Darstellungen von modulierten Graphen

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Geboren am 11. Juli 1993 in Waldbröl

13. März 2019

Bachelorarbeit Mathematik

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Zusammenfassung

Die Eigenschaften einer Algebra mit Hilfe einer Analyse ihrer Kategorie von Moduln zu studieren ist eine gewöhnliche Prozedur. Im Falle einer erblichen, endlich dimensionalen Algebra A über einem algebraisch abgeschlossenen Körper k lässt sich die Betrachtung von A-Moduln komplett in die Sprache von Köcherdarstellungen übersetzen [1, I.6, II.4]. Dies ist möglich, da sich für jede solche Algebra A ein Köcher Q konstruieren lässt, sodass die Kategorien von Moduln über der Pfadalgebra kQ und A äquivalent sind. Da des Weiteren $\mathsf{mod}(kQ)$ und die Kategorie der Darstellungen von Q äquivalent sind, sehen wir, dass die Darstellungstheorie von A komplett durch die von Q beschrieben wird.

Daher lassen sich einige starke Resultate aus der Darstellungstheorie von Köchern auf die Theorie von Algebren übertragen. Einen der zentralen Sätze stellt der Satz von Gabriel [9] dar, der alle Köcher endlichen Darstellungstyps klassifiziert, das heißt alle die Köcher deren Zahl an Isomorphismenklassen unzerlegbarer Darstellungen endlich ist.

Aber wie zuvor erwähnt, findet dieser Satz nur in der Darstellungstheorie von Algebren über algebraisch abgeschlossenen Körpern eine Anwendung. Folglich kann man sich die Frage stellen, ob es eine Verallgemeinerung der Theorie von Köcherdarstellungen gibt, die auf eine allgemeinere Klasse von Algebren anwendbar ist. Diese gesuchte Verallgemeinerung ist durch die Darstellungstheorie von Spezies gegeben. Gabriel führte sie zwar als erster ein [8], jedoch waren es Dlab und Ringel, die diese Theorie weiterentwickelten und so eine Erweiterung des Satzes von Gabriel beweisen konnten [6] [4], in welcher sie alle Spezies endlichen Darstellungstyps klassifizieren. In dieser Klassifizierung treten nun auch alle Dynkin Diagramme auf, die man aus der Klassifizierung halbeinfacher Lie Algebren kennt, und nicht bloß die des Typs $\mathbb{A}$, $\mathbb{D}$ und $\mathbb{E}$.

Das Ziel dieser Arbeit ist es die nötige Sprache und Werkzeuge zu entwickeln, um den Satz von Dlab und Ringel zu beweisen. Dabei folgen wir hauptsächlich [4]. Zu diesem Zweck fangen wir in Kapitel 2 bei Null an und beginnen mit einer Einführung in gewichtete Graphen (*valued graphs*), Spezies und deren Darstellungen. Anschließend führen wir den Tensor Ring $T(\mathcal{Q})$ einer Spezies $\mathcal{Q}$ ein, was uns zur einer Äquivalenz der Kategorie der Darstellungen von $\mathcal{Q}$ und der Kategorie endlich dimensionaler $T(\mathcal{Q})$-Rechtsmoduln führt. Diese Äquivalenz nutzen wir daraufhin um zu zeigen, dass die Kategorie von A-Rechtsmoduln, wobei A eine erbliche, endlich dimensionale Algebra über einem perfekten Körper k ist, komplett durch die Darstellungen einer assoziierten k-Spezies $\mathcal{Q}_A$ beschrieben werden kann.

Im folgenden Kapitel werden dann quadratische Formen gewichteter Graphen behandelt. Der erste wichtige Schritt ist dabei die Klassifizierung aller Graphen, deren quadratische Form positiv (semi-)definit ist. Wir werden sehen, dass diese Graphen genau durch die Dynkin Diagramme (beziehungsweise durch die Dynkin und erweiterten Dynkin Diagramme) gegeben sind. Es schließt sich

eine Analyse der Wurzelsysteme dieser Graphen an. Zu diesem Zweck führen wir Coxeter Transformationen ein, welche uns einen tieferen Einblick in die Struktur der betrachteten Wurzelsysteme erlauben.

In Kapitel 4 kehren wir zu Darstellungen von Spezies zurück und definieren Reflexionsfunktoren und mit diesen die Coxeter Funktoren auf eine ähnliche Weise, wie sie Bernstein, Gel'fand and Ponomarev [3] für gewöhnliche Köcher eingeführt haben. Anschließend verbinden wir unsere Konstruktion von Wurzelsystemen mit Hilfe von Coxeter Transformationen mit der Wirkung der Coxeter Funktoren auf unzerlegbaren Darstellungen, um den Hauptsatz dieser Arbeit zu beweisen — die Erweiterung von Dlab und Ringel zum Satz von Gabriel: Eine Spezies ist endlichen Darstellungstyps genau dann, wenn ihr unterliegender Graph Dynkin ist. In diesem Falle induziert der Dimensionvektor eine Bijektion zwischen den Isomorphieklassen unzerlegbarer Darstellungen und den positiven Wurzeln des unterliegenden Graphen.

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CHAPTER 1

Introduction

Studying the properties of an algebra by analysing its category of modules is a regular procedure. In the case of an hereditary, finite dimensional algebra A over an algebraically closed field k the analysis of A-modules fully translates into the language of quiver representations [1, I.6, II.4]. This is due to the fact that for each such algebra A one can construct a quiver Q such that the module categories of the path algebra kQ and A are equivalent and since on the other hand $\mathsf{mod}(kQ)$ and the category of representations of Q are equivalent we see that the representation theory of A is fully described by the one of Q.

Therefore one can transfer a lot of strong results from the study of quiver representations to the theory of algebras. One of the main results marks Gabriel's theorem which classifies all quivers of finite representation type, ie. those quivers featuring a finite number of isomorphism classes of indecomposable representations:

Theorem 1.1 (Gabriel's theorem [9])**.** *Let Q be a connected Quiver.*

1. *Then Q is of finite representation type if and only if Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$.*

2. *If Q is of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, the isomorphism classes of indecomposable representations of Q are in one-to-one correspondence with the positive roots of Q.*

But as mentioned above, this theorem only finds application in the representation theory of algebras over algebraically closed fields. So one could ask the question whether there is a generalisation of the theory of quiver representations which is applicable to more general algebras.

This generalisation is provided by the representation theory of species. Gabriel was the first to introduce this notion [8] but it were Dlab and Ringel who further developed this theory in order to prove a gerneralisation of Gabriel's theorem [6] [4] classifying all species of finite representation type. In this classification now all Dynkin diagrams form the classification of semisimple Lie algebras occur and not only those of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$.

The goal of this thesis is to establish all the needed language and tools in order to prove the theorem of Dlab and Ringel mainly following [4]. For this purpose starting from zero chapter 2 gives an introduction to valued graphs, species and their representations. After that we introduce the tensor ring $T(\mathcal{Q})$ of a species $\mathcal{Q}$ leading to an equivalence of the category of representations of $\mathcal{Q}$ and the category of finitely generated right $T(\mathcal{Q})$-modules. This equivalence is then used in order to show that the category of modules over a finite dimensional hereditary algebra A over a perfect field k is fully described by the representations of an associated k-species $\mathcal{Q}_A$.

The following chapter then treats quadratic forms of valued graphs. There the first important step will be the classification of all graphs leading to positive (semi-)definite quadratic forms. We will see that these valued graphs are exactly given by the Dynkin (respectively the Dynkin and extended Dynkin) diagrams. It then follows an analysis of the root systems of these graphs. For this purpose we will introduce Coxeter transformations giving us more insight into the structure of the considered root systems.

In chapter 4 we then come back to the representations of species and define reflection functors and with these the Coxeter functors in a similar way as it was done by Bernstein, Gel'fand and Ponomarev [3] for quiver representations. We then connect our construction of root systems via Coxeter transformations with the action of Coxeter functors on indecomposable representations in order to prove the main theorem of this thesis — Dlab and Ringel's extension of Gabriel's theorem: A species is of finite representation type if and only if its underlying graph is Dynkin. In this case the dimension vector induces a bijection between the isomorphism classes of indecomposable representations and the positive roots of the underlying graph.

CHAPTER 2

Species and their Representations

Instead of just stating all the needed definition right away, remember our motivation for studying a generalisation of quiver representations. So let us start with the definition of a quiver.

Definition 2.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and maps $s, t : Q_1 \to Q_0$ mapping an arrow $\alpha \in Q_1$ to its source $s(\alpha)$ resp. target $t(\alpha)$. Such an arrow $\alpha \in Q_1$ will be graphically represented by

$$s(\alpha) \xrightarrow{\quad\alpha\quad} t(\alpha).$$

As already outlined, we are aiming for a description of finite dimensional (hereditary) algebras A over a field k from which we want to demand as less as possible.
In the case where k is algebraically closed and A is basic we have $A/\operatorname{rad} A = \prod_{i=1}^{n} k$ where $\operatorname{rad} A$ is the Jacobson radical of A [1, I.6]. When constructing the quiver associated with A one can identify these n factors of k with the vertices and the number of arrows from i to j is given by the dimension of the k-vectorspace $e_i \left(A/(\operatorname{rad} A)^2\right) e_j$ where e_i is the idempotent element of the i-th factor in $A/\operatorname{rad} A$.
Now let k be a general field. In this setting, if we again assume A to be basic, we obtain $A/\operatorname{rad} A = \prod_{i=1}^{n} F_i$ where the F_i are now division rings in general [5]. Comparing with the special case before these n division rings should correspond to our vertices. But getting the idea of how to define arrows is first of all not obvious as $e_i \left(A/(\operatorname{rad} A)^2\right) e_j =: {}_iM_j$ now is an (F_i, F_j)-bimodule. So it arises the question whether one should take the dimension of ${}_iM_j$ as a left F_i-module or right F_j-module for the number of vertices from i to j. The answer is one should take both — in a sense leading to the definition of valued quivers. Note that if one computes the dimension of ${}_iM_j$ as a k-vectorspace, we obtain the relation

$$[F_i : k] \cdot \dim_{F_i} {}_iM_j = \dim_k {}_iM_j = [F_j : k] \cdot \dim_{F_j} {}_iM_j.$$

This relation will be incorporated in the definition of a valued quiver.

2.1 Valued Quivers

Having the above motivation in mind we can now give a first definition of a valued quiver. However, a reader who wants to skip the end of this chapter should be warned that there will be given a redefinition of valued quivers in 2.13 which lives up to findings being made in the following sections. A first working-definition is as follows.

Definition 2.2. A *(relative) valued graph* (Γ, d) is an undirected graph $\Gamma = (\Gamma_0, \Gamma_1, \psi)$ where Γ_0 is the set of vertices, Γ_1 the set of edges, and ψ a map $\Gamma_1 \to \{\{i,j\} | i,j \in \Gamma_0\}$ together with a valuation d which consists of positive integers d_{ij}^{α}, d_{ji}^{α} for all edges $\alpha : i \text{---} j \ \in \Gamma_1$ such that there exist strictly positive integers f_i for all $i \in \Gamma_0$ which satisfy

$$d_{ij}^{\alpha}\, f_j = d_{ji}^{\alpha}\, f_i \tag{2.1}$$

for all $\alpha : i \text{---} j \ \in \Gamma_1$. We will use the notation

$$i \ \overset{(d_{ij}^{\alpha}, d_{ji}^{\alpha})}{\text{------}} \ j$$

for edges $\alpha : i \text{---} j \ \in \Gamma_1$ and will omit $(d_{ij}^{\alpha}, d_{ji}^{\alpha})$ in the case it is equal to $(1,1)$.
A tuple (Γ, d) is a *(relative) valued quiver* if Γ is a quiver and d is a valuation on the underlying undirected graph. A sequence $i_1 \to i_2 \to \cdots \to i_k = i_1$ of arrows in a valued quiver is called an *oriented circuit*.
All graphs and quivers will be assumed to be finite in the sense that Γ_0 and Γ_1 are finite sets.

Remark 2.1. In this thesis we will denote relative valued quivers and graphs simply as valued quivers respectively graphs. For a discussion on the difference between absolute and relative valued quivers see [14].
By definition we assume the set of vertices Γ_0 of a valued graph to be finite. For convenience we will always assume that it is of the form $\Gamma = \{1, \dots, n\}$ and, if not stated otherwise, n will always denote the cardinality of Γ_0.

Example 2.1. Consider the following graphs together with the indicated valuation.

$$\Gamma: \ 1 \ \overset{(1,2)}{\text{------}} \ 2, \qquad \Gamma': \ 1 \ \overset{(3,1)}{\text{------}} \ 2 \ \overset{(2,4)}{\text{------}} \ 3, \qquad \Gamma'': \begin{tikzcd} 1 \arrow[r, dash, bend left, "{(1,2)}"] \arrow[r, dash, bend right] & 2 \end{tikzcd}$$

Then both Γ and Γ' are valued graphs as for Γ the choice $f_1 = 1$ and $f_2 = 2$ satisfies eq. (2.1). For Γ' one can convince oneself that $f_1 = 3$, $f_2 = 1$ and $f_3 = 2$ constitutes a possible choice.
In turn Γ'' does not define a valued graph as (2.1) would impose both $2f_1 = f_2$ and $f_1 = f_2$ which is a contradiction as the integers f_i are needed to be non-zero.

Note that in the first instance we allow parallel edges and arrows, loops, and possibly disconnected quivers in our definition. This is different to most other publications. The question why these cases can be left out of the discussion will be answered in section 2.6.

Example 2.2. In order to convince oneself that definition 2.2 of a valued quiver is a generalisation of the one of a quiver as given in 2.1 consider a quiver $Q = (Q_0, Q_1)$. By setting $d_{ij}^{\alpha} = d_{ji}^{\alpha} = 1$ for all $\alpha : i \to j \ \in Q_1$ the quiver Q becomes a valued quiver. We call this valuation the trivial valuation on Q.

Example 2.3. If (Γ, d) is a valued graph, we can define a new valuation $d^{\,\mathrm{op}}$ on Γ by setting

$$(d^{\,\mathrm{op}})^{\alpha}_{ij} := d^{\alpha}_{ji}$$

for all $\alpha : i \,\text{---}\, j \in \Gamma_1$. In order to convince ourselves that this indeed defines an allowed valuation let f_i be positive integers satisfying $d^{\alpha}_{ij} f_j = d^{\alpha}_{ji} f_i$ and use them to define

$$f_i' := \prod_{i \neq j \in \Gamma_0} f_j.$$

Then these integers satisfy

$$(d^{\,\mathrm{op}})^{\alpha}_{ij} f_j' = d^{\alpha}_{ji} f_i \left(\prod_{\substack{k \in \Gamma_0 \\ i \neq k \neq j}} f_k \right) = d^{\alpha}_{ij} f_j \left(\prod_{\substack{k \in \Gamma_0 \\ i \neq k \neq j}} f_k \right) = (d^{\,\mathrm{op}})^{\alpha}_{ji} f_i'$$

which implies that $(\Gamma, d^{\,\mathrm{op}})$ is a valued graph. A concrete non-trivial example of a pair of two valued graphs with opposite valuation are the Dynkin diagrams $\mathbb{B}_n$ and $\mathbb{C}_n$ given in fig. C.1.

Example 2.3 will be of importance at a later point. It turns out that the representation theory of (Γ, d) and $(\Gamma, d^{\,\mathrm{op}})$ runs quite analogously. Defining such representations of valued quivers will be the main topic of the subsequent sections.

2.2 Species

Actually, one does not talk about representations of valued quivers but rather about representations of species. Species are defined over a valued quiver and contain some more important data. Having the motivation from the beginning of the chapter in mind, we remember that aiming for a description of possibly general k-algebras one should associate division rings with the vertices and bimodules with the edges. These ideas plus some additional relations, whose importance become clear at a later point, are the ingredients of the following definition.

Definition 2.3. For (Γ, d) a valued graph a *modulation* $\mathbb{M}$ of (Γ, d) consists of a collection $\{F_i\}_{i \in \Gamma_0}$ of division rings and (F_i, F_j)-bimodules ${}_iM^{\alpha}_j$ and (F_j, F_i)-bimodules ${}_jM^{\alpha}_i$ for each edge $\alpha : i \,\text{---}\, j \in \Gamma_0$ such that

(i) there are isomorphisms of (F_i, F_j)-bimodules

$$ {}_jM^{\alpha}_i \cong \mathrm{Hom}_{F_i}({}_iM^{\alpha}_j, F_i) \cong \mathrm{Hom}_{F_j}({}_iM^{\alpha}_j, F_j), \tag{2.2}$$

(ii) $\dim_{F_j}({}_iM^{\alpha}_j) = d^{\alpha}_{ij}$ and $\dim_{F_i}({}_jM^{\alpha}_i) = d^{\alpha}_{ji}$

for every edge $\alpha : i \,\text{---}\, j \in \Gamma_1$. We will often call ${}_jM^{\alpha}_i$ the *dual bimodule* of ${}_iM^{\alpha}_j$ and vice versa. We will regularly use the notation

$$F_i \xrightarrow{\;{}_iM^{\alpha}_j,\, {}_jM^{\alpha}_i\;} F_j \tag{2.3}$$

for specifying the modulation on an edge $\alpha : i \text{ --- } j \ \in \Gamma_1$ and will sometimes omit specifying the dual bimodule ${}_jM_i^\alpha$ in such cases where its form can be deduced from the context.
A *species* is a pair $(\mathbb{M}, \Gamma)$ consisting of a relative valued quiver Γ together with a modulation $\mathbb{M}$ of its underlying undirected graph. Again we will use the graphical notation introduced in eq. (2.3) adding an orientation to the edges when specifying a species.

Remark 2.2.
(i) Given a modulation $\mathbb{M}$ of a valued graph (Γ, d) one can easily reconstruct (Γ, d) from $\mathbb{M}$. Therefore the only information which distinguishes a modulation from a species is that the latter additionally features an orientation of the underlying graph.

(ii) A brief introduction to the notion of bimodules is given in appendix A and division rings are discussed in appendix B. Regarding eq. (2.2) the (F_j, F_i)-bimodule structure on $\mathrm{Hom}_{F_i}({}_iM_j^\alpha, F_i)$ is defined via

$$(\mu\,\varphi\,\lambda)(m) := \varphi(m\,\mu)\,\lambda$$

for all $\lambda \in F_i$, $\mu \in F_j$, $m \in {}_iM_j^\alpha$ and $\varphi \in \mathrm{Hom}_{F_i}({}_iM_j^\alpha, F_i)$.

Analogously a (F_j, F_i)-bimodule structure on $\mathrm{Hom}_{F_j}({}_iM_j^\alpha, F_j)$ is given by

$$(\mu\,\psi\,\lambda)(m) := \mu\,\psi(\lambda\,m)$$

for all $\lambda \in F_i$, $\mu \in F_j$, $m \in {}_iM_j^\alpha$ and $\psi \in \mathrm{Hom}_{F_j}({}_iM_j^\alpha, F_j)$.

(iii) Given a modulation $\mathbb{M}$ of a valued graph containing a trivially valued edge $\alpha : i \xrightarrow{(1,1)} j$ we have $F_i \cong F_j$ as rings and ${}_iM_j^\alpha \cong {}_{F_i}F_{i\,F_i}$ as bimodules.

Proof. First note that since $\dim_{F_i} {}_jM_i^\alpha = 1$ there exists an isomorphism $\varphi : F_i \to {}_jM_i^\alpha$ of right F_i-modules. And since also $F_j \cong {}_iM_j^\alpha$ as right F_j-modules there exists an isomorphism

$$\psi' : F_j \xrightarrow{\cong} \mathrm{Hom}_{F_j}(F_j, F_j) \xrightarrow{\cong} \mathrm{Hom}_{F_j}({}_iM_j^\alpha, F_j) \xrightarrow{\cong} {}_jM_i^\alpha$$

of right F_j-modules. Now fix $m := \varphi(1_{F_i})$ and let $\lambda \in F_j$ such that $\psi'(\lambda) = m$. Then perform a rescaling $\psi := \psi'\lambda$. This map now satisfies

$$\psi(1_{F_j}) = \psi'(1_{F_j})\lambda = \psi'(\lambda) = m = \varphi(1_{F_i}).$$

If we now set $\Psi := \psi^{-1} \circ \varphi$ it is obvious that this map is bijective, additive, and sending 1_{F_i} to 1_{F_j}. In order to prove that it is an isomorphism of rings it is therefore only left to prove multiplicity. For this purpose let $a, b \in F_i$. Then using $m\Psi(a) = am$ one obtains

$$\psi(\Psi(a)\Psi(b)) = m(\Psi(a)\Psi(b)) = (m\Psi(a))\Psi(b) = (am)\Psi(b) = a(m\Psi(b)) = a(bm) = \varphi(ab)$$

which is equivalent to $\Psi(a)\Psi(b) = \Psi(ab)$ and thus $F_i \cong F_j$. $\square$

(iv) If F is a field and G a subfield the dual bimodule of ${}_FF_G$ is given by ${}_GF_F$. When giving examples of modulations we will therefore often simply omit specifying the dual bimodules as their definition is clear from the context. Such an example is given in the following.

Example 2.4. A modulation of the valued graph

$$\mathbb{B}_2: \ 1 \xrightarrow{(1,2)} 2$$

is given by

$$\mathbb{M}_1: \ \mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C}$$

since as required $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{R})$ as $(\mathbb{C},\mathbb{R})$-Bimodules, $\dim_{\mathbb{R}} \mathbb{C} = 2$, and $\dim_{\mathbb{C}} \mathbb{C} = 1$. Analogously one can convince oneself that also

$$\mathbb{M}_2: \ \mathbb{Q} \xrightarrow{\mathbb{Q}(\sqrt{2})} \mathbb{Q}(\sqrt{2})$$

defines a modulation on $\mathbb{B}_2$.

Note that different from our motivation we did not demand the division rings F_i to be k-algebras over some common field k. Such species are a special case.

Definition 2.4. Let k be a field. We call a species $(\mathbb{M},\Gamma)$ a k-*species* if the modulation $\mathbb{M} = (F_i, {}_iM_j^\alpha)$ satisfies the following properties.

(i) F_i is a finite dimensional k-division algebra for all $i \in \Gamma_0$.

(ii) k acts centrally on ${}_iM_j^\alpha$ for all edges $\alpha : i \text{ — } j$.

However, not every species is a k-species since there exist division rings which are infinite dimensional over their center. For an example see B.1 in the appendix or consult [13, pp. 216ff.]. A discussion on the difference between species and k-species is given in [14]. Even though some proofs are easier performed in the setting of k-species, we will entirely work in the more general setting in this thesis aiming towards the description of a special class of artinian rings and not only of certain k-algebras.

2.3 Representations and Morphisms

Given a species $(\mathbb{M},\Gamma)$ the important observation to make is that for every arrow $\alpha : i \to j \in \Gamma_1$ and X_i, X_j right F_i respectively F_j-modules the tensor product $X_i \otimes_{F_i} {}_iM_j^\alpha$ becomes a right F_j-module via

$$(x \otimes m)\mu := x \otimes (m\mu)$$

for $\mu \in F_j$ and $x \otimes m \in X_i \otimes_{F_i} {}_iM_j^\alpha$. This gives rise to F_j-linear maps $X_i \otimes_{F_i} {}_iM_j^\alpha \to X_j$ and it turns out assigning such a map to each arrow $\alpha \in \Gamma_1$ is just the right idea in order to define representations of a species.

Definition 2.5. A *representation* $\mathbf{X} = (X_i, {}_j\varphi_i^\alpha)$ of a species $(\mathbb{M},\Gamma)$ consists of finitely generated right F_i-modules X_i for all $i \in \Gamma_0$ and F_j-linear maps

$${}_j\varphi_i^\alpha : X_i \otimes_{F_i} {}_iM_j^\alpha \to X_j$$

for every arrow $\alpha : i \to j \ \in \Gamma_1$. We call $\underline{\dim}\,\mathbf{X} = (\dim_{F_i} X_i)_{i\in\Gamma_0} \in (\mathbb{Z}_{\geq 0})^n$ the *dimension vector* of $\mathbf{X}$. Regularly we will use the graphical notation

$$X_i \xrightarrow{{}_j\varphi_i^\alpha} X_j$$

for specifying $\mathbf{X}$ on an arrow $\alpha : i \to j \in \Gamma_1$.

Example 2.5. Let $\mathcal{Q}$ be the species $\mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C}$. In order to construct a representation of $\mathcal{Q}$ we need to assign vector spaces $X_1 \cong \mathbb{R}^k$ and $X_2 \cong \mathbb{C}^m$ to the vertices and a $\mathbb{C}$-linear map $\varphi : \mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}^m$ to the arrow, which means we can associate φ with an element in $\mathrm{Mat}^{m\times k}(\mathbb{C})$. Therefore

$$\begin{aligned}
\mathbf{X}: &\quad \mathbb{R} \xrightarrow{\quad 0 \quad} 0\\
\mathbf{X}': &\quad \mathbb{R} \xrightarrow{\begin{pmatrix}1\\ i\end{pmatrix}} \mathbb{C}^2\\
\mathbf{X}'': &\quad \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1 & 0 & 0\\ 0 & i & 0\end{pmatrix}} \mathbb{C}^2
\end{aligned}$$

all define representations of $\mathcal{Q}$. Their dimension vectors are $\underline{\dim}\,\mathbf{X} = (1, 0)$, $\underline{\dim}\,\mathbf{X}' = (1, 2)$, and $\underline{\dim}\,\mathbf{X}' = (3, 2)$.

Definition 2.6. Let $\mathbf{X} = (X_i, {}_j\varphi_i^\alpha)$, $\mathbf{Y} = (Y_i, {}_j\psi_i^\alpha)$ be representations of a species $\mathcal{Q} = (\mathbb{M}, \Gamma)$. Then a *morphism* $f : X \to Y$ of representations of $\mathcal{Q}$ consists of F_i-linear maps $f_i : X_i \to Y_i$ for each vertex $i \in \Gamma_0$ such that the diagram

$$\begin{tikzcd}
X_i \otimes_{F_i} {}_iM_j^\alpha \arrow[r, "{}_j\varphi_i^\alpha"] \arrow[d, "(f_i\otimes \mathrm{id}_{{}_iM_j^\alpha})"'] & X_j \arrow[d, "f_j"] \\
Y_i \otimes_{F_i} {}_iM_j^\alpha \arrow[r, "{}_j\psi_i^\alpha"] & Y_j
\end{tikzcd} \tag{2.4}$$

commutes for all arrows $\alpha : i \to j \ \in \Gamma_1$. Let $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ denote the set of morphisms between the representations $\mathbf{X}$ and $\mathbf{Y}$.

With this definition one sees that the representations of a species $\mathcal{Q}$ together with morphisms as defined above form a category denoted by $\mathsf{rep}(\mathcal{Q})$ where the composition of morphisms is defined component-wise. One can moreover prove that this category is abelian. However, due to lack of space this discussion is left out.

