The Vlasov-Poisson system in 2D

April 2, 2003

This simple case is studied for comparing different method and reconstruction for the numerical resolution. The Vlasov-Poisson system chosen with boundary conditions reads :

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(t,x)\frac{\partial f}{\partial v} = 0 \tag{1}$$

coupled with the normalized Poisson equation :

$$E(x,t) = -\frac{\partial \Phi(t,x)}{\partial x}, \qquad -\frac{\partial^2 \Phi(t,x)}{\partial x^2} = \int f(t,x,v)dv - 1 \tag{2}$$

1 The linear Landau damping in 1D

For the linear Landau damping in 1D the initial data are :

$$f(0,x,v) = \frac{1}{\sqrt{2\pi}}\exp(-v^2/2)(1+\alpha\cos(kx)) \quad \forall (x,v) \in [0,L]\times \mathbf{R} \tag{3}$$

i.e,

$$f(0,x,v) = f_0(v)(1+\alpha\cos(kx)) \quad \text{with } f_0(v) = \frac{1}{\sqrt{2\pi}}\exp(-v^2/2) \tag{4}$$

1.1 Linearised vlasov equation

If f and $\mathbf{E}$ are decomposed into their equilibrium part f_0 (resp. $\mathbf{E_0}$) and their fluctuation part $\tilde{f}$ (resp. $\tilde{\mathbf{E}}$), and if only the perturbations of first order are taken into account, then (1) becomes :

$$\frac{\partial f_0}{\partial t} + \frac{\partial \tilde{f}}{\partial t} + v_0\frac{\partial f_0}{\partial x} + \tilde{v}\frac{\partial f_0}{\partial x} + v_0\frac{\partial \tilde{f}}{\partial x} + E_0\frac{\partial f_0}{\partial v} + E_0\frac{\partial \tilde{f}}{\partial v} + \tilde{E}\frac{\partial f_0}{\partial v} = 0 \tag{5}$$

Then as f_0 is solution of the vlasov equation, the previous equation gives

$$\frac{\partial \tilde{f}}{\partial t} + \tilde{v}\frac{\partial f_0}{\partial x} + v_0\frac{\partial \tilde{f}}{\partial x} + E_0\frac{\partial \tilde{f}}{\partial v} + \tilde{E}\frac{\partial f_0}{\partial v} = 0 \tag{6}$$

Besides E_0 is assume equal to 0 because there is no fluctuation at the initial time and according to (3) $\frac{\partial f_0}{\partial x} = 0$ while $\frac{\partial f_0}{\partial v} = -vf_0$, then

$$\frac{\partial \tilde{f}}{\partial t} + v_0\frac{\partial \tilde{f}}{\partial x} - \tilde{E}vf_0 = 0 \tag{7}$$

Let the perturbations be written in the Fourier space as

$$\begin{array}{l} \tilde{f} = \sum_{k_x\omega} \tilde{f}_{k_x\omega}(v) \exp\left[i(k_x x - \omega t)\right] \\ \tilde{\Phi} = \sum_{k_x\omega} \tilde{\Phi}_{k_x\omega} \exp\left[i(k_x x - \omega t)\right] \end{array} \tag{8}$$

then (7) becomes :

$$-\omega \tilde{f}_{k_x\omega} + v k_x \tilde{f}_{k_x\omega} + k_x \tilde{\Phi}_{k_x\omega} v f_0 = 0 \tag{9}$$

For the following, the subscripts $k_x\omega$ are omitted.

$$(\omega - v k_x)\tilde{f} = k_x v f_0 \tilde{\Phi} \tag{10}$$

$$\tilde{f} = \frac{k_x v}{\omega - v k_x} f_0 \tilde{\Phi} \tag{11}$$

By integration, we obtain the linearised vlasov equation:

$$\tilde{n_i} = \langle \frac{k_x v}{\omega - k_x v} f_0 \rangle \tilde{\Phi} \qquad \text{with} \quad \langle \cdot \rangle = \int \cdot dv. \tag{12}$$

1.2 Linearised quasi-neutrality equation

The quasi-neutrality given by (2) can be written by decomposing in the equilibrium and fluctuation part as :

$$-\frac{\partial^2 \Phi_0}{\partial x^2} - \frac{\partial^2 \tilde{\Phi}}{\partial x^2} = \int \tilde{f}(t,x,v)dv + \int f_0(t,x,v)dv - 1 \tag{13}$$

$$-\frac{\partial^2 \tilde{\Phi}}{\partial x^2} = \int \tilde{f}(t,x,v)dv \tag{14}$$

and by projection in the Fourier space :

$$k_x^2 \tilde{\Phi} = \tilde{n_i} \tag{15}$$

1.3 Linearised dispersion relation

By (12) and (15), we can deduce the expression of the linearised dispersion relation :

$$k_x^2 \tilde{\Phi} = \langle \frac{k_x v}{\omega - k_x v} f_0 \rangle \tilde{\Phi} \tag{16}$$

