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Overview of Bloomberg Models for Interest Rate Derivatives in DLIB

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Abstract

This document provides an overview of the Bloomberg implementation of its Interest Rate models. Interest rate models are typically calibrated to the swaption markets and used to price exotic interest rate deals. This document specifically describes the implementation of the two factor Hull-White and shifted lognormal LIBOR market models, in a Monte Carlo simulation setting.

Keywords. Hull-White, LIBOR Market Model, Shifted Lognormal, Calibration, Monte Carlo.

Contents

1 Overview

This document provides an overview of pricing interest rate derivatives using each of the implementations of the DLIB supported interest rate models. The first model described is the Hull-White model. The second model described is the LIBOR Market model. Both of these models can use a constant or piecewise constant volatility structure.

This document is concerned almost exclusively with Monte Carlo pricing methods, and the calibration methods required for the determination of their corresponding model parameters. A calibration example using the swaption market is discussed, and the subsequent Monte Carlo re-pricing of the calibration instruments, thereby illustrating its quality-of-fit, is also described.

In what follows, we will describe the following three models for simulating future interest rates, beginning with a summary in §1.1 and §1.2:

1. Hull-White (HW1) one-factor model in which the instantaneous short-rate evolves according to a single mean-reverting normal process
2. Hull-White (HW2) two-factor model in which the instantaneous short-rate evolves according to a pair of mean-reverting correlated normal processes
3. LIBOR Market model (LMM) in which the LIBOR rates evolve according to a coupled system of correlated shifted-lognormal processes

1.1 Short-Rate Models

The following notation will be used throughout this document:

$r(t)$	The instantaneous risk-neutral interest rate, or *short-rate*, prevailing at time t
$\sigma(t)$	The volatility, possibly time-varying
$W(t)$	A standard Brownian motion

Hull-White One and Two Factor Models

If one models the continuously compounded growth of a risk-free investment by the deterministic equation

$$B'(t) = r(t)B(t),$$

then the instantaneous rate of growth $r(t)$ is called the *short-rate*. A more realistic model is required for pricing securities (and derivative claims contingent on them) whose cash flows depend on the changing levels of future interest rates, and for this purpose stochastic models for the short-rate have been introduced. The Hull-White two factor model simulates the short-rate $r(t)$ as a stochastic process decomposable into three components; a deterministic component $\theta(t)$, a mean reverting long-term stochastic process $x_1(t)$, and a second mean reverting short-term stochastic process $x_2(t)$. The SDE describing the Hull-White one-factor dynamics [BM, Ch.4] is given as:

$$\begin{aligned} r(t) &:= \theta(t) + x_1(t), && (1.1)\\ dx(t) &= -\kappa x(t)dt + \sigma(t)\ dW(t), \qquad x(0) = 0, && (1.2) \end{aligned}$$

where W is a standard Brownian motion under the risk neutral measure. Similarly, the system of SDE's describing the Hull-White two-factor dynamics is given as:

$$
\begin{aligned}
r(t) &:= \theta(t) + x_1(t) + x_2(t), && (1.3)\\
dx_1(t) &= -\kappa_1 x_1(t)dt + \sigma_1(t)\; dW_1(t), \qquad x_1(0) = 0, && (1.4)\\
dx_2(t) &= -\kappa_2 x_2(t)dt + \sigma_2(t)\; dW_2(t), \qquad x_2(0) = 0, && (1.5)\\
\rho\; dt &:= dW_1(t) \cdot dW_2(t), && (1.6)
\end{aligned}
$$

where W_1 and W_2 are standard Brownian motions under the risk neutral measure. Intuitively, the Hull-White two factor model for which $\kappa_2 \gg \kappa_1$ will mean revert to the one factor model, which in turn will mean revert to the deterministic $\theta(t)$, which directly reflects prevailing rates. The two factor model was introduced to overcome known limitations of the one factor model (for example, its inability to account for correlations between LIBOR rates), while still retaining the important feature of analytic tractability. In particular, solutions $r(t)$ are explicitly expressible by:

$$r(t) = \theta(t) + \int_0^t e^{-\kappa_1(t-u)}\sigma_1(u)dW_1(u) + \int_0^t e^{-\kappa_2(t-u)}\sigma_2(u)dW_2(u).$$

Note that the possibility of negative rates is not precluded in the distribution implied by $r(t)$, but their occurrence is of low probability. The Hull-White one and two factor models are described more fully in §2.

1.2 Market Models

The Hull-White short-rate model introduced above in §1.1, by virtue of its modeling the dynamical evolution of the *unobservable* instantaneous risk-free interest rate, necessarily trades off its dynamic tractability for certain difficulties when calibrated to market instruments. This is because market instruments are typically derived from actual observed rates of finite tenor; most typical among those market instruments being caps and swaptions, which are based on the observed finite tenor LIBOR rates. Models which directly postulate the dynamic evolution of *observable rates* permit a much easier interpretation of their parameters, and their calibration to observable market instruments is correspondingly simplified.

Furthermore, short-rate models are incapable of calibrating to the skew observed in the OTM swaption market. Also, although a two factor short-rate model can account for decorrelation in forward rates of different maturities, it is ineffective at modeling the observed rate of decorrelation with respect to increasing spread in maturities.

The LMM, on the other hand, which simulates a set of LIBOR rates and possesses a rich parameter structure, is free of these limitations. Table 1.1 below summarizes the salient comparisons between the shifted-lognormal LMM and the Hull-White two factor short-rate models.

	Hull-White One-Factor | Two-Factor	Shifted-Lognormal LMM
Underlying States	instantaneous short-rate	directly observable LIBOR rate
Model Dynamics	mean-reverting normal based	shifted-lognormal
Model Parameters	non-observable	realistic correlation structure observable volatility structure (ATM level) observable shift structure (OTM skew)
Calibration Instruments	caplets, ATM swaptions	full Swaption Matrix
Calibration Limitations	at least two swaptions per expiry no OTM swaption skew	any number of swaptions per expiry supports OTM swaption skew
Negative Rates	no bound on negativity	negativity bounded below[1]
Forward Swap Rates	unrealistically correlated swap rates	realistically decorrelated swap rates

Table 1.1: Contrasting Interest Rate Models

Shifted-Lognormal LMM

LIBOR Market Models come in many flavors, the most popular of which are the Lognormal, Shifted-Lognormal[2], and CEV. The shifted-lognormal process is essentially a hybrid of the lognormal and normal models, modulating between these two extremes by introducing an “additive shift parameter” ($\alpha \geq 0$). Unlike short-rate models and the lognormal LMM, the shifted-lognormal LMM can model a market skew exhibited in OTM swaption instruments. More specifically, a shifted-lognormal process models OTM skew by backing out a *strike-dependent* implied volatility σ_{SLN} from the Black-Scholes equation[3] in which σ_B denotes the Black volatility:

$$\mathrm{Bl}(F, K, \sigma_B(T)) \quad = \quad \mathrm{Bl}(F+\alpha, K+\alpha, \sigma_{SLN}(T)).$$

We will see below that the shifted-lognormal model behaves well in a negative interest rate environment, exhibiting an added benefit of using shifted rates beyond their skew-modeling utility.

In what follows, we will always consider a LIBOR Market Model of N forward rates

$$\{F_1(t), \ldots, F_N(t)\}$$

where $F_k(t)$ is the forward rate from T_k to T_{k+1}, evaluated at time t. Specifically, the LMM is specified by a set of times T_i, $i = 0 \ldots N+1$ relative to the present time $T_0 = 0$. Also, τ_j will denote the tenor of the j'th rate being modeled; *e.g.* $\tau_j \approx 0.25$ for the quarterly USD LIBOR index.

Characteristic of the LMM is to take these forwards $F_j(t)$ as the underlying stochastic variables, which we display diagrammatically as follows:

$$\text{today} = T_0 \longrightarrow T_1 \xrightarrow[\tau_1]{F_1} T_2 \xrightarrow[\tau_2]{F_2} \cdots \xrightarrow[\tau_{N-1}]{F_{N-1}} T_N \xrightarrow[\tau_N]{F_N} T_{N+1}$$

[1] Explicitly, each forward Libor rate F_k is bounded below by $-\alpha_k$, the negative of its corresponding shift parameter.
[2] In some treatments, the term Displaced Diffusion is used for Shifted-Lognormal.
[3] §A.6 derives an application of this formula to caplet pricing.

The dynamical equations used in the shifted-lognormal model describe the evolution of the underlying rates $F_j(t)$ in terms of their shift parameters α_j and volatility parameters $\sigma_j(t)$, and are expressed as follows:

$$dF_j(t) = \sigma_j(t)(F_j(t) + \alpha_j)dW_j^{Q_{T_j}}(t).$$

The various Brownian motions $dW_j^{Q_{T_j}}(t)$ are standard Brownian motions in their respective T_j-forward measures, and are correlated according to the "forward-forward" correlations $(\rho_{i,j})$, which are described by a two-parameter (but *not* two-factor) structured correlation matrix (ρ).

We will assume that the shifts, denoted by α_j, do not depend on time t and therefore constitute a collection of N constants. As observed, the quantities $(F_j(t)+\alpha_j)$ are lognormally distributed, and the requirement of positivity refers to the quantity $(F_j(t)+\alpha_j)$, which in turn requires that $F_j(t) > -\alpha_j$, thereby permitting negative rates. Indeed, negative rates are perfectly permissible in a normal model, which is approached for large values of the shifted-lognormal parameter α. A calibration to a negative rate market would be accommodated with sufficiently large shift parameters.

The volatilities $\sigma_j(t)$ do have a time-dependence, but only to the extent that they may take on different values when t progresses from one interval $[T_{k-1}, T_k]$ to the next $[T_k, T_{k+1}]$. Since each F_j has stopped evolving after time T_j, there is no volatility $\sigma_{j,k}$ for $k \geq j$, and hence this "piecewise-constant term-structure" of volatilities is represented by a collection of $\frac{N(N+1)}{2} = (1+2+\cdots+N)$ constants denoted $\sigma_{j,k}$.

We summarize below the complete set of model parameters associated with the volatility structure and the forward-forward correlation structure:

	Shift	Piecewise Constant Vol	Rebonato Full-Factor Correl
Function	$\alpha_i(t)$	$\sigma_i(t \in [T_{j-1}, T_j])$	$\rho_{i,j}(\rho_\infty, \rho_{decay})$
Parameters	α_i	$\sigma_{i,j}, \quad j \leq i$	$\rho_\infty, \rho_{decay}$
Number	N	$N(N+1)/2$	2

Table 1.2: Model Parameters

As explained above, all LMM models are based on simulating the evolution of a set of forward LIBOR rates, and thereby benefit from being directly observable in the market. In the context of the shifted-lognormal LMM, we find that the forward volatilities are intuitively associated with the volatility levels quoted in the ATM swaption market, and that the forward shifts are intuitively associated with the volatility skews quoted in the OTM swaption market. Finally, the forward-forward correlations can be associated with the CMS-spread options, and may even be explicitly specified based on historical data to reflect observed decorrelation of rates with respect to $|T_i - T_j|$.

1.3 Dual Curve Pricing

As described in [Mer1, Mer2], it is often appropriate to use distinct curves for discounting on the one hand, and for the interest rate curve on which underlyings and their associated cash flows are derived, on the other hand. One example is to use OIS discounting in combination with LIBOR rates when the underlyings are LIBOR based swap rates. In this case, it is the OIS forward rates that will be modeled (and evolved in the Monte Carlo simulation), but the basis adjustment represented by the spread of LIBOR over OIS will be applied in the calibration to LIBOR based market instruments, and also in the pricing of deals derived from LIBOR based underlyings.

More specifically, if we assume an OIS discount curve $P_t(T)$ giving the risk-free discounting to time T, evaluated at time t, from which one deduces an *OIS forward rate* from T_k to T_{k+1}:

$$F_k(t) := \frac{1}{\tau_k}\left[\frac{P_t(T_k)}{P_t(T_{k+1})} - 1\right],$$

then there will be an implied *LIBOR spread over OIS* given by $\beta_k(t) := L_k(t) - F_k(t)$ associated to each LIBOR rate $L_k(t)$. One consequence of this dual-curve setup is the modified expression for the swap rate from T_a to T_b:

$$S^{ab}(t) = \frac{\sum_{k=a}^{b-1} \tau_k P_t(T_{k+1})(F_k(t) + \beta_k(t))}{\sum_{k=a}^{b-1} \tau_k P_t(T_{k+1})} \neq \frac{P_t(T_a) - P_t(T_b)}{\sum_{k=a}^{b-1} \tau_k P_t(T_{k+1})}.$$

Whereas in the single-curve setup $\beta_k(t)$ is identically zero and there is no distinction between modeling LIBOR rates and OIS forward rates, the most general dual-curve setup must provide for modeling the joint evolution of rates $F_k(t)$ and $L_k(t)$ (or either with $\beta_k(t)$). By making the simplifying assumption of a *deterministic basis*, more specifically a time-independent basis, whereby the spread $\beta_k(t)$ does not evolve and retains its initial (non-zero) value $\beta_k(0)$ for all times, one need not evolve the $L_k(t)$ separately. Indeed, the evolution of the $F_k(t)$ suffice to recover the $L_k(t)$ by adding back the initial spread $\beta_k(0) = L_k(0) - F_k(0)$. For this reason, only the OIS forward rates are evolved, and only the SDE system (3.1) for F_k need be considered. As described in §A.8, calibrating model parameters for the OIS forwards from LIBOR dependent market instruments, as well as pricing LIBOR dependent deals from simulated OIS forward rates, involves no more than straightforward manipulations of the deterministic basis using the additive relationship $L_k(t) = F_k(t) + \beta_k(0)$.

