2(a,b) \mid \\ v \text{ is piecewise differentiable on } (a,b) \\ \text{and } \int_a^b |v'|^2 \, \mathrm{d}x < \infty\}.$$

Notation - H^1-Sobolev spaces

- The Sobolev space of weakly-differentiable is

$$H^1(a,b) := \{v \in C^0(a,b) \cap L^2(a,b) |$$

$$v \text{ is piecewise differentiable on } (a,b) \text{ and } \int_a^b |v'|^2 \, \mathrm{d}x < \infty\}.$$

(where v' is the piecewise derivative)

Example: if $v(x) = |x|$, then $v \in H^1(-1,1)$ with (weak) derivative

$$v'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

- The Sobolev space with zero boundary condition is

$$H_0^1(a,b) := \{v \in H^1(a,b) |\ v(a) = v(b) = 0\}.$$

General elliptic model problem in 1D Poisson

Given:

- source term: $f \in L^2(a, b)$,
- conductivity/diffusion coefficient:
  $k \in L^\infty(a, b)$ with $k(x) \geq k_0 > 0$,
- reaction coefficient:
  $p \in L^\infty(a, b)$ with $p(x) \geq p_0 \geq 0$.
- zero Dirichlet boundary values.

Find $u \in H_0^1(a, b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I := (a, b).$$

General elliptic model problem in 1D Poisson

$$-(ku')' + pu = f \qquad \text{in } I := (a, b).$$

If $u \in C^2(a, b)$ and all derivatives exist (in a classical sense), then u is called

a classical solution.

In this course we consider however variational or weak solutions.

What does that mean?

General elliptic model problem in 1D Poisson

$$-(ku')' + pu = f \qquad \text{in } I := (a, b).$$

Note: $k \in L^\infty(a, b)$ can be discontinuous, so the second derivative $(ku')'$ does not necessarily exist!

Still, the problem makes physically perfectly sense.

Mathematically, the conflict is resolved in the variational formulation.

Step 1: Variational problem formulation

Find $u \in H_0^1(a,b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I = (a,b).$$

As before: multiply equation with $v \in H_0^1(a,b)$ and integrate by parts:

$$\begin{aligned}
\int_I f\, v \,\mathrm{d}x &= \int_I (-(ku')' + pu)\, v \,\mathrm{d}x \\
&\overset{\text{part. int.}}{=} \int_I ku'v' \,\mathrm{d}x - (ku'v)(b) + (ku'v)(a) + \int_I puv \,\mathrm{d}x \\
&\overset{v \in H_0^1(a,b)}{=} \int_I ku'v' + puv \,\mathrm{d}x
\end{aligned}$$

Step 1: Variational problem formulation

Find $u \in H_0^1(a,b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I = (a,b).$$

Equivalent Variational Formulation:

$$\int_I ku'v' + puv = \int_I fv \qquad \text{for all } v \in H_0^1(a,b).$$

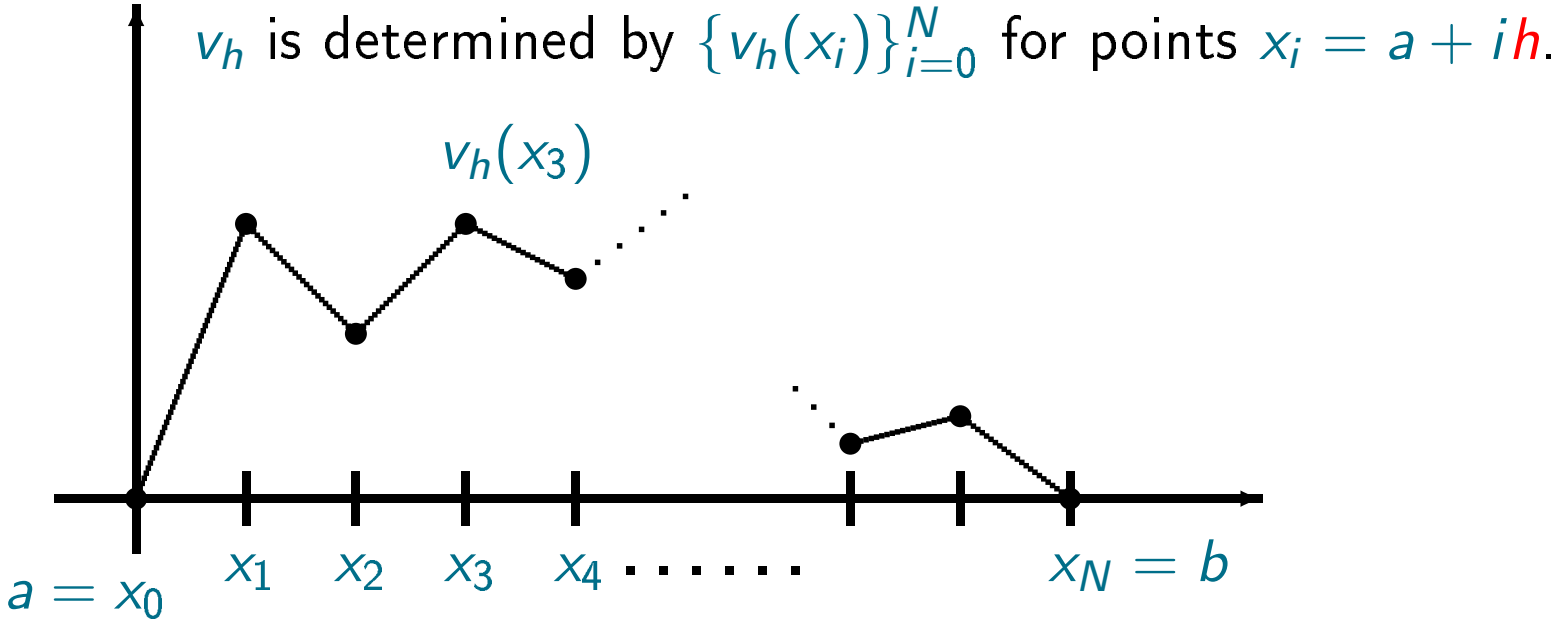
Next step:

