Higher quantum Airy structures in two-dimensional topological gravity

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I hereby declare that this thesis was formulated by myself and that no sources or tools other than those cited were used.

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Date Signature

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Contents

CHAPTER 1

Introduction

A quantum field theory is usually described by an action $S[\Phi]$. Given a set of observables $\{\tau_i\}$ we can encode the information about all their correlators in the partition function

$$Z(t) = \int \mathcal{D}\Phi \; e^{\hbar^{-1/2}\, S[\Phi] + \sum_i \tau_i\, t_i}$$

depending on the external currents $\{t_i\}$ and the coupling constant $\hbar^{1/2}$. Such a path integral description is in general not well-defined. However, for *topological* quantum field theories, $Z(t)$ can often be turned into a mathematically well-defined object. For some of these theories $Z(t)$ can then be proven to satisfy certain differential constraints

$$L_n\, Z(t) = 0 \qquad\qquad n \in I \tag{1.1}$$

where $\{L_n\}_{n\in I}$ is a set of differential operators depending on the variables t_i. For instance in theories related to matrix models such differential constraints naturally appear in the form of loop equations. These constraints give us a glimpse of how a non-perturbative solution for $Z(t)$ looks like. In some special cases it may even turn out that the differential equations (1.1) determine $Z(t)$ uniquely. Such a case is the theory of pure two-dimensional topological gravity and the statement that its associated partition function is uniquely determined by (1.1) is the famous Witten conjecture [1]. In this particular example $I = \{-1, 0, 1, 2, \ldots\}$ and the operators L_n satisfy

$$\big[L_m, L_n\big] = \hbar(m-n)\, L_{m+n}$$

Virasoro[1] commutation relations. It turns out that models related to pure two-dimensional topological gravity feature a similar property, for instance if one couples pure two-dimensional topological gravity to minimal matter [2, 3]. In a fascinating way these *Virasoro-like* differential constraints open the door for an analysis of the partition function that goes beyond perturbation theory.

To sum up, the idea of the approach is the following. Given a certain theory, derive differential constraints for its partition function Z and deduce information about Z from these. However, one could reverse the logic and ask: Which properties does a set of differential operators $\{L_i\}_{i\in I}$ have

[1] The unusual factor of $\hbar$ here is due to a certain normalisation of the operators L_n. The importance of this choice will become clear soon.

to satisfy such that it gives rise to a unique partition function-*like* solution Z of (1.1)? By this we mean that Z must admit an interpretation as a generating function for certain correlators. One can understand the notion of *quantum Airy structures* recently introduced by Kontsevich and Soibelman [4] and further developed in [5–8] as an answer to this question.
The properties which the differential operators have to satisfy are rather simple. We call a set of differential operators $\{L_i\}_{i\in I}$ acting on functions depending on variables $\{x^i\}_{i\in I}$ a *higher quantum Airy structure* if

$$L_i = \hbar\partial_{x_i} + \text{terms of degree } \geq 2 \qquad i \in I \tag{1.2}$$

and

$$[L_i, L_j] = \hbar\, f^k_{i,j}\, L_k \qquad i, j \in I \tag{1.3}$$

with $f^k_{i,j}$ a polynomial in $\{x^i, \hbar\partial_i, \hbar\}$ and $[\cdot,\cdot]$ the usual quantum commutator

$$[\hbar\partial_i, x^j] = \hbar\,\delta^j_i \qquad i, j \in I\,.$$

Of course, in order to make sense of property (1.2) one has to specify a degree grading:

$$\deg \hbar\partial_i := \deg x^i := \deg \hbar^{1/2} := 1\,.$$

Given such a higher quantum Airy structure $\{L_i\}_{i\in I}$ a central theorem of [4] tells us there exists a unique solution $Z(x)$ of the differential equations (1.1). Indeed, we will later learn that this Z features a lot of properties one expects from an actual partition function. And we will find even more: It turns out that the components $F_{g,n}$ occurring in the expansion

$$\log(Z) = \sum_{\substack{g,n\geq 0\\ 2g-2+n>1}} \frac{\hbar^{g-1}}{n!} F_{g,n}$$

can be computed recursively in $2g-2+n$ starting from the base cases $F_{0,3}$ and $F_{1,1}$. From the last sentence some readers probably feel reminded on the Chekov–Eynard–Orantin topological recursion [9–11] (see also [12]). This is no coincidence. Indeed, quantum Airy structures were introduced in [4] as an alternative take on the input data of the CEO topological recursion (see [5, Sec. 9]). Relating to the results of [7], special cases of *higher* quantum Airy structures should actually be interpreted in the context of the more general Bouchard–Eynard topological recursion [13].

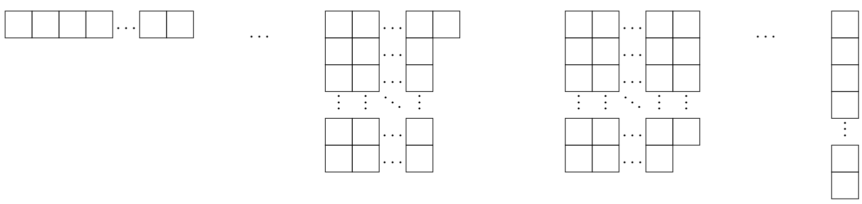
Even though the two defining properties of a (higher) quantum Airy structure seem innocent, the construction of such a set of operators with non-trivial commutation relations (1.3) is rather difficult. However, in [7] Borot, Bouchard, Chidambaram, Creutzig, and Noshchenko explained how a certain class of *Virasoro-like* constraints—also called $\mathcal{W}$-constraints—can be identified to be higher quantum Airy structures. This result is interesting from the physics point of view since, as already indicated, certain models related to pure two-dimensional topological gravity are known to satisfy $\mathcal{W}$-constraints. See for instance [3, 14, 15].
Now remember that we can associate a unique partition function Z to each higher quantum Airy structure. In some cases these Z can be identified to indeed be a *partition function* of an explicit (physical) enumerative problem. Interestingly, the number of higher quantum Airy structures constructed from $\mathcal{W}$-algebras in [7] is larger than the number of know explicit applications, ie. the

number of cases in which a higher Airy structure can be matched with an enumerative problem [7, Sec. 6]. Thus, since some of these Airy structures could so far not be matched with an explicit application (for instance in the context of matrix models or enumerative geometry) one might conjecture that this hints towards some not yet constructed theories. Indeed, (at least in a few cases) there is some evidence for this conjecture to be true [7, Sec. 6]. As we will see in this thesis, the list of $\mathcal{W}$-Airy structures of [7] can even be extended. This of course opens the door even wider for speculations about possible applications.
One can regard the construction of these new $\mathcal{W}$-Airy structures as the central outcome of this thesis. To be more precise, we will construct higher quantum Airy structures from the $\mathcal{W}(\mathfrak{gl}_r)$-algebra. This algebra can be understood as a higher spin extension of the Virasoro algebra with r bosonic currents. We refer to [16] for a review on $\mathcal{W}$-algebras in the context of conformal field theory. In mathematics $\mathcal{W}$-algebras are understood in the context of vertex operator algebras (see [17]).

Since the construction of an Airy structure from a $\mathcal{W}(\mathfrak{gl}_r)$-algebra—respectively from its *mode algebra*—is a non-trivial exercise, let us briefly outline the main steps of this construction.

- Since the elements of the mode algebra of $\mathcal{W}(\mathfrak{gl}_r)$ are abstract objects one first has to represent the modes as differential operators. This will be done by constructing so called *σ-twisted representations* of the bosonic Fock space which one then restricts to $\mathcal{W}(\mathfrak{gl}_r)$.
- In order to bring the differential operators into the correct form one performs a *dilaton shift*.
- We then pick a *subset* of the constructed differential operators that is of the form (1.2).

Then finally one checks whether property (1.3) is satisfied in which case the constructed set of operators is indeed a higher quantum Airy structure. One can thus interpret the choice of a σ-twisted representation, the dilaton shift, and the choice of a mode subset as the input data for the above construction. The main outcome of this thesis is a statement about sufficient conditions for this input data to provide us with an Airy structure. Since in order to state the precise conditions it would be necessary to introduce a lot more of notation, which would seem out of place here, we simply refer the reader to Theorem 4.19 and 4.20 for the precise results. One should remark that due to time restrictions the proof of the two theorems still contains a small gap. We will comment on this gap in Section 4.7.3 and postpone the full proof to [18].
Let us only briefly remark on how the construction presented in this thesis generalises the result of [7]. The key difference of the approach of this thesis is to allow for arbitrary twists σ, while in [7] only two special classes of twists were considered. One can illustrate the generalisation as follows. Using an explicit expression for the generators of $\mathcal{W}(\mathfrak{gl}_r)$ it was proven in [7, Thm. 3.16] that one can associate a distinct mode subalgebra of $\mathcal{W}(\mathfrak{gl}_r)$ to each Young diagram with r boxes. One can rephrase the result of [7, Thm. 4.9] in the way that the subalgebras of modes associated to



admit a representation in terms of differential operators forming an Airy structure. Those are diagrams with r boxes and at most one decrement. Of course, for large r the above diagrams are far from being all possible diagrams one could write down. It is thus natural to ask whether the subalgebras not in the above list can be represented as an Airy structure as well? One can understand the statements of Theorem 4.19 and 4.20 derived in this thesis as a positive answer to that question. This might hint that the construction of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures performed here is close to being optimal. The question whether the construction is further be generalisable is addressed in more detail in the conclusion.

Besides the construction of new $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures the second aim of this thesis is to put them into context by providing some explicit application of these structures. The thesis is thus structured as follows.
In Chapter 2 we start with a motivation from physics and review pure two-dimensional topological gravity in the context of integration on the moduli space of Riemann surfaces [1, 19] and in terms of matrix models. Especially, we will consider the Kontsevich matrix model which was used by Kontsevich in order to prove Witten's conjecture [20]. Certain aspects of this proof shall also be highlighted in order to emphasize the relation between matrix models and enumerative geometry. The statement of Witten's conjecture shall then be the motivating example for the introduction of higher quantum Airy structures in Chapter 3. Moreover, we will introduce the partition function associated to an Airy structure in this chapter and highlight certain aspects of the recursion relation satisfied by the correlators.
In Chapter 4, we then proceed with the construction of the already mentioned families of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures. Starting from zero we will begin by reviewing the definition of the $\mathcal{W}(\mathfrak{gl}_r)$-algebra in the language of vertex operator algebras. After that we will outline the construction of Airy structure-prototypes from the modes of this vertex algebra. In this part we will focus on the main steps of the construction which was carried out in full detail in [7]. In Theorem 4.19 and 4.20 we then present a list of sufficient conditions for the construction to indeed provide us with an Airy structure. The chapter is concluded with an outline of the main steps of the proof. The proof itself can be found in Appendix A.
In Chapter 5 we will present an overview about possible applications of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures and we will match the partition functions of some of these Airy structures with partition functions encoding certain intersection numbers or coming from matrix models. To be precise, first we will discuss r-spin theory [21] and its related matrix model analogue [3] in the context of Airy structures. Then, we will consider open intersection theory which studies intersections over the moduli space of Riemann surfaces *with boundaries*. The development of this theory was initiated in [22] and its matrix model description derived in [23]. At the end of the chapter we then present open r-spin intersection theory [24, 25] as the combination of both preceding theories. We will provide some evidence that the partition function encoding these intersection numbers might be associated to a certain $\mathcal{W}(\mathfrak{gl}_{r+1})$-Airy structure. We conclude with a discussion of open questions and possible research directions in Chapter 6.

CHAPTER **2**

Pure two-dimensional topological gravity

The central object of a (classical) theory of gravity is the Einstein-Hilbert action

$$\frac{1}{2\pi}\int_{\Sigma}\sqrt{g}\,R$$

from which one derives the equations of motion for the metric g, the Einstein field equations. If now the space Σ where g is defined on is a two-dimensional orientable surface the situation simplifies drastically. In this case due to the Gauss-Bonnet theorem the above action simply equals the Euler characteristic $\chi(\Sigma)$ of the surface which is a topological invariant of Σ. Thus the equations of motion are automatically satisfied and we could have as well considered the Lagrangian

$$\mathcal{L} = 0\,.$$

Even though classically two-dimensional gravity is trivial, it becomes a strikingly rich theory after quantisation. For a special choice of gauge symmetry, which we will speak about in more detail later, the resulting theory is called *pure two-dimensional topological gravity*. Gauge fixing the symmetry of the theory results in an interpretation of the path integral of the theory in terms of performing an integration over a suitable moduli space [26]. In this setting the observables of two-dimensional gravity will turn out to be well-defined geometrical objects. This point of view shall be developed in Section 2.1.
Another approach to solve two-dimensional quantum gravity is provided by thinking of the path integral $\int \mathcal{D}g$ over the metric as an integral over all possible geometries of Σ. We can approximate Σ by random triangulations, ie. assume that Σ is build up by glueing together triangles. Thus, in the limit where the number of triangles goes to infinity one would naively assume that the path integral is described by a sum over all possible triangulations

$$\int \mathcal{D}g \quad\longrightarrow \sum_{\substack{\text{random triangulations}\\ \text{of } \Sigma}}\,.$$

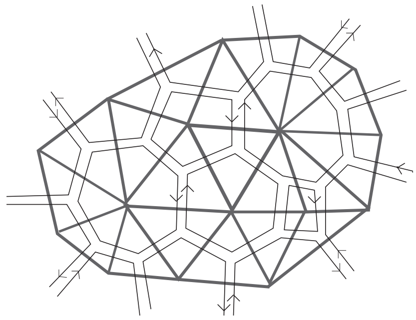


Figure 2.1: A triangulation (single lines) of a surface gives rise to a unique fatgraph (double lines). The image is taken from [27].

Now each triangulation gives rise to a fatgraph as depicted in Figure 2.1. Since fatgraphs are natural to appear in the Feynman expansion of hermitian matrix models this motivates that under the identification

$$\int \mathcal{D}g \;\longrightarrow\; \sum_{\substack{\text{random triangulations}\\ \text{of } \Sigma}} \;\longrightarrow\; \int_{\mathcal{H}_N} \mathrm{d}M$$

two-dimensional topological gravity admits a formulation in terms of hermitian matrix models. Since we will take a slightly different approach connecting two-dimensional gravity with matrix models we refer to [27] for a detailed review of the point view explained above.
In his breakthrough paper [1] Witten argues that the two points of view described above—the geometric and the matrix model one—indeed coincide leading him to a conjecture that the partition function of pure two-dimensional topological gravity is uniquely determined by a set of certain differential equations as we will see in Section 2.2. Shortly afterwards, the connection between the geometric and the matrix model interpretation of the theory was formulated rigorously by Kontsevich leading to his celebrated proof of Witten's conjecture [20]. We will review certain aspects of Kontsevich's proof in Section 2.3 and 2.4.

2.1 Intersection theory on the moduli space of Riemann surfaces

In order to quantise the Lagrangian $\mathcal{L} = 0$, one first has to specify the field content, which in our case is the metric g on Σ. Second, one needs to pick the symmetry of the theory which in the quantisation process gets gauge fixed. Obviously, the Lagrangian is invariant under arbitrary variations of the metric. One only has to take care that the signature stays positive. Following [26], in order to fix this symmetry one has to introduce suitable ghosts and ghosts-for-ghosts. The resulting theory is usually referred to as *pure two-dimensional topological gravity*. For convenience we will abbreviate this term by simply *pure gravity*.
After gauge fixing, the path integral of the gauge fixed Lagrangian transforms into an integral over the moduli of the surface Σ equipped with the complex structure induced by g besides an integration over the remaining auxiliary fields. The moduli space in question has a precise mathematical definition so that at the end we can write down a mathematically well-defined expression for the correlators of pure gravity.

We call a connected complex manifold Σ of complex dimension one a *Riemann surface*. Further let $\mathcal{M}_g$ denote the *moduli space of Riemann surfaces* of genus g

$$\mathcal{M}_{g,n} = \{\Sigma \mid \Sigma \text{ is a compact Riemann surface of genus } g\} /\sim \tag{2.1}$$

Its complex dimension is $3g-3$. Later we wish to consider observables $\tau_k(z)$ attached to certain points $z \in \Sigma$. In the path integral formulation this means that one must not divide by the full diffeomorphism group but only by those diffeomorphisms keeping z fixed. Thus on the geometric side we rather wish to integrate over $\mathcal{M}_{g,n}$ the moduli space of Riemann surfaces of genus g with n marked points, ie. $\mathcal{M}_{g,n}$ parametrizes $(\Sigma, z_1, \ldots, z_n)$ where Σ is a (compact) Riemann surface of genus g and $z_1, \ldots, z_n \in \Sigma$. Different from (2.1) we are not dividing by all biholomorphic maps but only by those respecting the marked points $z_1, \ldots, z_n$. Since each choice of a point increases the complex dimension of the moduli by one, clearly $\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3g-3+n$.

Example 2.1. Let us consider two examples [28, Sec. 6.1.1].

- First, let us consider $\mathcal{M}_{0,3}$. Since every compact Riemann surface of genus zero is isomorphic to the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the elements of this space are $[(\hat{\mathbb{C}}, z_1, z_2, z_3)] \in \mathcal{M}_{0,3}$. The automorphisms of $\hat{\mathbb{C}}$ are the Möbius transformations $z \mapsto \frac{a\,z+b}{c\,z+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$. Since there exists a unique Möbius transformation mapping z_1, z_2, z_3 to $0, 1, \infty$ we have $(\hat{\mathbb{C}}, z_1, z_2, z_3) \sim (\hat{\mathbb{C}}, 0, 1, \infty)$. This implies that $\mathcal{M}_{0,3}$ consists of only one equivalence class $[(\hat{\mathbb{C}}, 0, 1, \infty)]$ an thus $|\mathcal{M}_{0,3}| = 1$. This is in agreement with $\dim_{\mathbb{C}} \mathcal{M}_{0,3} = 0$.

- The example $\mathcal{M}_{1,1}$ is less trivial. In fact, due to $\dim_{\mathbb{C}} \mathcal{M}_{1,1} = 1$ we expect the space to depend on one complex modulus. Every Riemann surface of genus one is a torus which can biholomorphically be mapped to a parallelogram in $\mathbb{C}$ spanned by 1 and $\tau \in \mathbb{H}$ where the marked point is send to the origin. Two tori are equivalent if their underlying parallelograms described by τ and τ' are. That is the case if and only if $\tau' = \frac{a\,\tau+b}{c\,\tau+d}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Since two matrices $\pm A \in \mathrm{SL}(2,\mathbb{Z})$ induce the same transformation the automorphism group of the torus is $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$. We refer to standard literature on string theory (see for instance [29, 30]) that every $\tau \in \mathbb{H}$ is related to a unique element of the fundamental region

$$\mathcal{F} = \left\{-\frac{1}{2} \leq \operatorname{Re}\tau \leq 0,\ |\tau|^2 \geq 1\right\} \cup \left\{0 < \operatorname{Re}\tau < \frac{1}{2},\ |\tau|^2 > 1\right\}$$

  by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation. Thus, as expected $\mathcal{M}_{1,0} = \mathcal{F}$ is one-dimensional.

In order to perform integration on the space $\mathcal{M}_{g,n}$ we need to impose the stability condition

$$2g-2+n > 0\,. \tag{2.2}$$

The case where the above relation is met corresponds to the case where the automorphism group of the surfaces is finite. This is necessary in order to define an integration measure on the space (see [28, Sec. 6.1.2]).

In general the space $\mathcal{M}_{g,n}$ is non-compact. It can be compactified by allowing the surfaces Σ to have certain nodal singularities. We denote $\overline{\mathcal{M}}_{g,n}$ its Deligne-Mumford compactification [31]. For details on its construction we refer to [32] and the illustrated and less formal [28, Sec. 6.1.3].

So far we did not specify the observables of our theory. Witten argues [1] that the natural observables of two-dimensional topological gravity (on the geometric side) are the so called *ψ-classes* which are defined as follows. For $i \in \{1, \ldots, n\}$ let $\mathbb{L}_i$ be the line bundle over $\mathcal{M}_{g,n}$ whose fibre over a point $(\Sigma, z_1, \ldots, z_n) \in \mathcal{M}_{g,n}$ is the dual tangent space $T^*_{z_i}\Sigma$. The observables of interest are then $\psi_i := c_1(\mathbb{L}_i)$ the first Chern classes of these line bundles. Taking a suitable wedge product of the forms ψ_i matching up with the (virtual) dimension of $\overline{\mathcal{M}}_{g,n}$, we thus obtain the well-defined *intersection numbers*

$$\left\langle \tau_{k_1} \ldots \tau_{k_n} \right\rangle_g := \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{k_i} & \text{, if } 3g-3+n = \sum_{i=1}^n k_i \\ 0 & \text{, otherwise} \end{cases}$$

which are the correlators of the observables of pure gravity. In case the stability condition (2.2) is violated we set the above correlators to zero. One should interpret the notation in the way that the physical observables $\tau_k(z_i)$ are associated to powers ψ_i^k in the geometric picture of pure gravity. For the field theoretic origin of the observables τ_k we refer to [26, Sec. 6].

Example 2.2. Let us take up on Example 2.1.

- For $g = 0$ and $n = 3$ we only get a non-vanishing intersection number if $k_1 = k_2 = k_3 = 0$. In this case
$$\langle \tau_0\, \tau_0\, \tau_0 \rangle_0 = \int_{\overline{\mathcal{M}}_{0,3}} [1]$$
simply measures the volume of the moduli space. Since in Example 2.1 we have seen that $\mathcal{M}_{0,3}$ only consists of one point we can deduce that $\langle \tau_0\, \tau_0\, \tau_0 \rangle_0 = 1$.
- The only non-vanishing intersection number on $\overline{\mathcal{M}}_{1,1}$ is $\langle \tau_1 \rangle_1$. Its value is
$$\langle \tau_1 \rangle_1 = \frac{1}{24}$$
The rationality of this intersection number reflects the fact that $\overline{\mathcal{M}}_{1,1}$ is not a manifold. Following the arguments of Witten the $\frac{1}{12}$ in $\langle \tau_1 \rangle_1$ is due to the existence of an elliptic modular form of weight 12 with a simple zero at the cusp and the further factor $\frac{1}{2}$ is due to the fact that a generic elliptic curve has two symmetries [19].

2.2 Witten's conjecture

Comparing how ψ-classes behave under the pull-back by the map forgetting one of the marked points of $\overline{\mathcal{M}}_{g,n}$ one can show that the intersection numbers satisfy the two fundamental relations

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{k_i} \right\rangle_g = \sum_{j=1}^n \left\langle \tau_{k_j - 1} \prod_{i \neq j} \tau_{k_i} \right\rangle_g \tag{2.3}$$

$$\left\langle \tau_1 \prod_{i=1}^n \tau_{k_i} \right\rangle_g = (2g-2+n) \left\langle \prod_{i=1}^n \tau_{k_i} \right\rangle_g \tag{2.4}$$

referred to as the *string* and *dilaton equation*. From these equations using the initial values from Example 2.2 one is able to compute all intersection numbers $\left\langle \tau_{k_1} \ldots \tau_{k_n} \right\rangle_g$ for $g \leq 1$. For a more detailed discussion we refer to [33, Sec. 25.2].

It turns out, the intersection numbers satisfy further relations. In order to express these, it is convenient to introduce the generating function

$$F^{\mathrm{KW}}(t_0, t_1, \dots) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\dots,k_n \geq 0} \langle \tau_{k_1} \dots \tau_{k_n} \rangle_g \prod_{i=1}^{n} t_{k_i} \tag{2.5}$$

with formal variables (couplings) $\hbar, t_0, t_1, \dots$. We refer to this function as the *free energy* and denote $Z^{\mathrm{KW}} := \exp F^{\mathrm{KW}}$ the *partition function* of pure gravity.
The string and dilaton equation then translate into the differential constraints $L_{-1}\, Z = 0$ and $L_0\, Z = 0$ satisfied by Z^{KW} where

$$L_{-1} := \quad -\hbar\partial_{t_0} + \frac{1}{2} t_0^2 + \sum_{k=0}^{\infty} t_{k+1}\, \hbar\partial_{t_k}\,, \tag{2.6}$$

$$L_0 \;:= -\frac{3}{2}\,\hbar\partial_{t_1} + \sum_{k=0}^{\infty} \frac{2k+1}{2}\, t_k\, \hbar\partial_{t_k} + \frac{1}{16}\,. \tag{2.7}$$

Motivated by the analogy to matrix models Witten conjectured that F^{KW} satisfies further differential constraints. In his original formulation [19] conjectures that F^{KW} satisfies the so called KdV equations. This conjecture can equivalently be expressed in terms of differential operators acting on the partition function.

Witten's conjecture. *The partition function of Z^{KW} satisfies*

$$L_n\, Z^{KW} = 0 \qquad n \in \{-1, 0, 1, 2, \dots\} \tag{2.8}$$

where for $n > 0$

$$L_n := -\frac{(2n+3)!!}{2^{n+1}}\,\hbar\partial_{t_{n+1}} + \frac{1}{2} \sum_{\substack{j+k=n-1 \\ j,k\geq 0}} \frac{(2j+1)!!\,(2k+1)!!}{2^{n+1}} \hbar^2 \partial_{t_j}\partial_{t_k} + \sum_{k\geq 0} \frac{(2k+2n+1)!!}{2^{n+1}(2k-1)!!}\, t_k\, \partial_{k+n}\,. \tag{2.9}$$

These differential constraints determine Z^{KW} uniquely.

In equation (2.9) we have used the notation

$$(2k+1)!! = (2k+1)\cdot(2k-1)\cdot(2k-3)\cdot\ldots\cdot 3\cdot 1\,.$$

The differential operators $\{L_n\}_{n\geq -1}$ further satisfy

$$[L_m, L_n] = \hbar(m-n)\, L_{m+n} \qquad m,n \in \{-1, 0, 1, 2, \ldots\}\,, \tag{2.10}$$

ie. they are a representation of the Virasoro algebra in terms of differential operators over the field $\mathbb{C}_\hbar := \mathbb{C}((\hbar))$. Due to (2.10), one often refers to the differential constraints (2.8) as *Virasoro constraints*. The conjecture was first proven by Kontsevich [20] by making the analogy to matrix models explicit as we will review in the next section. One should also mention the proofs of Mirzakhani [34, 35], Kazarian–Lando [36], and Jarvis–Kimura–Vaintrob [37]. However, in the following we will focus on a discussion of certain aspects of Kontsevich's proof.

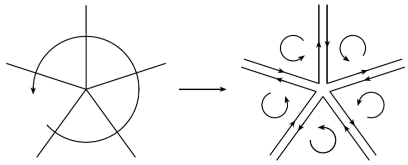


Figure 2.2: A cyclic ordering of edges as displayed in the left image is equivalently characterised by thin strips glued to the vertex. The figure was taken from [38].

2.3 A combinatorial formula for intersection numbers

The first step in Kontsevich's proof was the derivation of a diagrammatic formula for the intersection numbers using fatgraphs. Before we proceed, let us first fix a formal definition of what we mean by a fatgraph.

Definition 2.3. A *fatgraph* (also called *ribbon graph*) $\Gamma = (G, \sigma)$ is a graph G together with a cyclic ordering σ of the edges incident to a vertex.

That this definition of a fatgraph is equivalent to the interpretation found in most physics literature can be argued as follows. As illustrated in Figure 2.2 a cyclic ordering of edges incident to a vertex gives rise to a replacement of the edges by thin strips joining at the vertex with induced orientation on their boundaries. The area encircled by these strips is called a *face* of the fatgraph. If we attach an oriented disk to each such boundary component of a fatgraph Γ we obtain a surface $\Sigma(\Gamma)$ whose genus g satisfies

$$|V(\Gamma)| - |E(\Gamma)| + b = 2 - 2g$$

where $V(\Gamma)$ and $E(\Gamma)$ are the sets of vertices resp. edges and b is the number of boundary components of Γ as illustrated in Figure 2.3. For more details on fatgraphs we refer to [38].
An isomorphism of fatgraphs is a bijection between the vertices and edges of two graphs respecting the cyclic ordering. Further, we denote $\mathrm{Aut}(\Gamma)$ the automorphism group of a fatgraph Γ and let $\mathbb{RG}^3_{g,n}$ denote the set of isomorphism classes of closed connected trivalent fatgraphs of genus g with n marked faces labelled by formal variables $\lambda_1, \ldots, \lambda_n$. For an edge $e \in E(\Gamma)$ of a fatgraph Γ we set

$$\lambda(e) = \frac{1}{\lambda_i + \lambda_j} \tag{2.11}$$

where λ_i and λ_j are the labels of the two faces incident to e. Then Kontsevich's first step in order to prove Witten's conjecture was to show the following combinatorial formula for the intersection numbers.

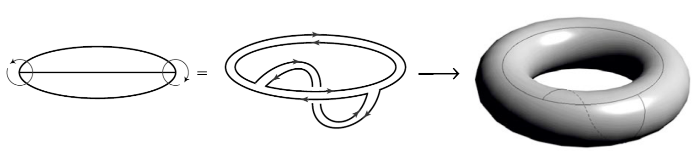


Figure 2.3: Example of a fatgraph drawn on its associated Riemann surface. The figures are taken from [38].

Theorem 2.4. [20] *Let $\lambda_1, \ldots, \lambda_n$ be formal variables. Then the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ satisfy*

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \left\langle \tau_{k_1} \ldots \tau_{k_n} \right\rangle_g \prod_{i=1}^{n} \frac{(2k_i-1)!!}{\lambda_i^{2k_i+1}} = \sum_{\Gamma \in \mathbb{RG}^3_{g,n}} \frac{2^{-|V(\Gamma)|+|E(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \lambda(e)\,. \tag{2.12}$$

In the proof of this theorem Kontsevich used a foliation of the moduli space $\overline{\mathcal{M}}_{g,n}$ by so called Strebel differentials whose trajectories assign a unique fatgraph to each surface. Thus, the foliation of $\overline{\mathcal{M}}_{g,n}$ admits a decomposition in terms of fatgraphs. Then integrating a suitable form containing the ψ-classes over this foliation one ends up with (2.12). For more details we refer to the original paper of Kontsevich [20] and the more pedagogic and illustrated book [28, Sec. 6.3].

2.4 The Kontsevich matrix model

We have already argued that pure gravity admits an interpretation in terms of matrix models since these may be interpreted as computing triangulations of Riemann surfaces. The approach Kontsevich took was different in the sense, that he constructed a matrix model evaluating the combinatorial formula (2.12) which was derived by a stratification of $\overline{\mathcal{M}}_{g,n}$ in terms of fatgraphs.
Given a diagonal matrix $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N)$ with $\lambda_i \in \mathbb{R}$, let us define the Gaussian measure

$$\mathrm{d}\mu_\Lambda(M) := c_\Lambda \mathrm{e}^{-\frac{\hbar^{-1/2}}{2}\mathrm{Tr}\,\Lambda M^2}\mathrm{d}M \qquad \mathrm{d}M = \left(\prod_{i=1}^{N} \mathrm{d}M_{ii}\right)\left(\prod_{i<j} \mathrm{d}\,\mathrm{Re}\,M_{ij}\;\mathrm{d}\,\mathrm{Im}\,M_{ij}\right) \tag{2.13}$$

on the space of hermitian $(N \times N)$-matrices $\mathcal{H}_N$ with c_Λ chosen such that $\int_{\mathcal{H}_N} \mathrm{d}\mu_\Lambda(M) = 1$. As usual we will write

$$\langle f(M) \rangle_\Lambda := \int_{\mathcal{H}_N} \mathrm{d}\mu_\Lambda(M)\, f(M)$$

for the expectation value of f with respect to the Gaussian measure $\mathrm{d}\mu_\Lambda$. Since we can write $\mathrm{Tr}(\Lambda M^2) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)\, M_{ij}M_{ji}$ the propagator is given by

$$\left\langle M_{ij} M_{kl} \right\rangle_\Lambda = \frac{2\,\delta_{i,l}\delta_{j,k}}{\lambda_i + \lambda_j}\,. \tag{2.14}$$

Remarkably, this propagator exactly resembles the weight for edges (2.11) in the combinatorial formula (2.12). Let us proceed by defining the *Kontsevich matrix model* whose partition function is

$$Z^{\mathrm{KMM}}(\Lambda) := \left\langle \exp\left(\mathrm{Tr}\,\tfrac{1}{3!}M^3\right)\right\rangle_\Lambda = c_\Lambda \int_{\mathcal{H}_N} \mathrm{d}M\; \mathrm{e}^{\hbar^{-1/2}\,\mathrm{Tr}\left(\frac{1}{3!}M^3 - \frac{1}{2}\Lambda M^2\right)}\,. \tag{2.15}$$

Expanding the potential $\exp \frac{1}{3!}M^3 = \sum_{k=0}^{\infty} \frac{1}{6^k k!} M^{3k}$ and applying Wick's theorem we can evaluate Z^{KMM} diagrammatically as the sum over all trivalent (possibly disconnected) fatgraphs weighting all edges with (2.14). The corresponding free energy $F^{\mathrm{KMM}} := \log Z^{\mathrm{KMM}}$ is computed in the same way but summing only over connected graphs. Eventually one arrives at

$$F^{\mathrm{KMM}}(\Lambda) = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\Gamma\in\mathbb{RG}^3_{g,n}}\sum_{a_1\dots a_n=1}^{N}\frac{2^{-|V(\Gamma)|+|E(\Gamma)|}}{|\operatorname{Aut}(\Gamma)|}\hbar^{g-1+n/2}\prod_{\{i,j\}=e\in E(\Gamma)}\frac{1}{\lambda_{a_i}+\lambda_{a_j}} \tag{2.16}$$

For a complete, pedagogic introduction to the computational steps sketched above we refer to [28, Sec. 2.2 & 6.3.6] and [39, Sec. 1.1].
By now it should be clear that the matrix model potential (2.15) was engineered so that its fatgraph expansion exactly reproduces the combinatorial formula (2.12). Let us thus proceed by plugging in equation (2.12) into (2.16) in order to obtain

$$F^{\mathrm{KMM}}(\Lambda) = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{g-1-n/2}}{n!}\sum_{a_1\dots a_n=1}^{N}\sum_{k_1,\dots,k_n=0}^{\infty}\left\langle\tau_{k_1}\dots\tau_{k_n}\right\rangle_g\prod_{i=1}^{n}\frac{(2k_i-1)!!}{\lambda_{a_i}^{2k_i+1}}\,.$$

The sum over the a_i can be carried out introducing Miwa's coordinates

$$t_k(\Lambda) := \hbar^{1/2}\,(2k-1)!!\,\operatorname{Tr}\Lambda^{-2k-1} \tag{2.17}$$

and comparing with the definition of F^{KW} given in (2.5) we finally obtain

$$F^{\mathrm{KMM}}(\Lambda) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\sum_{k_1,\dots,k_n\geq 0}\left\langle\tau_{k_1}\dots\tau_{k_n}\right\rangle_g\prod_{i=1}^{n}t_{k_i}(\Lambda) = F^{\mathrm{KW}}(t_0(\Lambda),t_1(\Lambda),\dots)\,.$$

Since in the large N limit all variables $t_0(\Lambda), t_1(\Lambda), \dots$ become algebraically independent we can thus interpret F^{KW} as the asymptotic expansion of F^{KMM} around $\Lambda^{-1}\to 0$ under the identification (2.17) as $N\to\infty$. Using this explicit matrix model interpretation Kontsevich prove Witten's conjecture. For the proof we refer interested readers to the Kontsevich's original paper [20] or to [40].

2.5 Motivation and outlook

To sum up, we have learned that pure gravity may be interpreted in different ways. First, it admits an interpretation in terms of enumerative geometry—integration over the moduli space $\overline{\mathcal{M}}_{g,n}$—and second, in terms of matrix models.
Further, we can interpret Witten's conjecture as third characterisation of pure gravity. In the formulation of Witten's conjecture presented in Section 2.2 the partition function Z^{KW} is uniquely characterised by differential constraints $L_n\,Z^{\mathrm{KW}} = 0$ imposed by the operators $\{L_n\}_{n\geq -1}$ satisfying Virasoro commutation relations. By this we mean, that one can translate $L_n\,Z^{\mathrm{KW}} = 0$ into certain relations satisfied by the correlators like in the case of the string and dilaton equation given in (2.3) and (2.4). Using these relations one is then able to compute all correlators recursively starting from the base cases $\langle\tau_0\,\tau_0\,\tau_0\rangle_0$ and $\langle\tau_1\rangle_1$. In this way we can say that pure gravity is fully characterised by the Virasoro constraints satisfied by its partition function. In the next chapter we will axiomatise this idea introducing *quantum Airy structures*.
For completeness let us remark that in its original formulation Witten's conjecture said that Z^{KW} is a tau-function of the KdV hierarchy (also satisfying the string equation). Thus as a fourth characterisation,

pure gravity admits an interpretation in terms of integrable hierarchies. However, this is beyond the scope of this thesis. In the following we will mainly restrict the discussion to the first three points of view mentioned above.

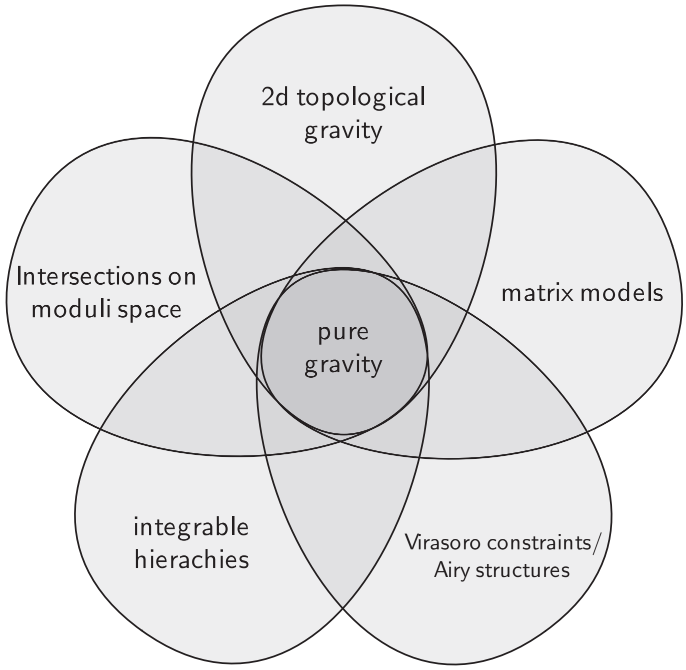


Figure 2.4: Illustration of the different characterisations of pure gravity developed in this chapter.