Example 2.6. Let us again consider the representations $\mathbf{X}$ and $\mathbf{X}'$ from example 2.5 and determine the set of morphisms $\mathrm{Hom}(\mathbf{X}', \mathbf{X})$ between them. A general element $f \in \mathrm{Hom}(\mathbf{X}', \mathbf{X})$ is of the form $f = (f_1, f_2)$ where f_1 is the multiplication with an element $a \in \mathbb{R}$ and $f_2 : \mathbb{C} \to 0$ must necessarily be the zero morphism. Moreover f must satisfy the commutation relation (2.4) which means that the outer square of

$$\begin{tikzcd}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, "\cong"] \arrow[d, "(f_1\otimes \mathrm{id}_{\mathbb{C}})"'] & \mathbb{C} \arrow[r, "\begin{pmatrix}1\\ i\end{pmatrix}"] \arrow[d, "a"] & \mathbb{C}^2 \arrow[d, "f_2=0"] \\
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, "\cong"] & \mathbb{C} \arrow[r, "0"] & 0
\end{tikzcd} \tag{2.5}$$

has to commute. However, this is satisfied for all $a \in \mathbb{R}$. Therefore $\mathrm{Hom}(\mathbf{X}', \mathbf{X}) \cong \mathbb{R}$ as a $\mathbb{R}$-vectorspace.
In general the set of morphisms between two representations does not carry a vectorspace structure. In this example this is due to the fact that the considered species was a $\mathbb{R}$-species.

Before we go on let us first convince ourself that the definitions given so far are really generalisations of representations of ordinary quivers. They are defined as follows.

Definition 2.7. A *representation* $\mathbf{X} = (X_i, \varphi_\alpha)$ of a quiver $Q = (Q_0, Q_1)$ over a field k consists of k-vector spaces X_i for each $i \in Q_0$ together with k-linear maps

$$\varphi_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$$

for each $\alpha \in Q_1$.

Remark 2.3. We have already seen that for $Q = (Q_0, Q_1)$ a quiver we can put a valuation on it by simply setting $d_{ij}^\alpha = d_{ji}^\alpha = 1$ for all $\alpha : i \to j \ \in Q_1$. We can further define a modulation $\mathbb{M}_k$ on Q by setting $F_i = {}_iM_j^\alpha = k$ for all $i \in Q_0$ and $\alpha : i \to j \in Q_1$.

Now let $\mathbf{X} = (X_i, \varphi_\alpha)$ be a representation of the quiver Q over the field k. Then this representation can be interpreted as a representation $(X_i, {}_j\varphi_i^\alpha)$ of the species $(\mathbb{M}_k, Q)$ via

$${}_j\varphi_i^\alpha : X_i \otimes_{F_i} \underbrace{{}_iM_j^\alpha}_{=k} \xrightarrow{\cong} X_i \xrightarrow{\varphi_\alpha} X_j.$$

The same identification relates morphisms of representations of Q with those of $(\mathbb{M}_k, Q)$.

So consequently we can deduce an equivalence between $\mathsf{rep}(\mathbb{M}_k, Q)$ the category of representations of the species $(\mathbb{M}_k, Q)$ and $\mathsf{quiv\text{-}rep}_k(Q)$ the category of representations of the quiver Q over k. Remembering that there is also an equivalence between $\mathsf{quiv\text{-}rep}_k(Q)$ and $\mathsf{mod}(kQ)$ the category of finitely generated right kQ-modules, where kQ is the path algebra associated to Q, one could ask whether there is a generalisation of the notion of a path algebra to the case of arbitrary species. This question will be answered in section 2.5.

2.4 Direct sums and indecomposable representations

This section shall introduce all notions needed to formally formulate the goal of this thesis. Let $\mathcal{Q} = (\mathbb{M}, \Gamma)$ be a species.

Definition 2.8. For $\mathbf{X} = (X_i, {}_j\varphi_i^\alpha)$ and $\mathbf{Y} = (Y_i, {}_j\psi_i^\alpha)$ representations of $\mathcal{Q}$ we define their *direct sum* $\mathbf{X} \oplus \mathbf{Y}$ to be

$$\left(X_i \oplus Y_i,\ {}_k\varphi_j^\alpha \oplus {}_k\psi_j^\alpha\right)_{i \in \Gamma_0,\, \alpha: j \to k \in \Gamma_1}$$

where the direct sums above are taken between F_i modules respectively F_i-linear maps. The sum of any finite number of representations $\mathbf{X}_1, \ldots, \mathbf{X}_m$ of $\mathcal{Q}$ is defined recursively by

$$\mathbf{X}_1 \oplus \cdots \oplus \mathbf{X}_n := (\mathbf{X}_1 \oplus \cdots \oplus \mathbf{X}_{m-1}) \oplus \mathbf{X}_m.$$

Remark 2.4. (i) The direct sum of two representations is itself a representation and satisfies the universal property of the coproduct in $\mathsf{rep}(\mathbb{M}_k, Q)$.

(ii) Taking direct sums is associative. Thus the order in the definition of the direct sum of arbitrary many representations is not important.

Example 2.7. Let $\mathcal{Q}$ be the species

$$\mathcal{Q}: \qquad \mathbb{C} \xleftarrow{\ \mathbb{C}\ } \mathbb{R} \xrightarrow{\ \mathbb{R}\ } \mathbb{R}$$

over the valued graph $1 \overset{(2,1)}{\textemdash\!\textemdash} 2 \textemdash\!\textemdash 3$. Then the direct sum of the representations

$$\mathbf{X}: \qquad \mathbb{C}^2 \xleftarrow{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}} \mathbb{R}^2 \xrightarrow{\ 0\ } 0$$

$$\mathbf{X}': \qquad \mathbb{C} \xleftarrow{\ (1)\ } \mathbb{R} \xrightarrow{\ (0\ 1)\ } \mathbb{R}$$

is given by

$$\mathbf{X} \oplus \mathbf{X}': \quad \mathbb{C}^2 \oplus \mathbb{C} \xleftarrow{\begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{R}^2 \oplus \mathbb{R} \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} 0 \oplus \mathbb{R}$$

which in turn is isomorphic to

$$\mathbb{C}^3 \xleftarrow{\begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\ (0\ 0\ 1)\ } \mathbb{R}\ .$$

Definition 2.9. (i) We say a representation $\mathbf{X} = (X_i, {}_j\varphi_i^\alpha)$ of $\mathcal{Q}$ is *indecomposable* if $\mathbf{X} \neq 0$ and there do not exist non-zero representations $\mathbf{X}_1, \mathbf{X}_2 \in \mathsf{rep}(\mathcal{Q})$ such that $\mathbf{X} \cong \mathbf{X}_1 \oplus \mathbf{X}_2$.

(ii) A species $\mathcal{Q}$ is said to be of *finite representation type* if the number of isomorphism classes of indecomposable representations in $\mathsf{rep}(\mathcal{Q})$ is finite.

Example 2.8. Consider the representations

$$\mathbf{X}: \qquad \mathbb{R} \xrightarrow{\ 0\ } 0$$

$$\mathbf{X}': \qquad \mathbb{R}^2 \xrightarrow{\ (1\ 0)\ } \mathbb{C}$$

of the species $\mathbb{R} \xrightarrow{\ \mathbb{C}\ } \mathbb{C}$. Then obviously $\mathbf{X}$ must be indecomposable while $\mathbf{X}'$ is not as it contains $\mathbf{X}$ as a direct summand.

With all the needed definitions at hand we can now formally state the goal of this thesis:

Find all species of finite representation type and classify their isomorphism classes of indecomposable representations.

One is interested in the isomorphism classes of indecomposable representations as these are the uniquely determinable building blocks of general representations. The uniqueness of the decomposition into indecomposable representations is due to the Krull-Schmidt theorem and will become clear in the next section when we will reinterpret the category of representations of a species as a module category.

Moreover it is now clear why we can demand quivers to be connected in order to classify all indecomposable representations. Since every representation of a species whose quiver is not connected can be decomposed into a direct sum of representations of the underlying connected components of the quiver, all indecomposable representations are already given by those of the connected components.

2.5 Tensor rings

We have already come up with the question whether there is a generalisation of the notion of a path algebra kQ of a quiver Q to the case of species $\mathcal{Q}$. The answer is yes, but before a definition can be given some previous constructions are needed. Let $\mathcal{Q} = (\Gamma, (F_i, {}_iM_j^\alpha)_{i\in\Gamma_0, \alpha:i-j\in\Gamma_1})$ be a species for which we set $D := \prod_{i=1}^n F_i$ and $M := \bigoplus_{\alpha:i\to j\in\Gamma_1} {}_iM_j^\alpha$.

Proposition 2.1. *Then D is a ring and M becomes a (D,D)-bimodule via*

$$D \times M \longrightarrow M,\ ((\lambda_i)_{i\in\Gamma_0}, (m_\alpha)_{\alpha\in\Gamma_2}) \longmapsto (\lambda_{s(\alpha)} m_\alpha)_{\alpha\in\Gamma_2}$$

and

$$M \times D \longrightarrow M,\ ((m_\alpha)_{\alpha\in\Gamma_2}, (\mu_i)_{i\in\Gamma_0}) \longmapsto (m_\alpha \mu_{t(\alpha)})_{\alpha\in\Gamma_2}.$$

Proof. This proof is a straight forward calculation and is therefore left out. □

Definition 2.10. We now define the *tensor ring* $T(\mathcal{Q})$ of the species $\mathcal{Q}$ to be

$$T(\mathcal{Q}) := \bigoplus_{m=0}^{\infty} T_m(\mathcal{Q}),$$

where $T_0(\mathcal{Q}) := D$ and

$$T_m(\mathcal{Q}) := T_{m-1}(\mathcal{Q}) \otimes_D M$$

for $m \geq 1$.

Proposition 2.2. *The tensor ring $T(\mathcal{Q})$ together with the tensor product between the summands*

$$T_k(\mathcal{Q}) \times T_m(\mathcal{Q}) \xrightarrow{\ \otimes\ } T_k(\mathcal{Q}) \otimes_D T_m(\mathcal{Q}) \xrightarrow{\ \cong\ } T_{k+m}(\mathcal{Q})$$

is indeed a ring.

Proof. By construction $T(\mathcal{Q})$ is a D-module and thus $(T(\mathcal{Q}), +)$ is an abelian group. Regarding the multiplication in $T(\mathcal{Q})$ associativity and distributivity follow from the associativity respectively distributivity of the tensor product between modules. The identity element is given by $(1_{F_i})_{i\in\Gamma_0} \in D \subseteq T(\mathcal{Q})$. □

Remark 2.5. Let Q be a quiver together with the trivial valuation and the modulation $\mathbb{M}_k$ over a field k as constructed in remark 2.3. Now observe that in this case $D = k^{Q_0}$ and $M = k^{Q_1}$. Let $m(\alpha) \in M$ denote the element whose entries $m(\alpha)_\beta$ are all zero for $\beta \neq \alpha$ and $m(\alpha)_\alpha = 1$. Then by definition of the bimodule structure on M and the product in $T(\mathbb{M}_k, Q)$ we find that

$$\begin{aligned}(m(\alpha)_\gamma \lambda_{t(\gamma)})_{\gamma \in Q_1} \cdot m(\beta) &= (m(\alpha)\lambda) \cdot m(\beta) \\ &= (m(\alpha)\lambda) \otimes_D m(\beta) \\ &= m(\alpha) \otimes_D (\lambda m(\beta)) \\ &= m(\alpha) \cdot (\lambda m(\beta)) \\ &= m(\alpha) \cdot (\lambda_{s(\gamma)} m(\beta)_\gamma)_{\gamma \in Q_1}\end{aligned}$$

for all $\lambda \in D$ and $\alpha, \beta \in Q_1$. So especially if we choose $\lambda = e_{t(\alpha)}$, where $\{e_i\}_{i \in Q_0}$ denotes the canonical basis of $D = k^{Q_0}$, we find

$$m(\alpha) \cdot m(\beta) = \big(m(\alpha)\, e_{t(\alpha)}\big) \cdot m(\beta) = m(\alpha) \cdot \big(e_{t(\alpha)}\, m(\beta)\big) \begin{cases} = 0 & , t(\alpha) \neq s(\beta) \\ \neq 0 & , t(\alpha) = s(\beta) \end{cases}. \tag{2.6}$$

Since the elements of the form $m(\alpha)$ from a basis of M we can deduce that elements $m(\alpha_1) \cdot m(\alpha_2) \cdot \ldots \cdot m(\alpha_k)$ where $\alpha_1 \alpha_2 \ldots \alpha_k$ is a path in Q form a basis of $T_k(\mathbb{M}_k, Q)$ for $k \geq 1$. So now it is obvious that the k-linear continuation of

$$\begin{array}{ccc} T(\mathbb{M}_k, Q) & \longrightarrow & kQ \\ m(\alpha_1) \cdot m(\alpha_2) \cdot \ldots \cdot m(\alpha_k) & \longmapsto & \alpha_1 \alpha_2 \ldots \alpha_k \\ e_i & \longmapsto & \epsilon_i \end{array}$$

is an isomorphism of rings, where ϵ_i is the lazy path at vertex $i \in Q_0$.
We can therefore deduce that the tensor ring is indeed the generalisation of a path algebra to the case of a general species. There is also a straight forward generalisation of the equivalence of the category of representations of a quiver and the one of finitely generated right modules over its associated path algebra. But first we require the following proposition.

Proposition 2.3. *The ring $T(\mathcal{Q})$ is a finitely generated D-module if $\mathcal{Q}$ is a species without oriented circuits.*

Proof. First note that in eq. (2.6) we have seen that for $m \in {}_{s(\alpha)}M^{\alpha}_{t(\alpha)}$ and $m' \in {}_{s(\beta)}M^{\beta}_{t(\beta)}$ we have $m \otimes m' = 0$ if $t(\alpha) \neq s(\beta)$. This means that ${}_{s(\alpha)}M^{\alpha}_{t(\alpha)} \otimes_D {}_{s(\beta)}M^{\beta} t(\beta) = 0$ if $t(\alpha) \neq s(\beta)$. Now let Γ_k denote the set of all paths $c = \alpha_1 \cdots \alpha_k$ of length k in the quiver Γ. Then define $M(c) := {}_{s(\alpha_1)}M^{\alpha_1}_{t(\alpha_1)} \otimes_D \cdots \otimes_D {}_{s(\alpha_k)}M^{\alpha_k}_{t(\alpha_k)}$ for such a $c \in \Gamma_k$ in order to see that each summand of the tensor ring decomposes into

$$T_k(\mathcal{Q}) = M \otimes_D M \otimes_D \cdots \otimes_D M = \bigoplus_{c \in \Gamma_k} M(c).$$

Since every ${}_{s(\alpha)}M^{\alpha}_{t(\alpha)}$ is a finitely generated D-module by definition of a modulation every $M(c)$ is finitely generated. Moreover since Γ_1 is finite by definition Γ_k must be finite too. Consequently

$T_k(\mathcal{Q})$ is finitely generated. As the quiver contains no oriented circuits, there exists an N such that $\Gamma_N = \varnothing$. Therefore since $\Gamma_* := \bigcup_{n=0}^{N} \Gamma_k$ is finite

$$T(\mathcal{Q}) = \bigoplus_{c \in \Gamma_*} M(c)$$

is a finitely generated D-module where we set $M(\epsilon_i) := F_i$ for the lazy paths ϵ_i. $\square$

Proposition 2.4. *Let $\mathcal{Q}$ be a species without oriented circuits. Then the categories* $\mathsf{rep}(\mathcal{Q})$ *and* $\mathsf{mod}(T(\mathcal{Q}))$ *are equivalent.*

Proof. The proof is mainly following [5, Proposition 10.1]. We define functors $F : \mathsf{mod}(T(\mathcal{Q})) \to \mathsf{rep}(\mathcal{Q})$ and $G : \mathsf{rep}(\mathcal{Q}) \to \mathsf{mod}(T(\mathcal{Q}))$ as follows.

Let X be a right $T(\mathcal{Q})$-module. Then since $D = \prod_{i=1}^{n} F_i$ is a subring of $T(\mathcal{Q}))$ the module X is also a D-module and factors uniquely into $X = \bigoplus_{i=1}^{n} X_i$ where $X_i = Xe_i$ for all $i \in \Gamma_0$. Obviously we want to assign these right F_i-spaces to the vertices. So we need to prove that they are finitely generated. Using that $\mathcal{Q}$ has no oriented circuits proposition 2.3 tells us that $T(\mathcal{Q})$ is a finitely generated D-module. So as X is a finitely generated $T(\mathcal{Q})$-module we obtain that Xe_i is a finitely generated F_i-module.

Now observe that since $M = \bigoplus_{\alpha: i \to j \in \Gamma_1} {}_iM_j^{\alpha}$ is a (D, D)-bimodule and the multiplication

$$\begin{aligned} X \times M &\longrightarrow \quad X \\ (x, m) &\longmapsto x \cdot m \end{aligned}$$

is D-balanced we obtain a right D-linear map $\varphi : X \otimes_D M \longrightarrow X$. Now we use that

$$\begin{aligned} X \otimes_D M &= \left(\bigoplus_{k=1}^{n} X_k \right) \otimes_D \left(\bigoplus_{\alpha \in \Gamma_1} {}_{s(\alpha)}M_{t(\alpha)}^{\alpha} \right) \\ &\cong \bigoplus_{k \in \Gamma_0, \alpha \in \Gamma_1} \left(X_k \otimes_D {}_{s(\alpha)}M_{t(\alpha)}^{\alpha} \right) \\ &= \bigoplus_{\alpha \in \Gamma_1} \left(X_{s(\alpha)} \otimes_D {}_{s(\alpha)}M_{t(\alpha)}^{\alpha} \right). \end{aligned}$$

as $X_k \otimes_D {}_{s(\alpha)}M_{t(\alpha)}^{\alpha} = 0$ for $k \neq s(\alpha)$. And therefore the D-linear map

$$\varphi : \bigoplus_{\alpha: i - j \in \Gamma_1} \left(X_i \otimes_D {}_iM_j^{\alpha} \right) \to \bigoplus_{i=1}^{n} X_i$$

induces F_j-linear maps ${}_j\varphi_i^{\alpha} : X_i \otimes_D {}_iM_j^{\alpha} \to X_j$ for all $\alpha : i \to j \in \Gamma_1$. We use this construction to define our functor $F(X) := (X_i, {}_k\varphi_j^{\alpha})_{i \in \Gamma_1, \alpha: j \to k \in \Gamma_1}$ on objects.

Regarding morphisms let $f : X \to Y$ be a morphism of right $T(\mathcal{Q})$-modules. Since f is especially D-linear we have $f(X_i) = f(Xe_i) = f(X)e_i \subseteq Y_i$ and thus $f : \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{n} Y_j$ induces F_i-linear maps $f_i : X_i \to Y_i$ for all $i \in \Gamma_0$. The fact that f is $T(\mathcal{Q})$-linear implies that, for $F(X) = (X_i, {}_k\varphi_j^{\alpha})_{i \in \Gamma_1, \alpha: j \to k \in \Gamma_1}$ and $F(Y) = (Y_i, {}_k\psi_j^{\alpha})_{i \in \Gamma_1, \alpha: j \to k \in \Gamma_1}$ the representations associated to X and Y, we have

$${}_j\psi_i^{\alpha}\big(f_i \otimes \mathrm{id}_{{}_iM_j^{\alpha}}(x \otimes m)\big) = {}_j\psi_i^{\alpha}(f_i(x) \otimes m) = f_i(x) \cdot m = f_j(x \cdot m) = f_j\big({}_j\varphi_i^{\alpha}(x \otimes m)\big)$$

for all $x \otimes m \in X_i \otimes_{F_i} {}_iM_j^\alpha$. Therefore $(f_i)_{i\in\Gamma_0}$ is indeed a morphisms of representations. So we set $F(f) := (f_i)_{i\in\Gamma_0}$.

Now we turn to the construction of the functor $G : \mathsf{rep}(\mathcal{Q}) \to \mathsf{mod}(T(\mathcal{Q}))$. Let $(X_i, {}_k\varphi_j^\alpha)$ be an object in $\mathsf{rep}(\mathcal{Q})$. Then $D = \prod_{i=1}^n F_i$ operates on X_i via the projection on F_i for all $i \in \Gamma_0$. So X_i becomes a D-module and we can define $X := \bigoplus_{i=1}^n X_i$. We make it a $T(\mathcal{Q})$-module by defining the multiplication on each summand of the tensor ring recursively. First note that on $T_1(\mathcal{Q}) = M$ we can define an action on X induced by

$$\varphi^{(1)} := \bigoplus_{\alpha:i\to j\in\Gamma_1} {}_j\varphi_i^\alpha \,:\, X \otimes_D M \cong \bigoplus_{\alpha:i\to j\in\Gamma_1} \left(X_i \otimes_D {}_iM_j^\alpha\right) \longrightarrow \bigoplus_{i=1}^n X_i = X.$$

With this map we can recursively define $\varphi^{(k+1)} := \varphi^{(1)} \circ (\varphi^{(k)} \otimes \mathrm{id}_M)$ which is of the form

$$\varphi^{(k+1)} : X \otimes_D T_{k+1}(\mathcal{Q}) \cong (X \otimes_D T_k(\mathcal{Q})) \otimes_D M \xrightarrow{\varphi^{(k)}\otimes \mathrm{id}_M} X \otimes_D M \xrightarrow{\varphi^{(1)}} X.$$

This defines a $T(\mathcal{Q})$-module structure on X. Moreover X is a finitely generated $T(\mathcal{Q})$-module since it already is finitely generated over D. So we can set $G(X_i, {}_k\varphi_j^\alpha) := X$.

Let $(f_i) : (X_i, {}_k\varphi_j^\alpha) \to (Y_i, {}_k\psi_j^\alpha)$ be a morphism of representations of $\mathcal{Q}$. Set $f := \bigoplus_{i=1}^n f_i : X \to Y$ where f_i is regarded as a D-linear map. In order to show that this map is right $T(\mathcal{Q})$-linear let $x = (x_i) \in X$ and $m = (m_\alpha) \in M$. Then with $\mathrm{pr}_j : X \to X_j$ the projection on the j-th summand the product of x and m in X is given by

$$\mathrm{pr}_j(x \cdot m) = \sum_{\substack{i\in\Gamma_0,\, \alpha\in\Gamma_1 \\ \text{st. } \alpha:i\to j}} {}_j\varphi_i^\alpha(x_i \otimes m_\alpha).$$

So if we now apply f on $x \cdot m$ we get

$$\begin{aligned}
\mathrm{pr}'_j\left(f(x\cdot m)\right) &= f_j(x \cdot m) \\
&= \sum_{\substack{i\in\Gamma_0,\, \alpha\in\Gamma_1 \\ \text{st. } \alpha:i\to j}} f_j\left({}_j\varphi_i^\alpha(x_i \otimes m_\alpha)\right) \\
&= \sum_{\substack{i\in\Gamma_0,\, \alpha\in\Gamma_1 \\ \text{st. } \alpha:i\to j}} {}_j\psi_i^\alpha\big(f_i(x_i) \otimes m_\alpha\big) \\
&= \mathrm{pr}'_j\left(f(x)\cdot m\right)
\end{aligned}$$

where $\mathrm{pr}'_j : Y \to Y_j$. Consequently, $f(x \cdot m) = f(x) \cdot m$. We can now inductively deduce that for an $m = m_k \otimes m_1 \in T_{k+1}(\mathcal{Q}) = T_k(\mathcal{Q}) \otimes_D M$ we have

$$f(x \cdot m) = f((x \cdot m_k)\cdot m_1) = f(x\cdot m_k)\cdot m_1 = (f(x)\cdot m_k)\cdot m_1) = f(x)\cdot m$$

since

$$x \cdot m = \varphi^{(k+1)}(x \otimes m) = \varphi^{(1)}\big(\varphi^{(k)}(x\otimes m_k) \otimes m_1\big) = (x \cdot m_k)\cdot m_1.$$

This finally proves that f is right $T(\mathcal{Q})$-linear and we can therefore set $G(f_i) := f$.

This concludes the construction of the functors F and G. They are mutually inverse by construction. □

Remark 2.6. Note that in the above proof the demand on $\mathcal{Q}$ not to contain any oriented circuits is crucial in order to ensure finite dimensionality of representations. One can extend the proposition to species containing oriented circuits by introducing admissible ideals of $T(\mathcal{Q})$, but we content ourself with the above result.

Earlier it has already been indicated that every representation can be uniquely decomposed into indecomposable blocks. Now with the reinterpretation of rep$(\mathcal{Q})$ as the category of right modules over $T(\mathcal{Q})$ we can finally proof this claim.

Theorem 2.5 (Krull-Schmidt Theorem)**.** *A representation* $\mathbf{X}$ *of a species* $\mathcal{Q}$ *exhibits a decomposition*

$$\mathbf{X} \cong \mathbf{X}_1 \oplus \ldots \mathbf{X}_m$$

into indecomposable representations $\mathbf{X}_i$ *which is unique up to the order of the direct summands.*

Proof. We can interpret $\mathbf{X}$ as an $T(\mathcal{Q})$-module $G(\mathbf{X})$ which exhibits a decomposition into indecomposable modules by the Krull-Schmidt theorem resulting in the demanded decomposition of $\mathbf{X}$ as the functors F and G are additive. For a proof of the Krull-Schmidt theorem for modules see [11, pp. 110-115]. □

We are now able to close the circle we spanned at the beginning of the chapter when motivating the definition of species. Remember that we aimed for a description of algebras A over possibly general fields k. It turns out we only need to demand the following property from k.

Definition 2.11. A field k is called *perfect* if either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and $k^p := \{a^p | a \in k\} = k$.