Replace u, v by $u_h, v_h \in V_{h,0} \subset H_0^1(a,b)$.

Step 2: discrete approximation space

Recall the space of piecewise linear functions $V_{h,0}$ with

- $N+1$: equidistant points x_i $\quad (0 \le i \le N)$;
- $h := (b-a)/N$: mesh size. $\quad (h = x_{i+1} - x_i)$
- $v_h \in V_{h,0}$ is continuous and piecewise linear.

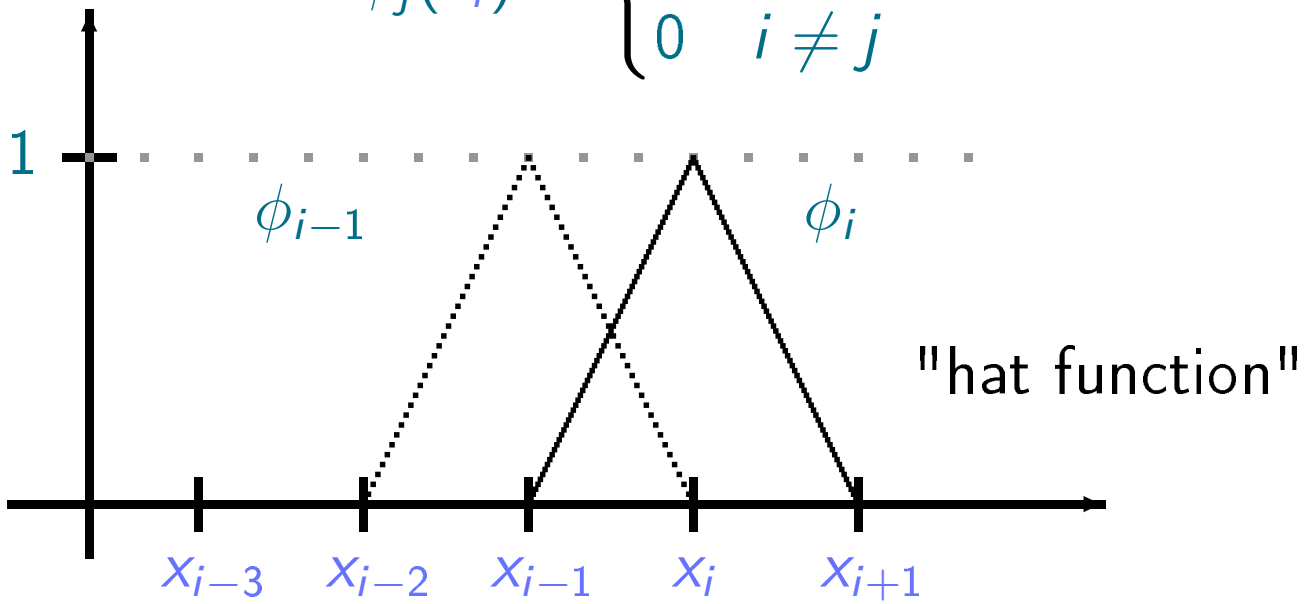


Step 2: discrete approximation space

Recall: simple representation of the space $V_{h,0}$.

