Abstracts

Gromov-Witten theory and the refined topological string

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(joint work with Andrea Brini)

Since the initiation of the field, developments in Gromov–Witten theory were often inspired by analogies with mathematical physics. Given a smooth Calabi–Yau threefold X and an effective curve class β, one such example is Gopakumar–Vafa or BPS integrality which predicts that there are a finite number of integer invariants underlying the generating series

$$\mathsf{GW}_\beta(X) \coloneqq \sum_{g\geq 0} u^{2g-2} \int_{[\overline{M}_g(X,\beta)]^{\mathrm{virt}}} 1 \in \mathbb{Q}(\!(u)\!)\,.$$

of Gromov–Witten invariants of all genera. For example when X is local $\mathbb{P}^2$ and $\beta = [L]$ the class of a line we find

$$\mathsf{GW}_{[L]}(K_{\mathbb{P}^2}) = 3 \cdot \left(2 \sin \tfrac{u}{2}\right)^{-2}\,. \tag{1}$$

Here, the leading factor 3 should be viewed as the single integer invariant — called *BPS invariant* — governing the Gromov–Witten invariants in every genus. In my talk I will introduce a refinement of this BPS integrality conjecture in the context of equivariant Gromov–Witten theory of Calabi–Yau fivefolds of the form $X \times \mathbb{C}^2$ concentrating on the case where $X = K_S$ is a local surface.

The refinement. Let S be a smooth projective surface. There is a natural action of $(\mathbb{C}^\times)^3$ on the fibres of $K_S \times \mathbb{C}^2 \to S$ where the ith $\mathbb{C}^\times$-factor scales the ith fibre direction with weight one. Via the embedding

$$T \coloneqq (\mathbb{C}^\times)^2 \hookrightarrow (\mathbb{C}^\times)^3,\ (t_1,t_2) \mapsto \left((t_1t_2)^{-1}, t_1, t_2\right)$$

we obtain a T-action on $K_S \times \mathbb{C}^2$ which fixes the holomorphic five-form. This lifts to an action on the moduli space of stable maps with this target.

We define the generating series of T-equivariant Gromov–Witten invariants

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2) \coloneqq \sum_{g\geq 0} {}^{T}\!\!\int_{[\overline{M}_g(K_S\times\mathbb{C}^2,\beta)]^{\mathrm{virt}}_T} 1$$

where the right-hand side integral is defined via virtual T-localisation. The sum is a well-defined element in the completed and localised T-equivariant Chow cohomology of a point. Since $H^T_*(\mathrm{pt}) \cong \mathbb{Q}[\epsilon_1,\epsilon_2]$ the expression can naturally be viewed as a power series in two parameters ϵ_1 and ϵ_2. In the one parameter limit $\epsilon_1 = -\epsilon_2 = \mathrm{i}u$, which we will refer to as the *unrefined limit*, we recover a quantity we have already encountered before:

Lemma 1. *We have*

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2)\Big|_{\epsilon_1=-\epsilon_2=\mathrm{i}u} = \mathsf{GW}_\beta(K_S)\,.$$

Refined BPS invariants. Now let S be a del Pezzo surface. In this case we can recursively define power series

$$\mathsf{BPS}_\beta(S)(\epsilon_1,\epsilon_2) \in \mathbb{Q}[\![\epsilon_1,\epsilon_2]\!]$$

labelled by effective curve classes β in S by demanding that these series satisfy

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2) =: \sum_{\substack{k\in\mathbb{Z}_{>0}\\ k\,\mid\,\beta}} \frac{1}{k}\, \frac{\mathsf{BPS}_{\beta/k}(S)(k\epsilon_1,k\epsilon_2)}{2\sinh\frac{k\epsilon_1}{2}\; 2\sinh\frac{k\epsilon_2}{2}}\,.$$

Conjecture 1. $\mathsf{BPS}_\beta(S)$ *lifts to an integer valued Laurent polynomial in* $\mathrm{e}^{\frac{\epsilon_1}{2}}, \mathrm{e}^{\frac{\epsilon_2}{2}}$.

