Designing finite automata II

Problem: *Design a DFA A such that $L(A)$ consists of all strings of □ and ◇ which are of length $3n$, for $n = 0, 1, 2, \ldots$*

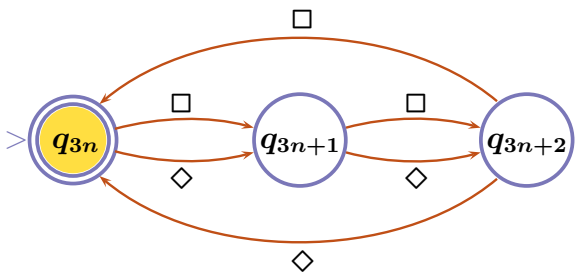
(1) Determine what to remember about the input string.

Assign a state to each of the possibilities: q_{3n}, q_{3n+1}, q_{3n+2}.

(2) Add the transitions telling how the possibilities rearrange,

select the initial state and the favourable states.

(3) Test the automaton.

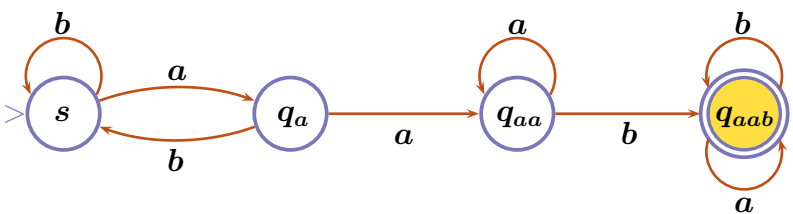


Pattern matching

Problem: *Design a DFA A such that $L(A)$ consists of all strings of a's and b's which contain aab as a substring*

There are four possibilities to remember:

- haven't seen any symbols of the pattern: (initial) state s
- have just seen an a: state q_a
- have just seen aa: state q_{aa} or
- have just seen the entire pattern aab: (favourable) state q_{aab}.



DFA as a theoretical model for programming

- unusual, different from conventional programming languages
- major concepts of imperative programming
  (e.g., sequences, branching, loops) can be simulated
- elegant, simple to use, has a thoroughly developed theory
- all pattern matching (e.g., WWW search engines) can be described
- limited expressive power (the automaton is not capable of increasing any of its resources during computation)

What about 'subprogram calls' ?

Combining two automata

Problem: *Design an automaton that accepts all the strings that have either two consecutive a's or two consecutive b's*

- On page 24 (4), there is a DFA A_4 accepting all the strings that have two consecutive a's. This automaton can easily be transformed into an automaton A'_4 that accepts all the strings with two consecutive b's.

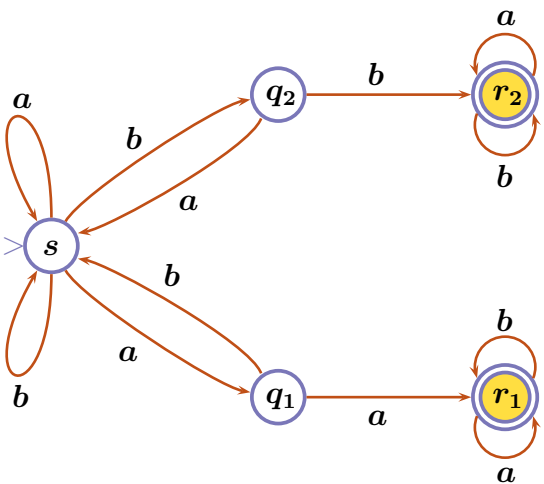
- Since we already have 'programs' that solve the problem for aa's and, respectively, bb's, an obvious idea would be to design a 'program' that would just 'call' the given programs as 'subprograms.' A possible solution would be to 'combine' the initial states of A_4 and A'_4.

Combining two automata (cont.)

Given: $L(A_4)$ = all words having two consecutive a's

$L(A'_4)$ = all words having two consecutive b's

Need: an automaton A such that $L(A) = L(A_4) \cup L(A'_4)$



Nondeterminism

Observe that the state transition diagram on the previous slide **no longer** represents a DFA: there is more than one arrow coming out of the initial state s with identical labels (say, a).