1.4 Linear growth rate

So find the linear growth rates is equivalent to find the zeros of the complex function $D(\omega)$ such that $D(\omega) = 0$ with

$$D(\omega) = k_x^2 + 1 + zZ(z) \quad \text{with} \quad z = \frac{\omega}{\sqrt{2}k_x} \tag{17}$$

where $Z(z)$ is the Fried et Conte relation which is defined by :

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{exp(-x^2)}{x - z} dx \tag{18}$$

Proof : $\Rightarrow$ *Computation of* $\langle \frac{kv}{\omega - kv} f_0 \rangle$

$$\begin{aligned} \langle \frac{kv}{\omega - kv} f_0 \rangle &= \langle \left[\frac{kv - \omega}{\omega - kv} + \frac{\omega}{\omega - kv} \right] f_0 \rangle \\ &= -\langle f_0 \rangle + \langle \frac{\omega}{\omega - kv} f_0 \rangle \\ &= -1 + \frac{1}{\sqrt{2\pi}} \int \exp\left(-\frac{v^2}{2} \right) \frac{\omega}{\omega - kv} dv \end{aligned}$$

Let $v' = \frac{v}{\sqrt{2}}$

$$I = -1 + \frac{1}{\sqrt{\pi}} \int \exp(-v'^2) \frac{1}{\left(1 - \frac{k\sqrt{2}v'}{\omega} \right)} dv'$$

Let $x = v'$ *and* $z = \frac{\omega}{\sqrt{2}k}$ *then*

$$\begin{aligned} I &= -1 + \frac{1}{\sqrt{\pi}} \int \exp(-x^2) \frac{1}{\left(1 - \frac{x}{z} \right)} dx \\ &= -1 - \frac{z}{\sqrt{\pi}} \int \exp(-x^2) \frac{1}{(x - z)} dx \\ &= -1 - zZ(z) \quad \textit{with} \quad Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{exp(-x^2)}{x - z} dx \end{aligned}$$

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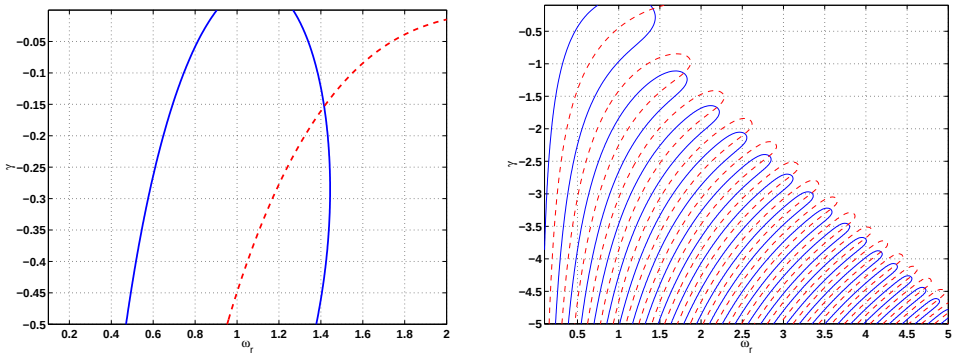
The resolution of (17) gives (cf. figures) for $k_x = 0.5$ and specially on the first



(a) (b)

Figure 1: Real and imaginary part of $D(\omega)$ defined by (17) with $\omega = \omega_r + i\gamma$.

figure (a) as predicted by the theory (cf [2]) we see that $\gamma = 0.153$ and $\omega = 1.4156$.

