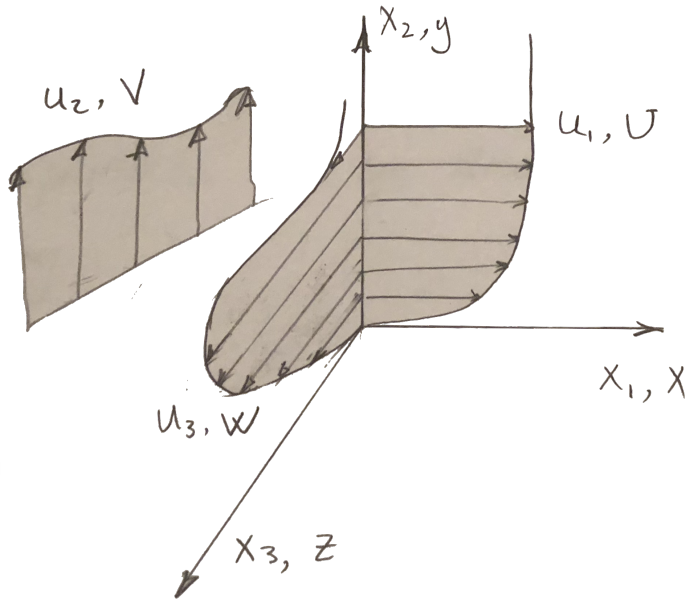
Definitions and notations



<u>Tensor notations:</u>

Space coordinates:

$$x_i = (x_1,\ x_2,\ x_3) = (x,\ y,\ z)$$

Velocity vector:

$$u_i = (u_1,\ u_2,\ u_3) = (u,\ v,\ w)$$

Einstein summation rule:

$$f_i\, g_i = f_1\, g_1 + f_2\, g_2 + f_3\, g_3$$

Nabla operator:

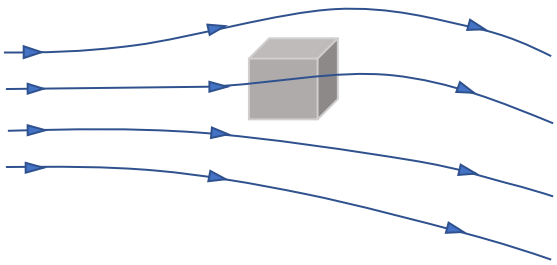
$$\nabla = \frac{\partial}{\partial x_i} = (\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z})$$

Laplace operator:

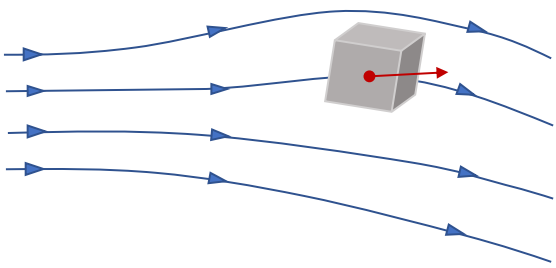
$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Kinematics: Description of motion without regard to forces.

The motion of a particle (fluid element) can be described in different ways.

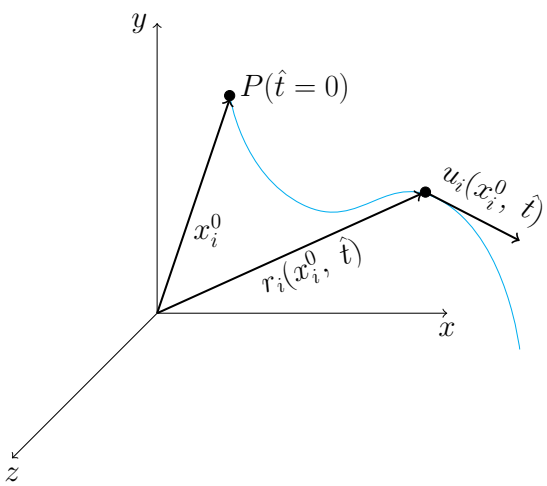


Eulerian coordinates: Infinitesimal fluid element fixed in space with the fluid moving through it.



Lagrangian coordinates: Infinitesimal fluid element moving along a streamline with velocity $\mathbf{u}$ equal to local flow velocity at each point.

Lagrangian coordinates: (classical mechanics)



Every particle is marked and followed in the flow.

Independent variables:

$$\begin{cases} x_i^0 : \text{initial position of the fluid particle} \\ \hat{t} : \text{time} \end{cases}$$

Particle path: $r_i = r_i(x_i^0,\ \hat{t})$

Particle velocity: $u_i = \dfrac{\partial r_i}{\partial \hat{t}}$

When we change x_i^0, we consider a new particle.