2 Hull-White One and Two Factor Models

We recall (1.1) and (1.3) in which $r(t)$ represents the instantaneous short rate, whose definition can be derived from other quantities, all derivable from the discount factor $P_0(t)$ giving today's value of \$1 delivered at time t. Denoting by $P_s(t)$ the stochastically evolving discount curve of bond prices prevailing at time $s \leq t$, and by $f_s(t)$ the stochastically evolving forward curve prevailing at time $s \leq t$, we write:

$$\begin{aligned} f_0(t) &:= \lim_{\Delta t \to 0} \frac{\frac{P_0(t)}{P_0(t+\Delta t)} - 1}{\Delta t} = -\frac{P_0'(t)}{P_0(t)} = -\left[\ln P_0(t)\right]', \\ f_s(t) &:= -\left[\ln P_s(t)\right]', \qquad s < t, \\ r(s) &:= f_s(s^+). \end{aligned} \tag{2.1}$$

It is also usual to introduce the continuously compounded interest rate, or *zero rate* $R(T)$, via

$$P_0(T) =: e^{-TR(T)},$$

hence:

$$R(T) = -\frac{\ln P_0(T)}{T} \overset{(2.1)}{=} \frac{1}{T}\int_0^T f_0(u)du,$$

and by analogy, the stochastically evolving zero-rate and an equivalent definition of the short rate:

$$\begin{aligned} R_s(T) &:= -\frac{\ln P_s(T)}{T-s} = \frac{1}{T-s}\int_s^T f_s(u)du, \quad s < T, \\ r(s) &:= R_s(s^+). \end{aligned}$$

Note that, from the fact $f_s(T)$ is a limit of the ratio of two tradable assets, namely $P_s(T)$–$P_s(T+\Delta t)$ and $P_s(T)$, and therefore a martingale with respect to the T-forward numeraire $P_s(T)$, we have

$$f_s(T) = \mathbb{E}^{\mathbb{Q}_T}\left[f_T(T)\middle|f_s\right] = \mathbb{E}^{\mathbb{Q}_T}\left[r(T)\middle|f_s\right],$$

giving an interpretation of $r(T)$ in the T-forward measure $\mathbb{Q}^T$.

Perhaps the most intuitive definition of the short rate, indeed the one used in earlier models having a deterministic interest rate $r(t)$, is via the differential equation describing the growth of the money market account:

$$B'(t) = r(t)B(t), \qquad B(0) = 1.$$

In this formulation $r(t)$ can be interpreted from the martingale property of the tradable $P_s(T)$ with respect to the numeraire $B(s)$,

$$P_0(T) = \mathbb{E}^{\mathbb{Q}_0}\left[B(T)^{-1}\right] = \mathbb{E}^{\mathbb{Q}_0}\left[e^{-\int_0^T r(u)du}\right], \tag{2.2}$$

where $\mathbb{Q}_0$ denotes the risk neutral measure.

2.1 Hull-White One-Factor Model Dynamics

We recall from §1.1 the SDE defining the one factor model of Hull-White for describing interest rate dynamics:

$$r(t) := \theta(t) + x(t), \tag{2.3a}$$

$$dx(t) = -\kappa x(t)dt + \sigma(t)\ dW(t), \qquad x(0) = 0, \tag{2.3b}$$

where $0 < \kappa$, $\theta(t), \sigma(t)$ are deterministic, and where W is a standard Brownian motions under the risk neutral measure. We also assume one of two parametric forms for $\sigma(t)$:

$$\sigma(t) = \begin{cases} \sigma_0 & \text{(constant volatility structure)}, \\ \sigma_j, \quad t_{j-1} \le t < t_j, \quad 1 \le j \le N & \text{(piecewise constant term structure)}. \end{cases}$$

Applying variation of parameters to (2.3b), and using the initial conditions, give:

$$x(t) = \int_0^t e^{-\kappa(t-u)}\sigma(u)\ dW(u),$$

from which we see that, letting $t \to \infty$, x mean reverts to its initial values of zero. Hence

$$r(t) \;=\; \theta(t) + \int_0^t e^{-\kappa(t-u)}\sigma(u)\; dW(u) \tag{2.4}$$

mean reverts to $\theta(t)$. It is not difficult to deduce that the mean and variance of $r(t)$, conditional on the state of the world at time $s < t$, are given by:

$$\begin{aligned} \mathbb{E}_s^{\mathbb{Q}_0}\left[r(t)\right] &= \theta(t) + x(s)e^{-\kappa(t-s)}, && (2.5)\\ \mathrm{Var}_s^{\mathbb{Q}_0}\left[r(t)\right] &= \int_s^t e^{-2\kappa(t-s)}\sigma^2(s)\; ds && (2.6)\\ &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-s)}\right) \qquad \text{when } \sigma \text{ is constant.} \end{aligned}$$

Note that (2.4) implies $r(t)$ is a Gaussian random variable, and so (2.2) and (2.5) with $s = 0$ imply:

$$\begin{aligned} P_0(T) &= \exp\left(-\,\mathbb{E}\left[\int_0^T r(u)\; du\right] + \tfrac{1}{2}\mathrm{Var}\left[\int_0^T r(u)\; du\right]\right),\\ f_0(T) &= \theta(T) - \frac{1}{2}\frac{d}{dt}\bigg|_T \mathrm{Var}\left[\int_0^T r(u)\; du\right], && (2.7) \end{aligned}$$

where we have used $\mathbb{E}\left(e^{\mathcal{N}(\mu,\nu)}\right) = \exp\left(\mu + \frac{1}{2}\nu\right)$ and then taken logarithmic derivatives. Thus (2.7) gives an explicit relationship between $\theta(t)$ and the initial forward curve f_0 (and κ, σ).

Since the one-factor model may be realized as a special case of the two-factor model described below in §2.2, namely by setting in (2.8)

$$x_1(t) = x(t), \quad \sigma_1(t) = \sigma(t), \quad \kappa_1 = \kappa, \quad \rho = \kappa_2 = \sigma_2 = 0,$$

we omit further discussion which may be obtained from specialization. Note that there are some subtle aspects in which the Hull-White one-factor model, although obtainable from the two-factor model by specialization, can give non-identical results (see §2.4).

2.2 Hull-White Two-Factor Model Dynamics

We recall from §1.1 the SDE system defining the two factor model of Hull-White for describing interest rate dynamics:

$$\begin{aligned} r(t) &:= \theta(t) + x_1(t) + x_2(t), && (2.8a)\\ dx_1(t) &= -\kappa_1 x_1(t)dt + \sigma_1(t)\; dW_1(t), \qquad x_1(0) = 0, && (2.8b)\\ dx_2(t) &= -\kappa_2 x_2(t)dt + \sigma_2(t)\; dW_2(t), \qquad x_2(0) = 0, && (2.8c) \end{aligned}$$

where $0 < \kappa_1 < \kappa_2$, $\theta(t), \sigma_1(t), \sigma_2(t)$ are deterministic, and where W_1 and W_2 are standard Brownian motions under the risk neutral measure. We assume that

$$dW_1(t) \cdot dW_2(t) \;=\; \rho\; dt, \tag{2.9}$$

where the correlation is assumed to be constant and $-1 \le \rho \le 1$.

Applying variation of parameters to (2.8b) and (2.8c), and using the initial conditions, gives:

$$x_i(t) = \int_0^t e^{-\kappa_i(t-u)}\sigma_i(u)\, dW_i(u), \qquad i = 1, 2,$$

from which we see that, letting $t \to \infty$, x_1 and x_2 mean revert to their initial values of zero. Hence

$$r(t) = \theta(t) + \int_0^t e^{-\kappa_1(t-u)}\sigma_1(u)\, dW_1(u) + \int_0^t e^{-\kappa_2(t-u)}\sigma_2(u)\, dW_2(u) \tag{2.10}$$

mean reverts to $\theta(t)$. It is not difficult to deduce that the mean and variance of $r(t)$, conditional on the state of the world at time $s < t$, are given by:

$$\mathbb{E}_s^{\mathbb{Q}_0}[r(t)] = \theta(t) + x_1(s)e^{-\kappa_1(t-s)} + x_2(s)e^{-\kappa_2(t-s)}, \tag{2.11}$$

$$\begin{aligned}
\mathrm{Var}_s^{\mathbb{Q}_0}[r(t)] &= \int_s^t e^{-2\kappa_1(t-u)}\sigma_1^2(u)\, du + \int_s^t e^{-2\kappa_2(t-u)}\sigma_2^2(u)\, du \\
&+ 2\int_s^t \rho e^{-(\kappa_1+\kappa_2)(t-u)}\sigma_1(u)\sigma_2(u) du \qquad (2.12)\\
&= \frac{\sigma_1^2}{2\kappa_1}\left(1 - e^{-2\kappa_1(t-s)}\right) + \frac{\sigma_2^2}{2\kappa_2}\left(1 - e^{-2\kappa_2(t-s)}\right) \\
&+ \frac{2\sigma_1\sigma_2\rho}{\kappa_1+\kappa_2}\left(1 - e^{-(\kappa_1+\kappa_2)(t-s)}\right), \qquad \text{when } \sigma_1, \sigma_2 \text{ are constant.}
\end{aligned}$$

Note that (2.10) implies $r(t)$ is a Gaussian random variable, and so the conclusions expressed in (2.7) similarly hold.

2.3 Calibration

We have seen in (2.7) one relationship between initial market data, namely the forward curve f_0, and the unknown parameters of the model. Additional relationships can be determined by matching volatilities to the caplet market, and both volatilities and correlations to the swaption and CMS spread markets. We indicate below the exact formula for caplet pricing, but remark that approximation formulas are also used due to the numerical complexity of their evaluation. In order to ease the notation, we introduce the following quantities, also used in §2.4, §A.9, §A.10:

$$h_{T,j}(u) := e^{-\kappa_j(T-u)}, \qquad j = 1, 2, \qquad 0 \le u \le T; \tag{2.13}$$

$$H_{T,j}(u) := \frac{1 - e^{-\kappa_j(T-u)}}{\kappa_j}, \qquad j = 1, 2, \qquad 0 \le u \le T; \tag{2.14}$$

$$\mathbf{x} := [x_1, x_2]', \qquad h_T := [h_{T,1}, h_{T,2}]', \qquad H_T := [H_{T,1}, H_{T,2}]' \tag{2.15}$$

so (2.12) can be expressed as:

$$\mathrm{Var}_s^{\mathbb{Q}_0}[r(T)] = \int_s^T [h_{T,1}, h_{T,2}] \cdot Q \cdot [h_{T,1}, h_{T,2}]'\, du = \int_s^T \langle h_T, Qh_T \rangle du, \tag{2.16}$$

where the covariance matrix Q is defined by

$$Q(u) := \begin{bmatrix} \sigma_1^2(u) & \rho\sigma_1(u)\sigma_2(u) \\ \rho\sigma_1(u)\sigma_2(u) & \sigma_2^2(u) \end{bmatrix}. \tag{2.17}$$

Accordingly, one derives the expected price at time s, of a zero coupon bond of maturity T, to be:

$$P_s(T) \quad = \quad \exp\left[-\big\langle H_T(s), \mathbf{x}(s)\big\rangle + \int_s^T \left[-\theta + \tfrac{1}{2}\big\langle H_T, QH_T\big\rangle\right] du\right], \tag{2.18}$$

where the implicit dependence of θ, H_T and Q on the integration parameter u is presumed. Moreover, the price of a caplet struck at K and expiring at time T and maturity T_M (tenor τ), is derived from the Black-Scholes formula, using an appropriate formula for Black volatility σ_B:

$$C(T,K) \quad = \quad P_0(T)\Phi(d_1) - (1+\tau K)P_0(T_M)\Phi(d_2), \tag{2.19a}$$

$$d_{1,2} \quad := \quad \frac{-\ln\left[(1+\tau K)P_0(T_M)/P_0(T)\right]}{\sigma_B} \pm \frac{\sigma_B}{2}, \tag{2.19b}$$

$$G(t;T,T_M) \quad := \quad [H_{T_M,1}(T)h_{T,1}(t), H_{T_M,2}(T)h_{T,2}(t)]' \tag{2.19c}$$

$$\sigma_B(T,T_M)^2 \quad := \quad \int_0^T \big\langle G(u;T,T_M), Q(u)G(u;T,T_M)\big\rangle du. \tag{2.19d}$$

Still more involved integration formulas can be obtained for model swaption prices and swap rate correlations (§ A.9, § A.10), which can be matched (as best possible) with the quoted swaption market and CMS spread option market. Additionally, swaption approximation formulas which do not involve numerical integration can also be derived. Specifically, one may calibrate the model to the ATM swaption market of normal volatilities using a swaption approximation formula.

Note that in the Hull-White one-factor model, $Q(u) = \left[\sigma^2(u)\right]$ in (2.17), and the inner-products in (2.16), (2.18) and (2.19) are replaced with scalar multiplication.

Model Parameters

The Hull-White two-factor model parameters, as specified in (2.8b), (2.8c), and (2.9), are

$$\rho, \kappa_1, \kappa_2, \sigma_1(t), \sigma_2(t).$$

The parameters ρ, κ_1, κ_2 are specified as fixed input (with default values provided), and although their values can be modified manually by the user, they are not *calibrated* parameters, which is to say their values are held fixed during the calibration process. Indeed, only the volatility functions $\sigma_1(t), \sigma_2(t)$ are calibrated to the market. Furthermore, the volatility functions are given a parametric form consistent with the term structure of expiries and tenors determined by the swaption market. In particular, there are two flavors[4] of volatility functions: *constant* volatility $\{\sigma_1, \sigma_2\}$, and the *piecewise constant* volatility $\{\sigma_{1,j}, \sigma_{2,j}\}$.

When using the constant volatility term structure we have $\sigma_1(t) \equiv \sigma_1, \sigma_2(t) \equiv \sigma_2$, in which case only two volatility parameters (one in the case of Hull-White one factor) must be specified. When using the piecewise constant volatility term structure with times $T_j, j = 1, \ldots, N$, we have

$$\sigma_k(t) := \sigma_{k,j} \text{ for } t \in [T_j, T_{j+1}), \qquad k = 1, 2,$$

in which case $2N$ model parameters (N in the case of Hull-White one factor) must be specified.

