# Problem 3: Transformation from a Uniform Distribution

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The Box-Muller transform is a popular sampling method for generating pairs of independent standard, normally distributed random variables from a pair of independent, standard uniformly distributed random numbers.

Specifically, Let $U_1 \sim Uniform(0,1)$ and $U_2 \sim Uniform(0,1)$ and define the random variables:

$$\Theta = 2\pi U_1$$

$$R = \sqrt{-2\ln(U_2)}$$

Show that the random variable vector

$$Z = \begin{pmatrix} R\cos(\Theta) \\ R\sin(\Theta) \end{pmatrix}$$

follows a standard bivariate normal distribution (i.e. $Z \sim \mathcal{N}(0, I_2)$).

Then, create a plot to verify that the transformed distribution is indeed Gaussian. You should start with `np.random.rand()` to sample from a uniform distribution.

## Solution

Suppose Z is the random vector

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

where $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$, which is literally just a point in 2D Cartesian space, that is represented in polar coordinates.

The goal is to show that the Joint Probability Density Function (PDF) $f_{X,Y}(x, y)$ produced by a Box Muller Transform is the exact same as the PDF of the standard bivariate normal distribution. Let's derive both.

The Covariance matrix is given as the identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means that $X$ and $Y$ and independent. So the joint probability density function is defined as

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} \tag{1}$$

So now we just have to show that the PDF resulting from the Box-Muller Transform is exactly that. So what is $f_{\Theta,R}(\theta, r)$, the polar version of $f_{X,Y}(x,y)$?

Since $U_1$ and $U_2$ are independent by assumption, $R$ and $\Theta$ must be independent because $\Theta = f(U_1)$ and $R = g(U_2)$, they are just some functions of the two variables. So

$$f_{\Theta,R}(\theta, r) = f_{\Theta}(\theta) f_R(r)$$

by the definition of independence for joint probability distributions.

We know that $U_1 \sim Uniform(0,1)$, so that $\Theta = 2\pi U_1$ must be $\Theta \sim Uniform(0, 2\pi)$, so its pdf is

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

For $R = \sqrt{-2\ln U_2}$, we can compute the CDF for $R$ and from there derive the pdf.

$$\begin{aligned} F_R(r) &= P(R \leq r) \\ &= P(\sqrt{-2\ln U_2} \leq r) \\ &= P(-2\ln U_2 \leq r^2) \\ &= P(2\ln U_2 \geq -r^2) \\ &= P(\ln U_2 \geq \frac{-r^2}{2}) \\ &= P(U_2 \geq e^{\frac{-r^2}{2}}) \\ &= 1 - P(U_2 \leq e^{\frac{-r^2}{2}}) \end{aligned}$$

Since we know that $U_2 \sim Uniform(0,1)$, its pdf is

$$f_{U_2}(u_2) = \begin{cases} 1 & 0 \leq u_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

which means the cumulative distribution function (cdf) is

$$F_{U_2}(u_2) = \begin{cases} u_2 & 0 \leq u_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} F_R(r) &= 1 - P(U_2 \leq e^{\frac{-r^2}{2}}) \\ &= 1 - e^{\frac{-r^2}{2}} \end{aligned}$$

Differentiating the cdf to get the pdf,

$$\begin{aligned}\frac{d}{dr}F_R(r) &= f_R(r)\\ &= re^{\frac{-r^2}{2}}, r \geq 0\end{aligned}$$

So in summary,

$$f_R(r) = re^{\frac{-r^2}{2}}, r \geq 0$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\implies f_{\Theta,R}(\theta, r) &= f_\Theta(\theta) f_R(r)\\ &= \frac{1}{2\pi} re^{\frac{-r^2}{2}} \qquad \text{where } 0 \leq \theta < 2\pi, r \geq 0\end{aligned}$$

