Mathematical Statistical Physics

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Contents

Chapter 1

Preliminary stuff

In the Heisenberg picture of quantum mechanics there are three core principles

1. Central objects are the observables, which ware realized as operators on a Hilbert space.
2. Time evolution acts on observables.
3. Auxiliary objects are the states, realized as vectors on the Hilbert space, used to compute expectation values of observables.

This is a notion heavily based on physical intuition. One can translate these three principles into a mathematical formulation in the following way.

1. The set of observables is a *C^*-algebra* $\mathcal{A}$, namely:
   - $\mathcal{A}$ is an associative algebra.
   - $\mathcal{A}$ is equipped with a norm such that $\|A \cdot B\| \leq \|A\| \cdot \|B\|$.
   - it is complete with respect to $\| \cdot \|$.
   - equipped with an involution $* : \mathcal{A} \to \mathcal{A}$ such that

$$(A^*)^* = A \tag{1.1}$$

$$(A + \lambda B)^* = A^* + \overline{\lambda} B^* \tag{1.2}$$

$$(AB)^* = B^* A^* \tag{1.3}$$

   - C^*-property, i.e. $\|A^* A\| = \|A\|^2$.

   Remarks 1.1. There is a lot to be said about C^*-algebras but we will keep it short.

   a) Observables $A \in \mathcal{A}$ are not required to be self-adjoint ($A = A^*$).
   b) In the quantum mechanic setting $\mathcal{H}$ isa Hilbert space, then one takes the "set of bounded operators" $\mathcal{A} = \mathcal{L}(\mathcal{H})$.
   c) Physically, there are unbounded observables. At least for self-adjoint ones $A = A'$, one can consider equivalently the unitary correspondents $U = e^{itA}$, since there is a one-to-one correspondence by Stokes' theorem.

d) $\mathcal{A}$ does not need to have a $\mathbb{1}$.

2. A pair $(\mathcal{A}, \tau)$ is a C^*-dynamical system if $\mathcal{A}$ is a C^*-algebra and $\mathbb{R} \ni t \mapsto \tau_t$ is a strongly continuous one-parameter group $*-$automorphism of $\mathcal{A}$:

   - $\tau_t : \mathcal{A} \to \mathcal{A}$ such that

$$\tau_t(A^*) = (\tau_t(A))^* \tag{1.4}$$
$$\tau_t(A + \lambda B) = \tau_t(A) + \lambda\tau_t(B) \tag{1.5}$$
$$\tau_t(AB) = \tau_t(A)\tau_t(B) \tag{1.6}$$
$$\|\tau_t(A)\| = \|A\| \; . \tag{1.7}$$

   - $\tau_0(A) = A; \tau_{t+s} = \tau_t(\tau_s(A))$.
   - For any $A \in \mathcal{A}$, $\|\tau_{t+\varepsilon}(A) - \tau_t(A)\| \to 0 (\varepsilon \to 0)$, i.e. no uniformity in A.

Remarks 1.2. τ is always generated by a $*$-derivation of the form

$$\delta_t : \mathcal{A} \longrightarrow \mathcal{A} \tag{1.8}$$
$$A \longmapsto t^{-1}(\tau_t(A) - A) \; . \tag{1.9}$$

The domain is defined as $D(\delta) = \{A \in \mathcal{A} \mid \lim_{t\to 0} \delta_t(A) \text{ exists}\}$ for which we then have

$$\delta : D(\delta) \longrightarrow \mathcal{A} \tag{1.10}$$
$$A \longmapsto \delta(A) = \lim_{t\to 0} \delta_t(A) \; . \tag{1.11}$$

Then δ is a closed, densely defined map such that

$$\begin{aligned} \mathbb{1} \ni D(\delta), \delta(\mathbb{1}) &= 0 \\ \delta(AB) &= \delta(A)B + A\delta(B) \\ \delta(A^*) &= \delta(A)^* \; . \end{aligned}$$

In fact, there is a one-to-one correspondence between τ_t and δ (Hille-Yoshida).