[4] See also Table 3.2 in §3.2 in the context of LMM.

Calibration Methodology

Calibration of the model parameters amounts to determining the $\mathbf{x} := [\sigma_{1,1}, \ldots, \sigma_{1,N}, \sigma_{2,1}, \ldots, \sigma_{2,N}]$ which minimize (best-fit) the objective function:

$$F(\mathbf{x}) := \sum_{1 \leq k \leq M} \omega_k \left(v_k - p_k(\mathbf{x})\right)^2 , \tag{2.20}$$

where M is the number of calibration instruments, v_k is the *market* value and ω_k is appropriate weight (*e.g.* Black vega weight ω_k applied to price v_k for Black-vol) of the k-th instrument (a swaption, caplet, or cap, of specific expiry, tenor, and strike), and $p_k(\mathbf{x})$ is its *model* value computed as a function of model parameters $\mathbf{x}$ (often using a closed-form approximation formula depending on the instrument).

The minimization of (2.20) is approached through a bootstrapping procedure, which takes advantage of the fact that $p_k(\mathbf{x})$ does not depend on all components of $\mathbf{x}$, but can rather be fully determined from the subset of $\sigma_{i,j}$ for which T_j is less than the maturity of the k-th instrument. In other words, $\sigma_{i,1}$ can be determined from the instrument(s) with expiry T_1, and then fixed in subsequent calibration of $\sigma_{i,2}$ when considering instruments of maturity T_2. In this way the $\sigma_{i,j}$ are *bootstrapped* by incrementally calibrating to instruments of increasing maturity.

As mentioned above, approximation formulas are employed to evaluate a model value for an instrument, (as is displayed in `Calibration:CalibrationAnalytics` screen under `Data set`), since performing Monte Carlo simulations is not feasible. When multiple instruments are included in the calibration, the optimizer may fall short (under-determined) of matching them simultaneously. In theory, it may also happen that multiple combinations of *approximate* solutions (over-determined) to the optimization algorithm give equally good minima of (2.20), yet re-calibrations with smoothly changing market values result in non-smoothly changing calibrated parameters. For vega calculations, stability of the calibration (smooth dependence of parameters on initial market conditions) is critical, and one of the important features of the Hull-White calibration methodology used is its demonstrable stability producing smooth *vega trends* in response to bumped market volatilities.

As for the optimization algorithm employed for the (bootstrapped) minimization of (2.20), Hull-White uses a gradient based approach which seeks an $\mathbf{x}$ where the gradient of the objective function vanishes. Gradient algorithms require only that the first derivatives $\partial F(\mathbf{x})/\partial x_i$ be evaluated.

Calibration Caveats

As already noted, if the selection of calibration instruments is limited to two instruments per expiry (or one in the case of Hull-White one factor), then the ability to find $\sigma_{i,j}$ producing prices that exactly match all instruments is theoretically achievable, and can be observed in `Calibration:CalibrationAnalytics` screen. When calibrating to multiple swaptions of the same expiry and tenor, the optimizer will not in general be able to find any combination of $\sigma_{i,j}$ making $F(\mathbf{x}) = 0$. In this sense Hull-White calibration is incapable of reproducing swaption smiles.

Furthermore, one should note that, while the model is capable of being calibrated to Black volatilities in a negative rate environment, it is often preferable to use Normal volatilities, as quotations in Normal Vols have better coverage from `VCUB` in extreme market conditions.

2.4 Monte Carlo Pricing

Monte Carlo simulations require that a particular numeraire be specified. The dynamic equations displayed in (2.8a), (2.8b) and (2.8c) are given with respect to the risk-neutral measure associated with the $B(s)$ numeraire. Although our simulations are performed using the Spot numeraire, it is instructive to present T-forward numeraire $P_s(T)$, and we therefore indicate the measure change as a transformation of the risk-neutral standard Brownian motions $W_1(t)$ and $W_2(t)$:

$$\begin{aligned} dW_1(t) \quad &\rightarrow \quad dW_1^{\mathbb{Q}_T} := dW_1(t) + \big[\sigma_1 H_{T,1}(t) + \rho\sigma_2 H_{T,2}(t)\big]dt, \\ dW_2(t) \quad &\rightarrow \quad dW_2^{\mathbb{Q}_T} := dW_2(t) + \big[\rho\sigma_1 H_{T,1}(t) + \sigma_2 H_{T,2}(t)\big]dt. \end{aligned}$$

For example, if $dZ^{\mathbb{Q}_T} := [\sigma_1 dW_1^{\mathbb{Q}_T}, \sigma_2 dW_2^{\mathbb{Q}_T}]'$, then one may derive the forward rate dynamics as

$$df_s(T) = \big\langle h_T, dZ^{\mathbb{Q}_T}\big\rangle,$$

the absence of drift being consistent with $f_s(T)$ being a martingale in the $\mathbb{Q}_T$ measure. Furthermore, the following explicit solution to the Hull-White SDE in the T-forward measure becomes:

$$r(t) \quad = \quad \theta(t) - \int_0^t \big\langle h_t, QH_T\big\rangle du + \int_0^t \big\langle h_t, dZ^{\mathbb{Q}_T}\big\rangle,$$

where the implicit dependence of h_t, H_T and Q on the integration parameter u is presumed. Comparing with (2.10), we recognize the appearance of the second term due to the change of measure. Conditioning on the initial condition $r(t_0) = \theta(t_0)+x_1(t_0)+x_2(t_0)$, we have the closed form solution

$$r(t;t_0) \quad = \quad \theta(t) + \big\langle h_t, \mathbf{x}\big\rangle(t_0) - \int_{t_0}^t \big\langle h_t, QH_T\big\rangle du + \int_{t_0}^t \big\langle h_t, dZ^{\mathbb{Q}_T}\big\rangle.$$

Using explicit formulas for the deterministic terms, and an $\mathcal{N}\left(0, \int_{t_0}^t \big\langle h_t, Qh_t\big\rangle du\right)$ for the stochastic term, an exact iterative formula for evolving $r(t_0 = t_k)$ to $r(t = t_k + \Delta t)$ can be obtained for path generation.

Simulation Caveats

One might expect that pricing with a one-factor model will always give identical results as pricing with a two-factor model which has been "degenerated" into a one-factor model by virtue of $\sigma_2 \equiv 0$. A subtle observation, however, is that since the random numbers generated for the models are inequivalent due to the difference in the number of factors (*e.g.* the second random number generated is assigned to $dW(t_2)$ in the former model, and to $dW_2(t_1)$ in the latter model), the prices produced by their respective simulations will not be identical. This difference is analogous to repricing using a different seed. Consequently, one can expect that the price difference observed when using Hull-White one-factor on the one hand, and Hull-White two-factor degenerated to a one-factor model on the other hand, will agree within one Monte Carlo standard deviation.

2.5 Forward-rate matching and Discount matching

Theoretically, when simulating a Hull-White model using an infinite number of Monte-Carlo paths, the simulated states should produce both discount factors and forward rates matching those implied from the initial yield curve. This can be seen from a simple floater valuation in which the forward rate is obtained from the difference between two zero-coupon bonds. Specifically, given the discount factors $D(t)$ from the initial yield curve, one has

$$D(t_1) = N(0)\mathbb{E}^{\mathbb{Q}}\left[\frac{N(t_2)^{-1}}{P(t_1,t_2)}\right], \tag{2.21a}$$

$$D(t_2) = N(0)\mathbb{E}^{\mathbb{Q}}\left[N(t_2)^{-1}\right], \tag{2.21b}$$

where $N(t)$ is the numeraire, $\mathbb{Q}$ denotes the probability measure, and $P(t_1,t_2)$ is t_1-value of the zero-coupon bond of maturity t_2. Both (2.21a) and (2.21b) are model-independent, and their difference gives:

$$\begin{aligned} D(t_1) - D(t_2) &= N(0)\mathbb{E}^{\mathbb{Q}}\left[\Big(\frac{1}{P(t_1,t_2)} - 1\Big)N(t_2)^{-1}\right] \\ &= N(0)\mathbb{E}^{\mathbb{Q}}\left[\tau L(t_1,t_2)N(t_2)^{-1}\right], \end{aligned} \tag{2.22}$$

where $L(t_1,t_2)$ is the t_1-fixed forward simple rate from t_1 to t_2, and τ is the coupon coverage. Thus, a simple floater is model-independent and any forward rate should match the curve-implied value if the discount factors are matched. However, in practice, we can simulate only a finite number of paths. Thus, numerically, limited sampling cannot produce an exact match in either $D(t_1)$ and $D(t_2)$. We therefore apply a moment matching technique by adding an adjustment to the simulated model states to match those discount factors required at any time t of interest. This discount matching is achieved by satisfying,

$$D(t) = N(0)\mathbb{E}^{\mathbb{Q}}\left[N(t)^{-1}\right]. \tag{2.23}$$

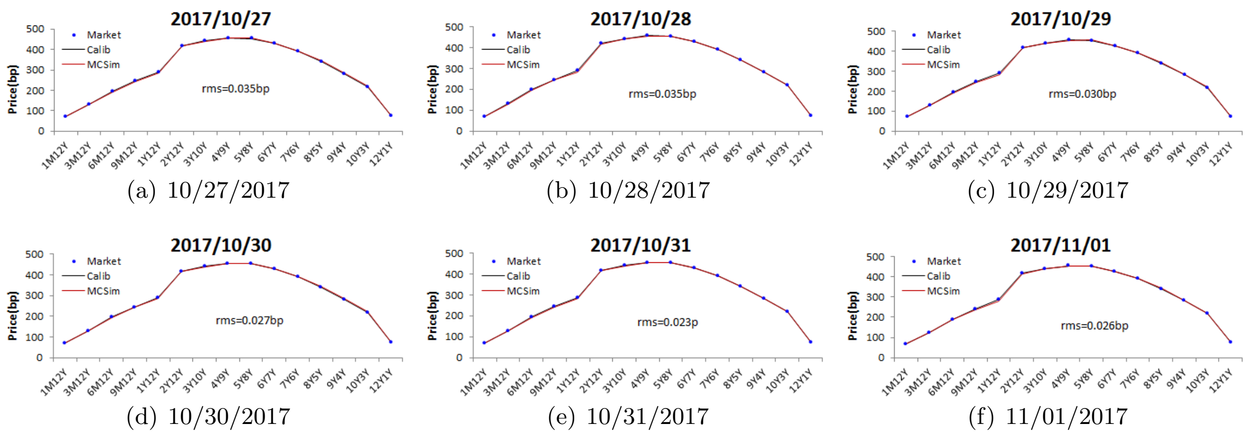
Note that $N(t)$ is a function of model states at t. Returning to the simple floater example, we match both $D(t_1)$ and $D(t_2)$ through (2.23). Hence, we have a perfect match of $D(t_2)$ through (2.21b), while for $D(t_1)$ one cannot simultaneously satisfy (2.21a). As a result, we cannot exactly match the forward rate by satisfying (2.22). In a model which is well-calibrated to real market data, the simulated forward rates have only a tiny difference with those implied by the forward curve. Of course, in extreme test situations, it is possible to generate artificially extreme model volatilities which will produce significantly large differences between these forward rates.

2.6 Calibration Fit to Monte-Carlo Simulation

An essential consideration in model validation, in addition to the verification of an individual price, is that of *stability*, which is to say the robustness of the pricing with respect to variations in the market inputs. Indeed, instabilities in daily pricing can become magnified in the calculation of derived quantities, and be responsible for unexpected changes in the sign of Vega or Theta.

An example of the the Hull-White calibration stability is presented below where a 15Y-NC5Y Bermudan swaption with fixed leg using 3.73% coupon, and floating leg using USD 3M Libor

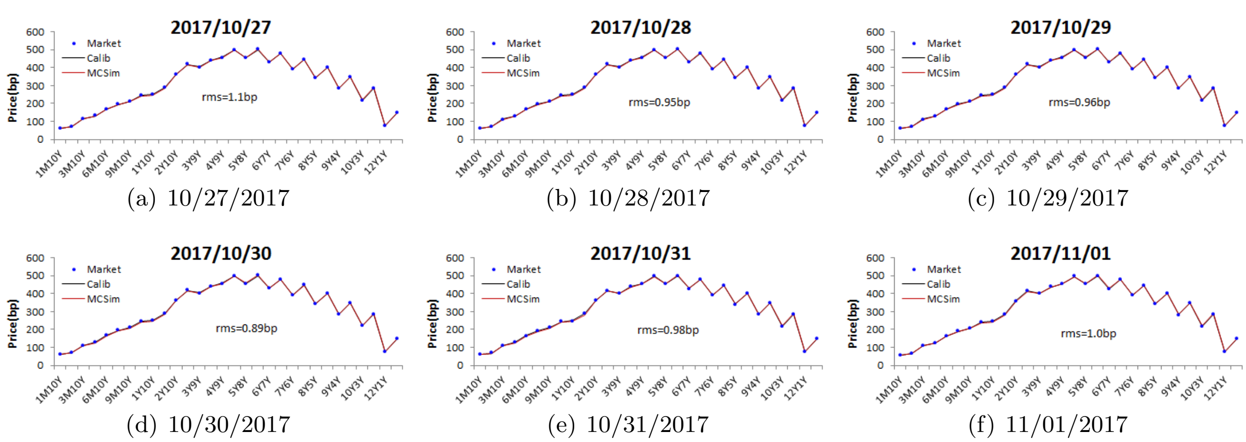
Index, is evaluated daily over a one week period. We present results using the Hull-White one-factor model in Figure 2.1, and using the Hull-White two-factor model in Figure 2.2. Using the model parameters obtained from the ATM calibration, we run a Monte Carlo simulation with 20,000 paths to see how well the original swaption market prices are reproduced. Further detail is provided in §3.5, where Figure 2.2(a) shows comparisons with LMM calibration pricing and stability.