<u>Eulerian coordinates:</u>

Consider a fixed point in space, the fluid flows past that point.

Independent variables:

$$\begin{cases} x_i : \text{space coordinate} \\ t : \text{time} \end{cases}$$

Fluid velocity: $u_i = u_i(x_i,\ t)$

Relation between Lagrangian and Eulerian coordinates,
i.e. relation between $(x_i^0,\ \hat{t})$ and $(x_i,\ t)$.

Particle position is expressed in the fixed space coordinates x_i

$\Rightarrow x_i = r_i(x_i^0,\ \hat{t})$ at time $t = \hat{t}$.

Question: *What is the rate of change in time of a given quantity, following the fluid particles, expressed in the Eulerian coordinates?*

Consider some property F following a particle:

$$F = F_L(x_i^0,\ \hat{t}) \quad \text{Lagrangian coord.}$$

$$F = F_E(x_i,\ t) \quad \text{Eulerian coord.}$$

$$x_i = r_i(x_i^0,\ \hat{t}) \Rightarrow F = F_E(r_i(x_i^0,\ \hat{t}),\ t)$$

$$\begin{aligned}
\Rightarrow \frac{\partial F}{\partial \hat{t}} &= \frac{\partial F_E}{\partial r_i}\frac{\partial r_i}{\partial \hat{t}} + \frac{\partial F_E}{\partial t}\frac{\partial t}{\partial \hat{t}} \\
&= \frac{\partial F_E}{\partial x_i}\frac{\partial r_i}{\partial \hat{t}} + \frac{\partial F_E}{\partial t} \\
&= \underbrace{\frac{\partial F_E}{\partial x_i}\ u_i}_{I} + \underbrace{\frac{\partial F_E}{\partial t}}_{II}
\end{aligned}$$

I : change due to motion with velocity u_i
II: local rate of change

Define: $\dfrac{\partial}{\partial \hat{t}} = \dfrac{\mathrm{D}}{\mathrm{D}t} = \dfrac{\partial}{\partial t} + u_i\dfrac{\partial}{\partial x_i} = \dfrac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)$

Called **material derivative**, **substantial derivative** or **total derivative**.

Examples

Velocity:

$$\frac{\mathrm{D}r_i}{\mathrm{D}t} = \{r_i = x_i \text{ in Eulerian coord.}\} =$$

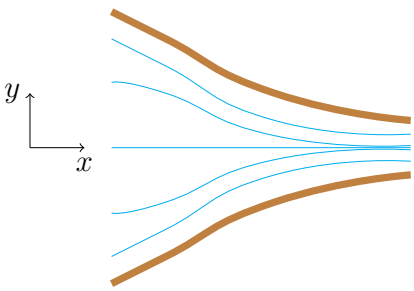
$$= \frac{\partial x_i}{\partial t} + u_j \frac{\partial x_i}{\partial x_j} = \{x_i \text{ is fixed in space}\} =$$

$$= \quad 0 \quad + u_j\, \delta_{ij} \quad = u_i$$

Acceleration:

$$a_i = \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

Consider a steady ($\partial/\partial t = 0$) converging river (2D flow):



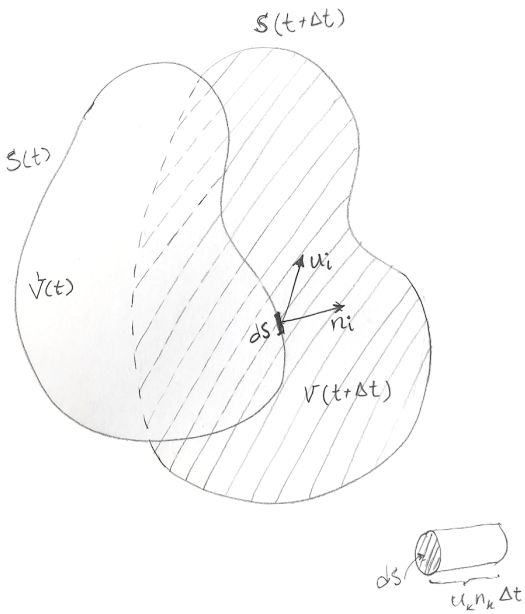
$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

assume $u \gg v$

$$\Rightarrow \quad a_x \approx u\frac{\partial u}{\partial x}$$

Acceleration is not zero even if the velocity at a fixed position does not change with time. A person sitting in a boat will feel the acceleration.