CHAPTER 3

Airy structures

In the previous chapter we have learned that the partition function of pure gravity is uniquely determined by a set of differential equations. It turns out, many other enumerative problems are also governed by such *Virasoro-like* constraints in the sense that the generating function Z associated to the enumerative problem is uniquely determined by the property

$$H_i Z = 0 \tag{3.1}$$

that it is annihilated by certain differential operators $\{H_i\}_{i \in I}$. In some cases these constraints even imply that Z may be computed recursively.

Quantum Airy structures are the answer to the following opposite question. Which properties does a set of differential operators $\{H_i\}$ have to satisfy such that there exists a unique solution Z of (3.1) which features properties expected from a partition function. The axiomatisation of such *Virasoro like constraints* in form of quantum Airy structures was developed in [4, 5]. However, a large class of interesting problems is characterised by a more general class of differential constraints than Virasoro constraints, namely $\mathcal{W}$-algebra constraints. The theoretical framework for these structures was developed in [7] and goes under the name higher quantum Airy structures.

This chapter is devoted to the introduction of these structures and we will highlight some of their general features. Along the way, we will explain how pure gravity is fitting into this picture.

3.1 Definition of higher quantum Airy structures

Let V be a $\mathbb{C}$-vector space (possibly countable infinite dimensional). Fix a basis $\{e_i\}_{i\in I}$ inducing linear coordinates x^i on V. We introduce the formal coupling parameter $\hbar$ and let

$$\mathcal{D}_V^\hbar := \mathbb{C}[\![\{x^i\}_{i\in I}, \{\hbar\partial_i\}_{i\in I}, \hbar]\!]$$

be the (completed) algebra of differential operators together with the quantum commutator

$$[\hbar\partial_i, x^j] = \delta_i^j \,.$$

We equip this space with the grading

$$\deg x^i = \deg \hbar\partial_i = \deg \hbar^{1/2} = 1$$

and we will write $P = Q + o(n)$ for two elements $P, Q \in \mathcal{D}_V^\hbar$ if they agree up to at least degree $n-1$.

Definition 3.1. We say a family $\{H_i\}_{i\in I} \subset \mathcal{D}_V^\hbar$ of differential operators on V is a *higher quantum Airy structure* if

(i) For all $i \in I$ the operator H_i is of the form

$$H_i = \hbar\partial_i + o(2)\,. \tag{3.2}$$

(ii) There exist $f_{i,j}^k \in \mathcal{D}_V^\hbar$ such that for all $i, j \in I$

$$[H_i, H_j] = \hbar\, f_{i,j}^k\, H_k\,. \tag{3.3}$$

We will usually refer to property (i) as the *degree one condition* and to (ii) as the *subalgebra condition*. If moreover the maximum degree of all H_i is r, we say that $\{H_i\}_{i\in I}$ is a *higher quantum r-Airy structure*. If we allow for powers $\hbar^{1/2}$ to appear in $\{H_i\}_{i\in I}$, we say these operators are a *crosscapped* higher quantum Airy structure if they satisfy (i) and (ii).

Remark 3.2. For convenience, we will often simply refer to the above defined (crosscapped) higher quantum Airy structures as *Airy structures*. The term *higher* refers to the fact that we allow the maximum degree r of the operators to be arbitrary. In the original definition [4, 5] the term quantum Airy structure just referred to the special case of quadratic operators $r = 2$.
The word *quantum* refers to the fact, that one can view the above defined operators as being the quantisation of a *classical* Airy structure. This point of view is emphasised in [4, 6] and shall not be further explained here. A basis independent definition of Airy structures is presented in [7].

Example 3.3. In Section 2.2 we introduced the differential operators

$$
\begin{aligned}
L_{-1} &= -\hbar\partial_{t_0} + \frac{1}{2}\, t_0^2 + \sum_{k=0}^{\infty} t_{k+1}\, \hbar\partial_{t_k}\,, \\
L_n \ &= -\frac{(2n+3)!!}{2^{n+1}}\, \hbar\partial_{t_{n+1}} + \frac{1}{2} \sum_{\substack{j+k=n-1 \\ j,k\geq 0}} \frac{(2j+1)!!\,(2k+1)!!}{2^{n+1}} \hbar^2 \partial_{t_j}\partial_{t_k} \\
&\qquad + \sum_{k\geq 0} \frac{(2k+2n+1)!!}{2^{n+1}(2k-1)!!}\, t_k\, \hbar\partial_{k+n} + \frac{\hbar}{16}\delta_{n,0}
\end{aligned}
$$

with $n \geq 0$ which by Witten's conjecture annihilate the partition function Z^{KW} of pure gravity. Clearly, after a suitable linear rescaling of the coordinates $t_0, t_1, \ldots$ the above equations satisfy the degree one condition (3.2). Regarding the subalgebra condition (3.3) note that the operators satisfy the commutation relations

$$
[L_m, L_n] = \hbar(m-n)\, L_{m+n}\,.
$$

Thus, after a suitable change of basis $\{L_n\}_{n\geq -1}$ is indeed a 2-Airy structure.

As we will see in a second, the statement that Z^{KW} is uniquely determined by the property that $L_n\, Z^{\mathrm{KW}} = 0$ for $n \geq -1$ naturally generalises to arbitrary Airy structures.

3.2 The associated partition function

Given an Airy structure $\{H_i\}$, the operators H_i naturally act on functions depending on $\{x_i\}_{i\in I}$ via differentiation. We can thus ask whether there exists a non-trivial solution Z to the set

$$
H_i\, Z = 0 \qquad\qquad i \in I \tag{3.4}
$$

of differential constraints. As already hinted Airy structures are exactly tailored so that we can give a positive answer to this question.

Theorem 3.4. [4, Thm. 2.4.2], [7, Prop. 2.20] *If $\{H_i\}_{i\in I}$ is a (crosscapped) higher quatum Airy structure then the differential equations* (3.4) *admit a unique solution Z of the form $Z = \exp(F)$ with*

$$
F = \sum_{\substack{g,n\geq 0 \\ 2g+2-n>0}} \frac{\hbar^{g-1}}{n!} \prod_{i_1\ldots i_n \in I} F_{g,n}[i_1 \ldots i_n]\, x_{i_1} \ldots x_{i_n} \tag{3.5}
$$

where the coefficents $F_{g,n}[i_1 \ldots i_n] \in \mathbb{C}$ are symmetric under permutation of $i_1, \ldots, i_n$. In case of a crosscapped Airy structure $g \in \frac{1}{2}\mathbb{N}_0$. Otherwise g is integer.

We will often call Z the *partition function*, F the *free energy*, and $F_{g,n}$ the *correlators* associated to the Airy structure $\{H_i\}_{i\in I}$.
The uniqueness of a solution Z is due to the specific form of the differential operators H_i, ie. the degree one condition 3.2. The subalgebra condition 3.3 ensures, that the tensors $F_{g,n}[i_1 \ldots i_n]$ are symmetric in their arguments. However, what is even more remarkable is that the $F_{g,n}$ may be computed recursively in $\chi_{g,n} := 2g + 2 - n$.

3.2.1 Topological recursion

For the rest of this section let $\{H_i\}_{i\in I}$ be a higher quantum r-Airy structure. Let us assume that the index set I is a subset of $\mathbb{N}$ and introduce the notation

$$J_k := \hbar\partial_k\,, \quad J_{-k} := k\,x_k \tag{3.6}$$

for $k \in I$ and set $\mathcal{I} := I \cup -I$. Then we can expand the operator H_i into a sum of normal ordered monomials in the Js and $\hbar$

$$H_i = J_i - \sum_{m=2}^{r} \sum_{\substack{j,n\geq 0\\ 2j+n=m}} \frac{\hbar^j}{n!} \sum_{a_1\ldots a_n\in\mathcal{I}} C^{(j)}[i|a_1,\ldots,a_n]\ \boldsymbol{:}J_{a_1}\ldots J_{a_n}\boldsymbol{:} \tag{3.7}$$

with coefficients $C^{(j)}[i|a_1,\ldots,a_n] \in \mathbb{C}$ and where $\boldsymbol{:}\ldots\boldsymbol{:}$ denotes the normal ordering of differential operators. By convention if $n=0$ the product $\boldsymbol{:}J_{a_1}\ldots J_{a_n}\boldsymbol{:}$ gets replaced by 1. For crosscapped Airy structures the sum over j in (3.7) runs over $\frac{1}{2}\mathbb{N}_0$. In the non-crosscapped case j is integer.
Let us now revisit the set of differential equations $H_i\,Z=0$ where $i \in I$. We know that these equations admit a solution $Z=\exp(F)$ with F being of the form 3.5. We can expand $Z^{-1}\,H_i\,Z$ into a linear combination of monomials in $\hbar$ and linear coordinates x^i. Necessarily all coefficents in this expansion need to vanish as $H_i\,Z=0$. Doing so, one finds that the $F_{g,n}$ satisfy a recursion relation defining them uniquely.
First, we need to introduce some more notation. Let $a=(a_1,\ldots,a_n)\in\mathcal{I}^n$ and $b=(b_2,\ldots,b_n)\in I^{n-1}$. We say $\boldsymbol{L}$ is a set partition of a if $\boldsymbol{L}$ is an unordered set of $||\boldsymbol{L}||$ pairwise disjoint non-empty subsets of a whose union is a. In this case we write $\boldsymbol{L}\vdash a$. Further, we say a map $\boldsymbol{\mu}:\boldsymbol{L}\to\mathfrak{P}(b)$ is a partition of b indexed by $\boldsymbol{L}$ if $(\mu_L)_{L\in\boldsymbol{L}}$ is a collection of pairwise disjoit possibly empty subsets of b whose union is b. We will write $\boldsymbol{\mu}\vdash_{\boldsymbol{L}} b$ in this case. Let us moreover formally introduce

$$F_{0,2}[a_1,a_2] := |a_1|\,\delta_{a_1,-a_2} \tag{3.8}$$

for $a_1,a_2\in\mathcal{I}$ and extend the definition of $F_{g,n}$ from I^n to $\mathcal{I}^n$ by setting $F_{g,n}[a_1,\ldots,a_n]=0$ if at least one $a_j<0$. One obtains that the $F_{g,n}$ are computed recursively from the coefficients $C^{(j)}[i|a]$ in the following way.

Theorem 3.5. [7, Corollary 2.16] *For all $b_1\in I$ and $b=(b_2,\ldots,b_n)\in I^{n-1}$ we have*

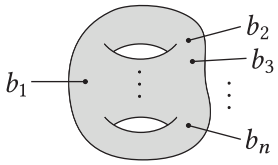
$$F_{g,n}\big[b_1,b\big] = \sum_{\substack{\ell,j\geq 0\\ 2\leq\ell+2j\leq r}} \frac{1}{\ell!} \sum_{a\in\mathcal{I}^\ell} C^{(j)}\big[b_1|a\big] \sum_{\boldsymbol{L}\vdash a} \sum_{\substack{h:\boldsymbol{L}\to\mathbb{N}_0\\ \ell+j+\sum_{L\in\boldsymbol{L}} h_L=g+||\boldsymbol{L}||}} \sideset{}{''}\sum_{\boldsymbol{\mu}\vdash_{\boldsymbol{L}} b} \left(\prod_{L\in\boldsymbol{L}} F_{h_L,|L|+|\mu_L|}\big[L,\mu_L\big]\right). \tag{3.9}$$

The double prime over the summation symbol indicates that terms with $h_L=0$, $|\mu_L|=0$ and $|L|\leq 2$ are excluded from the sum.

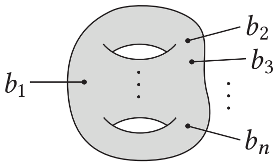
Remark 3.6. The original proof [7, Corollary 2.16] assumes that the Airy structure is non-crosscapped, ie. $j\in\mathbb{N}_0$ in the above formula. However, the same proof runs through in the crosscapped case if one extends the summation range to $j\in\frac{1}{2}\mathbb{N}_0$.

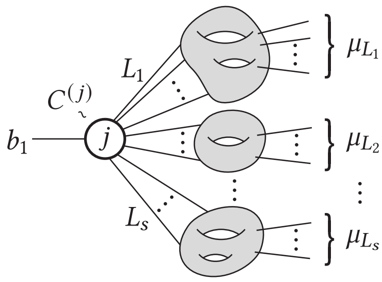
3.2.2 Intuition on the recursion formula

Let us develop some intuition for formula (3.9) in order to see that it indeed computes all the $F_{g,n}$ recursively in $\chi_{g,n} = 2g - 2 + n$. Intuitively, one should think of $F_{g,n}[b_1, \ldots, b_n]$ as a Riemann surface of genus g with n legs labelled by $b_1, \ldots, b_n \in \boldsymbol{I}$

$F_{g,n}[b_1, \ldots, b_n]$ "=" 

Informally speaking, one can interpret (3.9) as the sum over all splittings of $F_{g,n}[b_1, \ldots, b_n]$ into surfaces associated to $\{F_{h_L, |L|+|\mu_L|}\}_{L \in \boldsymbol{L}}$ while keeping the leg b_1 fixed



"=" $\sum_{\text{'all splittings'}} \sum_{\text{'intermediate states'}}$ 

(3.10)

In this picture one has to think about $L \in \boldsymbol{L} = \{L_1, \ldots, L_s\}$ as the set of incoming and μ_L as the set of outgoing legs of $F_{h_L, |L|+|\mu_L|}$. The vertex connecting b_1 with the $F_{h_L, |L|+|\mu_L|}$ carries a formal genus j and represents the factor $C^{(j)}[b_1|a]$ (times some symmetry factor). The sum constraint in (3.9) for the summation over $h : \boldsymbol{L} \to \mathbb{N}_0$ is equivalent to the demand that both sides in (3.10) feature the same characteristic, ie.

$$2g - 2 + n = 2\left(j + \sum_{L \in \boldsymbol{L}} h_L + \sum_{L \in \boldsymbol{L}} (|L| - 1)\right) - 2 + n\,.$$

Since terms including $F_{0,1}$ are absent on the right hand side of the recursion formula (3.9), we are only summing over splittings with $\chi_{h_L, |L|+|\mu_L|} \geq 0$ for all $L \in \boldsymbol{L}$. Thus clearly,

$$\chi_{h_L, |L|+|\mu_L|} < \chi_{g,n}$$

for all $L \in \boldsymbol{L}$ on the right hand side of (3.9). Therefore starting from level $\chi_{g,n} = 1$ with

$$F_{0,3}[b_1, b_2, b_3] = b_2 b_3\, C^{(0)}[b_1| - b_2, -b_3]\,, \qquad F_{1,1}[b_1] = C^{(1)}[b_1|\varnothing] \tag{3.11}$$

equation (3.9) tells us how to obtain all further $F_{g,n}$ with $\chi_{g,n} > 1$ recursively. For instance on the level $\chi_{g,n} = 2$ we find [7]

$$\begin{aligned} F_{0,4}[b_1, b_2, b_3, b_4] = b_2 b_3 b_4\, C^{(0)}[b_1| - b_2, -b_3, -b_4] + \sum_{a \in \boldsymbol{I}} \big(b_2\, C^{(0)}[b_1| - b_2, a]\, F_{0,3}[a, b_3, b_4] \\ + b_3\, C^{(0)}[b_1| - b_3, a]\, F_{0,3}[a, b_2, b_4] + b_4\, C^{(0)}[b_1| - b_4, a]\, F_{0,3}[a, b_2, b_3]\big)\,, \end{aligned} \tag{3.12}$$

and

$$F_{1,2}[b_1, b_2] = \sum_{a \in I} b_2\, C^{(0)}[b_1| - b_2, a]\, F_{1,1}[a] + \sum_{a_1, a_2 \in I} \tfrac{1}{2}\, C^{(0)}[b_1|a_1, a_2]\, F_{0,3}[b_2, a_1, a_2]$$
$$+ b_2\, C^{(1)}[b_1| - b_2]\,. \tag{3.13}$$

where we have already plugged in (3.8) for $F_{0,2}$.
The graphical interpretation (3.10) of the recursion formula can probably be made rigorous in a sense that the $F_{g,n}$ admit a Feynman diagram-like computation. In the case of quantum 2-Airy structures such a formulation has already been developed in [6, Sec. 1.3].

There is an interesting observation to make at this point. Considering the first sum in (3.9), we see that the number of terms entering the recursion is limited by the value of r which is the maximum degree of the monomials in the expansion of the $\{H_i\}_{i \in I}$. Thus, the larger the value of r the more fine tuned the coefficients $C^{(j)}[.]$ of the operators H_i have to be in order to produce symmetric $F_{g,n}$. However, in Chapter 4 we will learn that there exist higher quantum r-Airy structures for arbitrary large r despite this difficulty.
Moreover, notice that in the recursion relation (3.9) the first variable b_1 of $F_{g,n}$ has a special role which can explicitly be observed in equations (3.12) and (3.13). It is thus a priori not clear why the $F_{g,n}$ should be symmetric under permutation of their arguments. That this property is indeed satisfied is due to the subalgebra property of an Airy structure. See for instance the discussion in [5, Prop. 2.4] for the case $r = 2$.

3.3 Reduction

Let $\{H_i\}_{i \in I}$ be a higher quantum Airy structure on a vector space V and Z its corresponding partition function. Assume that the index subset $I_0 \subseteq I$ is so that $H_i = J_i$ for all $i \in I_0$. Then since $H_i\, Z = 0$, the partition function must be independent of $\{x_i\}_{i \in I_0}$. Let us thus consider the subspace

$$V_{\text{red}} := \{x \in V \mid \forall i \in I_0 : \ x_i = 0\}$$

and define

$$H_i\big|_{\text{red}} := H_i\big|_{V_{\text{red}}}$$

by formally setting all partials and coordinates not lying in V_{red} to zero.
Clearly, the operators $\{H_i|_{\text{red}}\}_{i \in I \setminus I_0}$ satisfy the degree one condition (3.2). Indeed, they allow for a unique solution Z_{red} solving

$$H_i|_{\text{red}}\, Z_{\text{red}} = 0$$

for all $i \in I \setminus I_0$ and not surprisingly $Z = Z_{\text{red}}$ [7, Lem. 2.18]. Nevertheless, the author has no knowledge about a proof that $\{H_i|_{\text{red}}\}_{i \in I \setminus I_0}$ in general satisfies the subalgebra property (3.3) making it an Airy structure again. However, in special cases this statement has already been proven to be true.

CHAPTER 4

Airy structures from $\mathcal{W}$-algebras

In Chapter 2 we have learned that the partition function Z^{KW} of pure gravity is uniquely determined by Virasoro constraints which we then in Chapter 3 identified to be an example of an Airy structure. Other models related to pure two-dimensional topological gravity feature the same property. Often these models satisfy so called $\mathcal{W}$-constraints which are a generalisation of Virasoro constraints. Indeed, also these $\mathcal{W}$-constraints fit into the framework of Airy structures as we will see in this chapter.
The first aim of this chapter is to introduce $\mathcal{W}$-algebras. They arise as subalgebras of certain chiral algebras. Under a chiral algebra we understand a CFT only depending on its holomorphic part, which means that all fields $\phi(z, \bar{z})$ only depend on the variable z. Those algebras are axiomatised as so called vertex operator algebras which for completeness we will introduce in Section 4.1. We will then use the introduced notation in order to formally define $\mathcal{W}$-algebras in Section 4.2. We then specialise to an analysis of the so called $\mathcal{W}(\mathfrak{gl}_r)$-algebra. Of course, our final aim is to construct Airy structures from this $\mathcal{W}$-algebra. The first step towards this goal is to represent the abstract $\mathcal{W}$-algebra (resp. its modes) in terms of differential operators which is done in Section 4.5.
Then finally, we will be able to introduce the anticipated $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures of [7] in Section 4.6. Moreover, in Section 4.7 we will present a new family of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures which was constructed under the supervision of Gaëtan Borot.

4.1 Lattice vertex algebras

Let us first introduce the notion of a vertex operator algebra as the axiomatisation of a chiral CFT.

Definition 4.1. A *vertex operator algebra* (VOA) $(V, Y, |0\rangle, |\omega\rangle)$ consists of

- a graded vector space $V = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n$ called the *state space*. For a $|v\rangle \in V_n$ we say $|v\rangle$ has *conformal weight n*.
- a *state-field correspondence* Y, which is a linear map

$$\begin{aligned} Y(\cdot, z): \ V &\longrightarrow \mathrm{End}(V)[\![z, z^{-1}]\!] \, , \\ |v\rangle &\longmapsto Y(|v\rangle, z) = \sum\nolimits_{n\in\mathbb{Z}} v_{(n)} z^{-n-1} \, . \end{aligned}$$

We call $v_{(n)} \in \mathrm{End}(V)$ the *modes* of $|v\rangle$. We demand that Y satisfies the *axiom of locality* which says that if $|u\rangle, |v\rangle \in V$ there exists a $N_{u,v} \in \mathbb{Z}$ such that

$$(z_1 - z_2)^{N_{u,v}} [Y(|u\rangle, z_1), Y(|v\rangle, z_2)] = 0 .$$

- a *vacuum state* $|0\rangle \in V$ which spans V_0 and moreover satisfies

  $$Y(|0\rangle, z) = \mathrm{id}_V$$

  and that for all $|v\rangle \in V$ we have

  $$Y(|v\rangle, z)\, |0\rangle - |v\rangle \in z\, V[\![z]\!]. \tag{4.1}$$

- a *conformal state* $|\omega\rangle \in V$ whose modes $Y(|\omega\rangle, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ form a representation of the Virasoro algebra

  $$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

  of central charge c. Moreover, if $|v\rangle \in V_n$ is homogenous, then $L_0\,|v\rangle = n\,|v\rangle$ and

  $$Y(L_{-1}\,|v\rangle, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(|v\rangle, z)$$

  for all $|v\rangle \in V$.

Remark 4.2. Let us halt for a second and convince ourselves that this definition is indeed an axiomatisation of the chiral part of a CFT. We have to think of V as our state space. In a CFT for each state $|\phi\rangle$ there exists an associated field $\phi(z) = Y(|\phi\rangle, z)$ acting on the state space. Equation (4.1) is the formalisation of the creation property $\phi_{(-1)}\,|0\rangle = |\phi\rangle$ and $|\omega\rangle$ has to be thought of as the state corresponding to the stress energy tensor $T(z) = Y(|\omega\rangle, z)$.

Definition 4.3. We define the *normal ordering* of two fields $A_1(z)$ and $A_2(z)$ as

$$\boldsymbol{:} A_1(z)\, A_2(z) \boldsymbol{:}\ := A_1(z)_+\, A_2(z) + A_2(z)\, A_1(z)_-$$

where for $A_i(z) = \sum_{n\in\mathbb{Z}} A_{i\,(n)} z^{-n-1}$ we set

$$A_1(z)_+ := \sum_{n<0} A_{1\,(n)} z^{-n-1} \text{ and } A_1(z)_- := \sum_{n\geq 0} A_{1\,(n)} z^{-n-1}$$

The definition is extended to multiple fields via

$$\boldsymbol{:} A_1(z) \dots A_n(z) \boldsymbol{:}\ :=\ \boldsymbol{:} A_1(z) \dots \boldsymbol{:} A_{n-1}(z)\, A_n(z) \boldsymbol{:} \dots \boldsymbol{:}\ .$$

The VOAs important for us are the so called lattice vertex algebras which we will introduce in this section. The content of this section can be found in most introductory textbooks on VOAs like [41, Sec. 3.5, 5.4], [42, Sec. 6.3, 6.4], and [43, Sec. 10]. Our exposition is close to the one found in [44]. Let Q be a lattice of finite rank together with a symmetric non-degenerate even bilinear form $\langle\cdot,\cdot\rangle$.

Denote $\mathfrak{h} := Q \otimes_{\mathbb{Z}} \mathbb{C}$ its complexification. The *Heisenberg Lie algebra* $\hat{\mathfrak{h}}$ is then defined to be

$$\hat{\mathfrak{h}} := \mathfrak{h}[t^{-1}, t] \oplus \mathbb{C}K$$

with Lie bracket

$$[\alpha_m, \beta_n] = \langle \alpha, \beta \rangle \, m \, \delta_{m+n,0} \, K$$

and central element K where we write $\alpha_m := \alpha \, t^m$ for all $\alpha \, t^m \in \mathfrak{h}[t^{-1}, t]$. Let $U(\hat{\mathfrak{h}})$ denote the universal enveloping algebra of $\hat{\mathfrak{h}}$. We write $\mathcal{F}$ for the $U(\hat{\mathfrak{h}})$-module generated by a single vector $|0\rangle$ such that for all $\alpha \in \mathfrak{h}$

$$K \, |0\rangle := |0\rangle \; , \quad \alpha_m \, |0\rangle := 0 \qquad \forall m \geq 0.$$

Then $\mathcal{F} \cong \mathrm{Sym}(t^{-1} \, \mathfrak{h}[t^{-1}]) \, |0\rangle$. We will later see that $\mathcal{F}$ (often called the *bosonic Fock space*) can be regarded as the state space of a certain VOA.
Informally speaking the state space of a lattice VOA contains a copy of $\mathcal{F}$ for each lattice point of Q. In order to define this state space, let us choose a bimultiplicative function $\epsilon : Q \times Q \to \{\pm 1\}$ satisfying

$$\epsilon(\alpha, \alpha) = (-1)^{|\alpha|^2 (|\alpha|^2 + 1)/2} \, , \qquad |\alpha| := \langle \alpha, \alpha \rangle \, .$$

If Q is the root lattice of type ADE such a map always exists. Take for example $\epsilon(\alpha, \beta) = (-1)^{\langle (1-\sigma)^{-1} \alpha, \beta \rangle}$ where σ is a Coxeter transformation. This map lets us introduce the so called twisted group algebra $\mathbb{C}_\epsilon[Q]$ spanned by $\{e^\alpha\}_{\alpha \in Q}$ with multiplication

$$e^\alpha \, e^\beta := \epsilon(\alpha, \beta) \, e^{\alpha + \beta} \, .$$

One can check that this product is associative using the bimultiplicativity of ϵ.
We will take $V_Q := \mathcal{F} \otimes \mathbb{C}_\epsilon[Q]$ as the state space of the *lattice vertex algebra* associated to Q. As vacuum vector we choose $|0\rangle \otimes e^0$ which we will also denote by $|0\rangle$. We now need to define a state-field correspondence on this space. In order to do so, we define the action of $\alpha_n \in \hat{\mathfrak{h}}$ on $v \otimes e^\beta \in V_Q$ as

$$\alpha_n \cdot (v \otimes e^\beta) := (\alpha_n \cdot v) \otimes e^\beta + \delta_{m,0} \, \langle \alpha, \beta \rangle \, v \otimes e^\alpha \, .$$

Moreover, let e^α act on the state space via

$$e^\alpha \cdot (v \otimes e^\beta) := \epsilon(\alpha, \beta) \, v \otimes e^{\alpha + \beta} \, .$$

This allows us to define the following state-field correspondence Y on V_Q

$$\phi(z) := Y(\phi_{-1} \, |0\rangle \otimes e^0, z) := \sum_{n \in \mathbb{Z}} \phi_n \, z^{-n-1} \, , \tag{4.2}$$

$$V^\alpha(z) := Y(|0\rangle \otimes e^\alpha, z) \qquad := e^\alpha z^\alpha \, \exp\Big(\sum\nolimits_{n<0} \alpha_{-n} \tfrac{z^n}{n}\Big) \, \exp\Big(\sum\nolimits_{n>0} \alpha_{-n} \tfrac{z^n}{n}\Big) \tag{4.3}$$

for $\phi \in \mathfrak{h}$ and $\alpha \in Q$ where $z^\alpha \cdot (v \otimes e^\beta) := z^{\langle \alpha, \beta \rangle} \, v \otimes e^\beta$.
Note that so far we have only defined the state-field correspondence for the vectors $\phi_{-1} \, |0\rangle \otimes e^0$ and $|0\rangle \otimes e^\alpha$. It turns out, one can extend Y in a unique way to a state-field correspondence on the whole space V_Q satisfying the properties of Definition 4.1 [41, Prop. 5.4] using the so called existence theorem of Frenkel–Kac–Radul–Wang [45, Prop. 3.1]. To be more precise, every vector in V_Q is

a linear combination of vectors of the form $\phi^1_{-n_1} \dots \phi^k_{-n_k}(|0\rangle \otimes e^\alpha)$. We then may compute Y on arbitrary vectors by linearly extending [42, Eq. 6.4.68]

$$Y\big(\phi^1_{-n_1} \dots \phi^k_{-n_k}(|0\rangle \otimes e^\alpha),\, z\big) = \,:\prod_{i=1}^{k} \left(\frac{1}{(n_i-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_i-1} \phi^i(z) \right) V^\alpha(z):\,. \tag{4.4}$$

Note that the state-field correspondence (4.2) restricted to $\mathcal{F}$ makes this space a subvertex algebra of V_Q called the *Heisenberg VOA* of $\mathfrak{h}$.
The conformal vector $|\omega\rangle$ of the Heisenberg VOA $\mathcal{F}$ is

$$|\omega\rangle := \frac{1}{2} \sum_{i=1}^{\mathrm{rank}(\mathfrak{h})} \alpha^i_{(-1)} \beta^i_{(-1)} |0\rangle\ . \tag{4.5}$$

where $\{\alpha^i\}$ and $\{\beta^i\}$ are bases of $\mathfrak{h}$ dual to each other with respect to $\langle\cdot\,,\cdot\rangle$. One can prove that the modes of this state indeed satisfy the Virasoro commutation relations with central charge $c = \mathrm{rank}(\mathfrak{h})$ [41, Sec. 3.5]. This completes the definition of the Heisenberg VOA $(\mathcal{F}, Y, |0\rangle, |\omega\rangle)$. One can lift the conformal vector of $\mathcal{F}$ to a conformal vector for V_Q. However, since this vector is of no interest for the further discussion we will skip its definition.
To sum up, we have defined the lattice VOA V_Q and the Heisenberg VOA $\mathcal{F}$ which is a sub-VOA of the first. Of course, we have not proven that the axioms from Definition 4.1 are indeed satisfied. We simply refer to the literature [41–43, 46] for a full treatment.

4.2 $\mathcal{W}(\mathfrak{g})$-algebras

4.2.1 Definition

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and let us further denote its set of roots by R and its root lattice by Q. We consider its associated lattice vertex algebra V_Q. In this setting we may identify $\mathfrak{h}$ with the Cartan subalgebra of $\mathfrak{g}$. We define the so called *screening charges*

$$V^\alpha_{(0)} = \mathrm{Res}_z\, V^\alpha(z) \qquad\qquad \alpha \in R$$

where $V^\alpha(z)$ is defined as in (4.3).

Definition 4.4. The $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ associated to $\mathfrak{g}$ is the vertex subalgebra of $\mathcal{F}$ defined as the intersection of all kernels of the screening charges $\{V^\alpha_{(0)}\}_{\alpha\in R}$ with $\mathcal{F}$, ie.

$$\mathcal{W}(\mathfrak{g}) := \left\{ v \in \mathcal{F} \,\middle|\, \forall \alpha \in R :\ V^\alpha_{(0)}(v) = 0 \right\} = \bigcap_{\alpha\in R} \ker V^\alpha_{(0)} \cap \mathcal{F}\,.$$

That the definition of $\mathcal{W}(\mathfrak{g})$ indeed defines a vertex subalgebra of $\mathcal{F}$, is for instance explained in [43, Rem. 10.39]. What makes $\mathcal{W}(\mathfrak{g})$-algebras so remarkable is that they are *freely generated.*

Definition 4.5. A VOA $(V, Y, |0\rangle, |\omega\rangle)$ is said to be *generated* by the elements $\{|v^i\rangle\}_{i\in I} \subset V$ if the vector space V is spanned by vectors

$$v^{i_1}_{(-k_1)} \dots v^{i_n}_{(-k_n)} |0\rangle$$

with $k_j > 0$ and $Y(|v^i\rangle, z) = \sum_{k\in\mathbb{Z}} v^i_{(k)} z^{-k-1}$. The VOA is called *freely generated* if the above vectors form a basis of V.

Theorem 4.6. [47, Theorem 4.6.9] *$\mathcal{W}(\mathfrak{g})$ is a VOA and contains the conformal vector* (4.5). *Moreover, there exist elements $|w^1\rangle, \ldots, |w^\ell\rangle$ of $\mathcal{W}(\mathfrak{g})$ of degree $d_1 + 1, \ldots, d_\ell + 1$ freely generating $\mathcal{W}(\mathfrak{g})$. Here, the d_is are the exponents of $\mathfrak{g}$.*

4.2.2 Examples

Example 4.7. Let us consider $\mathfrak{g} = \mathfrak{sl}_2$. The associated bosonic Fock space $\mathcal{F}$ is one-dimensional and spanned by the matrix $\chi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Its associated $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_2)$ is generated by the conformal vector

$$|\omega\rangle = \frac{1}{4}\chi_{(-1)}\chi_{(-1)}\,|0\rangle$$

Thus, $\mathcal{W}(\mathfrak{sl}_2)$ is nothing but the Virasoro vertex algebra with central charge 1 [7, Ex. 3.8].

Let us now specialize our discussion to $\mathfrak{g} = \mathfrak{gl}_r$ with $r > 1$. Formally, we have only defined $\mathcal{W}$-algebras associated to finite-dimensional simple Lie algebras so far. One obtains $\mathcal{W}(\mathfrak{gl}_r)$ from $\mathcal{W}(\mathfrak{sl}_r)$ by tensoring with the Heisenberg VOA of rank one $\mathcal{F}_0$, ie. $\mathcal{W}(\mathfrak{gl}_r) = \mathcal{W}(\mathfrak{sl}_r) \otimes \mathcal{F}_0$. In this case we can give an explicit expression for the generators introduced in Theorem 4.6 (cf. [7, Sec. 3.2.4]).

Example 4.8. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_r$ is spanned by the matrices $\chi^1, \ldots, \chi^r$ where χ^i is the matrix with vanishing entries except for a single 1 at position (i, i). $\mathcal{W}(\mathfrak{gl}_r)$ has central charge r and is freely generated by the vectors

$$|w^i\rangle = e_i\big(\chi^1_{(-1)}, \ldots, \chi^r_{(-1)}\big)\,|0\rangle\ , \quad i \in \{1, \ldots, r\}$$

where e_i is the i-th elementary symmetric polynomial in r variables [48, 49].

4.3 Constructing Airy structures from $\mathcal{W}(\mathfrak{gl}_r)$ — the conceptual idea

Before we proceed, let us motivate the remaining steps of this chapter. Remember, our final aim is to construct Airy structures from $\mathcal{W}(\mathfrak{gl}_r)$ introduced in Section 4.2. Let us explain what we mean by this taking up on Example 4.7. In the example we have learned that $\mathcal{W}(\mathfrak{sl}_2)$ is generated by a single vector $|\omega\rangle$ whose modes satisfy the Virasoro commutation relations. A priori these modes are abstract algebraic objects. Thus, in order to construct an Airy structure from $\mathcal{W}(\mathfrak{sl}_2)$ one first has to represent the modes in terms of differential operators. Then picking a suitable subset of modes which satisfies the subalgebra condition of an Airy structure, one might end up with an Airy structure. The Airy structure associated to pure gravity studied in Example 3.3 is an example for such a construction.
The ideas sketched above hold for $\mathcal{W}(\mathfrak{gl}_r)$ as well. Since it is easy to lose track of these conceptual ideas, let us give an outline of the further steps taken in this chapter:

(1.) First, in Section 4.4 we will give a classification of the subsets of modes of the generators $|w^i\rangle$ of $\mathcal{W}(\mathfrak{gl}_r)$ satisfying the subalgebra condition (3.3) of an Airy structure.

(2.) Second, we need to represent the modes of the generators of $\mathcal{W}(\mathfrak{gl}_r)$ in terms of differential operators. This is done in Section 4.5 by constructing so called twisted modules of the Heisenberg VOA on which $\mathcal{W}(\mathfrak{gl}_r)$ acts via differential operators. The input data for this construction is the choice of a twist automorphism σ (a permutation of the basis elements of $\mathfrak{h}$) and the outcome is a set of differential operators

$$W^{\sigma}_{i,k} \qquad\qquad i \in \{1, \ldots, r\},\ k \in \mathbb{Z}$$

which correspond to the modes of the generators $|w^i\rangle$ of $\mathcal{W}(\mathfrak{gl}_r)$ acting on the module.

(3.) Unfortunately, the set of operators $\{W^{\sigma}_{i,k}\}$ will not satisfy the degree one condition of an Airy structure since in general the operators do not feature a leading $\hbar\partial_*$ at degree one. Such a degree one term is produced by performing a so called *dilaton shift*.

$$W^{\sigma}_{i,k} \quad \xrightarrow{\text{dilaton shift}} \quad H^{\sigma}_{i,k}$$

This step will be done in Section 4.6 and Section 4.7 for two distinct choices of twist automorphisms σ respectively.

(4.) One then finally applies the analysis of the subalgebra condition done in step (1.) to the set of modes $\{H^{\sigma}_{i,k}\}$ and checks which choice of input data, ie. the choice of twist automorphism σ and dilaton shift, produces an Airy structure. This classifications will be presented in Theorem 4.16, 4.19, and 4.20.

4.4 The mode algebra and its subalgebras

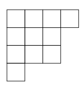
Let $\mathcal{A}$ be the associative algebra of modes of $\mathcal{W}(\mathfrak{gl}_r)$, ie. $\mathcal{A}$ is generated by the modes $\{w^i_{(k)}\}$ associated to the generators $\{|w^i\rangle\}$. Further, let $L(\mathcal{A})$ denote the set of possibly infinite sums of ordered monomials in $\mathcal{A}$ of bounded degree and conformal weight. It turns out, there exists an elegant way of constructing subsets

$$S \subset \{w^i_{(k)}\}_{i\in\{1\ldots r\},k\in\mathbb{Z}}$$

of the modes of the generators $|w^i\rangle$ of $\mathcal{W}(\mathfrak{gl}_r)$ satisfying that their left ideal $\mathcal{A}.S$ is a graded Lie subalgebra of $L(\mathcal{A})$ following [7, Sec. 3.3].