Let $\{\phi_j\}_{j=0}^{N} \subset V_{h,0}$ be functions so that for $i, j = 0, \ldots, N$:

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$



$$v_h(x) = \sum_{i=1}^{N-1} y_i \, \phi_i(x), \quad \text{where } y_i = v_h(x_i).$$

Step 2: discrete approximation space

Recall: head functions $\{\phi_j\}_{j=1}^{N-1}$ form a basis of $V_{h,0}$,

i.e. any $v_h \in V_{h,0}$ can be written as

$$v_h(x) = \sum_{i=1}^{N-1} y_i \, \phi_i(x), \quad \text{where } y_i = v_h(x_i).$$

We have

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: finite element method (FEM)

The finite element approximation $u_h \in V_{h,0}$ is the solution to

$$\int_I k u_h' v_h' + p u_h v_h = \int_I f v_h \qquad \text{for all } v_h \in V_{h,0}.$$

Since $\{\phi_i\}_{1 \le i \le N}$ are basis of $V_{h,0}$ this is the same as demanding

$$\int_I k u_h' \phi_i' + p u_h \phi_i = \int_I f \phi_i \qquad \text{for } 1 \le i \le N-1.$$

Step 3: finite element method (FEM)

Finite Element Approximation $u_h \in V_{h,0}$ is solution to

$$\int_I k u_h' \phi_i' + p u_h \phi_i = \int_I f \phi_i \qquad \text{for } 1 \le i \le N-1.$$

Write

$$u_h = \sum_{j=1}^{N-1} \xi_j \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

then for $1 \le i \le N-1$

$$\sum_{j=1}^{N-1} \xi_j \left(\int_I k \phi_j' \phi_i' + p \phi_j \phi_i \right) = \int_I f \phi_i.$$

Step 3: finite element method (FEM)

Finite Element Approximation

$$u_h = \sum_{j=1}^{N-1} \xi_j \, \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

solves

$$\sum_{j=1}^{N-1} \xi_j \underbrace{\left(\int_I k \, \phi_j' \phi_i' + p \, \phi_j \phi_i \right)}_{=\mathbf{A}_{ij}} = \underbrace{\int_I f \, \phi_i}_{\mathbf{b}_i} .$$

which can be written as system of equations

$$\begin{pmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1,N-1} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{N-1,1} & \cdots & \mathbf{A}_{N-1,N-1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N-1} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{N-1} \end{pmatrix}$$

Step 3: finite element method (FEM)

Summary: Finite element approximation

$$u_h = \sum_{j=1}^{N-1} \xi_j \, \phi_j \qquad \text{where } \xi_j = u_h(x_j) \text{ \underline{is unknown}},$$

solves

$$\boxed{\mathbf{A}\boldsymbol{\xi} = \mathbf{b},}$$

where $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\mathbf{b} \in \mathbb{R}^{N-1}$ are given by the entries

$$\mathbf{A}_{ij} := \int_I k \, \phi_j' \phi_i' + p \, \phi_j \phi_i \qquad \text{and} \qquad \mathbf{b}_i := \int_I f \, \phi_i$$

Step 3: finite element method (FEM)

We have $h = (b-a)/N$ and $x_i := a + i\,h$, i.e.,

$$a = x_0 < h = x_1 < 2h = x_2 < \cdots < ih = x_i < \cdots x_N = b.$$

<u>Goal:</u>
compute entries of matrix $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ with

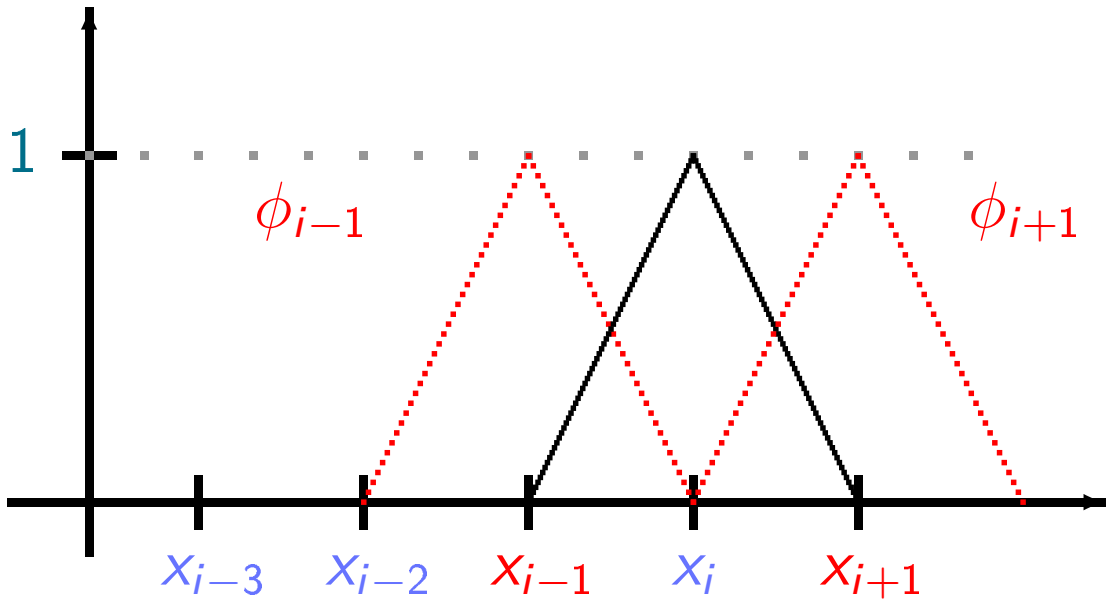
$$\mathbf{A}_{ij} := \int_I k\,\phi_j'\phi_i' + p\,\phi_j\phi_i.$$

We have 3 cases.