To come back to our earlier example, when $X = K_{\mathbb{P}^2}$ and $\beta = [L]$ a low-genus computer calculation yields

$$\mathsf{BPS}_{[L]}(\mathbb{P}^2) = \mathrm{e}^{\epsilon_1+\epsilon_2} + 1 + \mathrm{e}^{-\epsilon_1-\epsilon_2} + O(\epsilon_i^8)\,.$$

In the unrefined limit this expression specialises to

$$\mathsf{BPS}_{[L]}(\mathbb{P}^2)\Big|_{\epsilon_1=-\epsilon_2=\mathrm{i}u} = 3 + O(u^8)$$

which via Lemma 1 is consistent with equation (1). More generally, in the unrefined limit Conjecture 1 specialises to the original conjecture of Gopakumar and Vafa [3] which was proven by Ionel–Parker [5] and Doan–Ionel–Walpuski [2].

Geometric interpretation. We expect $\mathsf{BPS}_\beta(S)$ to have a geometric interpretation in terms of Gieseker stable sheaves on S with support β and fixed Euler characteristic one. If we denote by M_β the moduli space of such stable sheaves, then the Hilbert–Chow morphism induces a perverse grading on cohomology groups:

$$H^{i,j} := \mathrm{gr}_j^P H^i\big(M_\beta, \mathbb{Q}[\dim M_\beta]\big).$$

Conjecture 2. *Identifying* $q_\pm = \mathrm{e}^{\frac{\epsilon_1\pm\epsilon_2}{2}}$ *we have*

$$\mathsf{BPS}_\beta(S) = \sum_{i,j} q_+^i q_-^j\,(-1)^{i+j}\dim H^{i,j}\,. \tag{2}$$

In the unrefined limit the conjecture specialises to Maulik and Toda's proposal for Gopakumar–Vafa invariants [8].

Evidence. Orthogonal to the unrefined limit we have the following evidence for our conjectures.

Theorem 1. *Conjecture 1 and 2 hold for* $S = \mathbb{P}^2$ *in the limit* $\epsilon_2 = 0$.

Idea of the proof. The theorem is proven by passing through a correspondence with the Gromov–Witten theory of the surface relative a smooth anticanonical curve E. To be more precise, let us denote by $\overline{M}_g(S/E,\beta)$ the moduli stack of genus g, class β stable maps to S with a marking of maximal tangency $(E\cdot\beta)$ along E. Then via a degeneration to the normal cone argument one can show that

$$\epsilon_2\;\mathsf{GW}_\beta(K_S\times\mathbb{C}^2)\Big|_{\epsilon_2=0} = \frac{(-1)^{E\cdot\beta+1}}{E\cdot\beta}\sum_{g\geq 0}\epsilon_1^{2g-1}\int_{[\overline{M}_g(S/E,\beta)]^{\mathrm{virt}}}\lambda_g\,.$$

The theorem then follows from the work of Bousseau [1] (with the addition of [7]) who establishes the analogous statement of the theorem for the right-hand side of the last equation when $S = \mathbb{P}^2$. $\square$

Motivation & context. Conjecture 1 and 2 are informed by analogies with mathematical physics and expected correspondences with other curve counting theories. In each case these relations are natural refinements of their already known unrefined versions. To give more detail, if X is a smooth Calabi–Yau threefold which admits a non-trivial $\mathbb{C}^\times$-action we expect the equivariant Gromov–Witten theory of $X \times \mathbb{C}^2$ to be the counterpart of

- **the refined topological string on** X as studied in physics literature. Especially, Conjecture 1 is very much informed by Huang–Klemm's study of the anticipated B-model interpretation of this quantum field theory [4].
- **K-theoretic Pandharipande–Thomas theory of** X as introduced by Nekrasov and Okounkov [9]. Indeed, Kononov–Pi–Shen observed that for $X = K_{\mathbb{P}^2}$ in low degree these invariants can be matched with the right-hand side polynomial in equation (2) [6].

References

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