First we need to introduce some more notation. We say $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a *(descending) partition* of r if $\lambda_1 + \cdots + \lambda_p = r$ and λ is ordered such that $\lambda_j \geq \lambda_{j+1}$. In this case we write $\lambda \vdash r$. A partition λ of r can be characterised in two further equivalent ways. Often it is convenient to express equal blocks $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+n}$ simply as λ_j^n. Moreover, one can associate a unique Young diagram F^λ to each partition. For instance all notations

$$\lambda = (4, 3, 3, 1) \quad \longleftrightarrow \quad \lambda = (4, 3^2, 1) \quad \longleftrightarrow \quad F^\lambda =$$



characterise the same partition $\lambda \vdash 11$.

Given $\lambda \vdash r$, we set

$$\lambda(i) := \max\left\{ m \,\middle|\, \sum_{j=1}^{m} \lambda_j \geq i \right\}$$

and define the index set

$$I_\lambda := \left\{ (i,k) \in \{1,\dots,r\} \times \mathbb{Z} \,\middle|\, k \geq i - \lambda(i) \right\}. \tag{4.6}$$

We associate the subset $S_\lambda := \{w^i_{(k)}\}_{(i,k)\in I_\lambda}$ of modes of $|w^1\rangle, \dots, |w^r\rangle$ to such an index set.

Lemma 4.9. [7, Thm. 3.16] *For every descending partition $\lambda \vdash r$ the left ideal $\mathcal{A}.S_\lambda$ is a graded Lie subalgebra of $L(\mathcal{A})$. This means, there exist $f^{(k,t)}_{(i,r),(j,s)} \in L(\mathcal{A})$ such that*

$$\left[w^i_{(k)}, w^{i'}_{(k')} \right] = \sum_{(j,l)\in I_\lambda} f^{(j,l)}_{(i,k),(i',k')}\, w^j_{(l)} \tag{4.7}$$

for all $(i,k), (i',k') \in I_\lambda$.

In the next section we will proceed by representing the modes $w^i_{(k)}$ of the generators $|w^i\rangle$ in terms of differential operators. Then using Lemma 4.9, subsets of these differential operators automatically satisfy the subalgebra condition of an Airy structure given they correspond to a mode set S_λ induced by a partition $\lambda \vdash r$.

4.5 Representation in terms of differential operators via twisted modules

The algebra $\mathcal{W}(\mathfrak{gl}_r)$ is a subvertex algebra of the Heisenberg VOA $\mathcal{F}$ associated to the Cartan subalgebra $\mathfrak{h}$. We will explain the construction of certain twisted modules over $\mathcal{F}$ which we later then restrict to $\mathcal{W}(\mathfrak{gl}_r)$. We will see that $\mathcal{W}(\mathfrak{gl}_r)$ acts on these modules via differential operators.

4.5.1 The Coxeter case

Remember that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_r$ is spanned by $\chi^1, \dots, \chi^r$ where χ^i is the matrix with a single 1 at position (i,i). Then $\langle \chi^i, \chi^j \rangle = \mathrm{tr}(\chi^i \chi^j) = \delta_{i,j}$. The Coxeter element σ acts on this basis by permuting

$$\sigma : \quad \chi^1 \longrightarrow \chi^2 \longrightarrow \cdots \longrightarrow \chi^r \longrightarrow \chi^1 .$$

The order of σ is r and its eigenvalues are powers of the rth root of unity $\theta_r := e^{2\pi \mathrm{i}/r}$. It is straight forward to check that

$$v^a := \sum_{i=1}^{r} \theta_r^{-ai} \chi^i$$

are the eigenvectors of σ with eigenvalue θ_r^a. We represent these vectors as differential operators via

$$Y^\sigma(v^a_{(-1)}\,|0\rangle, z) := \sum_{k\in a/r+\mathbb{Z}} J_{rk} z^{-k-1} \tag{4.8}$$

where J_k is the differential operator already introduced in (3.6)

$$J_k := \hbar\partial_k\,, \quad J_{-k} := k\, x_k$$

acting on the space $\mathcal{T} := \mathbb{C}_{\hbar^{1/2}}[x_1, x_2, \dots]$. Let us extend this definition and set

$$J_0 := \hbar^{1/2}\, Q$$

where $Q \in \mathbb{C}$. Note that in the mode expansion (4.8) occur fractional powers of z. Thus, $(\mathcal{T}, Y^\sigma)$ is a so called twisted module of the Heisenberg VOA $\mathcal{F}$.

Definition 4.10. Let $(V, Y, |0\rangle, |\omega\rangle)$ be a VOA. We call (M, Y^M) a *σ-twisted module* of V if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded vector space with $M_n = 0$ for $n \ll 0$ and $\dim M_n < \infty$ for all n and Y^M is a linear map

$$\begin{aligned} Y^\sigma(\cdot, z):\ V &\longrightarrow \mathrm{End}(M)[\![z^{1/r}, z^{-1/r}]\!]\,, \\ |v\rangle &\longmapsto Y^\sigma(|v\rangle, z) = \textstyle\sum_{n\in\frac{1}{r}\mathbb{Z}} v_{(n)} z^{-n-1} \end{aligned}$$

such that

- If $|v\rangle \in V$ is an eigenvector $\sigma(|v\rangle) := \theta_r^a\ |v\rangle$ then
$$v_{(n)} = 0 \text{ unless } n \in \tfrac{a}{r} + \mathbb{Z}. \tag{4.9}$$
- The fields are local, ie. for all $|u\rangle, |v\rangle \in V$ there exists an $N_{u,v}$ such that
$$(z_1 - z_2)^{N_{u,v}}\ [Y^\sigma(|u\rangle, z_1), Y^\sigma(|v\rangle, z_2)] = 0\,.$$
- The fields obey the product formula
$$\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z_1^k}\Big\{(z_1 - z_2)^{N_{u,v}}\, Y^\sigma(|u\rangle, z_1)\, Y^\sigma(|v\rangle, z_2)\, |m\rangle\Big\}\Big|_{z_1=z_2=z} = Y^\sigma\big(u_{N-1-k}\,|v\rangle, z\big)\,|m\rangle \tag{4.10}$$
where $|u\rangle, |v\rangle \in V$ and $|m\rangle \in M$.

In our case the action of the Coxeter σ element on $\mathcal{F}$ is induced by its action on $\mathfrak{h}$. Notice that we have only defined Y^σ on the vectors $v^a_{(-1)}\,|0\rangle$. However, this is sufficient since these vectors are generators of $\mathcal{F}$ implying that the fields associated to vectors $v^1_{(k_1)} \dots v^l_{(k_l)}\,|0\rangle$ may be computed using a formula similar to (4.4) plus the product formula (4.10). For details see [50] and [7, Sec. 4.1.2]. Notice that due to property (4.9) the fields associated to vectors which are eigenvectors of σ with eigenvalue 1 admit an expansion in integer powers of z. Now notice that the generators

$$|w^i\rangle = e_i\big(\chi^1_{(-1)}, \dots, \chi^r_{(-1)}\big)\,|0\rangle\,, \quad i \in \{1, \dots, r\}$$

of $\mathcal{W}(\mathfrak{gl}_r) \subset \mathcal{F}$ are invariant under σ since the polynomials e_i are invariant under permutation of their arguments. This means that the fields $W_i(z) = Y^\sigma(|w_i\rangle, z)$ admit an expansion in terms of integer powers of z, implying that if one restricts the module $(\mathcal{T}, Y^\sigma)$ to $\mathcal{W}(\mathfrak{gl}_r)$ it becomes untwisted[1].

Definition 4.11. Nevertheless, we will call the modes

$$W^\sigma_{i,k} := r^{i-1}\, \mathrm{Res}_z\, z^k\, Y^\sigma(|w^i\rangle, z), \qquad i \in \{1, \dots, r\},\ k \in \mathbb{Z} \tag{4.11}$$

the *twist modes*.

One computes these modes by using a version of the reconstruction formula (4.4) for twisted modules together with the product formula (4.10) and eventually arrives at the following proposition.

Proposition 4.12. [7, Cor. 4.7] *The twist modes satisfy*

$$W^\sigma_{i,k} = \frac{1}{r} \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{i!\,\hbar^j}{2^j j!(i-2j)!} \sum_{\substack{p_{2j+1},\dots,p_i \in \mathbb{Z} \\ \sum_l p_l = r(k-i+1)}} \Psi^{(j)}_r(p_{2j+1}, p_{2j+2}, \dots, p_i) : \prod_{l=2j+1}^{i} J_{p_l} : \tag{4.12}$$

where $: \dots :$ *is the normal ordered product of differential operators and* $\Psi^{(j)}_r(p_{2j+1} \dots, p_i) \in \mathbb{Q}$ *are linear combinations of certain* r*th roots of unity introduced in* (A.3).

Example 4.13. For low values of i but arbitrary r these modes can explicitly be computed as [7, p. 39]

$$W^\sigma_{1,k} = J_{kr}, \tag{4.13}$$

$$W^\sigma_{2,k} = \frac{1}{2} \sum_{\substack{p_1,p_2 \in \mathbb{Z} \\ p_1+p_2=r(k-1)}} \left(r\delta_{r|p_1}\delta_{r|p_2} - 1\right) : J_{p_1} J_{p_2} : - \frac{(r^2-1)\hbar}{24}\, \delta_{k,1}, \tag{4.14}$$

$$\begin{aligned} W^\sigma_{3,k} = \frac{1}{6} \sum_{\substack{p_1,p_2,p_3 \in \mathbb{Z} \\ p_1+p_2+p_3=r(k-2)}} & \left(r^2\delta_{r|p_1}\delta_{r|p_2}\delta_{r|p_3} - r\delta_{r|p_1} - r\delta_{r|p_2} - r\delta_{r|p_3} + 2\right) : J_{p_1} J_{p_2} J_{p_3} : \\ & - \frac{(r-2)(r^2-1)\hbar}{24} J_{r(k-2)}, \end{aligned} \tag{4.15}$$

where $\delta_{r|a}$ is equal to one if r divides a and is zero otherwise. The main difficulty of computing the twist modes $W^\sigma_{i,k}$ is the computation of the values of $\Psi^{(j)}_r$.

What one should take away from the above expressions for the modes $W^\sigma_{i,k}$ is twofold: First, we have succeeded and represented the mode algebra $\mathcal{A}$ in terms of differential operators. Second, in the form above the operators $W^\sigma_{i,k}$ cannot form an Airy structure. Remember that the operators in an Airy structure necessarily start with a partial $\hbar\partial_*$, ie. with a single J_*, and contain monomials of degree larger equal two. The twist modes $W^\sigma_{i,k}$ are unfortunately of homogenous degree i. In Section 4.6 we will explain how to solve this problem, but first we will consider the case where σ is an arbitrary automorphism.

[1] Here, by *untwisted* we mean that $(\mathcal{T}, Y^\sigma)$ restricted to $\mathcal{W}(\mathfrak{gl}_r)$ is an $\mathrm{id}_{\mathcal{W}(\mathfrak{gl}_r)}$-twisted module, ie. an ordinary module of $\mathcal{W}(\mathfrak{gl}_r)$.

4.5.2 Twisting with an arbitrary automorphism

Let $\sigma \in \mathfrak{S}_r$ be an arbitrary element of the Weyl group of $\mathfrak{gl}_r$. It is a permutation of the elements χ^i and can thus be decomposed into $d \geq 1$ cycles

$$\sigma = \sigma_1 \dots \sigma_d$$

with each cycle σ_μ of length $r_\mu \geq 1$ such that $r_1 + \cdots + r_d = r$. Note that the case $d = 1$ refers to the already considered case in which σ is the Coxeter element.
After relabelling the χ^i we can assume that σ_μ acts as

$$\sigma_\mu : \quad \chi^{1+\boldsymbol{r}_{[\mu-1]}} \longrightarrow \chi^{2+\boldsymbol{r}_{[\mu-1]}} \longrightarrow \dots \longrightarrow \chi^{r_\mu+\boldsymbol{r}_{[\mu-1]}} \longrightarrow \chi^{1+\boldsymbol{r}_{[\mu-1]}}$$

leaving all other χ^i invariant where we introduced the notation $\boldsymbol{r}_{[\mu]} := \sum_{\nu=1}^{\mu} r_\nu$. It is easy to check that

$$v^{\mu,a} := \sum_{j=1}^{r_\mu} \theta_{r_\mu}^{-aj} \chi^{j+\boldsymbol{r}_{[\mu-1]}}\,, \qquad \mu \in \{1, \dots, d\},\ a \in 1, \dots, r_\mu$$

is an eigenvector of σ with eigenvalue $\theta_{r_\mu}^a$. Analogously to the Coxeter case we represent these eigenvectors as differential operators

$$Y^\sigma(v^{a,\mu}_{(-1)}\,|0\rangle\,, z) := \sum_{k \in a/r_\mu+\mathbb{Z}} J^\mu_{r_\mu k}\, z^{-k-1}$$

introducing the differential operators

$$J^\mu_k = \hbar\partial_{x^\mu_k}\,, \qquad J^\mu_0 = \hbar^{1/2} Q^\mu\,, \qquad J^\mu_{-k} = k\, x^\mu_k \tag{4.16}$$

with $k > 0$ and $Q^\mu \in \mathbb{C}$ acting on the space $\mathcal{T} := \mathbb{C}_{\hbar^{1/2}}[\{x^\mu_k\}_{\mu=1\dots d,\,k>0}]$. This makes $(\mathcal{T}, Y^\sigma)$ a twisted representation of $\mathcal{F}$ which restricted to $\mathcal{W}(\mathfrak{gl}_r)$ becomes untwisted due to the same arguments presented in Section 4.5.1. Again we define the *twist modes*

$$W^\sigma_{i,k} := \mathrm{Res}_z\, z^k\, Y^\sigma(|w^i\rangle\,, z)\,, \qquad i \in \{1, \dots, r\},\ k \in \mathbb{Z}\,.$$

It turns out, their computation reduces to the Coxeter case.

Proposition 4.14. [7, Lem 4.15] *For an automorphism σ with d cycles of respective lengths $r_1, \dots, r_d$ the twisted modes read*

$$W^\sigma_{i,k} = \sum_{M \subseteq \{1,\dots,d\}} \sum_{\substack{1 \leq i_\mu \leq r_\mu\ \mu \in M \\ \sum_\mu i_\mu = i}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^M \\ \sum_\mu k_\mu = k+1-|M|}} \prod_{\mu \in M} \frac{1}{r_\mu^{i_\mu-1}}\, W^\mu_{i_\mu,k_\mu} \tag{4.17}$$

where $W^\mu_{i_\mu,k_\mu} := W^{\sigma_\mu}_{i_\mu,k_\mu}$ are the modes (4.12) *with σ_μ interpreted as a Coxeter element of order r_μ and J_p replaced by J^μ_p.*

Example 4.15. In order to get familiar with equation (4.17), let us consider an explicit example. Assume $d = 2$, $r_1 = r - 1$, and $r_2 = 1$. Then for $i \leq 3$ the twist modes $W^1_{i,k}$ are given by (4.13) to

(4.15) if we replace r with r_1 and J_* with J^1_*. Further,

$$W^2_{i,k} = \delta_{i,1}\, J^2_k$$

which simplifies the combinatorics of formula (4.17) massively. It is straight forward to check that

$$\begin{aligned}
W^\sigma_{1,k} &= J^1_{(r-1)k} + J^2_k\,, \\
W^\sigma_{i,k} &= W^1_{i,k} + \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1+k_2=k-1}} (r-1)^{2-i}\, W^1_{i-1,k_1}\, J^2_{k_2} \qquad i\in\{2,\ldots,r-1\}\,, \\
W^\sigma_{r,k} &= \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1+k_2=k-1}} (r-1)^{2-r}\, W^1_{r-1,k_1}\, J^2_{k_2}\,.
\end{aligned} \tag{4.18}$$

Clearly, for the above operators we run into the problem encountered before that all $W^\sigma_{i,k}$ for $i>1$ have a vanishing first degree.

4.6 $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures from twisting with a Coxeter element

We explained that the modes of the generators $|w^i\rangle$ of $\mathcal{W}(\mathfrak{gl}_r)$ admit a representation in terms of differential operators $W^\sigma_{i,k}$. However, these operators are not of the desired form (3.2) in order to form an Airy structure since they are of homogenous degree

$$\deg(W^\sigma_{i,k}) = i\,.$$

This especially means that in general they do not feature a single partial as a leading term. One can produce such a leading partial by performing a so called *dilaton shift*. Again, we will first discuss the case of twisted representations coming from a twist with a Coxeter element and will later cover the case of twists with arbitrary automorphisms.
Let σ be a Coxeter transformation. The problem that all twist modes $W^\sigma_{i,k}$ are of homogenous degree is overcome by shifting the operator

$$J_{-s} \longrightarrow J_{-s} - t \tag{4.19}$$

by a $t\in\mathbb{C}^*$ while keeping all other J_a for $a\neq -s$ unchanged. This procedure is called the *dilaton shift*. Formally one defines

$$H^\sigma_{i,k} := \hat{T}\, W^\sigma_{i,k}\, \hat{T}^{-1}\,, \qquad \hat{T} := \exp\left(-\tfrac{J^\mu_s}{\hbar s}\, t\right)\,.$$

It follows from the Baker–Campbell–Hausdorff formula that conjugating with $\hat{T}$ means nothing but shifting the Js as in (4.19).
Performing the dilaton shift, breaks up the homogenity of $W^\sigma_{i,k}$ introducing terms of degree lower or equal to i in $H^\sigma_{i,k}$. It is thus left to check whether the obtained modes $H^\sigma_{i,k}$ indeed satisfy the degree one condition of Airy structures (3.2), which means they feature a single partial, ie. a single J_a with $a>0$, at degree one and have a vanishing degree zero component. Moreover, one has to restrict to a subset of modes $\{H^\sigma_{i,k}\}_{(i,k)\in I}$ such that all J_a with $a>0$ occur exactly once.
Having found such an index set I, one needs to check, whether the selected modes $\{H^\sigma_{i,k}\}_{(i,k)\in I}$ satisfy

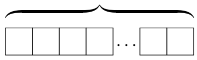
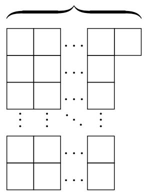
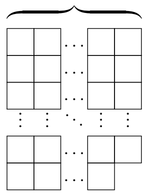
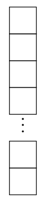
r, s	$s = 1$	$r = r's + 1$	$r = r's + s - 1$	$s = r + 1$
$\lambda_{r,s}$	r boxes 	$(r'+1)$ columns 	$(r'+1)$ columns 	

Table 4.1: The partitions $\lambda_{r,s}$ associated to different values s.

the subalgebra condition (3.3). Notice that since conjugation preserves the Lie bracket, the left ideals spanned by the $H^{\sigma}_{i,k}$ generate such a subalgebra if and only if the $W^{\sigma}_{i,k}$ do. This means, it suffices to prove that the index set I is induced by a descending partition of r using Lemma 4.9. Finally, one arrives at the following statement.

Theorem 4.16. [7, Thm. 4.9] *Let σ be a Coxeter transformation of order $r \geq 2$ and $s \in \{1, \dots, r+1\}$ with $r = \pm 1 \mod s$. Further assume that $Q = 0$ and $t \neq 0$. Then*

$$H^{\sigma}_{i,k} \qquad i \in \{1, \dots, r\},\ k \geq i - \lambda(i) + \delta_{i,1}$$

is higher quantum r-Airy structure on $\mathbb{C}[x_1, x_2, \dots]$ where $\lambda \vdash r$ is the partition

$$\lambda = \begin{cases} (r) & , s = 1 \\ (1^r) & , s = r+1 \\ \left((r'+1)^{r''}, (r')^{s-r''}\right) & , r = r's + r'',\ r'' \in \{1, s-1\}\,. \end{cases} \tag{4.20}$$

We call this Airy structure the $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure of type (r, s).

Before we continue, let us remark on the generality of Theorem 4.16. The partition λ, which was chosen in (4.20) to define the mode set of the Airy structure, characterises the subalgebra associated to this Airy structure via Lemma 4.9. The corresponding Young diagrams are depicted in Table 4.1. Conversely, one could say that the $\mathcal{W}(\mathfrak{gl}_r)$ mode subalgebras corresponding to these diagrams admit a representation in terms of differential operators forming an Airy structure. Now for large r the diagrams from Table 4.1 are far from being all diagrams of fixed weight r. Thus, one could ask whether the subalgebras associated to the diagrams in Table 4.1 are the only ones to admit a representation as an Airy structure. We will see the answer to this question is negative. Indeed, we will see that the opposite is true: every subalgebra characterised by a descending partition can be represented as an Airy structure. For this we need to consider σ-twisted representations where σ is allowed to be a permutation with arbitrary many cycles.
What makes Theorem 4.16 so remarkable is, that due to Theorem 3.4 each of these Airy structures having r, s (and the shift value t) as input parameters give rise to a unique partition function $Z^{(r,s)}$ satisfying

$$H^{\sigma}_{i,k}\, Z^{(r,s)} = 0 \qquad \forall i \in \{1, \dots, r\},\ k \geq i - \lambda(i) + \delta_{i,1}$$

whose correlators $F_{g,n}^{(r,s)}$ can be computed using the topological recursion from Theorem 3.5. In special cases these correlators admit an interpretation in terms of intersection numbers on certain moduli spaces or in terms of matrix models as we will see in Chapter 5. However, our running example pure two-dimensional topological gravity can already be classified in this list of Airy structures. This shall be done in the following Sections 4.6.1 and 4.6.2 reducing a certain $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure to $\mathcal{W}(\mathfrak{sl}_r)$.

4.6.1 The reduction to $\mathfrak{sl}_r$

Notice that the dilaton shifted operator $H^{\sigma}_{i,k}$ for $i = 1$ is of the form

$$H^{\sigma}_{1,k} = W^{\sigma}_{1,k} = J_{kr} \qquad\qquad k > 0$$

comparing with (4.13). Thus, the associated partition function $Z^{(r,s)}$ is independent of the variables $\{x_{rk}\}_{k\in\mathbb{N}}$ which means it makes sense to consider the reduced operators

$$H^{\sigma}_{i,k}\big|_{\mathrm{red}} := H^{\sigma}_{i,k}\big|_{J_{ra}=0,\, a\in\mathbb{Z}}$$

instead of $H^{\sigma}_{i,k}$. Indeed, $\{H^{\sigma}_{i,k}|_{\mathrm{red}}\}$ is again a higher Airy structure [7, Prop. 6.3]. One can consider it as an $\mathcal{W}(\mathfrak{sl}_r)$-Airy structure in the sense that $\mathfrak{sl}_r$ is generated by operators $\chi^1, \dots, \chi^r$ satisfying

$$0 = \chi^1 + \cdots + \chi^r = e_1(\chi^1, \dots, \chi^r)$$

implying the vanishing of $W^{\sigma}_{1,k}$ comparing with (4.11). We will thus refer to $\{H^{\sigma}_{i,k}|_{\mathrm{red}}\}$ as the $\mathcal{W}(\mathfrak{sl}_r)$-Airy structure of type (r, s).

Example 4.17. Let us consider the case $s = r + 1$ and calculate $H^{\sigma}_{i,k}|_{\mathrm{red}}$ for $i \le 3$ from the twist modes given in (4.13) to (4.15). For convenience let us first reduce the twist fields before performing the dilaton shift

$$\begin{aligned}
W^{\sigma}_{2,k}\big|_{J_{ra}=0} &= -\frac{1}{2} \sum_{\substack{p_1,p_2\in\mathbb{Z}\setminus r\mathbb{Z}\\ p_1+p_2=r(k-1)}} :J_{p_1}J_{p_2}: - \frac{(r^2-1)\hbar}{24}\,\delta_{k,1}\,,\\
W^{\sigma}_{3,k}\big|_{J_{ra}=0} &= \frac{1}{3} \sum_{\substack{p_1,p_2,p_3\in\mathbb{Z}\setminus r\mathbb{Z}\\ p_1+p_2+p_3=r(k-2)}} :J_{p_1}J_{p_2}J_{p_3}: \,.
\end{aligned}$$

In the second step we now perform the dilaton shift $J_p \to J_p - \delta_{p,-r-1}$ and obtain

$$H^{\sigma}_{2,k}\big|_{\mathrm{red}} = J_{rk+1} - \frac{1}{2} \sum_{\substack{p_1,p_2\in\mathbb{Z}\setminus r\mathbb{Z}\\ p_1+p_2=r(k-1)}} :J_{p_1}J_{p_2}: - \frac{(r^2-1)\hbar}{24}\,\delta_{k,1}\,, \tag{4.21}$$

$$H^{\sigma}_{3,k}\big|_{\mathrm{red}} = J_{rk+2} + \frac{1}{3} \sum_{\substack{p_1,p_2,p_3\in\mathbb{Z}\setminus r\mathbb{Z}\\ p_1+p_2+p_3=r(k-2)}} :J_{p_1}J_{p_2}J_{p_3}: \,. \tag{4.22}$$

In Chapter 5 we will meet the above operators again as (a part of) the differential constraints for r-spin intersection numbers. The case $r = 2$ is discussed in the following section.

4.6.2 Virasoro constraints of pure gravity and the $\mathcal{W}(\mathfrak{sl}_2)$-Airy structure

Let us take up on Example 4.17 and specify the analysis to the case $r = 2$, $s = 3$. In this case the only non-vanishing modes $H^{\sigma}_{i,k}|_{\mathrm{red}}$ come from $i = 2$. Setting $r = 2$ in (4.21) we obtain

$$H^{\sigma}_{2,k}\Big|_{\mathrm{red}} = J_{2k+1} - \frac{1}{2} \sum_{\substack{p_1,p_2 \in \mathbb{Z}\setminus 2\mathbb{Z} \\ p_1+p_2=2(k-1)}} :J_{p_1} J_{p_2}: - \frac{\hbar}{8}\delta_{k,1} \,.$$

Summing over $p_i = 2q_i + 1$ one can rewrite the above sum into

$$H^{\sigma}_{2,k}\Big|_{\mathrm{red}} = J_{2k+1} - \frac{1}{2} \sum_{\substack{q_1,q_2 \geq 0 \\ q_1+q_2=k-2}} J_{2q_1+1} J_{2q_2+1} - \sum_{q \geq \min\{0,1-k\}} J_{-(2q+1)} J_{2(q+k-1)+1)} - \frac{\delta_{k,0}}{2} J_{-1}J_{-1} - \frac{\hbar}{8}\delta_{k,1} \,.$$

Let us now perform a linear change of coordinates $\{x_q\}_{q\in 2\mathbb{Z}+1}$ to $\{t_q\}_{q\in\mathbb{N}_0}$ identifying

$$t_q = (2q+1)!!\, x_{2q+1} \,.$$

Then

$$\begin{aligned} H^{\sigma}_{2,k}\Big|_{\mathrm{red}} = (2k+1)!!\,\hbar\partial_{t_k} &- \frac{1}{2} \sum_{\substack{q_1,q_2 \geq 0 \\ q_1+q_2=k-2}} (2q_1+1)!!\,(2q_2+1)!!\,\hbar^2 \partial_{t_{q_1}} \partial_{t_{q_2}} \\ &- \sum_{q \geq \min\{0,1-k\}} \frac{(2(q+k-1)+1)!!}{(2q-1)!!} t_q\, \hbar\partial_{t_{q+k-1}} - \frac{\delta_{k,0}}{2} t_0^2 - \frac{\hbar}{8}\delta_{k,1} \,. \end{aligned}$$

Let us compare this to the differential operators $\{L_n\}_{n\geq -1}$ annihilating the partition function Z^{KW} of pure gravity as presented in Example 3.3. Remarkably, we find that indeed

$$H^{\sigma}_{2,k}\Big|_{\mathrm{red}} = -2^k\, L_{k-1} \qquad \text{for } k \geq 0 \text{ and } t_q = (2q+1)!!\, x_{2q+1} \,.$$

We have thus proven the following statement.

Proposition 4.18. *For $Z^{(2,3)}$ the partition function of the $\mathcal{W}(\mathfrak{sl}_2)$-Airy structure of type $(2,3)$ obtained by the dilaton shift $J_{-3} \to J_{-3} - 1$ we have*

$$Z^{(2,3)}(x_1, x_3, x_5, \ldots) = Z^{KW}(t_0, t_1, t_2, \ldots) \qquad \textit{with } t_q = (2q+1)!!\, x_{2q+1} \,.$$

On the level of correlators this identity implies

$$F^{(2,3)}_{g,n}\big[2k_1+1, \ldots, 2k_n+1\big] = \big\langle \tau_{k_1} \ldots \tau_{k_n} \big\rangle_g \prod_{i=1}^{n} (2k_i+1)!! \,.$$

This proposition marks the first classification of a model of two-dimensional topological gravity in the framework of $\mathcal{W}(\mathfrak{sl}_r)$-Airy structures. We will proceed with this classification in Chapter 5 but first we will outline the construction of more general $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures.

4.7 A generalisation to twists with arbitrary automorphisms

Now let $\sigma \in \mathfrak{S}_r$ be an arbitrary element of the Weyl group, ie. a permutation with d cycles of length $r_1, \ldots, r_d$ such that $r_1 + \cdots + r_d = r$. Then the differential operators $W^{\sigma}_{i,k}$ act on the space

$$\mathbb{C}[\{x^{\mu}_a\}_{\mu\in\{1\ldots d\},a>0}] \,.$$

Again, we will break up the homogeneity of $W^{\sigma}_{i,k}$ by performing a dilaton shift. There are d independent sets of variables $\{x^{\mu}_a\}_{a>0}$ labelled by $\mu \in \{1,\ldots,d\}$ in which we can perform the shift. Two distinct choices of shifts will provide us with Airy structures:

- one shifts in all d sets of variables.
- one shifts in all but one set of variables and the unshifted variables $\{x^{\mu}_a\}_{a>0}$ correspond to a fixed point $r_\mu = 1$.

4.7.1 A dilaton shift in all variables

In this section we present the first construction. The second case will be covered in Section 4.7.2. Let us choose $s_\mu > 0$ and $t_\mu \in \mathbb{C}^*$ for each $\mu \in \{1\ldots d\}$ and define

$$H^{\sigma}_{i,k} := \hat{T}\, W^{i}_{k}\, \hat{T}^{-1}\,, \qquad \hat{T} := \prod_{\mu=1}^{d} \exp\left(-\frac{J^{\mu}_{s_\mu}}{\hbar s_\mu}\, t_\mu\right) . \tag{4.23}$$

Remember that conjugating with $\hat{T}$ means nothing but shifting

$$J^{\mu}_{a} \longrightarrow J^{\mu}_{a} - t_\mu\, \delta_{a,-s_\mu} \qquad \forall \mu \in \{1,\ldots,d\},\ a>0\,.$$

It turns out, certain subsets of these operators $H^{\sigma}_{i,k}$ indeed form an Airy structure.

Theorem 4.19. *Let σ be a permutation with $d \geq 2$ cycles of length $r_1,\ldots,r_d$. Assume there exist μ_1, μ_2 such that $s_\mu = 1$ for all $\mu \notin \{\mu_1,\mu_2\}$ and*

$$\frac{r_{\mu_1}}{s_{\mu_1}} \geq r_\mu \geq \frac{r_{\mu_2}}{s_{\mu_2}}$$

for all $\mu \notin \{\mu_1,\mu_2\}$. Moreover, assume that s_{μ_1} satisfies

$$s_{\mu_1} \in \{1,\ldots,r_{\mu_1}+1\}, \quad r_{\mu_1} = -1 \mod s_{\mu_1}$$

and in case $r_{\mu_2} > 1$ we assume that s_{μ_2} satisfies

$$s_{\mu_2} \in \{1,\ldots,r_{\mu_2}\}, \quad r_{\mu_2} = +1 \mod s_{\mu_2}\,.$$

Then up to change of basis we obtain a family of crosscapped higher quantum r-Airy structures

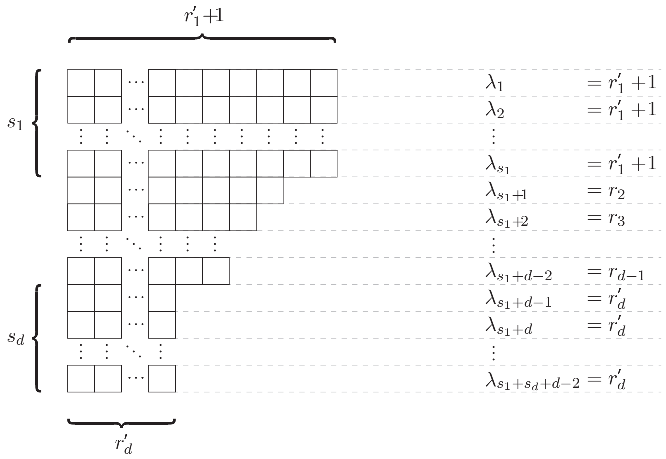
$$H^{\sigma}_{i,k} \qquad i \in \{1,\ldots,r\},\ k \geq i - \lambda(i) + \delta_{i,1}$$

where $\lambda \vdash r$ is the partition

$$\lambda = \begin{cases} \left((r_1'+1)^{s_1}, r_2, r_3, \ldots, r_{d-1}, {r_d'}^{s_d}\right) & , r_d \neq 1 \\ \left((r_1'+1)^{s_1}, r_2, r_3, \ldots, r_{d-1}\right) & , r_d = 1 \end{cases}$$

of r where without loss of generality we assumed $\mu_1 = 1$, $\mu_2 = d$ and $\frac{r_1}{s_1} \geq r_2 \geq \cdots \geq r_{d-1} \geq \frac{r_d}{s_d}$ and set $r_\mu' := \lfloor r_\mu / s_\mu \rfloor$. For the corresponding Young diagram see (4.24).
This family of Airy structures is parametrised by the shift values $t_1, \ldots, t_d \in \mathbb{C}^*$ and the charges $Q^1, \ldots, Q^d \in \mathbb{C}$ introduced in (4.16). *The first parameters are constrained by the requirement that if $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ for $\mu \neq \nu$ we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$. The scalars Q^μ need to add up to zero $Q^1 + \cdots + Q^d = 0$.*

The proof of the Theorem will be presented in Appendix A.1. It emerged under the supervision of Gaëtan Borot.
Before we proceed with a second construction leading to Airy structures let us briefly come back to the question raised in the last section: Do all subalgebras associated to a partition $\lambda \vdash r$ as constructed in Lemma 4.9 admit a representation as an Airy structure? The partition λ associated to the mode set in Theorem 4.19 can be depicted as



(4.24)

In the special case $r_d = 1$ the last block ${r_d'}^{s_d}$ is simply absent. Going through all cases, one thus finds that every $\lambda \vdash r$ is either of the form depicted in Table 4.1 or may be written in the form (4.24) which implies that indeed each subalgebra associated to a descending partition admits a representation as an Airy structure.

4.7.2 The fixed point case

Now let σ be a permutation with d cycles and at least one fixed point, ie. after a relabelling we can assume that $r_d = 1$. Let us now shift in all variable sets except the one associated to the fixed point. To be more explicit, let us choose $s_1, \dots, s_{d-1} > 0$ and $t_1, \dots, t_{d-1} \in \mathbb{C}^*$ and define

$$H^\sigma_{i,k} := \hat{T}\, W^i_k\, \hat{T}^{-1}\,, \qquad \hat{T} := \prod_{\mu=1}^{d-1} \exp\left(-\frac{J^\mu_{s_\mu}}{\hbar s_\mu} t_\mu\right)\,. \tag{4.25}$$

Also with this dilaton shift we obtain a result similar to the one in Theorem 4.19 where we twisted in all d variable sets.

Theorem 4.20. *Let σ be a permutation with $d \geq 2$ cycles of length $r_1, \dots, r_d$ and $r_d = 1$. Assume there exists a $\mu_1 \in \{1, \dots, d-1\}$ such that $s_\mu = 1$ and $\frac{r_{\mu_1}}{s_{\mu_1}} \geq r_\mu$ for all $\mu \neq \mu_1$. Moreover, we assume that s_{μ_1} satisfies*

$$s_{\mu_1} \in \{1, \dots, r_{\mu_1} + 1\} \qquad \text{and} \qquad r_{\mu_1} = -1 \mod s_{\mu_1}\,.$$

Then up to change of basis we obtain a family of crosscapped higher quantum r-Airy structures

$$H^\sigma_{i,k} \qquad i \in \{1, \dots, r\},\ k \geq i - \lambda(i) + \delta_{i,1}$$

where $\lambda \vdash r$ is the partition

$$\lambda = ((r_1' + 1)^{s_1}, r_2, r_3, \dots, r_{d-1})$$

of r where without loss of generality we assumed that $\mu_1 = 1$ and $\frac{r_1}{s_1} \geq r_2 \geq \cdots \geq r_{d-1}$. As before we set $r_1' := \lfloor r_1/s_1 \rfloor$.
This family of Airy structures is parametrised by the shift values $t_1, \dots, t_{d-1} \in \mathbb{C}^*$ and the charges $Q^1, \dots, Q^d \in \mathbb{C}$ introduced in (4.16). The first parameters are constrained by the requirement that if $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ for $\mu \neq \nu$ we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$. The scalars Q^μ need to add up to zero $Q^1 + \cdots + Q^d = 0$.

This theorem is a generalisation of [7, Thm. 4.16] from the case of $d = 2$ cycles to the case of arbitrary d. The proof of this theorem can be found in Appendix A.1. As for the proof of Theorem 4.19, the proof of the above theorem was worked out under the supervision of Gaëtan Borot. Let us introduce the following notation.

Notation. Let $\vec{r} = (r_1, \dots, r_d)$ and $\vec{s} = (s_1, \dots, s_{d-1})$ such that they satisfy all properties of Theorem 4.20. We call the Airy structures introduced in Theorem 4.20 the $\mathcal{W}(\mathfrak{gl}_r)$*-Airy structures of type* $(\vec{r}, \vec{s})$ and we use an analogous notation for their reduction to $\mathcal{W}(\mathfrak{sl}_r)$.