1.5 Energy computation

1.5.1 Expression of the kinetic and potential energy

If we defined the kinetic energy by :

$$E_{kin} = \int\int \frac{v^2}{2}(f - f_M)\, dxdv \tag{19}$$

then the potential energy can be written as :

$$E_{pot} = \frac{1}{2}\int \Phi(n_i - 1)dx \quad \text{with} \quad n_i = \int f dv \tag{20}$$

and the energy conservation is given by the fact that

$$\frac{\partial E_{tot}}{\partial t} = 0 \quad \text{with} \quad E_{tot} = E_{kin} + E_{pot} \tag{21}$$

Proof :

$$\begin{aligned}
\frac{\partial E_{kin}}{\partial t} &= \frac{1}{2}\int\int v^2 \frac{\partial f}{\partial t}\, dxdv \\
&= \frac{1}{2}\int\int v^2 \left(-v\frac{\partial f}{\partial x} - E(t,x)\frac{\partial f}{\partial v}\right) dxdv \quad \textit{because of the vlasov equation} \\
&= -\frac{1}{2}\int\int v^3 \frac{\partial f}{\partial x}\, dxdv + \frac{1}{2}\int\int v^2 \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v}\, dxdv \\
&= -\frac{1}{2}\left\{\int \left[v^3 f\right]_{xmin}^{xmax} dv - \int\int \frac{dv^3}{dx} f\, dxdv\right\} + \frac{1}{2}\int\int v^2 \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x}\, dxdv
\end{aligned}$$

The first term is equal to 0 *because* f *is periodic in* x *and the second is equal to* 0 *too because* v *does not depend on* x*.*

$$\begin{aligned}
I &= \frac{1}{2}\int\int v^2 \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v}\, dxdv \\
&= \frac{1}{2}\left\{\int \left[v^2 f\right]_{v=-vmax}^{v=+vmax} \frac{\partial \Phi}{\partial x}\, dx - \int\int 2v\frac{\partial \Phi}{\partial x} f\, dxdv\right\} \\
&= -\int\int v\frac{\partial \Phi}{\partial x} f\, dxdv \quad \textit{because } f(x,v) = f_0(v) \\
&\quad \textit{at the boundaries so } v^2 f \textit{ is the same for } v = -vmax \textit{ and } v = vmax
\end{aligned}$$

If we use again the Vlasov equation $\partial_t f + v\partial_x f + E\partial_v f = 0$*,*

$$\begin{aligned}
I &= -\int\int \Phi\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial v}\Phi\, dxdv \\
&= -\int\int \Phi\frac{\partial f}{\partial t}\, dxdv - \int\int E\frac{\partial f}{\partial v}\Phi\, dxdv \\
&= -\int\int \Phi\frac{\partial f}{\partial t}\, dxdv \quad \textit{because} \quad \int \frac{\partial f}{\partial v}\, dv = [f]_{v=-vmax}^{v=+vmax} = 0 \\
&\quad \textit{because } f(x,vmax) = f_0(vmax) = f_0(-vmax) = f(x,-vmax)
\end{aligned}$$

$$I = -\int\int \Phi \frac{\partial (f - f_0)}{\partial t}\, dxdv$$
$$= -\int \Phi \frac{\partial}{\partial t}\left[\int f\, dv - 1\right] dxdv$$

and if we remind the quasi-neutrality equation $-\frac{\partial^2 \Phi}{\partial x^2} = \int f\, dv - 1$ *:*

$$I = \int \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) dx$$
$$= \frac{1}{2}\int \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) dx + \frac{1}{2}\int \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) dx$$

Now

$$I_1 = \frac{1}{2}\int \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) dx = \frac{1}{2}\int \Phi \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Phi}{\partial t}\right) dx$$
$$= \frac{1}{2}\left[\Phi \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right)\right]_{xmin}^{xmax} - \frac{1}{2}\int \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right) dx$$

The first term is equal to 0 *because* Φ *is equal to* 0 *at the boundaries, so*

$$I_1 = -\frac{1}{2}\int \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right) dx = -\frac{1}{2}\left[\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial t}\right]_{xmin}^{xmax} + \frac{1}{2}\int \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial \Phi}{\partial t}\, dx$$
$$= \frac{1}{2}\int \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial \Phi}{\partial t}\, dx \quad \textit{because } \Phi \textit{ is equal to } 0 \textit{ at the boundaries}$$

So

$$I = \frac{1}{2}\int \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) dx + \frac{1}{2}\int \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial \Phi}{\partial t}\, dx = \frac{1}{2}\frac{\partial}{\partial t}\int \Phi \frac{\partial^2 \Phi}{\partial x^2}\, dx$$

and according to the quasi-neutrality equation

$$\frac{\partial E_{kin}}{\partial t} = -\frac{\partial}{\partial t}\left[\frac{1}{2}\int \Phi(\int f\, dv - 1)\, dx\right]$$

so

$$E_{pot} = \frac{1}{2}\int \Phi(n_i - 1)\, dx \quad with \quad n_i = \int f\, dv$$