Reynolds transport theorem

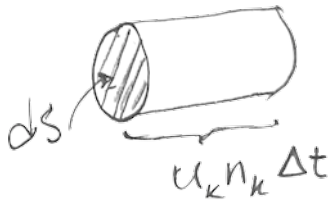


Changes in a material volume during time Δt. The shaded area with solid line corresponds to $V(t+\Delta t) - V(t)$.

Time derivative of a material volume integral:

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{V(t)} F\mathrm{dV} &= \lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\left[\int\limits_{V(t+\Delta t)} F(t+\Delta t)\,\mathrm{dV} - \int\limits_{V(t)} F(t)\,\mathrm{dV}\right]\right\} \\
&= \lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\left[\int\limits_{V(t+\Delta t)} F(t+\Delta t)\,\mathrm{dV} - \int\limits_{V(t)} F(t+\Delta t)\,\mathrm{dV}\right]\right. \\
&\qquad \left. + \frac{1}{\Delta t}\left[\int\limits_{V(t)} F(t+\Delta t)\,\mathrm{dV} - \int\limits_{V(t)} F(t)\,\mathrm{dV}\right]\right\} \\
&= \underbrace{\lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\int\limits_{V(t+\Delta t)-V(t)} F(t+\Delta t)\,\mathrm{dV}\right\}}_{I} + \int\limits_{V(t)} \frac{\partial F}{\partial t}\mathrm{dV}
\end{aligned}
$$

To compute the term I in the expression above we first look to a small volume element:



$$dV = u_k\, n_k\, \Delta t\, \mathrm{d}S$$

$$\begin{aligned}
\Rightarrow I &= \lim_{\Delta t \to 0} \left[\oint_{S(t)} F\left(t + \Delta t\right) u_k n_k \mathrm{d}S \right] = \oint_{S(t)} F u_k n_k \mathrm{d}S \\
&= \{\text{Gauss/divergence theorem}\} = \int\limits_{V(t)} \frac{\partial}{\partial x_k} \left(u_k F\right) \mathrm{dV}
\end{aligned}$$

$$\Rightarrow \boxed{\frac{\mathrm{D}}{\mathrm{D}t} \int\limits_{V(t)} F \mathrm{dV} = \int\limits_{V(t)} \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_k} \left(u_k F\right) \right] \mathrm{dV}}$$

Now, let $F = \rho$ (density)

$$\Rightarrow \frac{\mathrm{D}}{\mathrm{D}t} \int\limits_{V(t)} \rho \mathrm{d}V = \int\limits_{V(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \left(u_k \rho\right) \right] \mathrm{dV} = \{\text{mass conservation}\} = 0$$

Since the volume is arbitrary, the following must hold for the integrand

$$0 = \underbrace{\frac{\partial \rho}{\partial t}}_{1} + \underbrace{\frac{\partial}{\partial x_k} \left(u_k \rho\right)}_{2} = \frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} = \underbrace{\frac{\mathrm{D}\rho}{\mathrm{D}t}}_{3} + \underbrace{\rho \frac{\partial u_k}{\partial x_k}}_{4}$$

where we have used the definition of the material derivative in order to simplify the expression. The terms in the expression can be given the following interpretations:

① : accumulation of mass in a fixed volume
② : net flow rate of mass out of the volume
③ : rate of density change of material element
④ : volume expansion rate of material element

<u>Rate of change of given quantity per unit mass</u>

Let $F = \rho f$

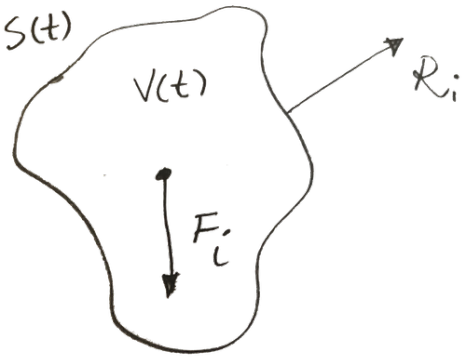
$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{V(t)} \rho f \mathrm{d}V &= \int\limits_{V(t)} \left[\frac{\partial}{\partial t}(\rho f) + \frac{\partial}{\partial x_k}(u_k \rho f)\right] \mathrm{dV} \\
&= \int\limits_{V(t)} \left[f\,\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial f}{\partial t} + \rho u_k\,\frac{\partial f}{\partial x_k} + f\,\frac{\partial}{\partial x_k}(\rho u_k)\right] \mathrm{dV} \\
&= \left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(u_k \rho) = 0 \quad \text{cont. eq.}\right\} \\
&= \int\limits_{V(t)} \rho\,\frac{\mathrm{D}f}{\mathrm{D}t}\mathrm{d}V
\end{aligned}
$$

$$
\Rightarrow \frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{V(t)} \rho f \mathrm{d}V = \int\limits_{V(t)} \rho\,\frac{\mathrm{D}f}{\mathrm{D}t}\mathrm{d}V.
$$

