Midterm Review Session

- → AR(1) vs RW
- → MA(2)
- → ARMA (1,1)
- → Trend / Seasonality

Part I

- → Forecast Evaluation.
- → optimal forecasts
- → Answer questions

Part II

$\varepsilon_T \varepsilon_S$ where $T \neq S$

$E[\varepsilon_t \varepsilon_{t-1}] = 0$

AR(1)

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t \quad ; \quad \varepsilon_t \sim N(0,1) \text{ iid}$$

$N(0,\sigma^2)$ iid

WN

$$y_{T+1} = \alpha + \phi y_T + \varepsilon_{T+1}$$

$$y_{T+1|T} = \alpha + \phi y_T \leftarrow \text{Forecast}$$

$(y_{T+1} - y_{T+1|T})$

$$E[(y_{T+1} - y_{T+1|T})^2] = E[(\varepsilon_{T+1})^2] = 1$$

$$y_{T+2} = \alpha + \phi y_{T+1} + \varepsilon_{T+2} = \alpha(1+\phi) + \phi^2 y_T + \varepsilon_{T+2} + \phi\varepsilon_{T+1}$$

$$y_{T+2|T} = \alpha + \phi y_{T+1|T} = \alpha + \phi[\alpha + \phi y_T]$$

$$y_{T+2|T} = \alpha + \alpha\phi + \phi^2 y_T = \alpha(1+\phi) + \phi^2 y_T$$

$$E[(y_{T+2} - y_{T+2|T})^2] = E[(\phi(y_{T+1} - y_{T+1|T}) + \varepsilon_{T+2})^2]$$

$E[\varepsilon_T \varepsilon_S] = 0 \ \forall T \neq S$ $\quad = 1 + \phi^2$

$$E[(\varepsilon_{T+2} + \phi\varepsilon_{T+1})^2] =$$
$$E[\varepsilon_{T+2}^2] + \phi^2 E[\varepsilon_{T+1}^2] +$$
$$2\phi E[\varepsilon_{T+1}\varepsilon_{T+2}] = 0$$

Recursive Substitution

$$y_{T+3} = \alpha + \phi y_{T+2} + \varepsilon_{T+3}$$
$$= \alpha + \phi[\alpha + \phi y_{T+1} + \varepsilon_{T+2}] + \varepsilon_{T+3}$$
$$= \alpha + \phi[\alpha + \phi[\alpha + \phi y_T + \varepsilon_{T+1}] + \varepsilon_{T+2}] + \varepsilon_{T+3}$$
$$= \alpha + \alpha\phi + \phi^2[\alpha + \phi y_T + \varepsilon_{T+1}] + \phi\varepsilon_{T+2} + \varepsilon_{T+3}$$
$$= \alpha + \alpha\phi + \alpha\phi^2 + \phi^3 y_T + \phi^2\varepsilon_{T+1} + \phi\varepsilon_{T+2} + \varepsilon_{T+3}$$
$$= \underbrace{\sum_{i=0}^{2}\phi^i\alpha + \phi^3 y_T}_{} + \underbrace{\sum_{i=0}^{2}\phi^i\varepsilon_{T+3-i}}_{}$$

$$y_{T+3|T} = \alpha + \phi y_{T+2|T} =$$
$$= \alpha + \phi[\alpha + \phi y_{T+1|T}]$$
$$= \alpha + \phi[\alpha + \phi[\alpha + \phi y_T]]$$
$$= \alpha + \alpha\phi + \alpha\phi^2 + \phi^3 y_T$$
$$= \alpha(1 + \phi + \phi^2) + \phi^3 y_T$$
$$= \underbrace{\sum_{i=0}^{2}\phi^i\alpha + \phi^3 y_T}_{} = \alpha\underbrace{\sum_{i=0}^{2}\phi^i}_{} + \phi^3 y_T$$

$$E[(y_{T+3} - y_{T+3|T})^2] = E[\underbrace{(\sum_{i=0}^{2}\phi^i\varepsilon_{T+3-i})^2}_{}]$$
$$= 1 + \phi^2 + \phi^4 + \phi^6 \dots \quad \frac{1}{1-\phi^2}$$

prove this

MSE $T \to \infty$ AR(1)

Can you forecast ∞ ?

$$\lim_{h\to\infty} y_{T+h} = \lim_{h\to\infty} \alpha\sum_{i=0}^{h}\phi^i + \phi^h y_T + \sum_{i=0}^{h}\phi^i\varepsilon_{T+h-i}$$
$$\lim_{h\to\infty} y_{T+h|T} = \lim_{h\to\infty} \alpha\sum_{i=0}^{h}\phi^i + \phi^h y_T$$
$$= \alpha \cdot \frac{1}{1-\phi} \Rightarrow \text{Convergence}$$

The above works for stationary processes i.e. $\phi < 1$. Now let's see what happens when $\phi = 1$; Unit Root Processes

$$y_t = \delta + y_{t-1} + \epsilon_t \quad ; \ \phi = 1$$

← Random walk with drift ; set $\delta = 0$

$$y_{T+1} = \delta + y_T + \epsilon_{T+1}$$

$$y_{T+1|T} = \delta + y_T$$

$$E[(y_{T+1} - y_{T+1|T})^2] = E[(\epsilon_{T+1})^2] = 1$$

$$y_{T+2} = \delta + y_{T+1} + \epsilon_{T+2}$$

$$= \delta + \delta + y_T + \epsilon_{T+1} + \epsilon_{T+2}$$

$$= 2\delta + y_T + \epsilon_{T+1} + \epsilon_{T+2}$$

$$y_{T+2|T} = \delta + y_{T+1|T}$$

$$= \delta + \delta + y_T = 2\delta + y_T$$

$$E[(y_{T+2} - y_{T+2|T})^2] = E[(\epsilon_{T+1} + \epsilon_{T+2})] = 2$$

$$y_{T+3} = 3\delta + y_T + \epsilon_{T+1} + \epsilon_{T+2} + \epsilon_{T+3}$$

$$y_{T+3|T} = 3\delta + y_T$$

$$E[(y_{T+3} - y_{T+3|T})^2] = 3$$

MSE increases linearly w/ horizon.