Step 3: finite element method (FEM)

Case 1: $|x_i - x_j| \geq 2h$ (i.e. $|i-j| \geq 2$) then $\phi_i(x) \cdot \phi_j(x) = 0$ and therefore

$$\mathbf{A}_{ij} = \int_I k\,\phi_j'\phi_i' + p\,\phi_j\phi_i = 0 \qquad \text{for } |i-j| \geq 2.$$



Step 3: finite element method (FEM)

Case 2: $|x_i - x_j| = h$, i.e. $j = i + / - 1$.

To compute

$$\mathbf{A}_{ij} := \int_I k\, \phi_j' \phi_i' + p\, \phi_j \phi_i,$$

we have $(1 \leq i \leq N - 1)$:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi_i'(x) = \begin{cases} \frac{1}{h}, & x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

We obtain:

$$\mathbf{A}_{i+1,i} = \mathbf{A}_{i,i+1} = \int_{x_i}^{x_{i+1}} k\, \phi_i' \phi_{i+1}' + p\, \phi_i \phi_{i+1} = \int_{x_i}^{x_{i+1}} \left(-\frac{k}{h^2} + p\, \phi_i \phi_{i+1} \right).$$

This is computed numerically (or by hand for simple k and p).

Step 3: finite element method (FEM)

Case 3: $|x_i - x_j| = 0$, i.e. $j = i$.

To compute

$$\mathbf{A}_{ij} := \int_I k\,\phi_j'\phi_i' + p\,\phi_j\phi_i,$$

we have $(1 \le i \le N-1)$:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi_i'(x) = \begin{cases} \frac{1}{h}, & x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

We obtain:

$$\mathbf{A}_{ii} = \int_{x_{i-1}}^{x_{i+1}} k\,|\phi_i'|^2 + p\,|\phi_i|^2 = \int_{x_{i-1}}^{x_{i+1}} \frac{k}{h^2} + p\,|\phi_i|^2,$$

which can be computed numerically using quadrature rules.

Step 3: finite element method (FEM)

We summarize:

$$\mathbf{A}_{ij} = \begin{cases} 0 & \text{if } |i-j| \geq 2 \\ \int_{x_i}^{x_{i+1}} -\frac{k}{h^2} + p\,\phi_i\phi_{i+1} & \text{if } i+1=j \\ \int_{x_{i-1}}^{x_{i+1}} \frac{k}{h^2} + p\,|\phi_i|^2 & \text{if } i=j \end{cases}$$

Sometimes, the entries can be computed by hand, for example if k and p are simple polynomials (or constant).

In general, an accurate numerical quadrature rule needs to be applied. In many cases, the composite trapezoidal rule is sufficient. For complicated coefficients k and p, high order Gauss quadratures might be needed.