Proposition 2.6. *Let* A *be a basic, hereditary, finite dimensional algebra over a perfect field* k*. Then there exists a* k*-species* $\mathcal{Q}$ *such that* $A \cong T(\mathcal{Q})$ *as* k*-algebras.*

Proof. For a full proof see [5, Prop. 10.2] or [2, Prop. 4.2.5]. Here we will only sketch it as done in [4]. For A basic $A/\operatorname{rad} A = \prod_{i=1}^n F_i$ with division algebras F_i over k satisfying $[F_i : k] < \infty$ as A is finite dimensional. Moreover $A/(\operatorname{rad} A)^2 = \prod_{i,j=1}^n {}_iM_j$ for (F_i, F_j)-bimodules ${}_iM_j$ for which we can assume that ${}_iM_j = 0$ if $i \geq j$ since A is hereditary [5, Prop. 10.2]. So we make the set of vertices $\Gamma_0 = \{1, \ldots, n\}$ a valued graph by specifying a valuation $i \xrightarrow{(d_{ij}, d_{ji})} j$ for $i < j$ by $d_{ij} := \dim_{F_j} {}_iM_j$, $d_{ji} := \dim_{F_i} {}_iM_j$ and setting all others to zero. This constitutes an allowed valuation because the integers $[F_i : k]$ satisfy

$$d_{ij}[F_j : k] = \dim_k {}_iM_j = d_{ji}[F_i : k].$$

Together with the set of arrows Γ_1 defined by $i \to j \in \Gamma_1 \Leftrightarrow i < j$ and ${}_iM_j \neq 0$ we obtain the valued quiver (Γ, d) associated to A.
The obvious choice for the modulation is $(F_i, {}_iM_j)$. That the modules ${}_iM_j$ satisfy the dual bimodule condition (2.2) is a direct consequence of them being k-modules. In this way we obtain the species $\mathcal{Q}_A$ associated to A satisfying $A \cong T(\mathcal{Q}_A)$. □

Remark 2.7. As for every algebra A we can find a basic algebra with equivalent module category we deduce that representation theory of species fully describes the representation theory of hereditary finite dimensional k-algebras for perfect fields k. If one drops the demand on A to be hereditary A is still isomorphic to a tensor algebra of some species $\mathcal{Q}$ modulo an admissible ideal of $T(\mathcal{Q})$ [7, Thm. 8.5.2]. That we demand k to be perfect is a necessary condition [14, Ex. 3.11].
In [4] Dlab and Ringel moreover argue that one can extend the proof to a special class of artinian rings using species instead of k-species.

2.6 Crushed valued quivers and oriented circuits

So far we allowed quivers to contain parallel arrows, ie. we allowed the existence of arrows $\alpha \neq \beta \in \Gamma_1$ such that $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$. Following ideas from [14, p. 19] we want to inspect possible differences between representations of a species $\mathcal{Q}$ with parallel arrows and the associated species $\mathcal{Q}^C$ where all parallel arrows got *crushed together* to only one arrow. Formally we define this species as follows.

Definition 2.12. Let Γ be a valued quiver. We define the new valued quiver Γ^C as follows:

- vertices: $\Gamma_0^C = \Gamma_0$.
- arrows: $\Gamma_1^C \subseteq \Gamma_1$, satisfying that for all $\alpha \in \Gamma_1$ there exists a unique $\beta \in \Gamma_1^C$ such that $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$.
- valuation: $(d^C)_{ij}^{\alpha} = \sum_{\beta: i \to j \in \Gamma_1} d_{ij}^{\beta}$ and $(d^C)_{ji}^{\alpha} = \sum_{\beta: i \to j \in \Gamma_1} d_{ji}^{\beta}$ for all $\alpha : i \to j \ \in \Gamma_1^C$ where the sum is taken over all paths connecting i and j.

We call Γ^C the *crushed (valued) quiver* of Γ. Visually one has to think of crushing a quiver as depicted here:

$$\begin{tikzcd}[column sep=huge] i \arrow[r, bend left=40, "{(d_{ij}^{\alpha_1}, d_{ji}^{\alpha_1})}"] \arrow[r, bend left=10, "{(d_{ij}^{\alpha_2}, d_{ji}^{\alpha_2})}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "{(d_{ij}^{\alpha_k}, d_{ji}^{\alpha_k})}"] & j \end{tikzcd} \quad \xmapsto{(\,)^C} \quad i \xrightarrow{(d_{ij}^{\alpha_1} + \dots + d_{ij}^{\alpha_k},\, d_{ji}^{\alpha_1} + \dots + d_{ji}^{\alpha_k})} j$$

For a species $\mathcal{Q} = (\mathbb{M}, \Gamma)$ its *crushed species* $\mathcal{Q}^C := (\mathbb{M}^C, \Gamma^C)$ is defined as follows:

- We set $F_i^C := F_i$ for all $i \in \Gamma_0^C = \Gamma_0$.
- ${}_i(M^C)_j^{\beta} := \bigoplus_{\beta: i \to j \in \Gamma_1} {}_iM_j^{\beta}$ for every $\alpha : i \to j \ \in \Gamma_1^C$ and analogously define ${}_j(M^C)_i^{\beta}$.

$$\begin{tikzcd}[column sep=huge] F_i \arrow[r, bend left=40, "{{}_iM_j^{\beta_1}}"] \arrow[r, bend left=10, "{{}_iM_j^{\beta_2}}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "{{}_iM_j^{\beta_k}}"] & F_j \end{tikzcd} \quad \xmapsto{(\,)^C} \quad F_i \xrightarrow{{}_iM_j^{\beta_1} \oplus \cdots \oplus {}_iM_j^{\beta_k}} F_j$$

Proposition 2.7. *In the setting of the above definition $\mathbb{M}^C$ is a modulation of Γ^C and therefore the crushed species $\mathcal{Q}^C := (\mathbb{M}^C, \Gamma^C)$ is indeed a species.*

Proof. First note that ${}_i(M^C)_j^\alpha$ is a (F_i, F_j)-bimodule for all $\alpha : i \to j \ \in \Gamma_1$ and that

$$\begin{aligned}
\operatorname{Hom}_{F_i}\left({}_i(M^C)_j^\alpha, F_i\right) &= \operatorname{Hom}_{F_i}\left(\bigoplus_{\beta:i\to j\,\in\Gamma_1} {}_iM_j^\beta, F_i\right) \\
&\cong \bigoplus_{\beta:i\to j\,\in\Gamma_1} \operatorname{Hom}_{F_i}\left({}_iM_j^\beta, F_i\right) \\
&\cong \bigoplus_{\beta:i\to j\,\in\Gamma_1} \operatorname{Hom}_{F_j}\left({}_iM_j^\beta, F_j\right) \\
&\cong \operatorname{Hom}_{F_j}\left(\bigoplus_{\beta:i\to j\,\in\Gamma_1} {}_iM_j^\beta, F_j\right) \\
&= \operatorname{Hom}_{F_j}\left({}_i(M^C)_j^\alpha, F_j\right)
\end{aligned}$$

where all isomorphisms are isomorphisms of (F_j, F_i)-bimodules. Regarding the dimension of the modules we find that

$$\dim_{F_j}\ {}_i(M^C)_j^\alpha = \dim_{F_j}\left(\bigoplus_{\beta:i\to j\,\in\Gamma_1} {}_iM_j^\beta\right) = \sum_{\beta:i\to j\,\in\Gamma_1} d_{ij}^\beta = (d^C)_{ij}^\alpha$$

and analogously $\dim_{F_i}\ {}_j(M^C)_i^\alpha = (d^C)_{ji}^\alpha$ which makes $\mathbb{M}^C$ a modulation of Γ^C. $\square$

Proposition 2.8. *Let $\mathcal{Q}$ be a species. Then $T(\mathcal{Q}) \cong T(\mathcal{Q}^C)$ as rings.*

Proof. We will proof the proposition by reviewing the construction of the tensor ring. First note that

$$T_0(\mathcal{Q}^C) = \prod_{i\in\Gamma_0^C} k_i^C = \prod_{i\in\Gamma_0} F_i = T_0(\mathcal{Q}).$$

Moreover for the next summand we find

$$T_1(\mathcal{Q}^C) = \bigoplus_{\alpha\in\Gamma_1^C} {}_i(M^C)_j^\alpha = \bigoplus_{\alpha\in\Gamma_1^C}\left(\bigoplus_{\beta:s(\alpha)\to t(\alpha)\,\in\Gamma_1} {}_iM_j^\beta\right).$$

while $T_1(\mathcal{Q}) = \bigoplus_{\alpha\in\Gamma_1} {}_{s(\alpha)}M_{t(\alpha)}^\alpha$. So because of the recursive definition of the tensor ring as given in 2.10 and its multiplication it now suffices to show that the bijection

$$\begin{aligned}
\varphi : \bigoplus_{\alpha\in\Gamma_1} {}_{s(\alpha)}M_{t(\alpha)}^\alpha &\longrightarrow \bigoplus_{\alpha\in\Gamma_1^C}\left(\bigoplus_{\beta:s(\alpha)\to t(\alpha)\,\in\Gamma_1} {}_{s(\alpha)}M_{t(\alpha)}^\beta\right) \\
(m_\alpha)_{\alpha\in\Gamma_1} &\longmapsto \left((m_\beta)_{\beta:i_1\to j_1\,\in\Gamma_1}, \ldots, (m_\beta)_{\beta:i_k\to j_k\,\in\Gamma_1}\right)
\end{aligned}$$

is an isomorphism of $D := T_0(\mathcal{Q}^{\mathcal{C}}) = T_0(\mathcal{Q})$ bimodules where for simplicity we let $\alpha_l : i_l \to j_l$ for $l = 1 \ldots k$ denote all arrows in Γ^C. Then for all $\lambda \in D$ and $m \in T_1(\mathcal{Q})$ we find

$$\begin{aligned}
\varphi(\lambda m) &= \varphi\big((\lambda_{s(\alpha)} m_\alpha)_{\alpha \in \Gamma_1}\big) \\
&= \Big((\lambda_{s(\beta)} m_\beta)_{\beta : i_1 \to j_1 \in \Gamma_1}, \ldots, (\lambda_{s(\beta)} m_\beta)_{\beta : i_k \to j_k \in \Gamma_1}\Big) \\
&= \Big((\lambda_{i_1} m_\beta)_{\beta : i_1 \to j_1 \in \Gamma_1}, \ldots, (\lambda_{i_k} m_\beta)_{\beta : i_k \to j_k \in \Gamma_1}\Big) \\
&= \Big(\lambda_{i_1} (m_\beta)_{\beta : i_1 \to j_1 \in \Gamma_1}, \ldots, \lambda_{i_k} (m_\beta)_{\beta : i_k \to j_k \in \Gamma_1}\Big) \\
&= \lambda \Big((m_\beta)_{\beta : i_1 \to j_1 \in \Gamma_1}, \ldots, (m_\beta)_{\beta : i_k \to j_k \in \Gamma_1}\Big) \\
&= \lambda\, \varphi(m)
\end{aligned}$$

and similarly $\varphi(m\lambda) = \varphi(m)\, \lambda$. So we can deduce that $T_1(\mathcal{Q}^{\mathcal{C}}) \cong T_1(\mathcal{Q})$ as (D, D)-bimodules. This concludes the proof. $\square$

Corollary 2.9. *For $\mathcal{Q}$ a species the categories* $\mathsf{rep}(\mathcal{Q})$ *and* $\mathsf{rep}(\mathcal{Q}^C)$ *are equivalent.*

Therefore in order to classify all species of finite representation type we can assume the underlying quiver to have no parallel arrows. This is already a huge simplification, but still quivers with anti-parallel arrows could occur. Anti-parallel arrows are a special case of valued quivers containing oriented circuits. Out of this class the valued graph $\tilde{\mathbb{A}}_n$ together with the orientation

$$\begin{tikzcd}[ampersand replacement=\&] (\tilde{\mathbb{A}}_n, \Omega): \& 1 \arrow[r] \& 2 \arrow[r] \& \ldots \arrow[r] \& n+1 \arrow[lll, bend left] \end{tikzcd}$$

is of special interest because its indecomposable representations can not be analysed with the tools which will later be developed. However, what we can say is that $(\tilde{\mathbb{A}}_n, \Omega)$ is not of finite representation type as the following example illustrates.

Example 2.9. Given a modulation $\mathbb{M} = (F_i, {}_iM_{i+1}^\alpha)_{i \in \Gamma}$ of $\tilde{\mathbb{A}}_n$ we can without loss of generality assume that $F = F_i$ and ${}_FF_F = {}_iM_{i+1}^\alpha$ for all vertices i by remark 2.2 (iii). We now want to construct an infinite number of indecomposable representations of the species $\mathcal{Q} := (\mathbb{M}, \tilde{\mathbb{A}}_n, \Omega)$ where the orientation of the arrows is defined as in (2.7). Then following an idea from [12, Prop. 5.3.2] we define a representation $\mathbf{X}(m)$ for $m \geq 2$ integer via

$$\begin{tikzcd}[ampersand replacement=\&] \mathbf{X}(m): \& F^m \arrow[r, "J_m"] \& F^m \arrow[r, "\mathrm{id}_{F^m}"] \& \ldots \arrow[r, "\mathrm{id}_{F^m}"] \& F^m \arrow[lll, bend left, "\mathrm{id}_{F^m}"] \end{tikzcd}$$

identifying $F^m \otimes_F F \cong F^m$ and where the map J_m is the Jordan-block with zero on the diagonal. In order to prove that $\mathbf{X}(m)$ is indeed indecomposable we use the following useful criterion: A finitely generated module over a ring is indecomposable if its endomorphismalgebra is local [11, p. 112]. Now using the equivalence of the categories $\mathsf{rep}(\mathcal{Q})$ and $\mathsf{mod}(T(\mathcal{Q}))$ it suffices to show that $\mathrm{End}(\mathbf{X}(m))$ is local.

So let $f = (A_1, \ldots, A_{n+1}) \in \mathrm{End}(\mathbf{X}(m))$ be given by matrices $A_i \in \mathrm{Mat}^{m\times m}(F)$. Then the commutation relation for morphisms (2.4) implies that all squares in

$$\begin{tikzcd}
\mathbf{X}(m): \arrow[d, "f"] & F^m \arrow[r, "J_m"] \arrow[d, "A_1"] & F^m \arrow[r, "\mathrm{id}_{F^m}"] \arrow[d, "A_2"] & \ldots \arrow[r, "\mathrm{id}_{F^m}"] & F^m \arrow[d, "A_{n+1}"] \arrow[lll, bend right, "\mathrm{id}_{F^m}"'] \\
\mathbf{X}(m): & F^m \arrow[r, "J_m"] & F^m \arrow[r, "\mathrm{id}_{F^m}"] & \ldots \arrow[r, "\mathrm{id}_{F^m}"] & F^m \arrow[lll, bend left, "\mathrm{id}_{F^m}"]
\end{tikzcd}$$

must commute which yields that $A := A_2 = A_3 = \cdots = A_{n+1} = A_1$ and $J_m A = A J_m$. The matrices A satisfying this commutation relation are exactly given by the upper triangular matrices with equal entries on each diagonal and thus

$$\mathrm{End}\left(\mathbf{X}(m)\right) \cong \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \ldots & 0 & a_1 \end{pmatrix} | a_1, \ldots a_m \in F \right\}$$

However, this ring of matrices is isomorphic to $F[t]/(t^m)$ which is local. This proves that $\mathbf{X}(m)$ is indecomposable. So we have constructed an infinite number of representations which moreover tells us that all species containing an oriented circuit admit an infinite number of indecomposable representations.

Moreover one can recover the representations of species with oriented circuits by considering representations of a species whose valued quiver has more vertices but does not contain oriented circuits.

Proposition 2.10. *Let $\mathcal{Q} = (\mathbb{M}, \Gamma)$ be a species such that Γ contains oriented circuits. Then there exists a species $\widetilde{\mathcal{Q}}$ without oriented circuits such that* $\mathsf{rep}(\mathcal{Q})$ *is equivalent to a subcategory of* $\mathsf{rep}(\widetilde{\mathcal{Q}})$.

Proof. Since the quiver Γ is finite by definition there may occur only finitely many oriented circuits in Γ. Thus it is sufficient to show that we can find a species $\widetilde{\mathcal{Q}}$ with one circuit less than $\mathcal{Q}$ which satisfies that $\mathsf{rep}(\mathcal{Q})$ is equivalent to a subcategory of $\mathsf{rep}(\widetilde{\mathcal{Q}})$ because in this case the claim follows iteratively.
So without loss of generality let

$$\begin{tikzcd}
1 \arrow[r, "(d_{1,2}^{\alpha_1}, d_{2,1}^{\alpha_1})"] & 2 \arrow[r, "(d_{2,3}^{\alpha_2}, d_{3,2}^{\alpha_2})"] & 3 \arrow[r, "(d_{3,4}^{\alpha_3}, d_{4,3}^{\alpha_3})"] & \cdots \arrow[r, "(d_{n-1,n}^{\alpha_{n-1}}, d_{n,n-1}^{\alpha_{n-1}})"] & n \arrow[llll, bend left, "(d_{n,1}^{\alpha_n}, d_{1,n}^{\alpha_n})"]
\end{tikzcd}$$

be an oriented circuit in Γ and define the quiver $\widetilde{\Gamma}$ by deflecting the arrow α_n to a new point $n+1$ and connecting 1 with $n+1$ as indicated below.

$$\begin{tikzcd}
1 \arrow[r, "(d^{\alpha_1}_{1,2}, d^{\alpha_1}_{2,1})"] \arrow[drr, bend right, "(1,1)"'] & 2 \arrow[r, "(d^{\alpha_2}_{2,3}, d^{\alpha_2}_{3,2})"] & 3 \arrow[r, "(d^{\alpha_3}_{3,4}, d^{\alpha_3}_{4,3})"] & \cdots \arrow[r, "(d^{\alpha_{n-1}}_{n-1,n}, d^{\alpha_{n-1}}_{n,n-1})"] & n \arrow[dll, bend left, "(d^{\alpha_n}_{n,n+1}, d^{\alpha_n}_{n+1,n})"] \\
 & & n+1 & &
\end{tikzcd}$$

So by construction the number of circuits in $\widetilde{\Gamma}$ is one less then in Γ. As a valuation $\tilde{d}$ on $\widetilde{\Gamma}$ we choose the same as for Γ and prescribing the new arrow the value $(1,1)$.

Now let $\mathbb{M} = (F_i, {}_iM_j^{\alpha})$ be the modulation of $\mathcal{Q}$. We define a valuation $\widetilde{\mathbb{M}} = (\widetilde{F}_i, {}_i\widetilde{M}_j^{\alpha})$ on $\widetilde{\Gamma}$ by leaving the modulation unchanged on all vertices and arrows except the new ones. There we define $F_{n+1} := F_1$ and ${}_{n+1}\widetilde{M}_1^{\alpha_{n+1}} := F_1$. It is now left to show that $\mathsf{rep}(\mathcal{Q})$ is equivalent to a subcategory of $\mathsf{rep}(\widetilde{\mathcal{Q}})$.

So let $\mathbf{X} = (X_i, {}_j\varphi_i^{\alpha})$ be a representation of $\mathcal{Q}$. It is simply promoted to a representation $\widetilde{\mathbf{X}} = (\widetilde{X}_i, {}_j\widetilde{\varphi}_i^{\alpha})$ of $\widetilde{\mathcal{Q}}$ by adding $\widetilde{X}_{n+1} := X_1$ and ${}_{n+1}\widetilde{\varphi}_1^{\alpha_n} : \widetilde{X}_1 \otimes_{\widetilde{F}_1} {}_1\widetilde{M}_{n+1}^{\alpha_n} \cong X_1 \to \widetilde{X}_{n+1} = X_1$ defined as ${}_{n+1}\widetilde{\varphi}_n^{\beta_n} := \mathrm{id}_{X_1}$ to $\mathbf{X}$.

For $f : \mathbf{X} \to \mathbf{Y}$ a morphism in $\mathsf{rep}(\mathcal{Q})$ we define a morphism $\tilde{f} : \widetilde{\mathbf{X}} \to \widetilde{\mathbf{Y}}$ of representations of $\widetilde{\mathcal{Q}}$ via $\tilde{f}_i := f_i$ for all $i \in \widetilde{\Gamma}_0 \diagdown \{n+1\} = \Gamma_0$ and $\tilde{f}_{n+1} := f_1$. This indeed defines a morphism of representations because since f is a morphism in $\mathsf{rep}(\mathcal{Q})$ the compatibility condition (2.4) is obviously fulfilled for all arrows in $\widetilde{\Gamma}_1 \diagdown \{\alpha_{n+1}\}$. But for the new arrow α_{n+1} just plugging in definitions also shows that

$$\begin{tikzcd}
\widetilde{X}_1 \otimes_{F_1} {}_1\widetilde{M}^{\alpha}_{n+1} = X_1 \otimes_{F_1} {}_1M^{\alpha}_{n+1} \arrow[r, "{}_1\widetilde{\varphi}^{\alpha}_{n+1} = \mathrm{id}_{X_1}"] \arrow[d, "\left(\tilde{f}_i \otimes \mathrm{id}_{{}_1\widetilde{M}^{\alpha}_{n+1}}\right) = \left(f_i \otimes \mathrm{id}_{{}_1M^{\alpha}_{n+1}}\right)"'] & \widetilde{X}_{n+1} = X_1 \arrow[d, "\tilde{f}_{n+1} = f_1"] \\
\widetilde{Y}_1 \otimes_{F_1} {}_1\widetilde{M}^{\alpha}_{n+1} = Y_1 \otimes_{F_1} {}_1M^{\alpha}_{n+1} \arrow[r, "{}_1\widetilde{\psi}^{\alpha}_{n+1} = \mathrm{id}_{Y_1}"] & \widetilde{Y}_{n+1} = Y_1
\end{tikzcd}$$

commutes. Consequently $\tilde{f}$ is a morphism of representations.

With these two constructions we define the functor

$$F : \mathsf{rep}(\mathcal{Q}) \longrightarrow \mathsf{rep}(\widetilde{\mathcal{Q}}) \,, \quad \begin{cases} \mathbf{X} \longmapsto \widetilde{\mathbf{X}} \\ f \longmapsto \tilde{f} \end{cases} .$$

On the subcategory of representations $(Z_i, {}_j\psi_i^{\alpha}) \in \mathsf{rep}(\widetilde{\mathcal{Q}})$ with $Z_1 = Z_{n+1}$ and ${}_{n+1}\psi_1^{\alpha_n+1} = \mathrm{id}_{Z_1}$ of $\mathsf{rep}(\widetilde{\mathcal{Q}})$ we can define an inverse G of F by simply forgetting the module at vertex. Therefore $\mathsf{rep}(\mathcal{Q})$ is indeed a subcategory of $\mathsf{rep}(\widetilde{\mathcal{Q}})$. $\square$

To conclude the findings of this section we have seen that in order to classify indecomposable representations of species we can always assume our valued quiver to have no parallel arrows. As oriented circuits directly lead to an infinite number of indecomposable representations we can also exclude such species from our search after all species of finite representation type.

Note that for a quiver without (anti-)parallel arrows an arrow $\alpha : i \to j$ is uniquely described by the touple (i, j). So labelling the bimodules ${}_iM_j^\alpha$ of a modulation with both i, j and α is not necessary. We will therefore just write ${}_iM_j$ instead of ${}_iM_j^\alpha$. Accordingly we will drop all unneeded labellings with arrows α in all our notation from now on.
In the light of the findings of this section we can give a redefinition of valued graphs and quivers where the unnecessary notion of multiple arrows between vertices is naturally absent.

Redefinition 2.13. A *valued graph* (Γ, d) is a finite set Γ of vertices together with non-negative integers d_{ij} for all $i, j \in \Gamma$ such that $d_{ii} = 0$ for all $i \in \Gamma$ and that there exist strictly positive integers f_i satisfying

$$d_{ij} f_j = d_{ji} f_i$$

for all $i \neq j \in \Gamma$. We say i and j are *neighbours* and call $\{i, j\}$ an *edge* if $d_{ij} \neq 0$. Graphically we will write

$$i \xrightarrow{(d_{ij}, d_{ji})} j$$

in this case.
The valued graph (Γ, d) will moreover always be assumed to be *connected* in the sense that for all $i, j \in \Gamma$ there exists a sequence $i = i_1, i_2, \ldots, i_k = j$ in Γ such that i_m and i_{m+1} are neighbours for all $m = 1, \ldots, k-1$.
An *orientation* of (Γ, d) is a subset $\Omega \subset \Gamma \times \Gamma$ such that for each edge i, j in Γ either $(i, j) \in \Omega$ or $(j, i) \in \Omega$ and for each $i \in \Gamma : (i, i) \notin \Omega$. If $(i, j) \in \Omega$ we will write $i \to j$.
A collection (Γ, d, Ω) of a valued graph (Γ, d) together with an orientation Ω is called a *valued quiver*. We say a sequence of vertices $(i_1, \ldots, i_m)$ is a *path* from i_1 to i_m in (Γ, d, Ω) if $i_r \to i_{r+1}$ lies in Ω for every $1 \leq r < m$.

With this redefinition we directly excluded quivers with (anti-)parallel arrows and loops from our discussion. Moreover the notion of an orientation will later turn out to be quite useful in practise when defining the reflection functors. Moreover note that we are now in accordance with the definitions of valued graphs and their representations as used in [4].

We want to close this section by finding an alternative characterisation of a circuit-free quiver. For this purpose we first need the following definitions.

Definition 2.14. Let (Γ, d, Ω) be a valued quiver and $k \in \Gamma$ a vertex. We define a new orientation $s_k\Omega$ on (Γ, d) via

$$(i, j) \in \Omega \quad \Leftrightarrow \quad \begin{cases} (i, j) \in s_k\Omega & \text{if } i \neq k \neq j \\ (j, i) \in s_k\Omega & \text{if } i = k \text{ or } j = k \end{cases}$$

ie. we obtain $s_k\Omega$ from Ω by reversing all arrows with target or source k and leaving all others unchanged.

Definition 2.15. We say a vertex $k \in \Gamma$ is a *sink* (*source*) if there are only incoming arrows to (outgoing arrows from) k, ie. for all $i \in \Gamma$ we have $(k, i) \notin \Omega$ ($(i, k) \notin \Omega$).

Lemma 2.11. *Let (Γ, d, Ω) be a valued quiver. Then the following statements are equivalent:*

(i) (Γ, d, Ω) *has no oriented circuits.*

(ii) *There exists a bijection* $\pi : \{1, \ldots, n\} \to \Gamma$ *such that* $\pi(t)$ *is a sink in* $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$ *for all* $t = 1, \ldots, n$.

(iii) *There exists a bijection* $\pi : \{1, \ldots, n\} \to \Gamma$ *such that* $\pi(t)$ *is a source in* $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$ *for all* $t = 1, \ldots, n$.

Proof. (i $\Rightarrow$ ii) We show the claim by induction in the number of vertices. For $|\Gamma| = 1$ we have $\Omega = \varnothing$. Thus the unique vertex in Γ is a sink.
Now let $|\Gamma| = n + 1$ and take an arbitrary $i \in \Gamma$. Then either i is a sink and we set $\pi(1) := i$ or there exists an arrow $i \to j$ in Ω. If so repeat this procedure with j, which means that we follow an arbitrary outgoing arrow until we reach a sink. This procedure terminates since the quiver is finite and contains no circuits. So we can assume that there exists a sink $\pi(1) \in \Gamma$. Now define a quiver on the vertices $\tilde{\Gamma} := \Gamma \diagdown \{\pi(1)\}$ by leaving out all arrows that end in $\pi(1)$. So explicitly written down we set $\tilde{\Omega} := \Omega \diagdown (\Gamma \times \{\pi(1)\})$. Then again $\tilde{\Gamma}$ has no oriented circuits with respect to $\tilde{\Omega}$ and $|\tilde{\Gamma}| = n$. Therefore we can use the induction hypothesis to find an ordering $\pi(2), \ldots, \pi(n+1)$ of $\tilde{\Gamma}$ such that $\pi(t)$ is a sink with respect to $s_{\pi(t-1)} \ldots s_{\pi(3)} s_{\pi(2)} \tilde{\Omega}$ for every $t = 2, \ldots, n+1$. Including $\pi(1)$ again it now still holds that $\pi(t)$ is a sink with respect to $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$ for all $t = 1, \ldots, n+1$ since all additional arrows $s_{\pi(1)} \Omega \diagdown \tilde{\Omega}$ are pointing away from $\pi(1)$ to a vertex in $\tilde{\Gamma}$.