$$\lim_{h \to \infty} y_{T+h} = \lim_{h \to \infty} \left(h\delta + y_T + \sum_{i=1}^{h} \epsilon_{T+i} \right) \to \infty$$

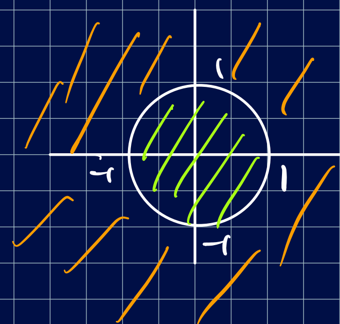
$$E[y_t] = E[\alpha + \phi y_{t-1} + \varepsilon_t] \quad \varepsilon_t \sim \mathcal{N}(0,1)$$

$$\eta = \alpha + \phi E[y_{t-1}] + 0$$

$$\eta = \alpha + \phi \eta$$

$$(1-\phi)\eta = \alpha$$

$$\eta = \frac{\alpha}{1-\phi} \iff \eta \to E[y_t]$$



$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$$

$$(1-\phi L) y_t = \varepsilon_t$$

↳ root of this expression

$$(1 - \phi L - \phi L^2) y_t$$

root of this needs to be outside the unit root circle

$$\lim_{h\to\infty} y_{T+h|T} = \lim_{h\to\infty} (h\delta + y_T) \to \infty$$ Set $\delta = 0$

$$\lim_{h\to\infty} E[(y_{T+h} - y_{T+h|T})^2] = \lim_{h\to\infty} h \to \infty.$$

This is why it is called a unit root and we say it is non-stationary.

Let's take the case $\phi = 0.3 < 1$ and $\alpha = 1$

$$y_t = 1 + 0.3\, y_{t-1} + \varepsilon_t$$

Let $y_T = 10$

$$y_{T+1} = 1 + 0.3\,(10) + \varepsilon_T$$

$$y_{T+1|T} = 1 + 0.3(10) = 1 + 3 = 4$$

$$MSE = E[(y_{T+1} - y_{T+1|T})^2] = E[\varepsilon_T^2] = 1$$

$$y_{T+2} = 1 + 0.3\, y_{T+1} + \varepsilon_{T+2}$$

$$y_{T+2|T} = 1 + 0.3\, y_{T+1|T}$$
$$= 1 + 0.3\,(4) = 1 + 1.2 = 2.2$$

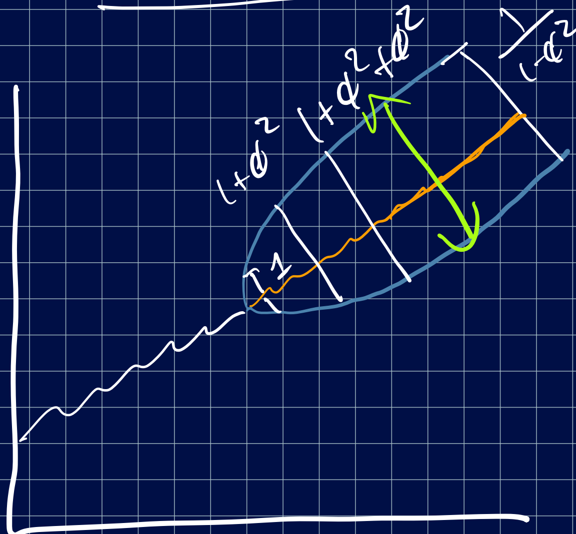
or $= (1 + 0.3) + (0.3)^2\,(10)$ $\quad y_{T+2} = \alpha(1+\phi) + \phi^2 y_T$, $\alpha = 1$, $\phi = 0.3$

$$= 1.3 + (0.09)\,10$$
$$= 1.3 + 0.9 = 2.2$$

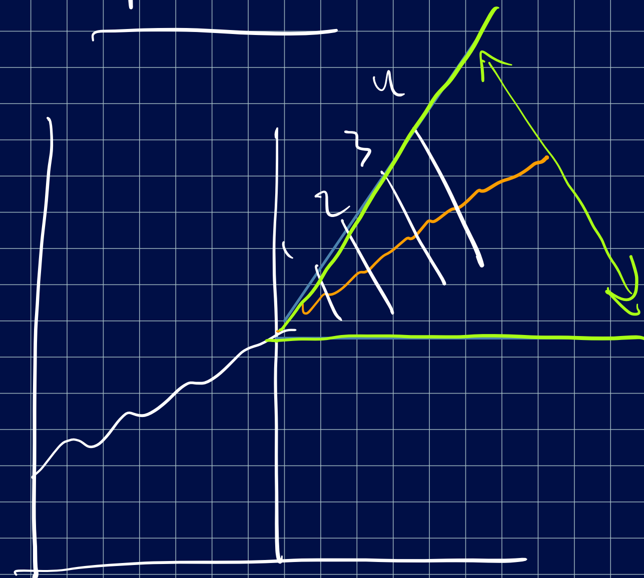
$$MSE = 1 + (0.3)^2 = 1.09$$

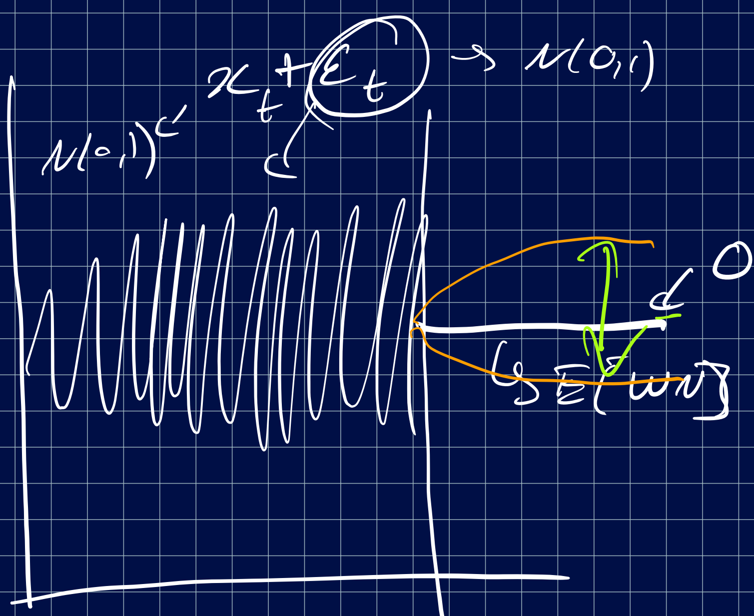
Now you can take this and apply to any α and ϕ as long as $\phi < 1$.