We say that this automaton is **nondeterministic** because, being in state s and reading a from the input tape, the device now has a choice: it can choose to move either to state q_1 or to state s.

What can affect this choice?

We may imagine that the automaton just tries to 'guess' which way will lead it finally to a favourable state.

But real computers cannot 'guess'...

This argument can hardly be refuted.

However, one can justify this variant of finite automata by the following:

- Nondeterministic finite automata are much **easier to design** than deterministic ones

  They allow us to model how 'programs' are built from 'subprograms'

- Whatever can be done by a nondeterministic finite automaton can also be done (though usually in a bit more complicated way) by a deterministic one.

  In other words, nondeterminism **does not increase the computational power** of finite automata

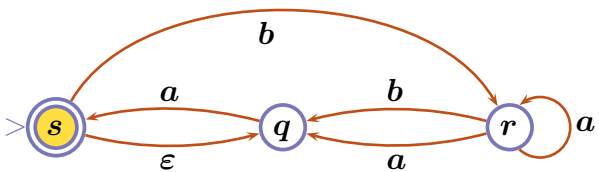
More nondeterminism

There are other features of nondeterministic finite automata (NFA):

- Just as from some states with some input symbols there may be several different possible next states, so with other combinations of states and input symbols there may be **no** possible moves at all (like in the example below, there is no a-arrow coming out of s).

- We also allow the machine to **'jump'** from a state to another state without consuming any input symbol. These 'jumps' will be indicated on the state transition diagrams by arrows labelled with the empty word ε.

  (Transition tables are no longer adequate to describe these automata.)



Acceptance of words by NFAs

A word w is **accepted** by an NFA A if **there exists** a computation of A on input w ending up with a configuration (q, ε), for some favourable state q.

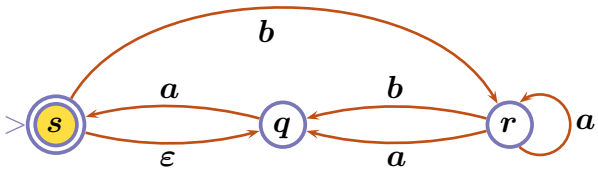
Otherwise, A **rejects** the input word.

So an NFA can reject an input word because **every** computation on this input is

- either 'stuck'
- or ends up with a configuration (q', ε), but q' is not a favourable state.

There can be **more than one** computation on an input word !!

Example: how an NFA works



- **Computations on input** $baba$:

  $(s, baba), (q, baba)$ — **stuck**

  $(s, baba), (r, aba), (r, ba), (q, a), (s, \varepsilon)$ — **accepted**

- **Computations on input** aa:

  $(s, aa), (q, aa), (s, a), (q, a), (s, \varepsilon)$ — **accepted**

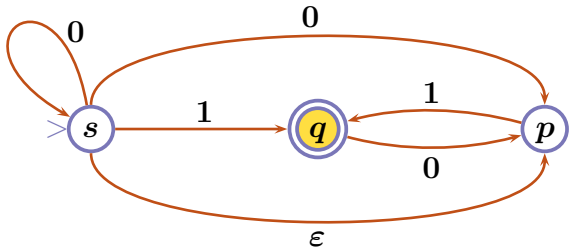
- **Computations on input** $babba$:

  $(s, babba), (q, babba)$ — **stuck**

  $(s, babba), (r, abba), (r, bba), (q, ba)$ — **stuck**

  $(s, babba), (r, abba), (q, bba)$ — **stuck**, no more $\rightsquigarrow$ **rejected**

NFA: another example



- **Computations on input** 11:

  $(s, 11), (q, 1)$ — stuck

  $(s, 11), (p, 11), (q, 1)$ — stuck, no more $\rightsquigarrow$ rejected

- **Computations on input** 00:

  $(s, 00), (s, 0), (s, \varepsilon)$; $(s, 00), (s, 0), (p, \varepsilon)$ — stuck

  $(s, 00), (s, 0), (p, 0)$; $(s, 00), (p, 0)$ — stuck

  $(s, 00), (p, 00)$ — stuck, no more $\rightsquigarrow$ rejected

- **Computations on input** 001:

  $(s, 001), (s, 01), (s, 1), (q, \varepsilon)$ — accepted

Nondeterministic Finite Automaton (NFA)

Formal definition:

A **nondeterministic finite automaton (NFA)** is a quintuple

$$A = (Q, \Sigma, \Delta, s, F) \quad \text{where}$$

- Q is a finite set of **states**
- Σ is a finite set, the **input alphabet**
- $s \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **favourable** (or **accepting**) **states**
- $\Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ is a ternary relation, the **transition relation**.

