Answers to More Exercises (2017)

1. A professor of logic meets 10 of his former students—Albert, Alice, Bob, Bertha, Clifford, Connie, David, Dora, Edgar, Edith—who have become five married couples. When asked about their husbands, the ladies gave the following answers:

   – Alice: My husband is Clifford, and Bob has married Dora.
   – Bertha: My husband is Albert, and Bob has married Connie.
   – Connie: Clifford is my husband, Bertha's husband is Edgar.
   – Dora: My husband is Bob, and David has married Edith.
   – Edith: Yes, David is my husband. And Alberts wife is Alice.

   Additional true information coming from the men was that every lady gave one correct and one wrong answer. This was sufficient to find out the truth. Reproduce the professors argument.

   Answer. Two cases are possible. Case 1: Dora's husband is Bob ($DB = 1$). Then $AC = 0$ (Alice's husband is NOT Clifford) and $ED = 0$ by statement 4. The last sentence gives $AA = 1$, the second one gives $CB = 1$, that is, Connie's husband is Bob, which is impossible in view of Case 1.

   Case 2: Dora's husband is not Bob, that is, $DB = 0$. Then $AC = 1$ (by 1), and so $CC = 0$, $BE = 1$ (by 3) and $BA = 0$, $CB = 1$ (by 2). By 5, $AA = 0$ and $ED = 1$. This leaves us with $DA = 1$.

2. Check the validity of the following arguments by using truth-tables.

   – Suppose $(P \vee Q) \to R$ and Q. Therefore R.
   – Suppose $P \to (Q \vee R)$ and P. Therefore R.

   Answer. (a) We are given that $((P \vee Q) \to R) = 1$ and $Q = 1$. Then $P \vee Q = 1$, and so $R = 1$. Thus, this argument is correct. (b) The second argument is not correct: indeed, let $P = 1$, $Q = 1$ $R = 0$. Then the premises of the argument are true, while the conclusion is false.

3. Formalise the following argument in propositional logic and demonstrate its validity. "If I graduate this semester, then I will have passed physics. If I do not study physics for 10 hours a week, then I will not pass physics. If I study physics for 10 hours a week, then I cannot play volleyball. Therefore, I will not graduate this semester if I play volleyball."

   Answer. Let G stand for 'I graduate this semester', P for 'I will have passed physics', S for 'I study physics for 10 hours a week', and V for 'I play volleyball'. Then we can represent the argument as follows: 'Suppose $G \to P$, $\neg S \to \neg P$, and $S \to \neg V$. Therefore, $V \to \neg G$.' To show that the argument is correct, one can construct the truth-table and observe that whenever the truth-values of the premises are all 1, the truth-value of the conclusion is 1 as well. Alternatively, suppose to the contrary that the argument is not valid. Then there are truth-values of the variables such that the premises are all 1, while $V \to \neg G$ is 0. Then $V = 1$ and $G = 1$. By the first premise, $P = 1$, by the third one, $S = 0$, which is impossible because in this case the second premise is false. Therefore, the argument is correct.

4. Answer true or false for each of the following statements:

   (a) $\{2, 4, 5\} \subseteq \{2n \mid n \in \mathbb{N}\}$.
   (b) $\{e, i, v, z\} \subseteq \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, r, s, t, u, v, x, y, z\}$.
   (c) $\{2n \mid n \in \mathbb{N}\} = \{2n + 1 \mid n$ is a natural number$\}$.
   (d) $\{1, 2, 3, 4, 5\} = \{5, 3, 2, 4, 1\}$.

   Answer.

   (a) False.
   (b) True.
   (c) False.
   (d) True.

5. Describe each of the following sets by listing its elements.

   (a) $\{n \mid n \in \mathbb{N}$ and $19 < n < 26\}$.
   (b) $\{2k + 1 \mid k$ is an even integer between -5 and $3\}$.
   (c) $\{x \mid x$ is a letter in the words EXAM COMMITTEE$\}$.

   Answer.

   (a) $\{20, 21, 22, 23, 24, 25\}$.
   (b) $\{-7, -3, 1, 5\}$.
   (c) $\{E, X, A, M, C, O, I, T\}$.

6. Describe each of the following sets in terms of a property of its elements (that is, using the 'description by properties' notation).

   (a) The set of dates in the month of July.
   (b) $\{1, 4, 9, 16, 25, 36, 49\}$.