(ii $\Rightarrow$ iii) Let $\pi : \{1, \ldots, n\} \to \Gamma$ be a bijection such that $\pi(t)$ is a sink in $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Gamma$ for all $t = 1, \ldots, n$ and define $\pi' : \{1, \ldots, n\} \to \Gamma$ via $\pi'(t) := \pi(n+1-t)$. Now since $\pi'(1) = \pi(n)$ is a sink in $s_{\pi(n-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$ this vertex must be a source in Ω as one can easily convince oneself that $\Omega = s_{\pi(n)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$. Using this last identity it further follows that

$$\begin{aligned} s_{\pi'(t-1)} \ldots s_{\pi'(2)} s_{\pi'(1)} \Omega &= s_{\pi(n+2-t)} \ldots s_{\pi(n-1)} s_{\pi(n)} \Omega \\ &= s_{\pi(n+2-t)} \ldots s_{\pi(n-1)} s_{\pi(n)} \left(s_{\pi(n)} s_{\pi(n-1)} \ldots s_{\pi(1)} \Omega \right) \\ &= s_{\pi(n+1-t)} s_{\pi(n-t)} \ldots s_{\pi(1)} \Omega. \end{aligned}$$

for all $t = 1 \ldots n$ which implies that $\pi'(t) = \pi(n+1-t)$ is indeed a source in the above orientation.

(iii $\Rightarrow$ i) Assume Γ has an oriented circuit, but it admits a bijection $\pi : \{1, \ldots, n\} \to \Gamma$ such that $\pi(t)$ is a source in $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Gamma$ for all $t = 1, \ldots, n$. Choose $i_1, \ldots, i_m$ such that $\pi(i_1), \pi(i_2), \ldots, \pi(i_m)$ is an oriented circuit and $i_1 < i_r$ for all $r = 2, \ldots, m$, ie. there exist arrows

$$\begin{tikzcd} \pi(i_1) \arrow[r] & \pi(i_2) \arrow[r] & \ldots \arrow[r] & \pi(i_m) \arrow[lll, bend left] \end{tikzcd}$$

in Ω. Now observe that the above oriented circuit is also contained in $s_{\pi(i_1-1)} s_{\pi(i_1-2)} \ldots s_{\pi(1)} \Omega$ because s_j acts on Ω by only changing arrows incoming to or outgoing from j and since we chose $i_1 < i_r$ for all $r = 2, \ldots, m$ minimal we have $\{\pi(1), \ldots, \pi(i_1-2), \pi(i_1-1)\} \cap \{\pi(i_1), \pi(i_2), \ldots, \pi(i_m)\} = \varnothing$. But the fact that $\pi(i_m) \to \pi(i_1)$ is an arrow lying in $s_{\pi(i_1-1)} s_{\pi(i_1-2)} \ldots s_{\pi(1)} \Omega$ is a contradiction since the vertex $\pi(i_1)$ is assumed to be a source with respect to $s_{\pi(i_1-1)} s_{\pi(i_1-2)} \ldots s_{\pi(1)} \Omega$. So consequently the quiver must have contained no oriented circuits at first. $\square$

Definition 2.16. Let (Γ, d, Ω) be a valued quiver. We call the orientation Ω *admissible* if it satisfies one of the equivalent properties given in lemma 2.11.
A bijection $\pi : \{1, \ldots, n\} \to \Gamma$ such that $\pi(t)$ is a sink in $s_{\pi(t-1)} \ldots s_{\pi(2)} s_{\pi(1)} \Omega$ for all $t = 1, \ldots, n$ is called an *admissible ordering* of the valued quiver.

CHAPTER 3

Quadratic Forms and Root Systems

For now we let representations of species rest and focus on the analysis of root systems associated to valued graphs. We will proceed in a similar way Bernstein et al. did in order to analyse the root system of ordinary quivers in [3] with the help of Coxeter transformations. Most concepts are easily transferred to the more general case of valued quivers.

3.1 Quadratic Forms of valued Graphs

Let (Γ, d) be a valued graph and as before assume that $\Gamma = \{1, \ldots, n\}$. We start defining its associated quadratic form by fixing positive integers f_i for all $i \in \Gamma_0$ such that $d_{ij}\, f_j = d_{ji}\, f_i$ and define $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ via

$$q(\mathbf{x}) := \sum_{i=1}^{n} f_i x_i^2 - \frac{1}{2} \sum_{i,j=1}^{n} d_{ij} f_j x_i x_j$$

for all $\mathbf{x} \in \mathbb{Z}^n$. Despite the factor $\frac{1}{2}$ the image of $\mathbf{x}$ under q lies in $\mathbb{Z}$ since for all $i, j \in \Gamma$

$$d_{ij} f_j x_i x_j + d_{ji} f_i x_j x_i = 2 d_{ij} f_j x_i x_j.$$

This shows that the map is well-defined.

We moreover associate a symmetric bilinear form $(\ ,\ ) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ to this valued graph which is defined as

$$(\mathbf{x}, \mathbf{y}) := q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) = 2 \sum_{i=1}^{n} f_i x_i y_i - \sum_{i,j=1}^{n} d_{ij} f_j x_i y_j \tag{3.1}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$. Note that we can recover q from $(\ ,\ )$ via

$$q(\mathbf{x}) = \frac{1}{2} (\mathbf{x}, \mathbf{x}). \tag{3.2}$$

3.2 Classification of positive (semi-)definite forms

A key step in order to prove Gabriel's theorem was to classify all quivers that feature a positive (semi-)definite quadratic form. This is not different from our case except that we have to work in a slightly more general setting. However, before we begin, an explicit definition of the definiteness of quadratic forms shall be given.

Definition 3.1. Let q be a quadratic form.

(i) We say q is *positive definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$.

(ii) q is called *positive semi-definite* if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq 0$.

Remark 3.1. Our definition (3.1) of the quadratic form q associated to a valued graph (Γ, d) depends on the choice of integers f_i which do not belong to the data of a valued graph. Therefore we have to show that the above definition of positive (semi-)definiteness does not depend on the choice of integers being made. So let f_i, f_i' be two sets of positive integers satisfying

$$d_{ij}\, f_j = d_{ji}\, f_i \quad \text{and} \quad d_{ij}\, f_j' = d_{ji}\, f_i' \tag{3.3}$$

and let q be the quadratic form defined with respect to the choice f_i and q' with respect to f_i'. We now want to show that q is positive (semi-)definite if and only if q' is positive (semi-)definite.
First we note that (3.3) implies that $f_j f_i' = f_i f_j'$. Now fix a $k \in \Gamma$ to observe that

$$f_k'\, q(\mathbf{x}) = \sum_{i=1}^{n} f_k' f_i x_i^2 - \frac{1}{2} \sum_{i,j=1}^{n} d_{ij} f_k' f_j x_i x_j = \sum_{i=1}^{n} f_i' f_k x_i^2 - \frac{1}{2} \sum_{i,j=1}^{n} d_{ij} f_j' f_k x_i x_j = f_k\, q'(\mathbf{x}).$$

So as both f_k and f_k' are positive and non-zero it follows that q is positive (semi-)definite if and only if q' is positive (semi-)definite.
So concluding one can make a statement about the definiteness of a quadratic form of a valued graph without specifying the choice of f_i.

Example 3.1. In order to calculate the quadratic form of the valued graph

$$\tilde{\mathbb{B}}_n : \quad 1 \xrightarrow{(1,2)} 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \xrightarrow{(2,1)} n+1$$

we first need to fix the integers f_i for every vertex satisfying (2.1). It is easy to see that $f_1 := f_{n+1} := 1$ and $f_i := 2$ for $i \neq 1, n+1$ do the job. So we can now compute

$$q(\mathbf{x}) = x_1^2 + \sum_{i=2}^{n} 2x_i^2 + x_{n+1}^2 - \sum_{i=1}^{n} 2x_i x_{i+1} = \sum_{i=1}^{n} \left(x_i^2 - 2x_i x_{i+1} + x_{i+1}^2\right) = \sum_{i=1}^{n} (x_i - x_{i+1})^2 .$$

This form is obviously positive semi-definite.

Let us compute the quadratic form of $\tilde{\mathbb{E}}_6$ as a second example. This graph is of the form

$$\begin{array}{lccccccccc} \tilde{\mathbb{E}}_6 : & 1 & \text{---} & 2 & \text{---} & 3 & \text{---} & 4 & \text{---} & 5 \\ & & & & & | & & & & \\ & & & & & 6 & & & & \\ & & & & & | & & & & \\ & & & & & 7 & & & & \end{array}$$

and allows the choice $f_i = 1$ for all i. The quadratic form then reads

$$q(\mathbf{x}) = \sum_{i=1}^{7} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_3x_6 - x_4x_5 - x_6x_7.$$

In contrast to $\tilde{\mathbb{B}}_n$ the graph $\tilde{\mathbb{E}}_6$ does not allow a direct statement about the definiteness of its quadratic form. Later it will also turn out to be positive semi-definite but we need some more tools in order justify this statement.

The rest of this section is now dedicated to the classification of all valued graphs that give rise to positive (semi-)definite quadratic forms. This will be done by generalising the steps made in [16, pp. 210 ff.] from the setting of quadratic forms of ordinary quivers to the ones of valued graphs.

Definition 3.2. Let $(\ ,\ )$ be the bilinear form associated to a quadratic form q as given by eq. (3.1). We define the *radical* of q to be the set

$$\operatorname{rad} q = \{\mathbf{x} \in \mathbb{Z}^n \,|\, \forall \mathbf{y} \in \mathbb{Z}^n : (\mathbf{x}, \mathbf{y}) = 0\}.$$

Remark 3.2. Again following the discussion in remark 3.1 one easily can convince oneself that $\operatorname{rad} q$ does not depend on the choice of integers f_i.

Lemma 3.1. *Let (Γ, d) be a valued graph and $(\ ,\ )$ its bilinear form. If there exists a vector $\delta \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$ such that $(\delta, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$ then*

(i) $\delta_i \neq 0$ *for all* $i \in \Gamma$.

(ii) q *is positive semi-definite.*

(iii) *For an* $\mathbf{x} \in \mathbb{Z}^n$ *the following statements are equivalent*

  (1) $q(\mathbf{x}) = 0$,

  (2) $\mathbf{x} \in \mathbb{Q}\delta$,

  (3) $\mathbf{x} \in \operatorname{rad} q$.

Proof. This proof is mainly following [16, Lem. 8.5].

(i) Especially $(\delta, \cdot)$ annihilates all basis vectors $\mathbf{e}_i \in \mathbb{Z}^n$. An explicit calculation then yields $0 = (\delta, \mathbf{e}_i) = 2f_i\delta_i - \sum_{j=1}^{n} d_{ji} f_i \delta_j$ which is equivalent to

$$\delta_i = \frac{1}{2} \sum_{j=1}^{n} d_{ji} \delta_j. \tag{3.4}$$

Now suppose there exists an $i \in \Gamma$ such that $\delta_i = 0$. Then it follows that for all neighbours j we have $\delta_j = 0$ since $d_{ij} > 0$ and all components of δ are positive. As by definition the graph (Γ, d) is connected it follows that $\delta = 0$ which is a contradiction.

(ii) We can use eq. (3.4) a second time in order to rewrite

$$2\sum_{i=1}^{n} f_i x_i^2 = \sum_{i=1}^{n} f_i \frac{x_i^2}{\delta_i} \sum_{j=1}^{n} d_{ji}\delta_j.$$

for an arbitrary $x \in \mathbb{Z}^n$. Further rearranging yields

$$\begin{aligned}2\sum_{i=1}^{n} f_i x_i^2 &= \frac{1}{2}\sum_{i,j=1}^{n}\Big(\underbrace{d_{ji}f_i}_{=d_{ij}f_i}\,\delta_j\frac{x_i^2}{\delta_i} + \underbrace{d_{ji}f_i\delta_j\frac{x_i^2}{\delta_i}}_{\text{swap } i \text{ and } j}\Big)\\ &= \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}f_i\delta_i\delta_j\left(\frac{x_i^2}{\delta_i^2}+\frac{x_j^2}{\delta_j^2}\right).\end{aligned}$$

This expression can now be used in order to rewrite the quadratic form evaluated on $\mathbf{x}$.

$$\begin{aligned}q(\mathbf{x}) &= \sum_{i=1}^{n} f_i x_i^2 - \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}f_j x_i x_j\\ &= \frac{1}{4}\sum_{i,j=1}^{n} d_{ij}f_i\delta_i\delta_j\left(\frac{x_i^2}{\delta_i^2}+\frac{x_j^2}{\delta_j^2}\right) - \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}f_j x_i x_j\\ &= \frac{1}{4}\sum_{i,j=1}^{n} d_{ij}f_i\delta_i\delta_j\left(\frac{x_i^2}{\delta_i^2}+\frac{x_j^2}{\delta_j^2} - 2\frac{x_i}{\delta_i}\frac{x_j}{\delta_j}\right)\\ &= \frac{1}{4}\sum_{i,j=1}^{n} d_{ij}f_i\delta_i\delta_j\left(\frac{x_i}{\delta_i}-\frac{x_j}{\delta_j}\right)^2. \end{aligned} \tag{3.5}$$

This shows that q is positive semi-definite.

(iii) Clearly (2) implies (3) and (3) implies (1) so it only remains to show that (1) implies (2). Considering (3.5) we see that $q(\mathbf{x}) = 0$ for an $\mathbf{x} \in \mathbb{Z}^n$ if and only if $\frac{x_i}{\delta_i} = \frac{x_j}{\delta_j}$ in the case that $d_{ij} \neq 0$. As by definition the graph is connected this is equivalent to $\frac{x_i}{\delta_i} = \frac{x_j}{\delta_j}$ for all $i, j \in \Gamma$. So if we choose $a := x_1$ and $b := \delta_1$ we obtain $\mathbf{x} = \frac{a}{b}\delta$.

□

Example 3.2. [16, p. 213] We want to use the above lemma to prove that the quadratic form of $\tilde{\mathbb{E}}_6$ calculated in example 3.1 is positive semi-definite. For this purpose we compute its associated bilinear form to be

$$\begin{aligned}(\mathbf{x},\mathbf{y}) = 2\sum_{i=1}^{7} x_i y_i &\\ -x_1y_2 - x_2y_3 - x_3y_4 - x_4y_5 - x_3y_6 - x_6y_7&\\ -y_1x_2 - y_2x_3 - y_3x_4 - y_4x_5 - y_6x_7 - y_3x_6.&\end{aligned}$$

Then plugging in $\delta = (1,2,3,2,1,2,1)$ for $\mathbf{x}$ yields

$$\begin{aligned}(\delta, \mathbf{y}) = {} & +2y_1 + 4y_2 + 6y_3 + 4y_4 + 2y_5 + 4y_6 + 2y_7 \\ & \qquad\quad - 1y_2 - 2y_3 - 3y_4 - 2y_5 - 3y_6 - 2y_7 \\ & - 2y_1 - 3y_2 - 2y_3 - 1y_4 \qquad\quad - 1y_6 \\ & \qquad\qquad\quad - 2y_3 \\ = {} & 0.\end{aligned}$$

Thus applying lemma 3.1 we learn that the quadratic form of $\tilde{\mathbb{E}}_6$ is positive semi-definite.

We see that lemma 3.1 provides a useful criterion in order to show that a graph is positive semi-definite. We can use it to obtain the following classification.

Theorem 3.2. *Let (Γ, d) be a valued graph.*

(i) *(Γ, d) is a Dynkin diagram if and only if its quadratic form is positive definite.*

(ii) *(Γ, d) is a Dynkin or extended Dynkin diagram if and only if its quadratic form is positive semi-definite.*

A list of all Dynkin and extended Dynkin diagrams can be found in figs. C.1 and C.2.

Proof. This proof is a generalisation of [16, Thm. 8.6]. We begin by showing that the quadratic form q of an extended Dynkin diagram is positive semi-definite. For this purpose we use lemma 3.1 which says that q is positive semi-definite if there exists a non-zero vector $\delta \in \mathbb{Z}^n$ with positive components such that $(\delta, \mathbf{x})$ vanishes for all $\mathbf{x} \in \mathbb{Z}^n$. The respective vectors δ for all extended Dynkin diagrams are displayed in fig. C.3. It is a straight forward calculation to show that these vectors indeed satisfy $(\delta, \cdot) = 0$ like we did in the case of $\tilde{\mathbb{E}}_6$ in example 3.2.

If we now conversely assume that q is positive semi-definite and (Γ, d) is neither a Dynkin nor an extended Dynkin diagram we can always choose a subgraph (Γ', d') which is of extended Dynkin type. If $\Gamma = \Gamma'$ the subgraph must contain less edges in order to be a proper subgraph. This means there exist $i, j \in \Gamma$ such that $d_{ij} \neq 0$ and $d'_{ij} = 0$. But this already implies that $0 = q'(\delta) > q(\delta)$ as δ is non-zero and positive. This is a contradiction to the assumption that q is positive semi-definite. Now if $\Gamma' \subsetneq \Gamma$ choose a vertex i_0 that does not lie in Γ' but has a neighbour say j_0 in Γ'. This is always possible as Γ is connected. Now define a vector $\mathbf{x} \in \mathbb{Z}^n$ by $x_i := 2\delta_i$ for all $i \in \Gamma'$, $x_{i_0} = 1$ and $x_j = 0$ for all other vertices j of Γ. Evaluating q on this vector yields

$$q(\mathbf{x}) \geq q'(2\delta) + f_{i_0} - 2d_{j_0 i_0} f_{i_0} \delta_{j_0} = f_{i_0}(1 - 2d_{j_0 i_0} \delta_{j_0})$$

and as $d_{j_0 i_0} \delta_{j_0} \geq 1$ it follows that $q(\mathbf{x}) < 0$ which is again a contradiction. Consequently if q is positive semi-definite (Γ, d) must already be Dynkin or extended Dynkin. If moreover q is positive definite (Γ, d) must be Dynkin as in the extended Dynkin case we have $q(\delta) = 0$.

So it is now left to show that the quadratic form q of a Dynkin diagram (Γ, d) is positive definite. Note that by extending (Γ, d) by one vertex $n+1$ and an edge we obtain an extended Dynkin diagram $(\bar{\Gamma}, \bar{d})$. Fix such a graph and let $\bar{q}$ be the quadratic form of it. Now suppose there exists an $\mathbf{x} \in \mathbb{Z}^n$ such that $q(\mathbf{x}) \leq 0$. Then define a vector $\bar{\mathbf{x}} \in \mathbb{Z}^{n+1}$ via $\bar{x}_i := x_i$ for $i \in \Gamma$ and $\bar{x}_{n+1} := 0$ for the new vertex $n+1 \in \bar{\Gamma} \setminus \Gamma$. It satisfies $\bar{q}(\bar{\mathbf{x}}) = q(\mathbf{x}) \leq 0$ and as $\bar{q}$ is positive semi-definite $\bar{q}(\bar{\mathbf{x}}) = 0$. Using lemma 3.1 it follows that $\bar{\mathbf{x}} = \frac{a}{b}\delta$ where $a, b \in \mathbb{Z}$ and δ is the non-zero vector satisfying $(\delta, \mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbb{Z}^{n+1}$. But as $\bar{x}_{n+1} = 0$ and $\delta_{n+1} \neq 0$ we already must have $a = 0$ which implies that $\mathbf{x} = 0$. So we can conclude that q is indeed positive definite. This closes the proof. □

3.3 Roots and Coxeter transformations

We now want to define a root system for a valued graph (Γ, d). This is done by first defining the following maps.

Definition 3.3. For $k \in \Gamma$ we define the linear transformation

$$\begin{aligned} s_k : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n, \\ \mathbf{x} &\longmapsto \mathbf{x} - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\mathbf{e}_k \end{aligned} \tag{3.6}$$

where $(\, , \,)$ is the bilinear form of (Γ, d) and $\mathbf{e}_k$ denotes the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ in $\mathbb{Z}^n$ which has all components equal zero except the k-th component. This map is well-defined, ie. its image lies in $\mathbb{Z}^n$ as can be seen by calculating $s_k\mathbf{x}$ for each component. For $i \neq k$

$$(s_k\mathbf{x})_i = x_i$$

and

$$(s_k\mathbf{x})_k = -x_k + \sum_{i=1}^{n} d_{ik}x_i \; \in \mathbb{Z}$$

Moreover we learn that s_k is independent of the choice of the integers f_i. We will refer to the maps s_k as *reflection maps*. The group $\mathcal{W} := \langle \{s_k\}_{k\in\Gamma} \rangle$ generated by these maps is called the *Weyl group*.

The attentive reader might have noticed that we have already used the symbol s_k in order to denote transformations of an orientation on (Γ, d). This is no coincidence and will be explained in the next chapter after developing some more insight.

Proposition 3.3. (i) *The map $s_k \in \mathcal{W}$ satisfies $(s_k)^2 = \mathrm{id}_{\mathbb{Z}^n}$.*

(ii) *Elements of the Weyl group conserve the bilinear form $(\, , \,)$, ie. we have*

$$(w\mathbf{x}, w\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

for all $w \in \mathcal{W}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$.

Proof. (i) It is a straight forward calculation that for all $x \in \mathbb{Z}^n$

$$\begin{aligned} (s_k)^2\mathbf{x} &= s_k\left(\mathbf{x} - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\mathbf{e}_k\right) \\ &= s_k\mathbf{x} - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\; s_k(\mathbf{e}_k) \\ &= \mathbf{x} - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\mathbf{e}_k + 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\mathbf{e}_k \\ &= \mathbf{x}. \end{aligned}$$

(ii) It is sufficient to show that the generators of the Weyl group conserve the bilinear form. So let $k \in \Gamma$. Then

$$\begin{aligned} (s_k\mathbf{x}, s_k\mathbf{y}) &= (\mathbf{x}, \mathbf{y}) - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}(\mathbf{e}_k, \mathbf{y}) - 2\frac{(\mathbf{y}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}(\mathbf{x}, \mathbf{e}_k) + 4\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\frac{(\mathbf{y}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}(\mathbf{e}_k, \mathbf{e}_k) \\ &= (\mathbf{x}, \mathbf{y}) \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$.

$\square$

Note that elements of the radical $\mathbf{x} \in \operatorname{rad} q$ are fixed points of the reflection maps s_k as $(\mathbf{x}, \mathbf{e}_k) = 0$. Consequently also for an arbitrary $w \in \mathcal{W}$ we have $w\mathbf{x} = \mathbf{x}$ and therefore w induces a map

$$\bar{w} : \mathbb{Z}^n / \operatorname{rad} q \longrightarrow \mathbb{Z}^n / \operatorname{rad} q.$$

Let $\overline{\mathcal{W}}$ denote the group of these induced maps. We will call this group the *reduced Weyl group*. Of course for a Dynkin diagram $\overline{\mathcal{W}} = \mathcal{W}$.

Lemma 3.4. *The reduced Weyl group* $\overline{\mathcal{W}}$ *of a valued graph* (Γ, d) *with positive semi-definite form* q *is finite.*

Proof. We follow the proof given in [4, Prop. 1.5]. Choose an $f \in \mathbb{Z}$ such that $f_i \leq f$ for all $i \in \Gamma$ and let $M \subset \mathbb{Z}^n$ be the set of vectors $\mathbf{x}$ which fulfil $q(\mathbf{x}) \leq f$. Now let $A \in \mathrm{GL}_n(\mathbb{Q})$ such that

$$Q = A^{\mathrm{T}} \operatorname{diag}(c_1, \ldots, c_r, 0, \ldots, 0)\, A$$

with $c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0}$ where $Q \in \mathrm{M}^{n \times n}(\mathbb{Z})$ is the matrix associated to q after fixing some numbering of Γ. Of course $n - r = \dim(\operatorname{rad} q)$ and the image of $\operatorname{rad} q$ under A consists of those vectors $\mathbf{y}$ with $y_i = 0$ for $i = 1, \ldots, r$.

Now let h be the common denominator of all entries of A. Then hA has integer entries and thus for all $\mathbf{x} \in M$ its image under hA has integer entries. Moreover $\mathbf{y} := A\mathbf{x}$ satisfies

$$c_i y_i^2 \leq \sum_{j=1}^{r} c_j y_j^2 = q(\mathbf{x}) \leq f.$$

So we have $|y_i| \leq \sqrt{f/c_i}$ for all $1 \leq i \leq r$. Thus the image of M under hA is modulo $hA(\operatorname{rad} q)$ finite. Therefore $\overline{M} := M / \operatorname{rad} q$ must already be finite too.

Now since the generators s_k of $\mathcal{W}$ preserve the quadratic form q as shown in proposition 3.3 all elements $\overline{w} \in \overline{\mathcal{W}}$ map $\overline{M}$ into $\overline{M}$. Moreover we note that since $q(\mathbf{e}_i) = f_i \leq f$ the set $\overline{M}$ contains a basis of $\mathbb{Z}^n / \operatorname{rad} q$. We finally conclude that since every map $\overline{w} \in \overline{\mathcal{W}}$ is fully described by its action on the basis elements we can embed $\overline{\mathcal{W}}$ into the symmetric group over the set $\overline{M}$. Thus $\overline{\mathcal{W}}$ has to be finite. $\square$

Definition 3.4. We say an element $\mathbf{x} \in \mathbb{Z}^n$ is a *root* if there exist a $k \in \Gamma$ and a $w \in \mathcal{W}$ such that $\mathbf{x} = w(\mathbf{e}_k)$. Moreover we say a root $\mathbf{x} \in \mathbb{Z}^n$ is *positive* if $x_i \geq 0$ for all $i \in \Gamma$. In this case we write $\mathbf{x} \geq 0$ and $\mathbf{x} \not\geq 0$ for an $\mathbf{x} \in \mathbb{Z}^n$ which is not positive.
We denote the set of all roots as Φ, the set of all positive roots as Φ^+, and call $\Phi^- := \{\mathbf{x} \in \mathbb{Z}^n | -\mathbf{x} \in \Phi^+\}$ the set of negative roots.

From lemma 3.4 we can directly conclude that the root system of a Dynkin diagram is finite. This is already a huge observation and will become important later when establishing a correspondence between irreducible representations of a species and the root system of its underlying graph.

Even though we have not developed enough tools in order to easily construct root systems explicitly we can already give a nice relation between the root systems of graphs with opposite valuation.