Remarks 1.3. In the quantum mechanic setting $\mathcal{A} = \mathcal{L}(\mathcal{H})$. The dynamics are generated by a $H \cdot H^*$ on $\mathcal{H}$, namely

$$\tau_t(A) = e^{itH} A e^{-itH} \; . \tag{1.12}$$

It is a $*$-automorphism by unitarity of e^{-itH} and a strongly continuous group because $t \mapsto e^{-itH}$ is so. The $*$-derivation is given by

$$\delta(A) = \frac{\mathrm{d}}{\mathrm{d}t}\tau_t(A) \mid_{t=0} = i[H, A] \; ,$$

sometimes written as $\tau_t(A) = e^{i[H,\cdot]t}(A)$.

3. Finally, a <u>state</u> over A is a positive, normalized linear functional over A

$$\begin{aligned} \omega : \mathcal{A} &\longmapsto \mathbb{C} \\ A &\longrightarrow \omega(A) \in \mathbb{C} \ , \end{aligned}$$

such that $\omega(A^*A) \geq 0$ (positivity) and $\|\omega\| := \sup \frac{\|\omega(A)\|}{\|A\|} = 1$ (normalization).

Remarks 1.4. Let us now try to establish some intuition

- The positivity of the quadratic function $\lambda \mapsto \omega((A+\lambda B)^*(A+\lambda B))$ implies
  a) $\omega(A^*B) = \omega(B^*A)$.
  b) $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$ (Cauchy-Schwarz inequality).
- In the quantum mechanic setting, any normalized vector $\psi \in \mathcal{H}$ defines a state by

$$\omega_\psi : \mathcal{A} \longrightarrow \mathbb{C} \tag{1.13}$$
$$A \longmapsto \langle \psi, A\psi \rangle \ . \tag{1.14}$$

- Also any density matrix $\varrho = \varrho^* \in \mathcal{L}(\mathcal{H})$ defines a state by

$$\omega_\varrho : A \longrightarrow \mathbb{C} \tag{1.15}$$
$$A \longmapsto \omega_\rho(A) = \mathrm{Tr}(\varrho A) \qquad (\mathrm{Tr}(\varrho) = 1) \ . \tag{1.16}$$

- If $\mathbb{1} \in \mathcal{A}$, then ω is normalized $\Leftrightarrow \omega(\mathbb{1}) = 1$.

It turns out that $\mathcal{A}$ may have inequivalent representations, corresponding to thermodynamically different situations. A representation of the C^*-algebra A on a Hilbert-space $\mathcal{H}$ is a $*$-morphisms $\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$, namely

$$\pi(A \cdot B) = \pi(A)\pi(B) \tag{1.17}$$
$$\pi(A + \lambda B) = \pi(A) + \lambda\pi(B) \tag{1.18}$$
$$\pi(A^*) = (\pi(A))^* \ . \tag{1.19}$$

π_1, π_2 are called equivalent if there is a unitary map

$$U : \mathcal{H}_1 \to \mathcal{H}_2 \text{ s.t. } U\Pi_1(A) = \Pi_2(A)U \quad \forall A \in \mathcal{A} \ . \tag{1.20}$$

Now given a π on $\mathcal{H}$ and any normlized vector $\xi \in \mathcal{H}$, then the map

$$\omega_\xi : \mathcal{A} \longrightarrow \mathbb{C} \qquad \omega_\xi(A) = \langle \xi, \pi(A)\xi \rangle \ , \tag{1.21}$$

defines a state on the algebra. Given a state ω on $\mathcal{A}$, there exists a $\mathcal{H}\omega$, a representation $\pi_\omega : \mathcal{A} \to \mathcal{H}\omega$ and a normalized $\Omega_\omega \in \mathcal{H}\omega$ such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \quad \text{"GNS construction".} \tag{1.22}$$

We will be mainly using two topologies

1. In A: $A_n \to A$ if $\|A_n - A\| \to 0$.
2. In a representation where A is represented as $\mathcal{L}(\mathcal{H})$ there: uniform convergence, strong convergence, weak convergence apply as usual, outside of representations, these terms do not make sense.