AR(1) => stationary



RW





$x \overset{c}{\sim} WN \to N(0,1)$

$E[x] = 0$

Moving Average (MA) Processes

Suppose we have an MA(2) process.

$$\Rightarrow y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim WN$$

$$y_{T+1} = \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1}$$

what happened to ε_{T+1} ?

$$y_{T+1|T} = \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1}$$

$$MSE_{T+1} = E[(\varepsilon_{T+1})^2] = 1$$

$$y_{T+2} = \underbrace{\varepsilon_{T+2} + \theta_1 \varepsilon_{T+1}}_{\text{you do not know this}} + \theta_2 \varepsilon_T$$

$$y_{T+2|T} = \theta_2 \varepsilon_T$$

$$MSE_{T+2} = E[(\varepsilon_{T+2} + \theta_1 \varepsilon_{T+1})^2] = 1 + \theta_1^2$$

$$y_{T+3} = \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2} + \theta_2 \varepsilon_{T+1}$$

$$y_{T+3|T} = 0 \qquad \Big| \; E[\overset{0}{\varepsilon_{T+3}} + \theta_1 \overset{0}{\varepsilon_{T+2}} + \theta_2 \overset{0}{\varepsilon_{T+1}} \,|\, I_T] = 0$$

$$MSE_{T+3} = E[(\varepsilon_{T+3} + \theta_1 \varepsilon_{T+2} + \theta_2 \varepsilon_{T+1})^2] = 1 + \theta_1^2 + \theta_2^2$$

What happens if we want to forecast 4 steps-ahead?

What is the forecast; what is the MSE?

Suppose we have the same model $\alpha = 0$

$$y_t = \alpha + \varepsilon_t + 0.1\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} \qquad \theta_1 = 0.1, \quad \theta_2 = 0.5$$

Let $\varepsilon_T = 10$, $\varepsilon_{T-1} = 5$; $\varepsilon_{T-2} = 1$

$$y_{T+1} = \varepsilon_{T+1} + 0.1\,(10) + 0.5\,(5)$$

$$y_{T+1|T} = 1 + 2.5 = 3.5$$

$$MSE_{T+1} = 1$$

$$y_{T+2} = \varepsilon_{T+2} + 0.1\,\varepsilon_{T+1} + 0.5\,\varepsilon_T$$

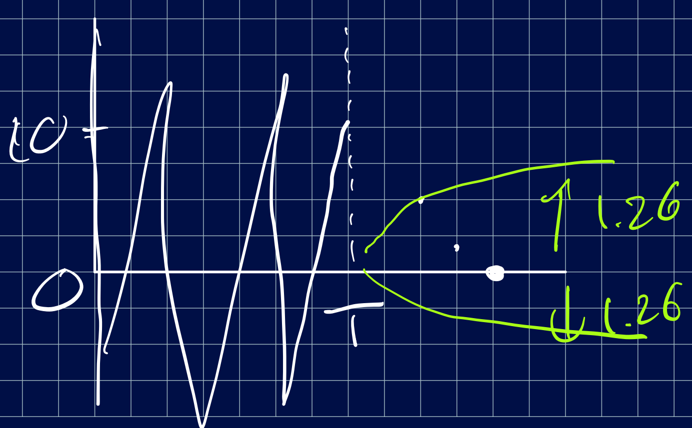
$$y_{T+2|T} = (0.5)(10) = 5$$

$$MSE_{T+2} = 1 + (0.1)^2 = 1.01$$

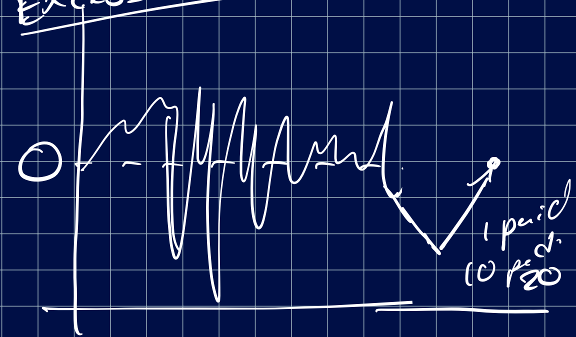
$$y_{T+3} = \varepsilon_{T+3} + 0.1\,\varepsilon_{T+2} + 0.5\,\varepsilon_{T+1}$$

$$y_{T+3|T} = 0$$

$$MSE_{T+3} = 1 + (0.1)^2 + (0.5)^2$$
$$= 1 + 0.01 + 0.25 = 1.26$$



Excess return



ARMA Processes

Let's take the case we have an ARMA (1,1)

$$\Rightarrow y_t = \underbrace{\alpha + \phi y_{t-1}}_{AR} + \underbrace{\varepsilon_t + \theta \varepsilon_{t-1}}_{MA\ part}$$

$$y_{T+1} = \alpha + \phi y_T + \varepsilon_{T+1} + \theta \varepsilon_T$$

$$\rightarrow y_{T+1|T} = \alpha + \phi y_T + \theta \varepsilon_T$$

$$MSE_{T+1} = E[(y_{T+1} - y_{T+1|T})^2]$$

$\varepsilon_t \sim wn\ N(0,1)$ iid.

