Problem Set B

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Problem 1.(a)

$$\begin{cases} -\Delta u(x) = f & x \in \Omega \\ \hat{n} \cdot \nabla u(x) = g & x \in \partial\Omega \end{cases}$$

A problem in differential equations is said to be well-posed if:
- a solution exists
- the solution is unique
- the solution changes continuously with the data.

The variational formulation of the problem is

Find $u \in H(\Omega)$ such that:

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \quad \forall v \in H^1(\Omega)$$

We choose v to be constant and that:

$$\int_\Omega f \, dx + \int_{\partial\Omega} g_N \, ds = 0 \qquad (*)$$

Thus the PDE is solvable if and only if f and g_N satisfy $(*)$, which is sometimes called the conservation property (or compatibility condition).

$\boxed{\text{The aboved discuss is about the existence of solution.}}$

Now, we suppose that the solution exists. Here, it is obvious that if u_i is a solution, then $u_k = u_i + C$ (C is a constant) is also a solution.

$$\begin{cases} -\Delta u_k = -\Delta(u_i + C) = -\Delta u_i = f \\ \hat{n} \cdot \nabla u_k = \hat{n} \cdot \nabla(u_i + C) = \hat{n} \cdot \nabla u_i = g \end{cases}$$

$\boxed{\text{The aboved discussion is about the uniqueness of solution}}$

(b) The author has tried to solve this in MATLAB. MATLAB still gives kind of "result" except the fact the the values of the result tends to infinity.

To solve this problem, a common practice is to enforce an additional constraint Such constraint is often chosen as mean zero constraint $\int_\Omega u \, d\Omega = 0$ which in terms of finite element approximation is translated into $\sum_{i=1}^{N_h} \xi_i \int \phi_i = 0$

Further detail of numerical implementation of such constraint could be found on page 95 of the TIA book, and has also been realized in the author's project Lab B.

Problem 2 $\quad -(k(x)u'(x))' + u'(x) = f(x) \quad , \quad x \in \Omega$

$\qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad , \quad x \in \partial\Omega$

(a) $V = H_0^1(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0 \}$

$$\int_\Omega -(k(x)u'(x))' v + \int_\Omega u'(x) v = \int_\Omega f(x) v$$

$$\underbrace{\int_{\partial\Omega} - k(x)u'(x) v}_{v \in H_0^1(\Omega)} + \int_\Omega k(x)u'(x) v' + \int_\Omega u'(x) v = \int_\Omega f(x) v$$

The variational formulation is: find $u \in V$ such that

$$a(u,v) = L(v) \qquad \forall v \in V$$

where $a(u,v) = \int_\Omega k u' v' + u' v \, dx$

$$L(v) = \int_\Omega f v = (f, v)$$

(b) $a(w,w) = \int_\Omega k(w')^2 + w'w \, dx \qquad \forall w \in V$

and $\int_\Omega w'w \, dx = \underbrace{\int_{\partial\Omega} ww \, dx}_{0} - \int_\Omega ww' \, dx$

$\Rightarrow 2\int_\Omega w'w \, dx = 0 \Rightarrow \int_\Omega w'w \, dx = 0$

Thus, $a(w,w) = \int_\Omega k(x)(w'(x))^2 dx \quad \forall w \in V$

$\|u\|_E^2 = a(u,u)$ is a norm:

① $\|u+v\|_E^2 = a(u+v, u+v)$

$= a(u,u) + a(v,v) + a(u,v) + a(v,u)$

Cauchy inequality $\rightarrow \leq \|u\|_E^2 + \|v\|_E^2 + \|u\|_E\|v\|_E + \|v\|_E\|u\|_E$

$= (\|u\|_E + \|v\|_E)^2$

$\Rightarrow \|u+v\|_E \leq \|u\|_E + \|v\|_E$

② $\|\lambda u\|_E^2 = a(\lambda u, \lambda u) = \lambda^2 a(u,u)$

$\Rightarrow \|\lambda u\|_E = |\lambda| \|u\|_E$

③ $\|u\| = \left(\int_\Omega k (u')^2\right)^{1/2} \geq 0 \quad \forall u \in V$

$\boxed{\text{definite positive}}$

If $\|u\| = 0 \Leftrightarrow \int_\Omega k(u')^2 = 0$

Meanwhile $k(u')^2 \geq 0$

Therefore $k(u')^2 = 0$, 2 cases:

Furthermore, $k(x) \geq k_0 > 0$

$\rightarrow (u')^2 = 0 \quad \rightarrow u' = 0$

since $u|_{\partial\Omega} = 0$

we thus have $u = 0$

(c) To prove the existence and uniqueness of solution of 2.(a), we need to verify all conditions of the Lax-Milgram theorem is verified: ($V = H_0^1(\Omega)$) from 2.(b) we have

$$\|w\|_V = \sqrt{\int_\Omega k(x)(w')^2} \geq \sqrt{k_0 \int_\Omega (w')^2} = \sqrt{k_0}\|\nabla w\| \Rightarrow \|w\|_V \geq C\|\nabla w\|$$

with $C \geq 0$

① $\mathcal{L}$ is a continuous on $V = H_0^1(\Omega)$

$$\mathcal{L}(v) = (f, v) \leq \|f\|\|v\| \underset{\text{poincaré - Friedrichs}}{\leq} C\|f\|\|\nabla v\| \leq C\|f\|\|v\|_V = C\|v\|_V$$

② $a(u,v) = \int_\Omega k u'v' + u'v \, dx$ is continuous on $H_0^1(\Omega)$

we need to prove $a(u,v) \leq C\|u\|_V\|v\|_V$

$$a(u,v) = k(\nabla u, \nabla v) + (\nabla u, v)$$

$$\leq \|k\|_{L^\infty(\Omega)}\|\nabla u\|\|\nabla v\| + \|\nabla u\|\|v\| \quad \text{Cauchy-Schwarz}$$

$$\leq C_1\|u\|_V\|v\|_V + C_2\|u\|_V\|\nabla v\| \quad \text{poincaré Friedrichs.}$$

$$= C_1\|u\|_V\|v\|_V + C_3\|u\|_V\|v\|_V$$

$$= C_4\|u\|_V\|v\|_V \quad \text{with } C_4 = C_1 + C_3$$

③ $a(u,v)$ is coercive

we need to prove $\exists\, C > 0$ such that $a(v,v) \geq \alpha \underbrace{\|v\|_V^2}_{a(v,v)}$ $\forall v \in V$

This is trivial, we take $\alpha = 1$

(d) → We introduce an admissible discretization $\mathcal{K}$ on $(0,1)$ with mesh size $h(x)$

→ We introduce the space of continuous piecewise linear functions on $(0,1)$ that have zero boundary conditions as

$$V_{h,0} = \{ v \in C^0(\Omega) \cap H_0^1(\Omega) \mid \forall K \in \mathcal{K}, v|_K \in \mathbb{P}^1(K) \}$$

Then the CG(1) formulation for the given BVP is

Find $u_h \in V_{h,0}$ such that

$$a(u_h, v) = \mathcal{L}(v) \quad \forall \; v \in V_{h,0}$$

(e) The Galerkin orthogonality tells us that

$$a(u - u_h, v) = 0 \quad \forall \; v \in V_{h,0}.$$

we first prove: $\|u - u_h\|_V \leq \|u - v\|_V \quad \forall v \in V_{h,0}$

$$\|u - u_h\|_V^2 = a(u - u_h, u - u_h)$$

$$= a(u - u_h, (u - v) + (v - u_h))$$

$$= a(u - u_h, u - v) + \underbrace{a(u - u_h, v - u_h)}_{0 \in V_{h,0}}$$

$$\leq C \|u - u_h\|_V \|u - v\|_V$$

continuity of a

$(C > 0)$

$$\Rightarrow \quad \|u-u_h\|_V \le \|u-v\|_V \quad \forall v \in V_{h,0}$$

$$\Rightarrow \quad \|u-u_h\|_V^2 \le \|u-v\|_V^2$$

take $v = \Pi u$ interpolation of u in $V_{h,0}$.

