Markov Chain Monte Carlo

As the question, the Markov Chain Monte Carlo in Metropolis-Hastings algorithm with probability density function given

$$f(x) = \frac{1}{2}\exp(-|x|)$$

where x takes values in the real line and $|x|$ denotes the absolute value of x to generate $x_1, x_1, \ldots, x_N$

First, conceptualize this distribution: The Laplace probability density function is defined as

$$f(x) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

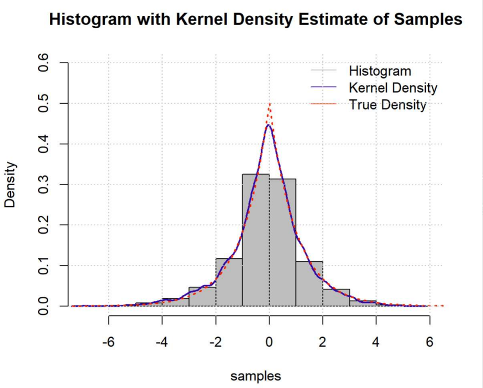
where μ is the location parameter here is equal to 0, and b is the scale parameter as 1, which is very similar to the given function. Therefore, the peak at $x = 0$.

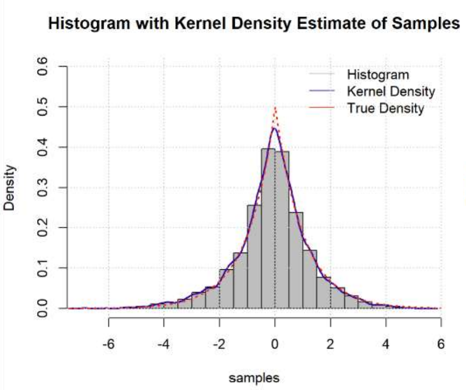
Because$|x|$ reaches its minimum value of 0 at $x = 0$, $exp(-|x|)$ reaches its maximum value of 1 at this point. Thus, $f(x)$reaches its maximum value of $\frac{1}{2}$ at $x = 0$. Theoretically, this distribution is symmetric about $x = 0$, with the shape on both sides of the distribution being mirror images of each other. The tails of the Laplace distribution decay faster than those of a normal distribution, and the Laplace distribution is sharper near the center.

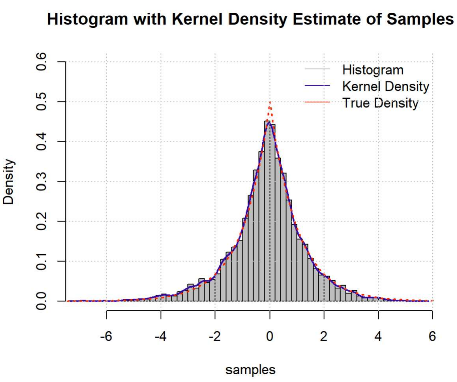
(a) Converged random walk Metropolis algorithm

In this plot, a histogram and a kernel density were constructed using the *random walk Metropolis algorithm*, with N set to *10000* and s set to *1*. We used the generated samples $(x_1, \ldots, x_N)$, and these plots provide estimates of $f(x)$.

The first histogram does not show an obvious concentration trend, although the high peak in the middle suggests a