   Answer. There can be several different solutions, here are some.

   (a) $\{x \mid x \in \mathsf{N}$ and $0 < x < 32\}$.
   (b) $\{n^2 \mid n \in \mathsf{N}$ and $1 \leq n \leq 7\}$.

7. Show that there are sets A, B, C such that $(A \cap B) \cup C \neq A \cap (B \cup C)$.

   Answer. There can be several different solutions, here is one of them.
   Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{3, 4\}$. Then $A \cap B = \{1\}$ and so $(A \cap B) \cup C = \{1, 3, 4\}$. On the other hand, $B \cup C = \{1, 2, 3, 4\}$, and so $A \cap (B \cup C) = \{1\}$. And $\{1, 3, 4\} \neq \{1\}$ as, for example, $3 \in \{1, 3, 4\}$ but $3 \notin \{1\}$.

8. Let X and Y be sets such that $|X| = m$ and $|Y| = n$ for some $m, n \in \mathbb{N}^+$. How many one-to-one functions are there from X to Y?

   Answer. There are two cases. If $m > n$ then there are no one-to-one functions from X to Y because of the following. Let $f : X \to Y$ be a function. For each $y \in Y$, take a box B_y. This way there are n boxes. We place the elements of X into these boxes: we put $x \in X$ into box B_y if $f(x) = y$. As $m > n$, by the pigeonhole principle, there will be at least one box containing two or more elements of X, so f is not one-to-one.

If $m \leq n$. Suppose the m elements of X are listed as $x_1, x_2, \ldots, x_m$. To have a one-to-one $X \to Y$ function, for each of these we need to find a different value from Y.

- The value for x_1 can be picked in n ways.
- The value for x_2 can be picked in $n-1$ ways (as the value used for x_1 cannot be used again).
- $\ldots$ and so on. The value for x_m can be picked in $n-(m-1)$ ways.

It follows that there are $n \cdot (n-1) \cdot \ldots \cdot (n-m+1)$ one-to-one functions from X to Y.

9. Determine whether each of these sets is the power set of a set, where a and b are distinct elements:

   - $\emptyset$;
   - $\{\emptyset, \{a\}\}$;
   - $\{\emptyset, \{a\}, \{\emptyset, a\}\}$;
   - $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

   Answer.

   - no;
   - yes;
   - no;
   - yes.

10. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find $A \times B$ and $B \times A$. Describe sets X and Y for which $X \times Y = Y \times X$.

    Answer. $A \times B = \{(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)\}$. $B \times A = \{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$. We have $X \times Y = Y \times X$ when $X = Y$.

11. Which of the following functions are 'onto'?

    - $f\colon \{a, b, c, d\} \to \{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$;
    - $g\colon \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = x^2$;
    - $g\colon \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = x + 1$;
    - $h\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $g(x, y) = 2x - y$;
    - $h\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $g(x, y) = x^2 - 4$.

    Answer.

    - yes;
    - no;

- yes;
- yes;
- no.

12. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$:

    - $f(x) = 2x + 1$;
    - $f(x) = x^2 + 1$;
    - $f(x) = x^3$.

    Answer.

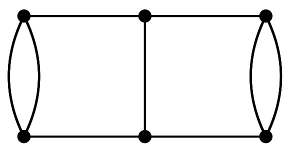
    - yes;
    - no;
    - yes.

13. In each of the following cases, either draw an undirected graph having the given property, or explain why no such graph exists.

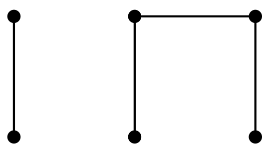
    (a) Having six vertices, each of degree 3.
    (b) Having six vertices and four edges.
    (c) Having four edges, and four vertices of degrees 1, 2, 3, 4, respectively.

    Answer.