(a) 10/27/2017 (b) 10/28/2017 (c) 10/29/2017

(d) 10/30/2017 (e) 10/31/2017 (f) 11/01/2017

Figure 2.1: Stability of Hull-White one-factor calibration and repricing of in-sample ATM swaption market performed over one week period. Simulations were performed using 20,000 paths.



(a) 10/27/2017 (b) 10/28/2017 (c) 10/29/2017

(d) 10/30/2017 (e) 10/31/2017 (f) 11/01/2017

Figure 2.2: Stability of Hull-White two-factor calibration and repricing of in-sample ATM swaption market performed over one week period. Simulations were performed using 20,000 paths.

3 Shifted Lognormal LIBOR Market Model

3.1 LIBOR Market Specification

The starting point for all pricing is the specification of a deal. Although the details of specifying a deal are not discussed here, certain elements of the deal's structure are required when pricing with the LIBOR Market Model. To the extent that specific underlying indexes which comprise a deal are related to the LIBOR rates (of specific tenor and currency), for example CMS rates or the LIBOR rates themselves, special attention must be given to the schedules implicit in the deal structure. In particular, if the deal requires LIBOR rates $\{F_1, \ldots, F_n\}$ which are fixed at fixing dates[5] $\{T_1^-, \ldots, T_n^-\}$ and are applied to accrual periods $\{T_1, \ldots, T_n, T_{n+1}\}$, then the following LIBOR Market would be diagrammatically specified as follows:

$$\text{today} = T_0 \longrightarrow T_1 \xrightarrow[\tau_1]{F_1} T_2 \xrightarrow[\tau_2]{F_2} \cdots \xrightarrow[\tau_{n-1}]{F_{n-1}} T_n \xrightarrow[\tau_n]{F_n} T_{n+1}$$

The time T_0 indicates the evaluation[6] date of the deal, and the coverages $\{\tau_1, \ldots, \tau_n\}$ are the accrual times given in units of year-fractions, whose exact values are deduced from the the day-count conventions specified by the deal. For example, the following data are associated to the quarterly USD LIBOR index, using the ACT/360 day-count convention and an effective date of December 11, 2012:

k	Reset Date T_k^-	Accrual Start T_k	Accrual End T_{k+1}	Days -	Coverage τ_k	Reset Rate F_k
1	03/11/2013	03/13/2013	06/13/2013	92	0.2555	0.2801
2	06/11/2013	06/13/2013	09/13/2013	92	0.2555	0.2865
3	09/11/2013	09/13/2013	12/13/2013	91	0.2527	0.3113

Table 3.1: US0003M (ACT/360) on 12/10/2012 (First Three LIBOR Periods)

3.2 Model Dynamics

Once the LIBOR Market grid has been specified, the LMM requires some additional model specification. As described briefly in §1.2, DLIB implements a shifted-lognormal model for the evolution of LIBOR rates (or OIS forward rates in the presence of a dual-curve setup[7]) using the *Rebonato two-parameter full-factor* correlation structure and either of the two supported volatility structure (constant and piecewise-constant).

The shifted-lognormal LMM models the dynamics of the LIBOR rates using the following coupled system of stochastic differential equations[8]:

$$dF_k(t) \quad = \quad \mu_k^Q(t)dt + \sigma_k(t)(F_k(t) + \alpha_k)dW_k^Q(t), \tag{3.1}$$

where the various Brownian motions $dW_k^Q(t)$ have the "forward-forward" correlation structure given by $\rho = \big(\rho_{i,j}\big)$:

[5] Fixing dates and forward rates are often referred to as reset dates and reset rates, respectively.
[6] The evaluation date is sometimes referred to as the settlement date, the pricing date, or the as-of date.
[7] When using a dual-curve setup, future references to LIBOR rates should be replaced with OIS forward rates.
[8] The superscript Q here indicates the Q-measure associated with a choice of numeraire.

$$\rho_{i,j} dt \quad := \quad \mathbb{E}\left[\langle dW_i^Q, dW_j^Q \rangle\right]. \tag{3.2}$$

The exact form of the *drift* terms $\mu_k^Q(t)$ appearing in (3.1) will depend on the measure $\mathbb{Q}$, which is to say the particular choice of numeraire, and is described elsewhere in §A.5. What concerns us in this section are the *diffusion* terms $\sigma_k(t)(F_k(t)+\alpha_k)dW_k^Q(t)$, and identifying the model parameters $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$ appearing in (3.1) and (3.2).

Strictly speaking, the model parameters are the functions $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$. However, inasmuch as we mandate parametric forms for the *volatility term structure* and the *correlation structure*, the model parameters are identified with the associated functional parameters.

The Rebonato (two-parameter, full-factor, time-independent) correlation structure is defined by:

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) e^{-\rho_d |T_i - T_j|},$$

and so only the two parameters ρ_d and ρ_∞ must be specified. Here, ρ_∞ is interpreted as a base correlation between arbitrary rates. As one moves away from the diagonal of ones on the correlation matrix ρ, the entries exponentially decay towards ρ_∞ at the rate ρ_d.

When using the constant volatility term structure we have $\sigma_i(t) = \sigma_i$, in which case N volatility parameters must be specified. When using the piecewise constant volatility term structure we have $\sigma_i(t) = \sigma_{i,j}$ for $t \in [T_j, T_{j+1})$, in which case $N(N+1)/2$ model parameters must be specified.

We summarize these two flavors of volatility structure in the table below:

	Constant	Piecewise Constant
$\sigma_i(t)$	σ_i	$\sigma_{i,j}, \quad t \in [T_{j-1}, T_j]$
Parameters	σ_i	$\sigma_{i,j}, \quad j \leq i$

Table 3.2: Volatility-Model Parameters

In summary, the starting point for constructing a shifted-lognormal LIBOR market model consists of specifying a schedule of LIBOR dates, a choice of volatility structure, and values for all of the corresponding model parameters $\{\alpha_k\}; \{\rho_\infty, \rho_d\}; \{\sigma_k\}$ or $\{\sigma_{i,j}\}$ or $\{\Phi_k, a, b, c, d\}$;. A discount curve must also be specified to fully define the LMM. In a single curve setup the initial values $F_i(t)$ of the LIBOR rates are deduced from the discount curve. On the other hand, if supplying a discount curve independently of the specification of forward rates is desired, then one may take advantage of the dual-curve support by specifying an OIS curve and a deterministic basis, as explained in §1.3.

3.3 Calibration

Model parameters are determined by fitting with market data. This is a matter of finding $\rho_{i,j}$ and $\sigma_{i,j}$ (or σ_k or $\{\Phi_k, a, b, c, d\}$) such that the price of a given instrument priced in the market agrees with the price determined by the solutions to the shifted-lognormal SDE (3.1).

Calibration refers to the particular method of finding values of model parameters which produce observable values (e.g. swaption volatilities or cap premiums or cms-spread correlations) of market data. It is generally not possible to find such values of the model parameters, and so one constructs a nonnegative-valued "objective function" of the model parameters which will be zero precisely when the market data is perfectly reproduced, and whose minimization is taken as a "best-fit" of the model parameters to market data. Our choice of objective function is the usual sum of squared differences between market data and model-implied data. Specifically, we construct a vector $\mathbf{e}(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^d$ where d is the number of model parameters being calibrated) whose k'th component is the signed-error in the k'th calibration instrument, so that the vector records all errors on an equal footing, and then take its Euclidean length. Note that if a weighting of the individual errors is desired (e.g. by the inverse of a bid-ask spread), one may replace the Euclidean length with a weighted norm $\sqrt{\sum_{1\leq k\leq M} \omega_k \mathbf{e}_k(\mathbf{x})^2}$ with weights $\omega_k > 0$, where $k = 1, \ldots, M$ and M is the number of instruments.

Since solutions are found by Monte Carlo simulation and the "fit" is in a least-squares sense, the process of calibrating cannot rely on the prohibitively costly procedure of performing Monte Carlo simulations to price a collection of market instruments (e.g. swaption-matrix) over hundreds of candidates for the model parameters. In order to make calibration feasible, approximation formulas have been developed to price these market instruments approximately for a given choice of model parameters without having to perform any Monte Carlo simulations. In this regard, we note that the pricing of caplets can be obtained from an exact formula as derived in §A.6. Furthermore, the pricing of swaptions can be approximated by making the assumption that the swap-rate, which is a stochastic average of its constituent forwards, itself follows shifted-lognormal dynamics. Detailed derivations of the swaption and cms-spread option approximation formulas can be found in [BM].

Calibration requires having market prices (or volatilities, or correlations) of a set of calibration instruments. In general, the choice of these instruments is tuned to the term-sheet being priced. The following instrument types are supported:

- Caplet Normal or Black Volatilities
- Swaption Normal or Black Volatilities
- Cap Normal or Black Volatilities
- ATM CMS-Spread Option Prices (Straddle)[9]
- OTM CMS-Spread Option Prices (Call, Put)[10]
- CMS-Spread Correlations[11]

More generally, users may provide their own choice of parameter values by explicitly overriding the internal defaults. Non-calibrated parameters typically arise when a user has already performed a calibration (in which all shifts, volatilities, and correlations are calibrated parameters, and subsequently wants to perform a quick recalibration of *only the volatilities* to a subset of market data. To explain this more fully, consider the following loosely accurate identifications:

- calibrate to ATM swaptions $\longleftrightarrow$ calibration of volatilities
- calibrate to OTM swaptions $\longleftrightarrow$ calibration of shifts
- calibrate to CMS-spread options $\longleftrightarrow$ calibration of correlations

[9] These may be quoted as Normal Volatility or Spot Premium or Forward Premium.

[10] OTM strikes must be quoted as absolute strikes.

[11] Though an historical correlation between CMS rates may be specified unambiguously, a contributor provided correlation used as a quoting mechanism must be qualified with the contributor's model dependent details.

A concrete example would be to perform a calibration to the ATM and OTM swaption markets and also the CMS-spread option market, which will result in all model parameters being calibrated. Alternatively, one could choose to calibrate the volatility parameters to the ATM swaption market only, while specifying the shifts and correlations as fixed non-calibrated parameters whose initial values would be specified under DLIB's Correlation tab.

CMS Spread Options

When calibrating to CMS spread options, it is important to include among the calibration instruments the swaption instruments which correspond to the reference swaps of each leg of the CMS spread. For example, calibrating to a CMS spread option 10/2 with expiry 3 years should include the 3x10 and the 3x2 swaptions. Also note that the ATM of the CMS spread straddle is a (shifted-lognormal) model-dependent convexity-corrected strike. Unless an absolute strike for the spread is supplied, the ATM strike is computed internally. One expedient pricing formula for the CMS spread premium uses a Gaussian bi-variate model whose ingredients are the normal volatilities (possibly converted from quoted ATM Black volatilities) of the reference swaps comprising the legs of the spread, and also their terminal correlation (which may be historically determined). The well known formula for the resulting normal volatility of the spread

$$\sigma_{spread} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

can be used (if not supplied directly) in a Bachelier pricing formula to price the straddle.

When internally calibrating to the CMS spread option price, whether provided directly as a Spot Premium, or Forward Premium, or indirectly as the implied normal volatility σ_{spread} or implied terminal correlation of the spread, the calibrator uses an approximation formula for the price as a function of model parameters, specifically the volatility and shift parameters of the constituent Libors belonging to the legs. Clearly, without calibration instruments which influence the model parameters associated with the CMS legs, such as swaptions on their underlying reference swaps, the leg volatilities will be indeterminate.

Reduced-Factor Correlation

For reasons of simulation performance (§3.4), it is beneficial to omit random number generation for those Brownian drivers which make essentially no statistically independent contribution. Removing such correlated factors is referred to as "Factor Reduction", and uses a mathematical technique called "Principal Component Analysis" (PCA). The PCA algorithm identifies a small set of eigen-directions (the principal components) to which the correlation matrix can be restricted with minimal loss of statistical information. The user may specify the number of factors to which the calibrated correlations will be reduced, which is used by the calibrator to perform the appropriate factor reduction when it applies PCA to a calibrated full-factor Rebonato correlation structure. It is this reduced-factor correlation structure which will be used by the Monte Carlo simulation for both drift calculations and generation of appropriately correlated random numbers.

Decoupled-Shift Calibration

The "Decoupled-Shift" calibration is an LMM methodology developed to address the regularity and robustness of the calibration of the shift parameters α_k. Although there are available techniques for imposing regularity on the model parameters (*e.g.* regularization with a penalty function), they are inadequate for correcting the erratic behavior visible in DLIB's Vega Scenario Analysis[12]. A smooth variation of calibrated model shifts with respect to systematic bumps to the market volatilities (VCUB) is necessary for an accurate Vega Scenario Analysis.

The Decoupled-Shift calibration method achieves the desired stability and regularity by separating the calibration of the shifts from the calibration of the volatilities. The key idea is to notice that the shift of the swap rate is, to a good approximation, a linear combination of LMM shifts per (A.14) and the swap rate shift, which can be directly computed by referring to the market swaption quotes at different strikes. Essentially, one collects all the equations (A.14) from the pairs of swaption quotes at different strikes, and then analytically solves the system of equations by using a constrained least-squares method. After mapping the market skew into LMM shifts by solving this system of equations, the standard calibration process is performed by fixing the shifts and calibrating the volatility and correlation parameters.

3.4 Monte Carlo Pricing

Once model parameters have been determined by the calibration phase, the Monte Carlo simulation can be performed to generate simulated LIBOR rates which will be used to determine pathwise payoff from which a final deal price can be computed as an expected value.

There are many popular options related to Monte Carlo simulations of discretized Ito processes, namely choice of seed for the random number generator (Mersenne Twister), choice of numeraire, the number of paths, the sampling interval, and additional options related to variance reduction.

Choosing the number of paths is the only option selectable by the DLIB user. With repsect to the choice of numeraire, the Spot Libor Numeraire is always used, and is described more fully in §A.1. With respect to variance reduction techniques, only moment matching is used, which involves adjusting the evolved LIBOR's at each time step to enforce agreement with known zero-coupon bonds, and is described more fully in §A.2.