Proposition 3.5. *Let* (Γ, d) *be a valued graph,* Φ *its set of roots,* $\mathcal{W}$ *its Weyl group, and* q *its quadratic form. Let* $\Phi^{\,\mathrm{op}}$ *be the set of roots of the valued graph* $(\Gamma, d^{\,\mathrm{op}})$ *as defined in example 2.3. Then the following holds.*

(i) *The map*

$$\Delta : \mathbb{Z}^n \diagdown \{\mathbf{x} | q(\mathbf{x}) = 0\} \longrightarrow \mathbb{Q}^n, \mathbf{x} \longmapsto \frac{2D\mathbf{x}}{q(\mathbf{x})} \tag{3.7}$$

induces a bijection between Φ *and* $\Phi^{\,\mathrm{op}}$ *respectively between the subsets* Φ^+ *and* $(\Phi^{\,\mathrm{op}})^+$.

(ii) *The map* Δ *commutes with the generators of the Weyl group in the sense that*

$$s_k^{\mathrm{op}} \Delta = \Delta s_k$$

for all vertices $k \in \Gamma$.

Proof. (ii) A direct computation shows that for an arbitrary $\mathbf{x} \in \Gamma$

$$\begin{aligned}
\left(s_k^{\mathrm{op}} \Delta \mathbf{x}\right)_k &= \frac{2}{q(\mathbf{x})} \left(s_k^{\mathrm{op}} D\mathbf{x}\right)_k \\
&= \frac{2}{q(\mathbf{x})} \left(-f_k x_k + \sum_{i=1}^{n} d_{ik}^{\mathrm{op}} f_i x_i\right) \\
&= \frac{2}{q(\mathbf{x})} \left(-f_k x_k + \sum_{i=1}^{n} d_{ik} f_k x_i\right) \\
&= \frac{2}{q(s_k \mathbf{x})} (D s_k \mathbf{x})_k \\
&= (\Delta s_k \mathbf{x})_k
\end{aligned}$$

for $k \in \Gamma$ where in the third step it was used that $d_{ik}^{\mathrm{op}} f_i = d_{ki} f_i = d_{ik} f_k$. Moreover for $i \neq k$

$$\left(s_k^{\mathrm{op}} \Delta \mathbf{x}\right)_i = (\Delta \mathbf{x})_i = \frac{f_i x_i}{2q(\mathbf{x})} = \frac{f_i \left(s_k \mathbf{x}\right)_i}{2q(s_k \mathbf{x})} = \left(\Delta s_k^{\mathrm{op}} \mathbf{x}\right)_i$$

which proves that $s_k^{\mathrm{op}} \Delta = \Delta s_k$.

(i) First note that the simple roots $\mathbf{e}_k \in \Phi$ are mapped to themselves again computing

$$\Delta \mathbf{e}_k = \frac{2D\mathbf{e}_k}{q(\mathbf{e}_k)} = \frac{2f_k \mathbf{e}_k}{2f_k} = \mathbf{e}_k.$$

Consequently an arbitrary root $s_{k_1} \dots s_{k_{n-1}} \mathbf{e}_n \in \Phi$ is mapped to

$$\Delta \left(s_{k_1} \dots s_{k_{n-1}} \mathbf{e}_n\right) = s_{k_1}^{\mathrm{op}} \dots s_{k_{n-1}}^{\mathrm{op}} \mathbf{e}_n$$

which is an element in $\Phi^{\,\mathrm{op}}$. Now obviously the map Δ is bijective on Φ. Moreover if $\mathbf{x} = w\mathbf{e}_k \in \Phi$ for some $w \in \mathcal{W}$ we have

$$\Delta \mathbf{x} = \frac{2D\mathbf{x}}{q(w\mathbf{e}_k)} = \frac{D\mathbf{x}}{f_k}.$$

Thus as all entries of D are positive we deduce that $\Delta \mathbf{x}$ is positive if and only if $\mathbf{x}$ is positive. $\square$

One often says that the root systems Φ and Φ^{op} are *dual* to one another. The correspondence for example already tells us that the root systems of $\mathbb{B}_n$ and $\mathbb{C}_n$ are bijective. Moreover we can use this relation to prove the following lemma which is a key observation implying some useful corollaries.

Lemma 3.6. [4, Lem. 1.4] *Let (Γ, d) be a valued graph with positive semi-definite quadratic form q. Then if $\mathbf{x} \in \Phi^+$ is a positive root and $k \in \Gamma$ we either find $s_k\mathbf{x}$ to be positive or $\mathbf{x} = \mathbf{e}_k$.*

Proof. This proof follows the one given in [4, Lem. 1.4]. First consider the case $q(\mathbf{x} - \mathbf{e}_k) = 0$. If (Γ, d) is Dynkin then $\mathbf{x} = \mathbf{e}_k$. If the graph is extended Dynkin lemma 3.1 tells us that $\mathbf{x} = \mathbf{e}_k + m\delta$ for some $m \in \mathbb{Z}$ where $\delta \in \operatorname{rad} q$ is the respective vector given in fig. C.3. As $\mathbf{x}$ is positive and every component of δ is larger or equal to 1 we need $m \geq 0$. In the case $m > 0$ the image of $\mathbf{x}$ under s_k is

$$s_k\mathbf{x} = m\delta - \mathbf{e}_k$$

as $s_k\delta = \delta$ and thus positive.

Now let $q(\mathbf{x} - \mathbf{e}_k) \neq 0$. As q is positive semi-definite using (3.2) we obtain

$$0 \leq (\mathbf{x} \pm \mathbf{e}_k, \mathbf{x} \pm \mathbf{e}_k) = (\mathbf{x}, \mathbf{x}) + (\mathbf{e}_k, \mathbf{e}_k) \pm 2(\mathbf{x}, \mathbf{e}_k).$$

Since $\mathbf{x}$ is a root it can be written as $\mathbf{x} = w\mathbf{e}_l$ for some $w \in \mathcal{W}$ and $l \in \Gamma$. Now plugging $(\mathbf{x}, \mathbf{x}) = (\mathbf{e}_l, \mathbf{e}_l) = 2f_l$ into the above equation and using that $q(\mathbf{x} - \mathbf{e}_k) > 0$ we get

$$-(f_l + f_k) \leq (\mathbf{x}, \mathbf{e}_k) < f_l + f_k.$$

In the case $(\mathbf{x}, \mathbf{e}_k) \leq 0$ the components of

$$s_k\mathbf{x} = \mathbf{x} - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)}\mathbf{e}_k$$

are all greater or equal to those of $\mathbf{x}$ and consequently $s_k\mathbf{x}$ is positive. If now $0 < (\mathbf{x}, \mathbf{e}_k) < f_l + f_k$ then the k-th component of $s_k\mathbf{x}$ satisfies

$$(s_k\mathbf{x})_k = x_k - 2\frac{(\mathbf{x}, \mathbf{e}_k)}{(\mathbf{e}_k, \mathbf{e}_k)} > x_k - \frac{1}{2}(1 - \frac{f_l}{f_k}).$$

In the case $f_k \geq f_l$ this yields $s_k\mathbf{x} \geq 0$ as all components are integer.

So we are left with the case $f_l > f_k$. Remember that on the valued graph $(\Gamma, d^{\mathrm{op}})$ where we set $(d^{\mathrm{op}})_{ij} := d_{ji}$ the choice $f'_i := (\prod_{j=1}^n f_j)/f_i$ satisfies

$$(d^{\mathrm{op}})_{ij}f'_j = (d^{\mathrm{op}})_{ji}f'_i.$$

and moreover $f'_l < f'_k$. We now use the bijection

$$\Delta : \Phi^+ \longrightarrow (\Phi^{\mathrm{op}})^+$$

from proposition 3.5 to find that $\Delta\mathbf{x}$ is a positive root of $(\Gamma, d^{\mathrm{op}})$. So since $f'_l < f'_k$ we can deduce that either $\Delta\mathbf{x} = \mathbf{e}_k$ or $s_k^{\mathrm{op}}\Delta\mathbf{x}$ is positive, which implies that either $\mathbf{x} = \Delta^{-1}\mathbf{e}_k = \mathbf{e}_k$ or $s_k\mathbf{x}$ is positive since $s_k^{\mathrm{op}}\Delta = \Delta s_k$.

□

Corollary 3.7. *Let (Γ, d) be of Dynkin or extended Dynkin type. Then for all roots $\mathbf{x} \in \Phi$ either $\mathbf{x}$ or $-\mathbf{x}$ is positive, which means that $\Phi = \Phi^+ \sqcup \Phi^-$.*

Proof. Let $\mathbf{x} = s_{k_t} \dots s_{k_1} \mathbf{e}_j \in \Phi$. Show the claim by induction in t. For $t = 0$ we have $\mathbf{x} = \mathbf{e}_j \geq 0$.

Now perform the induction step $t \to t+1$. We know that $s_{k_t} \dots s_{k_1} \mathbf{e}_j =: \mathbf{x}'$ must be either positive or negative. First let $\mathbf{x}'$ be positive. In this case we apply lemma 3.6 and deduce that either $\mathbf{x} = s_{k_{t+1}} \mathbf{x}' = s_{k_{t+1}} s_{k_t} \dots s_{k_1} \mathbf{e}_j$ is positive or $\mathbf{x}' = \mathbf{e}_{k_{t+1}}$ which means that $\mathbf{x} = s_{k_{t+1}} s_{k_t} \dots s_{k_1} \mathbf{e}_j = s_{k_{t+1}} e_{k_{t+1}} = -e_{k_{t+1}}$ is negative.

Now if $\mathbf{x}'$ is negative $-\mathbf{x}'$ is positive and we can apply lemma 3.6 again. So either $-s_{k_{t+1}}(\mathbf{x}')$ is positive which means $\mathbf{x} = s_{k_{t+1}} s_{k_t} \dots s_{k_1} \mathbf{e}_j$ is negative or $\mathbf{x}' = -e_{k_{t+1}}$. But the latter already yields that $\mathbf{x} = s_{k_{t+1}} s_{k_t} \dots s_{k_1} \mathbf{e}_j = s_{k_{t+1}}(-e_{k_{t+1}}) = e_{k_{t+1}}$ is positive. □

So far we know that a root system splits into a positive and a negative part but we do not have deeper insight into its structure. For this purpose one defines Coxeter transformations which provide a direct way of constructing a root system.

Definition 3.5. An element $c \in \mathcal{W}$ of the form

$$c = s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)}$$

where $\pi : \{1 \dots n\} \to \Gamma$ is a bijection is called a *Coxeter transformation*.

Remark 3.3. (i) If $c = s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)}$ is a Coxeter transformation, then its inverse $c^{-1} = s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(n)}$ is also a Coxeter transformation.

(ii) In lemma 3.4 it was shown that for Dynkin diagrams the Weyl group $\mathcal{W}$ is finite. Consequently the cyclic subgroup $\langle c \rangle \subseteq \mathcal{W}$ is finite and thus c must be of finite order which means there exists an $m > 0$ such that $c^m = \mathrm{id}_{\mathbb{Z}^n}$.

(iii) If Ω is an admissible orientation and π and π' are two different admissible orderings with respect to Ω as defined in definition 2.16 then

$$s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)} = s_{\pi'(n)} \dots s_{\pi'(2)} s_{\pi'(1)}, \tag{3.8}$$

which means that a Coxeter transformation does not depend on a particular admissible ordering chosen with respect to Ω.

Proof. One can prove the statement inductively in n. For $n = 1$ nothing is to show and for $n > 1$ note that if i and j are not neighbours then $s_i s_j = s_j s_i$. Now fix two admissible orderings π and π' and let $\pi(1) = \pi'(m)$. Then as $\pi(1)$ is a sink with respect to Ω by the definition of an admissible ordering there may only be arrows incoming to $\pi'(m)$. Therefore there are no edges between $\pi'(m)$ and $\pi'(1) \dots \pi'(m-1)$ since as π' is an admissible ordering only arrows from $\pi'(m)$ to $\pi'(1) \dots \pi'(m-1)$ may occur which is prohibited by our previous observation. Consequently

$$s_{\pi'(m)} \dots s_{\pi'(2)} s_{\pi'(1)} = s_{\pi'(m-1)} \dots s_{\pi'(2)} s_{\pi'(1)} s_{\pi'(m)}$$

and we can apply the induction hypothesis to the remaining $n-1$ elements in order to obtain eq. (3.8). □

Example 3.3. For the graph

$$\mathbb{B}_3: \quad 1 \xrightarrow{(1,2)} 2 \longrightarrow 3$$

the reflection maps can be computed to be

$$s_1 = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

For the ordering $1, 2, 3$ the associated Coxeter transformation is

$$c = s_3 s_2 s_1 = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

whereas

$$c' = s_3 s_1 s_2 = s_1 s_3 s_2 = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

The Coxeter transformations associated to the orderings $2, 1, 3$ and $2, 3, 1$ agree with each other because these orderings are both admissible orderings of the orientation $1 \to 2 \leftarrow 3$ while the ordering $1, 2, 3$ corresponds to the orientation $1 \leftarrow 2 \leftarrow 3$.

Lemma 3.8. *Let (Γ, d) be a valued graph with positive semi-definite quadratic form and $\mathbf{x} \in \mathbb{Z}^n$. Then $c\,\mathbf{x} = \mathbf{x}$ for a Coxeter transformation $c \in \mathcal{W}$ if and only if $\mathbf{x} \in \operatorname{rad} q$.*

Proof. Clearly an $\mathbf{x} \in \operatorname{rad} q$ is a fixed point of all generators s_k of the Weyl group and therefore $\mathbf{x}$ is also a fixed point of an arbitrary Coxeter transformation $c \in \mathcal{W}$. So it remains to prove the converse direction.

Let $c = s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)}$ be a Coxeter transformation such that $c\,\mathbf{x} = \mathbf{x}$. Note that s_k only affects the k-th component of its argument. Therefore since π is a bijection

$$(s_{\pi(t)}\mathbf{x})_{\pi(t)} = (s_{\pi(n)} \dots s_{\pi(t+1)} s_{\pi(t)}\mathbf{x})_{\pi(t)}$$

for all $1 \leq t \leq n$. So for $t = 1$ we find

$$(s_{\pi(1)}\mathbf{x})_{\pi(1)} = (s_{\pi(n)} \dots s_{\pi(1)}\mathbf{x})_{\pi(1)} = (c\,\mathbf{x})_{\pi(1)} = \mathbf{x}_{\pi(1)}.$$

So consequently $s_{\pi(1)}\mathbf{x} = \mathbf{x}$. Iteratively we can therefore deduce that $s_{\pi(1)}\mathbf{x} = \dots = s_{\pi(n)}\mathbf{x} = \mathbf{x}$, ie. we have $s_k\mathbf{x} = \mathbf{x}$ for all $k \in \Gamma$. Evaluating the k-th component of this expression yields

$$0 = x_k - (s_k\mathbf{x})_k = x_k - \left(-x_k + \sum_{i=1}^n d_{ik} z_i \right) = 2 \left(x_k - \frac{1}{2} \sum_{i=1}^n d_{ik} x_i \right).$$

Multiplying the above with $f_k x_k$ and summing over k then leads to

$$0 = \sum_{k=1}^n f_k x_k^2 - \frac{1}{2} \sum_{k,i=1}^n d_{ik} f_k x_i x_k = q(\mathbf{x})$$

and since q is positive semi-definite we finally obtain that $\mathbf{x} \in \operatorname{rad} q$. $\square$

Lemma 3.9. [4, Lem. 1.6] *Let $\mathbf{x} \in \Phi^+$ be a positive root of a Dynkin or extended Dynkin diagram (Γ, d) and let $c = s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)}$ be a Coxeter transformation. Then*

(i) $c\,\mathbf{x} \not\geq 0$ *if and only if* $\mathbf{x} = \mathbf{p}_{\pi(t)}$ *for* $1 \leq t \leq n$*, where*

$$\mathbf{p}_{\pi(t)} := s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(t-1)} \mathbf{e}_{\pi(t)}. \tag{3.9}$$

(ii) $c^{-1}\mathbf{x} \not\geq 0$ *if and only if* $\mathbf{x} = \mathbf{q}_{\pi(t)}$ *for* $1 \leq t \leq n$*, where*

$$\mathbf{q}_{\pi(t)} := s_{\pi(n)} s_{\pi(n-1)} \dots s_{\pi(t+1)} \mathbf{e}_{\pi(t)}. \tag{3.10}$$

Proof. We only proof (i) as (ii) runs analogously.

($\Rightarrow$) Let $c\,\mathbf{x} \not\geq 0$. Then there must exist a $t \in \{1 \dots n\}$ such that $s_{\pi(t-1)} \dots s_{\pi(1)} \mathbf{x} = \mathbf{e}_{\pi(t)}$ because otherwise as $\mathbf{x}$ is positive $s_{\pi(t)} \dots s_{\pi(1)} \mathbf{x}$ must be positive for all t by lemma 3.6. So especially $c\,\mathbf{x} = s_{\pi(n)} \dots s_{\pi(1)} \mathbf{x}$ is positive which is a contradiction. So immediately from $s_{\pi(t-1)} \dots s_{\pi(1)} \mathbf{x} = \mathbf{e}_{\pi(t)}$ for a suitable t we can deduce that $\mathbf{x} = s_{\pi(1)} \dots s_{\pi(t-1)} \mathbf{e}_{\pi(t)}$ since $s_k^2 = \mathrm{id}_{\mathbb{Z}^n}$ for all $k \in \Gamma$.

($\Leftarrow$) Now let $\mathbf{x} = s_{\pi(1)} \dots s_{\pi(t-1)} \mathbf{e}_{\pi(t)}$ for $1 \leq t \leq n$. Then $c\,\mathbf{x} = s_{\pi(n)} \dots s_{\pi(t)} \mathbf{e}_{\pi(t)}$. As a map s_k for $k \in \Gamma$ only changes the k-th component we find that

$$(c\,\mathbf{x})_{\pi(t)} = (s_{\pi(t)} \mathbf{e}_{\pi(t)})_{\pi(t)} = -1.$$

Consequently $c\,\mathbf{x}$ is not positive. □

Remark 3.4. If Ω is an admissible orientation the definition of $\mathbf{p}_i$ and $\mathbf{q}_i$ does not depend on the specific admissible ordering chosen with respect to Ω. This can be shown by an analogous argumentation as in remark 3.3 (iii).

3.4 The rootssystem of a Dynkin diagram

We now want to apply lemma 3.9 to the case of a Dynkin diagram in order to give a constructive description of its root system. We already know that this set is finite since $\mathcal{W}$ is finite. This fact also implies the following proposition.

Proposition 3.10. [4, Lem. 1.7] *Let $\mathbf{x} \in \mathbb{Z}^n$ be positive. Moreover let (Γ, d) be a Dynkin diagram and c a Coxeter transformation. Then there exists an integer $r \geq 0$ such that $c^r\mathbf{x} \not\geq 0$.*

Proof. If $\mathbf{x} = 0$ there is nothing to show. So let $\mathbf{x} \neq 0$. As (Γ, d) is Dynkin, the Coxeter transformation c is of finite order as discussed in remark 3.3. So let $m = \mathrm{ord}(c)$ and define $y := \sum_{t=1}^{m} c^t \mathbf{x}$. This vector satisfies $c\mathbf{y} = \mathbf{y}$ and must therefore vanish by lemma 3.8. So as $0 = \mathbf{x} + \sum_{t=1}^{m-1} c^t \mathbf{x}$ and $\mathbf{x}$ is positive non-zero there must already exist an $1 < r < m - 1$ such that $c^r\mathbf{x}$ has a negative component. □

We now have all the needed tools at hand to give a full description of all positive roots of a Dynkin diagram with the help of Coxeter transformations.

Corollary 3.11. [4, Prop. 1.9] *Let (Γ, d) be a Dynkin diagram, $c = s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)}$ a Coxeter transformation and let $a_{\pi(t)}$ be the largest integer such that $c^{-r}\mathbf{p}_{\pi(t)} \geq 0$ for all $0 \leq r \leq a_{\pi(t)}$ and $1 \leq t \leq n$. Then all positive roots are already given by*

$$\Phi^+ = \{c^{-r}\mathbf{p}_{\pi(t)} \mid 0 \leq r \leq a_{\pi(t)},\ 1 \leq t \leq n\} \tag{3.11}$$

where the vectors $\mathbf{p}_{\pi(t)}$ are defined as in lemma 3.9. Similarly if $b_{\pi(t)}$ is the largest integer such that $c^r\mathbf{q}_{\pi(t)} \geq 0$ for all $0 \leq r \leq b_{\pi(t)}$ and $1 \leq t \leq n$. Then all positive roots are already given by

$$\Phi^+ = \{c^{r}\mathbf{q}_{\pi(t)} \mid 0 \leq r \leq b_{\pi(t)},\ 1 \leq t \leq n\} \tag{3.12}$$

where again $\mathbf{q}_{\pi(t)}$ is defined as in lemma 3.9.

Proof. Only the first part of the proposition will be proven as the second part runs analogously.

By definition of the number $a_{\pi(t)}$ every root $c^{-r}\mathbf{p}_{\pi(t)}$ must lie in Φ^+ for all $0 \leq r \leq a_{\pi(t)}$ and $1 \leq t \leq n$. So it only remains to show that every positive root $\mathbf{x} \in \Phi^+$ must already be of this form. From proposition 3.10 we know that there exists an $r \geq 0$ such that $c^r\mathbf{x} \not\geq 0$. But then $c^{r-1}\mathbf{x}$ must already be of the form $c^{r-1}\mathbf{x} = \mathbf{p}_{\pi(t)}$ for a $1 \leq t \leq n$ as shown in lemma 3.9. So $\mathbf{x} = c^{-r+1}\mathbf{p}_{\pi(t)}$ is indeed of the desired form. ☐

One might ask now whether for different integers $s \neq t$ or vertices $i \neq j \in \Gamma$ the roots $c^{-s}\mathbf{p}_i$ and $c^{-t}\mathbf{p}_j$ are distinct. Of course we have

$$c^{-r}\mathbf{p}_i = c^{-(r+m)}\mathbf{p}_i$$

as c is of finite order m, but nevertheless the following still holds.

Proposition 3.12. *In the setting of corollary 3.11 the we have:*

(i) *Let $i, j \in \Gamma$, $1 \leq s \leq a_i$ and $1 \leq t \leq a_j$. Then $c^{-s}\mathbf{p}_i = c^{-t}\mathbf{p}_j \geq 0$ implies $s = t$ and $i = j$.*

(ii) *Let $i, j \in \Gamma$, $1 \leq s \leq b_i$ and $1 \leq t \leq b_j$. Then $c^{s}\mathbf{q}_i = c^{t}\mathbf{q}_j \geq 0$ implies $s = t$ and $i = j$.*

Proof. As always we will only proof the first part. Without loss of generality assume that $s \leq t$. Then from $c^{-s}\mathbf{p}_i = c^{-t}\mathbf{p}_j \geq 0$ we can deduce that

$$c^{t-s}\mathbf{p}_i = c^t c^{-s}\mathbf{p}_i = c^t c^{-t}\mathbf{p}_j = \mathbf{p}_j \geq 0$$

by applying lemma 3.9. But then lemma 3.6 already induces $t - s = 0$ since $c\mathbf{p}_i \not\geq 0$. So we arrive at $\mathbf{p}_i = \mathbf{p}_j$. Now suppose $i \neq j$. Let $\pi(a) = i$ and $\pi(b) = j$ and without loss generality assume that $a < b$. Then

$$0 = (\mathbf{p}_i)_j = (\mathbf{p}_j)_j = 1$$

which is a contradiction. So we must have had $i = j$ at first. ☐

Together with the observation that for each vertex $k = \pi(t) \in \Gamma$

$$\begin{aligned} c\,\mathbf{p}_{\pi(t)} &= s_{\pi(n)} \dots s_{\pi(2)} s_{\pi(1)} \left(s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(t-1)} \mathbf{e}_{\pi(t)}\right) \\ &= s_{\pi(n)} \dots s_{\pi(t+1)} s_{\pi(t)} \mathbf{e}_{\pi(t)} \\ &= -s_{\pi(n)} \dots s_{\pi(t+1)} \mathbf{e}_{\pi(t)} \\ &= -\mathbf{q}_{\pi(t)} \end{aligned} \tag{3.13}$$

we can thus think of the orbit of the root $\mathbf{p}_k$ under c^{-1} in the Dynkin case as follows:

$$\begin{tikzcd}
\Phi^+ : & \mathbf{p}_k \arrow[r, "c^{-1}"] & c^{-1}\mathbf{p}_k \arrow[r, "c^{-1}"] & \dots \arrow[r, "c^{-1}"] & c^{-a_k}\mathbf{p}_k = \mathbf{q}_{k'} \arrow[d, bend left, "c^{-1}"] \\
\Phi^- : & -\mathbf{q}_k \arrow[u, bend left, "c^{-1}"] & -c\,\mathbf{q}_k \arrow[l, "c^{-1}"'] & \dots \arrow[l, "c^{-1}"'] & -c^{a_{k'}}\mathbf{q}_k = -\mathbf{p}_{k'} \arrow[l, "c^{-1}"']
\end{tikzcd} \tag{3.14}$$

Example 3.4. We pick up on example 3.3 and compute the root system of $\mathbb{B}_3$. We fix the ordering $1, 2, 3$ of the vertices for which we have already calculated the Coxeter transformation c. Then we compute

$$\mathbf{p}_1 = \mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{p}_2 = s_1\mathbf{e}_2 = \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \mathbf{p}_3 = s_1 s_2 \mathbf{e}_3 = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

and with these vectors at hand we obtain

$$\begin{aligned}
c^{-1}\mathbf{p}_1 &= \begin{pmatrix}1\\1\\0\end{pmatrix}, & c^{-2}\mathbf{p}_1 &= \begin{pmatrix}1\\1\\1\end{pmatrix}, & c^{-3}\mathbf{p}_1 &= \begin{pmatrix}-1\\0\\0\end{pmatrix}\\
c^{-1}\mathbf{p}_2 &= \begin{pmatrix}2\\2\\1\end{pmatrix}, & c^{-2}\mathbf{p}_2 &= \begin{pmatrix}0\\1\\1\end{pmatrix}, & c^{-3}\mathbf{p}_2 &= \begin{pmatrix}-2\\-1\\0\end{pmatrix}\\
c^{-1}\mathbf{p}_3 &= \begin{pmatrix}0\\1\\0\end{pmatrix}, & c^{-2}\mathbf{p}_3 &= \begin{pmatrix}0\\0\\1\end{pmatrix}, & c^{-3}\mathbf{p}_3 &= \begin{pmatrix}-2\\-1\\-1\end{pmatrix}
\end{aligned}$$

Thus the set Φ^+ of positive roots of $\mathbb{B}_3$ is given by

$$\Phi^+ = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}.$$

Example 3.5. From proposition 3.5 we have already deduced that the root systems of $\mathbb{B}_n$ and $\mathbb{C}_n$ are bijective. We will take a look at this bijection in the case $n = 3$ now.
It makes sense to consider the sets of roots $\{c^{-r}\mathbf{p}_k \mid 0 \le r \le a_k\}$ for a fixed vertex k individually as the bijection $\Delta : \Phi^+_{\mathbb{B}_3} \to \Phi^+_{\mathbb{C}_3}$ given by eq. (3.7) turns out to be

$$\Delta\left(c^{-r}\mathbf{p}_k\right) = \frac{2D\left(c^{-r}\mathbf{p}_k\right)}{q(c^{-r}\mathbf{p}_k)} = \frac{2D\left(c^{-r}\mathbf{p}_k\right)}{2f_k} = \frac{1}{f_k} D\left(c^{-r}\mathbf{p}_k\right)$$

a linear map for fixed k where $D = \operatorname{diag}(f_1, f_2, f_3) = \operatorname{diag}(1, 2, 2)$. Moreover since Δ commutes with elements of the Weyl group we have $\Delta\left(c^{-r}\mathbf{p}_k\right) = \tilde{c}^{-r}\tilde{\mathbf{p}}_k$ where $\tilde{c}$ is the Coxeter transformation of $\mathbb{C}_3$ with respect to the same ordering as it was used in the definition of c. From example 3.4 we can therefore directly deduce the positive roots of $\mathbb{C}_3$ as listed in table 3.1.