  (If A, being in state $q \in Q$, has read the symbol $a \in \Sigma$, it enters one of the states q' for which (q, a, q') is in Δ. For any state q'' such that (q, ε, q'') is in Δ, A can 'jump' from state q to state q'' without reading any input symbol.)

Configurations of NFAs

Let $A = (Q, \Sigma, \Delta, s, F)$ be an NFA.

- A **configuration** of A is any pair of the form

  (state in Q, word over Σ)

- Configuration (q, w) **yields** configuration (q', w') **in one step** if
  - either $w = \sigma w'$ for some σ in Σ and the triple (q, σ, q') is in Δ
  - or $w = w'$ and the triple (q, ε, q') is in Δ (‘jump’).

Observe that in general a configuration can yield **several** (or **no**) configurations in one step. In other words, a configuration does not determine uniquely the configuration it yields in one step.

That is why the automaton A is called **nondeterministic**

Computations of NFAs

A **computation of A on input word w** is **any** sequence of configurations

$$(q_1, w_1), (q_2, w_2), \ldots, (q_k, w_k)$$

such that

- $(q_1, w_1) = (s, w)$ where s is the initial state of A
- every (q_i, w_i) yields (q_{i+1}, w_{i+1}) in one step, and
- either w_k is the empty word ε, or the configuration (q_k, w_k) yields no configuration in one step (**it gets stuck**)

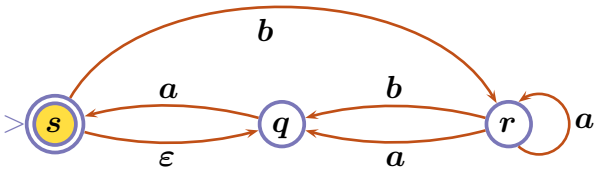
There can be **more than one** computation on an input word

NFAs and parallelism

Nondeterminism can be viewed as a kind of parallel computation wherein several 'processes' can be running concurrently.

- Suppose we are running an NFA on an input string and come to a state and read a symbol with multiple ways to proceed. Then the machine splits into multiple copies of itself and follows **all** the possibilities in parallel. If there are subsequent choices, the machine splits again.
- If the next input symbol does not appear on any arrows exiting the state occupied by a copy of the machine, that copy 'dies,' the branch of computation associated to it is 'stuck.'
- If a state with some exiting ε-arrows is encountered then, without reading any input, the machine splits into multiple copies: one following each exiting ε-arrow, and one staying at the current state.

Differences between NFAs and DFAs

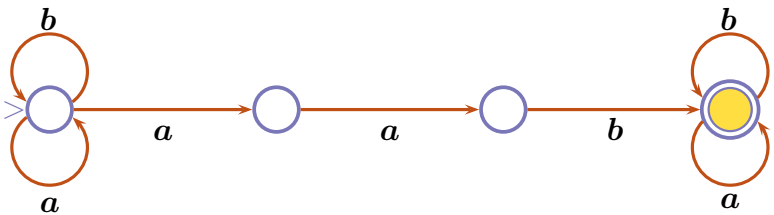


- There can be **more than one** arrow labelled by the same symbol coming out of a state (e.g., there are two a-arrows out of r).
- There can be **no** arrow labelled by some symbol coming out of a state (e.g., there is no b-arrow out of q).
- There can be arrows labelled by not symbols from the input alphabet but by the empty word ε. These arrows represent 'jumps' (e.g. there is a jump from s to q).

Every DFA is an NFA as well !