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1.5.2 Hamiltonian approach

Let H the Hamiltonian be defined by

$$H = \frac{v^2}{2} + \Phi \tag{22}$$

then the vlasov equation can be written as

$$\frac{\partial f}{\partial t} - [H, f] = 0 \tag{23}$$

where [., .] represent the poisson brackets, i.e.

$$[H, f] = \partial_x H \partial_v f - \partial_v H \partial_x f$$

The energy conservation can be deduced from :

$$\int\int dx\, dv H \partial_t f = \int\int dx\, dv H\, [H, f]$$

Replacing H by its expression in the left hand side, then the previous equation reads

$$\int\int\, dx\, dv\, \frac{v^2}{2}\frac{\partial f}{\partial t} + \int\int\, dx\, dv \Phi \frac{\partial f}{\partial t} = \gamma$$

where

$$\gamma = \int\int dx\, dv H\, [H, f]$$

$$\frac{\partial}{\partial t}\int\int\, dx\, dv\, \frac{v^2}{2} f + \int\int\, dx\, dv \Phi \frac{\partial f}{\partial t} = \gamma$$

And by analogy

$$\frac{\partial E_{kin}}{\partial t} + \frac{\partial E_{pot}}{\partial t} = \gamma$$

where the kinetic energy E_{kin} and the potential energy E_{pot} are defined by :

$$E_{kin} = \int\int\, dx\, dv\, \frac{v^2}{2} f \tag{24}$$

and (cf. proof)

$$E_{pot} = \frac{1}{2}\int\, dx \left|\frac{\partial \Phi}{\partial x}\right|^2 \tag{25}$$

Proof : $\Rightarrow \partial_t E_{pot} = \frac{1}{2}\frac{\partial}{\partial t}\int\, dx \left|\frac{\partial \Phi}{\partial x}\right|^2$

$$\begin{aligned}
\partial_t E_{pot} &= \int\int\, dx\, dv\, \Phi \frac{\partial f}{\partial t} \\
&= \int\, dx\, \Phi\, \partial_t \int f\, dv \quad \textit{and according to the quasi-neutrality equation} \\
&= \int\, dx\, \Phi \frac{\partial}{\partial t}\left(1 - \frac{\partial^2 \Phi}{\partial x^2}\right) = -\int\, dx\, \Phi \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) \\
&= -\int\, dx\, \Phi \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Phi}{\partial t}\right) \quad \textit{and by integration by parts} \\
&= -\left[\Phi \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right)\right]_{xmin}^{xmax} + \int\, dx \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right)
\end{aligned}$$

and the first term is equal to 0 *because* Φ *and* $\partial_x\Phi$ *are periodic, so :*

$$\partial_t E_{pot} = \frac{1}{2}\frac{\partial}{\partial t}\int dx \left|\frac{\partial \Phi}{\partial x}\right|^2$$

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The right hand side γ of the energy conservation equation is equal to 0 only if f is periodic in x. Indeed if f is not periodic, γ depends on the values of the distribution function at the boundaries (cf. proof)

$$\gamma = \int\int dx\, dv H\,[H, f] = -\int dv \frac{v^3}{2}\Delta f(v)$$

with $\Delta f(v) = f(xmax, v) - f(xmin, v)$.

Proof : *Computation of* $\gamma = \int\int dx\, dv H\,[H, f]$

$$\begin{aligned}
\gamma &= \int\int dx\, dv H\,[H, f] \\
&= \int\int dx\, dv H \frac{\partial H}{\partial x}\frac{\partial f}{\partial v} - \int\int dx\, dv H \frac{\partial H}{\partial v}\frac{\partial f}{\partial x} \\
&= \int dx \frac{\partial H}{\partial x}\,[H\ f]_{-vmax}^{vmax} - \int dx \frac{\partial H}{\partial x}\int dv\, f\frac{\partial H}{\partial v} \\
&\quad - \int dv \frac{\partial H}{\partial v}\,[H\ f]_{xmin}^{xmax} + \int dv \frac{\partial H}{\partial v}\int dx\, f\frac{\partial H}{\partial x} \\
&= \int dx \frac{\partial H}{\partial x}\,[H\ f]_{-vmax}^{vmax} - \int dv \frac{\partial H}{\partial v}\,[H\ f]_{xmin}^{xmax}
\end{aligned}$$

By replacing the hamiltonian H *by its expression the previous equality becomes :*

$$\gamma = \int dx \frac{\partial \Phi}{\partial x}\left[\left(\Phi + \frac{v^2}{2}\right)\ f\right]_{-vmax}^{vmax} - \int dv\, v\left[\left(\Phi + \frac{v^2}{2}\right)\ f\right]_{xmin}^{xmax}$$

