

Ramsey Theory with Ultrafilters

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Overview

- 1 Introduction
- 2 Ultrafilters
- 3 Infinite Ramsey's Theorem
- 4 Idempotent Ultrafilters and Finite Sums

Introduction to Ramsey Theory

What is Ramsey Theory?

A branch of ...combinatorics that focuses on the appearance of order in a substructure given a structure of a known size.

— *Wikipedia*

Complete disorder is impossible.

— *Theodore S. Motzkin*

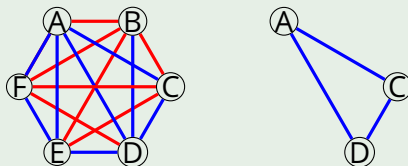
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Suppose we have a group of six random people. Any two given people are either friends or strangers. Can we always find at least three people who are either all mutual friends or all strangers?

Introduction to Ramsey Theory

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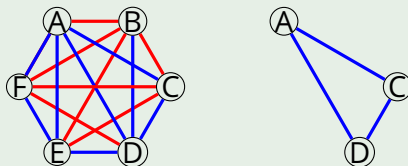
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In every case, the answer is yes!

Infinite Ramsey Theory

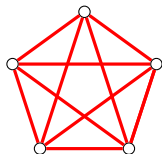
We can extend similar notions to the infinitary case:

If we colour the edges of an infinite complete graph either **red** or **blue**, can we find an infinite complete *monochromatic* subgraph?

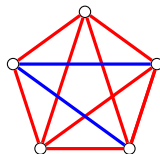
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Monochromatic

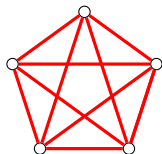


Not monochromatic

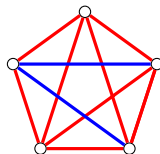
Infinite Ramsey Theory

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If we colour the edges of an infinite complete graph either **red** or **blue**, can we find an infinite complete *monochromatic* subgraph?



Monochromatic



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The answer is also yes - and we'll use ultrafilters as a tool to prove it!

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- 1 \mathcal{F} is non-empty and $\emptyset \notin \mathcal{F}$
- 2 If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
- 3 If $A \in \mathcal{F}$ and $B \subseteq \mathbb{N}$ satisfies $A \subseteq B$ then $B \in \mathcal{F}$

Definition

An ultrafilter on \mathbb{N} is a maximal filter \mathcal{U} on \mathbb{N} . i.e. if \mathcal{F} is another filter on \mathbb{N} such that $\mathcal{U} \subseteq \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$.

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Ultrafilters will be useful in proving the Infinite Ramsey Theorem.

Principal Ultrafilters

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Example

If a filter \mathcal{F} contains the singleton set $\{n\}$ for some $n \in \mathbb{N}$, then \mathcal{F} is principal. Moreover, $\mathcal{F} = \mathcal{U}_n$.

We specifically want to use *non-principal* ultrafilters for our results. We will show that such an ultrafilter does in fact exist.

Existence of a Non-Principal Ultrafilter on \mathbb{N}

First consider the Fréchet Filter, defined as the set of all co-finite subsets. (i.e. it contains all $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is finite.)

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Applying this to the Fréchet Filter is used to prove the existence of a non-principal ultrafilter. That is, there must be an ultrafilter \mathcal{U} containing the Fréchet Filter.

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Then if $A \subseteq \mathbb{N}$ is a finite set it cannot be in \mathcal{U} . This is since $\mathbb{N} \setminus A$ is in the Fréchet Filter, which is a subset of \mathcal{U} .

Therefore \mathcal{U} cannot be a principal ultrafilter. □

Important Property of Ultrafilters (1)

Let \mathcal{F} be a filter on \mathbb{N} . Then \mathcal{F} is an ultrafilter if and only if for every $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$.

Important Property of Ultrafilters (2)

Let \mathcal{U} be an ultrafilter on \mathbb{N} and $A_1, A_2, \dots, A_n \subseteq \mathbb{N}$.

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then there exists $1 \leq j \leq n$, such that $A_j \in \mathcal{U}$.

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then there exists $1 \leq j \leq n$, such that $A_j \in \mathcal{U}$.

If \mathcal{U} is non-principal, then we get a useful result:

If $A_1 \sqcup \dots \sqcup A_n = \mathbb{N}$ is a partition, then exactly one of $A_j \in \mathcal{U}$.

Infinite Ramsey Theorem

Theorem (Infinite Ramsey Theorem)

Given any 2-colouring $\gamma : \mathbb{N} \rightarrow \{\text{red}, \text{blue}\}$ of the edges of a complete graph on the natural numbers, there exists an infinite monochromatic subgraph.

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Fixing a particular vertex v we can partition \mathbb{N} into the sets:

- 1 v itself, $\{v\}$
- 2 $A_v =$ all vertices which share a red edge with v
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Now since \mathcal{U} is non-principal, the set $\{v\}$ is not in \mathcal{U} . So it must be that exactly one of A_v or B_v is in \mathcal{U} . So, suppose $B_v \in \mathcal{U}$.