Remarks 1.5. On the set of states over $\mathcal{A}$, designated by $\varepsilon(\mathcal{A})$, one has

- $\varepsilon(\mathcal{A})$ is a convex set. $\omega_1, \omega_2 \in \varepsilon(\mathcal{A})$ then $\omega = \lambda\omega_1 + \mu\omega_2 \in \varepsilon(\mathcal{A})$.
- $\varepsilon(\mathcal{A})$ is weakly-$*$ compact (if $\mathcal{A}$ has an identity), namely every sequence $(\omega_n)_{n\in\mathcal{N}}$ of states in $\mathcal{A}$ has convergent subsequences (in the weak-$*$-topology) $\exists(n_k)_{k\in\mathcal{N}}$: $\omega_{n_k}(A) \to \overline{\omega}(A)$.

In particular: A sequence of finite volume thermal equilibrium states always have infinite-volume limit points. As we shall see later, the uniqueness of the infinity point can be takes as a characterization of phase transitions.

Chapter 2

Ideal gases

The ideal gas is a gas of non-interacting particles. The thermodynamic limit (TDL) is usually obtained by taking N particles in a finite volume $\Lambda \in \mathbb{R}^d$ and letting $N \to \infty$, $\Lambda \to \mathbb{R}^d$ such that

$$N/|\Lambda| \to \varrho \qquad \text{with} \qquad 0 < \varrho < \infty. \tag{2.1}$$

There are two types of indistinguishable particles in nature

1. Bosons have a symmetric wave function.
2. Fermions have an antisymmetric wavefunction.

A Hilbert space carrying an arbitrary number of particles is a Fock space built upon the one-particle Hilbert space. Action of the symmetric group S_n on $\otimes^N \mathcal{H}$.

$$P_\sigma : \psi_1 \otimes \cdots \otimes \psi_N \longmapsto \psi_1 \sigma^{-1}(1) \otimes \cdots \otimes \psi_{\sigma^{-1}(N)} \;, \tag{2.2}$$

for any $\sigma \in S_n$ (check $P_{\sigma\sigma'} = P_\sigma \circ P_{\sigma'}$ and so on) and now

$$\mathcal{H}^N_{s/a} := \{\psi \in \otimes^N \mathcal{H} : P_\sigma \psi = \delta(\sigma)\psi \text{ for any } \sigma \in S_n\} \;, \tag{2.3}$$

$$\delta(\sigma) = \begin{cases} 1 & \text{if "s" (Bosons)} \\ -1 & \text{if "a" (Fermions)} \end{cases} \;. \tag{2.4}$$

Furthermore one has $\mathcal{H}^0_{s/a} := \mathbb{C}$ with unit vector Ω, called "vacuum". Now the Fock space is given by

$$\mathcal{F}_{s/a}(\mathcal{H}) = \bigoplus_{N \in \mathcal{N} \cup \{0\}} \mathcal{H}^N_{s/a} \;, \tag{2.5}$$

$\psi \in \mathcal{F}_{s/a}(\mathcal{H})$ is represented as $(\psi^N)_{N \in \mathcal{N} \cup \{0\}} = (\psi^0, \psi^1, \ldots)$ where $\psi^N \in \mathcal{H}^N_{s/a}$ with the norm given by

$$\|\psi\|^2_{\mathcal{F}_{s/a}(\mathcal{H})} = \sum_{N \geq 0} \|\psi^N\|^2_{\otimes^N \mathcal{H}} \;. \tag{2.6}$$

Remarks 2.1.
- This sum has to be convergent, thus the series elements have to decrease in value. Sinc ehte proability to have more than M particles is given by taking the sum $\sum_{N\geq M} \|\psi^N\|^2_{\otimes^N \mathcal{H}}$. So Fock space does not describe the infinite particle state as for $M \to \infty$ the state would always be zero. One can only take the TDL.