The first term is equal to 0 *because* $f(x, -vmax) = f(x, vmax) = f_M(vmax)$. *So*

$$\gamma = -\int dv\, v\left[\left(\Phi + \frac{v^2}{2}\right)\ f\right]_{xmin}^{xmax}$$

which gives

$$\gamma = -\int dv\, \frac{v^3}{2}\,[f]_{xmin}^{xmax}$$

because $\Phi(xmin) = \Phi(xmax) = 0$.
According to the previous equality we see that γ *is equal to* 0 *if* f *is periodic. This result could be find directly by using the permutation property of the poisson brackets in the integration. Indeed, let* f, g *and* h *be* 3 *functions, then*

$$\int\int f[g, h]\, dx\, dp = \int\int h[f, g]\, dx\, dp = \int\int g[h, f]\, dx\, dp$$

But in the case where f is a non-periodic function in x, then the term γ must be taken into account in the energy conservation equation because :

$$\gamma = -\int dv \frac{v^3}{2} \Delta f(v) \quad with \quad \Delta f(v) = f(xmax, v) - f(xmin, v).$$

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As seen before the expression of the energy conservation depend on the periodic or non-periodic conditions of the distribution function f. For summarising the energy conservation is given by

$$\begin{array}{ll} E_{kin} + E_{pot} = 0 & \text{if } f \text{ periodic in } x \\ E_{kin} + E_{pot} = -\Delta t \int dv \frac{v^3}{2} \Delta f(v) & \text{if } f \text{ non-periodic in } x. \end{array}$$

1.6 Results

1.6.1 The Linear Landau Damping in 1D

We consider the case $\alpha = 0.01$, the periodic length $Lx = 4\pi$, $k_x = 0.5$, $\Delta = 1/8$ and a mesh grid of $Nx = 128$ and $Nvpar = 128$. The Vlasov equation is solved by a time-splitting coupled to a semi-lagrangian method where the non-linearity due to the electric field is treated using a predictor-corrector scheme. Then with the evolution of the electric energy $\sum |E_i(t)|^2$, we find for the linear damping rate γ (cf figure 2 the same value as predicted by the theory (cf figure 1) i.e $\gamma = 0.153$. Besides we see that a complete period is approximatively equal to $T = 4.4$ which gives as numerical frequency $\omega_{num} = \frac{2\pi}{T} = 1.428$ close to the analytical results $\omega = 1.4156$. The behavior of the potential energy is similar to the one shown in Filbet's paper (cf [2] fig 3 p179).
Besides the conservation of ion number is close to 10^{-8} (cf. figure 4) and the energy conservation (cf. figure 3) is better than 2%.

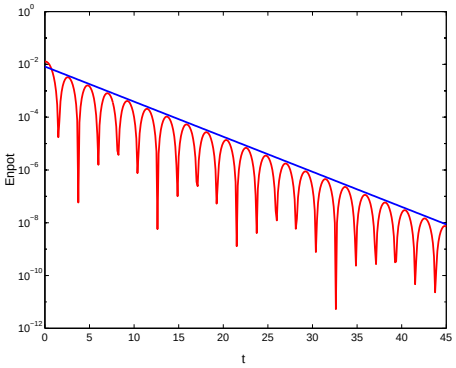


Figure 2: Evolution of the potential energy on logarithm scale in red. The blue line corresponds to $g(t) = E^0_{npot} * \exp(-2i\gamma t)$ with $\gamma = 0.153$

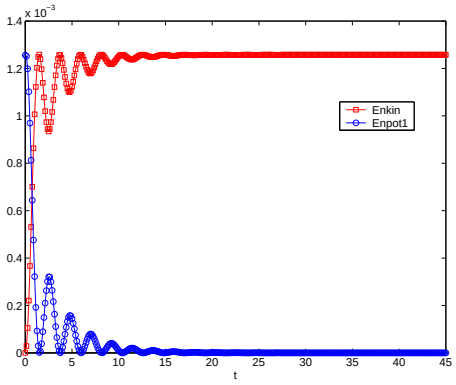


Figure 3: Evolution of the potential energy and the kinetic energy

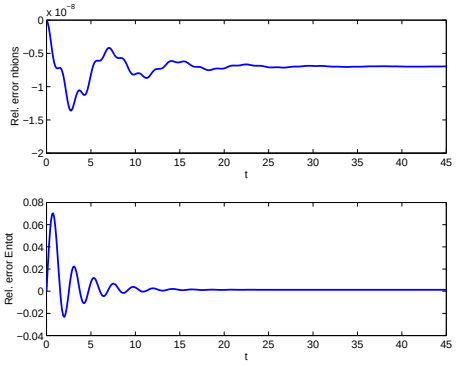


Figure 4: Evolution of the relative error of : a) the number of ions, b) the total energy for $vmax = 4.5$