Then, we can keep picking vertices v_i from B_v such that $B_{v_i} \in \mathcal{U}$ to obtain an infinite set of vertices which all share blue edges. Hence, we get a monochromatic infinite complete subgraph. □

Ultrafilter Addition

Definition

Given $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, define $A - n$ as the shifted subset given by

$$A - n = \{m \in \mathbb{N} : n + m \in A\}.$$

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We can extend this notion to define addition between ultrafilters:

Definition

Let \mathcal{U} and \mathcal{W} be ultrafilters on \mathbb{N} . We define $\mathcal{W} + \mathcal{U}$ such that for all $A \subseteq \mathbb{N}$, we have that $A \in \mathcal{W} + \mathcal{U}$ if and only if

$$\{n \in \mathbb{N} : A - n \in \mathcal{W}\} \in \mathcal{U}.$$

Properties of Ultrafilter Addition

Closure

If \mathcal{U} and \mathcal{W} are ultrafilters on \mathbb{N} , then $\mathcal{W} + \mathcal{U}$ is also an ultrafilter.

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If \mathcal{U}_n and \mathcal{U}_m are the principal ultrafilters associated to $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively, then $\mathcal{U}_n + \mathcal{U}_m = \mathcal{U}_{n+m}$.

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In this example, $\mathcal{U}_n + \mathcal{U}_m = \mathcal{U}_{n+m} = \mathcal{U}_m + \mathcal{U}_n$. Is this always the case?

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Non-Associativity

Ultrafilter addition is not associative. For any ultrafilters \mathcal{U} and \mathcal{W} , it is not generally true that $\mathcal{W} + \mathcal{U} = \mathcal{U} + \mathcal{W}$.

Idempotent Ultrafilters

Lemma (Ellis-Nakamura Lemma)

There exists a non-principal ultrafilter \mathcal{U} on \mathbb{N} such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$. That is, \mathcal{U} is said to be idempotent.

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Then if \mathcal{U} is idempotent with $A \in \mathcal{U}$:

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- $A \in \mathcal{U} + \mathcal{U}$
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So, we can define

$$A^* = A \cap \{n \in \mathbb{N} : A - n \in \mathcal{U}\},$$

and we have that $A^* \in \mathcal{U}$!

Definition

Given a subset $M \subseteq \mathbb{N}$, the finite sums of M is the subset $\text{FS}(M) \subseteq \mathbb{N}$ such that

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Example

Given $x, y \in \mathbb{N}$, the finite sums of x and y are

$$\text{FS}(\{x, y\}) = \{x, y, x + y\}.$$

Schur's Theorem

Theorem (Schur's Theorem)

Given any 2-colouring $\gamma : \mathbb{N} \rightarrow \{\text{red}, \text{blue}\}$, there exist distinct $x, y \in \mathbb{N}$ such that $\{x, y, x + y\}$ is a monochromatic subset.

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Proof. Let \mathcal{U} be an idempotent ultrafilter on \mathbb{N} and let $\gamma : \mathbb{N} \rightarrow \{\text{red}, \text{blue}\}$ be a 2-colouring on \mathbb{N} .

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So, we can define the sets $A = \{n \in \mathbb{N} : \gamma(n) = \text{red}\}$ and $B = \{n \in \mathbb{N} : \gamma(n) = \text{blue}\}$ and observe that $\mathbb{N} = A \sqcup B$.

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Then, exactly one of A or B is in \mathcal{U} . Without a loss of generality, suppose that $A \in \mathcal{U}$.

Proof of Schur's Theorem

Since \mathcal{U} is idempotent, we have $A^* = A \cap \{n \in \mathbb{N} : A - n \in \mathcal{U}\} \in \mathcal{U}$.
So, pick $x \in A^*$. We get that:

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- $A - x \in \mathcal{U}$

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Let $A_1 = A \cap (A - x) \in \mathcal{U}$ and pick $y \in A_1$. Then, we have:

- y is red, since $y \in A$
- $x + y \in A$, since $y \in A - x$

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Let $A_1 = A \cap (A - x) \in \mathcal{U}$ and pick $y \in A_1$. Then, we have:

- y is red, since $y \in A$
- $x + y \in A$, since $y \in A - x$

Therefore x, y , and $x + y$ are all γ -red!



Hindman's Theorem

We can carry on a (similar) construction indefinitely to obtain an *infinite* subset.

Theorem (Hindman's Theorem)

Let $\gamma : \mathbb{N} \rightarrow \{\text{red}, \text{blue}\}$ be a 2-colouring of \mathbb{N} . Then there exists an infinite set $M = \{x_1, x_2, \dots\} \subseteq \mathbb{N}$ such that $FS(M)$ is monochromatic.

Further Applications

What about finite cases?

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What about finite cases?

What about more than two colours?

References



David J. Fernandez-Bréton

Using ultrafilters to prove Ramsey-type theorems

American Mathematical Monthly 129 no. 2 (2022), 116-131



Jashan Bal

Ramsey theory



Timothy Gowers

How to use ultrafilters

https://www.tricki.org/article/How_to_use_ultrafilters



Veselin Jungić

Introduction to Ramsey Theory

Thank you!