$$= E[\varepsilon_{T+1}^2] = 1$$

$$y_{T+2} = \alpha + \phi y_{T+1} + \varepsilon_{T+2} + \theta \varepsilon_{T+1}$$

$$y_{T+2|T} = \alpha + \phi y_{T+1|T} + 0$$

$$MSE_{T+2} = E[(y_{T+2} - y_{T+2|T})^2]$$

$$= E[(\phi(y_{T+1} - y_{T+1|T}) + \varepsilon_{T+2} + \theta \varepsilon_{T+1})^2]$$

$$= E[(\phi \varepsilon_{T+1} + \theta \varepsilon_{T+1} + \varepsilon_{T+2})^2]$$

$$= E[(\varepsilon_{T+1}(\phi + \theta) + \varepsilon_{T+2})^2]$$

$$= (\phi + \theta)^2 + 1$$

Now let's see if we can use recursion.

$$\to y_{T+1} = \alpha + \phi y_t + \varepsilon_T + \theta \varepsilon_{T-1}$$

$$y_{T+2} = \alpha + \phi\, y_{T+1} + \varepsilon_{T+2} + \theta\, \varepsilon_{T+1}$$

$$= \alpha + \phi[\alpha + \phi y_T + \varepsilon_{T+1} + \theta\, \varepsilon_T]$$

$$+ \varepsilon_{T+2} + \theta\, \varepsilon_{T+1}$$

$$= \underbrace{\alpha(1+\phi) + \phi^2 y_T}_{} + \phi\, \varepsilon_{T+1} + \phi\theta\varepsilon_T$$

$$+ \varepsilon_{T+2} + \theta\, \varepsilon_{T+1}$$

AR(1)

$$= \boxed{\alpha(1+\phi) + \phi^2 y_T} + \varepsilon_{T+2} + (\phi+\theta)\varepsilon_{T+1} + \phi\theta\varepsilon_T$$

$$1 + (\phi+\theta)^2$$

$$y_{T+2|T} = \alpha + \phi\, y_{T+1|T}$$

$$= \alpha + \phi[\alpha + \phi y_T + \theta \varepsilon_T]$$

AR

$$= \boxed{\alpha(1+\phi) + \phi^2 y_T} + \boxed{\phi\theta\varepsilon_T}$$

$$y_{T+2} - y_{T+2|T} = \varepsilon_{T+2} + (\phi+\theta)\varepsilon_{T+1} \Rightarrow \text{forecast error}$$

$$E[(\varepsilon_{T+2} + (\phi+\theta)\varepsilon_{T+1})^2] = 1 + (\phi+\theta)^2$$

$$y_{T+3} = \alpha + \phi\, y_{T+2} + \varepsilon_{T+3} + \theta\, \varepsilon_{T+2}$$

$$= \alpha + \phi[\alpha + \phi[\alpha + \phi y_T + \varepsilon_{T+1} + \theta\varepsilon_T] + \varepsilon_{T+2} + \theta\varepsilon_{T+1}]$$

$$+ \varepsilon_{T+3} + \theta\, \varepsilon_{T+2}$$

$$= \alpha(1+\phi+\phi^2) + \phi^3 y_T + \phi[\phi\varepsilon_{T+1} + \phi\theta\varepsilon_T]$$

$$+ \phi\varepsilon_{T+2} + \phi\theta\varepsilon_{T+1} + \varepsilon_{T+3} + \theta\, \varepsilon_{T+2}$$

$$= \alpha \sum_{i=0}^{2} \phi^i + \phi^3 y_T + \underbrace{\phi^2 \varepsilon_{T+1}} + \theta\phi^2\varepsilon_T + \underbrace{\phi\varepsilon_{T+2}} + \underbrace{\phi\theta\varepsilon_{T+1}}$$

$$+ \varepsilon_{T+3} + \underbrace{\theta\varepsilon_{T+2}}$$

$$y_{T+3} = \alpha \sum_{i=0}^{2} \phi^i + \phi^3 y_T + \varepsilon_{T+3} + (\phi+\theta)\varepsilon_{T+2} + (\phi^2+\theta\phi)\varepsilon_{T+1} + \theta\phi^2\varepsilon_T$$

$$y_{T+3|T} = \alpha + \phi y_{T+2|T}$$
$$= \alpha + \phi[\alpha + \phi y_{T+1|T}]$$
$$= \alpha + \phi[\alpha + \phi[\alpha + \phi y_T + \theta\varepsilon_T]]$$
$$= \underbrace{\alpha \sum_{i=0}^{2} \phi^i + \phi^3 y_T}\, + \underbrace{\phi^2\theta\varepsilon_T}_{\text{MA component}}$$

$$y_{T+3} - y_{T+3|T} = \varepsilon_{T+3} + (\phi+\theta)\varepsilon_{T+2} + (\phi^2+\theta\phi)\varepsilon_{T+1}$$

$$MSE_{T+3} = 1 + (\phi+\theta)^2 + (\phi^2+\theta\phi)^2$$

Question: what is the limit of the forecast as $h \to \infty$. Hint think of AR processes.

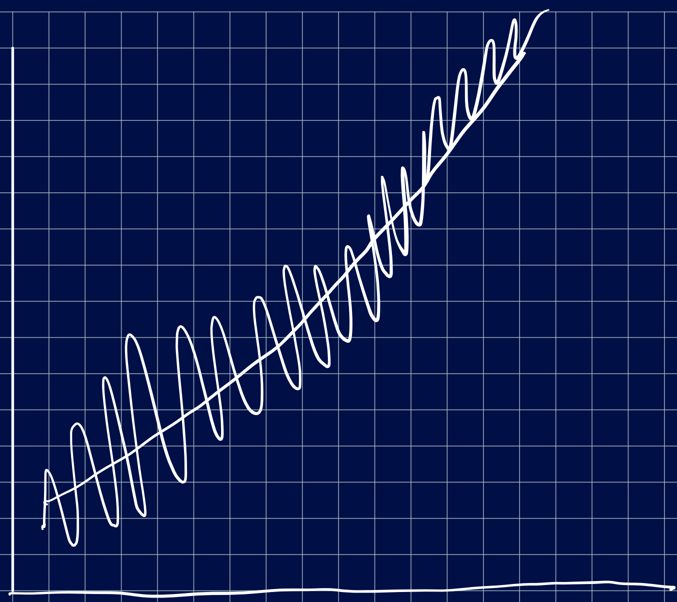
Solve Midterm Problems !!!

