Ramsey Theory with Ultrafilters

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Overview

What is Ramsey Theory?

> *A branch of … combinatorics that focuses on the appearance of order in a substructure given a structure of a known size.*
> *— Wikipedia*

> *Complete disorder is impossible.*
> *— Theodore S. Motzkin*

Example

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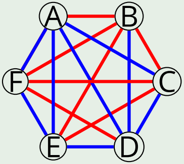

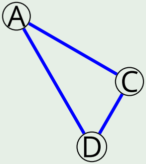

Introduction to Ramsey Theory

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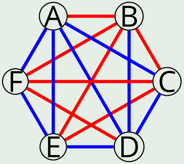

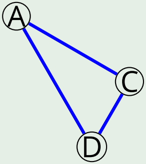

In every case, the answer is yes!

Infinite Ramsey Theory

We can extend similar notions to the infinitary case:

If we colour the edges of an infinite complete graph either red or blue, can we find an infinite complete *monochromatic* subgraph?

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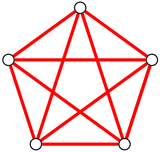

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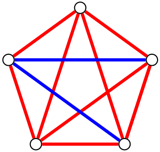

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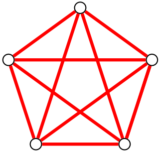

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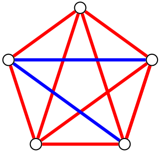

Not monochromatic

The answer is also yes - and we'll use ultrafilters as a tool to prove it!

Filters

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A filter $\mathcal{F}$ on $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ which satisfy the following conditions:

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A filter $\mathcal{F}$ on $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ which satisfy the following conditions:

1. $\mathcal{F}$ is non-empty and $\emptyset \notin \mathcal{F}$
2. If A, B $\in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $B \subseteq \mathbb{N}$ satisfies $A \subseteq B$ then $B \in \mathcal{F}$

Ultrafilters

Definition

An ultrafilter on $\mathbb{N}$ is a maximal filter $\mathcal{U}$ on $\mathbb{N}$. i.e. if $\mathcal{F}$ is another filter on $\mathbb{N}$ such that $\mathcal{U} \subseteq \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$.

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Ultrafilters will be useful in proving the Infinite Ramsey Theorem.

Principal Ultrafilters

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Fix $n \in \mathbb{N}$. The filter $\mathcal{U}_n$ is a principal ultrafilter if it contains all subsets of $\mathbb{N}$ that include the element n.

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Example

If a filter $\mathcal{F}$ contains the singleton set $\{n\}$ for some $n \in \mathbb{N}$, then $\mathcal{F}$ is principal. Moreover, $\mathcal{F} = \mathcal{U}_n$.

We specifically want to use *non-principal* ultrafilters for our results. We will show that such an ultrafilter does in fact exist.

Existence of a Non-Principal Ultrafilter on $\mathbb{N}$

First consider the Fréchet Filter, defined as the set of all co-finite subsets. (i.e. it contains all $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is finite.)

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Applying this to the Fréchet Filter is used to prove the existence of a non-principal ultrafilter. That is, there must be an ultrafilter $\mathcal{U}$ containing the Fréchet Filter.

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Then if $A \subseteq \mathbb{N}$ is a finite set it cannot be in $\mathcal{U}$. This is since $\mathbb{N} \setminus A$ is in the Fréchet Filter, which is a subset of $\mathcal{U}$.

Therefore $\mathcal{U}$ cannot be a principal ultrafilter. $\square$

Important Property of Ultrafilters (1)

Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then $\mathcal{F}$ is an ultrafilter if and only if for every $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$.

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If $\mathcal{U}$ is non-principal, then we get a useful result:

If $A_1 \sqcup \cdots \sqcup A_n = \mathbb{N}$ is a partition, then exactly one of $A_j \in \mathcal{U}$.

Theorem (Infinite Ramsey Theorem)

Given any 2-colouring $\gamma : \mathbb{N} \to \{red, blue\}$ of the edges of a complete graph on the natural numbers, there exists an infinite monochromatic subgraph.

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Fixing a particular vertex v we can partition $\mathbb{N}$ into the sets:

1. v itself, $\{v\}$
2. $A_v =$ all vertices which share a red edge with v
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Now since $\mathcal{U}$ is non-principal, the set $\{v\}$ is not in $\mathcal{U}$. So it must be that exactly one of A_v or B_v is in $\mathcal{U}$. So, suppose $B_v \in \mathcal{U}$.