concentration of data near zero. However, there is also a relatively large amount of data on both sides, making the overall distribution appear quite flat. However, more detail may also introduce noise issues, so it is more reasonable to use a break of 50. After increasing the number of groups, the middle graph, the histogram shows a clearer bimodal distribution, which may indicate that the sample data consists of two main clusters, each clustered around a central value. The right-side histogram and the kernel density estimate match very well, showing a clear unimodal trend with the peak at zero, indicating that most of the data is tightly clustered around zero.

```
> cat(" Monte Carlo estimates Mean:", sample_mean, "\n")
 Monte Carlo estimates Mean: -0.08561803
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> cat(" Montecarlo estimates Standard Deviation:",
sample_sd, "\n")
 Montecarlo estimates Standard Deviation: 1.348437
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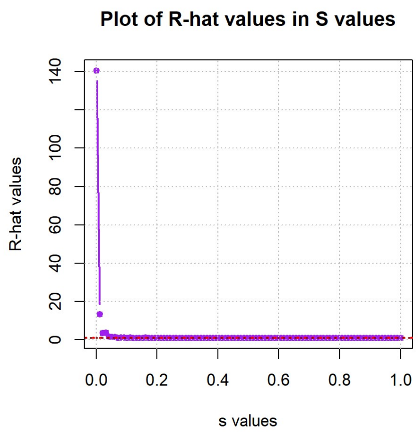
(b) Diagnostics in Convergence by $\hat{R}$ value.

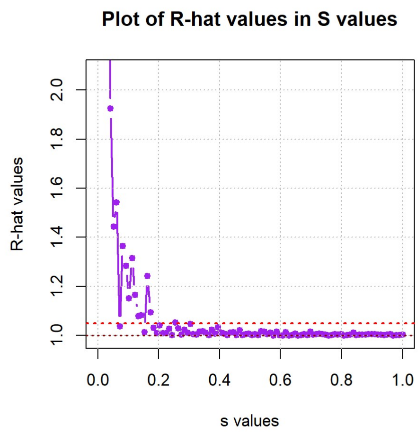
In general, values of$(\hat{R})$ close to 1 indicate *convergence*, and it is usually desired for $(\hat{R})$ to be lower than 1.05. Calculate the $(\hat{R})$ for the Random Walk Metropolis algorithm with $N = 2000$, $s = 0.001$ and $J = 4$.

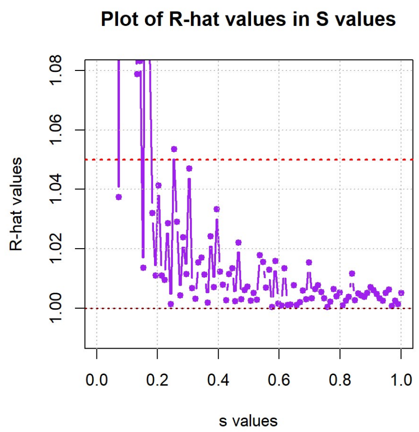
$$M_j = \frac{1}{N}\sum_{i=1}^{N} x_i^{(j)} \qquad W = \frac{1}{J}\sum_{j=1}^{J} V_j \qquad B = \frac{1}{J}\sum_{j=1}^{J}\left(M_j - M\right)^2$$

$$V_j = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^{(j)} - M_j\right)^2 \qquad M = \frac{1}{J}\sum_{j=1}^{J} M_j \qquad \hat{R} = \sqrt{\frac{B+W}{W}}$$

These three graphs illustrate the convergence behavior of the random walk Metropolis algorithm at different s values. As s increases, the $\hat{R}$ value gradually decreases and stabilizes, particularly when s is above *0.5*, where the $\hat{R}$ value is closer to 1, indicating that the *algorithm* is nearing ***convergence***.







The first graph shows the overall trend of $\hat{R}$ values as s ranges from 0 to 1. It can be seen that when s is small, the $\hat{R}$ value is very high; as s ***increases***, the $\hat{R}$ value drops rapidly and remains low and stable for most of the s range. The second graph provides a more detailed depiction of the changes in $\hat{R}$ values in the low s region. It shows that when s is extremely small, the $\hat{R}$ value fluctuates significantly, but as s increases, these fluctuations gradually diminish and tend to ***stabilize*** around a level close to the ***convergence*** criterion (*around 1.05*). The third graph focuses on a narrower range of $\hat{R}$ values (*from 1.0 to 1.08*), magnifying the subtle changes, thereby more intuitively reflecting the variations within a specific s value interval.