Momentum equation

Newton's second law:

Rate of change of momentum = *sum of forces*



F_i: body forces per unit mass (ex. gravity, centrifugal, Coriolis, electromagnetic)

R_i: Surface forces per unit area (ex. pressure, viscous forces)

The momentum conservation in integral form:

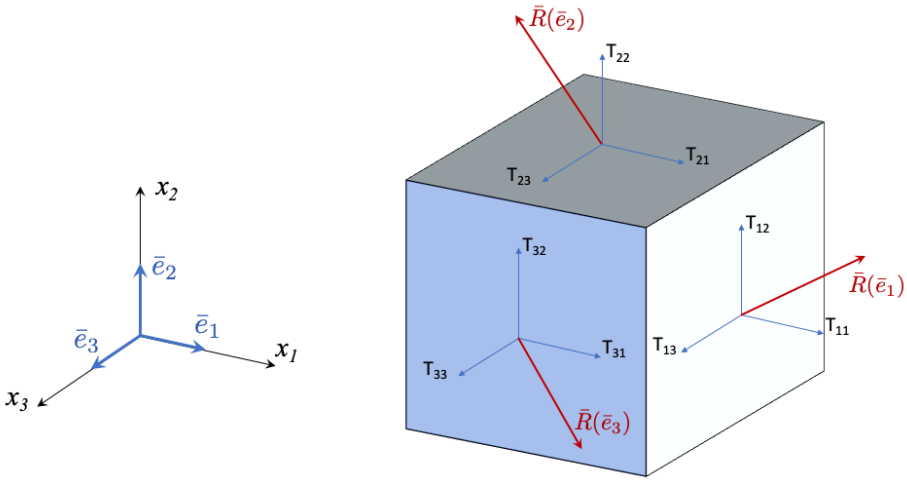
$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{V(t)} \rho u_i \,\mathrm{d}V = \int\limits_{V(t)} \rho F_i \,\mathrm{d}V + \int\limits_{S(t)} R_i \,\mathrm{d}S$$

Reynolds transport theorem gives

$$\int\limits_{V(t)} \rho \frac{\mathrm{D}u_i}{\mathrm{D}t} \,\mathrm{d}V = \int\limits_{V(t)} \rho F_i \,\mathrm{d}V + \int\limits_{S(t)} R_i \,\mathrm{d}S \quad (*)$$

Stress tensor

Consider a fluid element



Divide the surface forces $\bar{R}(\bar{e}_i)$ into components

$$\bar{R}(\bar{e}_i) = (T_{i1},\ T_{i2},\ T_{i3})$$

T_{ij} is called the *stress tensor*.

- diagonal components are called *normal stress*
- off-iagonal components are called *shear stress*
- is a symmetric tensor $T_{ij} = T_{ji}$

Now we can write

$$R_i = T_{ij} n_j = T_{ji} n_j$$

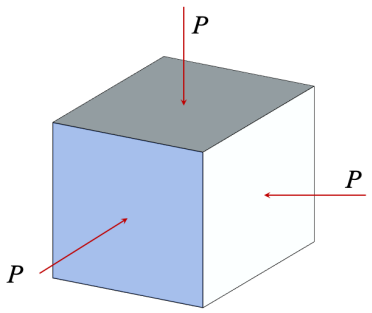
Replace R_i with $T_{ij} n_j$ in (*)

$$
\begin{aligned}
\int\limits_{V(t)} \rho \frac{\mathrm{D}u_i}{\mathrm{D}t} \, \mathrm{d}V &= \int\limits_{V(t)} \rho F_i \, \mathrm{d}V + \int\limits_{S(t)} T_{ij} n_j \, \mathrm{d}S \\
&= \{Gauss\ Theorem\} = \int\limits_{V(t)} \left(\rho F_i \ + \ \frac{\partial}{\partial x_j} T_{ij} \right) \, \mathrm{d}V
\end{aligned}
$$

Should hold for any $V(t)$

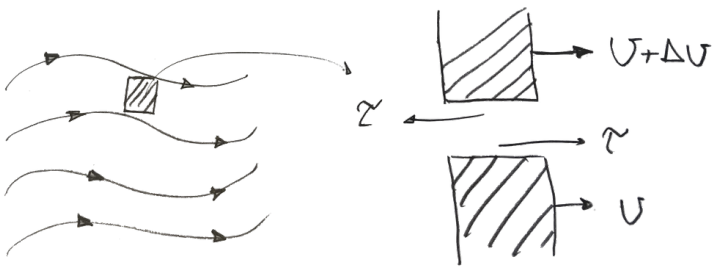
$$
\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho F_i \ + \ \frac{\partial T_{ij}}{\partial x_j}
$$

Pressure and viscous stress tensor



P: hydrodynamic pressure, directed inward.