- The number operator is given by $(N\psi)^N = N\psi^N$ for any $N \in \mathcal{N}$.

- Second quantization: If $A : \mathcal{H} \to \mathcal{H}$ is a linear operator then define $\Gamma(A), \mathrm{d}\Gamma(A) : \mathcal{H}^N_{s/a} \to \mathcal{H}^N_{s/a}$ for any N where

$$\Gamma(A) = A \otimes \cdots \otimes A \tag{2.7}$$

$$\mathrm{d}\Gamma(A) = \sum_{j=1}^{N} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{A}_{\text{j-th position}} \otimes \mathbb{1} \cdots \otimes \mathbb{1} \ , \tag{2.8}$$

  with which one has $N = \mathrm{d}\Gamma(\mathbb{1})$. Also if A is self-adjoint then $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(e^{itA}) \mid_{t=0}= i\mathrm{d}\Gamma(A)$ i.e. $\Gamma(e^{itA}) = e^{itd\Gamma(A)}$.

The core concepts of Fock spaces are those of annihilation and creation operators.

- Annihilation operators: For $\varphi \in \mathcal{H}$ one defines

$$b(\varphi) : \otimes^N \mathcal{H} \longrightarrow \otimes^{N-1}\mathcal{H} \tag{2.9}$$

$$b(\varphi)(\psi_1 \otimes \ldots \otimes \psi_N) = \sqrt{N}\langle \varphi, \psi_1\rangle_{\mathcal{H}}\ \psi_2 \otimes \ldots \otimes \psi_N \tag{2.10}$$

$$b(\varphi)\Omega = 0\ . \tag{2.11}$$

  Since $b(\varphi) : \mathcal{H}^N_{s/a} \to \mathcal{H}^{N-1}_{s/a}$ for any N, it extends to a map $b(\varphi) : \mathcal{F}_{s/a}(\mathcal{H}) \to \mathcal{F}_{s/a}(\mathcal{H})$.

- Creation operators: For $\varphi \in \mathcal{H}$ one defines

$$b^*(\varphi)\psi^{N-1} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\pm 1)^{k-1} P_{\pi_k}(\varphi \otimes \psi^{N-1}) \text{ where } \pm \text{ is for } s/a \text{ and} \tag{2.12}$$

$$\pi_k^{-1} = (k, 1, 2, \cdots, k-1, k+1, \cdot, N)\ . \tag{2.13}$$

  This can be interpreted as inserting a particle φ into each position and average over the possible positions, that's what P_{π_k} does, inserting into the k-th position. Proving this exactly is cumbersome.

Remarks 2.2.
- $b^*(\varphi) = (b(\varphi))^*$.

- $\varphi \mapsto b(\varphi)$ is anti-linear, i.e. $\varphi \mapsto b^*(\varphi)$ is linear.

- The following commutation law holds $Nb(\varphi) = b(\varphi)(N-1)$.

- If $U : \mathcal{H} \to \mathcal{H}$ is unitary: $\Gamma(U)b^*(\varphi)\Gamma(U)^* = b^*(U\varphi)$.

- The following commutation relations hold on $\mathcal{F}_s(\mathcal{H})$ (CCR)

$$[b(\varphi), b^*(\psi)] = \langle \varphi, \psi \rangle \mathbb{1} \tag{2.14}$$
$$[b(\varphi), b(\psi)] = [b^*(\varphi), b^*(\psi)] = 0 \; . \tag{2.15}$$

- Canonical anticommutatoin relation on $\mathcal{F}_a(\mathcal{H})$ (CAR)

$$\{b(\varphi), b^\dagger(\psi)\} = \langle \varphi, \psi \rangle \mathbb{1} \tag{2.16}$$
$$\{b(\varphi), b(\psi)\} = \{b^*(\varphi), b^*(\psi)\} = 0 \; , \tag{2.17}$$

where $\{A, B\} = AB + BA$.