In order to highlight certain difficulties showing up in the construction of these Airy structures, let us consider an explicit example.

Example 4.21. Let $d = 2$ and $r = 3$, ie. we assume that $r_1 = 2$ and $r_2 = 1$. For $i \leq 3$ the twist modes $W^{\sigma}_{i,k}$ of this particular example were already presented in Example 4.15. Plugging in the explicit expression for $W^1_{i_1,k_1}$ given in (4.14) we obtain

$$\begin{aligned}
W^{\sigma}_{1,k} &= J^1_{2k} + J^2_k\,,\\
W^{\sigma}_{2,k} &= \sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+p_2=k-1}} J^1_{2p_1}\,J^2_{p_2} + \frac{1}{2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+p_2=2(k-1)}} (-1)^{1+\delta_{2|p_1}} :J^1_{p_1}J^1_{p_2}: - \frac{\hbar}{8}\delta_{k,3}\,,\\
W^{\sigma}_{3,k} &= \frac{1}{4}\sum_{\substack{p_1,p_2,p_3\in\mathbb{Z}\\ p_1+p_2=2(k-p_3-2)}} (-1)^{1+\delta_{2|p_1}} :J^1_{p_1}J^1_{p_2}:J^2_{p_3} - \frac{\hbar}{16}J^2_{k-4}\,.
\end{aligned}$$

Now shifting $J^1_{-3} \to J^1_{-3} - 1$ yields

$$\begin{aligned}
H^{\sigma}_{1,k} &= J^1_{2k} + J^2_k\,,\\
H^{\sigma}_{2,k} &= J^1_{2k+1} + \sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+p_2=k-1}} J^1_{2p_1}\,J^2_{p_2} + \frac{1}{2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+p_2=2(k-1)}} (-1)^{1+\delta_{2|p_1}} :J^1_{p_1}J^1_{p_2}: - \frac{\hbar}{8}\delta_{k,3}\,,\\
H^{\sigma}_{3,k} &= -\frac{1}{4}J^2_{k+1} - \frac{1}{2}\sum_{p\in\mathbb{Z}} J^1_{2(k-p)-1}\,J^2_p + \frac{1}{4}\sum_{\substack{p_1,p_2,p_3\in\mathbb{Z}\\ p_1+p_2=2(k-p_3-2)}} (-1)^{1+\delta_{2|p_1}} :J^1_{p_1}J^1_{p_2}:J^2_{p_3} - \frac{\hbar}{16}J^2_{k-4}\,.
\end{aligned}$$

for $k \geq 0$. These operators are not yet in the normal from of an Airy structure. Only after taking a suitable linear combination

$$\begin{aligned}
H^{\sigma}_{1,k} + 4\,H^{\sigma}_{3,k-1} &= J^1_{2k} + o(2)\,,\\
H^{\sigma}_{2,k} &= J^1_{2k+1} + o(2)\,,\\
-4\,H^{\sigma}_{3,k} &= J^2_{k+1} + o(2)\,.
\end{aligned}$$

the operators satisfy the degree one condition of an Airy structure. This is what we mean when saying that $\{H^{\sigma}_{i,k}\}_{k\geq\delta_{i,1}}$ is an Airy structure *up to a change of basis.*

4.7.3 Outline of the proof

The proof of Theorem 4.19 and 4.20 contains a lot of technical, cumbersome steps. The proof is thus put in the appendix of this thesis. Here, we content ourselves with a brief explanation of the main steps of the proof.
Let us first discuss the proof of Theorem 4.19. The theorem states that a certain collection of operators $\{H^{\sigma}_{i,k}\}_{(i,k)\in I}$ obtained after a dilaton shift is an Airy structure. To prove the statement one proceeds as follows.

(1.) The operators of an Airy structure are necessarily of the form $J^{\mu}_a + o(2)$. It is thus necessary to first identify the degree zero and degree one term of $H^{\sigma}_{i,k}$ in order to prove this condition. It is a purely combinatorial task to identify the contributing terms in formula (4.17) after performing the dilaton shift.

(2.) In Example 4.21 we saw, that the degree one component of $H^{\sigma}_{i,k}$ might consist of more then one J^{*}_{*}. Thus, in general one will find that the degree one component is a linear combination of these J

$$H^{\sigma}_{i,k} = \sum_{\substack{1\leq\mu\leq d\\ a\in\mathbb{Z}}} \mathcal{M}_{(i,k),(\mu,a)}\, J^{\mu}_{a} + o(2)\,. \tag{4.26}$$

for some matrix $\mathcal{M}$. Moreover, this means that for generic i, k we may expect terms proportional to $J^{\mu}_{-a} = a\, x^{\mu}_{a}$. Therefore, we first need to construct an index set $I \subset \{1,\ldots,r\}\times\mathbb{Z}$ such that for all $(i,k)\in I$ we have $\mathcal{M}_{(i,k),(\mu,a)} = 0$ if $a \leq 0$.

(3.) Nevertheless, even restricted to such a subset I the degree one term (4.26) is in general a linear combination of multiple J^{μ}_{a}s for $a > 0$ (as we have seen for instance in Example 4.21 for $H^{\sigma}_{1,k}$). In order to bring the operators into the normal form of an Airy structure $J^{\mu}_{a} + o(2)$, one can show that the matrix $\mathcal{M}$ restricted to I is invertible under certain constraints on the dilaton shifts. This is the most cumbersome step of the proof. Admittedly, at this point the proof still contains a small gap since so far I have only proven that $\mathcal{M}$ admits a left inverse. That this left inverse is also a right inverse of $\mathcal{M}$, I could not yet prove in full generality. However, since in special cases the statement can be checked to be correct, I assume that with enough effort the statement can be proven in full generality.
Assuming we have found such an inverse $\mathcal{M}^{-1}$ one obtains operators

$$\tilde{H}^{\mu}_{a} := \sum_{(i,k)\in I} \left(\mathcal{M}^{-1}\right)_{(\mu,a),(i,k)} H^{\sigma}_{i,k} = \hbar\partial^{\mu}_{a} + o(2)$$

which are of the desired form. Equivalently, one can also perform a change of basis bringing the operators into the desired normal form.

(4.) The modes $\{\tilde{H}^{\mu}_{a}\}_{\mu\in\{1\ldots r\},\, a>0}$ satisfy the subalgebra condition if and only if the modes $\{W^{\sigma}_{i,k}\}_{(i,k)\in I}$ do. The latter is satisfied if I is induced by a descending partition of r as specified in Lemma 4.9. Once one has found the correct interpretation of I in terms of (possibly non-descending) partitions, this criterion thus allows for an easy check whether the mode set constructed in (3.) satisfies the subalgebra condition (3.3) of a higher quantum Airy structure.

Together the results of (3.) and (4.) then directly imply Theorem 4.19. The steps (1.) to (3.) are carried out in Section A.1.1 of the appendix and step (4.) is performed in Section A.1.2.
In Section A.1.3 we then discuss the fixed point case which will resemble the first case in many ways. Thus, the proof of Theorem 4.19 mostly relies on the same arguments used in the proof of the first theorem.

Having proven Theorem 4.19 and 4.20 we know that each of the constructed Airy structures comes with a unique partition function Z whose correlators are computed by a topological recursion as we have learned in Section 3.2.1. In Section A.2 we compute the base cases $F_{0,3}$, $F_{1/2,2}$, and $F_{1,1}$ of this recursion. This is on the one hand interesting in order to match the Airy structures with explicit enumerative problems in the future and on the other hand these correlators provide some information on the generality of the constraints imposed on the input data in Theorem 4.19 and 4.20. To be more precise, we may calculate $F_{0,3}$ and $F_{1/2,2}$ for arbitrary twists and dilaton shifts. If we find that for some choice of this input data the computed $F_{g,n}$ are non-symmetric we can deduce that the differential

operators associated to this input data *cannot* be an Airy structure.
In the future, we hope to generalise the sufficient conditions that provide us with an Airy structure in such a way that in the end they agree with the necessary conditions. Section A.2 contains all results obtained in this direction so far.

CHAPTER 5

Matching $\mathcal{W}(\mathfrak{sl}_r)$-Airy structures with models in two-dimensional topological gravity

That the theory of pure two-dimensional topological gravity is uniquely determined by Virasoro constraints, was our motivation to define Airy structures—a set of differential operators giving rise to a unique partition function. In Chapter 4 we then constructed a large family of these Airy structures, the $\mathcal{W}(\mathfrak{gl}_r)$ (resp. $\mathcal{W}(\mathfrak{sl}_r)$) Airy structures. This family of distinct Airy structures is indexed by partitions of r.
Out of this large class of Airy structures we have shown that the $\mathcal{W}(\mathfrak{sl}_2)$-Airy structure of type $(r,s) = (2,3)$ computes the correlators of pure gravity by identifying the respective partition functions $Z^{(2,3)}$ and Z^{WK} in Proposition 4.18. Thus, pure gravity can be interpreted in the three ways depicted below.

Intersection theory

$$Z^{\mathrm{KW}} = \exp\left(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\dots,k_n \geq 0} \left\langle \tau_{k_1} \dots \tau_{k_n} \right\rangle_g \prod_{i=1}^{n} t_{k_i} \right)$$

$t_k = (2k+1)!!\; x_{2k+1}$

Airy structure

$$Z^{(2,3)} = \exp\left(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\dots,k_n \geq 0} F_{g,n}[k_1,\dots,k_n] \prod_{i=1}^{n} x_{k_i} \right)$$

$t_k = \hbar^{1/2}\,(2k-1)!!\;\mathrm{Tr}\,\Lambda^{-2k-1}$

Matrix model (large N limit)

$$Z^{\mathrm{KMM}} = \mathrm{c}_\Lambda \int_{\mathcal{H}_N} \mathrm{d}M\, \exp\left(\hbar^{-1/2}\,\mathrm{Tr}\,\left(\tfrac{1}{3!}M^3 - \tfrac{1}{2}\Lambda M^2 \right) \right)$$

$x_{2k+1} = \frac{\hbar^{+1/2}}{2k+1}\;\mathrm{Tr}\,\Lambda^{-2k-1}$

Since all other Airy structures of the families constructed in Theorem 4.16, 4.19, and 4.20 also come with a unique partition function the obvious question is: Do these partition functions admit an interpretation in terms of enumerative geometry or matrix models? An answer to this question can only be given for some few special cases (compared to the large family of constructed Airy structures) which have already been identified in [7, Sec. 6]. This chapter we will discuss a selection of these.
The first generalisation of pure gravity we will consider is the so called r-spin theory developed by Witten [21] and their matrix model equivalent [3] in Section 5.1. We will see that the partition

functions of these theories are those of the $\mathcal{W}(\mathfrak{sl}_r)$-Airy structures of type $(r, r+1)$.
The second generalisation of pure gravity we will consider is to allow for world sheets (Riemann surfaces) with boundary. The intersection theory on the corresponding moduli space of Riemann surfaces with boundaries was mainly developed in [22, 51, 52] and is referred to as open intersection theory. This theory is rather *young* and its interplay with matrix models (and integrable hierarchies) has fuelled some interesting advances in this theory over the last years. We will review these developments in Section 5.2 and will connect open intersection theory to the $\mathcal{W}(\mathfrak{sl}_3)$-Airy structure of type $((2,1),(3))$ as done in [7]. In Section 5.3 we then introduce open r-spin theory [24, 25] as the combination of both preceding theories. So far, this theory has only been constructed for $g = 0$. One can thus only conjecture about possible connections to Airy structures.
One should mention that in the following we will *not* speak about the relation between Brézin–Gross–Witten theory and $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures of type $(r, 1)$ and the relation between $\mathcal{W}(\mathfrak{g})$-Airy structures and Fan–Jarvis–Ruan–Witten theory. We refer the interested reader to [7, Sec. 6] and references therein.

5.1 r-spin theory

5.1.1 Intersection theory

Through the study of minimal topological matter coupled to gravity [53] in terms of gauged WZW models Witten [2] found a natural generalisation of his interpretation of pure gravity in terms of enumerative geometry. This finally led him to a refinement of his original conjecture to this more general case. In the following, we will outline the construction of the correlators of this theory

$$\left\langle \tau_{i_1,k_1} \, \ldots \, \tau_{i_n,k_n} \right\rangle_g$$

in terms of intersection numbers on a suitable moduli space following [21]. The resulting theory is often referred to as *r-spin theory*. We refer the readers interested in the field theoretic origins of the theory to [2].
Let $r > 1$ be an integer and Σ be a (compact) Riemann surface of genus g and $z_1, \ldots, z_n \in \Sigma$. Further, choose integers $a_1, \ldots, a_n$ satisfying $0 \le a_i \le r-2$ which we call the *twists* of the marked points. Let ω_Σ denote the canonical bundle of Σ which is of degree $2g-2$. Then if $2g - 2 - \sum_{i=1}^n a_i$ is divisible by r the bundle $\omega_\Sigma(-\sum_{i=1}^n a_i\, z_i)$ admits for r-spin structures. By an r-spin structure we mean a line bundle $\mathcal{T}$ together with an isomorphism

$$\mathcal{T}^{\otimes r} \cong \omega_\Sigma\Big(- \sum\nolimits_{i=1}^n a_i\, z_i \Big) .$$

We denote $\overline{\mathcal{M}}^{1/r}_{g,\{a_1,\ldots,a_n\}}$ the compactified moduli space parametrising

$$(\Sigma, z_1, \ldots, z_n) \quad \text{with} \quad \mathcal{T}^{\otimes r} \cong \omega_\Sigma\Big(- \sum\nolimits_{i=1}^n a_i\, z_i \Big)$$

marked Riemann surfaces with spin structure. The correlators of the theory are defined via the integration of the so called *Witten class* $c_D(\mathcal{W})$ and the ψ-classes which we already introduced in Section 2.1. In genus zero the Witten class is the top Chern class of the bundle $\mathcal{W}$ (the Witten bundle)

with fiber $H^0(\Sigma, \mathcal{T})$ over $\overline{\mathcal{M}}^{1/r}_{g,\{a_1,\dots,a_n\}}$. In higher genus its construction satisfying the expected properties of [21] is highly non-trivial and is due to Polishchuk and Vaintrob [54, 55]. One should mention that today there exist various equivalent constructions of the Witten class (see the review in [24, Sec. 1] and references therein).
We now have everything at hand to define the correlators

$$\left\langle \tau_{k_1,a_1} \, \dots \, \tau_{k_n,a_n} \right\rangle_g^{1/r} := \frac{1}{r^g} \int_{\overline{\mathcal{M}}^{1/r}_{g,\{a_1,\dots,a_n\}}} c_D(\mathcal{W}) \prod_{i=1}^n \psi_i^{k_i} \tag{5.1}$$

of r-spin theory. The above integral vanishes unless the selection rule

$$\sum_{i=1}^n k_i + D = 3g - 3 + n \qquad \text{where } D = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}$$

is satisfied. We may interpret $\tau_{0,a}$ as the ath primary matter field and $\tau_{k,a}$ as its kth gravitational descendant [2]. For convenience let us introduce the partition function

$$Z^{1/r} := \exp\left(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{\{k_i,a_i\}} \left\langle \tau_{k_1,a_1} \, \dots \, \tau_{k_n,a_n} \right\rangle_g \prod_{i=1}^n t_{k_i,a_i} \right)$$

depending on the formal variables $t_{k,a}$ with $k \geq 0$ and $0 \leq a \leq r-2$.
It turns out, the correlators (5.1) satisfy similar recursion relations like those of pure gravity. The generalisations of the string and dilaton equations for instance are

$$\begin{aligned} \left\langle \tau_{0,0} \prod_{i=1}^n \tau_{k_i,a_i} \right\rangle_g &= \sum_{j=1}^n \left\langle \tau_{k_j-1,a_j} \prod_{i\neq j} \tau_{k_i,a_i} \right\rangle_g , \\ \left\langle \tau_{1,0} \prod_{i=1}^n \tau_{k_i,a_i} \right\rangle_g &= (2g-2+n) \left\langle \prod_{i=1}^n \tau_{k_i,a_i} \right\rangle_g . \end{aligned}$$

Equivalently, we can formulate these relations as differential constraints $L_n^{1/r} Z^{1/r} = 0$ for the partition function where [37, Thm. 5.4]

$$\begin{aligned} L_{-1}^{1/r} &= \qquad -\hbar\partial_{t_{0,0}} + \frac{1}{2}\sum_{a=0}^{r-2} t_{0,a}\, t_{0,r-2-a} + \sum_{k=1}^{\infty}\sum_{a=0}^{r-2} t_{k,a}\, \hbar\partial_{k-1,a} , \\ L_0^{1/r} &= -(r+1)\, \hbar\partial_{t_{1,0}} + \sum_{k=1}^{\infty}\sum_{a=0}^{r-2} (rk+a+1)\, t_{k,a}\, \hbar\partial_{k,a} + \hbar\, \frac{r^2-1}{24} . \end{aligned} \tag{5.2}$$

If we consider the above constraints for $r = 2$, we see that they coincide with those of pure gravity (2.6) and (2.7) under the identification of $t_{k,0}$ with t_k. This is no coincidence since for $r = 2$ the Witten class takes the values ± 1 depending on whether the spin structure is even or odd. Thus the integral over $\overline{\mathcal{M}}^{1/2}_{g,\{0,\dots,0\}}$ reduces to an integral over $\overline{\mathcal{M}}_{g,n}$ times the difference between the number of even and odd spin structures which cancels the factor 2^{-g} in the definition of the correlators [21, Sec. 1.7]. This led Witten to a refinement of his original conjecture presented in Section 2.2. He conjectured that

$Z^{1/r}$ is a tau function of the r-Gelfand-Dickey hierarchy which means that beside the string equation $Z^{1/r}$ satisfies further differential constraints generalising the ones introduced in (2.9). The conjecture was proven by Faber–Shadrin–Zvonkine [56]. For further references we see [7, Sec. 6.1].

5.1.2 The generalised Kontsevich matrix model

One calls matrix models of the form

$$Z^V(\Lambda) := \frac{\int_{\mathcal{H}_N} \mathrm{d}M \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr}\left(V(M+\Lambda) - V(\Lambda) - V'(\Lambda)M\right)}}{\int_{\mathcal{H}_N} \mathrm{d}M \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr} U_2(M,\Lambda)}} \tag{5.3}$$

generalised Kontsevich matrix models (GKMM). In the above equation V is a polynomial potential and U_2 represents the terms of the numerator potential quadratic in M, ie.

$$U_2(M,\Lambda) := \lim_{\epsilon \to 0} \frac{1}{\epsilon^2}\left(V(\epsilon M + \Lambda) - V(\Lambda) - V'(\Lambda)\epsilon M\right).$$

For the special choice of potential

$$V_r(M) := \frac{\sqrt{-r}}{r+1} M^{r+1}$$

for instance we have

$$U_2(M,\Lambda) = \frac{\sqrt{-r}}{2} \sum_{m=0}^{r-1} \Lambda^m M \Lambda^{r-1-m} M .$$

It turns out, in the special case $V = V_2$ the matrix integral transforms into

$$\begin{aligned} Z^{V_2}(\Lambda) &= \frac{\int_{\mathcal{H}_N} \mathrm{d}M \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr}\left(\frac{\sqrt{-2}}{3} M^3 + \sqrt{-2} M^2 \Lambda\right)}}{\int_{\mathcal{H}_N} \mathrm{d}M \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr} \sqrt{-2} M^2 \Lambda}} \\ &= \frac{\int_{\mathcal{H}_N} \mathrm{d}X \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr}\left(\frac{1}{3!} X^3 + \frac{\sqrt{-2}}{2} X^2 \Lambda\right)}}{\int_{\mathcal{H}_N} \mathrm{d}X \, \mathrm{e}^{-\hbar^{-1/2} \operatorname{Tr} \frac{\sqrt{-2}}{2} X^2 \Lambda}} \end{aligned}$$

where in the second step we substituted[1] $X = -\sqrt{-r}\, M$. Notice that the last expression (up to a rescaling of Λ) corresponds to the Kontsevich matrix model introduced in (2.15). This explains the name *generalised* Kontsevich matrix model for integrals of the form (5.3).
The matrix integral Z^{V_r} for $r \geq 2$ was studied in [3, 57] as a solution of certain so called $\mathcal{W}_r$-constraints. The solutions of these $\mathcal{W}_r$-constraints were moreover identified with the solutions of the r-Gelfand-Dickey hierarchy satisfying the string equation [3]. Therefore one may identify Z^{V_r} with

[1] Of course under this substitution one has to change the integration range from Hermitian to anti-Hermitian matrices. Actually, this agrees with [57] where is said that for the potential V_r the integrals in (5.3) must be either over the Hermitian or anti-Hermitian matrices if $r+1$ is either even or odd.

$Z^{1/r}$ in the large N limit. Let us introduce the times

$$T_k := \frac{\hbar^{1/2}}{k} \operatorname{Tr} \Lambda^{-k} .$$

Then under the identification of variables

$$T_{rk+a+1} \propto \frac{(-1)^k}{\sqrt{-r}\, \prod_{m=0}^{k}(m + \frac{a+1}{r})} \; t_{k,a} \qquad a \in \{0, \ldots, r-2\},\ k \in \mathbb{Z}_{\geq 0} \tag{5.4}$$

which is determined up to a proportionality constant[2], one has

$$Z^{V_r}(T_*) = Z^{1/r}(t_{*,*}) .$$

5.1.3 Identification of the associated $\mathcal{W}(\mathfrak{sl}_r)$-Airy structure

It turns out the $\mathcal{W}_r$-constraints of Adler–van Moerbeke [3] associated to r-spin theory fit into the framework of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures developed in Chapter 4. Comparing the form of the $\mathcal{W}_r$-operators of [3, Sec. 3 & 4] with the operators $W^{\sigma}_{i,k}$ of Example 4.17 one can deduce that the set of differential constraints governing r-spin theory is the $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure of type $(r, s) = (r, r+1)$. In order to find the correct value of the dilaton shift and the correct identification of variables some more effort is needed.
From [56] we know that $Z^{1/r}$ coincides with the total descendant potential of the A_{r-1}-singularity. Milanov [59] proved (using results of [14]) that the latter may be computed via the Bouchard–Eynard topological recursion applied to the spectral curve $x = \frac{z^r}{r}$, $y = -z$. Since it would go beyond the scope of this thesis to introduce the Bouchard–Eynard topological recursion we refer the interested reader to [13, 60] and simply use the results of [7, Sec. 5] as a dictionary between (local) spectral curves with their differentials $\omega_{g,n}$ and $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures with their associated $F_{g,n}$.
Indeed, inspecting Theorem 5.27 of [7] we see that the spectral curve $x = \frac{z^r}{r}$, $y = -z$ corresponds to the $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure obtained from the shift $J_{-r-1} \to J_{-r-1} - 1$. Further, equation (5.10) of [7] tells us that the differentials $\omega_{g,n}$ on this spectral curve are related to the correlators $F_{g,n}$ of the Airy structure via

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n>0} F^{(r,r+1)}_{g,n}\left[k_1 \ldots k_n\right] \prod_{i=1}^{n} \frac{\mathrm{d}z_i}{z_i^{k_i+1}} . \tag{5.5}$$

One can now identify the r-spin intersection numbers introduced in (5.1) with the differentials $\omega_{g,n}$ via the formula[3] [61]

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{0\leq a_1,\ldots,a_n\leq r-2 \\ k_1,\ldots,k_n\leq 0}} \left\langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \right\rangle_g \prod_{i=1}^{n} (rk_i + a_i + 1)!_r \, \frac{\mathrm{d}z_i}{z_i^{rk_i+a_i+2}} \tag{5.6}$$

[2] In (5.4) I am rather vague saying the identification holds up to some proportionality constant since I did not find an explicit expression in the literature. Starting from (5.4) one should be able to fix the coefficient explicitly comparing with the approach in [58, Sec. B]. The normalisation of the matrix integral (5.3) was adopted from [7, Sec. 6.1].

[3] Equation (5.5) differs from the expression in [61] by certain powers of r which is due to a different normalisations of the spectral curve used here [62].

where we introduced the notation

$$(rk_i + a_i + 1)!_r := (rk_i + a_i + 1) \cdot (r(k_i - 1) + a_i + 1) \cdot \ldots \cdot (r + a_i + 1) \cdot (a_i + 1)\,.$$

Comparing (5.5) with (5.6) one obtains

$$F_{g,n}^{(r,r+1)}\big[\{rk_i + a_i + 1\}_{i=1}^n\big] = \big\langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \big\rangle_g \prod_{i=1}^n (rk_i + a_i + 1)!_r\,.$$

We thus arrive at the following proposition.

Proposition 5.1. *Let $Z^{(r,r+1)}$ denote the partition function corresponding to the $\mathcal{W}(\mathfrak{sl}_r)$-Airy structure of type $(r, r+1)$ obtained by the shift $J_{-r-1} \to J_{-r-1} - 1$. Then under the identification of variables*

$$t_{k,a} = (rk + a + 1)!_r \, x_{rk+a+1} \qquad k \in \mathbb{Z},\ a \in \{0, \ldots, r-2\} \tag{5.7}$$

we have

$$Z^{1/r}(t_{*,*}) = Z^{(r,r+1)}(x_*)\,.$$

Notice that the above proposition is a natural generalisation of Proposition 4.18 where we identified the $\mathcal{W}(\mathfrak{sl}_2)$-Airy structure constraining pure gravity. In Example 4.17 we explicitly computed the modes $H^{\sigma}_{i,k}|_{\mathrm{red}}$ for $i = 2, 3$ of the above Airy structure. Indeed, performing the change of coordinates (5.7) one can easily show that $H^{\sigma}_{2,0}|_{\mathrm{red}}$ and $H^{\sigma}_{2,1}|_{\mathrm{red}}$ are nothing but the differential operators corresponding to the string and dilaton equation in (5.2). Explicitly, one has

$$H^{\sigma}_{2,k}|_{\mathrm{red}} = -L^{1/r}_{k-1} \qquad k \in \{0, 1\}\,.$$

Moreover, from the expansion of $H^{\sigma}_{i,k}$ into monomials of Js we can read of the values $F_{0,3}$ and $F_{1,1}$ which are [7, Prop. B.2 & Lem B.3]

$$F_{0,3}\big[p_1\, p_2\, p_3\big] = p_1 p_2 p_3\, \delta_{p_1+p_2+p_3,r+1}\,, \qquad F_{1,1}\big[p\big] = \frac{r^2-1}{24}\, \delta_{p,r+1}\,.$$

Using formula (5.5) we see that this is in accordance with the value of the associated r-spin correlators found in the literature [21, Eq. 3.1.3 & Sec. 1.7]

$$\big\langle \tau_{0,a_1}\, \tau_{0,a_2}\, \tau_{0,a_3} \big\rangle_0 = \delta_{a_1+a_2+a_3,r-2}\,, \qquad \big\langle \tau_{k,a} \big\rangle_1 = \frac{r-1}{24}\, \delta_{k,1}\, \delta_{a,0}\,.$$

5.2 Open intersection theory

From the physics point of view it is natural to ask whether the calculation of gravitational descendants can be generalised to the discussion of world sheets with boundaries. Let us be more precise what we mean with this. We call $(\Sigma, \partial\Sigma)$ a *Riemann surface with boundary* if Σ is obtained by removing finitely many non-intersecting open disks from a connected closed Riemann surface with $\partial\Sigma$ the union of the boundaries of the embedded disks. Necessarily these boundaries are assumed to be disjoint. What we desire is to study intersections on the (compactified) moduli space of these Riemann surfaces with boundary. However, there are serious difficulties defining such an integration. In the following

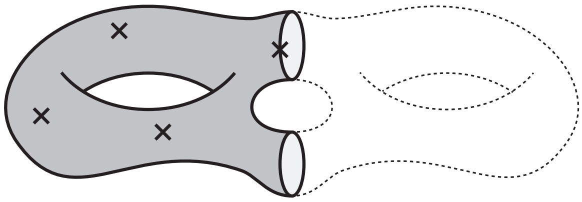


Figure 5.1: A Riemann surface with boundary $(\Sigma, \partial\Sigma)$ of doubled genus 3 with one boundary and 3 interior marked points together with its Schwarz reflection $\overline{\Sigma}$ in dashed lines.

these difficulties shall be reviewed and we will outline how Pandharipande, Solomon, and Tessler [22] solved them (in genus one). We will proceed with a discussion of the further developments in this field leading to a matrix model description of a *extended* version of these intersection numbers [63]. As a last point, we then present the link to Airy structures [7].

5.2.1 Intersection theory on the moduli space of Riemann surfaces with boundary

Let $(\Sigma, \partial\Sigma)$ be a Riemann surface with boundary. Via Schwarz reflection along the boundary one may construct a (canonical) double $D = \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ of Σ as depicted in Figure 5.1. We call the genus of this double the *doubled genus* of $(\Sigma, \partial\Sigma)$. We then define $\mathcal{M}_{g,k,n}$ to be the moduli space of Riemann surfaces with boundary $(\Sigma, \partial\Sigma)$ of doubled genus g with k marked boundary points $x_1, \ldots, x_k \in \partial\Sigma$ and n interior marked points $z_1, \ldots, z_n \in \Sigma \setminus \partial\Sigma$ as long as the stability condition

$$2g - 2 + k + 2n > 0$$

is met. Otherwise $\mathcal{M}_{g,k,n}$ is defined to be empty. There are several obstructions showing up here already:

- Of course, in general $\mathcal{M}_{g,k,n}$ is non-compact. Constructing a compactification $\overline{\mathcal{M}}_{g,k,n}$ one has to deal with the meeting of boundaries besides the usual nodal degenerations. These degenerations make $\overline{\mathcal{M}}_{g,k,n}$ a real orbifold with boundary $\partial\overline{\mathcal{M}}_{g,k,n}$ [22].

- For $g > 0$ the space $\mathcal{M}_{g,k,n}$ may be not orientable. See for instance [64, Sec. 3.2] for an illustrative argument for the non-orientability.

For a moment, let us disregard these problems. Since for interior marked points z_i we can define the cotangent line bundle $\mathbb{L}_i \to \mathcal{M}_{g,k,n}$ as in the closed case we naively would like to integrate products of $\psi_i = c_1(\mathbb{L}_i)$ over $\overline{\mathcal{M}}_{g,k,n}$. To be more precise we would like to define

$$\left\langle \tau_{k_1} \ldots \tau_{k_n} \sigma^k \right\rangle_g^o := \begin{cases} 2^{-\frac{g+k-1}{2}} \int_{\overline{\mathcal{M}}_{g,k,n}} \prod_{i=1}^n \psi_i^{k_i} & \text{, if } 3g - 3 + k + 2n = 2\sum_{i=1}^n k_i \\ 0 & \text{, otherwise.} \end{cases} \tag{5.8}$$

Note that in this notation the power of σ represents the number of boundary marked points. The factor of 2 in front of the integral is convention.
Let us take up on the difficulties dealing with $\overline{\mathcal{M}}_{g,k,n}$ mentioned above. In order to make sense of the integral on the right hand side of (5.8) one has to address the following points [22]

(i) Since $\overline{\mathcal{M}}_{g,k,n}$ in general has a boundary one needs to specify certain boundary conditions for the integrand along $\partial\overline{\mathcal{M}}_{g,k,n}$.

(ii) One must take care of the orientability-issues of $\overline{\mathcal{M}}_{g,k,n}$.

Problem (i) is addressed in [22] for genus zero as follows. If we set

$$E := \bigoplus_{i=1}^{n} \mathbb{L}_i^{\oplus k_i}$$

then the Euler class of this bundle $e(E) = \prod_{i=1}^{n} \psi_i^{k_i}$ corresponds to the desired integrand of (5.8). One then chooses a section $s : E \to \partial\overline{\mathcal{M}}_{g,k,n}$ satisfying certain compatibility conditions [22, Sec. 3] (see also [51, Sec. 2.4]) and defines the intersection numbers as

$$\left\langle \tau_{k_1} \ldots \tau_{k_n} \sigma^k \right\rangle_0^o := 2^{-\frac{g+k-1}{2}} \int_{\overline{\mathcal{M}}_{0,k,n}} e(E,s)\,. \tag{5.9}$$

Here, $e(E,s)$ denotes the Euler class of E relative to the section s. For details see the references mentioned above and [22, Sec. A]. In [51] Tessler mentions that he will give a generalisation of the construction to higher genus together with Solomon in a forthcoming paper [65].
Problem (ii) is solved for genus zero in [22] with a generalisation to higher genus to be given by Solomon and Tessler [65]. In [64, Sec. 3] Dijkgraaf and Witten interpret the non-orientability as a global anomaly of two-dimensional gravity on Riemann surfaces with boundary. In order to cancel the anomaly they propose to couple the theory to certain matter systems.
Let us assume all problems are overcome and the intersection numbers can be defined for arbitrary g. We define the *open free energy*

$$F^o(t_0, t_1, t_2, \ldots, s) := \sum_{g,k,m=0}^{\infty} \frac{\hbar^{(g-1)/2}}{n!} \sum_{k_1,\ldots,k_n \geq 0} \left\langle \tau_{k_1} \ldots \tau_{k_n} \sigma^k \right\rangle_g^o \frac{s^k}{k!} \prod_{i=1}^{n} t_{k_i}\,.$$

With F^{KW} the free energy of two-dimensional topological gravity for closed Riemann surfaces (Riemann surfaces *without* a boundary) defined in (2.5) we define the *full partition function*

$$Z^o := \exp\left(F^{\mathrm{KW}} + F^o\right).$$

In [22, Conj. 1] it is conjectured that this partition function is annihilated by the differential operators

$$\mathcal{L}_n := L_n + \hbar^{(n+2)/2} s \frac{\partial^{n+1}}{\partial s^{n+1}} + \hbar^{(n+2)/2} \frac{3n+3}{4} \frac{\partial^n}{\partial s^n}\,, \qquad n \geq -1 \tag{5.10}$$

where L_n is the same operator as in (2.6) to (2.9). The operators $\{\mathcal{L}_n\}_{n\geq -1}$ satisfy the same Virasoro commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = \hbar(m-n)\mathcal{L}_{m+n} \qquad m,n \in \{-1,0,1,2,\ldots\}\,.$$

In [22] this conjecture was proven for $g = 0$ and later proven in full generality by Buryak and Tessler [52]. The proof uses results of Buryak [66] stating that Z^o is a tau function of a certain integrable

hierarchy and a combinatorial formula for the open intersection numbers derived by Tessler [51] similar to the one of Kontsevich (2.12) derived for the closed theory.
Moreover, in [66] Buryak noticed that $F^o(t_0, t_1, \ldots, s)$ is the restriction of a so called *extended* open free energy

$$F^{o,\mathrm{ext}}\big(\{t_i, s_i\}_{i\geq 0}\big)$$

constraint by a larger set of differential constraints whose construction is natural from the point of view of integrable hierarchies. The two free energies are related by

$$F^o = F^{o,\mathrm{ext}}\big|_{s_0=s,\, s_{\geq 1}=0}\,. \tag{5.11}$$

Alexandrov later then identified the integrable hierarchy determining $F^{o,\mathrm{ext}}$ to be the MKP hierarchy allowing him to find a matrix model interpretation of open intersection theory.

5.2.2 The Kontsevich-Penner matrix model

In [23, 67, 68] Alexandrov studied the so called *Kontsevich-Penner matrix model*[4]

$$Z^Q_{\mathrm{KPMM}}(\Lambda) = \det(\Lambda)^Q\, c_\Lambda \int_{\mathcal{H}_N} \mathrm{d}M\, \mathrm{e}^{\hbar^{-1/2}\,\mathrm{Tr}\left(\frac{1}{3!}M^3 - \frac{1}{2}M^2\Lambda\right)}\, \det(\Lambda - M)^{-Q}\,. \tag{5.12}$$

where $Q \in \mathbb{C}$. In the large N limit it becomes a tau function of the so called MKP hierarchy [23] with respect to the times

$$T_k = \frac{\hbar^{1/2}}{k}\,\mathrm{Tr}(\Lambda) \qquad\qquad k > 0$$

For $Q = 0$ the above matrix model clearly reduces to the Kontsevich matrix model which we considered in (2.15). For $Q = 1$ Alexandrov proves [23] that the above matrix integral coincides with the full partition function of the extended open theory

$$\exp\left(F^{\mathrm{KW}} + F^{o,\mathrm{ext}}\right) = Z^1_{\mathrm{KPMM}} \tag{5.13}$$

under the identification of variables

$$t_k = (2k+1)!!\, T_{2k+1}\,, \qquad s_k = 2^{k+1}(k+1)!\, T_{2k+2}\,. \tag{5.14}$$

A (partly conjectural) interpretation of Z^Q_{KPMM} for arbitrary Q is given in [63]. Informally speaking, one can consider Q as a weight for the number of boundaries of the surfaces. To be more precise, let $\mathcal{M}_{g,k,n,b}$ be the submoduli of $\mathcal{M}_{g,k,n}$ containing only Riemann surfaces with b boundary components. Clearly, $\mathcal{M}_{g,k,n}$ decomposes into these submoduli. We define

$$\left\langle \tau_{k_1} \ldots \tau_{k_n}\, \sigma^k \right\rangle^o_{g,b} := 2^{-\frac{g+k-1}{2}} \int_{\overline{\mathcal{M}}_{g,k,n,b}} \prod_{i=1}^n \psi_i^{k_i} \tag{5.15}$$

[4] The expression for the matrix model was taken from [63, Eq. 1.7] and the $\hbar$-factors were adopted from [7, Sec. 6.1].

where one makes sense of the right hand side integral as in (5.9) via the choice of a boundary section. Then for arbitrary Q the free energy

$$F^{o,Q}(t_0, t_1, \ldots, s) := \sum_{g,k,m=0}^{\infty} \sum_{b=1}^{\infty} \frac{\hbar^{(g-1)/2} Q^b}{n!\, k!} \sum_{k_1,\ldots,k_n \geq 0} \left\langle \tau_{k_1} \ldots \tau_{k_n} \sigma^k \right\rangle_{g,b}^{o} \frac{s^k}{k!} \prod_{i=1}^{n} t_{k_i}$$

is conjectured to be computed by Z^Q_{KPMM} in the sense of (5.13) and (5.14) with $s_0 = s$ and $s_{\geq 1} = 0$. In the above representation it is clear that Q is a weight for the number of boundaries.