Consider two neighbouring layer of fluid particle:



$\tau = shear\ stress$
$[\tau] = force/area$

Differences in velocity causes stress.

For a fluid at rest only normal stresses are present, otherwise fluid elements would deform.

Divide the stress tensor as

$$T_{ij} = -P\delta_{ij} + \tau_{ij}$$

where τ_{ij} is viscous stress tensor and depends on motion of fluids.

$$\Rightarrow \rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho F_i$$

Cauchy momentum Equation (French mathematician 1789-1857).

Newtonian fluids

For a Newtonian fluid the viscous stresses are linear functions of deformation rate:

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

λ: second coefficient of viscosity

μ: dynamic viscosity

$\lambda + \frac{2}{3}\mu = \lambda_b$ (bulk viscosity)

Stokes hypotesis: $\lambda_b = 0 \;\Rightarrow\; \lambda = -\frac{2}{3}\mu$
(OK for monoatomic gases)

Then, for a Newtonian fluid the momentum equation reads:

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho F_i$$

This is called Navier-Stokes equation, named after Claude-Louis Navier (French, 1785-1836) George Gabriel Stokes (Irish, 1819-1903).

Energy equation

$$\underbrace{\textit{Rate of change of energy in the material element}}_{(I)} = \underbrace{\textit{Net flux of heat into the element}}_{(II)} + \underbrace{\textit{Rate of work done on the element by body and surface forces}}_{(III)}$$

$$(I) = \frac{\mathrm{D}}{\mathrm{D}t} \int\limits_{V(t)} \rho \left[e + \frac{1}{2} u_i u_i \right] \mathrm{d}V \quad (e \text{ is internal energy})$$

$$(II) = \underbrace{-\int\limits_{S(t)} q_i n_i \, \mathrm{d}S}_{\text{heat loss from surface}} \quad (q_i \text{ is heat flux vector, directed outward})$$

$$(III) = \underbrace{\int\limits_{V(t)} \rho F_i u_i \, \mathrm{d}V + \int\limits_{S(t)} R_i u_i \mathrm{d}S}_{\text{work rate} = \text{force} \cdot \text{velocity}}$$

$$\Rightarrow \frac{\mathrm{D}}{\mathrm{D}t} \int\limits_{V(t)} \rho \left[e + \frac{1}{2} u_i u_i \right] \mathrm{d}V = \int\limits_{V(t)} \rho F_i u_i \, \mathrm{d}V + \int\limits_{S(t)} (R_i u_i - n_i q_i) \, \mathrm{d}S$$

Remember $R_i = T_{ij} n_j = (\tau_{ij} - P\delta_{ij}) n_j$

Using Reynolds transport theorem and Gauss theorem yelds

$$\Rightarrow \rho \frac{\mathrm{D}}{\mathrm{D}t} \left(e + \frac{1}{2} u_i u_i \right) = \rho F_i u_i - \frac{\partial}{\partial x_i}(P u_i) + \frac{\partial}{\partial x_j}(\tau_{ij} u_i) - \frac{\partial q_i}{\partial x_i} \quad (*)$$

Take the dot product of momentum equation with u_i

$$\Rightarrow \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}u_i u_i\right) = \rho F_i u_i - u_i \frac{\partial P}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j} \quad (**)$$

then

$$(*) - (**) \Rightarrow \underbrace{\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -P\frac{\partial u_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}}_{\text{energy equation}}$$

Work of surface forces:

$$-\frac{\partial}{\partial x_i}(P u_i) = -P\frac{\partial u_i}{\partial x_i} - u_i \frac{\partial P}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}(\tau_{ij} u_j) = \underbrace{\tau_{ij}\frac{\partial u_j}{\partial x_i}}_{(I)} + \underbrace{u_j \frac{\partial \tau_{ij}}{\partial x_i}}_{(II)}$$

(I) Thermal terms (force · deformation): heat generated by compression and dissipation.

(II) Mechanical terms (velocity · force gradient): gradient accelerate fluid and increase kinetic energy.