2.1 Fermions

The (CAR) relatoins indicate that the fermionic creation/anihilation operators are a representation of the abstract algebra $CAR(\mathcal{H})$, defined as the C^*-algebra generated by $\mathbb{1}, a(\varphi), \varphi \in \mathcal{H}$ satisfying:

- $\varphi \mapsto a(\varphi)$ is antilinear.
- $\{a(\varphi), a(\psi)^*\} = \langle \varphi, \psi \rangle$; $\{a(\varphi), a(\psi)\} = 0 = \{a(\varphi)^*, a(\psi)^*\}$.

Since

$$(a(\varphi)^* a(\varphi))^2 = a(\varphi)^* \{a(\varphi), a(\varphi)^*\} a(\varphi) \tag{2.18}$$
$$= \|\varphi\|^2 a(\varphi)^* a(\varphi) \mid \text{ by the C*-property} \tag{2.19}$$
$$= \|a(\varphi)^* a(\varphi)\|^2 = \|\varphi\|^2 \|a(\varphi)^* a(\varphi)\| \tag{2.20}$$
$$\Rightarrow \|a(\varphi)^*\| = \|\varphi\| \; , \tag{2.21}$$

so that the map $\varphi \mapsto a(\varphi)^*$ is continuous.

Remarks 2.3.

- If h is a pre-Hilbert space, with $\overline{h} = \mathcal{H}$, then $CAR(h) = CAR(\mathcal{H})$.
- $CAR(\mathcal{H})$ is unique up to $*$-isomorphisms.

Consider Thermal equilibrium in a finite volume $\Lambda \subset C\mathbb{R}^d$. *Gibbs state* at inverse temperature $0 < \beta < \infty$ and chemical potential $\mu \in \mathbb{R}$ is given by

$$w_{\beta,\mu}(A) := Z^{-1}_{\beta,\mu} \mathrm{Tr}(e^{-\beta K_\mu} A) \; , \tag{2.22}$$

where $A \in \mathrm{CAR}(\mathcal{H}_1)$ and $K_\mu = \mathrm{d}\Gamma(H - \mu \mathbb{1}) = \mathrm{d}\Gamma(H) - \mu \mathcal{N}$ and

$$Z_{\beta,\mu} = \mathrm{Tr}(e^{-\beta K_\mu}) \; , \tag{2.23}$$

wheneer $\exp(-\beta K_\mu) \in I_1(\mathcal{F}_a(\mathcal{H}_1))$ (trace-class operator).

Lemma 2.1. $\exp(-\beta H) \in I_1(\mathcal{H}_1) \Leftrightarrow \exp(-\beta K_\mu) \in I_1(\mathcal{F}_a(\mathcal{H}_q)) \forall \mu \in \mathbb{R}$.

Example 2.1. $\Lambda = [-L/2, L/2], \mathcal{H}_1 = L^2(\Lambda), H = -\Delta_L$.

But $-\Delta$ on $\mathbb{R}^d$ has purely absolutely continuous spectrum so $\exp(\Delta)$ cannot be trace-class. Defining the activity Z as

$$Z = e^{\beta\mu} \ , \tag{2.24}$$

one can find the following relation

$$\exp(-\beta K_\mu) = \gamma(\exp(-\beta(H - \mu\mathbb{1}))) = Z\Gamma(\exp(-\beta H)) \tag{2.25}$$
$$e^{-\beta K_\mu} b^{\#}(\varphi) = z b^{\#}(e^{-\beta H}\varphi) e^{-\beta K_\mu} \ , \tag{2.26}$$

whereby the hashtag is a placeholder for nothing or dagger.