$k=$ \ $r=$	0	1	2		$k=$ \ $r=$	0	1	2
1	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\xmapsto{\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}}$	1	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\2\end{pmatrix}$
2	$\begin{pmatrix}2\\1\\0\end{pmatrix}$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\xmapsto{\begin{pmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{pmatrix}}$	2	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$
3	$\begin{pmatrix}2\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\xmapsto{\begin{pmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{pmatrix}}$	3	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$

(a) Positive roots $c^{-r}\mathbf{p}_k$ of $\mathbb{B}_3$.

(b) Positive roots $\tilde{c}^{-r}\tilde{\mathbf{p}}_k$ of $\mathbb{C}_3$.

Table 3.1: The positive roots of $\mathbb{C}_3$ obtained from those of $\mathbb{B}_3$.

3.5 The extended Dynkin case

In contrast to the prior section we will not give a full description of the root system of an extended Dynkin diagram, but we will only highlight certain aspects, which will later become important in order to classify all species of finite representation type.

Definition 3.6. Let (Γ, d) be a valued graph and Ω an orientation of it. The *Euler form* is the bilinear form $\langle\ ,\ \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} f_i x_i y_i - \sum_{(i,j)\in\Omega} d_{ij} f_j x_i y_j$$

where f_i are integers satisfying $d_{ij} f_j = d_{ji} f_i$.

Remark 3.5. Like the quadratic form of a valued quiver the Euler form also depends on the choice of f_i. Because we will not be interested in absolute values but rather need to know if $\langle \mathbf{x}, \mathbf{y} \rangle$ is zero, positive or negative for certain vectors $\mathbf{x}, \mathbf{y}$, one can argue as in remark 3.1 this property is independent of the choice of f_i.
In the case of a k-species one does not run in this problem since there is a natural choice $f_i = [F_i : k]$. With this choice one is moreover able to connect the Euler form to the homological form [10, Prop. 4.1]. This has the advantage that one can give more constructive proofs of the following propositions as done in [12, pp. 15-18]. But since we are working in the more general setting of species we can not take on this idea and need to perform direct calculations.

Example 3.6. Let us compute the Euler form of the valued quiver

$$(\tilde{\mathbb{B}}_3,\Omega): \qquad 1 \xrightarrow{(1,2)} 2 \longrightarrow 3 \xrightarrow{(2,1)} 4 \; .$$

Choose $f_1 = f_4 = 1$ and $f_2 = f_3 = 2$. Then

$$\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + 2x_2y_2 + 2x_3y_3 + x_4y_4 - 2x_1y_2 - 2x_2y_3 - 2x_3y_4.$$

The Euler form is related to the symmetric bilinear form $(\ ,\ )$ previously defined in (3.1) via

$$(\mathbf{x},\mathbf{y}) = \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle.$$

We have already seen that a Coxeter transformation c leaves $(\ ,\ )$ invariant. That the Euler form features the same property is first of all not obvious. In order to prove this we need the following proposition.

Proposition 3.13. *The sets $\{\mathbf{p}_k\}_{k\in\Gamma}$ and $\{\mathbf{q}_k\}_{k\in\Gamma}$ are both bases of $\mathbb{Z}^n$.*

Proof. Without loss of generality assume that $\pi = \mathrm{id}_{\{1,\dots,n\}}$. One can inductively show that $\mathbf{e}_1,\dots,\mathbf{e}_t$ lie in the span of $\mathbf{p}_1,\dots,\mathbf{p}_t$ for all $1 \le t \le n$. For $t = 1$ this is clear as $\mathbf{p}_1 = \mathbf{e}_1$ by definition. Now for the induction step $t \to t+1$ note that the vector $\mathbf{p}_{t+1}$ has the form

$$\mathbf{p}_{t+1} = (*,\dots,*,1,0,\dots,0)^{\mathrm{T}}$$

where the 1 is located in the $t+1$ row. So since $\mathbf{e}_1,\dots,\mathbf{e}_t$ lie in the span of $\mathbf{p}_1,\dots,\mathbf{p}_t$ one can find a linear combination such that

$$\mathbf{e}_{t+1} = \mathbf{p}_{t+1} + \sum_{i\le t} a_i\mathbf{p}_i. \tag{3.15}$$

This proves the first part of the claim. The second is shown by analogous arguments. □

Proposition 3.14. *Let $\mathbf{x},\mathbf{y} \in \mathbb{Z}^n$ and let Ω be admissible.*

(i) $\langle \mathbf{p}_k,\mathbf{x}\rangle = f_kx_k = \langle \mathbf{x},\mathbf{q}_k\rangle$ *for all $k \in \Gamma$.*

(ii) $\langle \mathbf{x},\mathbf{y}\rangle = -\langle \mathbf{y},c\,\mathbf{x}\rangle = \langle c\,\mathbf{x},c\mathbf{y}\rangle$.

Proof. (i) It suffices to prove the statement on all basis vectors $\mathbf{e}_l$ of $\mathbb{Z}^n$. As always we will only prove the first part and without loss of generality we assume that $\pi = \mathrm{id}_{\{1,\dots,n\}}$.

So we want to prove that $\langle \mathbf{p}_k,\mathbf{e}_l\rangle = f_k\delta_{k,l}$ for all k,l where $\delta_{k,l}$ denotes the Kronecker delta. By assumption Ω is an admissible orientation which means that $(i,j) \in \Omega$ if and only if $i > j$ and $d_{ij} \neq 0$. So we can rewrite $\langle \mathbf{p}_k,\mathbf{e}_l\rangle$ as follows

$$\langle \mathbf{p}_k,\mathbf{e}_l\rangle = f_l\,(\mathbf{p}_k)_l - \sum_{(i,l)\in\Omega} d_{il}f_l\,(\mathbf{p}_k)_i = f_l\left((\mathbf{p}_k)_l - \sum_{i>l} d_{il}\,(\mathbf{p}_k)_i\right). \tag{3.16}$$

First let $k \leq l$. In this case recalling the shape of $\mathbf{p}_k$ given in (3.15) we see that $(\mathbf{p}_k)_i = 0$ for all $i > l \geq k$. This means that the second term in (3.16) must always vanish. As moreover $(\mathbf{p}_k)_k = 1$ we see that

$$\langle \mathbf{p}_k, \mathbf{e}_l \rangle = f_l \left((\mathbf{p}_k)_l - \sum_{i>l} d_{il} (\mathbf{p}_k)_i \right) = f_l (\delta_{k,l} - 0) = f_k \delta_{k,l}.$$

So we are left with the case $k > l$. In this case the key observation to make is that

$$(\mathbf{p}_k)_i = (s_i \dots s_{k-1} \mathbf{e}_k)_i$$

for every $i < k$ which becomes obvious considering the definition of $\mathbf{p}_k$ given in (3.9). Using this (3.16) yields

$$\begin{aligned}
\frac{1}{f_l} \langle \mathbf{p}_k, \mathbf{e}_l \rangle &= (\mathbf{p}_k)_l - \sum_{i>l} d_{il} (\mathbf{p}_k)_i \\
&= (s_l \dots s_{k-1} \mathbf{e}_k)_l - \sum_{k \geq i > l} d_{il} (s_i \dots s_{k-1} \mathbf{e}_k)_i \\
&= -(s_{l+1} \dots s_{k-1} \mathbf{e}_k)_l + \sum_{j=1}^{n} d_{jl} (s_{l+1} \dots s_{k-1} \mathbf{e}_k)_j - \sum_{k \geq i > l} d_{il} (s_i \dots s_{k-1} \mathbf{e}_k)_i \\
&= 0 + \sum_{k \geq j > l} d_{jl} (s_j \dots s_{k-1} \mathbf{e}_k)_j - \sum_{k \geq i > l} d_{il} (s_i \dots s_{k-1} \mathbf{e}_k)_i \\
&= 0.
\end{aligned}$$

(ii) It suffices to show this statement for $\mathbf{x} = \mathbf{p}_k$ where k is an arbitrary vertex because in proposition 3.13 we have seen that these vectors span $\mathbb{Z}^n$. Now using that $c\,\mathbf{p}_k = -\mathbf{q}_k$ as shown in (3.13) we find

$$\langle \mathbf{p}_k, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{q}_k \rangle = -\langle \mathbf{y}, c\mathbf{p}_k \rangle.$$

□

So far we made no assumption on the valued graph except that the orientation needs to be admissible. Now we concentrate on the case where (Γ, d) is additionally extended Dynkin. We have seen that for these graphs there exists a vector δ spanning the radical of the associated quadratic form, ie. $(\delta, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$. This means that

$$\langle \delta, \mathbf{x} \rangle = -\langle \mathbf{x}, \delta \rangle$$

for an arbitrary $\mathbf{x}$. There is a "canonic" choice for δ given in fig. C.3 where one chooses one component to be equal to one. Then lemma 3.1 tells us that $\operatorname{rad} q = \mathbb{Z}\delta$.

Definition 3.7. For a vector $\mathbf{x} \in \mathbb{Z}^n$ we define its *defect* to be the integer

$$\partial \mathbf{x} := \langle \delta, \mathbf{x} \rangle.$$

Example 3.7. In example 3.6 we calculated the Euler form of $\tilde{\mathbb{B}}_3$ with respect to the orientation $1 \to 2 \to 3 \to 4$. With $\delta = (1, 1, 1, 1)$ the defect of a vector $\mathbf{x}$ is given by

$$\partial\mathbf{x} = x_1 + 2x_2 + 2x_3 + x_4 - 2x_2 - 2x_3 - 2x_4 = x_1 - x_4.$$

So we learn that for example $\mathbf{p}_4 = \mathbf{e}_4$ has negative and $\mathbf{q}_1 = \mathbf{e}_1$ has positive defect with respect to the orientation chosen.

It turns out that the above observation is no coincidence and that there is a deeper pattern which allows us to distinguish three classes of roots of extended Dynkin diagrams by their defect.

Proposition 3.15. *For a root $\mathbf{x}$ of an extended Dynkin diagram together with an admissible orientation the following holds:*

(i) $\mathbf{x} = c^{-r}\mathbf{p}_k$ *for some positive r and $k \in \Gamma$ if and only if $\partial\mathbf{x} < 0$.*

(ii) $\mathbf{x} = c^{r}\mathbf{q}_k$ *for some positive r and $k \in \Gamma$ if and only if $\partial\mathbf{x} > 0$.*

(iii) $c^r\mathbf{x} \geq 0$ *for all $r \in \mathbb{Z}$ if and only if $\partial\mathbf{x} = 0$.*

Proof. It suffices to prove the "only if" part of (i), (ii) and (iii) since then the other direction follows by a proof by contradiction. For the proof we will mostly follow ideas from [12, pp. 15-18].

So first let $\mathbf{x} = c^{-r}\mathbf{p}_k$. Then using proposition 3.14 one obtains

$$\partial(c^{-r}\mathbf{p}_k) = \langle \delta, c^{-r}\mathbf{p}_k \rangle = \langle \delta, \mathbf{p}_k \rangle = -f_k\delta_k < 0$$

where we used that $c\delta = \delta$. Analogously one finds $\partial(c^r\mathbf{q}_k) = f_k\delta_k > 0$.

So now let $\mathbf{x}$ be root such that $c^r\mathbf{x} \geq 0$ for all $r \in \mathbb{Z}$. Before calculating the defect of $\mathbf{x}$ we have to make the following observation. In lemma 3.4 it was proven that the reduced Weyl group $\overline{\mathcal{W}}$ which is obtained from $\mathcal{W}$ by modding out $\operatorname{rad} q$ is finite. This means that the element $\overline{c} \in \overline{\mathcal{W}}$ corresponding to c is of finite order m. Consequently $c^m\mathbf{x}$ may only vary from $\mathbf{x}$ by an element $\mathbf{v} \in \operatorname{rad}(q)$, ie. $c^m\mathbf{x} = \mathbf{x} + \mathbf{v}$. Suppose $\mathbf{v} \neq 0$ then for all $s \in \mathbb{Z}$

$$c^{sm}\mathbf{x} = \mathbf{x} + s\mathbf{v}.$$

Thus we can choose an s such that $c^{sm}\mathbf{x} \ngeq 0$ which is a contradiction. So $\mathbf{v}$ must vanish which implies that $c^m\mathbf{x} = \mathbf{x}$. Now set $\mathbf{y} := \sum_{r=1}^{m} c^r\mathbf{x}$. This vector is a fixed point of c, ie. $c\mathbf{y} = \mathbf{y}$. Then lemma 3.8 tells us that there exists an $a \in \mathbb{Z}$ such that $\mathbf{y} = a\delta$. Consequently the defect of $\mathbf{y}$ vanishes and

$$0 = a\langle \delta, \delta \rangle = \langle \delta, \mathbf{y} \rangle = \sum_{r=1}^{m} \langle \delta, c^r\mathbf{x} \rangle = m\langle \delta, \mathbf{x} \rangle$$

which implies that $\partial\mathbf{x} = \langle \delta, \mathbf{x} \rangle = 0$. □

From this proposition we learn a lot about the structure of the root system of an extended Dynkin diagram. In contrast to the finite c-orbit of $\mathbf{p}_k$ in the Dynkin case as pictured in (3.14) the orbit of $\mathbf{p}_k$ under c is now infinite because suppose $c^{-r}\mathbf{p}_k \ngeq 0$ for some minimal r, ie. $c^{-r+1}\mathbf{p}_k = \mathbf{q}_{k'}$ then

$$0 > \partial\mathbf{p}_k = \partial(c^{-r+1}\mathbf{p}_k) = \partial\mathbf{q}_{k'} > 0$$

which is a contradiction. Therefore $c^{-r}\mathbf{p}_k \geq 0$ for all $r \geq 0$. So for $n = |\Gamma|$ the number of vertices we have $2n$ infinite sequences of roots — n coming from $\mathbf{p}_k$ and n from $\mathbf{q}_k$. The roots with zero defect which are called *regular* roots have a finite c-orbit and are also of finite number, but they shall not be discussed further as they will not be of importance for us later.

CHAPTER 4

Indecomposable Representations

Now we are aiming to connect the language of roots and Coxeter transformations developed in the last chapter with the representation theory of species and especially with the indecomposable objects. This will be done by generalising the ideas of Bernstein et al. [3] from the case of quiver representations to representations of species. Just like in Gabriel's theorem we are aiming to prove a one-to-one correspondence between the indecomposable representations of a species and the positive roots of its underlying graph in the case the graph at hand is Dynkin and not surprisingly the bijection will be induced by taking the dimension vector of a representation.

This chapter mainly follows the approach of Dlab and Ringel to this problem as presented in [4, pp. 15-20].

4.1 Reflection functors

The first step in order to connect positive roots to indecomposable representations is to construct the so called reflection functors S_k^+ and S_k^- which correspond to the reflection maps s_k defined in (3.6). This section will mostly deal with the rather technical definition of these and to its end we will develop the indicated relation to the reflection maps.

So let us take a second step back and remember that in lemma 2.11 we have seen an equivalent characterisation of what it means for valued quiver (Γ, d, Ω) not to contain any oriented circuit. Key to this was the notion of the transformed orientation $s_k\Omega$ for a $k \in \Gamma$ where all arrows with target or source k got switched in direction with respect to the original orientation Ω while leaving the valuation unchanged.

Example 4.1. Consider for example the valued quiver

$$(\Gamma, d, \Omega) : \quad \begin{tikzcd} & & 3 \arrow[dr, "{(2,4)}"] & \\ 1 \arrow[r] & 2 \arrow[ur, "{(3,1)}"] \arrow[rr, "{(3,2)}"] & & 4. \end{tikzcd}$$

Then s_2 acts on Ω as shown below:

$$(\Gamma, d, s_2\Omega) : \quad \begin{tikzcd} & & 3 \arrow[dl, "{(3,1)}"'] \arrow[dr, "{(2,4)}"] & \\ 1 & 2 \arrow[l] & & 4. \arrow[ll, "{(3,2)}"'] \end{tikzcd}$$

Throughout this chapter let (Γ, d, Ω) be a valued quiver with an admissible orientation Ω and $\mathbb{M}$ a modulation of it. Further let $k \in \Gamma$ be a sink with respect to Ω. We now want to associate a representation $\mathbf{X}$ of the species $(\mathbb{M}, \Omega)$[1] to a representation of $(\mathbb{M}, s_k\Omega)$. We do this by constructing the *reflection functor*

$$S_k^+ : \mathsf{rep}(\mathbb{M}, \Omega) \longrightarrow \mathsf{rep}(\mathbb{M}, s_k\Omega) \,. \tag{4.1}$$

For a representation $\mathbf{X} = (X_i, {}_j\varphi_i)$ of $(\mathbb{M}, \Omega)$ we define an object $S_k^+\mathbf{X} =: \mathbf{Y} = (Y_i, {}_j\psi_i)$ in $\mathsf{rep}(\mathbb{M}, s_k\Omega)$ as follows. For all $i \in \Gamma$ with $i \neq k$ we simply set $Y_i := X_i$, but at the vertex k we have to be careful. Since k is a sink with respect to Ω there are only arrows incoming to k which in $\mathbf{X}$ correspond to linear maps with codomain X_k. Now with respect to the new orientation $s_k\Omega$ these maps have to be replaced by maps with domain Y_k. It therefore seems quite natural to choose Y_k to be the kernel in

$$0 \longrightarrow Y_k \xrightarrow{({}_j\kappa_k)_{j\in\Gamma}} \bigoplus_{j=1}^{n} \big(X_j \otimes {}_jM_k\big) \xrightarrow{({}_k\varphi_j)_{j\in\Gamma}} X_k \tag{4.2}$$

where we set ${}_k\varphi_j = 0$ for every j that is not a neighbour of k. Note that this definition is well-defined in the sense that Y_k is finite dimensional as Y_k is a submodule of the finite dimensional module $\bigoplus_{j=1}^{n}(X_j \otimes {}_jM_k)$ over the division ring F_k. Moreover via this construction we directly obtain F_k-linear maps

$${}_j\kappa_k : Y_k \longrightarrow \underbrace{X_j}_{=Y_j} \otimes_j M_k$$

for all $j \in \Gamma \diagdown \{k\}$. These maps can be used to construct the needed F_j-linear maps ${}_j\psi_k : Y_k \otimes_k M_j \to Y_j$ by using the following proposition.

Proposition 4.1. *Let $(F_i, {}_iM_j)$ be a modulation of $1 \xrightarrow{(d_{12}, d_{21})} 2$ and $(X_1)_{F_1}$, $(X_2)_{F_2}$ right modules. Then there exists an isomorphism of (F_2, F_1)-bimodules*

$$\begin{aligned} \Phi : \mathrm{Hom}_{F_2}\big(X_1 \otimes_{F_1} {}_1M_2, X_2\big) &\longrightarrow \mathrm{Hom}_{F_1}\big(X_1, X_2 \otimes_{F_2} {}_2M_1\big) \\ \varphi &\longmapsto \overline{\varphi} \end{aligned}$$

Proof. Since F_2 is a division ring its right module X_2 is free by lemma B.3. So writing $X_2 \cong \bigoplus_{i\in I} F_2$

[1] Note that from now on we are writing $(\mathbb{M}, \Omega)$ for a species and not $(\mathbb{M}, \Gamma, d, \Omega)$ anymore since the graph (Γ, d) can easily be reconstructed from the modulation $\mathbb{M}$.

we obtain the following isomorphism of (F_2, F_1)-bimodules

$$\begin{aligned}\operatorname{Hom}_{F_2}\big({}_1M_2, X_2\big) &\cong \bigoplus_{i\in I} \operatorname{Hom}_{F_2}\big({}_1M_2, F_2\big)\\ &\cong \bigoplus_{i\in I} \big(F_2 \otimes_{F_2} \operatorname{Hom}_{F_2}\big({}_1M_2, F_2\big)\big)\\ &\cong \left(\bigoplus_{i\in I} F_2\right) \otimes_{F_2} \operatorname{Hom}_{F_2}\big({}_1M_2, F_2\big)\\ &\cong X_2 \otimes_{F_2} \operatorname{Hom}_{F_2}\big({}_1M_2, F_2\big)\\ &\cong X_2 \otimes_{F_2} {}_2M_1.\end{aligned}$$

Then we use the tensor-hom adjunction from lemma A.3 in order to obtain

$$\operatorname{Hom}_{F_2}\big(X_1 \otimes_{F_1} {}_1M_2, X_2\big) \cong \operatorname{Hom}_{F_1}\big(X_1, \operatorname{Hom}_{F_2}({}_1M_2, X_2)\big) \cong \operatorname{Hom}_{F_1}\big(X_1, X_2 \otimes_{F_2} {}_2M_1\big).$$

Note that all isomorphisms are indeed isomorphisms of (F_2, F_1)-bimodules. □

Remark 4.1. Writing a bar over a morphism we both either refer to its image $\overline{\varphi} := \Phi(\varphi)$ under the above isomorphism Φ or to its image $\overline{\psi} := \Phi^{-1}(\psi)$ under the inverse Φ^{-1} of this isomorphism. From the situation at hand it will always be obvious which case is respectively meant at that point.

With this isomorphism we now obtain F_j-linear maps ${}_j\overline{\kappa}_k : Y_k \otimes {}_kM_j \to Y_j$ and set ${}_j\psi_k := {}_j\overline{\kappa}_k$. For all other arrows (i, j) with $i \neq k$ simply let ${}_j\psi_i := {}_j\varphi_i$ unchanged. This finishes the construction of a representation $S_k^+\mathbf{X} = (Y_i, {}_j\psi_i)$ of the species $(\mathbb{M}, s_k\Omega)$.

For a morphism $f = (f_i) : \mathbf{X} \to \mathbf{X}'$ of representations $\mathbf{X} = (X_i, {}_j\varphi_i)$ and $\mathbf{X}' = (X_i', {}_j\varphi_i')$ of $(\mathbb{M}, \Omega)$ we define a morphism $S_k^+ f = (g_i)$ in $\mathsf{rep}(\mathbb{M}, s_k\Omega)$ between $S_k^+\mathbf{X} = (Y_i, {}_j\psi_i)$ and $S_k^+\mathbf{X}' = (Y_i', {}_j\psi_i')$ as follows. For $k \neq i \in \Gamma$ we can simply set $g_i = f_i$ since by just plugging in definitions and using that f is a morphism we find that

$${}_j\psi_i' \circ (g_i \otimes \mathrm{id}_{{}_iM_j}) = {}_j\varphi_i' \circ (f_i \otimes \mathrm{id}_{{}_iM_j}) = f_j \circ {}_j\varphi_i = g_j \circ {}_j\psi_i.$$

At the vertex k we need to be careful again. In order to define $g_k : Y_k \to Y_k'$ consider the map

$$\tilde{g}_k := \left(\bigoplus_{j=1}^{n} \big(f_j \otimes \mathrm{id}_{{}_jM_k}\big)\right) \circ ({}_j\kappa_k)_{j\in\Gamma} \;:\; Y_k \longrightarrow \bigoplus_{j=1}^{n} (X_j' \otimes {}_jM_k).$$

It satisfies

$$\begin{aligned}({}_k\varphi_j')_{j\in\Gamma} \circ \tilde{g}_k &= ({}_k\varphi_j')_{j\in\Gamma} \circ \left(\bigoplus_{j=1}^{n} \big(f_j \otimes \mathrm{id}_{{}_jM_k}\big) \circ \big({}_j\kappa_k\big)_{j\in\Gamma}\right)\\ &= f_k \circ \underbrace{({}_k\varphi_j)_{j\in\Gamma} \circ ({}_j\kappa_k)_{j\in\Gamma}}_{=0}\\ &= 0\end{aligned}$$

and therefore by the universal property of the kernel there exists a unique $g_k : Y_k \to Y'_k = \ker(({}_j\varphi'_i)_{j\in\Gamma})$ such that the left diagram

$$
\begin{tikzcd}
Y_k \arrow[r, "{}_j\kappa_k"] \arrow[d, dashed, "\exists! g_k"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_j \otimes_{F_j} {}_jM_k \arrow[d, "f_j \otimes \mathrm{id}_{{}_jM_k}"] & \Leftrightarrow & Y_k \otimes_{F_k} {}_kM_j \arrow[r, "{}_j\overline{\kappa}_k = {}_j\psi_k"] \arrow[d, "g_k \otimes \mathrm{id}_{{}_kM_j}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_j \arrow[d, "g_j = f_j"] \\
Y'_k \arrow[r, "{}_j\kappa'_k"] & Y'_j \otimes_{F_j} {}_jM_k & & Y'_k \otimes_{F_k} {}_kM_j \arrow[r, "{}_j\overline{\kappa'}_k = {}_j\psi'_k"] & Y'_j
\end{tikzcd}
\tag{4.3}
$$

commutes for all neighbours j of k. Now under the isomorphism from proposition 4.1 this is equivalent to the statement that the right diagram commutes which means $S_k^+ f = (g_i)$ indeed defines a morphism between the representations $\mathbf{Y}$ and $\mathbf{Y}'$ which finishes our construction of S_k^+.

Proposition 4.2. *The reflection functor*

$$S_k^+ : \mathsf{rep}(\mathbb{M}, \Omega) \longrightarrow \mathsf{rep}(\mathbb{M}, s_k\Omega)$$

defined above is an additive functor.

Proof. In order to prove the functor properties one mainly uses the uniqueness of g_k in (4.3).