Trend and seasonality

Suppose we have a semi-annual series, we believe that it has some deterministic seasonal and trend component. we can write down our model as follows:

$$y_t = \alpha + \beta_1 \tau_t + \beta_2 \tau_t^2 + \underbrace{\omega_1 D_{1t} + \omega_2 D_{2t}} + \varepsilon_t$$

Every component of this model is deterministic.



Linear trend

quadratic

expected

$$\ln(y_t) = \alpha + \tau_t$$

$$y_t = e^{\alpha + \tau}$$

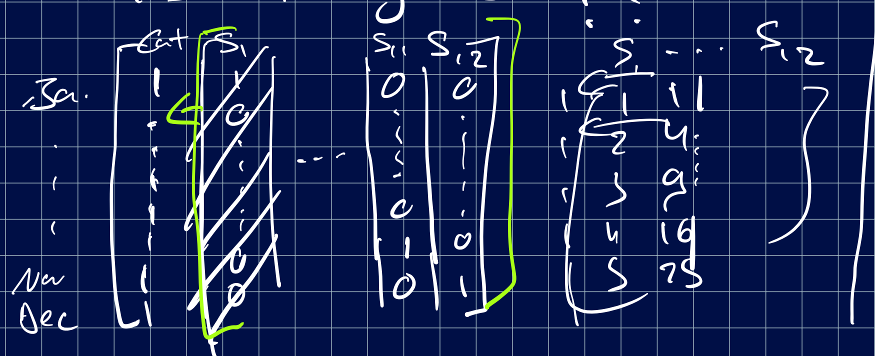
The effect at the same month of each year is the same. <=> I have a deterministic seasonal component.

$$y_t = \alpha + w_2 S_2 + w_3 S_3 + \dots + w_{12} S_{12} + \varepsilon_t$$

Mean of the variable in season 1.

w_2 is measure how different the mean is relative to season 1.

Why can't I have a constant and 12 seasonal dummies in the regression??



$$\hat{\beta} = (X'X)^{-1} X'Y$$

$$
\begin{array}{ccc}
A & B & C \\
\end{array}
$$
$$
\begin{bmatrix}
1 & 3 & 3 \\
1 & 2 & 3 \\
1 & 1 & 3 \\
1 & 5 & 3
\end{bmatrix}
$$

Column B has additional information ∴ therefore not multicollinear with A

Column C is multiple of A ∴ C is multicollinear w/ A.

$$R_t = \alpha_S + \alpha_B + \alpha_{PE} + \alpha_{PD} + \alpha_{Rec} + \varepsilon_t$$

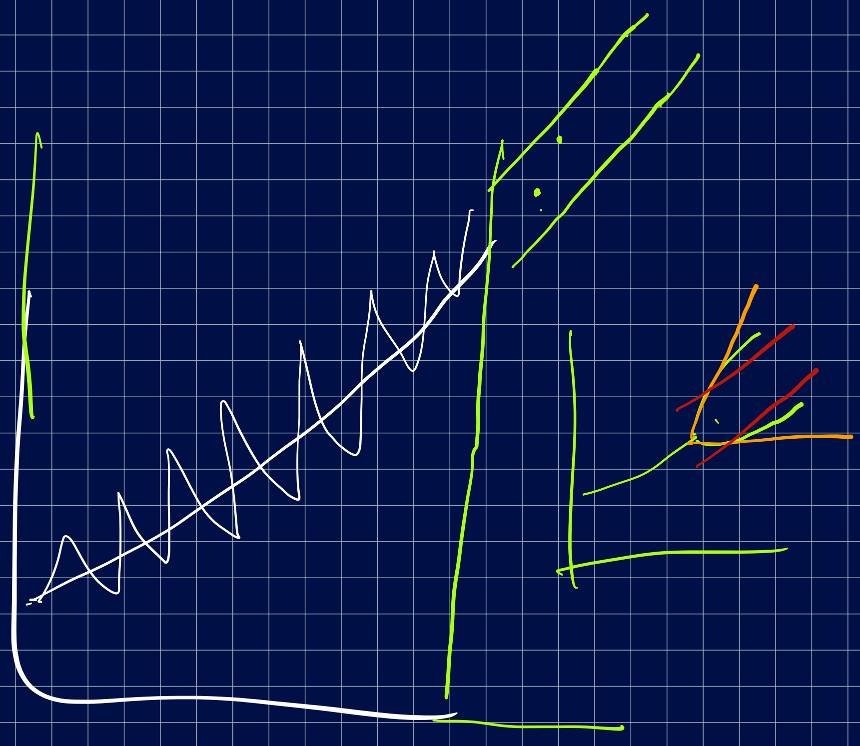
$$R_t = \underline{\underline{\alpha}} + \underline{\underline{\omega_B}} D_B + \omega_{PE} D_{PE} + \omega_{PC} D_{PD} + \omega_{Rec} D_{Rec}$$

$\alpha \to$ mean returns for stocks

$\omega_B \leftarrow$ deviation / relative rev of bond

$\omega_{PE} \leftarrow$ " " PE

⋮



Suppose we fit this model to our data, and we get the following estimates:

$\beta_1 = 4$ $\quad$ $w_1 = 2$ $\quad$ $D_{1t} = 1$ if H_1; 0 otherwise

$\beta_2 = 0.1$ $\quad$ $w_2 = 1$ $\quad$ $D_{2t} = 1$ if H_2; 0 otherwise

D_{it} is an indicator variable (binary).