Pattern matching revisited

Problem: *Design a finite automaton A such that $L(A)$ consists of all strings of a's and b's which contain aab as a substring*



- It is much easier to design an NFA than a DFA.
- We will show now that every NFA can be converted to a DFA accepting precisely the same words

Equivalence of DFAs and NFAs

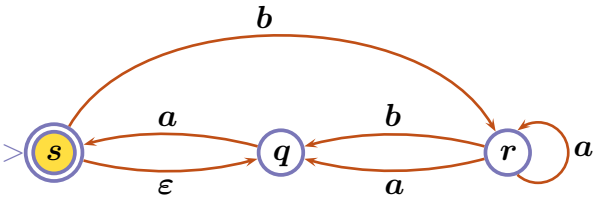
Two automata A and A' are said to be **equivalent** if they accept the same language

Claim: *For every NFA A, there is a DFA A' equivalent to A.*

Moreover, there is a 'mechanical method,' an **algorithm**, that carries out this conversion from a nondeterministic automaton to its deterministic equivalent

The Subset Construction

Task: given an NFA



construct an equivalent **DFA** (**deterministic !**)

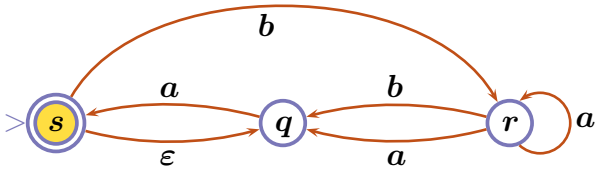
- watch out: in a DFA no choice, no 'getting stuck', no ε-jumps are allowed

Things to define:

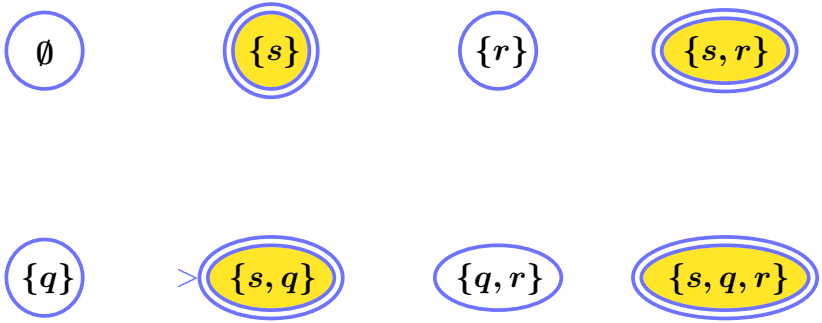
- new states ?
- new initial state ?
- new favourable states ?
- new transitions (arrows) ?

Subset construction: the new states

NFA:



- The new states: all the **subsets** of the NFA's states
- The new initial state: the set containing the NFA's initial state plus all those states that are reachable from it by some ε-jumps
- The new favourable states: those subsets that contain at least one of the NFA's favourable states

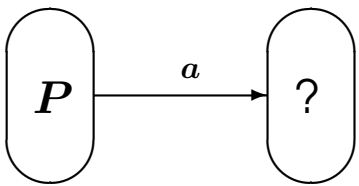


Subset construction: the new transition function

Recall: Being in a state and reading an input symbol, the transition function gives the automaton's next state.

And the new states are *subsets* of the 'old' states now.

General rule:



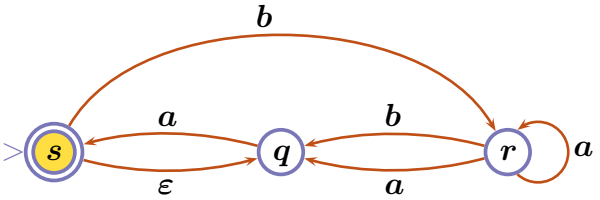
↑

the set of all those states that are reachable from some state in the set P

- either by an a-arrow,
- or by an a-arrow followed by possibly one or more ε-jumps

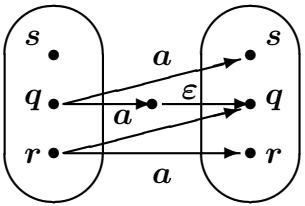
Example: computing the new transition function

NFA:



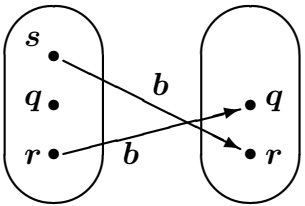
DFA: a-arrow leaving state $\{s, q, r\}$