So let us show that $S_k^+ \mathrm{id}_{\mathbf{X}} = \mathrm{id}_{S_k^+\mathbf{X}}$ for every $\mathbf{X} = (X_i, {}_j\varphi_i)$. For this set $(Y_i, {}_j\psi_i) := S_k^+\mathbf{X}$ and $(g_i) := S_k^+ \mathrm{id}_{\mathbf{X}}$. Then by definition $g_i = \mathrm{id}_{X_i} = \mathrm{id}_{Y_i}$ for $i \neq k$. In order to show that also $g_k = \mathrm{id}_{Y_k}$ note that g_k is the unique map which lets

$$
\begin{tikzcd}
Y_k \arrow[r, "{}_j\overline{\psi}_k"] \arrow[d, dashed, "\exists! g_k"'] & Y_j \otimes_{F_j} {}_jM_k \arrow[d, "\mathrm{id}_{Y_j} \otimes \mathrm{id}_{{}_jM_k}"] \\
Y_k \arrow[r, "{}_j\overline{\psi}_k"] & Y_j \otimes_{F_j} {}_jM_k
\end{tikzcd}
\tag{4.4}
$$

commute. But since also

$$
\begin{tikzcd}
Y_k \arrow[r, "{}_j\overline{\psi}_k"] \arrow[d, "\mathrm{id}_{Y_k}"'] & Y_j \otimes_{F_j} {}_jM_k \arrow[d, "\mathrm{id}_{Y_j} \otimes \mathrm{id}_{{}_jM_k}"] \\
Y_k \arrow[r, "{}_j\overline{\psi}_k"] & Y_j \otimes_{F_j} {}_jM_k
\end{tikzcd}
\tag{4.5}
$$

commutes we can directly deduce that $g_k = \mathrm{id}_{Y_k}$. Analogously one can argue that for morphisms $\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z}$ we have $S_k^+(g \circ f) = S_k^+ g \circ S_k^+ f$.

The functor S_k^+ is additive, ie. for representations $\mathbf{X} = (X_i, {}_j\varphi_i)$ and $\mathbf{Y} = (Y_i, {}_j\psi_i)$ we have $S_k^+(\mathbf{X} \oplus \mathbf{Y}) = S_k^+\mathbf{X} \oplus S_k^+\mathbf{Y}$, since

$$\ker\big(({}_k\varphi_j)_{j\in\Gamma} \oplus ({}_k\psi_j)_{j\in\Gamma}\big) \cong \ker\big(({}_k\varphi_j)_{j\in\Gamma}\big) \oplus \ker\big(({}_k\psi_j)_{j\in\Gamma}\big).$$

$\square$

Key to the construction was that the vertex k on which the action of S_k^+ was concentrated was a sink with respect to Ω. In the case where k is a source we can perform a similar construction in order to obtain a functor

$$S_k^- : \mathsf{rep}(\mathbb{M}, \Omega) \longrightarrow \mathsf{rep}(\mathbb{M}, s_k\Omega)\,. \tag{4.6}$$

This construction is dual to the prior one in the sense that as before we made regularly use of the universal property of the kernel we now exploit the universal property of the cokernel. Therefore we will not get into detail that much as in the definition of S_k^+ because all concepts are easily transferred to the dual case.

Let k be a source. For $\mathbf{X} = (X_i, {}_j\varphi_i)$ a representation of $(\mathbb{M}, \Omega)$ we define $S_k^-\mathbf{X} = (Y_i, {}_j\psi_i)$ as follows. Again for all $k \neq i \in \Gamma$ we set $Y_i := X_i$, but we now let Y_k be the cokernel of the diagram

$$X_k \xrightarrow{({}_k\overline{\varphi}_j)_{j\in\Gamma}} \bigoplus_{j=1}^n \left(X_j \otimes {}_jM_k\right) \xrightarrow{({}_k\pi_j)_{j\in\Gamma}} Y_k \longrightarrow 0 \tag{4.7}$$

and set ${}_k\psi_j := {}_k\pi_j$ for all neighbours j of k. For all other arrows $i \to j$ in Ω we define ${}_j\psi_i := {}_j\varphi_i$ as before.

For a morphism $f : \mathbf{X} \to \mathbf{X}'$ in $\mathsf{rep}(\mathbb{M}, \Omega)$ we define $S_k^- f = (g_i)$ by setting $g_i = f_i$ for all $i \neq k$ and letting g_k be the map uniquely defined by the universal property of the cokernel such that

$$\begin{tikzcd}
\bigoplus_{j=1}^n \left(X_j \otimes {}_jM_k\right) \arrow[r, "({}_k\pi_j)_{j\in\Gamma}"] \arrow[d, "\bigoplus_{j=1}^n (f_j \otimes \mathrm{id}_{{}_jM_k})"'] & Y_k \arrow[d, dashed, "\exists! g_k"] \\
\bigoplus_{j=1}^n \left(X_j' \otimes {}_jM_k\right) \arrow[r, "({}_k\pi_j)_{j\in\Gamma}"] & Y_k'
\end{tikzcd}$$

commutes. The argumentation that g indeed defines a morphism between $S_k^-\mathbf{X}$ and $S_k^-\mathbf{X}'$ runs analogously to the one given in the construction of S_k^+.

In the case of a general orientation Ω it is clear that applying s_k twice results in the same orientation Ω. That $S_k^\pm$ does not exhibit an functorial inverse is clear by considering the following types of representations.

Definition 4.1. For a vertex k we define the representation $\mathbf{S}(k) = (X_i, {}_j\varphi_i)$ of the species $(\mathbb{M}, \Omega)$ by choosing $X_k := F_k$ and $X_i := 0$ for all $i \neq k$ and simply setting all maps ${}_j\varphi_i = 0$ for all arrows $i \to j$ in Ω. We call $\mathbf{S}(k)$ the *simple representation* at vertex k.

For $k \in \Gamma$ a sink or source we respectively have $S_k^\pm\mathbf{S}(k) = 0$. It however turns out that in case of an indecomposable representation $\mathbf{X}$ non-isomorphic to $\mathbf{S}(k)$ the functors S_k^+ and S_k^- are mutually inverse which is proven in the following proposition.

Lemma 4.3. [4, Prop. 2.1] *Let* $\mathbf{X}$ *a representation of* $(\mathbb{M}, \Omega)$.

(i) *Let* $k \in \Gamma$ *be a sink. Then there exists a canonical monomorphism*

$$\mu : S_k^- S_k^+\mathbf{X} \longrightarrow \mathbf{X}$$

of representations and $\mathbf{im}(\mu)$ *has a complement in* $\mathbf{X}$ *which is a direct sum of copies of* $\mathbf{S}(k)$. *Thus, if* $\mathbf{X}$ *is indecomposable, then either*

$$\mathbf{X} \cong \mathbf{S}(k) \text{ and } S_k^+\mathbf{X} = 0$$

or μ *is an isomorphism in which case*

$$\underline{\dim}\big(S_k^+\mathbf{X}\big) = s_k\big(\underline{\dim}(\mathbf{X})\big).$$

(ii) *Let $k \in \Gamma$ be a source. Then there exists a canonical epimorphism*

$$\varepsilon : \mathbf{X} \longrightarrow S_k^+ S_k^- \mathbf{X}$$

of representations and ϵ has a section $\epsilon' : S_k^+ S_k^- \mathbf{X} \to \mathbf{X}$ and $\mathbf{X}$ is a direct sum of $\mathbf{im}(\epsilon')$ and copies of $\mathbf{S}(k)$. Thus, if $\mathbf{X}$ is indecomposable, then either

$$\mathbf{X} \cong \mathbf{S}(k) \text{ and } S_k^- \mathbf{X} = 0$$

or ϵ is an isomorphism in which case

$$\underline{\dim}\big(S_k^-\mathbf{X}\big) = s_k\big(\underline{\dim}(\mathbf{X})\big).$$

Proof. The proof is following the one given in [4, Prop. 2.1].

(i) Define $\mu = (\mu_i)_{i\in\Gamma}$ as follows. For $i \neq k$ we can choose $\mu_i := \mathrm{id}_{X_i}$ as $(S_k^- S_k^+ \mathbf{X})_i = X_i$ and regarding vertex k consider the short exact sequence

$$0 \longrightarrow \big(S_k^+\mathbf{X}\big)_k = \ker({}_k\varphi_j)_{j\in\Gamma} \xrightarrow{({}_j\kappa_k)_{j\in\Gamma}} \bigoplus_{j=1}^n \big(X_j \otimes {}_jM_k\big) \xrightarrow{({}_k\pi_j)_{j\in\Gamma}} \big(S_k^- S_k^+\mathbf{X}\big)_k \longrightarrow 0.$$

As by definition ${}_k\varphi_j\, {}_j\kappa_k = 0$ and $(S_k^- S_k^+ \mathbf{X})_k = \operatorname{coker}({}_j\kappa_k)_{j\in\Gamma}$ there exists a unique $\mu_k : \operatorname{coker}({}_j\kappa_k)_{j\in\Gamma} \to X_k$ such that

$$\begin{tikzcd}
0 \arrow[r] & \big(S_k^+\mathbf{X}\big)_k \arrow[r, "({}_j\kappa_k)_{j\in\Gamma}"] & \bigoplus\limits_{j=1}^n \big(X_j \otimes {}_jM_k\big) \arrow[r, "({}_k\pi_j)_{j\in\Gamma}"] \arrow[d, "({}_k\varphi_j)_{j\in\Gamma}"'] & \big(S_k^- S_k^+\mathbf{X}\big)_k \arrow[r] \arrow[dl, dashed, "\exists! \mu_k"] & 0 \\
& & X_k & &
\end{tikzcd} \tag{4.8}$$

commutes. In order to show that μ is a morphism of representations we need to proof that ${}_j\varphi_i(\mu_i \otimes \mathrm{id}_{{}_iM_j}) = \mu_j\, {}_j\psi_i$ for all arrows $i \to j$ in Ω where the ${}_j\psi_i$ are the F_j-linear maps of $S_k^- S_k^+ \mathbf{X}$. Clearly this is satisfied for $j \neq k$ since by definition ${}_j\psi_i = {}_j\varphi_i$ and μ_i and μ_j are the identity maps. Also using that the diagram (4.8) commutes we find that also in the remaining case

$${}_k\varphi_i(\mu_i \otimes \mathrm{id}_{{}_iM_j}) = {}_k\varphi_i\, \mathrm{id}_{X_i \otimes {}_iM_j} = \mu_k\, {}_k\pi_i = \mu_k\, {}_k\psi_i.$$

It now must be shown that μ is a monomorphism in $\mathtt{rep}(\mathbb{M}, \Omega)$ as claimed. So let $f : \mathbf{Z} \to S_k^- S_k^+ \mathbf{X}$ be a morphism with $\mu f = 0$. If we consider this composition component-wise we directly obtain $0 = \mu_i f_i = \mathrm{id}_{X_i} f_i = f_i$ for $i \neq k$. At the vertex k some more work is necessary. First observe that since $({}_k\pi_j)_{j\in\Gamma}$ is surjective there exists a unique $f' : Z_k \to \bigoplus_{j=1}^n X_j \otimes {}_jM_k$ such that $f_k = ({}_k\pi_j)_{j\in\Gamma}\, f'$. Now since (4.8) commutes we have

$$({}_k\varphi_j)_{j\in\Gamma}\, f' = \mu_k\, ({}_k\pi_j)_{j\in\Gamma}\, f' = \mu_k\, f_k = 0$$

and as $(S_k^+\mathbf{X})_k = \ker({}_k\varphi_j)_{j\in\Gamma}$ there exists a unique $\hat{f} : Z_k \to (S_k^+\mathbf{X})_k$ such that $f' = ({}_j\kappa_k)_{j\in\Gamma}\,\hat{f}$. Therefore

$$f_k = \underbrace{({}_k\pi_j)_{j\in\Gamma}\,({}_j\kappa_k)_{j\in\Gamma}}_{=0}\,\hat{f} = 0$$

which finally shows that μ is a monomorphism.

Now if $({}_k\varphi_j)_{j\in\Gamma}$ is not surjective the F_k-module X_k may be decomposed into $Y_k := \operatorname{im}({}_k\varphi_j)_{j\in\Gamma}$ and copies of F_k by extension of basis, ie. $X_k \cong Y_k \oplus (F_k)^r$ for some positive r. If we now set $Y_i := X_i$ for the other $i \neq k$ then $\mathbf{Y} = (Y_i, {}_j\varphi_i)$ is a representation of $(\mathbb{M}, \Omega)$ and moreover $\mathbf{X}$ decomposes into $\mathbf{X} \cong \mathbf{Y} \oplus \bigoplus_{i=0}^r \mathbf{S}(k)$.

Therefore if $\mathbf{X}$ is indecomposable either $\mathbf{X} = \mathbf{S}(k)$ in which case $S_k^+\mathbf{X} = 0$ or $({}_k\varphi_j)_{j\in\Gamma}$ must be surjective. In the latter case μ_k is an isomorphism which means that $\mathbf{X} \cong S_k^- S_k^+ \mathbf{X}$.

To conclude part (i) of the proof it is left to show the transformation behaviour of the dimension vector under S_k^+ in the case where μ is an isomorphism. Observe that in this case we obtain a short exact sequence

$$0 \longrightarrow \left(S_k^+\mathbf{X}\right)_k \longrightarrow \bigoplus_{j=1}^n \left(X_j \otimes {}_jM_k\right) \longrightarrow X_k \longrightarrow 0$$

from (4.8). Applying the dimension formula to this sequence we can compute the dimension of $(S_k^+\mathbf{X})_k$ to be

$$\begin{aligned}
\dim_{F_k}\left(S_k^+\mathbf{X}\right)_k &= \dim_{F_k}\left(\bigoplus_{j=1}^n \left(X_j \otimes {}_jM_k\right)\right) - \dim_{F_k} X_k \\
&= -\dim_{F_k} X_k + \sum_{j=1}^n \dim_{F_j}(X_j)\,\dim_{F_k}({}_jM_k) \\
&= -\dim_{F_k} X_k + \sum_{j=1}^n d_{jk}\dim_{F_j} X_j \\
&= \left(s_k(\underline{\dim}\,\mathbf{X})\right)_k
\end{aligned}$$

where we used the identity $\dim_{F_k} {}_jM_k = d_{jk}$. As additionally $(S_k^+\mathbf{X})_i = X_i$ for $i \neq k$ we finally obtain

$$\underline{\dim}\left(S_k^+\mathbf{X}\right) = s_k\left(\underline{\dim}\,\mathbf{X}\right).$$

(ii) The proof of the second statement runs analogously to the first one. The essence here is to define $\varepsilon = (\varepsilon_i)$ again as $\varepsilon_i := \mathrm{id}_{X_i}$ for $i \neq k$ and to choose ε_k to be the unique map that lets

$$\begin{tikzcd}
 & & X_k \arrow[dll, dashed, "\exists!\varepsilon_k"'] \arrow[d, "({}_k\overline{\varphi}_j)_{j\in\Gamma}"] & & \\
0 \arrow[r] & \left(S_k^+S_k^-\mathbf{X}\right)_k \arrow[r] & \bigoplus_{j=1}^n \left(X_j \otimes {}_jM_k\right) \arrow[r] & \left(S_k^-\mathbf{X}\right)_k \arrow[r] & 0
\end{tikzcd}$$

commute. All further arguments are dual to the ones given in the first part.

$\square$

Example 4.2. Let us consider the trivially valued quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

together with the trivial modulation $\mathbb{M}_k$ as defined in remark 2.3 for a field k. We want to observe the action of S_3^+ on the representation

$$\mathbf{X}: \quad \begin{tikzcd} & k^2 \arrow[dr, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] & \\ k \arrow[ur, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[rr, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & & k^2. \end{tikzcd}$$

Thus in order to determine $(S_3^+\mathbf{X})_3$ via eq. (4.2) we need to calculate the kernel of the map

$$k \oplus k^2 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} k^2$$

which is given by

$$0 \longrightarrow k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}} k \oplus k^2 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} k^2.$$

Thus we deduce that $S_3^+\mathbf{X}$ is of the form

$$S_3^+\mathbf{X}: \quad \begin{tikzcd} & k^2 & \\ k \arrow[ur, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & & k^2. \arrow[ul, "\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[ll, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \end{tikzcd}$$

For the dimension vectors of the two representations we find

$$\underline{\dim}\left(S_3^+\mathbf{X}\right) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = s_3\left(\underline{\dim}\,\mathbf{X}\right).$$

This is no contradiction to lemma 4.3 since $\mathbf{X}$ contains $\mathbf{S}(3)$ as a direct summand which is killed under the action of S_3^+.

On the other hand if a representation $\mathbf{Y}$ does not contain $\mathbf{S}(k)$ as a direct summand lemma 4.3 tells us that $\underline{\dim}(S_k^+\mathbf{Y}) = s_k(\underline{\dim}\,\mathbf{Y})$ always holds. Moreover we learn that the functors $S_k^\pm$ are almost invertible in the following sense.

Corollary 4.4. *Let $\mathcal{Q}$ be a species with a sink k in its underlying valued quiver and let $(\mathsf{rep}(\mathcal{Q})-\mathbf{S}(k))$ denote the full subcategory of $\mathsf{rep}(\mathcal{Q})$ whose objects do not contain $\mathbf{S}(k)$ as summands. Then S_k^+ induces an equivalence of categories*

$$S_k^+ : \big(\mathsf{rep}(\mathcal{Q}) - \mathbf{S}(k)\big) \longrightarrow \big(\mathsf{rep}(S_k^+\mathcal{Q}) - \mathbf{S}(k)\big)$$

with its inverse functor induced by S_k^-.

4.2 The indecoposable projective and injective Representations

After this last rather technical section it will get more intuitive in this one. We have already seen that the simple representations $\mathbf{S}(k)$ correspond to the simple roots $\mathbf{e}_k$ under application of the dimension vector. We now want to go on and construct the representations corresponding to the roots $\mathbf{p}_k$ and $\mathbf{q}_k$ defined in (3.9) respectively (3.10).

For the considered species $\mathcal{Q} = (\mathbb{M}, \Omega)$ let $\pi : \{1, \ldots, n\} \to \Gamma$ be an admissible ordering with respect to Ω. Having the relation between the reflection funtors and reflection maps in mind the following definition is quite obvious.

Definition 4.2. We define the representations $\mathbf{P}(\pi(t))$ and $\mathbf{I}(\pi(t))$ of $(\mathbb{M}, \Omega)$ as

$$\mathbf{P}(\pi(t)) := S^-_{\pi(1)} S^-_{\pi(2)} \ldots S^-_{\pi(t-1)} \mathbf{S}(\pi(t))$$

where $\mathbf{S}(\pi(t))$ is the simple representation at $\pi(t)$ of $(\mathbb{M}, s_{\pi(t)} s_{\pi(t+1)} \ldots s_{\pi(n)} \Omega)$ and

$$\mathbf{I}(\pi(t)) := S^+_{\pi(n)} S^+_{\pi(n-1)} \ldots S^+_{\pi(t+1)} \mathbf{S}(\pi(t))$$

where $\mathbf{S}(\pi(t))$ is the simple representation at $\pi(t)$ of $(\mathbb{M}, s_{\pi(t)} s_{\pi(t-1)} \ldots s_{\pi(1)} \Omega)$.

Remark 4.2. An important observation to make is that the definitions of the representations $\mathbf{P}(i)$ and $\mathbf{I}(i)$ are independent of the chosen admissible ordering π. This is due to arguments as used in remark 3.3 (iii) and the fact that $S_i^\pm S_j^\pm = S_j^\pm S_i^\pm$ for i and j both sinks respectively sources with respect to Ω.

Proposition 4.5. *For all $i \in \Gamma$ we have*

(i) $\underline{\dim}\, \mathbf{P}(i) = \mathbf{p}_i$

(ii) $\underline{\dim}\, \mathbf{I}(i) = \mathbf{q}_i$

where $\mathbf{p}_i$ and $\mathbf{q}_i$ are defined as in eq. (3.9) *respectively* (3.10)*.*

Proof. Let t such that $i = \pi(t)$. Then note that for all $r = 1 \ldots t$ we have

$$S^-_{\pi(r)} S^-_{\pi(r+1)} \ldots S^-_{\pi(t-1)} \mathbf{S}(\pi(t)) \neq 0$$

since

$$\Big(S^-_{\pi(r)} S^-_{\pi(r+1)} \ldots S^-_{\pi(t-1)} \mathbf{S}(\pi(t))\Big)_{\pi(t)} = F_{\pi(t)}.$$

So we can make repetitive use of lemma 4.3 in order to find that

$$\begin{aligned}\underline{\dim}\,\mathbf{P}(\pi(t)) &= \underline{\dim}\left(S^-_{\pi(1)}\dots S^-_{\pi(t-1)}\mathbf{S}(\pi(t))\right)\\ &= s_{\pi(1)}\dots s_{\pi(t-1)}\,\underline{\dim}\,\mathbf{S}(\pi(t))\\ &= s_{\pi(1)}\dots s_{\pi(t-1)}\mathbf{e}_{\pi(t)}\\ &= \mathbf{p}_{\pi(t)}.\end{aligned}$$

The second statement is proven similarly. □

We can provide some more intuition on these representations by computing their form explicitly. For this purpose we need to state some definitions. For a path $c = (i_1, i_2 \dots, i_k)$ in the valued quiver (Γ, d, Ω) we set $M(c) := {}_{i_1}M_{i_2} \otimes_{F_{i_2}} \dots \otimes_{F_{i_{k-1}}} {}_{i_{k-1}}M_{i_k}$ and let $\Omega(i, j)$ denote the set of all paths from i to j. For the lazy paths ϵ_i we set $M(\epsilon_i) := F_i$ and $\Omega(i, i) = \{\epsilon_i\}$. Moreover we define the opposite path $c^{\,\mathrm{op}}$ of $c = (i_1, i_2 \dots, i_k)$ to be $c^{\,\mathrm{op}} = (i_k, i_{k-1} \dots, i_1)$.

Proposition 4.6. *Let $i \in \Gamma$. Then $\mathbf{P}(i) \cong (Y_j, {}_j\varphi_i)$ with*

$$Y_j = \bigoplus_{c\in\Omega(i,j)} M(c)$$

and

$$\begin{aligned}{}_k\varphi_j : Y_j \otimes_{F_j} {}_jM_k &\longrightarrow Y_k,\\ x\otimes m &\longmapsto x\otimes m.\end{aligned}$$

Analogously for $\mathbf{I}(i) = (Z_j, {}_j\psi_i)$ we have

$$Z_j = \bigoplus_{c\in\Omega(j,i)} M(c^{\,\mathrm{op}})$$

and

$$\begin{aligned}{}_k\psi_j : \quad Z_j \otimes_{F_j} {}_jM_k &\longrightarrow Z_k,\\ ((m_1\otimes\dots\otimes m_l)\otimes m) &\longmapsto \begin{cases}(m_1\otimes\dots\otimes m_{l-1})\,m_l(m) & , m_l\in {}_kM_j\\ 0 & , \text{otherwise}\end{cases}\end{aligned}$$

where m_l is understood as an element of $\mathrm{Hom}_{F_k}({}_jM_k, F_k)$.

Proof. After relabelling the indices we can without loss of generality assume that the admissible ordering is of the form $\pi = \mathrm{id}_{\{1,\dots,n\}}$. In this case $(i,j)\in\Omega$ if and only if $i > j$ and ${}_iM_j \neq 0$ since the orientation is admissible. Consequently for $j > i : \Omega(i,j) = \varnothing$ and

$$\left(\mathbf{P}(i)\right)_j = \left(S^-_1\dots S^-_{i-1}\mathbf{S}(i)\right)_j = 0 = \bigoplus_{c\in\Omega(i,j)} M(c) = Y_j.$$

In the case $i = j$ we find

$$\left(\mathbf{P}(i)\right)_i = \left(S^-_1\dots S^-_{i-1}\mathbf{S}(i)\right)_i = \left(\mathbf{S}(i)\right)_i = F_i = M(\epsilon_i) = Y_i.$$

So we are left with the case $i > j$ for which we will show the claim inductively. We perform the induction step $j \to j-1$ by noting that

$$\big(\mathbf{P}(i)\big)_{j-1} = \big(S_1^- \dots S_{i-1}^- \mathbf{S}(i)\big)_{j-1} = \left(S_{j-1}^- \underbrace{\big(S_j^- \dots S_{i-1}^- \mathbf{S}(i)\big)}_{=:\mathbf{X}}\right)_{j-1}.$$

and that we can compute $(S_{j-1}^- \mathbf{X})_{j-1}$ from the defining short exact sequence (4.7). Plugging in $X_{j-1} = 0$ yields the exact sequence

$$0 \longrightarrow \bigoplus_{k=1}^{n} \big(X_k \otimes {}_kM_{j-1}\big) \xrightarrow{({}_{j-1}\pi_k)_{k\in\Gamma}} \big(S_{j-1}^- \mathbf{X}\big)_{j-1} \longrightarrow 0$$

which in turn implies that $(\mathbf{P}(i))_{j-1}$ is isomorphic to

$$\begin{aligned}
\bigoplus_{k=1}^{n} \big(X_k \otimes {}_kM_{j-1}\big) &= \bigoplus_{k=j}^{i} \big(X_k \otimes {}_kM_{j-1}\big) \\
&= \bigoplus_{k=j}^{i} \left(\left(\bigoplus_{c\in\Omega(i,k)} M(c)\right) \otimes {}_kM_{j-1}\right) \\
&= \bigoplus_{(k,j-1)\in\Omega,\, c\in\Omega(i,k)} M(c) \otimes {}_kM_{j-1} \\
&= \bigoplus_{c\in\Omega(i,j-1)} M(c)
\end{aligned}$$

where in the first line we used that $X_k = 0$ for $k \notin \{j, \dots, i\}$ and in the second line we made use of the induction hypothesis. And since ${}_{j-1}\varphi_k = {}_{j-1}\pi_k : X_k \otimes {}_kM_{j-1} \to X_{j-1}, x \otimes m \mapsto x \otimes m$ we have already proven the first part.

As the second part only uses arguments dual to the ones above this part of the proof is skipped. □

The above form of the representations $\mathbf{P}(i)$ and $\mathbf{I}(i)$ makes it clear that these are the generalisations of the indecomposable projective respectively injective representations in the representation theory of ordinary quivers. Therefore it is no surprise that these representations feature the same properties.

Proposition 4.7. *The representation* $\mathbf{P}(i)$ *is projective and* $\mathbf{I}(i)$ *is injective.*

Proof. Let $f : \mathbf{X} \to \mathbf{Y}$ be a epimorphism in rep($\mathcal{Q}$), ie. for each vertex k the linear map f_k is surjective, and $g : \mathbf{P}(i) \to \mathbf{Y}$ a morphism. We need to construct a morphism $g' : \mathbf{P}(i) \to \mathbf{X}$ such that $f \circ g' = g$.