If we increase the value $vmax$ to 6 instead of 4.5, we see that no difference in the energy conservation but the conservation of the number of particle is increase by a factor 1000 (relative error of 10^{-12} (cf. figure (5)).

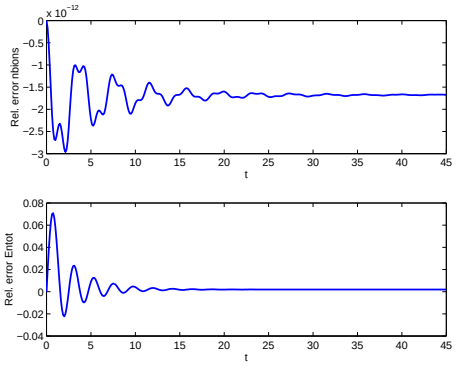


Figure 5: Evolution of the relative error of : a) the number of ions, b) the total energy for $vmax = 6$.

The energy conservation can be increased if we decrease the time step for example if we decrease dt by a factor 10, i.e. $dt = 0.0125$ then the energy conservation is increased by a factor 10 (cf. figure (6)).

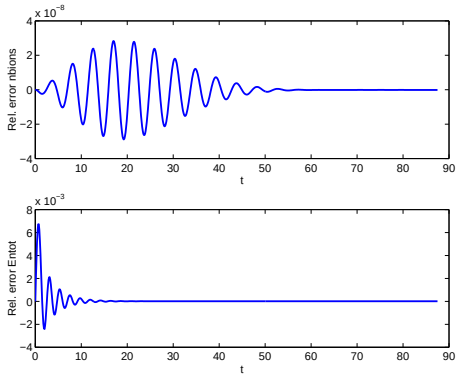
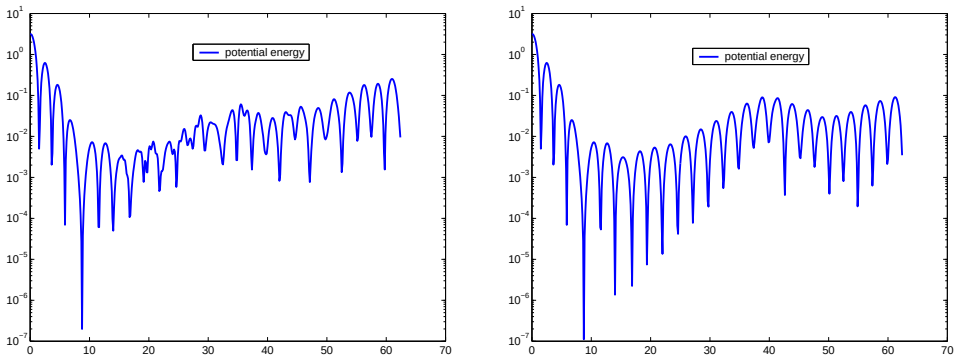


Figure 6: Evolution of the relative error of : a) the number of ions, b) the total energy for $vmax = 6.$, with $dt = 0.0125$

1.6.2 The Strong Landau Damping in 1D

In this example, the amplitude of the initial perturbation is increased. α is taken equal to 0.5 and $v_{max} = 6$. The number of cells is $N_x = 32$ and $N_v = 64$ or $N_v = 128$.
When we look at the evolution of the potential energy we notice that in this case the semi-lagrangian method with 64 points in the velocity direction is not sufficient to describe correctly the phenomena (cf fig. 7).