'goes' to $\{s, q, r\}$



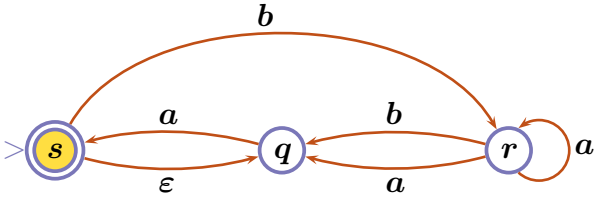
b-arrow leaving state $\{s, q, r\}$

'goes' to $\{q, r\}$

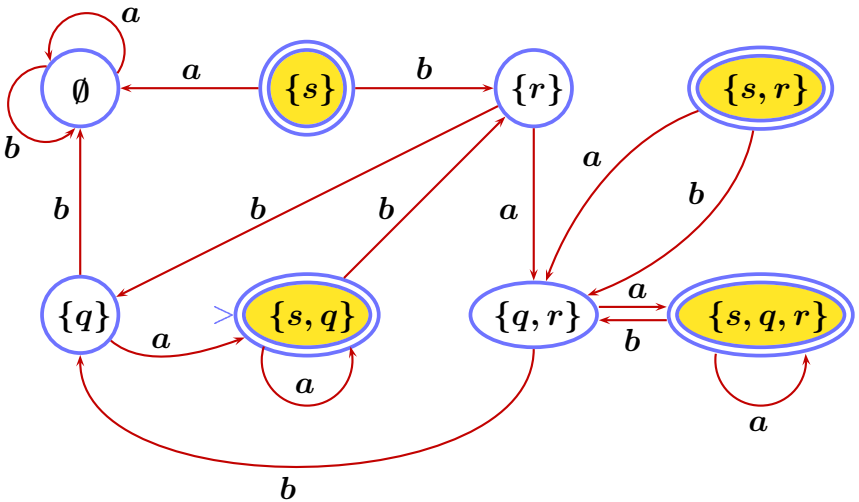


Subset construction: the new transition function

NFA:

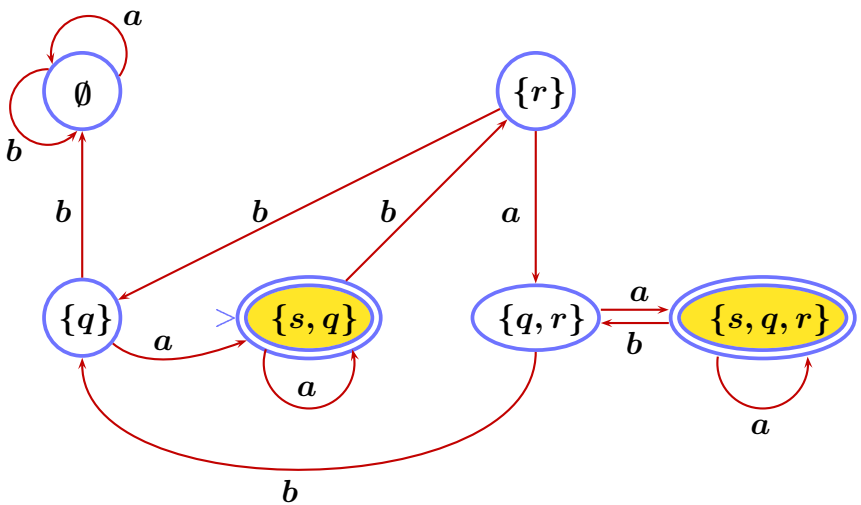


DFA:



Removing the unreachable states

Observe that there is no computation going through the states $\{s\}$ and $\{s, r\}$, so we can safely remove them together with their outgoing arrows:



The initial state is never removable! (all computations start there)

State minimisation problem

We saw that it is very easy to design nondeterministic finite automata for, say, pattern matching problems. However, an NFA cannot be directly implemented as a computer program. One must convert it into a DFA. But the number of states of the resulting DFA may increase significantly

$2^{number\ of\ states\ of\ the\ original\ NFA}$ in the worst case

An important field where finite automata are being used is **hardware design**: some components of hardware are based on simulation of DFA. An important objective here is to use finite automata that have a **minimum** possible number of states. There are methods for doing this. (The 'subset construction' is just the initial phase of design.)