In order to convert $f_{\Theta,R}(\theta, r) \rightarrow f_{X,Y}(x, y)$, we perform the change of variables

$$X = R\cos\Theta, \quad Y = R\sin\Theta$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Theta} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \Theta} \end{vmatrix} = \begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = R(\cos^2\Theta + \sin^2\Theta) = R$$

In order to perform $\{\theta, r\} \rightarrow \{x, y\}$, we need to multiply the new distribution with the jacobian of the transformation, which becomes

$$\begin{aligned}f_{\Theta,R}(\theta, r) &= f_{X,Y}(x, y) \cdot r\\ \frac{1}{2\pi} re^{\frac{-r^2}{2}} &= f_{X,Y}(x, y) \cdot r\\ \frac{1}{2\pi} e^{\frac{-r^2}{2}} &= f_{X,Y}(x, y)\end{aligned}$$

Since $x = rcos(\theta)$, $y = rsin(\theta)$, $r = \sqrt{x^2 + y^2}$ according to rectangular to polar transformation, so

$$\frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}} = f_{X,Y}(x, y)$$

which was exactly the pdf of the standard bivariate normal distribution (equation (1)) defined in the beginning.

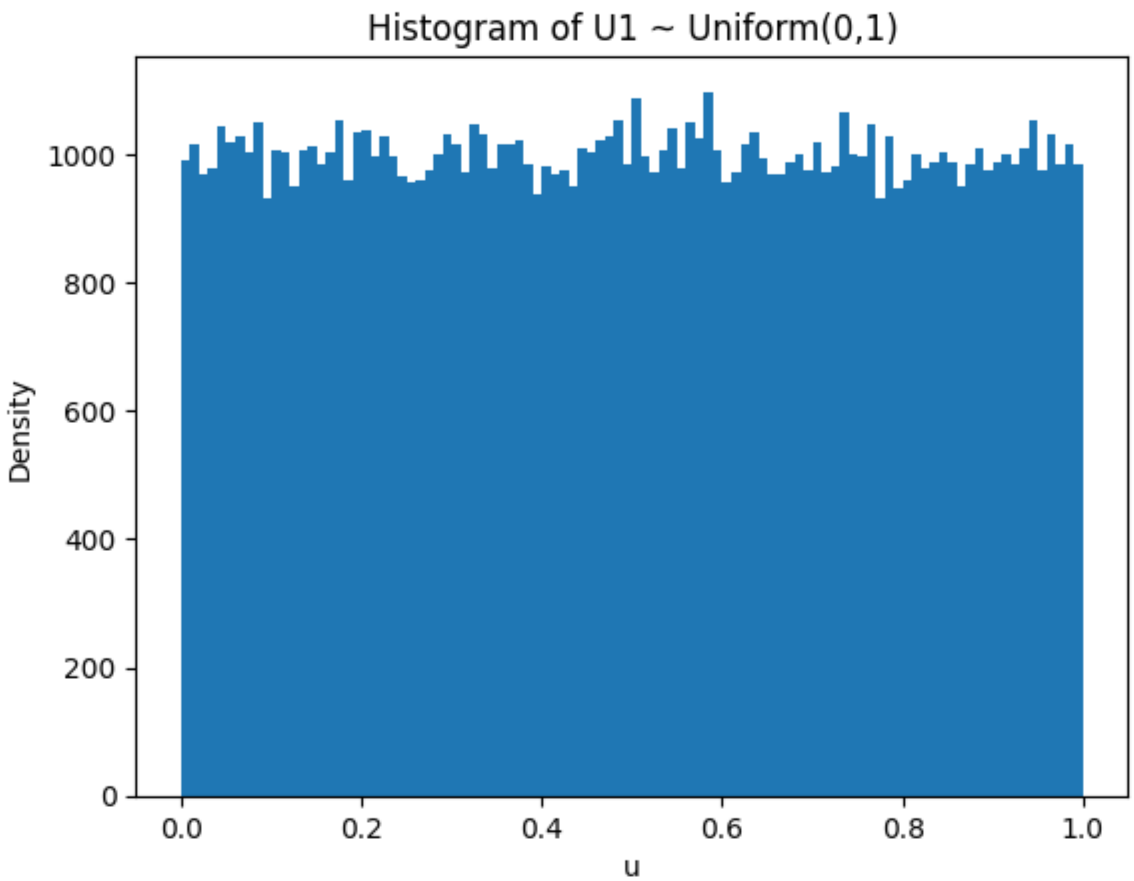
# Plotting

In [44]:
```
import numpy as np
import matplotlib.pyplot as plt
```

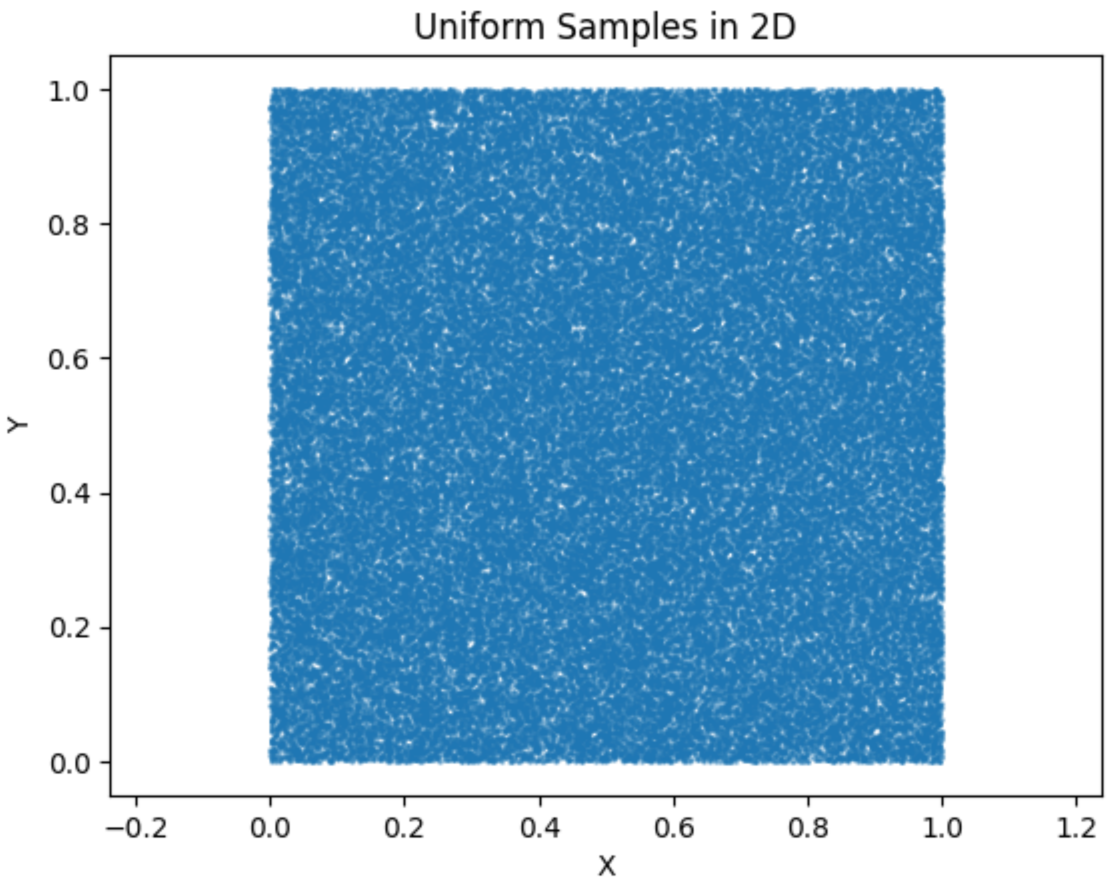
```python
# Number of Samples
N = 100000

U1 = np.random.rand(N)
U2 = np.random.rand(N)
```