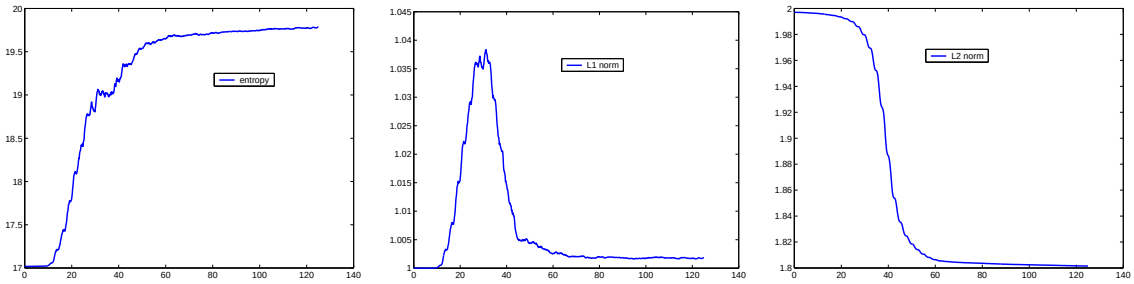


(a) (b)

Figure 7: Time evolution of the potential energy in logarithm scale for the semi-lagrangian scheme ; the cubic spline interpolation with : a) 32×64 and b) 32×128 unknowns for strong landau damping

In this problem according to the vlasov equation the kinetic entropy $H(t) = \int\int dxdvf(x,v,t)\ln(f(x,v,t))$ and the L^p-norms $L^p = \int\int dxdv|f(x,v,t)|^p$ for $p = 1, 2$ are theoretically conserved. So the behavior of these values give important informations on the quality of the numerical scheme. An increase of the entropy proves the presence of numerical dissipation. The variation of the L^1-norm represents the rate of negative values since the global mass $\int f(x,v)\,dxdv$ is preserved while the dissipation of the L^2-norm is an another characterization of the scheme dissipation. The following results (cf. fig. 8) concerning the strong

landau damping show firstly that the semi-lagrangian scheme does not preserve the positivity and the amplitude of spurious oscillations increases when non-linear effects occur. In this figure 8 we present L^1/Lx and $\sqrt{(}L^2)$ instead of directly the L^1-norm and L^2-norm for comparing the results with the one shown in figure 6 in Filbet's paper [2]. These results are totally comparable and we see that the semi-lagrangian method is less dissipative than a PFC scheme which is more conservative. (F. Filbet writes that this weak dissipation favor the introduction of numerical errors (cf. p82 [3])).



(a) (b) (c)

Figure 8: Time evolution of : a) the entropy, b) L^1/Lx and c) $\sqrt{(}L^2)$, for cubic spline interpolation with 32×128 for strong landau damping.

As seen before the results obtain are comparable to the one shown by F. Filbet [2].

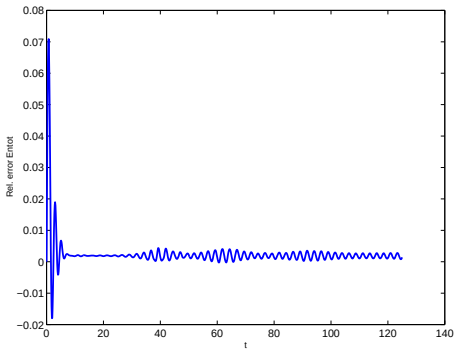


Figure 9: Time evolution of the relative error of the total energy ; for cubic spline interpolation with 32×128 for strong landau damping.

The total energy is conserved with an error smaller than 2% (cf. figure 9) while the relative error of the number of ions is equal to 10^{-5} (cf. fig. 11). The conservation of the number of ions is better if if we increase the $vmax$ value or the number of points in v-direction ($\simeq 10^{-8}$ for $vmax = 6.$ and $Nv = 256$; $\simeq 10^{-11}$ for $vmax = 8.$ and $Nv = 256$), but the energy conservation stays of the same order. So the error made on the energy seems not to come from a problem of integral computation but really from the resolution of the Vlasov equation. The good resolution of the quasi-neutrality equation only depends on the good computation of $\int f\, dv$ because the Fourier resolution in 1D is trivial. The mean

value of the relative error on the total energy is equal to $\epsilon_{mean} = 2.72 \times 10^{-3}$ with a maximum value equal to $\epsilon_{max} = 7.1 \times 10^{-2}$. If we decrease the Δt by a factor 2 (i.e $\Delta t = 1/16$) then $\epsilon_{mean} = 7.75 \times 10^{-4}$ and $\epsilon_{max} = 3.4 \times 10^{-2}$.

Questions :

- Is really the scheme of second order ?