So our model looks like this:

$$y_t = 4\tau_t + 0.1\tau_t^2 + 2D_{1t} - 5D_{2t} + \varepsilon_t \sim WN$$

T : December 2020 : 20 = T

We are going to forecast:

T+1 : June 2021 $\quad$ At T+2; $\tau = 21$ forecasting June

T+2 : Dec. 2021

$$y_{T+1|T} = 4(21) + 0.1(21)^2 + 2 = 128.1 + 2 = 130.1$$

$$y_{T+2|T} = 4(22) + 0.1(22)^2 + 1 = 137.4$$

$$MSE_{T+1} = E[(y_{T+1} - y_{T+1|T})^2] = E[\varepsilon_{T+1}^2] = 1$$

MSE is going to be the same b/w T+1 and T+2, b/c this is a deterministic model.

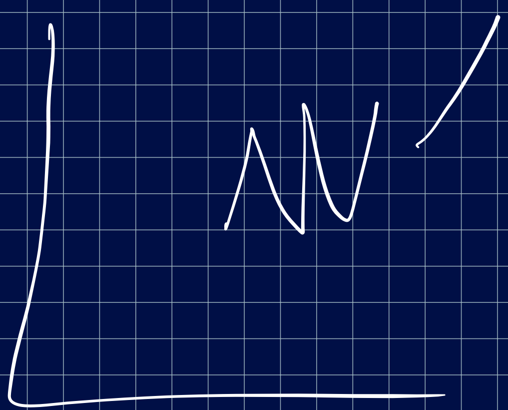
All I need is which time period I am on.

There are no dynamics, and no recursions.

$$y_{T+3|T} = (23)4 + 0.1(23)^2 + 2$$

$$= 92 + 52.9 + 2$$

$$= 146.9$$



PART II

$e = y - f_3$

Forecast Evaluation

Suppose you have 2 forecasters. They each submit their best model

	Y	F1	F2	e_1	e_2
T+1	10	7	9	3	1
T+2	8	9	6	-1	2
T+3	-5	0	-1	-5	-4
T+4	3	1	1	2	2
T+5	0	-5	2	5	-2

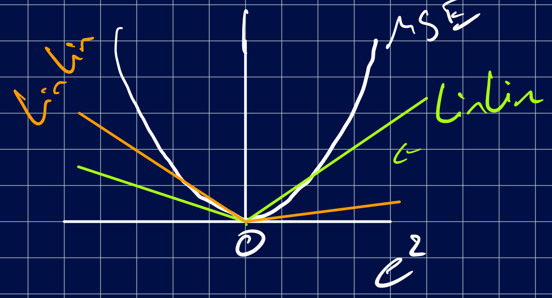
$$MSE_1 = \frac{3^2 + 1^2 + 5^2 + 2^2 + 5^2}{5} = 12.8$$

$$MSE_2 = \frac{1^2 + 2^2 + 4^2 + 2^2 + 2^2}{5}$$

$$= 5.8$$

$MSE_2 < MSE_1$ ∴ F_2 better forecast

$$MSE = L((y - \hat{y})^2)$$



FE(1):

$\begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix}$

$$L(e) = \exp(e) - e - 1$$

$$L(FE_1) = \exp(-2) + 2 - 1 + \exp(1) - 1 - 1 + \exp(-2) + 2 - 1 + \exp(1) - 1 - 1$$

$$2[\exp(-2) + 2 - 1] + 2[\exp(1) - 2]$$

$$2[0.1353 + 2 - 1] + 2[2.7183 - 2]$$

$$2.2706 + 1.4366 = 3.7072$$

$$\frac{3.7072}{4} = 0.9268$$

FE(2)

$\begin{pmatrix} 2 \\ 0 \\ -1 \\ -3 \end{pmatrix}$

$$L(e) = \exp(e) - e - 1$$

$$\left.\begin{aligned} \exp(2) - 2 - 1 &= 4.39 \\ \exp(0) - 0 - 1 &= \underline{\underline{0}} \\ \exp(-1) \cancel{+1-1} &= 0.368 \\ \exp(-3) + 3 - 1 &= 2.04978707 \end{aligned}\right\} / 4$$

$$= \underline{1.70}18005$$

1.70

Forecast Optimality

An optimal forecast needs to have 0 bias and need to be correlated 1 to 1 w/ the actual value forecasted.

We can test these optimality conditions using the Mincer - Zarnowitz regression.

Suppose we generate forecasts using a general model:

$$f_{T+1|T} = g(z_t)$$

We would like to use the MZ regression to test for optimality.

$$y_{T+1} = \alpha + \phi f_{T+1|T} + \varepsilon_{T+1}$$

Under optimality it must be that:

$H_0: \alpha = 0$ & $\phi = 1$

Further suppose we get the following results

$\rightarrow$ $y_{T+1} = 4 + 0.4 f_{T+1|T} + \varepsilon_{T+1}$ $\leftarrow$

Are the forecasts optimal?

$$y_{T+1} = 0 + 1 f_{T+1|T} + \varepsilon_{T+1} \leftarrow$$

Is this optimal?

YES

$$y_{T+1} = \alpha + \phi y_T + \varepsilon_{T+1}$$

$$y_{T+1|T} = \alpha + \phi y_T$$

$$y_{T+1} - y_{T+1|T} = \varepsilon_{T+1}$$

$$y_{T+2} - y_{T+2|T+1} = \varepsilon_{T+2}$$

$$y_{T+3} - y_{T+3|T+2} = \varepsilon_{T+3}$$

$\varepsilon_T \sim N(0, \hat{\sigma}^2)$

Cask t = 0

$$E[\varepsilon_{T+1}\, \varepsilon_{T+2}] = 0$$

no serial correlation

$$\begin{bmatrix} y & \hat{y}^{\text{youk.}} \\ & \end{bmatrix}$$

Can you tell me if my forecasts are optimal?