3.5 Examples

Calibrating to Swaptions and CMS Spread Straddles

In this example we consider calibration using a piecewise-constant volatility model, and using the OTM swaption market but limiting the smiles to [ATM, ATM ±25bp, ATM±50bp].

As explained in §3.3, we have only to select a settlement date, a currency, and a discount curve. We choose the following:

[12] The Vega Scenario Analysis in DLIB involves multiple calibrations to swaption skews by parallel shifting the market volatility data.

```
SettlementDate = 20120227,  Currency = EUR,  DiscountCurve = S45.
```

Finally, we choose a LIBOR Market structure of forty six-month LIBORs consistent with the EUR swaption market and (T+2) convention, and starting one year from settlement (so the 1×20 swaption is included):

$$T_0 \longrightarrow T_1 \xrightarrow{\tau_1} T_2 \xrightarrow{\tau_2} T_3 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{38}} T_{39} \xrightarrow{\tau_{39}} T_{40} \xrightarrow{\tau_{40}} T_{41}$$

$$\text{2-27-2012} \longrightarrow \text{2-28-2013} \xrightarrow{0.5083} \text{8-30-2013} \xrightarrow{0.5056} \text{2-28-2013} \quad \cdots \quad \text{2-27-2032} \xrightarrow{0.5167} \text{8-31-2032} \xrightarrow{0.5028} \text{2-28-2033}$$

For reference, we reproduce the ATM Swaption Market from Feb 29, 2012 in Table 3.3, the OTM Swaption Market from Feb 29, 2012 in Table 3.4, and the ATM spread option strikes and premiums in Table 3.5.

Exp×Ten	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	25y	30y
1m	54.43	40.97	42.33	41.15	39.95	37.75	36.44	35.23	34.54	34.18	30.92	31.18	32.3	33.54
3m	56.83	43.64	44.19	42.69	41.45	38.59	36.78	35.73	35.15	34.88	31.68	31.99	33.13	34.37
6m	61.9	45.42	45.78	43.3	42.18	39.38	37.66	36.61	36.05	35.77	32.51	32.69	33.95	35.25
9m	61.88	45.72	44.86	42.49	41.44	38.8	37.15	36.29	35.84	35.62	32.47	32.55	33.87	35.18
1y	60.65	45.59	43.85	41.78	40.86	38.34	36.74	35.92	35.44	35.19	32.47	32.78	34.0	35.14
2y	60.33	46.01	41.6	38.25	36.88	35.48	34.31	33.62	33.15	32.84	30.88	31.26	32.54	33.59
3y	49.4	40.05	36.71	34.21	33.16	32.05	31.2	30.67	30.25	30.0	28.61	29.56	30.89	31.85
4y	40.79	34.48	32.57	31.29	30.41	29.56	28.85	28.43	28.18	28.04	27.05	27.96	29.47	30.55
5y	34.6	30.38	29.42	28.77	28.22	27.72	27.28	26.98	26.8	26.73	26.31	27.23	28.63	29.48
6y	30.45	28.02	27.44	26.91	26.44	26.08	25.85	25.73	25.73	25.83	25.77	26.48	27.78	28.54
7y	27.98	26.63	26.07	25.55	25.07	24.9	24.84	24.89	25.04	25.22	25.36	26.14	27.19	27.71
8y	26.66	25.47	24.95	24.47	24.15	24.13	24.22	24.41	24.63	24.9	24.92	25.47	26.54	27.12
9y	25.37	24.38	23.91	23.63	23.46	23.6	23.84	24.09	24.4	24.73	25.03	25.37	26.04	26.55
10y	24.26	23.42	23.19	23.07	23.06	23.35	23.65	24.0	24.36	24.77	24.97	25.46	25.91	26.01
15y	24.45	24.5	24.73	24.96	25.24	25.89	26.5	26.89	27.27	27.6	26.6	25.84	24.99	24.76
20y	28.16	28.53	28.82	29.16	29.35	29.71	29.85	30.06	30.1	30.17	27.27	25.26	24.42	23.58
25y	32.41	31.88	31.85	31.42	31.16	30.93	30.7	30.57	30.33	30.07	26.14	24.28	22.71	22.14
30y	29.81	28.8	28.49	27.86	27.25	27.21	27.25	27.32	27.41	27.5	24.14	22.37	21.19	20.17

Table 3.3: ATM Swaption Market Feb 29, 2012

(Exp x Ten)×Vol	ATM	-200bp	-100bp	-50bp	-25bp	ATM	+25bp	+50bp	+100bp	+200bp
1y x 2y	45.59	-	31.242	6.112	2.019	45.59	-0.87	-1.124	-0.833	0.487
1y x 5y	40.86	-	13.641	4.625	1.93	40.86	-1.364	-2.311	-3.377	-3.89
1y x 10y	35.19	37.032	9.928	3.894	1.727	35.19	-1.354	-2.394	-3.743	-4.671
1y x 20y	32.78	31.817	9.47	3.751	1.66	32.78	-1.276	-2.218	-3.308	-3.635
1y x 30y	35.14	40.04	11.57	4.612	2.053	35.14	-1.602	-2.807	-4.264	-4.913
5y x 2y	30.38	18.296	4.992	1.895	0.826	30.38	-0.629	-1.098	-1.684	-2.031
5y x 5y	28.22	17.04	5.247	2.096	0.936	28.22	-0.745	-1.329	-2.113	-2.71
5y x 10y	26.73	16.626	5.313	2.133	0.951	26.73	-0.749	-1.324	-2.054	-2.46
5y x 20y	27.23	20.87	6.334	2.492	1.099	27.23	-0.841	-1.461	-2.186	-2.42
5y x 30y	29.48	25.937	7.469	2.912	1.28	29.48	-0.978	-1.701	-2.556	-2.9
10y x 2y	23.42	11.445	3.644	1.458	0.651	23.42	-0.516	-0.916	-1.445	-1.805
10y x 5y	23.06	13.645	4.456	1.796	0.802	23.06	-0.631	-1.113	-1.717	-2.015
10y x 10y	24.77	16.179	5.097	2.045	0.913	24.77	-0.722	-1.28	-2.004	-2.461
10y x 20y	25.46	22.06	6.245	2.429	1.069	25.46	-0.82	-1.432	-2.178	-2.552
10y x 30y	26.01	23.945	6.506	2.532	1.118	26.01	-0.869	-1.533	-2.387	-2.968

Table 3.4: OTM Swaption Market Feb 29, 2012

Exp	Strike	Premium
1y	1.29	46.10
2y	1.17	59.10
3y	0.92	69.00
4y	0.67	73.90
5y	0.50	78.10
6y	0.37	85.30
7y	0.31	91.20
8y	0.26	95.50
9y	0.18	99.80
10y	0.06	106.90
12y	-0.02	107.70
15y	-0.06	108.40

Table 3.5: CMS 10-2 Spread Straddles

Exp	ATM Strike	Straddle Premium Fit (bp)
1.0y	1.29	0.01
2.0y	1.17	−0.55
3.0y	0.92	−0.35
4.0y	0.67	−0.13
5.0y	0.5	−0.59
6.0y	0.37	−0.06
7.0y	0.31	−0.25
8.0y	0.26	−0.13
9.0y	0.18	−0.02
10.0y	0.06	−0.16

Table 3.6: CMS 10-2 Spread Straddle Calibration Fit

Fit to Monte-Carlo Simulation

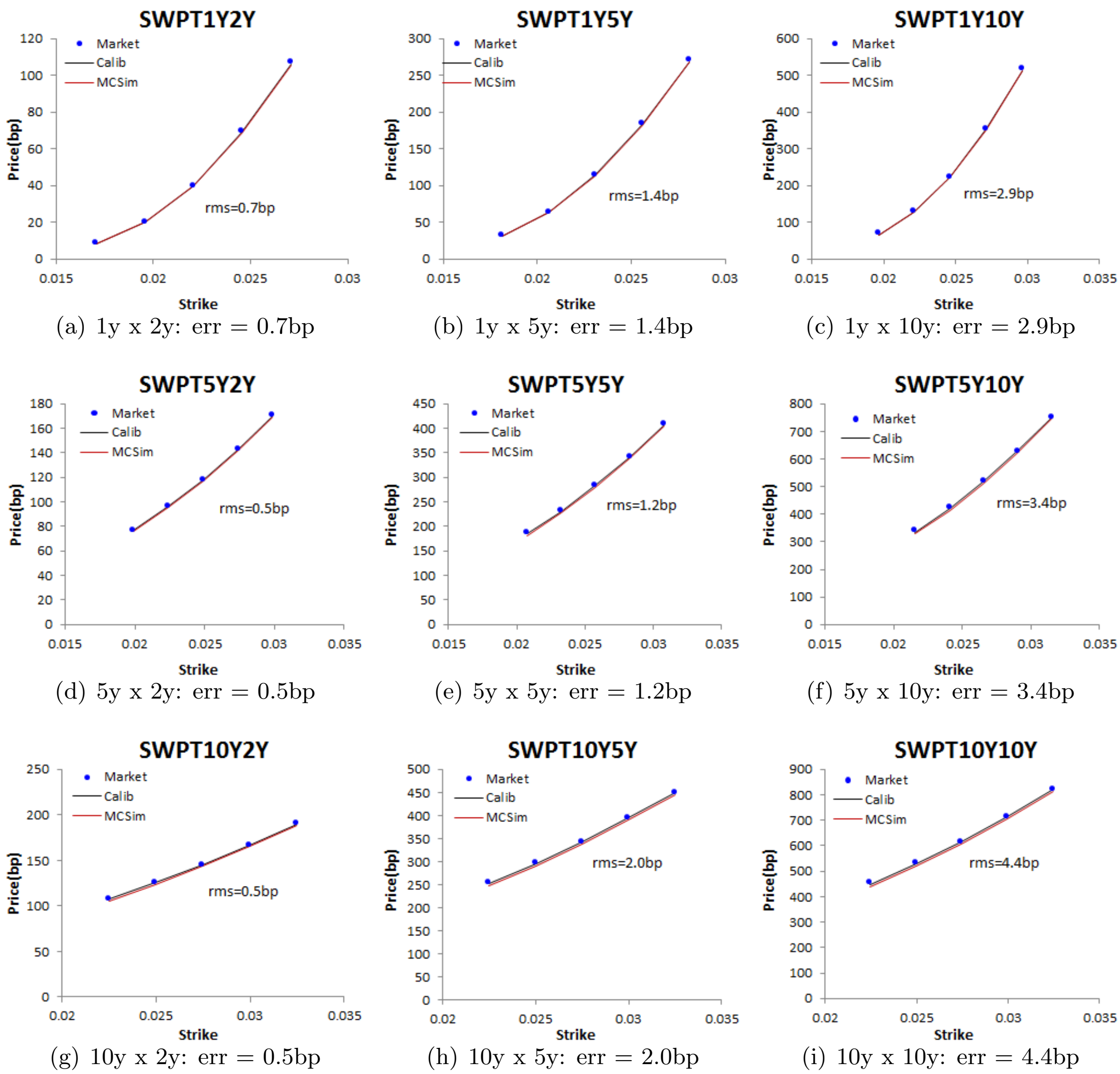
Using the model parameters obtained in the above calibration, we may run a Monte Carlo simulation to see how well the original swaption market prices can be reproduced. Indeed, using 20,000 paths, we obtain a fit of 2.3bp in price. The smile errors in price (omitting the 1x20 for lack of space) are presented in Figure 3.1.

Calibration and Pricing Stability

As already described in §2.6, sability is an essential consideration in model validation, In the figures below, we present a stability analysis of calibration and pricing over a one week period. The deal we consider is a 15Y-NC5Y Bermudan swaption with fixed leg using 3.73% coupon, and floating leg using USD 3M Libor Index. This deal was evaluated using the LMM model, and also with Hull-White for comparison, by calibrating to the *Upper Triangular* section of the swaption-matrix consisting of both ATM and OTM[13] swaptions with a maturity of less than 15 years. Calibration and pricing of the Bermudan swaption was performed on each day using LMM, and also with the Hull-White one-factor (calibrated to 1Y swaptions), and with Hull-White two-factor (calibrated to 1Y and 2Y swaptions) for comparison, as shown in Figure 3.2. Note that all simulations were performed using 20,000 paths.

In Figure 3.3 we illustrate the strikingly consistent re-pricing of the in-sample upper-triangle (ATM and OTM) of calibration instruments evident in daily snapshots over a one week period.

[13]The OTM strike was set to the deal's fixed coupon of 3.73%.



(a) 1y x 2y: err = 0.7bp

(b) 1y x 5y: err = 1.4bp

(c) 1y x 10y: err = 2.9bp

(d) 5y x 2y: err = 0.5bp

(e) 5y x 5y: err = 1.2bp

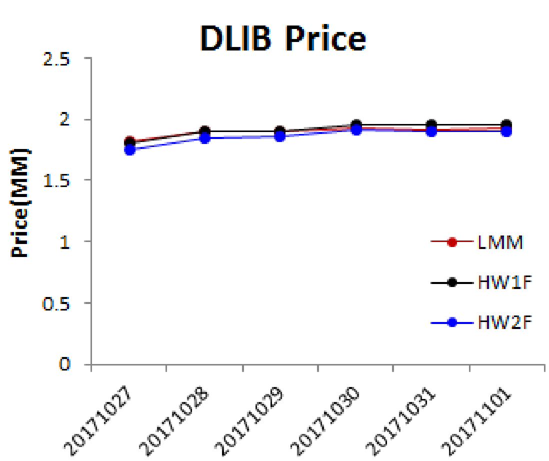
(f) 5y x 10y: err = 3.4bp

(g) 10y x 2y: err = 0.5bp

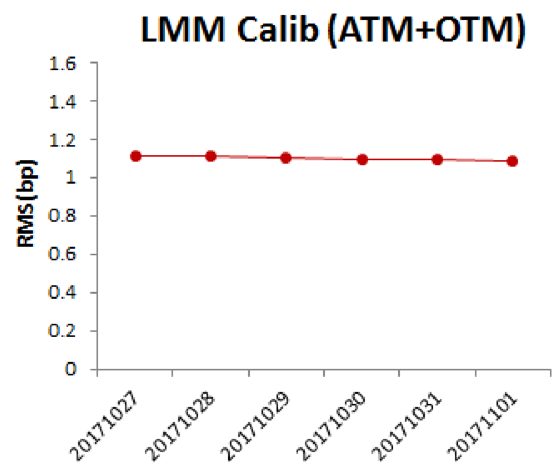
(h) 10y x 5y: err = 2.0bp

(i) 10y x 10y: err = 4.4bp

Figure 3.1: OTM smile errors (rms over strikes: ATM + [-50, 25, 0, 25, 50]bp). Data is obtained on 2017/12/01 from the USD BVOL swaption market. Simulations performed using 20,000 paths.