In [45]:
```python
plt.figure()
plt.hist(U1, bins=100)
plt.xlabel("u")
plt.ylabel("Density")
plt.title("Histogram of U1 ~ Uniform(0,1)")
plt.show()
```



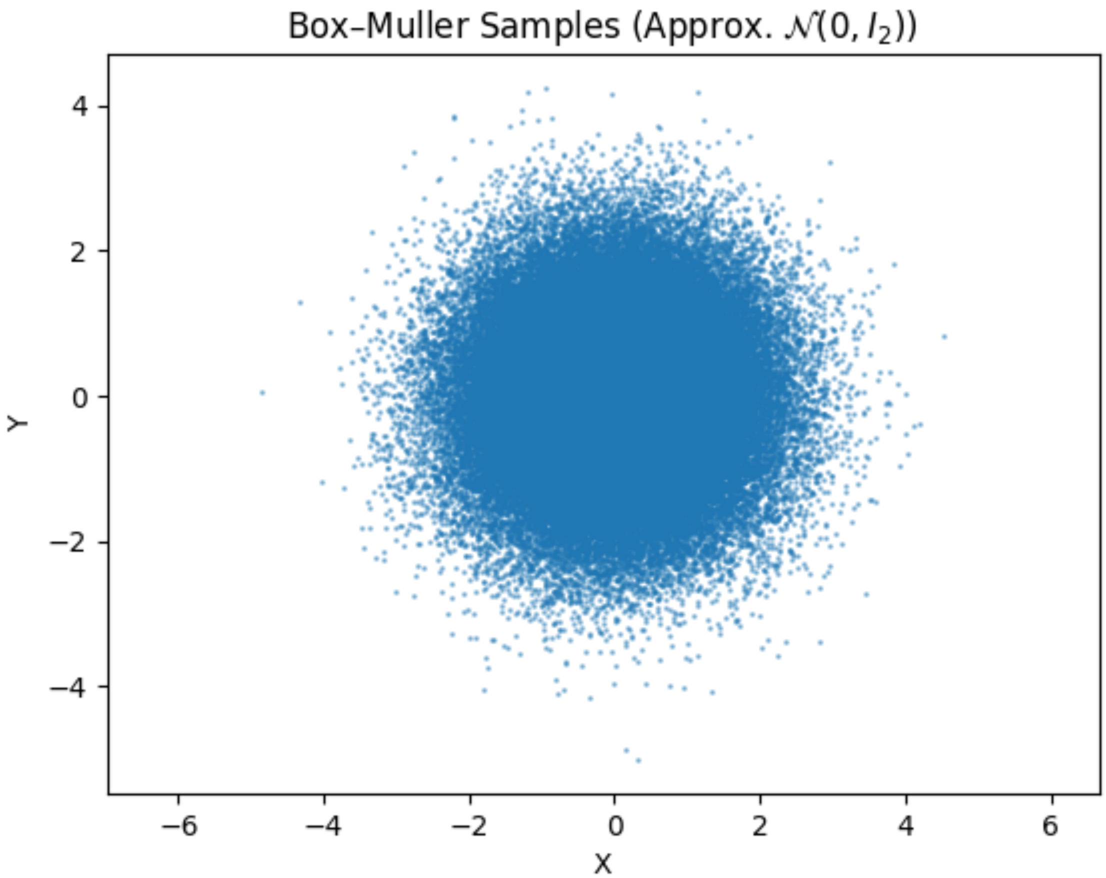

In [54]:
```python
# 2D scatter plot (joint distribution)
plt.figure()
plt.scatter(U1, U2, s=1, alpha=0.4)
plt.xlabel("X")
plt.ylabel("Y")
plt.title("Uniform Samples in 2D")
plt.axis("equal")
plt.show()
```