Point	dt	'Landau 1D'	'strong Landau 1D'
0	0.00125	5.54e-07	2.513e-06
1	0.0125	9.2e-06	1.07e-05
2	0.125	0.0014	0.0092
3	1.25	0.2	0.15

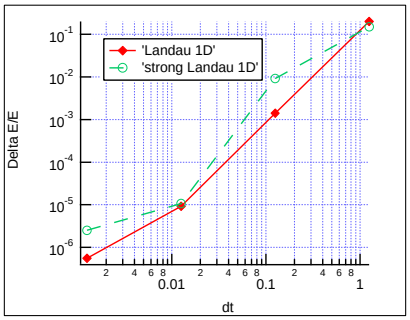


Figure 10:

- What appends with the energy conservation if the distribution function is not periodic or null in x ?
- What does it means $\Delta t = 1/16$ for the 4D code ?

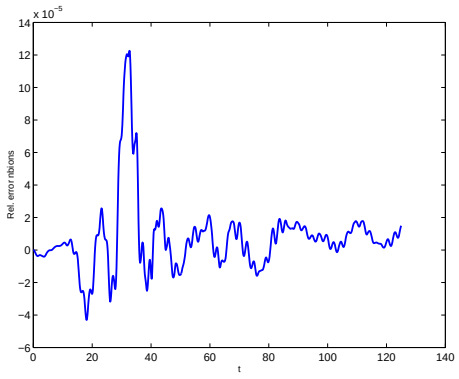
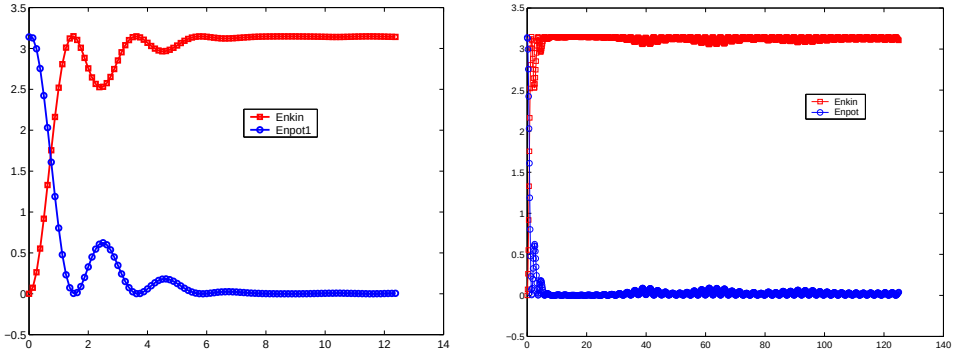


Figure 11: Time evolution of the relative error of the ion number ; for cubic spline interpolation with 32×128 for strong landau damping.



(a) (b)

Figure 12: Time evolution of the potential and kinetic energy : a) at the begin , b) along all the simulation ; for cubic spline interpolation with 32×128 for strong landau damping

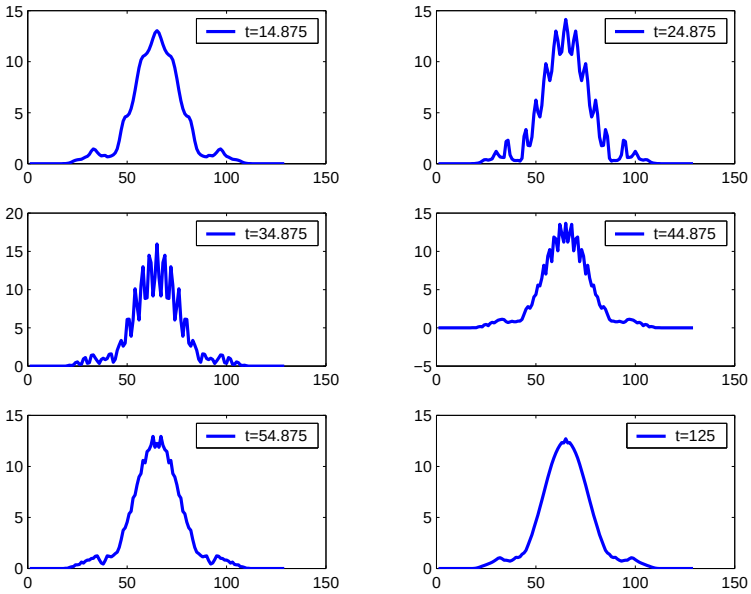


Figure 13: Time development of the spatially integrated distribution function for the SL scheme ; for strong landau damping with 32×128 mesh points; at different times : $t \simeq 15\Omega_p$, $t \simeq 25\Omega_p$, $t \simeq 35\Omega_p$, $t \simeq 45\Omega_p$, $t \simeq 55\Omega_p$ and at the end of the simulation.

Questions :

- Why do the distribution function relax to a Maxwellian profile (cf. figure 13) ?

References

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