Energy equations in alternative form

$q_i = -K\dfrac{\partial T}{\partial x_i}$, K is the thermal conductivity

Enthalpy $h = e + \dfrac{P}{\rho} \;\Rightarrow \dfrac{\mathrm{D}h}{\mathrm{D}t} = \dfrac{\mathrm{D}e}{\mathrm{D}t} + \dfrac{1}{\rho}\dfrac{\mathrm{D}P}{\mathrm{D}t} - \dfrac{P}{\rho^2}\dfrac{\mathrm{D}\rho}{\mathrm{D}t}$

Consider a calorically perfect gas: $h = c_p T$ (c_p is specific heat coeff. at constant pressure)

Energy eq. $\Rightarrow \;\rho c_p \dfrac{\mathrm{D}T}{\mathrm{D}t} = \dfrac{\mathrm{D}P}{\mathrm{D}t} + \tau_{ij}\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial}{\partial x_i}\left(K\dfrac{\partial T}{\partial x_i}\right)$

Complete system of equations

So far we have 5 equations:

- continuity equation
- 3 momentum equations (in x, y and z directions)
- energy equation

and 6 unknowns:

$$u,\ v,\ w,\ \rho,\ P \text{ and } T$$

The last equation needed to close the system is the equation of state:

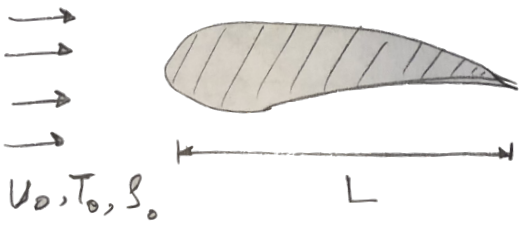
$$P = \rho T \Re \qquad (\Re \text{ is the gas constant})$$

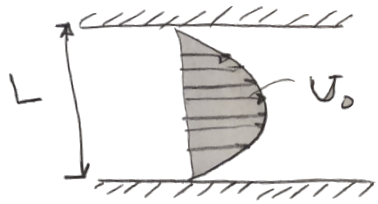
Equations in non-dimensional form

Often is convenient to work with non-dimensional equations.

Let us introduce following reference values:

L	ref. length	$x_i = x_i^* \, L$	$\Rightarrow \quad \dfrac{\partial}{\partial x_i} = \dfrac{1}{L}\dfrac{\partial}{\partial x_i^*}$
U_0	ref. velocity	$u_i = u_i^* \, U_0$	
L/U_0	ref. time	$t_i = t^* \, L/U_0$	
ρ_0	ref. density	$\rho = \rho^* \, \rho_0$	
$\rho_0 U_0^2$	ref. pressure	$p = p^* \, \rho_0 U_0^2$	
μ_0	ref. viscosity	$\mu = \mu^* \, \mu_0$	





Introduce above into the continuity equations

$$\frac{\rho_0 \, U_0}{L} \frac{\partial \rho^*}{\partial t^*} + \frac{\rho_0 \, U_0}{L} \frac{\partial}{\partial x_i^*}(\rho^* u_i^*) = 0.$$

Divide the equation by $\dfrac{\rho_0 \, U_0}{L}$

$$\Rightarrow \frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x_i^*}(\rho^* u_i^*) = 0 \qquad \text{continuity equation in nondimensional form}$$

Introduce nondimensional variables into the momentum equations

$$\frac{\rho_0\, U_0^2}{L}\,\rho^*\frac{\mathrm{D}u_i^*}{\mathrm{D}t^*} = -\frac{\rho_0\, U_0^2}{L}\frac{\partial p^*}{\partial x_i^*} + \frac{\mu_0 U_0}{L^2}\frac{\partial}{\partial x_j^*}\left[\lambda^*\frac{\partial u_k^*}{\partial x_k^*}\delta_{ij} + \mu^*\left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*}\right)\right]$$

Define Reynolds number

$$Re = \frac{\rho_0 U_0 L}{\mu_0} \qquad \left(\frac{\dfrac{\rho_0 U_0^2}{L}}{\dfrac{\mu_0 U_0}{L_0^2}} \quad \frac{\text{inertia forces}}{\text{viscous forces}}\right)$$

Division by $\dfrac{\rho_0\, U_0^2}{L}$ gives

$$\rho^*\frac{\mathrm{D}u_i^*}{\mathrm{D}t^*} = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re}\frac{\partial}{\partial x_j^*}\left[\lambda^*\frac{\partial u_k^*}{\partial x_k^*}\delta_{ij} + \mu^*\left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*}\right)\right]$$