Proposition 2.1. *If* $\exp(-\beta H) \in I_1(\mathcal{H}_1)$*, then the Gibbs states are given by*

$$\omega_{\beta,\mu}(a^*(\varphi)a(\psi)) = \langle \psi, Ze^{-\beta H}(1 + Ze^{-\beta H})^{-1}\varphi\rangle \tag{2.27}$$
$$\omega_{\beta,\mu}(a^*(\varphi)) = 0 = \omega_{\beta,\mu}(a(\varphi)) \ . \tag{2.28}$$

Proof. By Eq. (2.26) one has

$$\omega_{\beta,\mu}(a^*(\tilde{\varphi})a(\psi)) = Z_{\beta,\mu}^{-1}\mathrm{Tr}(b^*(Ze^{-\beta H}\tilde{\varphi})e^{-\beta K_\mu}b(\varphi)) \mid \text{cyclicity of trace} \tag{2.29}$$
$$= \omega_{\beta,\mu}(a(\psi)a^*(Ze^{-\beta H}\tilde{\varphi})) \mid \text{use CAR} \tag{2.30}$$
$$= -\omega_{\beta,\mu}(a^*(Ze^{-\beta H}\tilde{\varphi})a(\psi))\langle\psi, Ze^{-\beta H}\tilde{\varphi}\rangle \tag{2.31}$$
$$\Rightarrow \omega_{\beta,\mu}(a^*((\mathbb{1} + Ze^{-\beta H})\tilde{\varphi})a(\psi)) = \langle\psi, Ze^{-\beta H}\tilde{\varphi}\rangle \ , \tag{2.32}$$

and hence the first term of the claim, since $Ze^{-\beta H} > 0$ implies that $(\mathbb{1} + Ze^{-\beta H})$ is invertible.
If ψ is such that $\mathcal{N}\psi = N\psi$ (i.e. finite particle number) then

$$\langle\psi, e^{-\beta K_\mu}\psi\rangle = N^{-1}\langle\psi, e^{-\beta K_\mu}b(\varphi)\mathcal{N}\psi\rangle \tag{2.33}$$
$$= N^{-1}\langle\psi, (\mathcal{N}+1)e^{-\beta K_\mu}b(\varphi)\psi\rangle \tag{2.34}$$
$$= \frac{N+1}{N}\langle\psi, e^{-\beta K_\mu}b(\varphi)\psi\rangle \tag{2.35}$$
$$\Rightarrow \langle\psi, e^{-\beta K_\mu}\psi\rangle = 0 \ . \tag{2.36}$$

Computing $\mathrm{Tr}(e^{-\beta K_\mu}b(\varphi))$ in a basis such vectors proves the claim. Now: By a similar argument

$$\omega_{\beta,\mu}(a^*(\varphi_1)\cdots a^*(\varphi_n)a(\psi_m)\cdot a(\psi_1)) = \delta_{n,m}\det\left(\{\langle\psi_i, \varrho\psi_j\rangle\}_{i,j=1}^n\right) \ , \tag{2.37}$$

where $\varrho = Ze^{-\beta H}(\mathbb{1} + Ze^{-\beta H})^{-1}$. □

Remarks 2.4. The *2n-point function* is expressed in terms of the 2-point function $\langle\psi, Ze^{-\beta H}(\mathbb{1} + Ze^{-\beta H})^{-1}\varphi\rangle$ called *Gaussian state* or *quasi-free state.*

Remarks 2.5. The crucial property used in this calculation, expressed in Eq. (2.26), is the *KMS condition*:

$$\omega_{\beta,\mu}(BA) = \omega_{\beta,\mu}(A \underbrace{\tau_{i\beta}(B)}_{e^{\beta K_\mu} B e^{-\beta K_\mu}}) \ . \tag{2.38}$$

Take the thermodynamic limit, which means here that $\mathcal{H} = L^2(\mathbb{R}^d)$, then one has to take the limit of the operators

$$(H\psi)(x) = (-\Delta\psi)(x) = (2\pi)^{-d/2} \int |\xi|^2 \hat{\psi}(\xi) e^{i\xi\cdot x} \mathrm{d}\xi \ . \tag{2.39}$$