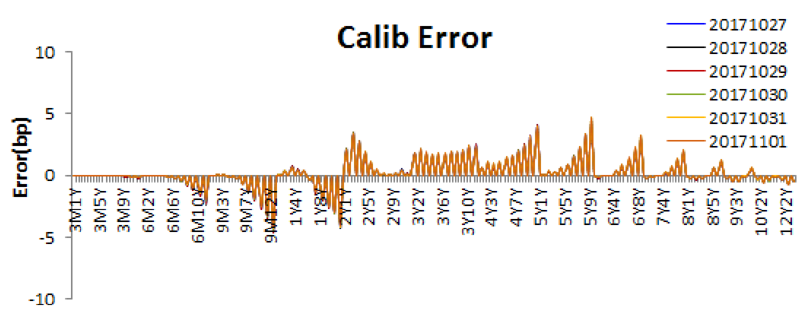


(a) Comparison of daily re-pricing of 15YNC2 Bermudan swaption using LMM, HW1F, and HW2F models.

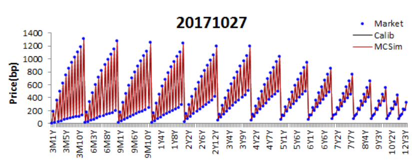


(b) LMM calibration quality in terms of RMS error.

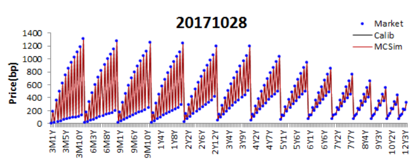
Figure 3.2: Comparison of consistency in daily pricing and calibration for LMM.



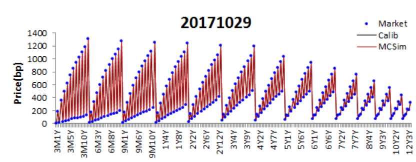
(a) Daily calibration errors per calibration instruments over one week period are so stable as to be indistinguishable day to day.



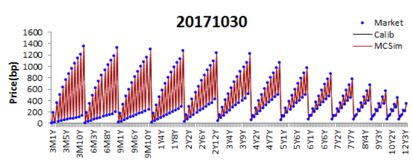
(b) 10/27/2017



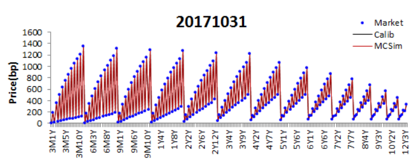
(c) 10/28/2017



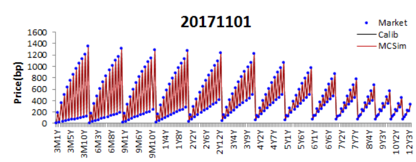
(d) 10/29/2017



(e) 10/30/2017



(f) 10/31/2017



(g) 11/01/2017

Figure 3.3: Stability of calibration and repricing of in-sample market (ATM, OTM) over one week period. Simulations were performed using 20,000 paths.

A Appendices

A.1 Numeraires

Recall that a numeraire is a choice of asset price used to normalize other derivative asset prices. In this section we describe the numeraire used in the LMM pricing. Notationally, $P_T(t)$ denotes the time t value of a zero-coupon bond of maturity T (necessarily a pathwise quantity for $t > 0$), so $P_T := P_T(0)$ is today's value of the bond maturing at time T which is determined from the initial discount curve; and $P_n := P_{T_n}(0)$ is today's value of the bond maturing at the Libor date T_n.

The Spot-Libor numeraire, also called the "discretely rebalanced bank account" numeraire, is a discrete version of the risk-neutral numeraire associated with the continuously compounded money market account. Specifically, its value at time T_0 is \$1. ; at time T_1 has increased in value to P_1^{-1} at which time the LIBOR from T_1 to T_2 has been fixed at $F_1(T_1)$; at time T_2 the numeraire has increased in value to $P_1^{-1}(1 + \tau_1 F_1(T_1))$; and so on, until Libor date T_k at which time its value becomes

$$\begin{aligned} \mathcal{N}(T_k) \quad &:= \quad P_1^{-1} \cdot \prod_{j=1}^{k-1}(1 + \tau F_j(T_k)) \\ &= \quad P_1^{-1} \cdot \prod_{j=1}^{k-1}(1 + \tau_j F_j(T_j)) \qquad \text{(since } F_j \text{ freezes at } T_j). \end{aligned}$$

In other words, starting with a \$1. purchase of bonds paying P_1^{-1} at maturity T_1, one cashes out at each LIBOR date T_k and uses the proceeds to purchase new bonds paying a return of $(1+\tau_k F_k(T_k))$ at maturity T_{k+1}.

A more complete description must account for cashing out the bonds at an intermediate time $T_k \leq t < T_{k+1}$. Using the γ-notation to indicate, at an arbitrary time, the Libor index of the first rate "not yet frozen":

$$\gamma(t) = k + 1 \Longleftrightarrow T_k^- \leq t < T_{k+1}^-,$$

we write more generally (by discounting the overshoot from $(1 + F_k(T_k))$ by $P_{k+1}(t)$):

$$\begin{aligned} \mathcal{N}(t) \quad &:= \quad \mathcal{N}(T_{k+1}) \cdot P_{k+1}(t) \qquad (T_k \leq t < T_{k+1}) \\ &= \quad P_1^{-1} \cdot \prod_{j=1}^{\gamma(t)-1} (1 + \tau_j F_j(T_j)) \cdot P_{\gamma(t)}(t). \end{aligned} \tag{A.1}$$

Strictly speaking $P_{\gamma(t)}(t)$, the time t value of the bond maturing at time T_{k+1} whose rate was reset at $F_k(T_k)$, is not mandated by the discrete rates $F_k(t)$ modeled in the Libor Market Model. A fuller discussion of these off-grid "stub discounts", and their related "stub rates"

$$F(t; t, T_{\gamma(t)}) := \frac{P_{\gamma(t)}(t)^{-1} - 1}{\tau_k}$$

is given in §A.3.

A.2 Unbiasing Algorithm

Unbiasing, also called moment-matching, is a technique used to enforce agreement between simulated quantities, known to be martingales, with their initial values. For example, zero-coupon bonds are tradable assets, paying \$1 at maturity, and when discounted by (any) numeraire their expected values (with respect to the numeraire's associated measure) will, in theory, agree with their initial values. In practice, however, when using a reasonable number of Monte Carlo paths, one finds

$$P_T \;\neq\; \left\langle \mathcal{N}(T)^{-1} \right\rangle ,$$

where we use the notation $\langle\cdot\rangle$ to indicate the sample mean over all paths.

The unbiasing methodology replaces $F_k(T) \to \hat{F}_k(T)$ so that

$$\hat{\mathcal{N}}(T) \;:=\; P_1^{-1} \prod_{k=1}^{\gamma(T)-1} (1+\tau_k \hat{F}_k) \cdot P_{\gamma(T)}(T), \tag{A.2a}$$

$$P_T \;=\; \left\langle \hat{\mathcal{N}}(T)^{-1} \right\rangle . \tag{A.2b}$$

Unbiasing formulas are developed so that all fixed-coupon bonds (with arbitrary coupon schedules) and on-grid float-coupon bonds are exactly priced, while the off-grid float-coupon bonds are priced very nearly exactly.

A.3 Market Aligned Libor Grid

There are certain scheduling dates that are naturally included when determining the Libor grid. One perspective is that T_0 is the deal's evaluation date, and T_n is the deal's horizon date. Another view is that T_0 is today's date, and T_n is the maturity of the furthest dated calibration instrument. These views have specific implications for calibration and simulation, and in the following paragraphs we describe the latter "market-aligned" perspective used in DLIB.

The "market aligned" approach is more natural for the purist, whose view is that the unadulterated quotes of the liquid instruments in today's market are the most reliable inputs to which model parameters should be calibrated. In this view, the Libor grid should agree with that of the prevailing swap whose tenor encompasses the market instruments relevant to the deal. This choice will, for the most part, give consistency between the Libor grid and the calibration instruments, and hence will be straightforward for the calibration phase. On the other hand, if the deal dates, which will include fixing and accrual schedules of Libor and CMS underlyings relevant to the deal, is misaligned with today's market (for example, when pricing an aged deal), then some interpolation methodology at the pricing phase will have to account for the mismatch between those underlying Libors being priced and the internally simulated Libor states.

The interpolation scheme used in adapting a market-aligned Libor grid to the pricing of off-grid Libor rates amounts to the evaluation of "short-dated bonds" $P_{\gamma(t)}(t)$ and their associated "stub rates" $L(t; T, T_{\gamma(T)})$, namely the forward rate from $T \to T_{\gamma(T)}$ evaluated at time t. This approach suffers from some performance degradation during the pricing phase, but is completely flexible in that it does not impose any constraint on the set of pricing dates. It should be remarked that when

$$t < T_k < T < T_{k+1} \quad \gamma(T) = k+1,$$

the evaluation of the interpolated stub rate $L(t; T, T_{k+1})$ is determined by the arbitrage-free requirement to be a multiple (whose value depends only on the initial discount curve) of the simulated state $[L(t; T_k, T_{k+1}) + \alpha_k]$:

$$[L(t; T, T_{k+1}) + \alpha_k] \quad = \quad c \cdot [L(t; T_k, T_{k+1}) + \alpha_k]\,. \tag{A.3}$$

On the other hand, when $T_k \le t \le T < T_{k+1}$, which includes the special case when $t = T$ relevant to evaluating $P_{\gamma(T)}(T)$, the right hand side of (A.3) has been frozen at L_k's fixing time T_k^-, and so the stub rate will suffer from the *premature freezing* of $L(T_k; T_k, T_{k+1})$ which has no evolution between T_k and t. The method of [Wer] was adopted to overcome this premature freezing by allowing L_k to continue evolving for $T_k \le t < T_{k+1}$, and was found to have many advantages over competing algorithms.

Finally, it should be noted that the evaluation of stub rates described here is critical to the time t evaluation of discount factors, in particular the numeraire $\mathcal{N}^S(t)$ discussed in §A.1, when t is not a grid point T_k.

A.4 Shifted-Lognormal LMM Distribution

As described in §1.2, in the shifted-lognormal LMM each forward rate F_j evolves under the corresponding forward measure $Q^{T_{j+1}}$ as a shifted geometric Brownian motion. This implies

$$d(F_j(t) + \alpha_j) \quad = \quad \sigma_j(t)(F_j(t) + \alpha_j)\; dW_j^j(t),$$

from which it follows that the forward rate F_j can be explicitly written as

$$F_j(T) \quad = \quad -\alpha_j + (F_j(t) + \alpha_j) e^{-\frac{1}{2}\int_t^T \sigma_j^2(u)\,du + \int_t^T \sigma_j(u)\,dW_j^j(u)} \qquad t < T \le T_j. \tag{A.4}$$

The distribution of $F_j(T)$, conditional on $F_j(t)$, $t < T \le T_{j-1}$, is then shifted-lognormal with density

$$p_{F_j(T)|F_j(t)}(x) \quad = \quad \frac{1}{(x + \alpha_j) U_j(t,T)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \frac{x+\alpha_j}{F_j(t)+\alpha_j} + \frac{1}{2} U_j^2(t,T)}{U_j(t,T)} \right)^2 \right\}, \tag{A.5}$$

for $x > -\alpha_j$, where $U_j(t,T)$ is the cumulative volatility defined by

$$U_j(t,T) \quad := \quad \sqrt{\int_t^T \sigma_j^2(u) du}\;.$$

A.5 Shifted-Lognormal LMM Drift Terms

We give below the dynamics of the forwards $F_j(t)$ in the Spot LIBOR measure Q_S associated with the Spot LIBOR numeraire (A.1).

Recalling (3.1), we write

$$\begin{aligned} dX_j(t) &= \mu_j^Q(t)dt + \sigma_j(t)X_j(t)dW_j^Q(t), && \text{(A.6a)} \\ X_j(t) &:= F_j(t) + \alpha_j. && \text{(A.6b)} \end{aligned}$$

The drifts μ_j^Q of the forwards F_j are computed by requiring that the forwards be martingales in their own T_j-forward measure, *i.e.* by applying the change-of-numeraire technique to

$$dX_j(t) = \sigma_j(t)X_j(t)dW_j^{Q_{T_j}}(t). \quad \text{(A.7)}$$

Derivations of the explicit expression for μ_j^Q are provided in [BM, §6.3], and give the following:

$$\mu_j^{Q_S}(t) = \sigma_j(t)X_j(t)\sum_{i=\gamma(t)}^{j}\frac{\rho_{i,j}\sigma_i(t)\tau_i X_i(t)}{1+\tau_i F_i(t)}. \quad \text{(A.8)}$$

A.6 Shifted-Lognormal LMM Caplet and Swaption Pricing

In this section we first show that caplet prices in the shifted-lognormal LMM can be calculated in closed form. Consider a caplet whose payoff, fixed at T_j and paid at T_{j+1}, is given by

$$\tau_j[F_j(T_j) - K]^+$$

where K is its strike price. Then, assuming unit notional, the price of the caplet can be calculated as follows:

$$\begin{aligned} \mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) &= \tau_j P(0, T_{j+1})\mathbb{E}^{Q_{j+1}}\left[(F_j(T_j) - K)^+\right] \\ &= \tau_j P_{j+1}\mathbb{E}^{Q_{j+1}}\left[\big(F_j(T_j) + \alpha_j - (K + \alpha_j)\big)^+\right]. && \text{(A.9)} \end{aligned}$$

Since, according to (3.1), $X_j(T_j) = F_j(T_j) + \alpha_j$ is a lognormal random variable in the Q_j measure, the last expectation in (A.9) yields the adjusted Black caplet price that corresponds to a shifted geometric Brownian motion:

$$\mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) = \tau_j P_{j+1}\mathrm{Bl}\big(K + \alpha_j, F_j + \alpha_j, V_j(T_j)\big) \quad \text{(A.10)}$$

where the terminal volatility up to time T_j is given by

$$V_j(T_j) := \sqrt{\int_0^{T_j} \sigma_j^2(s)\, ds}\ ,$$

and

$$\mathrm{Bl}(K,F,v) := F\cdot\Phi\left(\frac{\ln(F/K)+v^2/2}{v}\right) - K\cdot\Phi\left(\frac{\ln(F/K)-v^2/2}{v}\right) \tag{A.11}$$

is the Black-Scholes formula with Φ denoting the standard normal distribution function.