5.2.3 A $\mathcal{W}(\mathfrak{sl}_3)$-Airy structure for the extended open theory

In [63] Alexandrov derives certain $\mathcal{W}(\mathfrak{sl}_3)$-constraints for the Kontsevich-Penner matrix model (5.12). In [7] these constraints were identified to correspond to a particular Airy structure.

Proposition 5.2. [7, Prop. 6.3] *The extended open partition function $Z^Q_{KPMM}(T_*)$ is the partition function associated to the $\mathcal{W}(\mathfrak{sl}_3)$-Airy structure of type $(\vec{r}, \vec{s}) = ((2,1),(3))$ obtained from the dilaton shift $J^1_{-3} \to J^1_{-3} - 1$ under the identification of variables $T_k = x^1_k$ for $k > 0$ and $J^1_0 = \hbar^{1/2} Q$.*

The Airy structure described above was constructed in Example 4.21 in the unreduced case. Reducing from $\mathcal{W}(\mathfrak{gl}_3)$ to $\mathcal{W}(\mathfrak{sl}_3)$ one formally has to set $W^{\sigma}_{1,k} = J^1_{2k} + J^2_k$ equal to zero. Since at this point there would be no gain for the further discussion to derive these operators we refer to the original literature [7, Prop. 6.3] for their explicit construction and the match with those derived by Alexandrov.

5.3 Open r-spin theory

After our presentation of r-spin theory and open intersection theory as two possible generalisations of pure gravity it is natural to ask the following question. Can one formulate an *open r-spin theory* as the combination of the preceding two? Recently, a positive answer to this question (in genus zero) was given by Buryak, Clader, and Tessler [24, 25].

5.3.1 The geometric setting

Given a Riemann surface with boundary $(\Sigma, \partial\Sigma)$ in the sense of Section 5.2.1 with internal marked points $z_1, \ldots, z_n \in \Sigma \setminus \partial\Sigma$ and boundary marked points $x_1, \ldots, x_k \in \partial\Sigma$. Then the double $D = \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ comes with an involution ϕ exchanging Σ with $\overline{\Sigma}$ and $\mathrm{Fix}(\phi) = \partial\Sigma$. Then roughly speaking an r-spin structure over $(\Sigma, \{z_i\}, \{x_i\})$ is an r-spin structure $\mathcal{T}$ over the double D with marked points $z_i, \phi(z_i), x_j$ together with an involution $\tilde{\phi} : \mathcal{T} \to \mathcal{T}$ lifting ϕ. The twists of the z_i lie in the range $a_i \in \{0, \ldots, r-1\}$. The twists of boundary points are fixed to $r-2$. In [24] Buryak–Clader–Tessler give a rigorous definition of r-spin structures over disks (ie. the genus zero case) and construct a (compact) moduli space $\overline{\mathcal{M}}^{1/r}_{0,k,\{a_1,\ldots,a_n\}}$ of these so called *r-spin disks*.
In order to define a descendant integration over this moduli space one has to address the same problems (i), (ii) which already occurred in the case of open intersection theory in Section 5.2.1. As in (5.9) the fact that $\overline{\mathcal{M}}^{1/r}_{0,k,\{a_1,\ldots,a_n\}}$ has a boundary is addressed by the choice of a *canonical* boundary condition s. Working out these difficulties in [24] Buryak–Clader–Tessler moreover give a definition of an open

analogue of the Witten bundle $\mathcal{W}$ resulting in the definition of the intersection numbers

$$\left\langle \tau_{k_1,a_1} \, \ldots \, \tau_{k_n,a_n} \, \sigma^k \right\rangle_0^{1/r,o} := \int_{\overline{\mathcal{PM}}^{1/r}_{0,k,\{a_1,\ldots,a_n\}}} e\left(\mathcal{W} \oplus \bigoplus_{i=1}^n \mathbb{L}_i^{\oplus k_i}, \, s \right)$$

with $\overline{\mathcal{PM}}^{1/r}_{0,k,\{a_1,\ldots,a_n\}}$ a certain subset of $\overline{\mathcal{M}}^{1/r}_{0,k,\{a_1,\ldots,a_n\}}$. Of course, these correlators vanish unless a certain selection rule, which we will omit here, is satisfied. It turns out that indeed open 2-spin intersection numbers coincide with those introduced in (5.1). This is the analogue of the same correspondence in the closed case. For details we refer to [24].

In [25, Thm. 1.3] Buryak–Clader–Tessler further make the observation that the associated genus zero free energy

$$F_0^{1/r,o}(t_{*,*}, s) := \sum_{k+2n>2} \frac{1}{n!} \sum_{\substack{0 \le a_1,\ldots,a_n \le r-1 \\ k_1,\ldots,k_n \ge 0}} \left\langle \tau_{k_1,a_1} \, \ldots \, \tau_{k_n,a_n} \, \sigma^k \right\rangle_0^{1/r,o} \frac{s^k}{k!} \prod_{i=1}^n t_{k_i,a_i} \tag{5.16}$$

is related to the so called *extended closed r-spin* free energy $F_0^{1/r,\mathrm{ext}}(t_{*,*})$ via

$$F_0^{1/r,o}(t_{*,0}, \ldots, t_{*,r-1}, s) = -\frac{1}{r} F_0^{1/r,\mathrm{ext}}\big|_{t_{k,r-1} \to t_{k,r-1} - r\,\delta_{k,0} s} + \frac{1}{r} F_0^{1/r,\mathrm{ext}}. \tag{5.17}$$

where $F_0^{1/r,\mathrm{ext}}(t_{*,*})$ is the generating function of r-spin intersection numbers (of Riemann surfaces without a boundary)

$$F_0^{1/r,ext}(t_{*,*}) = \sum_{n \ge 2} \frac{1}{n!} \sum_{\substack{0 \le a_1,\ldots,a_n \le r-1 \\ k_1,\ldots,k_n \ge 0}} \left\langle \tau_{0,-1} \, \tau_{k_1,a_1} \, \ldots \, \tau_{k_n,a_n} \, \sigma^k \right\rangle_0^{1/r} \prod_{i=1}^n t_{k_i,a_i} \tag{5.18}$$

where exactly one of the twists is equal to -1. In [69] it is shown that in this special case the construction of the Witten class is still possible. One should note that the free energies (5.16) and (5.18) depend on the times $t_{*,r-1}$ unlike the closed r-spin intersection numbers defined in (5.1).

The relation (5.17) is remarkable in two ways. First, it relates open intersection theory to closed intersection theory in an unexpected way and second as $F_0^{1/r,\mathrm{ext}}$ is proven in [69] to be the genus zero part of a special solution of the so called extended r-reduced KP hierarchy of Bertola and Yang [70] we know $F_0^{1/r,o}$ has the same property. Of course, due to the latter observation it is natural to conjecture that once a theory for genus $g > 0$ is found the associated free energy will be a solution to this integrable hierarchy. This is so to say the open r-spin analogue of Witten's conjecture and allows for a prediction of the higher genus correlators. See [25, Sec. 6] for its detailed formulation.

5.3.2 A conjecture about a possible $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure

On the other hand this conjecture is highly interesting for our program matching $\mathcal{W}(\mathfrak{sl}_r)$-Airy structures with applications in enumerative geometry. One might hope that the correspondence for closed r-spin theory and open intersection theory with $\mathcal{W}(\mathfrak{sl}_r)$-Airy structures generalise to open r-spin theory in the following sense.

Question 5.1. [7, Question 6.4] *Let $Z^{((r,1),(r+1))}$ be the partition function of the $\mathcal{W}(\mathfrak{sl}_{r+1})$-Airy structure of type $((r,1),(r+1))$ obtained from shifting $J^1_{-r-1} \to J^1_{-r-1} - 1$ and choosing $Q^1 = 1$. Does it coincide with the tau function of the extended r-reduced KP hierarchy of Bertola and Yang* [70]*?*

Let us provide some evidence that the above question might have a positive answer. From (4.18) it is straight forward to compute the operator $H^{\sigma}_{2,k}$ for $k \geq 0$ of the above Airy structure explicitly. Upon reduction to $\mathcal{W}(\mathfrak{sl}_{r+1})$ by formally setting $W^{\sigma}_{1,k} = J^1_{rk} + J^2_k$ equal to zero one obtains

$$H^{\sigma}_{2,k}\big|_{\text{red}} = J^1_{rk+1} + \frac{1}{2} \sum_{p_1+p_2=r(k-1)} :J^1_{p_1} J^1_{p_2}: - \frac{(r^2-1)\hbar}{24}\,\delta_{k,1}\,.$$

Then plugging in the definition of the Js (4.16) yields

$$\begin{aligned}
H^{\sigma}_{2,0}\big|_{\text{red}} &= \hbar\partial_{x_1} - \frac{1}{2}\sum_{p=0}^{r-1} p(r-p)\,x_p x_{r-p} - \sum_{p=r+1}^{\infty} p\,x_p\,\hbar\partial_{x_{p-r}} - \hbar^{1/2} Q^1 r\,x_r\,, \\
H^{\sigma}_{2,1}\big|_{\text{red}} &= \hbar\partial_{x_{r+1}} - \sum_{p=1}^{\infty} p\,x_p\,\hbar\partial_{x_p} - \hbar\left(\frac{r^2-1}{24} + \frac{(Q^1)^2}{2}\right), \\
H^{\sigma}_{2,k}\big|_{\text{red}} &= \hbar\partial_{x_{rk+1}} - \sum_{p=1}^{\infty} p\,x_p\,\hbar\partial_{x_{r(k-1)+p}} - \frac{1}{2}\sum_{p=1}^{r(k-1)-1} \hbar^2\,\partial_p\partial_{r(k-1)-p} + \hbar^{3/2}\partial_{x_{r(k-1)}}\,, \qquad k > 1.
\end{aligned} \tag{5.19}$$

writing $x_p := x^1_p$. Remarkably, the first operator $H^{\sigma}_{2,0}|_{\text{red}}$ equals the operator L_{-1} associated to the string equation satisfied by the particular tau function of Bertola and Yang [70, Eq. 1.23] up to some factors[5]. This is a first evidence for Question 5.1 to have a positive answer.

5.3.3 A matrix model candidate

From the matrix model point of view it is tempting to assume that open r-spin theory is related to Kontsevich–Penner matrix models which we introduced in (5.12) with higher order potentials as considered in (5.3).

Such matrix models were for instance analysed in [71]. Note that the Virasoro constraints derived by Ashok and Troost [71, eq. 3.6] exactly equal expression (5.19) once we set $Q^1 = 1$ and shift $J^1_{-r-1} \to J^1_{-r-1} - r\alpha$. It is thus natural to ask the following.

Question 5.2. *Do the matrix models considered in* [71, Sec. 2.3.2] *admit an interpretation as the partition function of the $\mathcal{W}(\mathfrak{gl}_r)$-Airy structure of type $((r,1),(r+1))$ reduced to $\mathcal{W}(\mathfrak{sl}_r)$ obtained from shifting $J^1_{-r-1} \to J^1_{-r-1} - r\alpha$, choosing $Q^1 = 1$, and identifying $r = p$?*

[5] Notice that the deviating factors are consistent with the different normalisation of the (closed) string equation [70, Eq. 1.11].

CHAPTER 6

Conclusion and open questions

The focus of this thesis laid on the study of $\mathcal{W}(\mathfrak{gl}_r)$-higher Airy structures—both as a mathematically interesting theory in its own right and also in its interplay with enumerative geometry and matrix models. In the following we will summarize the main results and discuss open questions and possible further research directions.

Without question, the central outcome of this thesis are the Theorems 4.19 and 4.20 where we found a new family of $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures building up on the work of [7]. Remember that the construction of the differential operators $H^{\sigma}_{i,k}$ forming the Airy structure $\{H^{\sigma}_{i,k}\}_{(i,k)\in I}$ depends on the following input data:

- A choice of twist σ,
- a choice of dilaton shift $J^{\mu}_{a} \to J^{\mu}_{a} - \delta_{a,-s_{\mu}}\, t_{\mu}$,
- and a choice of index set I.

Then Theorem 4.19 and 4.20 both provide us with sufficient conditions for the above input data to indeed produce an Airy structure. Remarkably, we have also learned that every mode subalgebra of $\mathcal{W}(\mathfrak{gl}_r)$ indexed by a descending partition in the sense of Lemma 4.9 admits a representation in terms of differential operators forming an Airy structure.

Of course, one has to remark that at this point the proof of Theorem 4.19 and 4.20 contains a small gap which of course has to be closed in the future. In detail this was discussed in Section 4.7.3 point (3.). I postpone the completion of the proof to [18]. Once, the proof is completed one can ask whether it is possible to loosen the constraints on the input data so that it still produces an Airy structure.

Question 6.1. *Can the classification of $\mathcal{W}(\mathfrak{gl}_r)$-higher Airy structures obtained from a twist with an arbitrary permutation and a suitable dilaton shift initiated in Theorem 4.19 and 4.20 be completed to a classification of all higher Airy structures admitted by the $\mathcal{W}(\mathfrak{gl}_r)$-algebra?*

In order to address this questions, it is essential to inspect the necessary conditions on the input data to produce an Airy structure. These conditions can for instance be deduced from the constraint that the correlators $F_{g,n}$ associated to an Airy structure must be symmetric in their arguments. As $F_{0,3}$ and $F_{1/2,2}$ are the correlators easiest to be accessed they are best suited for this analysis. Thus, the completion of the computation of $F_{1/2,2}$ in the most general setting possible might be an important step towards finding an answer for Question 6.1. Note, that so far this computation has (partly conjectural) only been carried out in the case where σ has two cycles.

As we emphasized in Chapter 5, there is a close relation between certain $\mathcal{W}(\mathrm{gl}_r)$-Airy structures (respectively their reduction to $\mathcal{W}(\mathrm{sl}_r)$) and enumerative geometry, matrix models, and integrable hierarchies. It is natural to expect a similar correspondence to exist for the new-found Airy structures.

Question 6.2. *Do the higher Airy structures of Theorem 4.19 and 4.20 admit an interpretation in terms of enumerative geometry, matrix models, and integrable hierarchies?*

In the special case of the $\mathcal{W}(\mathrm{gl}_{r+1})$-Airy structure of type $(\vec{r}, \vec{s}) = ((r, 1), r+1)$ we have already teased a possible connection to open r-spin intersection theory and the extended r-reduced KP hierarchy in Question 5.1 and introduced a possible matrix model candidate in Question 5.2. For all other new found Airy structures there is no such (conjectural) correspondence known yet. In order to match the different fields the two most promising starting-points are probably the following.

- Identify the (structure of the) correlators $F_{g,n}$ of the Airy structure with the correlators of the conjectured partner theory.

- Identify the $\mathcal{W}$-constraints governing the respective theories.

Regarding the first approach (and also regarding the second in case one wants to identify explicit expressions for the differential constraints) it is necessary to determine closed expressions for the coefficients $C^{(j)}[.]$ in the expansion of $H^{\sigma}_{i,k}$ in (3.7) resp. (A.39). These coefficients together with the topological recursion relation of Theorem 3.5 determine the correlators $F_{g,n}$. So far, closed expressions for these coefficients are only known in certain special cases.

In [7, Sec. 5] the $\mathcal{W}(\mathfrak{gl}_r)$-Airy structures of type (r, s) (Theorem 4.16) were matched with the Bouchard–Eynard topological recursion for so called *admissible* spectral curves. This correspondence was teased in Section 5.1.3 but a full discussion would have gone beyond the scope of this thesis.

Question 6.3. *Do the Airy structures of Theorem 4.19 and 4.20 allow for an interpretation in terms of the Bouchard–Eynard topological recursion on more general spectral curves?*

Especially this question will be addressed in a forthcoming paper of N. Aghaei, G. Borot, R. Kramer, and the author [18]. Assuming such a description in terms of spectral curves exists one can ask the following.

Question 6.4. *Can these spectral curves be interpreted as the B-model of a certain A-model geometry and do the multidifferentials associated to the spectral curve encode invariants of the A-model in the sense of* [72]*?*

We thus see there is still a plethora of questions to address and there are more then enough research directions worth to be taken.

APPENDIX **A**

$\mathcal{W}(\mathfrak{gl}_r)$-Airy structures

Section A.1 of this appendix is devoted to the proof of Theorem 4.19 and 4.20. An outline of the proof was already presented in the main part, Section 4.7.3.
In Section A.2 we then compute the three correlators $F_{0,3}$, $F_{1/2,2}$ and $F_{1,1}$ for the Airy structures of Theorem 4.19. These correlators are of special interest as they constitute the base case of the recursion for all higher $F_{g,n}$.

A.1 Proof of the main theorems

Let $\sigma \in \mathfrak{S}_r$ be a permutation with d cycles of respective length r_μ such that $r = r_1 + \cdots + r_d$. Since the σ-twisted modes $W^\sigma_{i,k}$ and their dilaton shifted partners $H^\sigma_{i,k}$ are the central objects of study, let us briefly remind ourselves that

$$W^\sigma_{i,k} = \sum_{M \subseteq \{1,\ldots,d\}} \sum_{\substack{1 \le i_\mu \le r_\mu \ \mu \in M \\ \sum_\mu i_\mu = i}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^M \\ \sum_\mu k_\mu = k+1-|M|}} \prod_{\mu \in M} \frac{1}{r_\mu^{i_\mu - 1}} W^\mu_{i_\mu, k_\mu} \qquad i \in \{1, \ldots, r\},\ k \in \mathbb{Z} \quad \text{(A.1)}$$

where we set

$$W^\mu_{i_\mu,k_\mu} = \frac{1}{r_\mu} \sum_{j_\mu=0}^{\lfloor i_\mu/2 \rfloor} \frac{i_\mu!\, \hbar^{j_\mu}}{2^{j_\mu} j_\mu! (i_\mu - 2j_\mu)!} \sum_{\substack{p^\mu_{2j+1}, \ldots, p^\mu_{i_\mu} \in \mathbb{Z} \\ \sum_l p^\mu_l = r_\mu(k_\mu - i_\mu + 1)}} \Psi^{(j_\mu)}_{r_\mu}\left(p^\mu_{2j_\mu+1}, p^\mu_{2j_\mu+2}, \ldots, p^\mu_{i_\mu}\right) :\prod_{l=2j_\mu+1}^{i_\mu} J^\mu_{p^\mu_l}: . \quad \text{(A.2)}$$

Here, $\Psi^{(j)}_r(\ldots)$ are certain rational numbers admitting a representation in terms of sums over r-roots of unity:

$$\Psi^{(j)}_r(a_{2j+1}, \ldots, a_i) := \frac{1}{i!} \sum_{\substack{m_1, \ldots, m_i = 0 \\ m_l \neq m_{l'}}}^{r-1} \left(\prod_{l'=1}^{j} \frac{\theta^{m_{2l'-1} + m_{2l'}}}{(\theta^{m_{2l'}} - \theta^{m_{2l'-1}})^2} \prod_{l=2j+1}^{i} \theta^{-m_l a_l} \right) \quad \text{(A.3)}$$

where $\theta = e^{2\pi \mathbf{i}/r}$. In the following exposition we will regularly meet the following cases.

Lemma A.1. *Let $r, s > 0$ coprime and $i \in \{1, \ldots, r\}$. Then*

$$\Psi_r^{(0)}\big(\underbrace{-s, \ldots, -s}_{i\text{-times}}\big) = (-1)^{r+1}\,\delta_{r,i}\,, \tag{A.4}$$

$$\Psi_r^{(0)}\big(\underbrace{-s, \ldots, -s}_{(i-1)\text{-times}}, (i-1)s\big) = (-1)^{i-1}\,\frac{r}{i}\,. \tag{A.5}$$

Fixing $s_\mu > 0$ and $t_\mu \in \mathbb{C}^*$ for all $\mu \in \{1, \ldots, d\}$, we define the dilaton shifted modes

$$H^\sigma_{i,k} := \hat{T}\, W^\sigma_{i,k}\, \hat{T}^{-1} \qquad \hat{T} := \prod_{\mu=1}^{d} \exp\left(-\frac{J^\mu_{s_\mu}}{\hbar s_\mu} r_\mu t_\mu\right)\,.$$

Notice the extra factor of r_μ in the definition of $\hat{T}$ different from our exposition in Section 4.7.1, equation (4.23). It simply amounts to a rescaling of t_μ not affecting the results but simplifying the calculation enormously.

A.1.1 The degree one condition

Notation. For $n \in \mathbb{N}$ we will write $[n] := \{1, \ldots, n\}$ and by convention set $[0] := \emptyset$.

As motivated in Section 4.7.3 let us begin by identifying the degree one component of $H^\sigma_{i,k}$. Let π_1 be the projection to degree one.

Lemma A.2. *Let r_μ and s_μ be coprime and $t_\mu \neq 0$ for all $\mu \in [d]$. Then for $i \in \{1, \ldots, r\}$ and $k \in \mathbb{Z}$*

$$\pi_1\left(H^\sigma_{i,k}\right) = \sum_{\mu \in [d]} \sum_{\substack{M \subseteq [d]\setminus\{\mu\} \\ 1 \le i - \boldsymbol{r}_M \le r_\mu}} (-1)^{|M|}\, t_\mu^{i-1} \left(\prod_{\nu \in M} \left(\frac{t_\nu}{t_\mu}\right)^{r_\nu}\right) J^\mu_{\Pi_\mu(i-\boldsymbol{r}_M, k-\boldsymbol{r}_M+\boldsymbol{s}_M)} \tag{A.6}$$

where we introduced the notation $\boldsymbol{r}_M := \sum_{\nu \in M} r_\nu$, $\boldsymbol{s}_M := \sum_{\nu \in M} s_\nu$ and

$$\Pi_\mu(i, k) := r_\mu k - (r_\mu - s_\mu)(i-1)\,.$$

Proof. Considering equations (A.1) and (A.2), we notice that $W^\sigma_{i,k}$ is the linear combination of monomials

$$\hbar^{\sum_\nu j_\nu} \prod_{\nu \in M} : \prod_{l=2j_\nu+1}^{i_\nu} J^\nu_{p^\nu_l} :$$

for $M \subseteq [d]$. After performing the dilaton shift, these will contribute to the degree one component of $H^\sigma_{i,k}$ if $j_\nu = 0$ for all $\nu \in M$ and there exists a $\mu \in M$ and $l_0 \in \mathbb{Z}$ such that for all $l \neq l_0$ we have $p^\mu_l = -s_\mu$ and similarly for all $\nu \neq \mu$ and $l \in \mathbb{Z}$ we have $p^\nu_l = -s_\nu$. In this case, the constraint from the sum over the p^μs forces

$$p^\mu_{l_0} = r_\mu k_\mu - (r_\mu - s_\mu)(i_\mu - 1) =: \Pi_\mu(i_\mu, k_\mu)$$

and analogously the sum-constraint for the p^νs with $\nu \neq \mu$ implies

$$r_\nu(k_\nu + 1) = (r_\nu - s_\nu)i_\nu \,,$$

which since k_ν and i_ν are integer with $1 \leq i_\nu \leq r_\nu$ and $\gcd(r_\nu, s_\nu) = 1$ implies that necessarily $i_\nu = r_\nu$ and $k_\nu = r_\nu - s_\nu - 1$. If we plug the expression for i_ν into the sum-constraint over the is, we find that i_μ is fixed to the value $i_\mu = i - \boldsymbol{r}_{M'_\mu}$ where $M'_\mu := M \setminus \{\mu\}$. Analogously, plugging the expression for k_ν into the k sum-constraint we find $k_\mu = k - \boldsymbol{r}_{M'_\mu} + \boldsymbol{s}_{M'_\mu}$ which results in

$$p^\mu_{l_0} = \Pi_\mu(i - \boldsymbol{r}_{M'_\mu}, k - \boldsymbol{r}_{M'_\mu} + \boldsymbol{s}_{M'_\mu}) \,.$$

Now summing up the contributions from all $\mu \in [d]$ we obtain

$$\begin{aligned}\pi_1\Big(H^\sigma_{i,k}\Big) = \sum_{\mu\in[d]} \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\\ 1\leq i-\boldsymbol{r}_{M'_\mu}\leq r_\mu}} \prod_{\nu\in M'_\mu} &\left(\frac{1}{r_\nu^{r_\nu}} \Psi^{(0)}_{r_\nu}\big(\underbrace{-s_\nu, \ldots, -s_\nu}_{r_\nu\text{-times}}\big)\ (-r_\nu t_\nu)^{r_\nu} \right) \\ &\times \frac{\boldsymbol{i}_{M'_\mu}}{r_\mu^{\boldsymbol{i}_{M'_\mu}}} \Psi^{(0)}_{r_\mu}\big(\underbrace{-s_\mu, \ldots, -s_\mu}_{(\boldsymbol{i}_{M'_\mu}-1)\text{-times}}, s_\mu(\boldsymbol{i}_{M'_\mu} - 1)\big)\ \big(-r_\mu t_\mu\big)^{\boldsymbol{i}_{M'_\mu}-1}\ J^\mu_{\Pi_\mu(\boldsymbol{i}_{M'_\mu}, \boldsymbol{k}_{M'_\mu})}\end{aligned}$$

where we used the notation $\boldsymbol{i}_M := i - \boldsymbol{r}_M$ and $\boldsymbol{k}_M := k - \boldsymbol{r}_M + \boldsymbol{s}_M$. The values of the Ψ-coefficients can be read of from Lemma A.1. Plugging these in and summing over M'_μ instead of M then yields expression (A.6). ◻

Remark A.3. Regarding the generality of our approach, let us remark the following. We could have also allowed for s_μ such that $d_\mu := \gcd(r_\mu, s_\mu) \neq 1$ in Lemma A.2. In this case we would have obtained a result similar to the one in (A.6) which is also a linear combiation of $J^\mu_{\Pi_\mu(i',k')}$ for some i', k'. However, the image of Π_μ is contained in $d_\mu\mathbb{Z}$. Thus, if $d_\mu \neq 1$ the operators $H^\sigma_{i,k}$ can never form an Airy structure due to condition (3.2).

We can rephrase the result of Lemma A.2 in the sense that

$$\pi_1\Big(H^\sigma_{i,k}\Big) = \sum_{\mu\in[d],\, a\in\mathbb{Z}} \mathcal{M}_{(i,k),(\mu,a)}\, J^\mu_a$$

where $\mathcal{M}$ is a matrix with entries

$$\mathcal{M}_{(i,k),(\mu,a)} = \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ 1\leq i-\boldsymbol{r}_M\leq r_\mu}} (-1)^{|M|}\, t_\mu^{i-1} \left(\prod_{\nu\in M} \left(\frac{t_\nu}{t_\mu}\right)^{r_\nu} \right) \delta_{a,\,\Pi_\mu(i-\boldsymbol{r}_M, k-\boldsymbol{r}_M+\boldsymbol{s}_M)} \,. \tag{A.7}$$

Notice that since $\Pi_\mu(i,k)$ is increasing in k for fixed i, there must exist a $K \in \mathbb{Z}$ such that for all $k \geq K$ we have $\mathcal{M}_{(i,k),(\mu,a)} = 0$ for all $a \leq 0$. For fixed i let $k_{\min}(i)$ be minimal with this property, ie.

$$k_{\min}(i) := \min\left\{K \in \mathbb{Z} \mid \forall k \geq K,\, \mu \in [d],\, a \leq 0 :\ \mathcal{M}_{(i,k),(\mu,a)} = 0\right\} \,.$$

One can derive an approximate formula for this lower bound. Notice that $\mathcal{M}_{(i,k),(\mu,a)} = 0$ for a fixed a unless

$$a = \Pi_\mu(i - \boldsymbol{r}_M, k - \boldsymbol{r}_M + \boldsymbol{s}_M)$$

for some $M \subseteq [d] \setminus \{\mu\}$ with $1 \leq i - \boldsymbol{r}_M \leq r_\mu$. Thus if $\Pi_\mu(i - \boldsymbol{r}_M, k - \boldsymbol{r}_M + \boldsymbol{s}_M) > 0$ for all $M \subseteq [d] \setminus \{\mu\}$ we have $\mathcal{M}_{(i,k),(\mu,a)} = 0$ for all $a \leq 0$. Now let us use that for $i' \in \{1, \ldots, r_\mu\}$ and $k \in \mathbb{Z}$ we have $\Pi_\mu(i', k') > 0$ if and only if

$$k' \geq i' - 1 - \left\lfloor \frac{s_\mu(i'-1)}{r_\mu} \right\rfloor + \delta_{i',1}$$

which holds since $r_\mu, s_\mu > 0$ are coprime. We thus see that

$$k_{\min}(i) \leq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) \tag{A.8}$$

where

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) := \max\left\{ i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} \;\middle|\; {\scriptstyle \mu\in[d],\, M\subseteq[d]\setminus\{\mu\} \atop 1\leq i-\boldsymbol{r}_M\leq r_\mu} \right\}. \tag{A.9}$$

Notice that for generic $\{t^\nu\}$ we even have $k_{\min}(i) = \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$. However, for special values of $\{t^\nu\}$ the sum in (A.7) may vanish even if it is non-empty.
One could ask now what we have gained by transcending from $k_{\min}(i)$ to $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ because (A.9) still looks quite untraceable. It turns out, $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ allows for a nice closed formula.

Proposition A.4. *If* $\frac{r_1}{s_1} \geq \cdots \geq \frac{r_d}{s_d}$ *and* $\gcd(r_\mu, s_\mu) = 1$ *for all* $\mu \in [d]$ *we have*

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) = \begin{cases} i - 1 - \left\lfloor \frac{s_1(i-1)}{r_1} \right\rfloor + \delta_{i,1} & , 1 \leq i \leq r_1 \\ i - 1 - \left\lfloor \frac{s_2(i-r_1-1)}{r_2} \right\rfloor - s_1 + \delta_{i,1+r_1} & , r_1 < i \leq \boldsymbol{r}_{[2]} \\ \vdots & \vdots \\ i - 1 - \left\lfloor \frac{s_d(i-\boldsymbol{r}_{[d-1]}-1)}{r_d} \right\rfloor - \boldsymbol{s}_{[d-1]} + \delta_{i,1+\boldsymbol{r}_{[\mu-1]}} & , \boldsymbol{r}_{[d-1]} < i \leq r\,. \end{cases} \tag{A.10}$$

Proof. Notice that by writing $i = i' + \boldsymbol{r}_{[\lambda-1]}$ for $i' \in \{1, \ldots, r_\lambda\}$ we can rewrite the claim (A.10) into

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) = i' - 1 - \left\lfloor \frac{s_\lambda(i'-1)}{r_\lambda} \right\rfloor + \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{s}_{[\lambda-1]} + \delta_{i',1}\,. \tag{A.11}$$

Clearly,

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) \geq i' - 1 - \left\lfloor \frac{s_\lambda(i'-1)}{r_\lambda} \right\rfloor + \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{s}_{[\lambda-1]} + \delta_{i',1}$$

is true since the right hand side corresponds to the choice $\mu = \lambda$ and $M = [\lambda - 1]$ in (A.9). It is thus left to prove the reverse direction. We do this by induction in d.
For $d = 1$ the claim trivially holds. Now let $d > 1$. First we will consider the case $i \leq r_1$, ie. $\lambda = 1$.

For $\mu \in [d]$ and $M \subseteq [d] \setminus \{\mu\}$ such that $1 < i - \boldsymbol{r}_M \leq r_\mu$ we find that

$$\begin{aligned} i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &\leq i - 1 - \left\lfloor \frac{s_1(i - \boldsymbol{r}_M - 1)}{r_1} \right\rfloor - \boldsymbol{s}_M \\ &\leq i - 1 - \left\lfloor \frac{s_1(i-1)}{r_1} - \boldsymbol{s}_M \right\rfloor - \boldsymbol{s}_M \\ &\leq i - 1 - \left\lfloor \frac{s_1(i-1)}{r_1} \right\rfloor + \delta_{i,1} \,. \end{aligned} \tag{A.12}$$

The case $1 = i - \boldsymbol{r}_M$ needs some extra care. Clearly, if $M = \varnothing$ and $i = 1$ the relation (A.12) holds. If $M \neq \varnothing$ one can argue that

$$\begin{aligned} i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &= i - \boldsymbol{s}_M \\ &< i - \frac{s_1 \boldsymbol{r}_M}{r_1} \qquad \text{(A.13)} \\ &= i - \frac{s_1(i-1)}{r_1} \,. \qquad \text{(A.14)} \end{aligned}$$

In the second line we used that $\frac{r_1}{s_1} \geq \frac{r_\nu}{s_\nu}$ by assumption. This relation indeed explains the strict inequality in line (A.13) because assume we had $\boldsymbol{s}_M = \frac{s_1 \boldsymbol{r}_M}{r_1}$. Then for all $\nu \in M$ necessarily $\frac{r_1}{s_1} = \frac{r_\nu}{s_\nu}$ which since $\gcd(r_{\nu'}, s_{\nu'}) = 1$ implies that $r_1 = r_\nu$. Using that $i \leq r_1$, this in turn leads to

$$i - \boldsymbol{r}_M = i - |M|\, r_1 < 1$$

which is a contradiction to our assumption $1 = i - \boldsymbol{r}_M$. This explains the strict inequality in (A.13). Let us go on with (A.14) implying that

$$\begin{aligned} i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &\leq i - \left\lceil \frac{s_1(i-1)}{r_1} \right\rceil \\ &= i - 1 - \left\lfloor \frac{s_1(i-1)}{r_1} \right\rfloor + \delta_{r_1 | s_1(i-1)} \,. \end{aligned}$$

Now since r_1 and s_1 are coprime and $i \leq r_1$, we see that r_1 divides $s_1(i-1)$ if and only if $i = 1$ which finally implies that relation (A.12) holds even if $i - \boldsymbol{r}_M = 1$. This concludes the case $i \leq r_1$.
Now assume $i > r_1$ and choose $\lambda > 1$ and $i' \in \{1, \ldots, r_\lambda\}$ such that $i = i' + \boldsymbol{r}_{[\lambda-1]}$. We need to prove that

$$i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} \leq i - 1 - \left\lfloor \frac{s_\lambda(i'-1)}{r_\lambda} \right\rfloor - \boldsymbol{s}_{\lambda-1} + \delta_{1,i'}$$

for all $\mu \in [d]$ and $M \subseteq [d] \setminus \{\mu\}$ such that $1 < i - \boldsymbol{r}_M \leq r_\mu$. By our induction hypothesis we know that the above formula holds for $\frac{r_2}{s_2} \geq \frac{r_3}{s_3} \geq \cdots \geq \frac{r_d}{s_d}$ if we substitute $[\lambda]$ with $\{2, \ldots, \lambda\}$. Let us introduce the shorthand notation $(1, \lambda] := \{2, \ldots, \lambda\}$ for this integer interval.