How can one evaluate this? The dynamics on $CAR(\mathcal{H})$ are given by $\tau_t(a^{\#}(\varphi)) = a^{\#}(e^{-itH}\varphi)$. As one does per usual one takes the well understood finite case $f(-\Delta_L)$ which converges to $f(-\Delta)$ strongly for any bounded, continuous function f. This applies to e^{-itx}, $e^{-\beta x}$, $Ze^{-\beta x}(1 + Ze^{-\beta x})^{-1}$(for $x > 0$).

Theorem 2.1. *Let $\omega^2_{\beta,\mu}$ be the Gibbs state associated to $H_l = -\Delta_L$ on $CAR(\mathcal{H}_L)$. Then*

$$\omega^L_{\beta,\mu}(A) \longrightarrow \omega_{\beta,\mu}(A) \qquad (L \to \infty) \ ,$$

for any $A \in CAR(\mathcal{H}_L)$ and any L, where $\omega_{\beta,\mu}$ is the quasi-free state over $CAR(\mathcal{H})$ with the two-point function

$$\omega_{\beta,\mu}(a^*(\varphi)a(\psi)) = (2\pi)^{-d/2} \int \overline{\psi(\xi)} \frac{Ze^{-\beta|\xi|^2}}{1 + Ze^{-\beta|\xi|^2}} \hat{\varphi}(\xi) \mathrm{d}\xi \ . \tag{2.40}$$

Proof. It suffices to proe the statement for the two-point function. But this follows from the weak convergence of

$$Ze^{-\beta(-\Delta_L)}(\mathbb{1} + Ze^{-\beta(-\Delta_L)}) \xrightarrow{w} Ze^{-\beta(-\Delta)} \left(\mathbb{1} + Ze^{-\beta(-\Delta)}\right)^{-1} \ , \tag{2.41}$$

for a proof refer to exercise sheet 1. □

Remarks 2.6. This theorem has obviously a lot of interesting implications, amongst other things

- Given β, μ, the limit of $\omega^L_{\beta,\mu}$ exists and it is unique and $\omega_{\beta,\mu}$ defines the thermal equilibrium state on $CAR(L^2(\mathbb{R}^d))$.
- Density of the free Fermi gas

$$\begin{aligned} \varrho(\beta,\mu) &= \lim_{L\to\infty} L^{-d} \sum_{n\geq 1} \omega^L_{\beta,\mu}(\underbrace{a^\dagger(\psi_n)a(\psi_n)}_{\mathcal{N}}) \qquad (2.42) \\ &= (2\pi)^{d/2} \int \frac{Ze^{-\beta}|\xi|^2}{1 + Ze^{-\beta|\xi|^2}} \mathrm{d}\xi \ , \qquad (2.43) \end{aligned}$$

where $(\psi_n)_{n\in\mathcal{N}}$ is a basis of $\mathcal{H}_L$. This means that the momentum density distribution

$$0 \le \frac{Ze^{-\beta|\xi|^2}}{1+Ze^{-\beta|\xi|^2}} \le 1 \; , \tag{2.44}$$

and its limit as $\beta \to \infty$ (i.e. $T \to 0$)

$$\lim_{\beta\to\infty} \frac{1}{1+e^{\beta(|\xi|^2-\mu}} = \begin{cases} 0: & \text{if } |\xi|^2 > \mu \\ 1: & \text{if } |\xi|^2 < \mu \; . \end{cases} \tag{2.45}$$

This is illustrated in Fig. 2.1.