$MSE_{\text{Yahi}} = 0.02$

$MSE_{\text{Linlin}} = 0.01$

$$y_{T+1} = \beta_1 \cdot x_{1,t} + \varepsilon_{1,t+1}$$

$$y_{T+1} = \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \varepsilon_{T+1}$$

$U_t \sim WN(0,1)$

1) → $\varepsilon_t^{gdm} = \alpha + U_t$; $\alpha = 0$

$H_0 : \alpha = 0$ ← No bias condition

$H_1 : \alpha \neq 0$

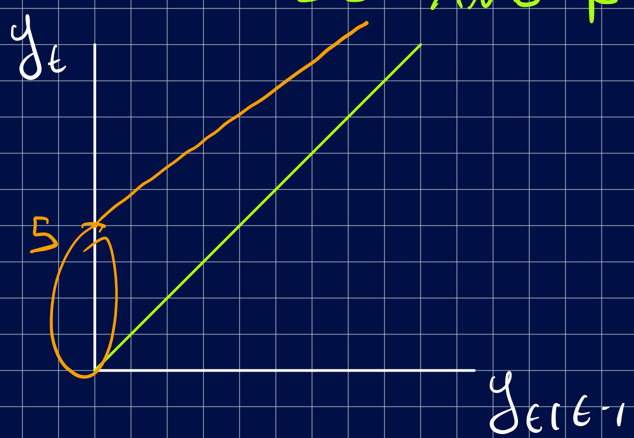
2) $[y_t - y_{t|t-1}] = \alpha + U_t$

$y_t - \beta y_{t|t-1} = \alpha + U_t$

$\beta = 1$: Assumption

2) $y_t = \alpha + \beta y_{t|t-1} + U_t$

$\alpha = 0$ AND $\beta = 1$



$\alpha = 5$

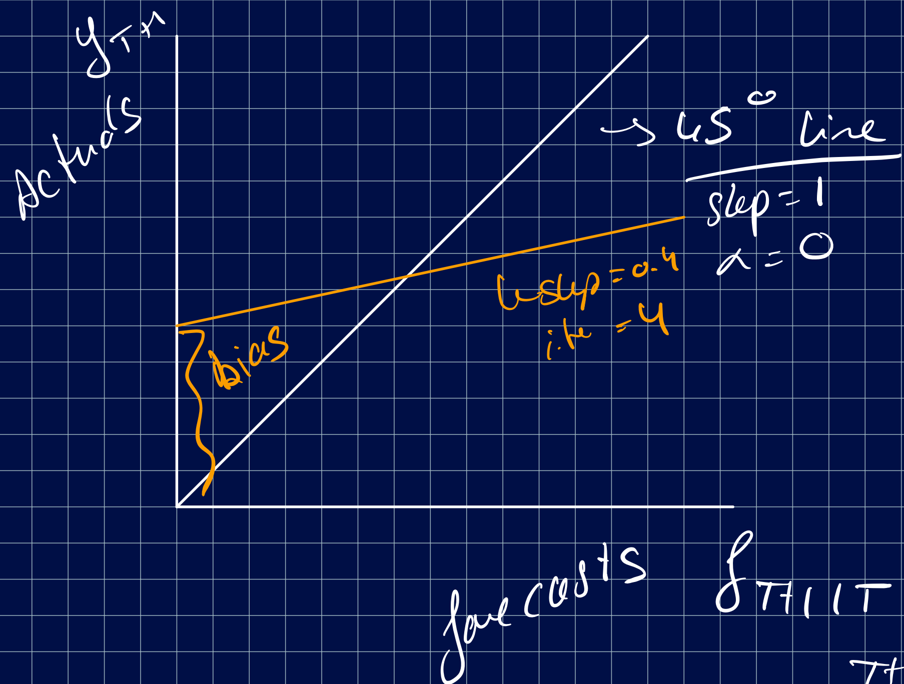
$\beta = 0.8$

$$\boxed{\text{Under MSE loss} \quad \alpha = 0, \quad \beta = 1}$$

Optimal forecast

$f_{T+1|T}$; $\overbrace{f_{T+2|T+1}}^{h=1}$; $\overbrace{f_{T+2|T}}^{h=2}$

Not this



How can you make my forecasts optimal?

$T+1 = 10$

$4 + 0.4(10)$

$$\boxed{f^*_{T+1|T} = 4 + 0.4 f_{T+1|T}}$$

$f_{T+1\|T}$	$f^*_{T+1\|T}$
10	8
8	7.2
7	6.8
6	6.4
5	6

$$f^*_{t+1|t} = \hat{\alpha} + \hat{\phi} f_{t+1|t}$$

$\hat{\alpha} = 0$

$\hat{\phi} = 1$

$$\Rightarrow f^*_{t+1|t} = f_{t+1|t}$$

Objective is to forecast & inflation

NN(3) [6 ; 3 ; 2] ←

$$
\hookrightarrow \begin{array}{cc} y & f \end{array} \\
\left[\begin{array}{cc}
y_0 & f_0 \\
y_1 & f_1 \\
y_2 & f_2 \\
y_3 & f_3 \\
y_4 & f_4
\end{array}\right]
$$

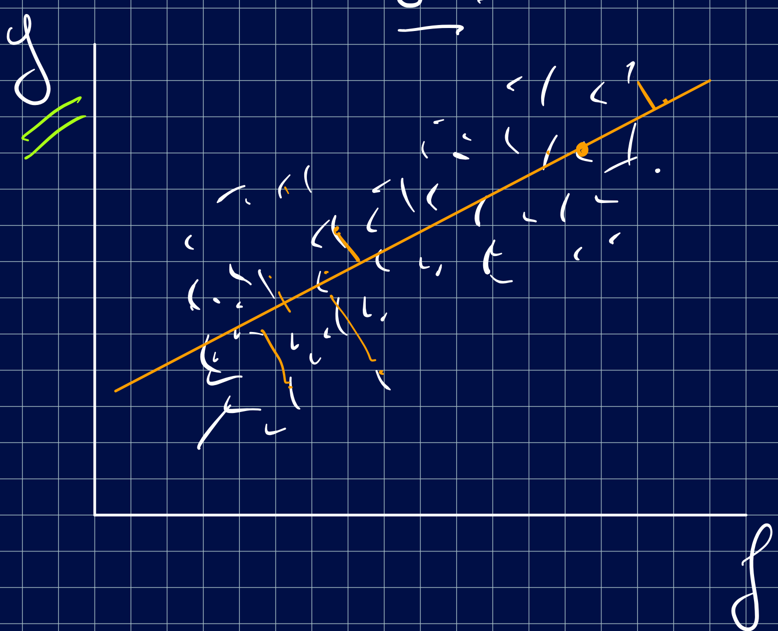
$$y_t = \alpha + \beta f_t + \varepsilon_t$$

$$\hat{\alpha} = 6$$

$$\hat{\beta} = 0.8$$

w.r.t. f $\quad \hat{\alpha} = 0$

$\hat{\phi} = 1$



State Space Models

We have seen the following model:

$$y_{t+1} - \mu = \phi (y_t - \mu) + \varepsilon_{t+1} , \varepsilon_{t+1} \sim WN(0,1)$$