We distinguish between the following cases. First, assume $1 \in M$. Then necessarily $\mu \neq 1$ and

$$\begin{aligned}
i-1-&\left\lfloor \frac{s_\mu(i-\boldsymbol{r}_M-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} \\
&= (i-r_1)-1-\left\lfloor \frac{s_\mu\big((i-r_1)-\boldsymbol{r}_{M\setminus\{1\}}-1\big)}{r_\mu} \right\rfloor - \boldsymbol{s}_{M\setminus\{1\}} + \delta_{1,(i-r_1)-\boldsymbol{r}_{M\setminus\{1\}}} + r_1 - s_1 \\
&\leq (i-r_1)-1-\left\lfloor \frac{s_\lambda\big((i-r_1)-\boldsymbol{r}_{(1,\lambda-1]}-1\big)}{r_\lambda} \right\rfloor - \boldsymbol{s}_{(1,\lambda-1]} + \delta_{1,(i-r_1)-\boldsymbol{r}_{(1,\lambda-1]}} + r_1 - s_1 \\
&= i-1-\left\lfloor \frac{s_\lambda(i'-1)}{r_\lambda} \right\rfloor - \boldsymbol{s}_{[\lambda-1]} + \delta_{1,i'}
\end{aligned}$$

where in the third line we used the induction hypothesis.
Now consider the case where $1 \notin M$. As long as $r_1 < i - \boldsymbol{r}_M$, we can reduce this case to the prior one by arguing that

$$\begin{aligned}
i-1-\left\lfloor \frac{s_\mu(i-\boldsymbol{r}_M-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &= i-1-\left\lfloor \frac{s_\mu\big(i-r_1-\boldsymbol{r}_M-1\big)}{r_\mu} + \frac{s_\mu r_1}{r_\mu} \right\rfloor - \boldsymbol{s}_M \\
&\leq i-1-\left\lfloor \frac{s_\mu\big(i-\boldsymbol{r}_{M_{+1}}-1\big)}{r_\mu} \right\rfloor - \boldsymbol{s}_{M_{+1}}
\end{aligned}$$

by writing $M_{+1} := M \cup \{1\}$. In the second line we again used that $\frac{r_1}{s_1} \geq \frac{r_\mu}{s_\mu}$.
Next, let us consider the case $1 < i - \boldsymbol{r}_M \leq r_1$ where $1 \notin M$. We cannot use the same argument as before since $i - \boldsymbol{r}_{M_{+1}} < 1$. The idea now is to construct a set $M' \subseteq M_{+1}$ such that $1 \in M'$ and $1 \leq i - \boldsymbol{r}_{M'} \leq r_\nu$ for some $\nu \notin M'$ and

$$i-1-\left\lfloor \frac{s_\mu(i-\boldsymbol{r}_M-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} \leq i-1-\left\lfloor \frac{s_\mu(i-\boldsymbol{r}_{M'}-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_{M'} + \delta_{1,i-\boldsymbol{r}_{M'}}. \quad \text{(A.15)}$$

If we succeed, we would have reduced this case again to the one considered before.
We construct M' as follows. Let us set $\alpha := \min\{\mu, \lambda\}$. Then

$$\begin{aligned}
i-1-\left\lfloor \frac{s_\mu(i-\boldsymbol{r}_M-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &\leq i-1-\left\lfloor \frac{s_\alpha(i-\boldsymbol{r}_M-1)}{r_\alpha} \right\rfloor - \boldsymbol{s}_M \\
&\leq i-1-\left\lfloor \frac{s_\alpha\big(i-\boldsymbol{r}_{M_{+1}}-1\big)}{r_\alpha} \right\rfloor - \boldsymbol{s}_{M_{+1}}
\end{aligned}$$

using that $\frac{r_\nu}{s_\nu} \geq \frac{r_{\nu'}}{s_{\nu'}}$ for $\nu \leq \nu'$ twice. Now let us label the elements of $M = \{\nu_1, \ldots, \nu_m\}$ such that $\nu_1 < \cdots < \nu_m$ and choose ℓ such that $\nu_{\ell-1} < \alpha \leq \nu_\ell$ (formally set $\nu_0 := 0$). We then find that

$$\begin{aligned}
i-1-\left\lfloor \frac{s_\alpha\big(i-\boldsymbol{r}_{M_{+1}}-1\big)}{r_\alpha} \right\rfloor - \boldsymbol{s}_{M_{+1}} &\leq i-1-\left\lfloor \frac{s_{\nu_\ell}\big(i-\boldsymbol{r}_{M_{+1}}-1\big)}{r_{\nu_\ell}} \right\rfloor - \boldsymbol{s}_{M_{+1}} \\
&= i-1-\left\lfloor \frac{s_{\nu_\ell}\big(i-\boldsymbol{r}_{M_{+1}\setminus\{\nu_\ell\}}-1\big)}{r_{\nu_\ell}} \right\rfloor - \boldsymbol{s}_{M_{+1}\setminus\{\nu_\ell\}} \quad \text{(A.16)}
\end{aligned}$$

where in the first line we used $i - \boldsymbol{r}_{M_{+1}} < 1$. Then either $1 \leq i - \boldsymbol{r}_{M_{+1}\setminus\{\nu_\ell\}} \leq r_{\nu_\ell}$ in which case we set $M' := M_{+1} \setminus \{\nu_\ell\}$ and $\nu := \nu_\ell$ or we have $i - \boldsymbol{r}_{M_{+1}\setminus\{\nu_\ell\}} < 1$ and apply the step from line one to line two in (A.16) again pulling out $\nu_{\ell+1}$. We can iteratively pull out a ν_j with $j \geq \ell$ for at most $(m-\ell+1)$ times. Let us assume that after the $(m-\ell+1)$th step we still have not found a suitable M'. Then $i - \boldsymbol{r}_{\tilde{M}} < 1$ where $\tilde{M} = \{1, \nu_1, \ldots, \nu_{\ell-1}\} \subseteq [\lambda - 1]$. This is a contradiction since

$$i - \boldsymbol{r}_{\tilde{M}} = i' + \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{r}_{\tilde{M}} = i' + \boldsymbol{r}_{[\lambda-1]\setminus\tilde{M}} \geq 0\,.$$

Consequently, the iteration must terminate meaning there exists a $j \in \{0, \ldots, m-\ell\}$ such that for $M' = M_{+1} \setminus \{\nu_\ell, \ldots, \nu_{\ell+j}\}$ and $\nu = \nu_{\ell+j}$ equation (A.15) holds and $1 \leq i - \boldsymbol{r}_{M'} \leq r_\nu$.
The last case to cover is the one where $1 = i - \boldsymbol{r}_M$ and $1 \notin M$. To cover this case, let us split M into $M_1 := M \cap [\lambda - 1]$ and $M_2 := M \setminus [\lambda - 1]$ to find

$$\begin{aligned}
i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &= i - \boldsymbol{s}_{M_2} - \boldsymbol{s}_{M_1} \\
&< i - \frac{s_\lambda \boldsymbol{r}_{M_2}}{r_\lambda} - \boldsymbol{s}_{M_1} \\
&= i - \frac{s_\lambda(i'-1)}{r_\lambda} - \frac{s_\lambda \boldsymbol{r}_{[\lambda-1]\setminus M_1}}{r_\lambda} - \boldsymbol{s}_{M_1} \\
&\leq i - \frac{s_\lambda(i'-1)}{r_\lambda} - \boldsymbol{s}_{[\lambda-1]}
\end{aligned}$$

where from the first to the second line we used the same trick as in (A.13). Therefore

$$\begin{aligned}
i - 1 - \left\lfloor \frac{s_\mu(i - \boldsymbol{r}_M - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_M + \delta_{1,i-\boldsymbol{r}_M} &\leq i - \left\lceil \frac{s_\lambda(i'-1)}{r_\lambda} \right\rceil - \boldsymbol{s}_{[\lambda-1]} \\
&\leq i - 1 - \left\lfloor \frac{s_\lambda(i'-1)}{r_\lambda} \right\rfloor - \boldsymbol{s}_{[\lambda-1]} + \delta_{i',1}
\end{aligned}$$

which was to be shown. $\square$

From now on we will always assume that $\frac{r_1}{s_1} \geq \ldots \geq \frac{r_d}{s_d}$ if not stated otherwise. The modes selected with the help of $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ via

$$H^{\sigma}_{i,k} \qquad i \in \{1, \ldots, r\},\ k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$$

shall later be shown to be an Airy structure for some certain r_μ, s_μ. For future reference, let us therefore define the following index set.

Definition A.5. We define the index set I to be

$$I := \left\{(i,k) \in \{1 \ldots r\} \times \mathbb{Z} \mid k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)\right\} \tag{A.17}$$

with $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ as in (A.10).

With equation (A.8) and (A.10) we have two different characterisations of I. The first equation tells us that $H^{\sigma}_{i,k}$ is a linear combination of J^{μ}_{a} with $a > 0$ only while the characterisation in (A.10) will be important later in order to check whether the modes satisfy the subalgebra condition. In the following we will need yet a third characterisation of I.

Proposition A.6. *If* $\gcd(r_\mu, s_\mu) = 1$ *for all* $\mu \in [d]$ *then for* $i \in \{1, \ldots, r\}$ *and* $k \in \mathbb{Z}$ *we have* $(i,k) \in I$ *if and only if there exists a* $\mu \in [d]$ *such that*

$$\Pi_\mu(i,k) - \Delta_\mu > 0$$

where $\Delta_\mu := \boldsymbol{r}_{[\mu-1]} s_\mu - r_\mu \boldsymbol{s}_{[\mu-1]}$.

Proof. Remember that

$$\Pi_\mu(i,k) = r_\mu k - (r_\mu - s_\mu)(i-1)\,.$$

First, let $c := \Pi_\mu(i,k) - \Delta_\mu > 0$. We choose $i' \in \{1, \ldots, r_\mu\}$ and $k' \in \mathbb{Z}$ such that $\Pi_\mu(i',k') = c$. Then

$$\Pi_\mu(i,k) = \Pi_\mu(i' + \boldsymbol{r}_{[\mu-1]}, k' + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]})\,.$$

Now using that $\Pi_\mu(i,k) = \Pi_\mu(j,l)$ with $i, j \in \{1, \ldots, r\}$ and $k, l \in \mathbb{Z}$ if and only if there exists an $m \in \mathbb{Z}$ such that $i = j + m r_\mu$ and $k = l + m(r_\mu - s_\mu)$, we see that (i,k) has to be of the form

$$i = i' + m r_\mu + \boldsymbol{r}_{[\mu-1]}\,, \qquad k = k' + m(r_\mu - s_\mu) + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]}$$

for some $m \in \mathbb{Z}$. Remember that we need to prove that $k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$. First assume $m \geq 0$ and choose $\lambda \geq \mu$ such that $\boldsymbol{r}_{[\lambda-1]} < i \leq \boldsymbol{r}_{[\lambda]}$. Let us write $[\mu, \lambda-1] := \{\mu, \mu+1, \ldots, \lambda-1\}$ for $\mu < \lambda$ and by convention set $[\lambda, \lambda-1] := \varnothing$. Since $c > 0$ we know that

$$k' \geq i' - 1 - \left\lfloor \frac{s_\mu(i'-1)}{r_\mu} \right\rfloor + \delta_{i',1}$$

and if $i - \boldsymbol{r}_{[\lambda-1]} \neq 1$ we thus find that

$$\begin{aligned}
\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) &= i - 1 - \left\lfloor \frac{s_\lambda(i - \boldsymbol{r}_{[\lambda-1]} - 1)}{r_\lambda} \right\rfloor - \boldsymbol{s}_{[\lambda-1]} \\
&\leq i - 1 - \left\lfloor \frac{s_\mu(i' + mr_\mu - \boldsymbol{r}_{[\mu,\lambda-1]} - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_{[\lambda-1]} \\
&\leq i - 1 - \left\lfloor \frac{s_\mu(i' - 1)}{r_\mu} \right\rfloor - ms_\mu - \boldsymbol{s}_{[\mu-1]} \\
&\leq k' + m(r_\mu - s_\mu) + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]} - \delta_{i',1} \\
&\leq k\,.
\end{aligned}$$

In the second and third line we used that $\frac{r_\nu}{s_\nu} \geq \frac{r_{\nu'}}{s_{\nu'}}$ for $\nu \leq \nu'$ and in line four we plugged in the expression for i and used (A.6). In the case $i - \boldsymbol{r}_{[\lambda-1]} = 1$ we have to be more careful due to the additional contribution from the Kronecker delta in $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$. We find

$$\begin{aligned}
k &\geq i' - 1 - \left\lfloor \frac{s_\mu(i' - 1)}{r_\mu} \right\rfloor + \delta_{i,1} + m(r_\mu - s_\mu) + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]} \\
&> \boldsymbol{r}_{[\lambda-1]} - \frac{s_\mu(i' - 1)}{r_\mu} - ms_\mu - \boldsymbol{s}_{[\mu-1]} \\
&= \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{s}_{[\lambda-1]} - \frac{s_\mu(i' - 1)}{r_\mu} - ms_\mu + \boldsymbol{s}_{[\mu,\lambda-1]} \\
&\geq \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{s}_{[\lambda-1]} - \frac{s_\mu}{r_\mu}(i' + mr_\mu - \boldsymbol{r}_{[\mu,\lambda-1]} - 1) \\
&= \boldsymbol{r}_{[\lambda-1]} - \boldsymbol{s}_{[\lambda-1]} \\
&= \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) - 1\,.
\end{aligned}$$

In this calculation we used the arguments from the prior one and identified $i - \boldsymbol{r}_{[\lambda-1]} = 1$ in line three and five. This closes the case $m \geq 0$. The case $m < 0$ is left for the reader.

Now let $(i, k) \in I$. We choose $\mu \in [d]$ and $i' \in \{1, \ldots, r_\mu\}$, $k' \in \mathbb{Z}$ such that $i = i' + \boldsymbol{r}_{[\mu-1]}$ and $k = k' + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]}$. Then we immediately find that $\Pi_\mu(i, k) = \Pi_\mu(i', k') + \Delta_\mu$ which means it suffices to show that $\Pi_\mu(i', k') > 0$. But this follows from $k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ which written out is nothing but

$$k \geq i' - 1 - \left\lfloor \frac{s_\mu(i' - 1)}{r_\mu} \right\rfloor + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]} + \delta_{i',1}$$

using Proposition A.4. Plugging $k = k' + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]}$ into the above expression, one finds that k' satisfies (A.6) which is equivalent to our claim that $\Pi_\mu(i', k') > 0$. □

Remember that we defined the matrix $\mathcal{M}_{(i,k),(\mu,a)}$ to be the collection of coefficients

$$\pi_1\left(H^\sigma_{i,k}\right) = \sum_{\mu\in[d],\, a\in\mathbb{Z}} \mathcal{M}_{(i,k),(\mu,a)}\, J^\mu_a$$

of the projection to the first degree. Its explicit values are given in equation (A.7). Now that we know which modes feature only partials in degree one, the next step is to bring the operators into the normal form (3.2) of an Airy structure, ie. to find the inverse of $\mathcal{M}$. We will do so in two steps.

Proposition A.7. *Let $\mu, \nu \in \{1, \dots, d\}$ and $a, b \in \mathbb{Z}$. Then*

$$\sum_{\substack{1 \le i \le r, k \in \mathbb{Z} \\ \Pi_\mu(i,k) = a + \Delta_\mu}} t_\mu^{1-i}\, \mathcal{M}_{(i,k),(\nu,b)} = \delta_{\mu,\nu}\, \Upsilon^\mu_{b-a} \tag{A.18}$$

where we set $\Delta_\mu := \boldsymbol{r}_{[\mu-1]} s_\mu - r_\mu \boldsymbol{s}_{[\mu-1]}$ and $\Upsilon^\mu_b \in \mathbb{C}$ is vanishing for $b < 0$ and $\Upsilon^\mu_0 \neq 0$ if and only if for all $\mu \neq \nu$ such that $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$. Explicitly, Υ^μ_b is given for $b \geq 0$ by

$$\Upsilon^\mu_b := \sum_{\substack{M \subseteq [d] \setminus \{\mu\} \\ b = \Delta_{M,\mu}}} (-1)^{|M|} \prod_{\nu \in M} \left(\frac{t_\nu}{t_\mu}\right)^{r_\nu} \tag{A.19}$$

where we introduced the notation

$$\Delta_{M,\mu} := \sum_{\nu \in M} (r_\mu s_\nu - r_\nu s_\mu) - \sum_{\nu=1}^{\mu-1} (r_\mu s_\nu - r_\nu s_\mu)\,.$$

Proof. To simplify the notation set

$$A_{(\mu,a),(\nu,b)} := \sum_{\substack{1 \le i \le r, k \in \mathbb{Z} \\ \Pi_\mu(i,k) = a + \Delta_\mu}} t_\mu^{1-i}\, \mathcal{M}_{(i,k),(\nu,b)}\,. \tag{A.20}$$

First, let us rewrite the sum constraint. Let $i' \in \{1, \dots, r_\mu\}$ and $k' \in \mathbb{Z}$ such that $\Pi_\mu(i', k') = a$. We thus obtain that

$$a + \Delta_\mu = \Pi_\mu(i' + \boldsymbol{r}_{[\mu-1]}, k' + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]})\,.$$

Now using that $\Pi_\mu(i,k) = \Pi_\mu(j,l)$ with $i, j \in \{1, \dots, r\}$ and $k, l \in \mathbb{Z}$ if and only if there exists an $m \in \mathbb{Z}$ such that $i = j + m r_\mu$ and $k = l + m(r_\mu - s_\mu)$ we see that (A.20) can be rewritten as

$$A_{(\mu,a),(\nu,b)} = \sum_{\substack{m \in \mathbb{Z} \\ 1 \le i_{(m)} \le r}} t_\mu^{1-i_{(m)}}\, \mathcal{M}_{(i_{(m)},k_{(m)}),(\nu,b)} \tag{A.21}$$

where we used the shorthand notation

$$i_{(m)} := i' + m r_\mu + \boldsymbol{r}_{[\mu-1]}\,, \qquad k_{(m)} = k' + m(r_\mu - s_\mu) + \boldsymbol{r}_{[\mu-1]} - \boldsymbol{s}_{[\mu-1]}\,.$$

Now let us plug in the explicit value of $\mathcal{M}$ given in (A.7) to obtain

$$A_{(\mu,a),(\nu,b)} = \sum_{\substack{m \in \mathbb{Z} \\ 1 \le i_{(m)} \le r}} \sum_{\substack{M \subseteq [d] \setminus \{\nu\} \\ 1 \le i_{(m)} - \boldsymbol{r}_M \le r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m)}-1} \left(\prod_{\lambda \in M} \left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b, \Pi_\nu(i_{(m)} - \boldsymbol{r}_M, k_{(m)} - \boldsymbol{r}_M + \boldsymbol{s}_M)} \tag{A.22}$$

Let us first consider the case $\nu \neq \mu$. We can split the sum over the set M into the cases where $\mu \in M$

and $\mu \notin M$ leading to

$$
\begin{aligned}
&A_{(\mu,a),(\nu,b)} \\
&= \sum_{\substack{m\in\mathbb{Z} \\ 1\leq i_{(m)}\leq r}} \sum_{\substack{M\subseteq[d]\setminus\{\nu,\mu\} \\ 1\leq i_{(m)}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m)}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\,\Pi_\nu(i_{(m)}-\boldsymbol{r}_M,k_{(m)}-\boldsymbol{r}_M+\boldsymbol{s}_M)} \\
&\quad + \sum_{\substack{m\in\mathbb{Z} \\ 1\leq i_{(m)}\leq r}} \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\setminus\{\nu\} \\ 1\leq i_{(m)}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m)}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\,\Pi_\nu(i_{(m)}-\boldsymbol{r}_M,k_{(m)}-\boldsymbol{r}_M+\boldsymbol{s}_M)} \\
&= - \sum_{\substack{m\in\mathbb{Z} \\ 1\leq i_{(m)}\leq r}} \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\setminus\{\nu\} \\ 1\leq i_{(m+1)}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m+1)}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\,\Pi_\nu(i_{(m+1)}-\boldsymbol{r}_M,k_{(m+1)}-\boldsymbol{r}_M+\boldsymbol{s}_M)} \\
&\quad + \sum_{\substack{m\in\mathbb{Z} \\ 1\leq i_{(m)}\leq r}} \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\setminus\{\nu\} \\ 1\leq i_{(m)}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m)}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\,\Pi_\nu(i_{(m)}-\boldsymbol{r}_M,k_{(m)}-\boldsymbol{r}_M+\boldsymbol{s}_M)}
\end{aligned}
\tag{A.23}
$$

where in the third line we used that $i_{(m)} - \boldsymbol{r}_M = i_{(m+1)} - \boldsymbol{r}_{M\cup\{\mu\}}$ for $\mu \notin M$ and a similar equality for $k_{(m)}$ and thus substituted M by $M \cup \{\mu\}$ in the first sum which leads to an overall minus sign for this sum. From the last line we see that both sums cancel each other except for possible boundary terms $m \in \{m_{\min}, m_{\max}\}$ where

$$
m_{\min} = \left\lceil \frac{1}{r_\mu}\left(1 - i' - \boldsymbol{r}_{[\mu-1]}\right)\right\rceil , \quad m_{\max} = \left\lfloor \frac{1}{r_\mu}\left(\boldsymbol{r}_{\{\mu\ldots d\}} - i'\right)\right\rfloor .
$$

More precisely, we get

$$
\begin{aligned}
&A_{(\mu,a),(\nu,b)} \\
&= - \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\setminus\{\nu\} \\ 1\leq i_{(m_{\max}+1)}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m_{\max}+1)}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\Pi_\nu(i_{(m_{\max}+1)}-\boldsymbol{r}_M,k_{(m_{\max}+1)}-\boldsymbol{r}_M+\boldsymbol{s}_M)} \\
&\quad + \sum_{\substack{\{\mu\}\subseteq M\subseteq[d]\setminus\{\nu\} \\ 1\leq i_{(m_{\min})}-\boldsymbol{r}_M\leq r_\nu}} (-1)^{|M|} \left(\frac{t_\nu}{t_\mu}\right)^{i_{(m_{\min})}-1} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\nu}\right)^{r_\lambda}\right) \delta_{b,\Pi_\nu(i_{(m_{\min})}-\boldsymbol{r}_M,k_{(m_{\min})}-\boldsymbol{r}_M+\boldsymbol{s}_M)} .
\end{aligned}
$$

However, also these terms vanish due to the sum constraints. Note that for all $M \subseteq [d] \setminus \{\nu\}$ we have

$$
i_{(m_{\max}+1)} - \boldsymbol{r}_M > r - \boldsymbol{r}_M = \sum_{\lambda\notin M} r_\lambda \geq r_\nu
$$

which implies that the first sum is empty. Analogously, one can show that also the second sum vanishes. This finally leads to the conclusion that $A_{(\mu,a),(\nu,b)} = 0$ for $\nu \neq \mu$.
Let us now consider the case $\nu = \mu$. In this case the expression for $A_{(\mu,a),(\nu,b)}$ given by equation

(A.22) simplifies significantly to

$$\begin{aligned} A_{(\mu,a),(\mu,b)} &= \sum_{\substack{m\in\mathbb{Z}\\ 1\le i_{(m)}\le r}} \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ 1\le i_{(m)}-\boldsymbol{r}_M\le r_\mu}} (-1)^{|M|} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\mu}\right)^{r_\lambda}\right) \delta_{b,\Pi_\mu(i_{(m)}-\boldsymbol{r}_M,k_{(m)}-\boldsymbol{r}_M+\boldsymbol{s}_M)} \\ &= \sum_{M\subseteq[d]\setminus\{\mu\}} (-1)^{|M|} \left(\prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\mu}\right)^{r_\lambda}\right) \delta_{b,a+\Delta_{M,\mu}} \end{aligned}$$

by writing

$$\Delta_{M,\mu} = \sum_{\lambda\in M}(r_\mu s_\lambda - r_\lambda s_\mu) - \sum_{\lambda=1}^{\mu-1}(r_\mu s_\lambda - r_\lambda s_\mu)\,.$$

Since by assumption $\frac{r_1}{s_1}\ge\frac{r_2}{s_2}\ge\ldots\ge\frac{r_d}{s_d}$, this sum satisfies $\Delta_{M,\mu}\ge 0$ for all $M\subseteq[d]$. Thus, we indeed find that

$$A_{(\mu,a),(\mu,b)} = \delta_{\mu,\nu}\,\Upsilon^\mu_{b-a}$$

with $\Upsilon^\mu_b = 0$ for $b<0$ and

$$\Upsilon^\mu_b = \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ b=\Delta_{M,\mu}}} (-1)^{|M|} \prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\mu}\right)^{r_\lambda}$$

for $b\ge 0$. Especially, for $b=0$ we find that

$$\begin{aligned} \Upsilon^\mu_0 &= \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ \forall\lambda\in M:\,\frac{r_\mu}{s_\mu}=\frac{r_\lambda}{s_\lambda}}} (-1)^{|M|} \prod_{\lambda\in M}\left(\frac{t_\lambda}{t_\mu}\right)^{r_\lambda} \\ &= \prod_{\substack{\lambda\in[d]\setminus\{\mu\}\\ \frac{r_\mu}{s_\mu}=\frac{r_\lambda}{s_\lambda}}} \left(1-\left(\frac{t_\lambda}{t_\mu}\right)^{r_\lambda}\right) \end{aligned}$$

which is non-zero if and only if $t_\mu^{r_\mu}\ne t_\lambda^{r_\lambda}$ for all $\mu\ne\lambda$ such that $\frac{r_\mu}{s_\mu}=\frac{r_\lambda}{s_\lambda}$. Here, we used that r_λ and s_λ are coprime for all λ which implies that necessarily $r_\mu=r_\lambda$ and $s_\mu=s_\lambda$ if $\frac{r_\mu}{s_\mu}=\frac{r_\lambda}{s_\lambda}$. □

The proof of the following proposition contains a small gap which needs to be closed in the future. Since for a large number of special cases the statement can be proven to be true, I strongly conjecture that the gap can as well be closed in the most general case with enough effort. Therefore, I postpone the full proof of the following proposition to [18] and present those parts of the proof which have already been worked out completely.

Proposition A.8. *Assume r_μ and s_μ are coprime for all $\mu\in[d]$ and $t_\mu^{r_\mu}\ne t_\nu^{r_\nu}$ for all $\mu\ne\nu$ with $\frac{r_\mu}{s_\mu}=\frac{r_\nu}{s_\nu}$. Then the modes*

$$H^\sigma_{i,k} \qquad (i,k)\in I$$

corresponding to the index set I defined in (A.17) *satisfy the degree one condition* (3.2) *of a higher quantum Airy structure up to a change of coordinates.*

(Incomplete) Proof. Let us define

$$\mathcal{N}_{(\mu,a),(i,k)} := t_\mu^{1-i} \sum_{m\geq 0} (-1)^m \left(\Upsilon_0^\mu\right)^{-m-1} \sum_{\substack{\gamma_1,\ldots,\gamma_m>0 \\ \sum_\ell \gamma_\ell = \Pi_\mu(i,k)-\Delta_\mu - a}} \prod_{\ell=1}^m \Upsilon^\mu_{\gamma_\ell} \tag{A.24}$$

for $\mu \in \{1,\ldots,d\}$, $a > 0$ and $(i,k) \in I$ where we again set $\Delta_\mu = \boldsymbol{r}_{[\mu-1]} s_\mu - r_\mu \boldsymbol{s}_{[\mu-1]}$ and $\Upsilon^\mu_{\delta_\ell}$ as in (A.19). For $m = 0$ one has to interpret the last sum as $\delta_{\Pi_\mu(i,k),a+\Delta_\mu}$. Note that for fixed $(\mu,a),(i,k)$ the sum on the right hand side of (A.24) is well defined since due to the sum constraint on the γ_ℓ only finitely many terms contribute. Using Proposition A.7, it is straightforward to see that this matrix satisfies

$$\sum_{(i,k)\in I} \mathcal{N}_{(\mu,a),(i,k)}\, \mathcal{M}_{(i,k),(\nu,b)} = \delta_{\mu,\nu}\, \delta_{a,b}\,. \tag{A.25}$$

Indeed, due to Proposition A.7 we already know that

$$\sum_{\substack{1\leq i\leq r,k\in\mathbb{Z} \\ \Pi_\mu(i,k)=c+\Delta_\mu}} t_\mu^{1-i}\, \mathcal{M}_{(i,k),(\nu,b)} = \delta_{\mu,\nu}\, \Upsilon^\mu_{b-c}$$

for $b, c > 0$. Since $c > 0$ using Proposition A.6 we can substitute the sum over $1 \leq i \leq r$ and $k \in \mathbb{Z}$ with just a sum over $(i,k) \in I$ and arrive at

$$\sum_{\substack{(i,k)\in I \\ \Pi_\mu(i,k)=c+\Delta_\mu}} t_\mu^{1-i}\, \mathcal{M}_{(i,k),(\nu,b)} = \delta_{\mu,\nu}\, \Upsilon^\mu_{b-c} \tag{A.26}$$

If we now use that

$$\delta_{a,b} = \sum_{c>0} \Upsilon^\mu_{b-c} \sum_{m\geq 0} (-1)^m \left(\Upsilon_0^\mu\right)^{-m-1} \sum_{\substack{\gamma_1,\ldots,\gamma_m>0 \\ \sum_\ell \gamma_\ell = c-a}} \prod_{\ell=1}^m \Upsilon^\mu_{\gamma_\ell}\,,$$

then multiplying both sides of (A.26) with

$$\sum_{m\geq 0} (-1)^m \left(\Upsilon_0^\mu\right)^{-m-1} \sum_{\substack{\gamma_1,\ldots,\gamma_m>0 \\ \sum_\ell \gamma_\ell = c-a}} \prod_{\ell=1}^m \Upsilon^\mu_{\gamma_\ell}$$

and summing over $c > 0$ we obtain (A.25). This proves that $\mathcal{N}$ is a left inverse of $\mathcal{M}$.
On the other hand, to prove that $\mathcal{N}$ is also a right inverse, ie. that

$$\sum_{\substack{\mu\in[d] \\ a>0}} \mathcal{M}_{(i,k),(\mu,a)}\, \mathcal{N}_{(\mu,a),(j,l)} = \delta_{i,j}\, \delta_{k,l}$$

for all $(i,k),(j,l) \in I$ needs more effort. Indeed, this is the gap which was mentioned in the introduction of this proposition. So far, I did not succeed to prove that $\mathcal{N}$ is also a right inverse. However, since the statement is correct for all special cases I considered so far, I am convinced that

this gap in the proof can be closed in the future.
Let us assume that $\mathcal{M}$ has a two sided inverse. We deduce that

$$\pi_1\Big(H^\sigma_{i,k}\Big) = \sum_{\mu\in[d],\, a>0} \mathcal{M}_{(i,k),(\mu,a)}\ \hbar\partial_{x^\mu_a}$$

is a basis of the linear span $\mathbb{C}\langle\{\hbar\partial_{x^\mu_a}\}_{\mu\in[d],\, a>0}\rangle$. We can thus perform a change of basis

$$y_{(i,k)} := \sum_{\substack{\mu\in[d]\\ a>0}} \mathcal{N}_{(\mu,a),(i,k)}\ x^\mu_a \qquad (i,k)\in I$$

after which $H^\sigma_{i,k} = \hbar\partial_{y_{(i,k)}} + o(2)$ for all $(i,k)\in I$ satisfies the degree one condition (3.2) of a higher quantum Airy structure. Here, we implicitly assumed that the degree zero component of $H^\sigma_{i,k}$ is vanishing. Let us quickly convince ourselves that this is indeed true for $(i,k)\in I$.
Following the combinatorics in the proof of Lemma A.2, one finds that the projection to degree zero of $H^\sigma_{i,k}$ for arbitrary i and k is

$$\pi_0\Big(H^\sigma_{i,k}\Big) = \sum_{\substack{M\subseteq[d]\\ i=\boldsymbol{r}_M,\, k=\boldsymbol{r}_M-\boldsymbol{s}_M-1}} (-1)^{|M|} \prod_{\nu\in M} t_\nu^{r_\nu}\,.$$

Therefore, it suffices to show that for all $M\subseteq[d]$ we have $(\boldsymbol{r}_M,\, \boldsymbol{r}_M-\boldsymbol{s}_M-1)\notin I$. Thus, let $M\subseteq[d]$ and choose $\mu\in\{1,\dots,d\}$ and $i'\in\{1,\dots,r_\mu\}$ such that $\boldsymbol{r}_M = i' + \boldsymbol{r}_{[\mu-1]}$. Then by Proposition A.4 we have

$$\begin{aligned}
\mathfrak{d}_{\mathbf{r},\mathbf{s}}(\boldsymbol{r}_M) &= \boldsymbol{r}_M - 1 - \left\lfloor \frac{s_\mu(\boldsymbol{r}_M - \boldsymbol{r}_{[\mu-1]} - 1)}{r_\mu} \right\rfloor - \boldsymbol{s}_{[\mu-1]} + \delta_{i',1} \\
&\geq \boldsymbol{r}_M - 1 - \left\lfloor \boldsymbol{s}_M - \boldsymbol{s}_{[\mu-1]} - \frac{s_\mu}{r_\mu} \right\rfloor - \boldsymbol{s}_{[\mu-1]} + \delta_{i',1} \\
&> \boldsymbol{r}_M - \boldsymbol{s}_M - 1
\end{aligned} \tag{A.27}$$

where from the first to the second line we used that $\frac{r_1}{s_1}\geq\cdots\geq\frac{r_d}{s_d}$. Consequently by definition of I, we deduce that $(\boldsymbol{r}_M,\, \boldsymbol{r}_M-\boldsymbol{s}_M-1)\notin I$ which finally proves that $\pi_0(H^\sigma_{i,k})=0$ for all $(i,k)\in I$. □

A.1.2 The subalgebra condition

Having proven the degree one condition for the modes

$$H^\sigma_{i,k} \qquad (i,k)\in I$$

with index set I as defined in (A.17), we need to check whether these modes generate a graded Lie subalgebra as demanded for an Airy structure. In order to do so, we use Lemma 4.9 stating that if I is *induced by a descending partition* then $\{H^\sigma_{i,k}\}_{(i,k)\in I}$ generate a graded Lie subalgebra. By this we mean that there exists a descending partition $\lambda=(\lambda_1,\dots,\lambda_p)\vdash r$ such that $I = I_\lambda\setminus\{(1,0)\}$ where

$$I_\lambda := \Big\{(i,k)\in\{1,\dots,r\}\times\mathbb{Z} \mid 1\leq i\leq r,\ k\geq i-\lambda(i)\Big\}$$

and we set

$$\lambda(i) := \max\left\{ m \;\middle|\; \sum_{j=1}^{m} \lambda_j \geq i \right\}.$$

Explicitly, in our case at hand this means that there must exist a $\lambda \vdash r$ such that $\lambda(i) = i - \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i) + \delta_{i,1}$. Writing $i = i' + \boldsymbol{r}_{[\nu-1]}$ for $i' \in \{1, \ldots, r_\nu\}$ and again assuming that $\frac{r_1}{s_1} \geq \cdots \geq \frac{r_d}{s_d}$, we can write out $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ using Proposition A.4 and obtain

$$\lambda(i) = 1 + \left\lfloor \frac{s_\nu(i'-1)}{r_\nu} \right\rfloor + \boldsymbol{s}_{[\nu-1]} - \delta_{i',1}\,\delta_{\nu>1}\,. \tag{A.28}$$

The case $d = 1$ was discussed in [7] resulting in the following correspondence.

Proposition A.9. [7, Prop. B.1] *Let $r > 1$ and $s \in \{1, \ldots, r+1\}$ be coprime. Then there exists a descending partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ such that*

$$\lambda(i') = 1 + \left\lfloor \frac{s(i'-1)}{r} \right\rfloor \qquad i' \in \{1, \ldots, r\} \tag{A.29}$$

if and only if $r = \pm 1 \mod s$. In this case, λ is given by

$$\lambda_1 = \cdots = \lambda_{r''} = r' + 1, \qquad \lambda_{r''+1} = \cdots = \lambda_p = r', \qquad p = s - \delta_{s,r+1} \tag{A.30}$$

writing $r = r's + r''$ with $r'' \in \{1, s-1\}$. In particular, we have $\lambda = (r)$ for $s = 1$ and $\lambda = (1^r)$ for $s = r+1$.

This proposition has the following generalisation to the case $d > 1$.

Proposition A.10. *Let $d > 1$. Given $\frac{r_1}{s_1} \geq \cdots \geq \frac{r_d}{s_d}$ with r_μ and s_μ coprime for all μ, then there exists a descending partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of $r = r_1 + \cdots + r_d$ such that* (A.28) *is satisfied if and only if the following holds true*

(i) $s_1 \in \{1, \ldots, r_1 + 1\}$ *and* $r_1 = -1 \mod s_1$,

(ii) $s_\mu = 1$ *for all* $\mu \in \{2, \ldots, d-1\}$,

(iii) If $r_d > 1$ *we have* $s_d \in \{1, \ldots, r_d\}$ *and* $r_d = +1 \mod s_d$.

In this case λ is given by

$$\lambda = \begin{cases} ((r_1'+1)^{s_1}, r_2, r_3, \ldots, r_{d-1}, {r_d'}^{s_d}) & , r_d \neq 1 \\ ((r_1'+1)^{s_1}, r_2, r_3, \ldots, r_{d-1}) & , r_d = 1 \end{cases} \tag{A.31}$$

where $r'_\mu := \lfloor r_\mu / s_\mu \rfloor$.

Proof. First, let us prove that (i), (ii) and (iii) are necessary such that

$$\mu(i) := 1 + \left\lfloor \frac{s_\nu(i'-1)}{r_\nu} \right\rfloor + \boldsymbol{s}_{[\nu-1]} - \delta_{i',1}\,\delta_{\nu>1} \tag{A.32}$$

admits a descending partition λ such that $\mu(i) = \lambda(i)$. As always, we write $i = i' + \boldsymbol{r}_{[\nu-1]}$. We begin by presenting a few general statements regarding the construction of λ following the lines of [7, Prop. B.1].

Assume $\mu : \{1, \ldots, r\} \to \mathbb{N}$ is weakly increasing with $\mu(1) = 1$ and

$$\mu(i+1) - \mu(i) \in \{0, 1\} . \tag{A.33}$$

Let $\kappa_1 < \cdots < \kappa_{p-1}$ be a complete list of jumps of μ in the sense that

$$\mu(\kappa_j + 1) - \mu(\kappa_j) = 1 .$$

Additionally, set $\kappa_0 := 0$ and $\kappa_p := r$. If we further define

$$\lambda_j := \kappa_j - \kappa_{j-1} \qquad j \in \{1, \ldots, t\}$$

then by construction

$$\mu(i) = \max \left\{ m \;\middle|\; \sum_{j=1}^{m} \lambda_j \geq i \right\} .$$

Notice that in general the partition $\lambda := (\lambda_1, \ldots, \lambda_p)$ must not be decreasing. What one should take away from this construction is, that λ_j measures the length of the interval between the $(j-1)$-th and j-th jump of μ.

Let us get back to our case at hand where μ is given by (A.32). The first constraint on $r_1, \ldots, r_d$ and $s_1, \ldots, s_d$ comes from the requirement that $\mu(i+1) - \mu(i) \in \{0, 1\}$. At the value $i = 1 + \boldsymbol{r}_{[\nu-1]}$ with $\nu > 1$, we find for $r_\nu > 1$ that

$$\begin{aligned}\mu(2 + \boldsymbol{r}_{[\nu-1]}) - \mu(1 + \boldsymbol{r}_{[\nu-1]}) &= 1 + \left\lfloor \frac{s_\nu}{r_\nu} \right\rfloor + \boldsymbol{s}_{[\nu-1]} - \delta_{2,1} - \big(1 + \boldsymbol{s}_{[\nu-1]} - \delta_{1,1}\big) \\ &= 1 + \left\lfloor \frac{s_\nu}{r_\nu} \right\rfloor \end{aligned} \tag{A.34}$$

implying that necessarily $s_\nu < r_\nu$. For $r_\nu = 1$ and $1 < \nu < d$, one finds that

$$\mu(2 + \boldsymbol{r}_{[\nu-1]}) - \mu(1 + \boldsymbol{r}_{[\nu-1]}) = s_\nu$$

implying $s_\nu = 1$. Notice that we do not get any restrictions for s_d in the case $r_d = 1$ since μ actually does not depend on s_d.

Regarding s_1, by writing $s_1 = s_1' \, r_1 + s_1''$ with $s_1'' \in \{0, \ldots, r_1 - 1\}$ we find for $i < r_1$ that

$$\mu(i+1) - \mu(i) = s_1' + \left\lfloor \frac{s_1'' \, i}{r_1} \right\rfloor - \left\lfloor \frac{s_1'' \, (i-1)}{r_1} \right\rfloor .$$

This implies that either $s_1' = 0$ in which case $s_1'' \in \{0, \ldots, r_1 - 1\}$ may be arbitrary or $s_1' = 1$ and

$$\left\lfloor \frac{s_1'' \, i}{r_1} \right\rfloor \leq \left\lfloor \frac{s_1'' \, (i-1)}{r_1} \right\rfloor$$

which can only hold if $s_1'' \in \{0,1\}$ since otherwise for increasing i the right hand side jumps earlier from zero to one than the left hand side which violates the inequality. The two cases translate into the constraint that $s_1 \leq r_1 + 1$. If $r_1 = 1$, the last statement is also true since in this case

$$\mu(r_1+1) - \mu(r_1) = \left\lceil \frac{s_1}{r_1} \right\rceil - 1$$

implies that $s_1 \in \{1,2\}$. To summarize, the demand that $\mu(i+1) - \mu(i) \in \{0,1\}$ gives us the constraints

(I) $s_1 \in \{1,\dots,r_1+1\}$,

(II) $s_\mu \in \{1,\dots,r_\mu - 1 + \delta_{r_\mu,1}\}$ for $\mu \in \{2,\dots,d-1\}$,

(III) $s_d \in \{1,\dots,r_d\}$ if $r_d \neq 1$.