Skip *

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i}(\rho u_i) = 0 \\[2ex] \rho\dfrac{\mathrm{D}u_i}{\mathrm{D}t} = -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{Re}\dfrac{\partial}{\partial x_j}\left[\lambda\dfrac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)\right] \end{cases}$$

Non-dimensional energy equation

Introduce T_0, κ_0 and c_{p_0} as reference values. It yields

$$T = T^* \, T_0, \quad \kappa = \kappa^* \, \kappa_0, \quad c_p = c_p^* \, c_{p_0},$$

introduce them in to the energy equation

$$\Rightarrow \rho^* c_p^* \frac{\mathrm{D}T^*}{\mathrm{D}t^*} = \frac{U_0^2}{c_{p_0} T_0} \left[\frac{\mathrm{D}p^*}{\mathrm{D}t^*} + \frac{\mu_0}{\rho_0 L U_0} \tau_{ij}^* \frac{\partial u_i^*}{\partial x_j^*} \right] + \frac{\kappa_0}{\rho_0 L U_0 c_{p_0}} \frac{\partial}{\partial x_i^*} \left(\kappa^* \frac{\partial T^*}{\partial x_i^*} \right)$$

We can write

$$\frac{U_0^2}{c_{p_0} T_0} = \left\{ c_{p_0} = \frac{\gamma}{(\gamma - 1)} \Re \right\} = \frac{U_0^2 (\gamma - 1)}{\gamma \Re T_0} = \frac{U_0^2 (\gamma - 1)}{a^2} = (\gamma - 1) M_0^2$$

where a is speed of sound and M_0 the Mach number. Further

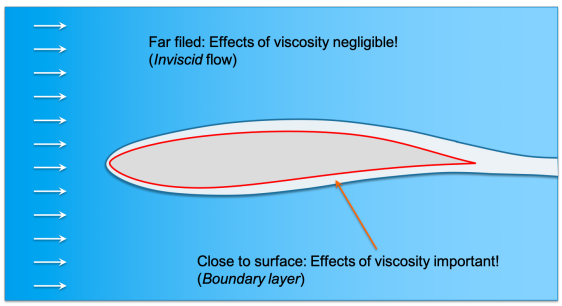
$$\frac{\kappa_0}{\rho_0 L U_0 c_{p_0}} = \frac{\kappa_0}{\mu_0 c_{p_0}} \frac{\mu_0}{\rho_0 L U_0} = \frac{1}{Pr} \frac{1}{Re}$$

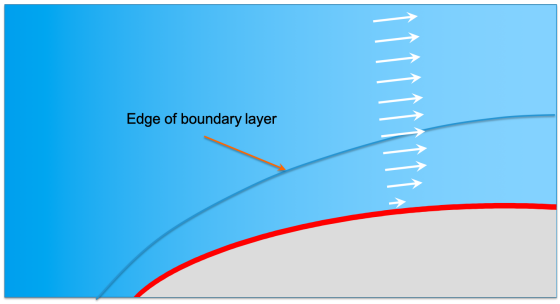
where Pr is the Prandtl number.

Skip $*$

$$\Rightarrow \rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = (\gamma - 1) M_0^2 \left[\frac{\mathrm{D}p}{\mathrm{D}t} + \frac{1}{Re} \tau_{ij} \frac{\partial u_i}{\partial x_j} \right] + \frac{1}{Pr} \frac{1}{Re} \frac{\partial}{\partial x_i} \left(\kappa \frac{\partial T}{\partial x_i} \right)$$

Boundary-layer equations





Far from the wall effect of viscosity is negligible (Inviscid flow):

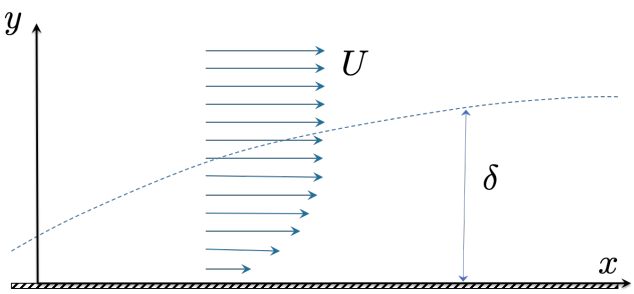
$$Re \;\rightarrow\; \infty \;\Rightarrow\; \text{momentum eq.: } \rho\,\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{\partial p}{\partial x_i} \quad \text{(Euler eq.)}$$

Can we make a similar simplification of equations in the regions close to the wall?

For simplicity, consider a 2D and steady flow ($\partial/\partial t = 0$).