We can represent the above model as: capacitian form

State equation

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

obs equation

$$y_t = \mu + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \end{bmatrix}$$

$$\xi_t = \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \end{bmatrix} ; \quad F = \begin{bmatrix} \phi_1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} ; \quad e_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

$$\xi_{t+1} = F \cdot \xi_t + e_{t+1} \quad : \text{State equation}$$

$$y_t = \mu + H \xi_t \quad : \text{observation equation.}$$

Let's do an example:

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 (y_t - \mu) \\ y_t - \mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

$$y_{t+1} - \mu = \phi(y_t - \mu) + \varepsilon_{t+1} \Leftarrow$$

$$y_t - \mu = y_t - \mu \Leftarrow$$

Suppose we want to model Exchange Rates and inflation. We can look at it from the perspective of the consumer.

AR(1) → RW w/drift

$$XR_t = 1 + XR_{t-1} + \varepsilon_{XR,t} \quad \text{: State equation}$$

$$\pi_t = \boxed{1.1}\, XR_t + \varepsilon_{\pi,t} \quad \text{: obs. eqn.}$$

!!!

$$XR_T = 3$$

$$XR_{T+1|T} = 1 + XR_T = 4$$
$$XR_{T+2|T} = 1 + XR_{T+1|T} = 5$$
$$XR_{T+3|T} = 1 + XR_{T+2|T} = 6$$

Step 1: get forecasts of state

$$\pi_{T+1|T} = (1.1)\, XR_{T+1|T} = 4.4$$
$$\pi_{T+2|T} = (1.1)\, XR_{T+2|T} = 5.5$$
$$\pi_{T+3|T} = (1.1)\, XR_{T+3|T} = 6.6$$

Step 2: plug in obs. eqn. to get forecasts

Questions ?

$T=1$

$$y_{T+1} = 1+2+\frac{1}{2}2^2+\frac{1}{2}1$$

$$y_{T+1|T} = 3+2+0.5 = 5.5$$

$(0,0,0)$ → $y_t = \alpha$

$(1,0,0)$ → $y_t = \alpha + \phi y_{t-1}$

$(2,0,0)$ → $y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2}$

$(0,0,1)$ → $y_t = \alpha + \varepsilon_t + \theta \varepsilon_{t-1}$

$(0,0,2)$ → $y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$

$(1,0,1)$ → $y_t = \alpha + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$

$(1,0,2)$

$(2,0,1)$

$(2,0,2)$

$$y_t = \alpha + \cancel{\phi_1 y_{t-1}} + \phi_2 y_{t-2}$$

$$y_t = \alpha + \phi y_{t-2}$$

$$y_t = \alpha + \varepsilon_t + \theta \varepsilon_{t-2}$$

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-2}$$

$$y_{t+1} = \mu_{S_{t+1}} + \sigma_{S_{t+1}} \varepsilon_{t+1}$$

$$Pr(S_{t+1} = 1 \mid S_{t=1}) = P_{11} = 0.6 \quad \Big| \quad P_{12} = 0.4$$

$$Pr(S_{t+1} = 2 \mid S_{t=2}) = P_{22} = 0.4 \quad \Big| \quad P_{21} = 0.6$$

$$\mu_1 = -10 \; ; \; \sigma_1 = 5$$

$$\mu_2 = 6 \; ; \; \sigma_2 = 2$$

$$Pr(S_T = 1 \mid I_t) = 0$$

$$Pr(S_T = 2 \mid I_t) = 1$$

$$y_{T+1|T} = 0.6[-10] + 0.4(6)$$

$$= -6 + 2.4 = \boxed{-3.6}$$

$$y_{T+2|T} = 0.6^2(-10) + (0.4)^2(6)$$

$$+ 0.6 \times 0.4(6) + 0.4 \times 0.6(-10)$$

$$= .36(-10) + (.16)(6)$$

$$+ .24(-10) + .24(6)$$

$$= -10(.6) + 6(.4)$$

$$= -6 + 2.4 = -3.6$$

$$P_{11} \; ; \; P_{12} \; ; \; P_{21} \; ; \; P_{11}. \quad \leftarrow \quad P1 = 0.5$$

$$P2 = 0.2$$

$$y_{T+1} = P1(P_{11} \times y_T) + P_1(P_{12}\, y_T)$$

$$y_{T+2} = P_1(P_{11} \times P_{11}\ y_T) +$$
$$P_1(P_{11} \times P_{12}\ y_T) +$$
$$P_2(P_{22} \times P_{22}\ y_T) +$$
$$P_2(P_{22} \times P_{21}\ y_T) +$$
$$P_2(P_{21} \times P_{12}\ y_T) +$$

$$P_1 \times P_{11} \times P_{11}\ y_T$$
$$P_1\ \ P_{12}\ \ P_{22}\ \ y_T$$
$$P_1\ \ P_{11}\ \ P_{12}\ \ y_T$$
$$P_1\ \ P_{12}\ \ P_{21}\ \ y_T$$

$$P_2 \times P_{22} \times P_{22}\ y_T$$
$$+$$
$$P_2 \times P_{21} \times P_{11}\ y_T$$
$$+$$
$$P_2 \times P_{22} \times P_{21}\ y_T$$
$$P_2 \times P_{21} \times P_{21}\ y_T$$

$$E[y_T]$$

$$+ P_2\, y_T$$

$$P_{22}\, y_T \qquad P_{21}\, y_T$$