In order to derive a formula for the price of a swaption similar to (A.10), it is not possible to follow the above argument exactly since the sum of shifted-lognormal processes is not itself a shifted-lognormal process, On the other hand, a similar analysis can nonetheless be used to develop an *approximate formula* for pricing a swaption. One strives to model the swap rate S^{ab} as a shifted-lognormal process in the forward swap measure Q^{ab} associated with the annuity numeraire A^{ab}, by determining scalar parameters σ^{ab}, α^{ab}, and a single Q^{ab}-Brownian motion $W^{ab}(t)$, such that

$$\begin{aligned} X^{ab} &:= S^{ab}+\alpha^{ab}, \\ dX^{ab} &\approx \sigma^{ab}(t)\cdot X^{ab}(t)\cdot dW^{ab}(t). \end{aligned}$$

Once this is done, the caplet pricing formula (A.10) is easily adapted to a swaption pricing formula by making the following substitutions:

$$T_j \longrightarrow T_a, \qquad \tau_j P_{j+1} \longrightarrow A^{ab}, \qquad F_j \longrightarrow S^{ab}, \qquad \alpha_j \longrightarrow \alpha^{ab}, \qquad \sigma_j \longrightarrow \sigma^{ab}.$$

Specifically, consider the following quantities from which the swap rate from T_a to T_b is derived:

$$X_k(t) := F_k(t)+\alpha_k, \tag{A.12a}$$

$$A^{ab}(t) := \sum_{k=a}^{b-1}\tau_k P_t(T_{k+1}), \qquad \text{(annuity)} \tag{A.12b}$$

$$\omega_k^{ab}(t) := \frac{\tau_k P_t(T_{k+1})}{A_{ab}(t)} \qquad k=a,\dots,b-1, \tag{A.12c}$$

$$S^{ab}(t) := \sum_{k=a}^{b-1}\omega_k^{ab}(t)F_k(t). \qquad \text{(swap rate)} \tag{A.12d}$$

Following [BM, §6.15], one derives the following approximate shifted-lognormal dynamics of S^{ab}:

$$dS^{ab}(t) \approx \big(S^{ab}(t)+\alpha^{ab}\big)\cdot\big\langle\gamma_{ab}, dW_{ab}\big\rangle, \tag{A.13}$$

where α^{ab} is the "shift of the swaption"

$$\alpha^{ab} := \sum_{k=a}^{b-1}\omega_k^{ab}\alpha_k,$$

and dW_{ab} is the "row vector of correlated Brownian motions"

$$dW_{ab}(t) := \big[dW_a(t),\dots,dW_{b-1}(t)\big],$$

and γ_{ab} is the "vectorized volatility of the swaption"

$$\gamma_{ab}(t) \quad := \quad \frac{\left[\tau_a P_{a+1} X_a \sigma_a(t), \ldots, \tau_{b-1} P_b X_{b-1} \sigma_{b-1}(t)\right]}{\sum_{k=a}^{b-1} \tau_k P_{k+1} X_k},$$

and the constants $X_k = X_k(0)$ and $P_k = P_0(T_k)$ are values frozen at time T_0. Furthermore, by giving the explicit expression for the variance of the one dimensional Brownian motion $\langle \gamma_{ab}, dW_{ab} \rangle$ in terms of a quadratic form in the model parameters:

$$\begin{aligned}
\rho_{ab} \quad &:= \quad \left(\rho_{i,j}\right)_{i=a,\ldots,b-1; j=a,\ldots,b-1} \\
\langle \gamma_{ab}, dW_{ab} \rangle^2 \quad &= \quad \langle \gamma_{ab}, dW_{ab} \rangle \cdot \langle dW_{ab}, \gamma_{ab}{}' \rangle = \gamma_{ab} \cdot dW'_{ab} \cdot dW_{ab} \cdot \gamma_{ab}{}', \\
\mathbb{E}\left[\langle \gamma_{ab}, dW_{ab} \rangle^2\right] \quad &\overset{(3.2)}{=} \quad \left(\gamma_{ab} \cdot \rho_{ab} \cdot \gamma_{ab}{}'\right)\, dt = \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle dt,
\end{aligned}$$

we can simplify (A.13) by introducing the "volatility of the swaption"

$$\sigma^{ab}(t) \quad := \quad \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle^{\frac{1}{2}}. \tag{A.14}$$

Using the above we can write the dynamics of $S^{ab}(t)$ in terms of (approximate) shifted-lognormal parameters α^{ab} and $\sigma^{ab}(t)$ and Q^{ab}-Brownian motion $dW^{ab}(t)$:

$$\begin{aligned}
X^{ab}(t) \quad &:= \quad S^{ab}(t) + \alpha^{ab}, \\
dW^{ab}(t) \quad &:= \quad \langle \gamma_{ab}, dW_{ab} \rangle / \sigma^{ab}, \\
dX^{ab}(t) \quad &\approx \quad \sigma^{ab}(t) \cdot X^{ab}(t) \cdot dW^{ab}(t).
\end{aligned} \tag{A.15}$$

We can thus price swaptions by following the same procedure as in the previous caplet case. To this end, we consider a European payer swaption with maturity T_a and strike K, whose underlying swap pays on times $T_{a+1}, \ldots, T_b$. Assuming unit notional, the swaption price at time zero can be calculated from the expectation (of a ratio of tradables) in the forward swap measure as follows:

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\frac{\left(P_{T_a}(T_a) - P_{T_a}(T_b) - \sum_{k=a}^{b-1} \tau_k P_{T_a}(T_{k+1}) K\right)^+}{\sum_{k=a}^{b-1} \tau_k P_{T_a}(T_{k+1})}\right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(S^{ab}(T_a) - K\right)^+\right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(X^{ab}(T_a) - (K + \alpha^{ab})\right)^+\right].
\end{aligned} \tag{A.16}$$

As (A.15) expresses $X^{ab}(t)$ as an approximate geometric Brownian motion under Q^{ab}, we obtain in analogy to (A.10):

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &\approx \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab}, X^{ab}(0), \Gamma^{ab}(T_a)\big) \\
&= \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab}, S^{ab}(0) + \alpha^{ab}, \Gamma^{ab}(T_a)\big),
\end{aligned} \tag{A.17}$$

where[14]

$$\Gamma^{ab}(T_a) := \sqrt{\int_0^{T_a} \left(\sigma^{ab}\right)^2 ds} = \sqrt{\sum_{i,j=a}^{b-1} \rho_{i.j} \int_0^{T_a} \gamma_{ab,i}(s)\ \gamma_{ab,j}(s)\ ds}.$$

[14]That the variance $\mathbb{E}\left[\left(\int_0^{T_a} \sigma^{ab}(s)\ dW^{ab}(s)\right)^2\right]$ of $X^{ab}(T_a)$ is $\left(\Gamma^{ab}(T_a)\right)^2$ follows from the Itô-isometry property.

A.7 Shifted-Lognormal LMM CMS-SO Pricing Approximation

In this section we present an approximation formula for the CMS spread option price, which gives straddle, call, and put prices as a function of the strike and the model parameters $\alpha_k, \sigma_k, \rho_{i,j}$.

The basic tool for pricing the difference of two instruments is the Margrabe spread option pricing formula [Mar]. Now the Margrabe formula applies to the difference between two processes which are lognormal in a common measure. The first issue to confront is adapting the Margrabe formula to the difference between two shifted-lognormal processes, which is easily addressed. Less straightforward is the second issue, namely that the formulas for the CMS rates are naturally derived in their respective annuity measures Q^{ab} and Q^{ac}, and therefore these formulas must be adapted to a common measure which we choose to be that of the T_a-Forward. Thirdly, it will turn out that the CMS rates are no longer shifted-lognormal processes once adapted to this shared Q^{T_a} measure, and so we will need to invoke the technique of moment-matching to determine genuine shifted-lognormal process which best approximate (in the sense of agreement with their first two moments) the CMS rates.

Summarizing, in order to price CMS spread options we need to consider the two CMS rates $S^{ab}(t)$ and $S^{ac}(t)$, $a < b < c$, at time T_a under the same measure Q^{T_a}. Having derived the dynamics of the CMS rate processes in their natural annuity measures, the adapted formulas giving their moments in the common Q^{T_a} measure can be deduced by applying the change-of-measure technique. Next, one performs moment-matching to determine a new shifted-lognormal processes under the Q^{T_a} measure which possess the same first and second moments as the original processes. Finally, one invokes a variant of the Margrabe spread option pricing formula adapted to two shifted-lognormal underlying processes.

We mandate a new shifted-lognormal process under the Q^{T_a} measure of the following form:

$$\hat{S}^{ab}(T_a) = -\alpha^{ab} + (\hat{S}^{ab}(0) + \alpha^{ab})e^{-\frac{1}{2}T_a(\hat{\sigma}^{ab})^2 + \hat{\sigma}^{ab}W_{cms1}(T_a)}$$

Here, we use the same shift α^{ab} as that of $S^{ab}(T_a)$ in the Q^{ab} measure.

Since $\mathbf{E}^{T_a}[S^{ab}(T_a)]$ and $\mathbf{E}^{T_a}[S^{ab}(T_a)^2]$ can be derived from their moments $\mathbf{E}^{ab}[S^{ab}(T_a)^n]$ in the annuity measure Q^{ab} using the Linear Swap Model [Pel], we have just determined the parameters $\hat{S}^{ab}(0)$ and $\hat{\sigma}^{ab}$ in our moment matched process $\hat{S}^{ab}(T_a)$ in terms of $\alpha^{ab}, \sigma^{ab}, \rho$, which is to say in terms of the shifted-lognormal LMM model parameters. A similar calculation produces the moment matched process of the second CMS rate $\hat{S}^{ac}(T_a)$.

The final step before applying the Margrabe spread option formula is to establish the correlation between $W_{cms1}(T_a)$ and $W_{cms2}(T_a)$:

$$\begin{aligned}
\rho &:= Correl\left(\ln\left(\frac{\hat{S}^{ab}(T_a)+\alpha^{ab}}{\hat{S}^{ab}(0)+\alpha^{ab}}\right), \ln\left(\frac{\hat{S}^{ac}(T_a)+\alpha^{ac}}{\hat{S}^{ac}(0)+\alpha^{ac}}\right)\right) \\
&\approx \frac{\int_0^{T_a}\left\langle\gamma_{ab}, \gamma_{ac}\rho_{ac}^{ab}\right\rangle dt}{\sqrt{\int_0^{T_a}\left\langle\gamma_{ab}, \gamma_{ab}\rho_{ab}\right\rangle dt \ \int_0^{T_a}\left\langle\gamma_{ac}, \gamma_{ac}\rho_{ac}\right\rangle dt}}
\end{aligned} \tag{A.18}$$

where ρ^{ab}_{ac} is the rectangular sub-matrix of ρ given by

$$\begin{pmatrix} \rho_{a,a} & \cdots & \rho_{a,b} \\ \vdots & \rho_{i,j} & \vdots \\ \rho_{c,a} & \cdots & \rho_{c,b} \end{pmatrix}.$$

By making the additional definitions where the symbol x indicates an integration parameter:

$$\begin{aligned} \widetilde{K} &= K + \alpha^{ab} - \alpha^{ac}, \\ X_1(x) &= \hat{S}^{ab}(0) \exp\left(\hat{\sigma}^{ab} x \sqrt{T_a} - \frac{1}{2} T_a (\hat{\sigma}^{ab})^2\right), \\ X_2(x) &= \hat{S}^{ac}(0) \exp\left(\hat{\sigma}^{ac} x \rho \sqrt{T_a} - \frac{1}{2} \rho^2 T_a (\hat{\sigma}^{ac})^2\right), \end{aligned}$$

we can express the price of a call option on a CMS spread of strike K using the Margrabe spread option formula[15]:

$$\begin{aligned} \mathbf{CMSSC}(0; T_a, K, \hat{S}^{ab}, \hat{\sigma}^{ab}, \alpha^{ab}, \hat{S}^{ac}, \hat{\sigma}^{ac}, \alpha^{ac}, \rho) &= B_a(0) \cdot \mathbf{E}^{T_a}\left[\left(\hat{S}^{ac}(T_a) - \hat{S}^{ab}(T_a) - K\right)^+\right] \\ &\approx B_a(0) \cdot \int_{-\infty}^{\infty} \mathrm{Bl}(X_2(x), X_1(x) + \widetilde{K}, \sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac}) \frac{e^{-x^2/2} dx}{\sqrt{2\pi}}. \end{aligned}$$

The price of a put option on a CMS spread of strike K is derived similarly to (A.19):

$$\begin{aligned} \mathbf{CMSSP}(0; T_a, K, \hat{S}^{ab}, \sigma^{ab}, \alpha^{ab}, \hat{S}^{ac}, \sigma^{ac}, \alpha^{ac}, \rho) &= B_a(0) \cdot \mathbf{E}^{T_a}\left[\left(-\hat{S}^{ac}(T_a) + \hat{S}^{ab}(T_a) + K\right)^+\right] \\ &\approx B_a(0) \cdot \int_{-\infty}^{\infty} \mathrm{Bl}(X_1(x) + \widetilde{K}, X_2(x), \sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac}) \frac{e^{-x^2/2} dx}{\sqrt{2\pi}}. \end{aligned}$$

A.8 Dual Curve Calculations

When calculating in a dual-curve setup, one makes the distinction between the OIS forward rates derived from the OIS discount curve $P_t(T)$

$$F_k(t) := \frac{1}{\tau_k}\left[\frac{P_t(T_k)}{P_t(T_{k+1})} - 1\right], \tag{A.19}$$

and the LIBOR rates $L_k(t)$ derived from the quoted LIBOR index. In particular, the implied LIBOR spread over OIS, associated with each LIBOR rate $L_k(t)$, is given by

$$\beta_k(t) := L_k(t) - F_k(t). \tag{A.20}$$

[15]Strictly speaking, the Margrabe formula applies only to the exchange of one lognormally distributed asset for another, which is to say when the strike K of their difference is equal to zero. An adjustment to the formula for non-zero strike, which replaces Margrabe's closed form solution with an integration formula, is presented in [BM, §App.E]. The shifts of the shifted lognormal processes $\hat{S}^{ab}, \hat{S}^{ac}$ are absorbed into the strike K, and account for the strike $\widetilde{K}$.