Let us introduce following *scales*:

L :	length scale for	x
δ :	”	y
U_0 :	velocity scale for	u
V_0 :	”	v
ρ_0 :	density scale	



Introduce above scalings in continuity equation:

$$\underbrace{\frac{\partial}{\partial x}(\rho u)}_{\mathcal{O}\left(\frac{\rho_0 U_0}{L}\right)} + \underbrace{\frac{\partial}{\partial y}(\rho v)}_{\mathcal{O}\left[\frac{\rho_0 V_0}{\delta}\right]} = 0$$

Balance of terms: $\dfrac{\rho_0 U_0}{L} \sim \dfrac{\rho_0 V_0}{\delta} \Rightarrow V_0 \sim \dfrac{\delta}{L} U_0$

Cont. eq:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\mathcal{O}\left(\frac{\rho_0 U_0}{L}\right) \quad \mathcal{O}\left(\frac{\rho_0 V_0}{\delta}\right)$$

Balance of terms: $\frac{\rho_0 U_0}{L} \sim \frac{\rho_0 V_0}{\delta} \Rightarrow V_0 \sim \frac{\delta}{L} U_0$

momentum eq.

$$x: \quad \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] +$$

$$\sim \frac{\rho_0 U_0^2}{L} \quad \rho_0 \frac{\delta}{L} \frac{U_0 U_0}{\delta} \quad \sim \frac{\rho_0 U_0^2}{L} \quad 1 \qquad \sim \frac{U_0 \mu_0}{L^2} \frac{1}{Re}$$

$$1 \qquad 1 \qquad + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$\sim \frac{U_0 \mu_0}{L^2} \qquad \sim \frac{\mu_0 U_0}{\delta^2} \quad \frac{\mu_0 U_0}{L^2}$$

$$\frac{1}{Re} \qquad \frac{1}{Re}\left(\frac{L}{\delta}\right)^2 \quad \frac{1}{Re}$$

let $\frac{1}{R}\left(\frac{L}{\delta}\right)^2 = 1 \Rightarrow \frac{\delta}{L} = \frac{1}{\sqrt{Re}}$

other choices do not give a consistent eq

$\Rightarrow$ drop terms of order $\frac{1}{R}$ and higher:

$$\Rightarrow \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y-momentum eq:

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left[\lambda'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] +$$

$$\rho_0 \frac{U_0}{L}\frac{\delta}{L}U_0 \quad \rho_0 \frac{\delta}{L}U_0 \frac{\delta}{L}U_0 \frac{1}{\delta} \quad \frac{\rho_0 U_0^2}{\delta}(1) \quad \frac{\mu_0}{\delta}\frac{U_0}{L}\left(\frac{1}{Re}\right) \quad \frac{\mu_0}{\delta}\frac{\delta}{L}\frac{U_0}{\delta}\left(\frac{1}{Re}\right)$$

$$\left(\frac{\delta}{L}\right)^2 \quad \left(\frac{\delta}{L}\right)^2 + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)\right]$$

multiply by $\frac{\delta}{\rho_0 U_0^2}$

$$\frac{\mu_0}{L}\frac{\delta}{L}U_0\frac{1}{L} \quad \frac{\mu_0}{L}\frac{U_0}{\delta} \quad \frac{\mu_0}{\delta}\frac{\delta}{L}U_0\frac{1}{\delta}$$

$$\frac{1}{Re}\left(\frac{\delta}{L}\right)^2 \quad \left(\frac{1}{Re}\right) \quad \left(\frac{1}{Re}\right)$$

$\Rightarrow$

$Re \gg 1$ and $\frac{\delta}{L} \ll 1$

Balance of $O(1)$ terms gives:

$$\frac{\partial P}{\partial y} = 0$$

pressure is constant inside the boundary layer

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \\ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y}\right] \\ \text{Thermal Boundary-Layer eq. (homework)} \end{cases}$$

these equation are parabolic and can be solve by marching along x-direction

Solution Procedure:

1. Solve inviscid (Euler) eq to get pressure and velocity on the surface. (Same as the pressure at edge of the boundary layer)

2- Solve boundary-layer eqs. with initial and boundary conditions

I.C. : $u(x = x_0, y) = u_{in}(y)$

B.C. : $u(x, y=0) = 0$

$v(x, y=0) = 0$ (or $= v_w$ if suction/blowing)

$u(x, y \to \infty) = u_e(x)$

Boundary-layer eqs as $y \to \infty$ gives

$$\rho_e u_e \frac{\partial u_e}{\partial x} = -\frac{dP}{dx}$$