Then, we can keep picking verticles v_i from B_v such that $B_{v_i} \in \mathcal{U}$ to obtain an infinite set of vertices which all share blue edges. Hence, we get a monochromatic infinite complete subgraph. □

Ultrafilter Addition

Definition

Given $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, define $A - n$ as the shifted subset given by

$$A - n = \{m \in \mathbb{N} : n + m \in A\}.$$

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Given $A = \{1, 2, 3, 4, 5\}$ and $n = 2$, we have $A - n = \{1, 2, 3\}$.

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We can extend this notion to define addition between ultrafilters:

Definition

Let $\mathcal{U}$ and $\mathcal{W}$ be ultrafilters on $\mathbb{N}$. We define $\mathcal{W} + \mathcal{U}$ such that for all $A \subseteq \mathbb{N}$, we have that $A \in \mathcal{W} + \mathcal{U}$ if and only if

$$\{n \in \mathbb{N} : A - n \in \mathcal{W}\} \in \mathcal{U}.$$

Properties of Ultrafilter Addition

Closure

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If $\mathcal{U}_n$ and $\mathcal{U}_m$ are the principal ultrafilters associated to $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively, then $\mathcal{U}_n + \mathcal{U}_m = \mathcal{U}_{n+m}$.

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Non-Associativity

Ultrafilter addition is not associative. For any ultrafilters $\mathcal{U}$ and $\mathcal{W}$, it is not generally true that $\mathcal{W} + \mathcal{U} = \mathcal{U} + \mathcal{W}$.

Idempotent Ultrafilters

Lemma (Ellis-Nakamura Lemma)

There exists a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$. That is, $\mathcal{U}$ is said to be idempotent.

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Then if $\mathcal{U}$ is idempotent with $A \in \mathcal{U}$:

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So, we can define

$$A^* = A \cap \{n \in \mathbb{N} : A - n \in \mathcal{U}\},$$

and we have that $A^* \in \mathcal{U}$!

Finite Sums

Definition

Given a subset $M \subseteq \mathbb{N}$, the finite sums of M is the subset $\mathrm{FS}(M) \subseteq \mathbb{N}$ such that

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Example

Given $x, y \in \mathbb{N}$, the finite sums of x and y are

$$\mathrm{FS}(\{x, y\}) = \{x, y, x + y\}.$$

Schur's Theorem

Theorem (Schur's Theorem)

Given any 2-colouring $\gamma : \mathbb{N} \to \{red, blue\}$*, there exist distinct* $x, y \in \mathbb{N}$ *such that* $\{x, y, x + y\}$ *is a monochromatic subset.*

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Proof. Let $\mathcal{U}$ be an idempotent ultrafilter on $\mathbb{N}$ and let $\gamma : \mathbb{N} \to \{\text{red}, \text{blue}\}$ be a 2-colouring on $\mathbb{N}$.

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So, we can define the sets $A = \{n \in \mathbb{N} : \gamma(n) = \text{red}\}$ and $B = \{n \in \mathbb{N} : \gamma(n) = \text{blue}\}$ and observe that $\mathbb{N} = A \sqcup B$.

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Then, exactly one of A or B is in $\mathcal{U}$. Without a loss of generality, suppose that $A \in \mathcal{U}$.

Proof of Schur's Theorem

Since $\mathcal{U}$ is idempotent, we have $A^* = A \cap \{n \in \mathbb{N} : A - n \in \mathcal{U}\} \in \mathcal{U}$.
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- y is red, since $y \in A$
- $x + y \in A$, since $y \in A - x$

Therefore x, y, and $x + y$ are all γ-red! $\square$

We can carry on a (similar) construction indefinitely to obtain an *infinite* subset.

Theorem (Hindman's Theorem)

Let $\gamma : \mathbb{N} \to \{red, blue\}$ *be a 2-colouring of* $\mathbb{N}$. *Then there exists an infinite set* $M = \{x_1, x_2, ...\} \subseteq \mathbb{N}$ *such that FS(M) is monochromatic.*

Further Applications

What about finite cases?

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What about finite cases?

What about more than two colours?

References

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Thank you!