In the case $s_1 = r_1 + 1$ one can argue that the above conditions already imply that (i) to (iii) hold. Indeed, since we assume that $\frac{r_1}{s_1} \geq \frac{r_\mu}{s_\mu}$ for all $\mu \in \{2,\dots,d\}$ the case $s_1 = r_1 + 1$ forces $\frac{r_\mu}{s_\mu} < 1$ for all $\mu \geq 2$. Thus (II) forces $d = 2$ and due to (III) necessarily $r_d = 1$. This case is clearly covered in (i) to (iii).

Now assume $s_1 \leq r_1$. By assumption there exists a descending partition $\lambda = (\lambda_1,\dots,\lambda_p)$ with $\mu(i) = \lambda(i)$. In order to find a description of λ in terms of (A.10), notice the similarity between (A.32) and (A.29) for fixed ν and $i' \in \{1,\dots,r_\nu\}$. Except for the constant shift $\boldsymbol{s}_{[\nu]}$ and the $\delta_{i',1}$ the two maps coincide, which means that except for the transition values $i = \boldsymbol{r}_{[\nu-1]} \to \boldsymbol{r}_{[\nu-1]} + 1$ they jump at the same value i'. At the transition points we find

$$\mu(1+\boldsymbol{r}_{[\nu]}) - \mu(\boldsymbol{r}_{[\nu]}) = \left\lceil \frac{s_\nu}{r_\nu} \right\rceil - 1 = 0$$

for all $\nu < d$ and if $r_{\nu+1} > 1$ we moreover have

$$\mu(2+\boldsymbol{r}_{[\nu]}) - \mu(1+\boldsymbol{r}_{[\nu]}) = 1$$

as discussed in (A.34). This means that if we let $\lambda^\mu = (\lambda_1^\mu,\dots,\lambda_{s_\mu}^\mu)$ denote the partition satisfying

$$\lambda^\mu(i') = 1 + \left\lfloor \frac{s_\mu(i'-1)}{r_\mu} \right\rfloor \qquad i' \in \{1,\dots,r_\mu\}$$

and set $\lambda^\mu := (1)$ if $r_\mu = 1$ then the partition

$$\begin{aligned}
\lambda = \big(\lambda_1^1 \quad &, \lambda_2^1, \dots, \lambda_{s_1-1}^1, \lambda_{s_1}^1 + 1, \\
\lambda_1^2 - 1 &, \lambda_2^2, \dots, \lambda_{s_2-1}^2, \lambda_{s_2}^2 + 1, \\
\lambda_1^3 - 1 &, \lambda_2^3, \dots, \lambda_{s_3-1}^3, \lambda_{s_3}^3 + 1, \\
&\vdots \\
\lambda_1^d - 1 &, \lambda_2^d, \dots, \lambda_{s_d-1}^d, \lambda_{s_d}^d\big)
\end{aligned}$$

is the one satisfying $\mu(i) = \lambda(i)$. Note that in case $s_\mu = 1$ the μth line

$$(\ldots, \lambda_1^\mu - 1, \lambda_2^\mu, \ldots, \lambda_{s_\mu - 1}^\mu, \lambda_{s_\mu}^\mu + 1, \ldots)$$

must be replaced by $(\ldots, r_\mu, \ldots)$. Since we assume λ to be a descending partition necessarily λ^μ needs to be descending as well for all $\mu \in \{1, \ldots, d\}$. Consequently, Proposition A.9 tells us that $r_\mu = \pm 1 \mod s_\mu$. Note that at the transition between two parts of the partition the constraint $\lambda_{s_\mu}^\mu + 1 \geq \lambda_1^{\mu+1} - 1$ is always satisfied, because if we consider the explicit value of λ_j^μ given in (A.30), we find that

$$\lambda_{s_\mu}^\mu + 1 = \left\lfloor \frac{r_\mu}{s_\mu} \right\rfloor + 1 \geq \left\lfloor \frac{r_{\mu+1}}{s_{\mu+1}} \right\rfloor + 1 > \left\lfloor \frac{r_{\mu+1}}{s_{\mu+1}} \right\rfloor + 1 - 1 = \lambda_1^{\mu+1} - 1$$

for $s_\mu, s_{\mu+1} \neq 1$. Here, we used that by assumption $\frac{r_\mu}{s_\mu} \geq \frac{r_\nu}{s_\nu}$ for all $\mu \leq \nu$. There are similar arguments for the case where s_μ or $s_{\mu+1}$ is equal one.
One obtains further restrictions on the choice of s_μ considering the constraint that

$$\lambda_1^{\mu+1} - 1 \geq \lambda_2^{\mu+1}, \qquad \lambda_{s_\mu - 1}^\mu \geq \lambda_{s_\mu}^\mu + 1 \tag{A.35}$$

for all $\mu < d$. Assume for example that $s_1 > 1$ and $r_1 = +1 \mod s_1$, ie. $r_1 = r_1' s_1 + 1$. Then (A.30) tells us that $\lambda_1^1 = r_1' + 1$ and $\lambda_2^1 = \ldots = \lambda_{s_1}^1 = r_1'$. But this contradicts (A.35) for $\mu = 1$ since $\lambda_{s_1 - 1}^1 < \lambda_{s_1}^1 + 1$. Consequently we are left with $r_1 = -1 \mod s_1$ which together with (I) implies (i). It is straightforward to see that condition (A.35) checked for arbitrary ν induces (ii) and (iii).

Now let us briefly argue why (i), (ii) and (iii) are sufficient such that $\mu(i)$ is induced by a descending partition. Following the proceeding analysis of the necessary conditions, it is straightforward to see that if $r_1, \ldots, r_d$ and $s_1, \ldots, s_d$ satisfy (i) to (iii), the partition (A.31) corresponding to the diagram (4.24) is descending and indeed satisfying $\mu(i) = \lambda(i)$. □

We now have everything at hand to prove Theorem 4.19.

Proof of Theorem 4.19. Without loss of generality assume that $s_2 = \ldots = s_{d-1} = 1$ and $\frac{r_1}{s_1} \geq r_2 \geq \ldots \geq r_{d-1} \geq \frac{r_1}{s_1}$. Notice that the selected modes

$$H_{i,k}^\sigma \qquad i \in \{1, \ldots, r\},\ k \geq i - \lambda(i) + \delta_{i,1} \tag{A.36}$$

with the partition

$$\lambda = \begin{cases} ((r_1' + 1)^{s_1}, r_2, r_3, \ldots, r_{d-1}, r_d'^{\,s_d}) & , r_d \neq 1 \\ ((r_1' + 1)^{s_1}, r_2, r_3, \ldots, r_{d-1}) & , r_d = 1 \end{cases}$$

exactly correspond to the modes $\{H_{i,k}^\sigma\}_{(i,k) \in I}$ where I is the index set defined in (A.17) by performing the identification of index sets via Proposition A.10. Thus, Proposition A.8 tells us that after a change of basis the modes (A.36) satisfy the degree one condition.
Since by assumption

$$H_{1,0}^\sigma = W_{1,0}^\sigma = J_0^1 + \ldots + J_0^d = \hbar^{1/2}\,(Q^1 + \ldots + Q^d) = 0\,,$$

the modes (A.36) satisfy the subalgebra condition if the modes $\{H_{i,k}^\sigma\}_{(i,k) \in I_\lambda}$ do. Here I_λ is defined as in (4.6). Now using that $H_{i,k}^\sigma$ is obtained from $W_{i,k}^\sigma$ via conjugation, the claim immediately follows from Lemma 4.9 since conjugation preserves the Lie bracket. □

A.1.3 The fixed point case

Before we restrict our analysis to the fixed point case, let us consider modes $H^{\sigma}_{i,k}$ where we shift in every J^{μ} except for one μ. After a relabelling, we can assume we are not shifting J^{d}. To be more precise, choose $s_1, \dots, s_{d-1} > 0$ and $t_1, \dots, t_{d-1} \in \mathbb{C}^*$ and set

$$H^{\sigma}_{i,k} := \hat{T}\, W^{\sigma}_{i,k}\, \hat{T}^{-1}\,, \qquad \hat{T} := \prod_{\mu=1}^{d-1} \exp\left(-\frac{J^{\mu}_{s_\mu}}{\hbar\, s_\mu} r_\mu t_\mu\right) . \tag{A.37}$$

Of course, the case considered in Theorem 4.20 is the one where $r_d = 1$. Analysing the degree one component of $H^{\sigma}_{i,k}$ in this more general setting $r_d \geq 1$, we will see that the case $r_d = 1$ is the only one in which we can expect the modes (A.37) to form an Airy structure.

Proposition A.11. *Let r_μ and s_μ be coprime for all $\mu \in [d-1]$. Then for $i \in \{1, \dots, r\}$ and $k \in \mathbb{Z}$*

$$\begin{aligned}\pi_1\Big(H^{\sigma}_{i,k}\Big) = &\sum_{\mu \in [d-1]} \sum_{\substack{M \subseteq [d-1] \setminus \{\mu\} \\ 1 \leq i - \boldsymbol{r}_M \leq r_\mu}} (-1)^{|M|}\, t_\mu^{i-1} \left(\prod_{\nu \in M} \left(\frac{t_\nu}{t_\mu}\right)^{r_\nu}\right) J^{\mu}_{\Pi_\mu(i - \boldsymbol{r}_M, k - \boldsymbol{r}_M + \boldsymbol{s}_M)} \\ &+ \sum_{\substack{M \subseteq [d-1] \\ i = 1 + \boldsymbol{r}_M}} (-1)^{|M|} \left(\prod_{\nu \in M} t_\nu^{r_\nu}\right) J^{d}_{r_d(k - \boldsymbol{r}_M + \boldsymbol{s}_M)} \,. \end{aligned} \tag{A.38}$$

Proof. Notice that the first line in equation (A.38) coincides with the formula for the degree one part for the modes shifted in all variables J^{μ} as derived in (A.6) upon substituting $[d]$ with $[d-1]$. This is due to the fact that in both cases the combinatorics how one can produce a single J^{μ}_{*} are the same except that in our case at hand we can only substitute $J^{\mu}_{-s_\mu}$ for $\mu \in \{1, \dots, d-1\}$ with a factor $(-r_\mu t_\mu)$. We therefore refer to Lemma A.2 explaining the first line of equation (A.38).

Let us now explain the second line of (A.38) using the same arguments as in Lemma A.2. Consider equations (A.1) and (A.2). By performing the shift (A.37) we get a contribution from $M \subseteq [d]$ to the coefficient in front of J^{d}_{*} only if $d \in M$ and $j_\nu = 0$ for all $\nu \in M$. Moreover, necessarily $i_d = 1$ and $p^{\nu}_{l} = -s_\nu$ for all $\nu \neq d$ and $l \in \mathbb{Z}$. The sum constraint on the p^{ν}s for $\nu \neq d$ then becomes

$$r_\nu\,(k_\nu + 1) = (r_\nu - s_\nu)\, i_\nu$$

forcing $i_\nu = r_\nu$ since $\gcd(r_\nu, s_\nu) = 1$. In the case $\nu = d$ we get

$$p^{d}_{1} = r_d\, k_d - (r_d - s_d)(i_d - 1)\,.$$

Inserting the sum constraint $k + 1 - |M| = \sum_\mu k_\mu$ and $i_d = 1$ then results in

$$p^{d}_{1} = r_d\left(k - \boldsymbol{r}_{M'_d} + \boldsymbol{s}_{M'_d}\right)$$

where $M'_d := M \setminus \{d\}$. Replacing $J^{\nu}_{-s_\nu}$ by $(-r_\nu t_\nu)$ for all $\nu \neq d$ in (A.1) for the above described contributions leads to

$$\pi_1\left(H^{\sigma}_{i,k}\right)\Big|_{J^1_*=\ldots=J^{d-1}_*=0} = \sum_{\substack{d\in M\subseteq[d]\\ i=1+\mathbf{r}_{M'_d}}} \left(\prod_{\nu\in M'_d} \frac{1}{r_\nu^{r_\nu}} \Psi^{(0)}_{r_\nu}\big(\underbrace{-s_\nu,\ldots,-s_\nu}_{r_\nu\text{-times}}\big)\ (-r_\nu t_\nu)^{r_\nu} \right)$$
$$\times \frac{1}{r_d}\Psi^{(0)}_{r_d}(0)\ J^d_{r_d(k-\mathbf{r}_{M'_d}+\mathbf{s}_{M'_d})}\,.$$

The explicit values for the Ψ-coefficients can be read of from Lemma A.1. Summing over M'_d instead of M then exactly produces the second line in equation (A.38). ◻

Remark A.12. Before we proceed with the construction of the associated Airy structure let us briefly remark on the generality of our approach. There is an important observation one should make considering equation (A.38). Let us assume that $r_d > 1$. In this case it is easy to check that the sum constraints imply that

$$\pi_1\left(H^{\sigma}_{i,k}\right) = 0$$

for $i > r_1 + \ldots + r_{d-1} + 1$. Consequently, we cannot hope to obtain an Airy structure from these differential operators since some have a vanishing first degree. For us the only interesting case thus is $r_d = 1$ where $\pi_1(H^{\sigma}_{i,k}) \neq 0$ for arbitrary i.
Imagine we would have performed dilaton shifts in even less J^μ. Let us say we shifted only $J^1_{-s_1},\ldots,J^\lambda_{-s_{\lambda-1}}$ keeping $J^{\lambda+1}_*,\ldots,J^d_*$ unchanged where $\lambda < d$. Following the proof of Proposition A.11 it is easy to see that

$$\pi_1\left(H^{\sigma}_{i,k}\right) = 0$$

for $i > r_1 + \ldots + r_{\lambda-1} + 1$. Consequently, also this case will not produce us any Airy structures.

Remark A.12 clearly explains that in order to obtain Airy structures we need to restrict ourselves to the fixed point case $r_d = 1$. Due to the similarity between the degree one components of (A.38) and (A.6) we can use the results from the preceding sections in order to prove Theorem 4.20.

Proof of Theorem 4.20. Without loss of generality assume $s_2 = \ldots = s_{d-1} = 1$ and $\frac{r_1}{s_1} \geq r_2 \geq \ldots \geq r_{d-1}$. We have

$$H^{\sigma}_{1,0} = \hbar^{1/2}(Q^1 + \ldots + Q^d) = 0$$

which vanishes by assumption. Thus the selected modes

$$H^{\sigma}_{i,k} \qquad i\in\{1,\ldots,r\},\ k \geq i - \lambda(i) + \delta_{i,1}$$

where we chose the partition

$$\lambda = ((r'_1+1)^{s_1}, r_2, r_3, \ldots, r_{d-1})$$

must already satisfy the subalgebra condition using Lemma 4.9.
It remains to show that after a suitable change of basis the operators can be brought into the normal form of a higher quantum Airy structure. First, let us argue that $\pi_1(H^{\sigma}_{i,k})$ for $k \geq i - \lambda(i) + \delta_{i,1}$ is a linear combination of J^μ_as with $a > 0$ only. Using Proposition A.10, we see that $i - \lambda(i) + \delta_{i,1} = \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$

where

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(i' + \boldsymbol{r}_{[\mu-1]}) = (i' + \boldsymbol{r}_{[\mu-1]}) - 1 - \left\lfloor \frac{s_\mu (i'-1)}{r_\mu} \right\rfloor - \boldsymbol{s}_{[\nu-1]} + \delta_{i',1} \qquad i' \in \{1, \dots, r_\mu\},\ \mu < d$$

and $\mathfrak{d}_{\mathbf{r},\mathbf{s}}(r) = \boldsymbol{r}_{[d-1]} - \boldsymbol{s}_{[d-1]} + 1$. Comparing this expression with (A.10) and its relation to $k_{\min}(i)$ in (A.8) indeed shows that the first line of (A.38) is a linear combination of J^μ_a with $a > 0$ only. That the remaining part

$$\pi_1\left(H^\sigma_{i,k}\right)\Big|_{J^1_*=\ldots=J^{d-1}_*=0} = \sum_{\substack{M\subseteq[d-1]\\ i=1+\boldsymbol{r}_M}} (-1)^{|M|} \left(\prod_{\nu\in M} t_\nu^{r_\nu}\right) J^d_{k-\boldsymbol{r}_M+\boldsymbol{s}_M}$$

is a linear combination of $\{J^d_a\}_{a>0}$, follows from the fact that for all $M \subseteq [d-1]$ with $i = 1 + \boldsymbol{r}_M$ we have

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}(1 + \boldsymbol{r}_M) > \boldsymbol{r}_M - \boldsymbol{s}_M$$

following the same line of reasoning as in (A.27). Thus, for all $k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(1 + \boldsymbol{r}_M)$ we have

$$k - \boldsymbol{r}_M + \boldsymbol{s}_M > 0\,.$$

Further, we notice that $\pi_0(H^\sigma_{i,k}) = 0$ for all $k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$ since $\pi_0(H^\sigma_{i,k})$ coincides with the degree zero projection computed in (A.8) for the case where we shifted in all cycles.

Now let us make the essential observation that for $i = r$ equation (A.38) reduces to

$$\pi_1\left(H^\sigma_{r,k}\right) = (-1)^{d-1} \left(\prod_{\nu=1}^{d-1} t_\nu^{r_\nu}\right) J^d_{k-\boldsymbol{r}_{[d-1]}+\boldsymbol{s}_{[d-1]}}$$

since only the second line gets a non-vanishing contribution from $M = [d-1]$. Therefore, except for a linear rescaling of the coordinates x^d_a the differential operators

$$H^\sigma_{r,k} = (-1)^{d-1} \left(\prod_{\nu=1}^{d-1} t_\nu^{r_\nu}\right) J^d_{k-\boldsymbol{r}_{[d-1]}+\boldsymbol{s}_{[d-1]}} + o(2) \qquad k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(r) = \boldsymbol{r}_{[d-1]} + \boldsymbol{s}_{[d-1]} + 1$$

directly satisfy the degree one condition (3.2). Moreover, notice that we can eliminate all J^d_a in $\pi_1(H^\sigma_{i,k})$ for $i < r$ by taking linear combinations with $\pi_1(H^\sigma_{r,k})$. In other words, there exists a change of basis of $\{x^\mu_a\}$ to $\{\tilde{x}^\mu_a\}$ under which (A.38) transforms into

$$\pi_1\left(H^\sigma_{i,k}\right) = \sum_{\mu\in[d]} \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ 1\leq i-\boldsymbol{r}_M\leq r_\mu}} (-1)^{|M|}\, t_\mu^{i-1} \left(\prod_{\nu\in M} \left(\frac{t_\nu}{t_\mu}\right)^{r_\nu}\right) \tilde{J}^\mu_{\Pi_\mu(i-\boldsymbol{r}_M,\,k-\boldsymbol{r}_M+\boldsymbol{s}_M)} \qquad k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}(i)$$

for $i < r$. This expression is of course nothing else but the degree one projection of the operators considered in (A.6) where we shifted in all cycles. From Proposition A.8 we know that after a further change of basis these modes can be brought into the normal form (3.2). □

A.2 Computation of the correlators

The quantum Airy structures constructed in Section 4.7.1 and 4.7.2 fall into the class of *crosscapped* higher quantum Airy structures since the modes $H^{\sigma}_{i,k}$ are in general build up from monomials

$$\hbar^{j/2} :J^{\mu_1}_{a_1} \dots J^{\mu_\ell}_{a_\ell}:$$

with fractional powers of $\hbar$. Also in this case Theorem 3.4 tells us that one can associate a unique partition function $Z = \exp F$ to this Airy structure satisfying

$$H^{\sigma}_{i,k} \cdot Z = 0 \qquad (i,k) \in I$$

where

$$F(x^1_1, \dots, x^d_1, x^1_2, \dots) = \sum_{\substack{g \in \mathbb{N}/2, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\{(\mu_i, a_i)\}} F_{g,n} \left[\begin{smallmatrix} \mu_1 & \cdots & \mu_n \\ a_1 & \cdots & a_n \end{smallmatrix} \right] x^{\mu_1}_{a_1} \dots x^{\mu_n}_{a_n}$$

and the $F_{g,n}$ are symmetric under permutation of indices and transform as a tensor under a change of basis [7, Prop. 2.20]. It follows that the $F_{g,n}$, which we will refer to as *correlators*, can be computed recursively from the coefficients of $H^{\sigma}_{i,k}$. Let

$$H^{\sigma}_{i,k} = \sum_{\mu=1}^{d} \sum_{a>0} \mathcal{M}_{(i,k),(\mu,a)} J^{\mu}_a - \sum_{m \geq 2} \sum_{\substack{\ell, j \geq 0 \\ \ell + j = m}} \frac{\hbar^{j/2}}{\ell!} \sum_{\alpha \in \mathcal{I}^{\ell}} C^{(j)}[i,k|\alpha] :J_{\alpha_1} \dots J_{\alpha_\ell}: \tag{A.39}$$

introducing $\mathcal{I} := \{1, \dots, d\} \times \mathbb{Z}_{\neq 0}$ and writing $J_\alpha := J^{\mu}_a$ for $\alpha = (\mu, a)$. Then since the $F_{g,n}$ transform as tensors the correlators of the base case $2g - 2 + n = 1$ of the recursion are

$$F_{0,3} \left[\begin{smallmatrix} \mu_1 & \mu_2 & \mu_3 \\ a_1 & a_2 & a_3 \end{smallmatrix} \right] = a_2\, a_3\, C^{(0)} \left[\begin{smallmatrix} \mu_1 \\ a_1 \end{smallmatrix} \middle| \begin{smallmatrix} \mu_2 & \mu_3 \\ -a_2 & -a_3 \end{smallmatrix} \right], \tag{A.40}$$

$$F_{1/2,2} \left[\begin{smallmatrix} \mu_1 & \mu_2 \\ a_1 & a_2 \end{smallmatrix} \right] = a_2\, C^{(1/2)} \left[\begin{smallmatrix} \mu_1 \\ a_1 \end{smallmatrix} \middle| \begin{smallmatrix} \mu_2 \\ -a_2 \end{smallmatrix} \right], \tag{A.41}$$

$$F_{1,1} \left[\begin{smallmatrix} \mu_1 \\ a_1 \end{smallmatrix} \right] = C^{(1)} \left[\begin{smallmatrix} \mu_1 \\ a_1 \end{smallmatrix} \middle| \varnothing \right]. \tag{A.42}$$

Here, we simplified the notation introducing

$$C^{(j)} \left[\begin{smallmatrix} \mu_1 \\ a_1 \end{smallmatrix} \middle| \begin{smallmatrix} \mu_2 & \cdots & \mu_n \\ a_2 & \cdots & a_n \end{smallmatrix} \right] := \sum_{(i,k) \in I} \left(\mathcal{M}^{-1} \right)_{(\mu_1, a_1),(i,k)} C^{(j)} \left[i, k \middle| \begin{smallmatrix} \mu_2 & \cdots & \mu_n \\ a_2 & \cdots & a_n \end{smallmatrix} \right].$$

where $\mathcal{M}^{-1}$ is the inverse of $\mathcal{M}$ constructed in (A.24). This section will be devoted to the computation of the correlators $F_{0,3}$, $F_{1/2,2}$ and $F_{1,1}$ associated to the Airy structures from Theorem 4.19.

Throughout the whole section, let $\sigma \in \mathfrak{S}_r$ be a permutation with $d > 1$ cycles of respective length r_μ such that $r_1 + \dots + r_d = r$. For $\mu \in \{1, \dots, d\}$ choose $t_\mu \in \mathbb{C}^*$ and $s_\mu > 0$ coprime with r_μ. Further let $Q^1, \dots, Q^d \in \mathbb{C}$ denote the values of $J^{\mu}_0 = \hbar^{1/2} Q^{\mu}$. Then associated to this input data we let $H^{\sigma}_{i,k}$ denote the dilaton shifted modes constructed in (A.4) and we always let I denote the index set defined in (A.17).

If the input data $(r_\mu, s_\mu, t_\mu, Q^\mu)_{\mu \in [d]}$ satisfies the constraints from Theorem 4.19 we say they satisfy the *sufficient Airy condition*. If all constraints are met the same theorem tells us that $\{H^{\sigma}_{i,k}\}_{(i,k) \in I}$

is an Airy structure which in turn implies that the correlators (A.40) to (A.42) are symmetric under permutation of (μ_i, a_i).
However, we can compute the left hand side of equation (A.40) to (A.42) also if $(r_\mu, s_\mu, t_\mu, Q^\mu)_{\mu\in[d]}$ do not satisfy the sufficient Airy condition. If we find that in this case the expressions are non-symmetric we can deduce that $\{H^\sigma_{i,k}\}_{(i,k)\in I}$ cannot be an Airy structure for this input data. This provides us with a possibility to check the generality of our approach.
Nevertheless, throughout the whole section we will assume that r_μ and s_μ are coprime and that we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ if $r_\mu = r_\nu$ for $\mu \neq \mu$. Otherwise, the correlators may not be well-defined. Thus, even if we say that we are considering *arbitrary input data* we implicitly always assume that the just mentioned constraints hold.

A.2.1 The computation of $F_{0,3}$

Proposition A.13. *If the input data satisfies the sufficient Airy condition from Theorem 4.19 then*

$$F_{0,3}\left[\begin{smallmatrix}\mu_1 & \mu_2 & \mu_3\\ a_1 & a_2 & a_3\end{smallmatrix}\right] = \begin{cases} +\frac{r_1'+1}{r_1 t_1}\, a_1\, a_2\, a_3\, \delta_{s_1, a_1+a_2+a_3} & \text{, if } \mu_1 = \mu_2 = \mu_3 = 1 \text{ and } r_1 > 1 \\ -\frac{r_d'}{r_d t_d}\, a_1\, a_2\, a_3\, \delta_{s_d, a_1+a_2+a_3} & \text{, if } \mu_1 = \mu_2 = \mu_3 = d \text{ and } r_d > 1 \\ 0 & \text{, otherwise} \end{cases}$$

where $r'_\mu := \lfloor r_\mu / s_\mu \rfloor$ and without loss of generality we assumed that $s_2 = \cdots = s_{d-1} = 1$ and $\frac{r_1}{s_1} \geq r_2 \geq \ldots \geq r_{d-1} \geq \frac{r_d}{s_d}$.
If for arbitrary input data we have $r_\mu \neq \pm 1 \mod s_\mu$ for any μ, then there exist $a_1, a_2, a_3 > 0$ such that

$$a_2\, a_3\, C^{(0)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\middle|\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right] \neq a_1\, a_3\, C^{(0)}\left[\begin{smallmatrix}\mu_2\\ a_2\end{smallmatrix}\middle|\begin{smallmatrix}\mu_1 & \mu_3\\ -a_1 & -a_3\end{smallmatrix}\right] .$$

Proof. First, let us introduce some more notation. To an $a_1 \in \mathbb{Z}$ we will always associate the uniquely determined $i_1 \in \{1, \ldots, r_{\mu_1}\}$ and $k_1 \in \mathbb{Z}$ such that $a_1 = \Pi_{\mu_1}(i_1, k_1)$ and vice versa. The index μ_1 to take here will always be clear from the context. Analogously, we always let $i_1^- \in \{1, \ldots, r_{\mu_1}\}$ and $k_1^- \in \mathbb{Z}$ denote the integers satisfying $-a_1 = \Pi_{\mu_1}(i_1^-, k_1^-)$.
First, we need to extract $C^{(0)}\left[i, k \middle| \begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]$ from $H^\sigma_{i,k}$. We start from the twist modes $W^\sigma_{i,k}$ given in (A.1) build up from $W^\mu_{i_\mu,k_\mu}$ defined in (A.2). In the case $\mu_2 = \mu_3 =: \mu$ we produce a term proportional to $J^\mu_{-a_2} J^\mu_{-a_3}$ from $W^\mu_{i_\mu,k_\mu}$ if $(i_\mu - 2)$ of the p^μs are equal to $-s_\mu$ and the two remaining are equal to $-a_2$ and $-a_3$. Moreover, we can only produce a term proportional to $J^\mu_{-a_2} J^\mu_{-a_3}$ if in $W^\nu_{i_\nu,k_\nu}$ for $\nu \neq \mu$ all p^νs are equal to $-s_\nu$. One then applies the same arguments as in Lemma A.2 leading to

$$C^{(0)}\left[i, k \middle| \begin{smallmatrix}\mu & \mu\\ -a_2 & -a_3\end{smallmatrix}\right] = - \sum_{\substack{M \subseteq [d]\setminus\{\mu\} \\ 2 \leq i - \boldsymbol{r}_M \leq r_\mu}} \left(\prod_{\nu\in M} -t_\nu^{r_\nu}\right) (-t_\mu)^{i - \boldsymbol{r}_M - 2}\, \frac{(i - \boldsymbol{r}_M)(i - \boldsymbol{r}_M - 1)}{r_\mu^2}$$
$$\times\, \Psi^{(0)}_{r_\mu}\big(\underbrace{-s_\mu, \ldots, -s_\mu}_{(i - \boldsymbol{r}_M - 2)\text{-times}}, -a_2, -a_3\big)\; \delta_{s_\mu, \Pi_\mu(i - \boldsymbol{r}_M, k - \boldsymbol{r}_M + \boldsymbol{s}_M) + a_2 + a_3} . \tag{A.43}$$

Notice that the above coefficient vanishes if $r_\mu = 1$. We can extend the sum over M to $i - \boldsymbol{r}_M = 1$

since the contributing term is vanishing and rewrite

$$C^{(0)}\left[i,k\,|\,{}^{\mu}_{-a_2}\;{}^{\mu}_{-a_3}\right] = -\sum_{a_1>0} \mathcal{M}_{(i,k),(\mu,a_1)}\; t_\mu^{-1}\; (-1)^{i_1-2}\frac{i_1\,(i_1-1)}{r_\mu^2} \times \Psi^{(0)}_{r_\mu}\big(\underbrace{-s_\mu,\ldots,-s_\mu}_{(i_1-2)\text{-times}},-a_2,-a_3\big)\;\delta_{s_\mu,\,a_1+a_2+a_3}$$

where $\mathcal{M}$ is the matrix defined in (A.7) and we associate $a_1 = \Pi_\mu(i_1,k_1)$. The summation only runs over $a_1 > 0$ and not over the all integers since we chose the index set I so that for all $(i,k)\in I$ we have $\mathcal{M}_{(i,k),(\mu,a)} = 0$ if $a \le 0$. Thus, we obtain

$$C^{(0)}\left[{}^{\mu_1}_{a_1}\,|\,{}^{\mu}_{-a_2}\;{}^{\mu}_{-a_3}\right] = \begin{cases} -t_\mu^{-1}\;(-1)^{i_1-2}\;\frac{i_1\,(i_1-1)}{r_\mu^2}\;\Psi^{(0)}_{r_\mu}\big(\underbrace{-s_\mu,\ldots,-s_\mu}_{(i_1-2)\text{-times}},-a_2,-a_3\big)\;\delta_{s_\mu,a_1+a_2+a_3} & ,\ \mu_1=\mu \\ 0 & ,\ \mu_1\neq\mu\,. \end{cases}$$

For $\mu_1 = \mu$ the above expression exactly equals the coefficient $C^{(0)}[a_1\,|\,-a_2,\,-a_3]$ from [7, Prop. B.2] if we set t_μ equal to one. Thus, using the result of [7, Prop. B.2] we immediately get

- if $r_\mu = -1 \mod s_\mu$ then

$$F_{0,3}\left[{}^{\mu}_{a_1}\;{}^{\mu}_{a_2}\;{}^{\mu}_{a_3}\right] = +\frac{r'_\mu+1}{r_\mu t_\mu}\;a_1\,a_2\,a_3\;\delta_{s_\mu,a_1+a_2+a_3}\,.$$

- if $r_\mu = +1 \mod s_\mu$ then

$$F_{0,3}\left[{}^{\mu}_{a_1}\;{}^{\mu}_{a_2}\;{}^{\mu}_{a_3}\right] = -\frac{r'_\mu}{r_\mu t_\mu}\;a_1\,a_2\,a_3\;\delta_{s_\mu,a_1+a_2+a_3}\,.$$

- if $r_\mu \neq \pm1 \mod s_\mu$ then there exist $a_1,a_2,a_3>0$ such that

$$a_2\,a_3\,C^{(0)}\left[{}^{\mu}_{a_1}\,|\,{}^{\mu}_{-a_2}\;{}^{\mu}_{-a_3}\right] \neq a_1\,a_3\,C^{(0)}\left[{}^{\mu}_{a_2}\,|\,{}^{\mu}_{-a_1}\;{}^{\mu}_{-a_3}\right]\,.$$

Thus, if the input data satisfies the sufficient Airy condition we especially have

$$F_{0,3}\left[{}^{\mu}_{a_1}\;{}^{\mu}_{a_2}\;{}^{\mu}_{a_3}\right] = 0$$

for all $\mu\in\{2,\ldots,d-1\}$ since $s_2=\ldots=s_{d-1}=1$.
What is now left to prove is that

$$C^{(0)}\left[{}^{\mu_1}_{a_1}\,|\,{}^{\mu_2}_{-a_2}\;{}^{\mu_3}_{-a_3}\right] = 0$$

for μ_1,μ_2,μ_3 pairwise distinct and the input data satisfying the sufficient Airy condition. For this we first need to compute $C^{(0)}\left[i,k\,|\,{}^{\mu_2}_{-a_2}\;{}^{\mu_3}_{-a_3}\right]$. Following the same line of reasoning as in Lemma A.2 one obtains

$$C^{(0)}\left[i,k\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] = -\sum_{M\subseteq[d]\setminus\{\mu_2,\mu_3\}}\left(\prod_{\nu\in M}(-t_\nu^{r_\nu})\right) t_{\mu_2}^{i_2^- -1}\ t_{\mu_3}^{i_3^- -1}\ \delta_{i,\,i_2^-+i_3^-+\boldsymbol{r}_M}\ \delta_{k,\,k_2^-+k_3^-+\boldsymbol{r}_M-\boldsymbol{s}_M+1}$$

$$=: \sum_{M\subseteq[d]\setminus\{\mu_2,\mu_3\}} C_M^{(0)}\left[i,k\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right]\,. \tag{A.44}$$

We will take a rather indirect path to compute $C^{(0)}\left[{}^{\mu_1}_{a_1}\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right]$ by only performing half of the basis change as in Proposition A.7. More explicitly, the proposition tells us that

$$\sum_{b\geq a_1}\Upsilon^{\mu_1}_{b-a_1}\,C^{(0)}\left[{}^{\mu_1}_{b}\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] = \sum_{\substack{1\leq i\leq r,\,k\in\mathbb{Z}\\ \Pi_{\mu_1}(i,k)=a_1+\Delta_{\mu_1}}} t_{\mu_1}^{1-i}\,C^{(0)}\left[i,k\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] \tag{A.45}$$

where Υ^{μ}_{b} is defined as in (A.19) and

$$\Delta_\mu = \boldsymbol{r}_{[\mu-1]}s_\mu - r_\mu\boldsymbol{s}_{[\mu-1]} \geq 0\,.$$

The left hand side of (A.45) is well defined since $\Upsilon^{\mu_1}_{b-a_1}\neq 0$ for only finitely many $b>0$. In turn, the right hand side of (A.45) can be rewritten according to equation (A.21) into

$$\sum_{b\geq a_1}\Upsilon^{\mu_1}_{b-a_1}\,C^{(0)}\left[{}^{\mu_1}_{b}\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] = \sum_{\substack{m\in\mathbb{Z}\\ 1\leq i_{(m)}\leq r}} t_{\mu_1}^{1-i_{(m)}}\,C^{(0)}\left[i_{(m)},k_{(m)}\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] \tag{A.46}$$

where we wrote

$$i_{(m)} := i_1 + mr_{\mu_1} + \boldsymbol{r}_{[\mu_1-1]}\,,\qquad k_{(m)} = k_1 + m(r_{\mu_1}-s_{\mu_1}) + \boldsymbol{r}_{[\mu_1-1]} - \boldsymbol{s}_{[\mu_1-1]}$$

with $a_1 = \Pi_{\mu_1}(i_1,k_1)$. Let us now make the important observation that for $M\subseteq[d]\setminus\{\mu_1,\mu_2,\mu_3\}$ we have

$$C_M^{(0)}\left[i,k\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right] = -t_{\mu_1}^{r_{\mu_1}}\,C^{(0)}_{M\cup\{\mu_1\}}\left[i+r_{\mu_1},k+r_{\mu_1}-s_{\mu_1}\,|\,{}^{\mu_2}_{-a_2}\ {}^{\mu_3}_{-a_3}\right]\,. \tag{A.47}$$

Thus, expanding the right hand side of (A.46) leads to

$$
\begin{aligned}
&\sum_{b\geq a_1} \Upsilon^{\mu_1}_{b-a_1}\, C^{(0)}\left[\begin{smallmatrix}\mu_1\\ b\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\quad= \sum_{\substack{m\in\mathbb{Z}\\ 1\leq i_{(m)}\leq r}} \sum_{M\subseteq[d]\setminus\{\mu_1,\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m)}}\, C^{(0)}_M\left[i_{(m)},k_{(m)}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\qquad+ \sum_{\substack{m\in\mathbb{Z}\\ 1\leq i_{(m)}\leq r}} \sum_{\{\mu_1\}\subseteq M\subseteq[d]\setminus\{\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m)}}\, C^{(0)}_M\left[i_{(m)},k_{(m)}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\quad= -\sum_{\substack{m\in\mathbb{Z}\\ 1\leq i_{(m)}\leq r}} \sum_{\mu_1\in M\subseteq[d]\setminus\{\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m+1)}}\, C^{(0)}_M\left[i_{(m+1)},k_{(m+1)}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\qquad+ \sum_{\substack{m\in\mathbb{Z}\\ 1\leq i_{(m)}\leq r}} \sum_{\{\mu_1\}\subseteq M\subseteq[d]\setminus\{\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m)}}\, C^{(0)}_M\left[i_{(m)},k_{(m)}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\quad= -\sum_{\mu_1\in M\subseteq[d]\setminus\{\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m_{\max}+1)}}\, C^{(0)}_M\left[i_{(m_{\max}+1)},k_{(m_{\max}+1)}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\\
&\qquad+ \sum_{\{\mu_1\}\subseteq M\subseteq[d]\setminus\{\mu_2,\mu_3\}} t_{\mu_1}^{1-i_{(m_{\min})}}\, C^{(0)}_M\left[i_{(m_{\min})},k_{(m_{\min})}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right]\,.
\end{aligned}
$$

Here, in the first step we split up the sum over M into the cases $\mu_1 \notin M$ and $\mu_1 \in M$. In the second step we then used (A.47). The first and second sum cancel each other except for m being either one of

$$
m_{\min} = \left\lceil \tfrac{1}{r_\mu}\left(1 - i' - \boldsymbol{r}_{[\mu-1]}\right)\right\rceil\,, \quad m_{\max} = \left\lfloor \tfrac{1}{r_\mu}\left(\boldsymbol{r}_{\{\mu\ldots d\}} - i'\right)\right\rfloor\,.
$$

But also these remaining terms vanish. Consider for example the first term. For all $M \subseteq [d]\setminus\{\mu_2,\mu_3\}$ we have

$$
i_{(m_{\max}+1)} - \boldsymbol{r}_M > r - \boldsymbol{r}_M = \sum_{\nu\notin M} r_\nu \geq r_{\mu_2} + r_{\mu_3} \geq i_2^- + i_3^-
$$

implying that all $C^{(0)}_M$ vanish. The same holds for the second line which now finally implies that

$$
\sum_{b\geq a_1} \Upsilon^{\mu_1}_{b-a_1}\, C^{(0)}\left[\begin{smallmatrix}\mu_1\\ b\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2 & \mu_3\\ -a_2 & -a_3\end{smallmatrix}\right] = 0\,.
$$

Since by assumption $\Upsilon^{\mu_1}_0 \neq 0$, we can multiply the above with

$$
\sum_{m\geq 0} (-1)^m \left(\Upsilon^{\mu}_0\right)^{-m-1} \sum_{\substack{\gamma_1,\ldots,\gamma_m>0\\ \sum_\ell \gamma_\ell = a_1-c}} \prod_{\ell=1}^{m} \Upsilon^{\mu}_{\gamma_\ell}
$$

and sum over $a_1 > 0$. Then using that

$$\delta_{b,c} = \sum_{a_1>0} \Upsilon^{\mu}_{b-a_1} \sum_{m\geq 0} (-1)^m \left(\Upsilon^{\mu}_0\right)^{-m-1} \sum_{\substack{\gamma_1,\dots,\gamma_m>0 \\ \sum_\ell \gamma_\ell = a_1-c}} \prod_{\ell=1}^{m} \Upsilon^{\mu}_{\gamma_\ell}$$

for $b, c > 0$, this finally yields $C^{(0)}\left[\begin{smallmatrix}\mu_1\\ c\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\ \begin{smallmatrix}\mu_3\\ -a_3\end{smallmatrix}\right] = 0$ for all $c > 0$. □

Remark A.14. For input data not satisfying the sufficient Airy condition in some cases one can find that

$$a_2\, a_3\, C^{(0)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\ \begin{smallmatrix}\mu_1\\ -a_3\end{smallmatrix}\right] \neq a_1\, a_3\, C^{(0)}\left[\begin{smallmatrix}\mu_2\\ a_2\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\mu_1\\ -a_1\end{smallmatrix}\ \begin{smallmatrix}\mu_1\\ -a_1\end{smallmatrix}\right] = 0$$

for certain $\mu_1 \neq \mu_2$ and $a_1, a_2, a_3 > 0$ and generic t^{μ}. I am unsure about the explicit constraints this puts on the input data. Anyway, I think the constraints coming from the symmetry of $F_{1/2,2}$ are stronger. Thus, it seems more promising to analyse the constraints coming from the symmetry of $F_{1/2,2}$.