As explained in §1.3, we make the simplifying assumption that $\beta_k(t)$ is time independent, hence $L_k(t) = F_k(t) + \beta_k$ where $\beta_k := L_k(0) - F_k(0)$. As only the OIS forward rates $F_k(t)$ are evolved in the Monte Carlo simulation, we need only explore the implications of the basis adjustment to calibration and deal pricing. Using the notation from (A.12), consider the following expression for the swaption price $\mathbf{PS}(t; a, b, K)$ on a payer swap from T_a to T_b struck at K:

$$\begin{aligned}
S^{ab}(t) \quad &:= \quad \sum_{k=a}^{b-1} \omega_k^{ab}(t) L_k(t) \\
&\overset{(A.20)}{=} \quad \sum_{k=a}^{b-1} \omega_k^{ab}(t)(F_k(t) + \beta_k), \qquad \text{(swap rate[16])} \\
\mathbf{PS}(t; a, b, K) \quad &= \quad N_{\mathbb{Q}}(t)\ \mathbb{E}_t^{\mathbb{Q}} \left[\frac{A^{ab}(T)\big(S^{ab}(T) - K\big)^+}{N_{\mathbb{Q}}(T)} \right],
\end{aligned}$$

which, when priced with respect to the forward swap measure $\mathbb{Q} = Q^{ab}$ at $t = 0, T = T_a$, gives

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}} \left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[F_k(T_a) - (K - \beta_k)\big] \right)^+ \right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}} \left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[(X_k(T_a) - (K + \alpha_k - \beta_k)\big] \right)^+ \right] \\
&= \quad A^{ab}(0)\ \mathbb{E}^{Q^{ab}} \left[\left(X^{ab}(T_a) - \sum_{k=a}^{b-1} \omega_k^{ab}(T_a)(K + \alpha_k - \beta_k) \right)^+ \right]. \qquad \text{(A.21)}
\end{aligned}$$

Comparing (A.21) with the single-curve pricing of a swaption with respect to the shifted lognormal LMM as in (A.16), we obtain the analog of (A.17) in the dual-curve setup:

$$\begin{aligned}
\mathbf{PS}(0; a, b, K) \quad &\approx \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, X^{ab}(0), \Gamma^{ab}(T_a)\big) \\
&= \quad A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, S^{ab}(0) + \alpha^{ab} - \beta^{ab}, \Gamma^{ab}(T_a)\big) \qquad \text{(A.22)}
\end{aligned}$$

where

$$\beta^{ab} := \sum_{k=a}^{b-1} \omega^{ab}(0)\beta_k(0).$$

Setting $b = a + 1$ in (A.22) gives the analog of (A.10) for dual-curve caplet pricing. In other words, the dual-curve calibrated shifts are essentially the single-curve shifts $(\alpha_k - \beta_k)$, calibrated to the dual-curve quoted swaptions using the dual-curve ATM, and then adjusted by the LIBOR spreads β_k. As explained earlier, the Monte Carlo simulation evolves the OIS forward rates $F_k(t)$, or more precisely the $\log(F_k(t) + \alpha_k)$. Therefore, with respect to pricing, when a pathwise LIBOR underlying is required for evaluating a payoff, the returned rate is obtained from the OIS forward by the addition of the basis adjustment $F_k \to F_k + \beta_k$.

[16] As noted in §1.3, this expression differs from the single-curve formula by the quantity $\sum_{k=a}^{b-1} \omega_k^{ab}\beta_k$.

A.9 Hull-White Swaption Pricing

Referring to [BM, §4.2.4] for detailed derivations, we present below the exact formula for pricing a European swaption with fixing time T_{ex}, strike K, and swap schedule $\mathcal{T} = \{t_0 = a, t_1, \ldots, t_n = b\}$ and corresponding coverages $\{\alpha_1, \ldots, \alpha_n\}$:

$$\begin{aligned}\mathbf{PS}(0; a, b, K) &= P_0(T_{ex})\mathbb{E}^{Q_{T_{ex}}}\left[\left(P_0(t_0) - \sum_{i=1}^{n} c_i P_0(t_i)\right)^+\right] \\ &= \frac{1}{\sqrt{2\pi\nu_{22}}}\int_{-\infty}^{\infty}\left[J_0(z) - \sum_{i=1}^{n} c_i J_i(z)\right]dz, \end{aligned} \tag{A.23}$$

where $P_0(t)$ from (2.18) is the discount to time t, and

$$\begin{aligned}\boldsymbol{H}_i &\overset{(2.15)}{:=} H_{t_i}(T_{ex}) = [H_1(t_i - T_{ex}), H_2(t_i - T_{ex})], \\ \nu_{i,j} &\overset{(2.16)}{:=} \int_0^{T_{ex}} h_{T_{ex},i}(u) q_{i,j}(u) h_{T_{ex},j}(u) du, \\ \boldsymbol{Q} &:= \begin{bmatrix}\nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22}\end{bmatrix}, \\ \boldsymbol{q} &:= [\nu_{21}, \nu_{22}], \\ W_i(z) &:= \frac{\det(\boldsymbol{q}, \boldsymbol{x}(z)) + H_1(t_i - T_{ex}) \det \boldsymbol{Q}}{\sqrt{\nu_{22} \det \boldsymbol{Q}}}, \\ J_i(z) &:= P_0(t_i)\Phi\left(-W_i(z)\right)\exp\left[-\frac{\left(z + \langle \boldsymbol{q}, \boldsymbol{H}_i\rangle\right)^2}{2\nu_{22}}\right],\end{aligned}$$

and $\boldsymbol{x}(z) := [x(z), z]$ uniquely solves the equation

$$P_0(t_0) = \sum_{i=1}^{n} c_i P_0(t_i) \exp\left[-p_i(\boldsymbol{x}) + p_0(\boldsymbol{x})\right], \tag{A.24}$$

where

$$c_i := \alpha_i K + \delta_i^n, \tag{A.25}$$

$$p_i(\boldsymbol{x}) := \langle \boldsymbol{H}_i, \boldsymbol{x}\rangle + \tfrac{1}{2}\langle \boldsymbol{H}_i, \boldsymbol{Q}\boldsymbol{H}_i\rangle. \tag{A.26}$$

Hull-White one-factor swaption pricing

In the Hull-White one-factor model, (A.23) takes the simpler [BM, §3.3] form

$$\mathbf{PS}(0; a, b, K) = J_0(z) - \sum_{i=1}^{n} c_i J_i(z), \tag{A.27}$$

where

$$\begin{aligned}\nu &:= \int_0^{T_{ex}} h^2_{T_{ex}}(u)\sigma^2(u)du,\\ W_i(z) &:= \frac{z + (H(t_i) - H(t_0))\,\nu}{\sqrt{\nu}},\\ J_i(z) &:= P_0(t_i)\Phi\left(-W_i(z)\right),\end{aligned}$$

and z uniquely solves (A.24) using the scalar version of (A.26):

$$p_i(z) \quad := \quad H_i \cdot z + \tfrac{1}{2}H_i^2 \cdot \nu.$$

Note that in the case of a caplet where $n{=}1$, $t_0{=}T_{ex}$, and $t_1{=}T_M$, (2.19) follows from (A.27):

$$c_1 = (1+\tau K), \qquad H_{t_0} = 0, \qquad H_{t_1} = H(T_M - T_{ex}),$$

$$z \cdot H_{t_1} \overset{(A.24)}{=} \ln\left[(1+\tau K)P_0(T_M)/P_0(T_{ex})\right] - \tfrac{1}{2}H_{t_1}^2\nu,$$

$$W_0(z) = \frac{z}{\sqrt{\nu}} \overset{(2.19b)}{=} -d_1, \qquad W_1(z) = \frac{z + H_{t_1}\nu}{\sqrt{\nu}} \overset{(2.19b)}{=} -d_2$$

$$\mathbf{PS}(0; T_{ex}, T_M, K) \overset{(A.27)}{=} J_0(z) - (1+\tau K)J_1(z) = P_0(T_{ex})\Phi(d_1) - (1+\tau K)P_0(T_M)\Phi(d_2).$$

A.10 Hull-White Correlation of CMS Rates

Starting with the dynamics of the zero coupon bond of maturity T, conditioned on its state $\boldsymbol{x}(t)$ at time t:

$$d\mathbb{P}\left(t,T;\boldsymbol{x}(t)\right) \quad = \quad (\dots)\,dt - \mathbb{P}\left(t,T;\boldsymbol{x}\right)(t)\big\langle H_T(t), dZ(t)\big\rangle,$$

where $dZ(t)\!:=\![\sigma_1 dW_1, \sigma_2 dW_2]$, and using $dZ(t)\cdot dZ'(t) = Q(t)dt$, we may write

$$\frac{d\mathbb{P}\left(t,T_1;\boldsymbol{x}(t)\right)}{\mathbb{P}\left(t,T_1;\boldsymbol{x}(t)\right)} \cdot \frac{d\mathbb{P}\left(t,T_2;\boldsymbol{x}(t)\right)}{\mathbb{P}\left(t,T_2;\boldsymbol{x}(t)\right)} \quad = \quad \big\langle H_{T_1}(t), Q(t)H_{T_2}(t)\big\rangle\,dt.$$

Let

$$\mathcal{T}\!:=\!\{t_0, t_1, \dots, t_n\}, \qquad \mathcal{S}\!:=\!\{s_0, s_1, \dots, s_m\},$$

be the respective schedules of two CMS rates, with corresponding coverages

$$\{\alpha_1, \dots, \alpha_n\}, \qquad \{\beta_1, \dots, \beta_m\}.$$

By denoting

$$P_i(t) = \mathbb{P}\left(t, t_i; \boldsymbol{x}(t)\right), \qquad Q_i(t) = \mathbb{P}\left(t, s_i; \boldsymbol{x}(t)\right),$$

the two CMS rates are explicitly given as

$$\begin{aligned} R_1(t) &= \frac{P_0(t) - P_n(t)}{\sum_{i=1}^n \alpha_i P_i(t)},\\ R_2(t) &= \frac{Q_0(t) - Q_m(t)}{\sum_{i=1}^m \beta_i Q_i(t)}.\end{aligned}$$

Thus, if we define the vector quantities $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ by

$$\begin{aligned}\boldsymbol{p}(t) &= \sum_{i=1}^{n} \alpha_i P_i(t) \left[P_0(t) H_{t_0}(t_i) - P_n(t) H_{t_n}(t_i) \right], \\ \boldsymbol{q}(t) &= \sum_{i=1}^{n} \alpha_i Q_i(t) \left[Q_0(t) H_{t_0}(t_i) - Q_n(t) H_{t_n}(t_i) \right],\end{aligned}$$

then we have

$$\begin{aligned}dR_1(t) \cdot dR_2(t) &= \big\langle \boldsymbol{p}(t), \boldsymbol{q}(t) \big\rangle \left(\sum_{i=1}^{n} \alpha_i P_i(t) \right)^{-2} \left(\sum_{i=1}^{m} \beta_i Q_i(t) \right)^{-2}, \\ |\, dR_1(t)|^2 &= \big\langle \boldsymbol{p}(t), \boldsymbol{p}(t) \big\rangle \left(\sum_{i=1}^{n} \alpha_i P_i(t) \right)^{-4}, \\ |\, dR_2(t)|^2 &= \big\langle \boldsymbol{q}(t), \boldsymbol{q}(t) \big\rangle \left(\sum_{i=1}^{m} \beta_i Q_i(t) \right)^{-4}.\end{aligned}$$

The instantaneous correlation between the two CMS rates $R_1(t)$ and $R_2(t)$ will therefore be

$$\rho_{inst}(R_1, R_2; t) = \frac{dR_1(t) \cdot \, dR_2(t)}{|\, dR_1(t)| \cdot |\, dR_2(t)|} = \frac{\big\langle \boldsymbol{p}(t), \boldsymbol{q}(t) \big\rangle}{|\, \boldsymbol{p}(t)||\, \boldsymbol{q}(t)|}. \tag{A.28}$$

The terminal correlation between the two CMS rates is obtained by integration:

$$\rho_{term}(R_1, R_2; T_{ex}) = \int_0^{T_{ex}} \left\langle \frac{\boldsymbol{p}(t)}{|\, \boldsymbol{p}(t)|}, \frac{\boldsymbol{q}(t)}{|\, \boldsymbol{q}(t)|} \right\rangle dt. \tag{A.29}$$

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