$$\Rightarrow \quad \|u-u_h\|_V^2 \le \|u-\Pi u\|_V^2$$

$$\le \sum_{K\in\mathcal{K}} k_0 \|D(u-\Pi(u))\|_{L^2(K)}^2$$

$$\le \sum_{K\in\mathcal{K}} k_0 C h_K^2 \|D^2 u\|_{L^2(K)}^2$$

interpolation error estimate

$$\Rightarrow \text{Thus: } \|u-u_h\|_V \le C \|h u''\|_{L^2(0,1)}$$

(f) An a posteriori error estimate takes the form of

$$\|u-u_h\|_V \le \sum_{K\in\mathcal{K}} R_K(u_h)$$

where $R_K(u_h)$ are computable error indicators which measure the error on each element.

An adaptive algorithm for local mesh refinement consists of the following steps:

1. construct initial mesh $\mathcal{K}$
2. obtain FEM approximation, u_h
3. Compute local error indicators, $R_K(u_h)^2$
4. Compute maximum: $m := \max_{K \in \mathcal{K}} R_K(u_h)^2$
5. Mark elements with error over $\gamma \cdot m$, $0 < \gamma < 1$
6. Refine those marked elements and get a new mesh $\mathcal{K}$
7. Return to step 2

stopping criteria: $\|u - u_h\|_V \leq TOL$.

Adaptivity is important because refining the the whole mesh is computationally expensive and we only want to refine the mesh in region where the error is large.

Problem 3

$$\begin{cases} \dot{u} + \Delta u + u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega,\ t > 0 \\ u = u_0 & x \in \Omega,\ t = 0 \end{cases}$$

$$\int_\Omega \frac{\partial u}{\partial t} v + \Delta u v + u v = \int_\Omega f v$$

$$\|$$

$$\cancel{\int_{\partial\Omega} \nabla u v}^{\,0} - \int_\Omega \nabla u \nabla v + \int_\Omega \frac{\partial u}{\partial t} v + \int_\Omega u v = \int_\Omega f v$$

Variational formulation: Find $u(t) \in H_0^1(\Omega)$ such that

$$\left(\frac{\partial u}{\partial t}, v\right) - (\nabla u, \nabla v) + (u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), t \in (0, T_f]$$

and $u(\cdot, 0) = u_0$

We introduce an admissible triangulation T_h of Ω with mesh size h

We define finite element space $V_{h,0}$.

For $\mathbb{P}^1$ Lagrange space, we have

$$V_{h,0} := \{ v \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) : \forall K \in T_h : v|_K \in \mathbb{P}^1(K) \}$$

Semi-discrete formulation: Find $u_h(t) \in V_{h,0}$ such that:

$$\left(\frac{\partial u_h}{\partial t}, v\right) - (\nabla u_h, \nabla v) + (u_h, v) = (f, v) \quad \forall\, v \in V_{h,0}, t \in (0, T_f]$$

with $u_h(\cdot, 0) = u_{0,h} \in V_{h,0}$

We write $u_h(x,t) = \sum_{j \in N_h} \xi_j(t)\, \phi_j(x)$

basis functions of $V_{h,0}$

N_h is the number of interior nodes $\mathcal{N}_{h,0}$

By replacing u_h and v in the semi-discretization formulation by ϕ_i: we have

$$\sum_{j=1}^{N_h} \frac{\partial \xi_j}{\partial t}(\phi_j, \phi_i) - \sum_{j=1}^{N_h} \xi_j (\nabla\phi_j, \nabla\phi_i) + \sum_{j=1}^{N_h} \xi_j (\phi_j, \phi_i) = (f, \phi_i) \quad \forall\, i \in N_h$$

$$\xi_j(0) = u_{0,h}(z_j) \qquad z_j \in \mathcal{N}_{h,0}$$

In algebraic system form:

$$M\dot{\xi}(t) + A\xi(t) + M\xi(t) = b(t)$$
$$\xi(0) = U_{0,h}$$

where: $M_{i,j} = (\phi_j, \phi_i)$

$A_{i,j} = -(\nabla\phi_j, \nabla\phi_i)$

$b_i = (f(t), \phi_i)$

$U_{0,h}(j) = u_{0,h}(z_j)$