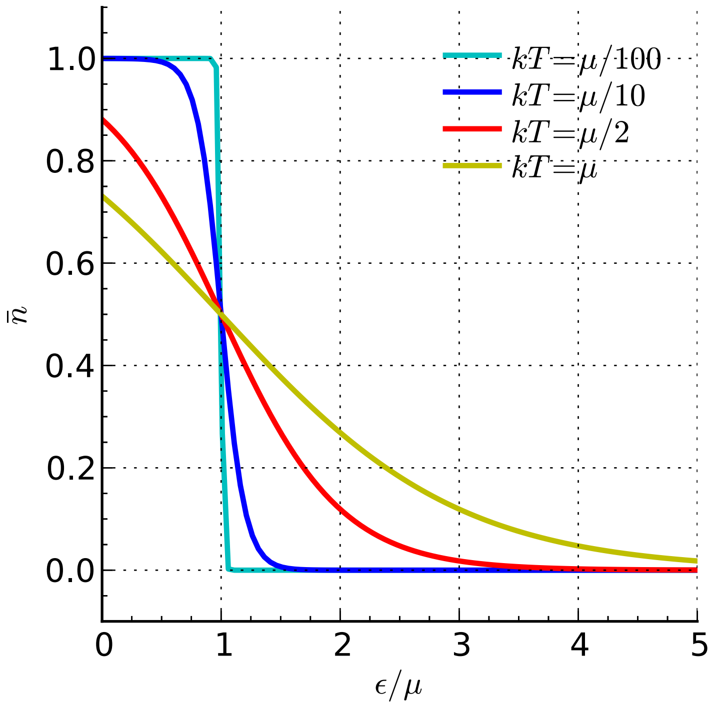

Figure 2.1 Fermi-Dirac distribution [stolen; By Krishnavedala - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=15478733`].

- $\omega_{\beta,\mu}$ having an infinite number of particles cannot be given by a vector or density matrix in $\mathcal{F}_a(L^2(\mathbb{R}^d))$. There is the famous Araki-Wyss representation associated to $\omega_{\beta,\mu}$, namely to $\varrho = \frac{Ze^{-\beta H}}{\mathbb{1}+Ze^{-\beta H}}$.

$$H_\varrho = \mathcal{F}_a(\mathcal{H}) \otimes \mathcal{F}_a(\mathcal{H}) \qquad \omega_\varrho = \omega \otimes \omega \tag{2.46}$$

$$\pi(a^*(\varphi)) = b^*(\sqrt{1-\varrho}\varphi) \otimes \mathbb{1} + (-\mathbb{1})^{\mathcal{N}} \otimes b(\sqrt{\varrho}\varphi) \; . \tag{2.47}$$

If the density is a projection, i.e. $\varrho = \varrho^2$ (for us that means $\beta = \infty$, as is clearly visible in Fig. 2.1., then the two Fock-spaces have the interpretation: particles $\otimes$ antiparticles. Check $(\mathcal{H}_\varrho, \Omega_\varrho, T_\varrho)$ is the GNS representation of $\omega_{\beta,\mu}$:

$$\langle \Omega_\varrho, \pi_\varrho(a^*(\varphi)a(\varphi))\Omega_\varrho\rangle = \langle \Omega \otimes \Omega, \left((-\mathbb{1})^{\mathcal{N}} \otimes b(\sqrt{\varrho}\varphi)\right)\left((-\mathbb{1})^{\mathcal{N}} \otimes b^*(\sqrt{\varrho}\psi)\right)\Omega \otimes \Omega\rangle \tag{2.48}$$

$$= \langle \Omega, b(\sqrt{\varrho}\varphi)b^*(\sqrt{\varrho}\psi)\Omega\rangle \tag{2.49}$$

$$= \langle \sqrt{\varrho}\psi, \sqrt{\varrho}\varphi\rangle + \langle \Omega, b^*(\sqrt{\varrho}\psi)b(\sqrt{\varrho}\varphi)\Omega\rangle \tag{2.50}$$

$$= \langle \psi, \varrho\varphi\rangle = \omega_{\beta,\mu}(a^*(\varphi)a(\varphi)) \tag{2.51}$$