Part 1 - Finite Elements in 1D

1.4. 1D elliptic problems with general boundary conditions

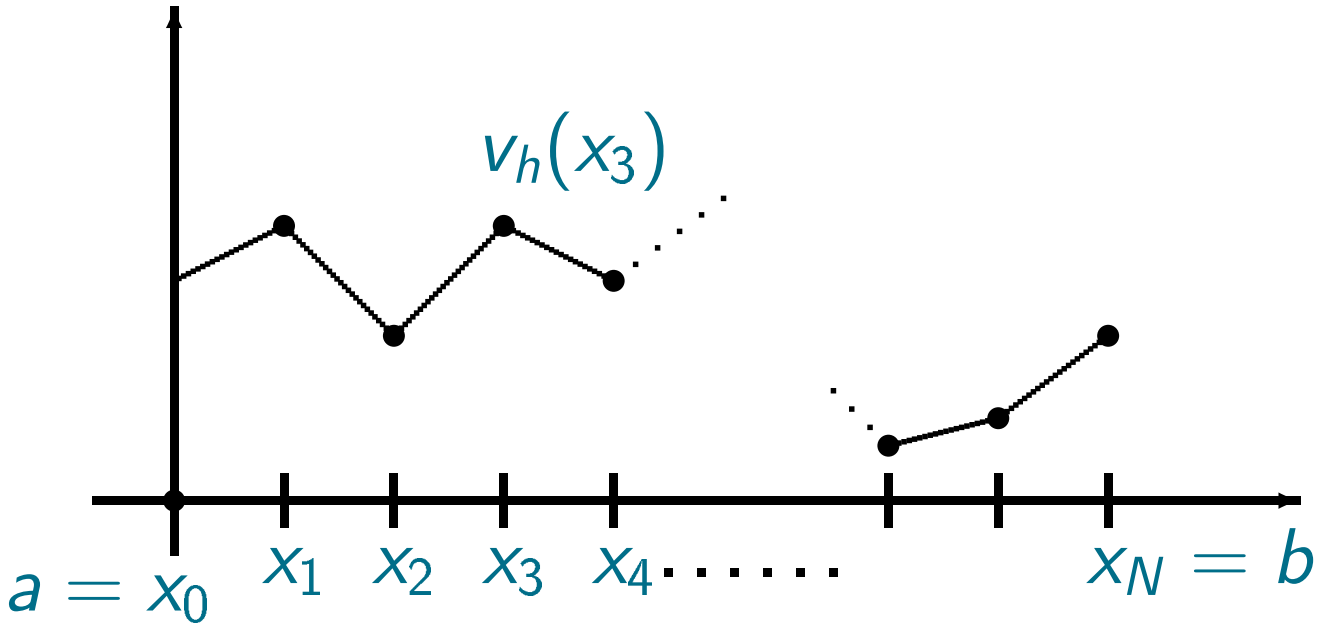
General Dirichlet boundary condition

Remark: For simplicity, we will drop the reaction term pu in the following slides. The treatment with or without reaction term works analogously.

Finite element space without boundary values

As before: we consider $N+1$ points x_i and let $h := (b-a)/N$ be the mesh size.

The finite element space V_h is defined as before, except that we do not force zero boundary values:



1D Dirichlet boundary value problem

Given:

- source term: $f \in L^2(a, b)$,
- conductivity/diffusion coefficient: $k \in L^\infty(a, b)$ with $k(x) \geq k_0 > 0$,
- Dirichlet boundary values g_a and g_b.

Find $u \in H^1(a, b)$ solving the differential equation

$$-(ku')' = f \qquad \text{in } I = (a, b)$$

and with

$$u(a) = g_a \qquad \text{and} \qquad u(b) = g_b.$$

1D Dirichlet boundary value problem

Find $u \in H^1(a,b)$ solving the differential equation

$$-(ku')' = f \qquad \text{in } I = (a,b)$$

and with $\quad u(a) = g_a \quad$ and $\quad u(b) = g_b$.

To obtain the Variational Formulation, we multiply with $v \in H_0^1(a,b)$ and integrate by parts as before:

$$\begin{aligned}
\int_I f\,v &= -\int_I (ku')'\,v \\
&= \int_I ku'\,v' \;-\; k(b)u'(b)v(b) + k(a)u'(a)v(a). \\
&= \int_I ku'\,v'.
\end{aligned}$$

1D Dirichlet boundary value problem

Find $u \in H^1(a,b)$ solving the differential equation

$$-(ku')' = f \qquad \text{in } I = (a,b)$$

and with $\quad u(a) = g_a \quad$ and $\quad u(b) = g_b$.