(a)

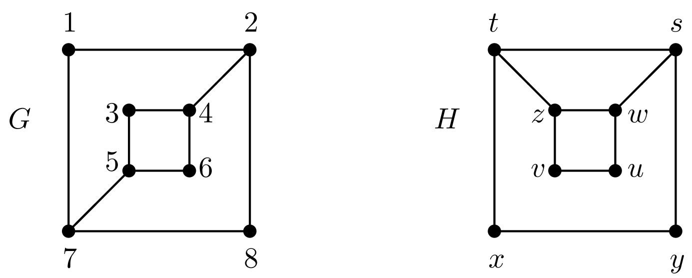


(b)



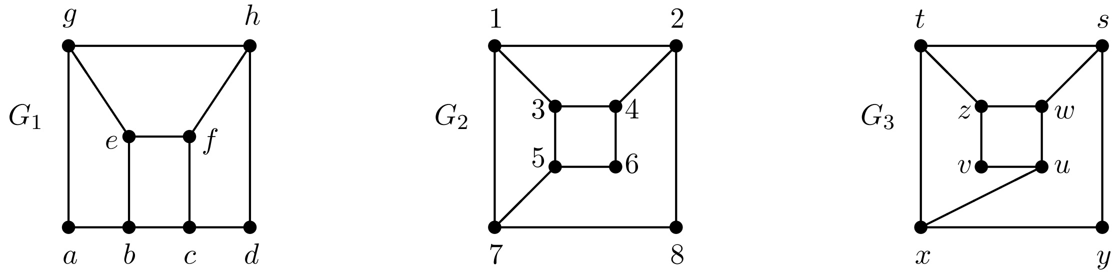
(c) There is no such undirected graph because, if a graph has four vertices of degrees 1, 2, 3, 4, then it must have $\frac{1+2+3+4}{2} = 5$ edges by the Handshaking theorem on page 34 `http://www.dcs.bbk.ac.uk/~michael/foc/slides/FoC-3.pdf`.

14. Determine whether the graphs below are isomorphic:



Answer. They are not isomorphic because t, s, w and z in H are all of degree 3 and form a cycle, but there are no such edges in G. (Show that this property is an invariant.)

15. Of the three following graphs, two are isomorphic and the third is not. Determine which pair are isomorphic and which one is the third nonisomorphic one. Explain your answers.



Answer. G_1 and G_2 are isomorphic, and G_3 is the nonisomorphic third one. There can be several different explanations, using different functions and invariants, here is just one of each.

All three graphs have 8 vertices, 11 edges, 2 vertices of degree 2 and the rest are of degree 3. However, consider the following property: "*There is a path of length* 4 *connecting the two vertices of degree* 2". It is not hard to see that this property is an invariant. And G_3 has this property: (v, u, w, s, y) is such a path. On the other hand, say, G_1 does not have this property: any path between a and d is either of length 3 or longer than 4.

A function f showing that G_1 and G_2 are isomorphic can be defined as

$$\mathsf{f}(a) = 6, \quad \mathsf{f}(b) = 4, \quad \mathsf{f}(c) = 2, \quad \mathsf{f}(d) = 8, \quad \mathsf{f}(e) = 3, \quad \mathsf{f}(f) = 1, \quad \mathsf{f}(g) = 5, \quad \mathsf{f}(h) = 7\,.$$

16. Let $\Sigma = \{x, y\}$. For each of the following languages over Σ, find a regular expression representing it:

(a) All strings that do not contain the substring xx.

(b) All strings that contain at least one y and do not end with xx.

Answer.

(a) $y^*(xyy^*)^* \cup y^*(xyy^*)^*x$;

(b) $y \cup (x \cup y)^*(yy \cup xy \cup yx)$.

17. Give context-free grammars for the following languages

– $L = \{a^n b^m \mid n \neq m\}$.

– $L = \{a^n b^m c^k \mid n = m + k\}$.

– $L = \{a^n b^m c^k \mid n \neq m + k\}$.

Answer.

(a) Either $m < n$ or $m > n$, which gives the cases X and Y:
$S \to X, \quad S \to Y,$
$X \to aXb, \quad X \to Xb, \quad X \to b,$
$Y \to aYb, \quad Y \to aY, \quad Y \to a.$

(b) $S \to aSc, \quad S \to B, \quad B \to bBc, \quad B \to \varepsilon$

(c) Either $k < n + m$ or $k > n + m$.
$S \to EcC, \quad S \to aAE, \quad S \to AU,$
$A \to aA, \quad A \to \varepsilon, \quad B \to bB, \quad B \to \varepsilon, \quad C \to cC, \quad C \to \varepsilon,$
$E \to aEc, \quad E \to F,$
$F \to bFc, \quad F \to \varepsilon,$
$U \to aUc, \quad U \to V,$
$V \to bVc, \quad V \to bB.$