A.2.2 The computation of $F_{1/2,2}$

Computations for an arbitrary number of cycles

Different from the computation of $F_{0,3}$ in the computation of $F_{1/2,2}$ we can not circumvent the explicit computation of entries of the matrix $\mathcal{M}^{-1}$ in order to get from $C^{(j)}[i,k|\dots]$ to $C^{(j)}[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}|\dots]$. In the most general setting possible I could so far obtain the following result.

Proposition A.15. *For arbitrary input data we have*

$$\begin{aligned} C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] = &-\frac{Q^{\mu_1}}{t_{\mu_1}}\, \frac{r_{\mu_1} - i_1 + 1}{r_{\mu_1}}\ \delta_{\mu_1,\mu_2}\, \delta_{s_{\mu_1}, a_1+a_2} \\ &- \sum_{\lambda\neq\mu_2}\ \sum_{M\subseteq[d]\setminus\{\mu_2,\lambda\}} Q^{\lambda} \left(\prod_{\nu\in M} (-t_\nu^{r_\nu})\right) t_{\mu_2}^{i_2^- -1}\ \left(\mathcal{M}^{-1}\right)_{(\mu_1,a_1),\,(i_2^- + \boldsymbol{r}_M+1,\, k_2^- + \boldsymbol{r}_M - \boldsymbol{s}_M + 1)}\, . \end{aligned} \tag{A.48}$$

Proof. We proceed as for the computation of $F_{0,3}$. First we extract $C^{(1/2)}\left[i,k \,\middle|\, \begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right]$ which is the coefficient in front of $\hbar^{1/2} J^{\mu_2}_{-a_2}$. The factor $\hbar^{1/2}$ necessarily comes from $J^{\lambda}_0 = \hbar^{1/2} Q^{\lambda}$ where $Q^{\lambda} \in \mathbb{C}$. We produce a term proportional to $J^{\lambda}_0 J^{\mu_2}_{-a_2}$ from $W^{\sigma}_{i,k}$ as follows. First, let $\lambda = \mu_2$. The contributions from $W^{\mu_2}_{i_{\mu_2},k_{\mu_2}}$ come from these terms in (A.2) where $(i_{\mu_2} - 2)$ of the p^{μ_2}s are equal to $-s_{\mu_2}$ and the two remaining ones take the values 0 and $-a_2$. The contributions from the other $W^{\nu}_{i_\nu,k_\nu}$ with $\nu \neq \mu_2$ come from the terms with all p^{ν} equal to $-s_\nu$ leading to

$$\begin{aligned} H^{\sigma}_{i,k} = &\sum_{\substack{M\subseteq[d]\setminus\{\mu_2\} \\ 2\leq i-\boldsymbol{r}_M \leq r_{\mu_2}}} \left(\prod_{\nu\in M} -t_\nu^{r_\nu}\right)\ (-t_{\mu_2})^{i-\boldsymbol{r}_M-2}\ \frac{(i-\boldsymbol{r}_M)(i-\boldsymbol{r}_M-1)}{r_{\mu_2}^2} \\ &\times \Psi^{(0)}_{r_{\mu_2}}\big(\underbrace{-s_{\mu_2},\dots,-s_{\mu_2}}_{(i-\boldsymbol{r}_M-2)\text{-times}}, -a_2, 0\big)\ \delta_{s_{\mu_2}, \Pi_{\mu_2}(i-\boldsymbol{r}_M, k-\boldsymbol{r}_M+\boldsymbol{s}_M)+a_2}\ J^{\mu_2}_0 J^{\mu_2}_{-a_2} + \dots \end{aligned}$$

Notice that the last expression exactly corresponds to the contribution calculated in (A.43) setting $a_3 = 0$. It is therefore not surprising that for $\lambda \neq \mu_2$ the coefficient in front of $J_0^{\lambda} J_{-a_2}^{\mu_2}$ can be read of from (A.44) setting $a_3 = 0$ and identifying $\mu_3 = \lambda$:

$$H_{i,k}^{\sigma} = \sum_{M \subseteq \{\mu_2, \lambda\}} \left(\prod_{\nu \in M} (-t_{\nu}^{r_\nu}) \right) t_{\mu_2}^{i_2^- - 1} \, \delta_{i, i_2^- + r_M + 1} \, \delta_{k, k_2^- + r_M - s_M + 1} \, J_0^{\lambda} J_{-a_2}^{\mu_2} + \dots$$

Here, we set $-a_2 = \Pi_{\mu_2}(i_2^-, k_2^-)$. Summing up all contributions we get

$$\begin{aligned} C^{(1/2)} \left[i, k \,|\, {\textstyle{\mu_2 \atop -a_2}} \right] = &- \sum_{\substack{a_1 > 0 \\ s_{\mu_2} = a_1 + a_2}} \mathcal{M}_{(i,k),(\mu_2,a_1)} \, \frac{Q^{\mu_2}}{t_{\mu_2}} \, (-1)^{i_1 - 2} \, \frac{i_1(i_1 - 1)}{r_{\mu_2}^2} \, \Psi_{r_{\mu_2}}^{(0)} \big(\overbrace{-s_{\mu_2}, \dots, -s_{\mu_2}}^{(i_1 - 2)\text{-times}}, -a_2, 0 \big) \\ &- \sum_{\lambda \neq \mu_2} \sum_{M \subseteq \{\mu_2, \lambda\}} Q^{\lambda} \left(\prod_{\nu \in M} (-t_{\nu}^{r_\nu}) \right) t_{\mu_2}^{i_2^- - 1} \, \delta_{i, i_2^- + r_M + 1} \, \delta_{k, k_2^- + r_M - s_M + 1} \end{aligned} \tag{A.49}$$

where we associate $a_1 = \Pi_{\mu_2}(i_1, k_1)$. We can simplify the first line by plugging in the explicit value of Ψ.

$$\begin{aligned} \Psi_{r_{\mu_2}}^{(0)} \big(-s_{\mu_2}, \dots, -s_{\mu_2}, -a_2, 0 \big) &= \frac{r_{\mu_2} - i_1 + 1}{i_1} \, \Psi_{r_{\mu_2}}^{(0)} \big(\underbrace{-s_{\mu_2}, \dots, -s_{\mu_2}}_{(i_1 - 2)\text{-times}}, -a_2 \big) \\ &= (-1)^{i_1 - 2} \, \frac{r_{\mu_2}(r_{\mu_2} - i_1 + 1)}{i_1(i_1 - 1)} \, \delta_{r_{\mu_2} | a_1 + a_2 - s_{\mu_2}} \, . \end{aligned}$$

The first line follows directly from the definition (A.3) of Ψ and the second line from using [7, Lem. A.6]. Plugging in this expression yields the claimed formula for $C^{(1/2)} \left[{\textstyle{\mu_1 \atop a_1}} \,|\, {\textstyle{\mu_2 \atop -a_2}} \right]$ after multiplying with $\mathcal{M}^{-1}$. ◻

The following passage is rather speculative and is partly relying on evidences coming from numerical calculations. What makes the computation of $F_{1/2,2}$ more difficult than the one of $F_{0,3}$ is the fact that we need to calculate the values of the matrix $\mathcal{M}^{-1}$ in (A.48) explicitly. We will later see that the contributions coming from the last line in (A.48) are essential in order to ensure the symmetry of the correlators in the case where the input data satisfies the sufficient Airy condition. On the other hand, since the first line of (A.48) is vanishing for $\mu_1 \neq \mu_2$ the contributions from the second line may allow for a check of symmetry

$$a_2 \, C^{(1/2)} \left[{\textstyle{\mu_1 \atop a_1}} \,|\, {\textstyle{\mu_2 \atop -a_2}} \right] \overset{?}{\longleftrightarrow} a_1 \, C^{(1/2)} \left[{\textstyle{\mu_2 \atop a_2}} \,|\, {\textstyle{\mu_1 \atop -a_1}} \right]$$

in the case of arbitrary input data. I expect the constraints coming from this symmetry constraint to be stronger than the ones coming from the symmetry of $F_{0,3}$ as specified in remark A.3. However, this characterisation of symmetry I have not derived yet. The following will be an exposition of what I know so far featuring some closed formulas for $F_{1/2,2}$ in the case of $d = 2$ cycles. What is rather easy to compute is the first line of equation (A.48).

- In the case $r_\mu = -1 \mod s_\mu$ we have

$$a_2\, C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] = -\frac{(r_{\mu_1} - (r'_{\mu_1}+1)\,a_1)Q^{\mu_1}}{r_{\mu_1} t_{\mu_1}}\, a_2\;\delta_{\mu_1,\mu_2}\,\delta_{s_{\mu_1},a_1+a_2} + \ldots \tag{A.50}$$

- In the case $r_\mu = +1 \mod s_\mu$ we have

$$a_2\, C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] = -\frac{r'_{\mu_1} Q^{\mu_1}}{r_{\mu_1} t_{\mu_1}}\, a_1\, a_2\;\delta_{\mu_1,\mu_2}\,\delta_{s_{\mu_1},a_1+a_2} + \ldots \tag{A.51}$$

Here, one expresses i_1 in terms of a_1 using that $a_1 < s_\mu$. While (A.51) is symmetric under permutation of indices (A.50) is in general obviously non-symmetric. In the case of input data satisfying the sufficient Airy condition, we will see that the terms making (A.50) non-symmetric are exactly cancelled by the contributions from the second line in (A.48).

Computations for the case $d = 2$

Regarding the second line of (A.48), let us consider the case $d = 2$ in which we only get a contribution from $M = \varnothing$ and thus

$$C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] = \ldots - \left(\sum_{\lambda\neq\mu_2} Q^\lambda\right) t_{\mu_2}^{i_2^- -1}\;\left(\mathcal{M}^{-1}\right)_{(\mu_1,a_1),\,(i_2^-+1,\;k_2^-+1)}\,.$$

Now we plug in the expression for $\mathcal{M}^{-1}$ from (A.24) and obtain

$$C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] = \cdots - \tilde{Q}^{\mu_2}\,\frac{t_{\mu_2}^{i_2^- -1}}{t_{\mu_1}^{i_2^-}}\;\sum_{m\geq 0}(-1)^m\left(\Upsilon_0^{\mu_1}\right)^{-m-1}\sum_{\substack{\gamma_1,\ldots,\gamma_m>0\\ \sum_\ell \gamma_\ell = s_{\mu_1}+\Pi_{\mu_1}(i_2^-,k_2^-)-\Delta_{\mu_1}-a_1}}\;\prod_{\ell=1}^m \Upsilon_{\gamma_\ell}^{\mu_1}$$

where we set $\tilde{Q}^\mu := \sum_{\lambda\neq\mu} Q^\lambda$. The explicit value for $\Upsilon_a^{\mu_1}$ for our case at hand is

$$\Upsilon_a^1 = \delta_{a,0} - \left(\frac{t_2}{t_1}\right)^{r_2}\delta_{a,\Delta}\,, \qquad \Upsilon_a^2 = -\left(\frac{t_1}{t_2}\right)^{r_1}\delta_{a,0} + \delta_{a,\Delta}$$

where $\Delta = r_1 s_2 - r_2 s_1 \geq 0$. First let us consider the case $\Delta = 0$. In this case, all terms in (A.50) vanish except for $m = 0$ if we have

$$s_{\mu_1} = a_1 + a_2$$

where we used that $r_1 = r_2$, $s_1 = s_2$ due to $\Delta_\mu = 0$. We obtain

$$\begin{aligned} C^{(1/2)}\left[\begin{smallmatrix}\mu_1\\ a_1\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\mu_2\\ -a_2\end{smallmatrix}\right] &= \ldots - \tilde{Q}^{\mu_2}\,\frac{t_{\mu_2}^{i_2^- -1}}{t_{\mu_1}^{i_2^-}}\;\left(\Upsilon_0^{\mu_1}\right)^{-1}\delta_{s_{\mu_1},\,a_1+a_2} \\ &= \ldots - \tilde{Q}^{\mu_2}\,\frac{t_{\mu_2}^{i_2^- -1}}{t_{\mu_1}^{i_2^-}}\;\frac{1}{\left(1-\left(\frac{t_\nu}{t_{\mu_1}}\right)^{r_\nu}\right)}\,\delta_{s_{\mu_1},\,a_1+a_2} \end{aligned} \tag{A.52}$$

where we chose $\nu \neq \mu_1$. At first sight this seems very bad since the expression multiplied with a_2 looks not at all symmetric if $\mu_1 \neq \mu_2$. It turns out, in the case where the input data satisfies the sufficient Airy condition everything works out fine at the end.
Remember we are working under the assumption $\Delta = 0$ implying $r_1 = r_2$ and $s_1 = s_2$. If now the input data satisfies the sufficient Airy condition we have both

$$r_1 = +1 \mod s_1 \quad \text{and} \quad r_1 = -1 \mod s_1 \tag{A.53}$$

where $s_1 \in \{1, \ldots, r_1 + 1\}$. This leaves us with the following possibilities:

- $s_1 = 1$ and r_1 is arbitrary. In this case (A.52) vanishes.
- $s_1 = 2$ and r_1 is odd. Expression (A.52) always vanishes unless $a_1 = a_2 = 1$. Now using that

  $$-1 = \Pi_{\mu_2}(r_{\mu_2} - r'_{\mu_2}, r_{\mu_2} - s_{\mu_2} - r'_{\mu_2})$$

  with $r'_\mu := \lfloor r_\mu/2 \rfloor$, we plug $i_2^- = r_{\mu_2} - r'_{\mu_2}$ into (A.52) which yields

  $$C^{(1/2)}\left[\begin{smallmatrix}\mu_1 \\ a_1\end{smallmatrix}\middle|\begin{smallmatrix}\mu_2 \\ -a_2\end{smallmatrix}\right] = \cdots + \begin{cases} Q^{\mu_2}\, \dfrac{1}{t_{\mu_1}\left(1-\left(\frac{t_\nu}{t_{\mu_1}}\right)^{r_\nu}\right)}\, \delta_{2,\, a_1+a_2} & ,\ \mu_1 = \mu_2 \\ Q^{\mu_2}\, \dfrac{t_{\mu_1}^{r'_{\mu_1}}\, t_{\mu_2}^{r'_{\mu_2}}}{t_{\mu_1}^{r_{\mu_1}} - t_{\mu_2}^{r_{\mu_2}}}\, \delta_{2,\, a_1+a_2} & ,\ \mu_1 \neq \mu_2\,. \end{cases}$$

  Here, we used $\tilde{Q}^{\mu_2} = -Q^{\mu_2}$ and $r_\mu = 2r'_\mu + 1$. Remember that since the input data satisfies the sufficient Airy condition we have $Q^1 = -Q^2$. Therefore, we indeed find that

  $$a_2\, C^{(1/2)}\left[\begin{smallmatrix}\mu_1 \\ a_1\end{smallmatrix}\middle|\begin{smallmatrix}\mu_2 \\ -a_2\end{smallmatrix}\right] = a_1\, C^{(1/2)}\left[\begin{smallmatrix}\mu_1 \\ a_1\end{smallmatrix}\middle|\begin{smallmatrix}\mu_2 \\ -a_2\end{smallmatrix}\right]$$

  for $\mu_1 \neq \mu_2$.

Let us summarize the results for the case where the sufficient Airy condition is satisfied.

Proposition A.16. *If $d = 2$, $r_1 = r_2$, $s_1 = s_2$, and the sufficient Airy condition is satisfied we have*

- *If $s_\mu = 1$ we have*

  $$F_{1/2,2}\left[\begin{smallmatrix}\mu_1 & \mu_2 \\ a_1 & a_2\end{smallmatrix}\right] = 0\,.$$

- *If $s_\mu = 2$ we have*

  $$F_{1/2,2}\left[\begin{smallmatrix}\mu_1 & \mu_2 \\ a_1 & a_2\end{smallmatrix}\right] = \begin{cases} \dfrac{Q^{\mu_1}}{t_{\mu_1}}\left(\dfrac{\tilde{r}'}{\tilde{r}} + \dfrac{1}{1-\left(\frac{t_\nu}{t_{\mu_1}}\right)^{\tilde{r}}}\right)\delta_{2,\, a_1+a_2} & ,\ \mu_1 = \mu_2 \\ +Q^{\mu_1}\, \dfrac{t_{\mu_1}^{\tilde{r}'}\, t_{\mu_2}^{\tilde{r}'}}{t_{\mu_1}^{\tilde{r}} - t_{\mu_2}^{\tilde{r}}}\, \delta_{2,\, a_1+a_2} & ,\ \mu_1 \neq \mu_2 \end{cases}$$

  where we wrote $\tilde{r} := r_1$, $\tilde{r}' := \lfloor \tilde{r}/2 \rfloor$, and chose $\nu \neq \mu_1$.

□

In the case where the input data does not satisfy the sufficient Airy condition symmetry is clearly failed if $Q^1 \neq -Q^2$ and $s_1 > 1$ considering equation (A.52).
If (A.53) is not satisfied, I assume that (A.52) will not be satisfied for generic t_μ as well. I did not do the calculation but I assume probing (A.52) for $a_1 = 1$ and $a_2 = s_\mu - 1$ will prove asymmetry.

The second case we need to consider is $\Delta > 0$, ie. $\frac{r_1}{s_1} > \frac{r_2}{s_2}$. In this case $\Delta_\mu = \delta_{2,\mu}\Delta$ implying that (A.50) gives a non-vanishing contribution if and only if Δ divides

$$s_{\mu_1} - a_2' - \Delta_{\mu_1} - a_1 \geq 0$$

in which case m is equal to the above expression divided by Δ. Here, we wrote $a_2' := -\Pi_{\mu_1}(i_2^-, k_2^-)$. Remember that $\Pi_{\mu_2}(i_2^-, k_2^-) = -a_2$. This leaves us with

$$C^{(1/2)}\left[\begin{smallmatrix}\mu_1 \\ a_1\end{smallmatrix} \middle| \begin{smallmatrix}\mu_2 \\ -a_2\end{smallmatrix}\right] = \cdots - \frac{\tilde{Q}^{\mu_2}}{t_{\mu_2}} \left(\Upsilon_0^{\mu_1}\right)^{(s_{\mu_1}-\Delta_{\mu_1}-a_1-a_2')/\Delta-1} \left(-\Upsilon_\Delta^{\mu_1}\right)^{(s_{\mu_1}-\Delta_{\mu_1}-a_1-a_2')/\Delta} \delta_{\Delta|(s_{\mu_1}-a_1-a_2')}$$

for $s_{\mu_1} - a_2' - \Delta_{\mu_1} - a_1 \geq 0$ and zero otherwise. Let us focus on the case $\mu_1 = \mu_2 =: \mu$. Then $a_2' = a_2$ and plugging in the values for $\Upsilon_a^{\mu_1}$ we obtain

$$C^{(1/2)}\left[\begin{smallmatrix}\mu \\ a_1\end{smallmatrix} \middle| \begin{smallmatrix}\mu \\ -a_2\end{smallmatrix}\right] = \cdots - \frac{\tilde{Q}^{\mu}}{t_\mu} \left(\frac{t_2}{t_1}\right)^{r_\nu(s_\mu-a_1-a_2)/\Delta} \delta_{\Delta|(s_\mu-a_1-a_2)}\, \delta_{s_\mu-a_1-a_2-\Delta_\mu\geq 0}\,. \tag{A.54}$$

So far the calculation holds for arbitrary input data. Now let us restrict ourself to the case where the input data satisfies the sufficient Airy condition and sum up the contributions (A.50) resp. (A.51) with (A.54).

- First let $\mu = 1$. In this case $r_1 = -1 \mod s_1$ which means we have to sum up (A.50) and (A.54). What is highly interesting is that the term making (A.50) non-symmetric is exactly cancelled from the contribution $s_1 - a_1 - a_2 = 0$ in (A.54) using $\Delta_1 = 0$. We obtain

$$\begin{aligned} F_{1/2,2}\left[\begin{smallmatrix}1 & 1 \\ a_1 & a_2\end{smallmatrix}\right] = {} & \frac{(r_1'+1)Q^1}{r_1 t_1}\, a_1\, a_2\, \delta_{s_1,a_1+a_2} \\ & -a_2\, \frac{Q^1}{t_1} \left(\frac{t_2}{t_1}\right)^{r_2(s_1-a_1-a_2)/\Delta} \delta_{\Delta|(s_1-a_1-a_2)}\, \delta_{s_1-a_1-a_2-\Delta\geq 0}\,. \end{aligned}$$

- Now consider $\mu = 2$ where we have $r_1 = +1 \mod s_1$. This time we sum up (A.51) and (A.54) and obtain

$$F_{1/2,2}\left[\begin{smallmatrix}2 & 2 \\ a_1 & a_2\end{smallmatrix}\right] = -\frac{r_2' Q^2}{r_2 t_2}\, a_1\, a_2\, \delta_{s_2,a_1+a_2} + a_2\, \frac{Q^2}{t_2} \left(\frac{t_2}{t_1}\right)^{r_1(s_2-a_1-a_2)/\Delta} \delta_{\Delta|(s_2-a_1-a_2)}\, \delta_{s_2-a_1-a_2-\Delta\geq 0}\,.$$

Obviously, the first terms in the above two equations are symmetric under the exchange of a_1 and a_2. But considering the second term, it seems we again run into the problem of having a non-symmetric contribution since this term is only linearly dependent on a_2 and not on a_1.
I have tested the constraint coming from the Kronecker delta for all input data satisfying the Airy condition with $r_1, r_2 \leq 50$ numerically and found that the second term gives a non-vanishing contribution only if $\mu = 2$ and $a_1 = a_2 = 1$. Of course this is only a **observation**. But if this statement

holds for arbitrary input data satisfying the sufficient Airy condition the symmetry problem would be solved, since in this case we had the following.

Conjecture A.17. *If $d = 2$, $\frac{r_1}{s_1} > \frac{r_2}{s_2}$, and the sufficient Airy condition is satisfied then*

$$F_{1/2,2}\left[\begin{smallmatrix} 1 & 1 \\ a_1 & a_2 \end{smallmatrix}\right] = \frac{(r_1' + 1)Q^1}{r_1 t_1}\, a_1\, a_2\ \ \delta_{s_1, a_1+a_2}$$

and

$$F_{1/2,2}\left[\begin{smallmatrix} 2 & 2 \\ a_1 & a_2 \end{smallmatrix}\right] = -\frac{r_2' Q^2}{r_2 t_2}\, a_1\, a_2\ \ \delta_{s_2, a_1+a_2} + \frac{Q^2}{t_2}\left(\frac{t_2}{t_1}\right)^{r_1}\ \delta_{2,\, a_1+a_2}\ \delta_{s_2,\, \Delta+2}\,.$$

So far I did not try to compute $F_{1/2,2}\left[\begin{smallmatrix} \mu_1 & \mu_2 \\ a_1 & a_2 \end{smallmatrix}\right]$ for $\mu_1 \neq \mu_2$.

A.2.3 The computation of $F_{1,1}$

Computations for an arbitrary number of cycles

Regarding the computation of $F_{1,1}$, one meets the same problems as in the computation of $F_{1/2,2}$. In the most general setting possible one obtains the following.

Proposition A.18. *For arbitrary input data we have*

$$\begin{aligned} C^{(1)}\left[\begin{smallmatrix} \mu \\ a \end{smallmatrix}\middle|\varnothing\right] &= \left(\frac{(r_\mu^2 - 1)}{24\, r_\mu^2\, t_\mu} - \frac{(r_\mu - 1)}{r_\mu}\ \frac{(Q^\mu)^2}{t_\mu}\right)\ \delta_{a, s_\mu} \\ &\quad - \frac{1}{2}\sum_{\mu_1 \neq \mu_2}\ \sum_{M \subseteq [d] \setminus \{\mu_1, \mu_2\}} Q^{\mu_1}\, Q^{\mu_2}\ \left(\prod_{\nu \in M} (-t_\nu)^{r_\nu}\right)\left(\mathcal{M}^{-1}\right)_{(\mu,a),\,(2+\boldsymbol{r}_M,\,1+\boldsymbol{r}_M - \boldsymbol{s}_M)}\,. \end{aligned} \tag{A.55}$$

Proof. We need to compute the coefficient $C^{(1)}[i,k|\varnothing]$. Comparing with the computation of $C^{(1)}\left[i,k\middle|\begin{smallmatrix} \mu_2 \\ a_2 \end{smallmatrix}\right]$ it is clear that the contribution from terms $J_0^{\mu_1} J_0^{\mu_2} = \hbar\, Q^{\mu_1} Q^{\mu_2}$ is obtained from (A.49) setting $a_2 = 0$ and summing over μ_2

$$\begin{aligned} C^{(1)}[i,k|\varnothing] = &- \sum_{\substack{\mu_1 \in \{1,\ldots d\} \\ a_1 > 0}} \mathcal{M}_{(i,k),(\mu_1,a_1)}\ (-1)^{i_1 - 2}\frac{(Q^{\mu_1})^2}{t_{\mu_1}}\ \frac{i_1(i_1-1)}{r_{\mu_1}^2}\ \Psi^{(0)}_{r_{\mu_1}}\big(\underbrace{-s_{\mu_1}, \ldots, -s_{\mu_1}}_{(i_1-2)\text{-times}}, 0, 0\big)\ \delta_{s_{\mu_1}, a_1} \\ &- \frac{1}{2}\sum_{\mu_1 \neq \mu_2}\ \sum_{M \subseteq [d] \setminus \{\mu_1,\mu_2\}} Q^{\mu_1} Q^{\mu_2}\ \left(\prod_{\nu \in M} (-t_\nu)^{r_\nu}\right)\ \delta_{i,\,2+\boldsymbol{r}_M}\ \delta_{k,\,1+\boldsymbol{r}_M - \boldsymbol{s}_M} \\ &+ \ldots \end{aligned}$$

where the dots indicate a contribution not included yet. Now using that $a_1 = \Pi_{\mu_1}(i_1, k_1) = s_{\mu_1}$ if $i_1 = 2$ and $k_1 = 0$ and plugging in

$$\Psi^{(0)}_{r_{\mu_1}}(0,0) = \frac{1}{2} r_{\mu_1}(r_{\mu_1} - 1)$$

the expression simplifies to

$$
\begin{aligned}
C^{(1)}\,[i,k|\varnothing] = &- \sum_{\mu_1\in\{1,\dots d\}} \mathcal{M}_{(i,k),(\mu_1,s_{\mu_1})}\, \frac{(r_{\mu_1}-1)}{r_{\mu_1}}\, \frac{(Q^{\mu_1})^2}{t_{\mu_1}} \\
&-\frac{1}{2}\sum_{\mu_1\neq\mu_2}\ \sum_{M\subseteq[d]\setminus\{\mu_1,\mu_2\}} Q^{\mu_1}Q^{\mu_2}\left(\prod_{\nu\in M}(-t_\nu)^{r_\nu}\right)\delta_{i,\,2+\boldsymbol{r}_M}\,\delta_{k,\,1+\boldsymbol{r}_M-\boldsymbol{s}_M} \\
&+\dots
\end{aligned}
\tag{A.56}
$$

The second contribution to $C^{(1)}\,[i,k|\varnothing]$ comes from monomials

$$
\hbar^{j_\mu} : J^{\mu}_{p_{2j_\mu+1}} \dots J^{\mu}_{p_{i_\mu}} :
$$

in $W^{\mu}_{i_\mu,k_\mu}$ with $j_\mu = 1$. In order to contribute the $(i_\mu - 2)$ indices p^μ must be equal to $-s_\mu$. For $\nu \neq \mu$ the contributing terms from $W^{\nu}_{i_\nu,k_\nu}$ are those with $j_\nu = 0$ and all $p^{\nu}_{\ell} = -s_\nu$. Summing up all these contributions for $\mu \in \{1,\dots,d\}$, one obtains

$$
H^{\sigma}_{i,k} = \sum_{\mu=1}^{d}\ \sum_{\substack{M\subseteq[d]\setminus\{\mu\}\\ 2\le i-\boldsymbol{r}_M\le r_\mu}} \left(\prod_{\nu\in M}(-t_\nu^{r_\nu})\right)\binom{i-\boldsymbol{r}_M}{2}\Psi^{(1)}_{r_\mu}\big(\underbrace{-s_\mu,\dots,-s_\mu}_{(i-\boldsymbol{r}_M-2)\text{-times}}\big)\frac{(-t_\mu)^{i-\boldsymbol{r}_M-2}}{r_\mu^2}\,\hbar+\dots
$$

If we now insert

$$
\Psi^{(1)}_{r_\mu}\big(\underbrace{-s_\mu,\dots,-s_\mu}_{(i-\boldsymbol{r}_M-2)\text{-times}}\big) = \begin{cases} -\frac{r_\mu(r_\mu^2-1)}{24} & ,\ i-\boldsymbol{r}_M = 2 \\ 0 & ,\ i-\boldsymbol{r}_M \neq 2 \end{cases}
$$

we get the missing contribution

$$
C^{(1)}\,[i,k|\varnothing] = \sum_{\substack{\mu\in[d]\\ a>0}} \mathcal{M}_{(i,k),(\mu,a)}\,\frac{r_\mu(r_\mu^2-1)}{24}\,\frac{1}{t_\mu r_\mu^2}\,\delta_{s_\mu,a}+\dots
$$

which together with (A.56) proves the claimed expression for $C^{(1)}\,[\,{}^{\mu}_{a}\,|\varnothing]$ multiplying $C^{(1)}\,[i,k|\varnothing]$ with $\mathcal{M}^{-1}$. □

Computations for the case $d = 2$

What makes (A.55) difficult to handle is that the second line involving $\mathcal{M}^{-1}$ which is in general non-vanishing. In the special case $d = 2$ one can write down an explicit formula.

Proposition A.19. *If $d = 2$ then for $\frac{r_1}{s_1} > \frac{r_2}{s_2}$ we have*

$$
C^{(1)}\,[\,{}^{\mu}_{a}\,|\varnothing] = \frac{1}{t_\mu}\left(\frac{(r_\mu^2-1)}{24\,r_\mu^2} - \frac{(r_\mu-1)}{r_\mu}\,(Q^\mu)^2\right)\delta_{a,s_\mu} - \frac{1}{t_\mu}\,Q^1\,Q^2\left(\frac{t_2}{t_1}\right)^{r_\nu(s_\mu-a)/\Delta}\delta_{\Delta|(s_\mu-a)}\,\delta_{s_\mu-a-\Delta_\mu\ge 0}
$$

where $\nu \neq \mu$. In the case $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ we have

$$C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right] = \frac{1}{t_\mu}\left(\frac{(r_\mu^2-1)}{24\,r_\mu^2} - \frac{(r_\mu-1)}{r_\mu}\,(Q^\mu)^2 - Q^1\,Q^2\,\frac{1}{1-\left(\frac{t_\nu}{t_\mu}\right)^{r_\mu}}\right)\delta_{a,s_\mu}\,.$$

If the input data satisfies the sufficient Airy condition we further identify $Q^1 = -Q^2$ and

$$F_{1,1}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\right] = C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right]\,.$$

Proof. What makes the case $d = 2$ easy to handle is that $[d] \setminus \{\mu_1, \mu_2\} = \varnothing$ in the second line of (A.55). Therefore, we only get a contribution from $M = \varnothing$ which is

$$C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right] = \cdots - Q^1\,Q^2\,\left(\mathcal{M}^{-1}\right)_{(\mu,a),(2,1)}\,.$$

Now plugging in the explicit value of $\mathcal{M}^{-1}$ given in (A.24), one obtains

$$C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right] = \cdots - Q^1\,Q^2\,t_\mu^{-1}\sum_{m\geq 0}(-1)^m\left(\Upsilon_0^\mu\right)^{-m-1}\sum_{\substack{\gamma_1,\ldots,\gamma_m>0\\ \sum_\ell \gamma_\ell = s_\mu-\Delta_\mu-a}}\prod_{\ell=1}^{m}\Upsilon^\mu_{\gamma_\ell}\,.$$

We have to distinguish between the following cases. First, let $r_1 = r_2$ and $s_1 = s_2$. In this case $\Delta_\mu = 0$ for all μ and

$$\Upsilon_a^\mu = \left(1-\left(\frac{t_\nu}{t_\mu}\right)^{r_\nu}\right)\,\delta_{a,0}$$

with $\nu \neq \mu$. Therefore, we only get a contribution for $m = 0$ which is

$$C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right] = \cdots - Q^1\,Q^2\,\frac{1}{t_\mu\left(1-\left(\frac{t_\nu}{t_\mu}\right)^{r_\mu}\right)}\,\delta_{s_\mu,a}\,.$$

In the second case $\frac{r_1}{s_1} > \frac{r_2}{s_2}$ we have

$$\Upsilon_a^1 = \delta_{a,0} - \left(\frac{t_2}{t_1}\right)^{r_2}\delta_{a,\Delta}\,, \qquad \Upsilon_a^2 = -\left(\frac{t_1}{t_2}\right)^{r_1}\delta_{a,0} + \delta_{a,\Delta}$$

if we set $\Delta := r_1 s_2 - r_2 s_1 > 0$. Consequently, in (A.19) we will only get a contribution if $s_\mu - \Delta_\mu - a$ is a positive multiple of Δ in which case $m = (s_\mu - \Delta_\mu - a)/\Delta$. Let us assume $s_\mu - \Delta_\mu - a \geq 0$. Then

$$C^{(1)}\left[\begin{smallmatrix}\mu\\ a\end{smallmatrix}\middle|\varnothing\right] = \cdots - Q^1\,Q^2\,t_\mu^{-1}\,\left(\Upsilon_0^\mu\right)^{(s_\mu-\Delta_\mu-a)/\Delta-1}\,\left(-\Upsilon_\Delta^\mu\right)^{(s_\mu-\Delta_\mu-a)/\Delta}\,\delta_{\Delta|(s_\mu-a)}\,.$$

Then plugging in the explicit values of Υ_a^μ yields the claim expression. □

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