Variational formulation: find $u \in H^1(a,b)$ with

$$u(a) = g_a \qquad \text{and} \qquad u(b) = g_b$$

and such that

$$\int_I k u' v' = \int_I f v \qquad \text{for all } v \in H_0^1(a,b).$$

1D Dirichlet boundary value problem

Variational formulation: find $u \in H^1(a,b)$ with

$$u(a) = g_a \qquad \text{and} \qquad u(b) = g_b$$

and such that

$$\int_I k\,u'\,v' = \int_I f\,v \qquad \text{for all } v \in H_0^1(a,b).$$

Finite element approximation: find $u_h \in V_h$ with

$$u_h(a) = g_a \qquad \text{and} \qquad u_h(b) = g_b$$

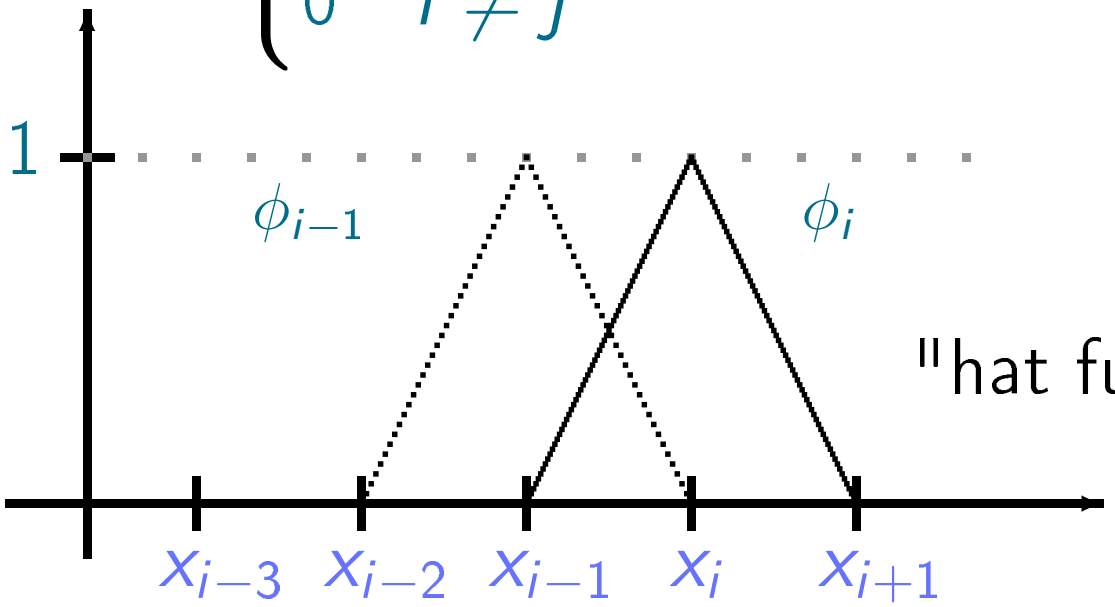
and such that

$$\int_I k\,u_h'\,v_h' = \int_I f\,v_h \qquad \text{for all } v_h \in V_{h,0}.$$

1D Dirichlet boundary value problem

Recall $\{\phi_j\}_{j=0}^{N} \subset V_h$ as the hat functions with:

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{for } i, j = 0, \cdots, N$$



"hat function"

$$u_h(x) = \sum_{i=0}^{N} \xi_i \, \phi_i(x), \quad \text{where } \xi_i = u_h(x_i).$$

1D Dirichlet boundary value problem

Finite element approximation: find $u_h \in V_h$ with

$$u_h(a) = g_a \qquad \text{and} \qquad u_h(b) = g_b$$

and such that

$$\int_I k\, u_h'\, v_h' = \int_I f\, v_h \qquad \text{for all } v_h \in V_{h,0}.$$

Since ϕ_i are basis for $V_{h,0}$ this is the same as

$$\int_I k\, u_h'\, \phi_i' = \int_I f\, \phi_i \qquad \text{for all } 1 \le i \le N-1.$$

1D Dirichlet boundary value problem

Finite Element Approximation $u_h = \sum_{j=0}^{N} \xi_j \, \phi_j$ is characterized by the following $N+1$ conditions:

1. $$\sum_{j=0}^{N} \xi_j \int_I k \, \phi_j' \, \phi_i' = \int_I f \, \phi_i \qquad \text{for } 1 \le i \le N-1$$

We do not have this equation for $i = 0$ and $i = N$.

Here we can use the BC $u_h(a) = g_a$ and $u_h(b) = g_b$:

2. $\xi_0 = g_a \qquad \leftarrow$ left boundary condition

3. $\xi_N = g_b \qquad \leftarrow$ right boundary condition

1D Dirichlet boundary value problem

The equations

1. $$\sum_{j=0}^{N} \xi_j \int_I k\, \phi_j' \phi_i' = \int_I f\, \phi_i \qquad \text{for } 1 \le i \le N-1$$

2. $$\xi_0 = g_a$$

3. $$\xi_N = g_b$$

can be written as a system

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \int_I k\, \phi_0' \phi_1' & \int_I k\, \phi_1' \phi_1' & \cdots & \int_I k \phi_N' \phi_1' \\ \vdots & & & \vdots \\ \int_I k\, \phi_0' \phi_{N-1}' & \int_I k\, \phi_1' \phi_{N-1}' & \cdots & \int_I k \phi_N' \phi_{N-1}' \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{N-1} \\ \xi_N \end{pmatrix} = \begin{pmatrix} g_a \\ \int_I f \phi_1 \\ \vdots \\ \int_I f \phi_{N-1} \\ g_b \end{pmatrix}$$

1D Dirichlet boundary value problem

As before, we can simplify this to:

$$\mathbf{A} := \begin{pmatrix} 1 & 0 & & & \cdots & 0 \\ -\frac{1}{h^2}\int_{x_0}^{x_1} k & \frac{1}{h^2}\int_{x_0}^{x_2} k & -\frac{1}{h^2}\int_{x_1}^{x_2} k & 0 & \cdots & 0 \\ 0 & -\frac{1}{h^2}\int_{x_1}^{x_2} k & \frac{1}{h^2}\int_{x_1}^{x_3} k & -\frac{1}{h^2}\int_{x_2}^{x_3} k & \cdots & 0 \\ \vdots & & & & \cdots & \\ 0 & \vdots & 0 & -\frac{1}{h^2}\int_{x_{N-2}}^{x_{N_1}} k & \frac{1}{h^2}\int_{x_{N-2}}^{x_N} k & -\frac{1}{h^2}\int_{x_{N-1}}^{x_N} k \\ 0 & \cdots & & & 0 & 1 \end{pmatrix}$$

With $\mathbf{b} := \begin{pmatrix} g_a \\ \int_I f\phi_1 \\ \vdots \\ \int_I f\phi_{N-1} \\ g_b \end{pmatrix}$ we can solve $\mathbf{A}\boldsymbol{\xi} = \mathbf{b}$.