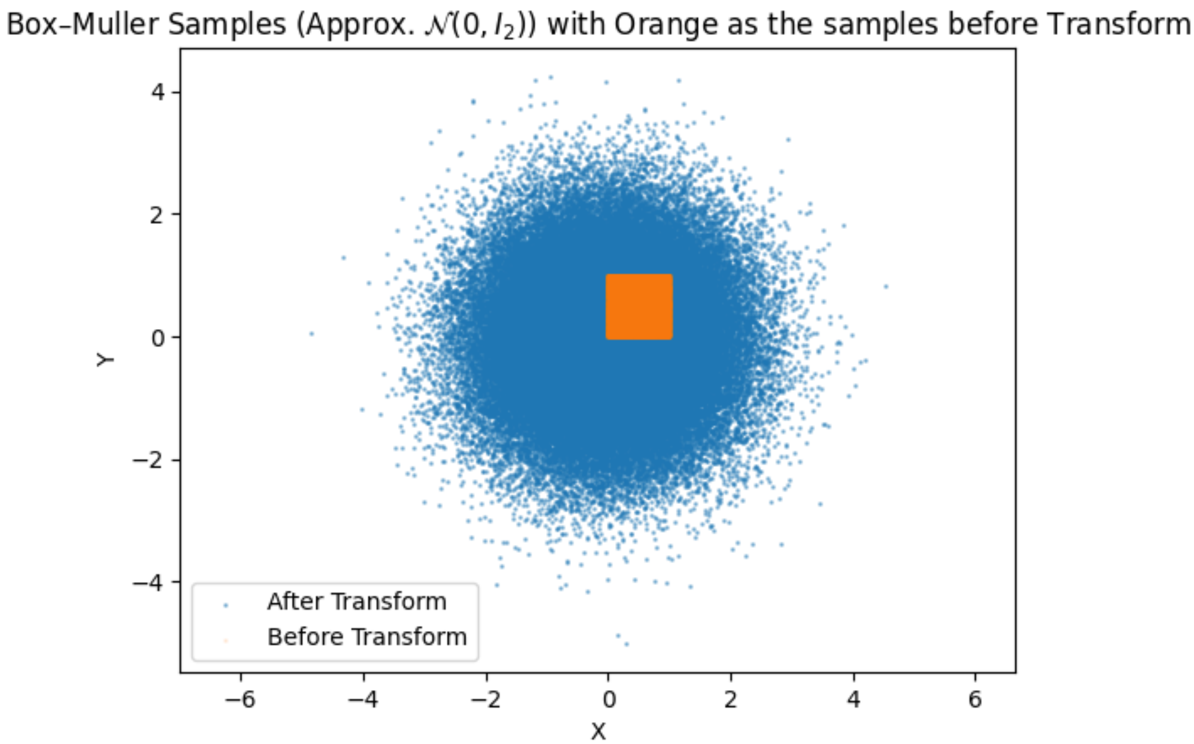

Indeed, these 100,000 draws looks pretty uniformly distributed to me.

In [47]:
```
# Box-Muller Transform
Theta = 2 * np.pi * U1
R = np.sqrt(-2 * np.log(U2))
X = R * np.cos(Theta)
Y = R * np.sin(Theta)
```

In [53]:
```
# 2D scatter plot (joint distribution)
plt.figure()
plt.scatter(X, Y, s=1, alpha=0.4)
plt.xlabel("X")
plt.ylabel("Y")
plt.title("Box-Muller Samples (Approx. $\mathcal{N}(0, I_2)$)")
plt.axis("equal")
plt.show()
```




In [59]:
```
# 2D scatter plot (joint distribution)
plt.figure()
plt.scatter(X, Y, s=1, alpha=0.4)
plt.scatter(U1, U2, s=1, alpha=0.1)
plt.xlabel("X")
plt.ylabel("Y")
plt.legend(["After Transform", "Before Transform"])
plt.title("Box–Muller Samples (Approx. $\mathcal{N}(0, I_2)$) with Orange as the sa
plt.axis("equal")
plt.show()
```




The little orange square of uniformly distributed sample points became a symmetrically distributed gaussian blob of points in this 2D cartesian plane, verifying that Box-Muller algorithm is correct, simple, and fast.