We obtain $u_h = \sum_{j=0}^{N} \xi_j \phi_j$.

Neumann boundary conditions

<u>Remark:</u> If we consider the elliptic equation

$$(ku')' = f$$

with Neumann boundary condition on both ends, i.e.,

$$k(a)u'(a) = n_a \qquad \text{and} \qquad k(b)u'(b) = n_b,$$

then the problem cannot have a unique solution anymore, because if $u(x)$ is a solution, $u(x) + c$ is still a solution for any constant $c \in \mathbb{R}$.

Therefore it is important to consider the problem with a reaction term pu with $p > 0$, where this problem cannot occur.
Alternatively we could also consider mixed Dirichlet-Neumann boundary conditions.

1D Neumann boundary value problem

- source term: $f \in L^2(a,b)$,
- conductivity/diffusion coefficient:
  $k \in L^\infty(a,b)$ with $k(x) \geq k_0 > 0$,
- reaction coefficient:
  $p \in L^\infty(a,b)$ with $p(x) \geq p_0 > 0$.
- Neumann boundary values n_a and n_b.

Find $u \in H^1(a,b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I = (a,b)$$

and $\quad k(a)u'(a) = n_a \quad$ and $\quad k(b)u'(b) = n_b$.

1D Neumann boundary value problem

Find $u \in H^1(a,b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I = (a,b)$$

and with $\quad k(a)u'(a) = n_a \quad$ and $\quad k(b)u'(b) = n_b$.

To obtain the Variational Formulation, we multiply with $v \in H^1(a,b)$ and integrate by parts as before:

$$\begin{aligned}
\int_I f\,v &= \int_I -(ku')'\,v + pu\,v \\
&= \int_I (ku'\,v' + pu\,v) \;- k(b)u'(b)v(b) + k(a)u'(a)v(a). \\
&= \int_I (ku'\,v' + pu\,v) \;- n_b v(b) + n_a v(a).
\end{aligned}$$

1D Neumann boundary value problem

Find $u \in H^1(a,b)$ solving the differential equation

$$-(ku')' + pu = f \qquad \text{in } I = (a,b)$$

and with $\quad k(a)u'(a) = n_a \quad$ and $\quad k(b)u'(b) = n_b$.

Variational formulation: find $u \in H^1(a,b)$ with

$$\int_I ku'\,v' + pu\,v = \int_I f\,v \; - n_a v(a) + n_b v(b)$$

for all $v \in H^1(a,b)$.

1D Neumann boundary value problem

Variational formulation: find $u \in H^1(a,b)$ with

$$\int_I k u' v' + p u v = \int_I f v \; - n_a v(a) + n_b v(b)$$

for all $v \in H^1(a,b)$.

Finite element approximation: find $u_h \in V_h$ with

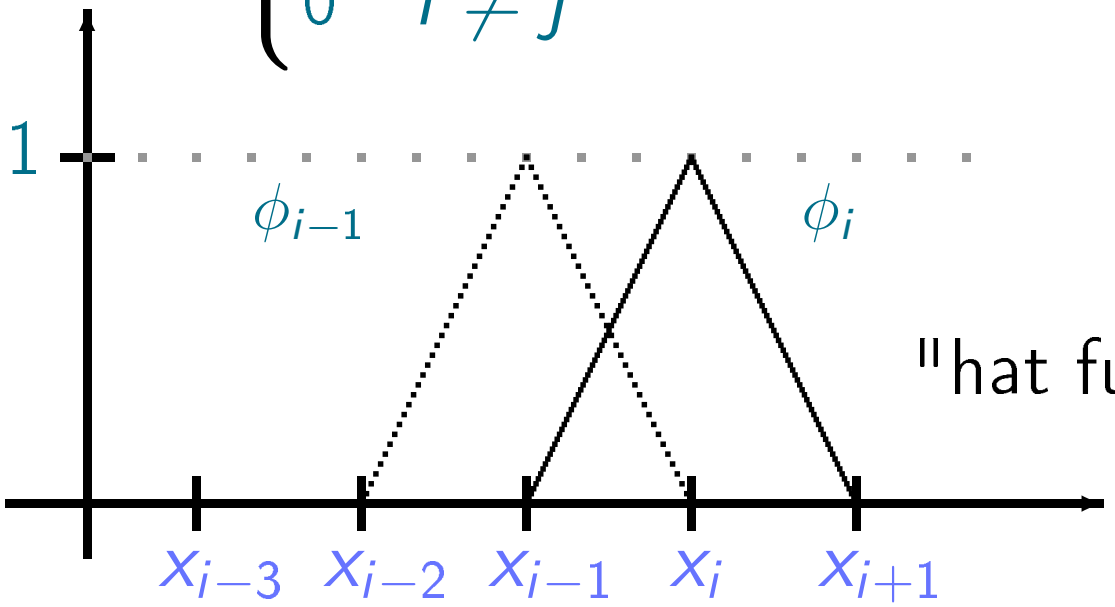
$$\int_I k u_h' v_h' + p u_h v_h = \int_I f v_h \; - n_a v_h(a) + n_b v_h(b)$$

for all $v_h \in V_h$.

1D Neumann boundary value problem

Recall $\{\phi_j\}_{j=0}^{N} \subset V_h$ as the hat functions with:

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{for } i, j = 0, \cdots, N$$



"hat function"

$$u_h(x) = \sum_{i=0}^{N} \xi_i \, \phi_i(x), \quad \text{where } \xi_i = u_h(x_i).$$

1D Neumann boundary value problem

Finite element approximation: find $u_h \in V_h$ with

$$\int_I k u_h' v_h' + p u_h v_h = \int_I f v_h \; - n_a v_h(a) + n_b v_h(b)$$

for all $v_h \in V_h$.

Since ϕ_i are basis for V_h this is the same as

$$\int_I k u_h' \phi_i' + p u_h \phi_i = \int_I f \phi_i \; - n_a \phi_i(a) + n_b \phi_i(b)$$

for all $0 \leq i \leq N$.

1D Neumann boundary value problem

Finite element approximation: find $u_h \in V_h$ with

$$\int_I k u_h' \phi_i' + p u_h \phi_i = \int_I f \phi_i \; - n_a \phi_i(a) + n_b \phi_i(b)$$

for all $0 \leq i \leq N$.

We observe:

- If $i = 0$, then $\phi_0(a) = 1$ and $\phi_0(b) = 0$.
- If $i = N$, then $\phi_N(a) = 0$ and $\phi_N(b) = 1$.
- If $0 < i < N$, then $\phi_i(a) = 0$ and $\phi_i(b) = 0$.

1D Neumann boundary value problem

Finite Element Approximation $u_h = \sum_{j=0}^{N} \xi_j \, \phi_j$ is characterized by the following $N+1$ conditions:

1. $$\sum_{j=0}^{N} \xi_j \int_I \left(k \, \phi_j' \, \phi_i' + p \, \phi_j \, \phi_i \right) = \int_I f \, \phi_i \qquad \text{for } 1 \leq i \leq N-1$$

And for the boundary terms $i = 0$ and $i = N$:

2. $$\sum_{j=0}^{N} \xi_j \int_I \left(k \, \phi_j' \, \phi_0' + p \, \phi_j \, \phi_0 \right) = \int_I (f \, \phi_0) \; - n_a$$

3. $$\sum_{j=0}^{N} \xi_j \int_I \left(k \, \phi_j' \, \phi_N' + p \, \phi_j \, \phi_N \right) = \int_I (f \, \phi_N) \; + n_b$$

1D Neumann boundary value problem

The equations can be written as a system

$$\mathbf{A}\boldsymbol{\xi} = \mathbf{b},$$

where the entries of $\mathbf{A} \in \mathbb{R}^{(N+1)\times(N+1)}$ are given by

$$\mathbf{A}_{ij} := \int_I k\,\phi_j'\,\phi_i' + p\,\phi_j\,\phi_i$$

and

$$\boldsymbol{\xi} := \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{N-1} \\ \xi_N \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} := \begin{pmatrix} \int_I f\,\phi_0 & - n_a \\ \int_I f\,\phi_1 & \\ \vdots & \\ \int_I f\,\phi_{N-1} & \\ \int_I f\,\phi_N & + n_b \end{pmatrix}$$

Mixed Dirichlet-Neumann boundary condition

In the case of elliptic problems

$$(ku')' = f$$

with mixed Dirichlet-Neumann boundary conditions, e.g.,

$$u(a) = g_a \qquad \text{and} \qquad k(b)u'(b) = n_b,$$

we have to combine the previous two case. This means that we need to work in the space

$$H^1_{0,a}(a,b) = \{v \in H^1(a,b) |\ v(a) = 0\}$$

for the Variational Formulation and similarly with

$$V_{h,0,a} = \{v_h \in V_h |\ v_h(a) = 0\}$$

for the finite element method.