First we will show that g is fully described by the value $g_i(1) =: y$. Clearly for any $a \in (\mathbf{P}(i))_i = F_i$ we have $g_i(a) =: ya$. Now let $x \in (\mathbf{P}(i))_j$. Without loss of generality assume there exists a path $c = (i, i_2, \dots, i_k, j)$ such that $x \in M(c)$ which means x is of the form $x = m_1 \otimes m_2 \otimes \dots \otimes m_k$. Now let $\mathbf{P}(i) = (P(i)_j, {}_j\varphi_i)$ and $\mathbf{Y} = (Y_i, {}_j\psi_i)$ and set

$$\varphi_c := {}_j\varphi_{i_k} \circ \big({}_{i_k}\varphi_{i_{k-1}} \otimes \mathrm{id}_{{}_{i_k}M_j}\big) \circ \dots \circ \big({}_{i_2}\varphi_i \otimes \mathrm{id}_{{}_{i_2}M_{i_3}} \otimes \dots \otimes \mathrm{id}_{{}_{i_k}M_j}\big) \tag{4.9}$$

Then $x = \varphi_c(x)$ and since g is a morphism of representations

$$g_j(x) = g_j(\varphi_c(x)) = (\psi_c \circ (g_i \otimes \mathrm{id}_{iM_{i_2}} \otimes \cdots \otimes \mathrm{id}_{i_k M_j}))(x) = \psi_c(g_i(1) \otimes x) = \psi_c(y \otimes x)$$

which implies that g is fully described by $y = g_i(1)$. On the other hand given an element $y \in Y_i$ we can construct a morphism $g^y = (g^y_j)$ as follows. On $P(i)_i$ let g^y_i be given by $g^y_i(a) := ya$ and on $P(i)_j$ we define g^y_j on the summand $M(c)$ to be $g^y_j(x) := \psi_c(y \otimes x)$. This clearly makes g^y a morphism of representations.

We now make use of this construction as follows. First fix $y := g_i(1)$ and choose an $x \in X_i$ such that $f_i(x) = y$ which we can do since f_i is surjective by assumption. We use this element to obtain a morphism $g^x : \mathbf{P}(i) \to \mathbf{X}$ that satisfies $g^x_i(1) = x$. Then since $(f \circ g^x)_i(1) = f_i(g^x_i(1)) = y$ and $f \circ g^x$ is fully described by the value at that point we hence deduce that $f \circ g^x = g$. Thus $\mathbf{P}(i)$ is projective.

As one can use similar arguments to prove that $\mathbf{I}(i)$ is injective this part of the proof will be skipped. □

Proposition 4.8. *The representations* $\mathbf{P}(i)$ *and* $\mathbf{I}(i)$ *are indecomposable.*

Proof. This proof is a direct generalisation of [16, Prop. 2.8]. Suppose that $\mathbf{P}(i) = \mathbf{X} \oplus \mathbf{Y}$. Then since $P(i)_i = F_i$ we may assume that $X_i = P(i)_i$ and $Y_i = 0$. Now consider the space $P(i)_j$ whose elements are sums of elements $x \in M(c)$ where $c = (i, i_2, \ldots, i_k, j)$ is a path from i to j. Then with $\mathbf{P}(i) = (P(i)_j, {}_j\varphi_i)$ and using the notation introduced in (4.9) we see that since

$$\varphi_c : (X_i \oplus 0) \otimes M(c) \longrightarrow X_j \oplus Y_j$$

must factor over its summands the map φ_c sends $1 \otimes x \in X_i \otimes M(c)$ to an element in X_j. But $\varphi_c(1 \otimes x) = x$ and thus since x was an arbitrary element of $M(c) \subset P(i)_j$ every element of $P(i)_j$ must lie in X_j. Consequently $X_j = P(i)_j$ and $Y_j = 0$. So we deduce that $\mathbf{X} = \mathbf{P}(i)$ and $\mathbf{Y} = 0$. Thus $\mathbf{P}(i)$ must be indecomposable.

The proof for $\mathbf{I}(i)$ uses similar arguments. □

Remark 4.3. For an indecomposable representation $\mathbf{X}$ one can even proof the converse direction: If $\mathbf{X}$ is projective it must already be isomorphic to $\mathbf{P}(i)$ for some vertex i. Since this fact is not needed in the following we refer to [4, Prop. 2.3, 2.4] for a full discussion.

Moreover we can find an easy way to calculate the vectors $\mathbf{p}_i$ and $\mathbf{q}_i$ without having to compute them from the reflection maps by using proposition 4.6.

Corollary 4.9. *The components of* $\mathbf{p}_i$ *and* $\mathbf{q}_i$ *are*

$$(\mathbf{p}_i)_j = \sum_{(i,i_2,\ldots,i_k,j)\in\Omega(i,j)} d_{ii_2} d_{i_2 i_3} \ldots d_{i_k j},$$
$$(\mathbf{q}_i)_j = \sum_{(j,i_2,\ldots,i_k,i)\in\Omega(j,i)} d_{ii_k} d_{i_k i_{k-1}} \ldots d_{i_2 j}$$

for every $i \neq j \in \Gamma$ *and* $(\mathbf{p}_i)_i = (\mathbf{q}_i)_i = 1$.

Proof. Using proposition 4.5 it is a straight forward calculation that

$$\begin{aligned}
(\mathbf{p}_i)_j &= \dim_{F_j} \left(\mathbf{P}(i)\right)_j \\
&= \dim_{F_j} \left(\bigoplus_{c \in \Omega(i,j)} M(c) \right) \\
&= \sum_{(i,i_2,\ldots,i_k,j) \in \Omega(i,j)} \dim_{F_j} \left({}_iM_{i_2} \otimes {}_{i_2}M_{i_3} \otimes \cdots \otimes {}_{i_k}M_j \right) \\
&= \sum_{(i,i_2,\ldots,i_k,j) \in \Omega(i,j)} d_{ii_2} d_{i_2 i_3} \ldots d_{i_k j}.
\end{aligned}$$

□

Example 4.3. Consider the valued quiver

$$\begin{tikzcd}
 & & 3 \arrow[dr, "{(2,4)}"] & \\
1 \arrow[r, "{(1,3)}"] & 2 \arrow[ur, "{(3,1)}"] \arrow[rr, "{(3,2)}"] & & 4
\end{tikzcd}$$

Then independent of a specific modulation $\mathbb{M}$ chosen the dimension vector of $\mathbf{P}(2)$ is always

$$\left(\underline{\dim}\, \mathbf{P}(2)\right)^{\mathrm{T}} = \mathbf{p}_2^{\mathrm{T}} = (0, 1, 3, 3 \cdot 2 + 3) = (0, 1, 3, 9)$$

and the one of $\mathbf{I}(3)$ is

$$\left(\underline{\dim}\, \mathbf{I}(3)\right)^{\mathrm{T}} = \mathbf{q}_3^{\mathrm{T}} = (1 \cdot 3, 1, 1, 0) = (3, 1, 1, 0).$$

4.3 The Coxeter functor

The last ingredient in order to connect indecomposable representations to roots is to find a suitable object corresponding to the Coxeter transformation defined in definition 3.5 which is a product $c = s_{\pi(n)} \ldots s_{\pi(2)} s_{\pi(1)}$ of the reflection maps.

Definition 4.3. Let $\pi : \{1, \ldots, n\} \to \Gamma$ be an admissible ordering of Γ with respect to Ω. Then the functors

$$C^+ := S^+_{\pi(n)} \ldots S^+_{\pi(2)} S^+_{\pi(1)} \text{ and } C^- := S^-_{\pi(n)} \ldots S^-_{\pi(2)} S^-_{\pi(1)}$$

from the category $\mathtt{rep}(\mathbb{M}, \Omega)$ into itself are called the *Coxeter functors* of $\mathtt{rep}(\mathbb{M}, \Omega)$.

Remark 4.4. Again by the arguments given in remark 4.2 one can show that the definition of the Coxeter functors does not depend on the particular admissible ordering π chosen but only on the admissible orientation Ω.

One should think of C^+ as c and C^- as c^{-1}. We will later see that these functors are like this association already suggests almost mutually inverse to one another. For this we will first prove that the representations $\mathbf{P}(i)$ and $\mathbf{I}(i)$ behave under the Coxeter functors just like the roots $\mathbf{p}_i$ respectively $\mathbf{q}_i$ do under the Coxeter transformation as shown in lemma 3.9.

Proposition 4.10. [4, Prop. 2.4] *Let C^+ and C^- be the Coxeter functors of* $\mathsf{rep}(\mathbb{M},\Omega)$ *and* $c : \mathbb{Z}^n \to \mathbb{Z}^n$ *the corresponding Coxeter transformation and let* $\mathbf{X}$ *be an indecomposable representation of* $(\mathbb{M},\Omega)$.

(i) *Then the following three statements are equivalent*
   (1) $\mathbf{X} \cong \mathbf{P}(k)$ *for some* $k \in \Gamma$
   (2) $C^+\mathbf{X} = 0$
   (3) $c(\underline{\dim}\,\mathbf{X}) \ngeq 0$.

(ii) *Dually also the following statements are equivalent*
   (1) $\mathbf{X} \cong \mathbf{I}(k)$ *for some* $k \in \Gamma$
   (2) $C^-\mathbf{X} = 0$
   (3) $c^{-1}(\underline{\dim}\,\mathbf{X}) \ngeq 0$.

Proof. We will only prove part (i) of the proposition as the other part can be shown by similar arguments.

$(1 \Rightarrow 2)$ Let π be an admissible ordering of Γ with respect to the orientation Ω. Since $\mathbf{X} \cong \mathbf{P}(k)$ is indecomposable we can apply lemma 4.3 to find that either $S^+_{\pi(1)}\mathbf{P}(\pi(t)) = 0$ or

$$S^+_{\pi(1)}\mathbf{P}(\pi(t)) = S^+_{\pi(1)}S^-_{\pi(1)}S^-_{\pi(2)}\dots S^-_{\pi(t-1)}\mathbf{S}(\pi(t)) \cong S^-_{\pi(2)}\dots S^-_{\pi(t-1)}\mathbf{S}(\pi(t)).$$

So iteratively we obtain that either directly $C^+\mathbf{P}(\pi(t)) = 0$ or

$$C^+\mathbf{P}(\pi(t)) = S^+_{\pi(n)}\dots S^+_{\pi(2)}S^+_{\pi(1)}S^-_{\pi(1)}S^-_{\pi(2)}\dots S^-_{\pi(t-1)}\mathbf{S}(\pi(t)) \cong S^+_{\pi(n)}\dots S^+_{\pi(t)}\mathbf{S}(\pi(t)) = 0.$$

So in both cases $C^+\mathbf{P}(\pi(t)) = 0$ as claimed.

$(2 \Rightarrow 1)$ Conversely, if we assume that $C^+\mathbf{X} = 0$, let t be minimal such that

$$S^+_{\pi(t)}\dots S^+_{\pi(2)}S^+_{\pi(1)}\mathbf{X} = 0.$$

Then again using lemma 4.3 we obtain $S^+_{\pi(t-1)}\dots S^+_{\pi(2)}S^+_{\pi(1)}\mathbf{X} \cong \mathbf{S}(\pi(t))$. Therefore

$$\mathbf{X} \cong S^-_{\pi(1)}S^-_{\pi(2)}\dots S^-_{\pi(t-1)}\mathbf{S}(\pi(t)) \cong \mathbf{P}(\pi(t)).$$

$(2 \Leftrightarrow 3)$ First let $C^+\mathbf{X} = 0$. Choose t minimally such that

$$S^+_{\pi(t)}\dots S^+_{\pi(2)}S^+_{\pi(1)}\mathbf{X} = 0.$$

then $S^+_{\pi(t-1)}\dots S^+_{\pi(1)}\mathbf{X} \cong \mathbf{S}(\pi(t))$ and

$$\begin{aligned} s_{\pi(t)}\dots s_{\pi(1)}\big(\underline{\dim}\,\mathbf{X}\big) &= s_{\pi(t)}\Big(\underline{\dim}\,\big(S^+_{\pi(t-1)}\dots S^+_{\pi(1)}\mathbf{X}\big)\Big) \\ &= s_{\pi(t)}\big(\underline{\dim}\,\mathbf{S}(\pi(t))\big) \\ &= s_{\pi(t)}\mathbf{e}_{\pi(t)} \\ &= -\mathbf{e}_{\pi(t)}. \end{aligned}$$

Therefore also $c(\underline{\dim}\,\mathbf{X}) \not\geq 0$.

Conversely, if $C^+\mathbf{X} \neq 0$, then as a direct consequence of lemma 4.3 we obtain

$$c\big(\underline{\dim}(\mathbf{X})\big) = s_{\pi(n)} \dots s_{\pi(1)}\big(\underline{\dim}(\mathbf{X})\big) = s_{\pi(n)} \dots s_{\pi(2)}\left(S^+_{\pi(1)}\mathbf{X}\right) = \dots = \underline{\dim}\left(C^+\mathbf{X}\right) \geq 0.$$

□

Corollary 4.11. [4, Prop. 2.2] *Let C^+ and C^- be the Coxeter functors of* $\mathsf{rep}(\mathbb{M}, \Omega)$ *and $c : \mathbb{Z}^n \to \mathbb{Z}^n$ the Coxeter transformation with respect to Ω. Let $\mathbf{X}$ be an indecomposable representation of $(\mathbb{M}, \Omega)$.*

(i) *Then either*

$$\mathbf{X} \cong \mathbf{P}(k) \textit{ for a } k \in \Gamma \textit{ and } C^+\mathbf{X} = 0$$

or

$$\mathbf{X} \cong C^-C^+\mathbf{X} \textit{ and } \underline{\dim}\left(C^+\mathbf{X}\right) = c\big(\underline{\dim}\,\mathbf{X}\big).$$

(ii) *And dually we then either have*

$$\mathbf{X} \cong \mathbf{I}(k) \textit{ for a } k \in \Gamma \textit{ and } C^-\mathbf{X} = 0$$

or

$$\mathbf{X} \cong C^+C^-\mathbf{X} \textit{ and } \underline{\dim}\left(C^-\mathbf{X}\right) = c^{-1}\big(\underline{\dim}\,\mathbf{X}\big).$$

Proof. Let $\mathbf{X}$ be irreducible and fix an admissible ordering π of Ω. If $C^+\mathbf{X} = 0$ we know from proposition 4.10 that there must exist a $k \in \Gamma$ such that $\mathbf{X} = \mathbf{P}(k)$. So now assume that $C^+\mathbf{X} \neq 0$. Consequently for all $1 \leq t \leq n$

$$0 \neq S^+_{\pi(t)}S^+_{\pi(t-1)} \dots S^+_{\pi(1)}\mathbf{X}.$$

Thus we can repetitively make use of lemma 4.3 in order to obtain

$$\underline{\dim}\left(C^+\mathbf{X}\right) = s_{\pi(n)}\left(\underline{\dim}\left(S^+_{\pi(n-1)} \dots S^+_{\pi(1)}\mathbf{X}\right)\right) = s_{\pi(n)} \dots s_{\pi(1)}\big(\underline{\dim}(\mathbf{X})\big) = c\big(\underline{\dim}(\mathbf{X})\big).$$

and moreover that

$$\begin{aligned} \mathbf{X} \cong S^-_{\pi(1)}S^+_{\pi(1)}\mathbf{X} &\cong S^-_{\pi(1)}S^-_{\pi(2)}S^+_{\pi(2)}S^+_{\pi(1)}\mathbf{X} \\ &\vdots \\ &\cong S^-_{\pi(1)} \dots S^-_{\pi(n)}S^+_{\pi(n)} \dots S^+_{\pi(1)}\mathbf{X} \\ &\cong C^-C^+\mathbf{X}. \end{aligned}$$

The second part of the corollary can be proven by similar arguments. □

Thus we can deduce that C^+ and C^- are mutually inverse to one another if one restricts the functors to the category whose objects do not contain the indecomposable projective representations $\mathbf{P}(i)$ respectively the indecomposable injective representations $\mathbf{I}(i)$ as direct summands.

Definition 4.4. We say an indecomposable representation $\mathbf{X}$ of a species $(\mathbb{M}, \Omega)$ is *preprojective* if $\mathbf{X} \cong (C^-)^r\mathbf{P}(i)$ and *preinjective* if $\mathbf{X} \cong (C^+)^r\mathbf{I}(i)$ for a vertex i and some positive integer r.

Again the preprojective and -injective representations show a similar behaviour like their corresponding roots $c^{-r}\mathbf{p}_i$ and $c^{+r}\mathbf{q}_i$ and are pairwise different for different values of i and r.

Proposition 4.12. *Let C^+ and C^- be the Coxeter functors of* $\mathsf{rep}(\mathbb{M}, \Omega)$.

(i) *If for some positive integers s, t and vertices $i, j \in \Gamma$ we have $(C^-)^s\mathbf{P}(i) \cong (C^-)^t\mathbf{P}(j) \neq 0$, then already $s = t$ and $i = j$.*

(ii) *Analogously $(C^+)^s\mathbf{I}(i) \cong (C^+)^t\mathbf{I}(j) \neq 0$ implies that $s = t$ and $i = j$.*

Proof. Using proposition 4.5 and corollary 4.11 we have

$$c^{-s}\mathbf{p}_i = \underline{\dim}\left((C^-)^s\mathbf{P}(i)\right) = \underline{\dim}\left((C^-)^t\mathbf{P}(j)\right) = c^{-t}\mathbf{p}_j$$

and by a direct consequence of proposition 3.12 we obtain $s = t$ and $i = j$. □

Indecomposable reps	Positive roots
$\mathbf{S}(i)$	$\mathbf{e}_i$
$\mathbf{P}(i)$	$\mathbf{p}_i$
$\mathbf{I}(i)$	$\mathbf{q}_i$
$S_k^{\pm}$	s_k
C^+	c
C^-	c^{-1}

Table 4.1: Association between the indecomposable representations of a species $\mathcal{Q}$ and the positive roots Φ^+ of its underlying valued graph. For the above three rows the correspondence is induced by the dimension vector. The lower three are associated because of the similarity of their action on the above elements.

4.4 Classification of graphs of finite representation type

We are now fully equipped in order to prove the generalisation of Gabriel's theorem by Dlab and Ringel. Before going into its proof let us pause for a moment and look back on the correspondence between the indecomposable representations of a circuit-free species and the positive roots of its underlying valued graph as summarized in section 4.4. With this association and the construction of the root system of a Dynkin graph via Coxeter transformations from corollary 3.11 in mind the following result is no big surprise.

Theorem 4.13 (Dlab, Ringel [6],[4])**.** *Let $(\mathbb{M},\Omega)$ be a species of a valued graph (Γ,d).*

1. *$(\mathbb{M},\Omega)$ is of finite representation type if and only if (Γ,d) is a Dynkin diagram.*
2. *If (Γ,d) is a Dynkin diagram then the map $\underline{\dim} : \mathrm{Ob}\big(\mathsf{rep}(\mathbb{M},\Omega)\big) \to \mathbb{Z}^n$ induces a one-to-one correspondence between the isomorphism classes of indecomposable representations in $\mathsf{rep}(\mathbb{M},\Omega)$ and the positive roots of (Γ,d).*

Proof. This proof mainly uses ideas from [12, pp. 17-19] and generalises them from quiver representations to the representation theory of species.

2. We start by proving the second part of the theorem. We first of all need to show that for an irreducible representation $\mathbf{X}$ its dimension vector is a positive root. In order to see this note that there exists an $r > 0$ such that $(C^+)^r\mathbf{X} = 0$ because suppose not then by corollary 4.11 we have
$$0 \leq \underline{\dim}\left((C^+)^t\mathbf{X}\right) = c^t\left(\underline{\dim}\,\mathbf{X}\right)$$
for all $t > 0$ which is a contradiction to the fact shown in proposition 3.10 that in case of a Dynkin diagram for all positive $\mathbf{x} \in \mathbb{Z}^n$ there exists an integer r such that $c^r\mathbf{x} \ngeq 0$.

   So let $r > 0$ be minimal such that $(C^+)^r\mathbf{X} = 0$. Then again using corollary 4.11 we deduce that $(C^+)^{r-1}\mathbf{X} \cong \mathbf{P}(k)$ for some vertex i and moreover
$$\begin{aligned} c^{r-1}\left(\underline{\dim}\,\mathbf{X}\right) &= \underline{\dim}\left((C^+)^{r-1}\mathbf{X}\right) \\ &= \underline{\dim}\,\mathbf{P}(i) \\ &= \mathbf{p}_i \end{aligned}$$

where again it was made use of lemma 4.3 and corollary 4.11. Recalling that all positive roots are of the form $c^{-t}\mathbf{p}_k$ for some positive integer t and $k \in \Gamma$ as shown in corollary 3.11 we deduce that $\underline{\dim}\,\mathbf{X} = c^{-(r-1)}\mathbf{p}_i$ is indeed a positive root.

In order to show the injectivity of the map note that another interesting outcome of the above argumentation is that an arbitrary indecomposable representation $\mathbf{X}$ of a species whose graph is Dynkin must already be preprojective.

So now let $\mathbf{X} \cong (C^-)^r\mathbf{P}(i)$ and $\mathbf{Y} \cong (C^-)^t\mathbf{P}(j)$ be two indecomposable representations with equal dimension vector. Then applying proposition 3.12 on

$$c^{-r}\mathbf{p}_i = \underline{\dim}\left((C^-)^r\mathbf{P}(i)\right) = \underline{\dim}\left((C^-)^t\mathbf{P}(j)\right) = c^{-t}\mathbf{p}_j$$

we can directly deduce that $r = t$ and $i = j$ and thus that $\mathbf{X} \cong \mathbf{Y}$.

That the map is surjective is obvious since as every positive root $\mathbf{x}$ is of the form $\mathbf{x} = c^{-t}\mathbf{p}_k$ for some positive integer t and $k \in \Gamma$ we have

$$\underline{\dim}\left((C^-)^t\mathbf{P}(k)\right) = c^{-t}\mathbf{p}_k = \mathbf{x}$$

with $(C^-)^t\mathbf{P}(k)$ being an indecomposable representation.

1. The prior part already implies that species whose valued graph is Dynkin are of finite representation type since their set of positive roots is finite by corollary 3.11. So it remains to show that all other species are of infinite representation type.

   First let (Γ, d) be extended Dynkin and Ω be admissible. Then for each vertex i the representations $(C^-)^r\mathbf{P}(i)$ are pairwise non-isomorphic for different values of r by proposition 4.12. If now $(C^-)^r\mathbf{P}(i) \neq 0$ for all $r \in \mathbb{Z}_{\geq 0}$ we have found an infinite number of non-isomorphic representations. So suppose $(C^-)^r\mathbf{P}(i) = 0$ for some minimal r then by proposition 4.10 there exists a $k \in \Gamma$ such that $(C^-)^{r-1}\mathbf{P}(i) \cong \mathbf{I}(k)$. But then

   $$c^{-(r-1)}\mathbf{p}_i = \underline{\dim}\left((C^-)^{r-1}\mathbf{P}(i)\right) = \underline{\dim}\left(\mathbf{I}(k)\right) = \mathbf{q}_k$$

   which implies that both $c^{-(r-1)}\mathbf{p}_i$ and $\mathbf{q}_k$ have the same defect. This is a contradiction because $c^{-(r-1)}\mathbf{p}_i$ must have negative and $\mathbf{q}_k$ positive defect by proposition 3.15.

   So far we only discussed extended Dynkin diagrams with an admissible orientation. This means we have to cover the case $(\Gamma, d) = \tilde{\mathbb{A}}_n$ together with a non-admissible orientation separately. But as we have already seen in example 2.9, also in this case one can construct an infinite number of indecomposable representations.

   This leaves us with the case of (Γ, d) being neither a Dynkin nor an extended Dynkin diagram. Now simply choose a subgraph (Γ', d') of (Γ, d) which is extended Dynkin and modulate it by restricting $(\mathbb{M}, \Omega)$ to it. About this species we already know that it features an infinite number of indecomposable representations $\{\mathbf{X}'(i)\}_{i \in I}$. We can now promote these to representations $\{\mathbf{X}(i)\}_{i \in I}$ of $(\mathbb{M}, \Omega)$ by setting $X(i)_j := X'(i)_j$ for all $j \in \Gamma'$ and filling up the remaining vertices $j \in \Gamma \setminus \Gamma'$ with $X(i)_j := 0$. In this way we obtain an infinite number of indecomposable representations of $(\mathbb{M}, \Omega)$.

$\square$

Remark 4.5. So finally we could make use of our detailed analysis of the root system of a Dynkin graph in order to find a constructive description of the indecomposable representations of Dynkin type species. One could ask now whether there is a correspondence between the indecomposable representations of a species whose underlying graph is an extended Dynkin diagram and its positive roots too.
Clearly under application of the dimension vector the preprojective representations are send to roots of negative defect and dimension vector of preinjectives is a positive root with positive defect in the extended Dynkin case. What remains is the analysis of indecomposable representations which are neither preprojective nor -injective. These are referred to as *regular* representations since their dimension vector is a regular root. A full description of these regular representations was given by Dlab and Ringel in [4, pp. 20-37].

APPENDIX A

Modules over non-commutative rings

In this thesis we are mostly dealing with associative, unital rings which in general do not need to be commutative. Therefore the notion of modules over such rings shall be developed in this chapter and we will state all important lemmas used in this thesis. We will mainly follow [16, pp. 117-118, pp. 193-194]. Rings will assumed to be associative with $1 \neq 0$ throughout the chapter.

Definition A.1. (i) A *right A-module M* is an abelian group $(M, +)$ together with an operation

$$\begin{aligned} M \times A &\longrightarrow M \\ (m, a) &\longmapsto ma \end{aligned}$$

such that for all $m, m' \in M$ and all $a, a' \in A$ we have

$$(m + m')a = ma + m'a, \tag{A.1}$$
$$m(a + a') = ma + ma', \tag{A.2}$$
$$m(aa') = (ma)a', \tag{A.3}$$
$$m1 = m. \tag{A.4}$$

(ii) Analogously one defines a *left A-module M*. One only defines the multiplication with A to be from the left ie. the ring elements in (A.1) to (A.4) are multiplied from the left. Only in the case of (A.3) one has to take care of the correct ordering

$$(aa')m = a(a'm). \tag{A.5}$$

(iii) With these definitions at hand for A, B rings one says a left A-module and right B-module M is an *(A, B)-bimodule* if it satisfies the compatibility condition

$$a(m\,b) = (a\,m)b \tag{A.6}$$

for all $a \in A$, $b \in B$ and $m \in M$.

In order to simplify notations M_A will denote that M is a right A-module and accordingly ${}_AM$ and ${}_AM_B$ shall be the shorthand notations indicating that M is a left A-module respectively an (A, B)-bimodule.

Definition A.2. For M and N two right A-modules we say a map $\varphi : M \to N$ is a *morphism* of right A-modules if for all $m, m' \in M$ and all $a \in A$ we have

$$\begin{aligned} \varphi(m+m') &= \varphi(m) + \varphi(m'), \\ \varphi(ma) &= \varphi(m)\, a. \end{aligned}$$

The set of all such morphisms is denoted by $\operatorname{Hom}_A(M,N)$. One analogously defines morphisms of left A-modules.

Example A.1. In certain cases we can equip the set of morphisms between modules with a module structure. Let A, B and C be rings.

(i) Let ${}_AM_B$, ${}_AN_C$ be bimodules. Then the space $\operatorname{Hom}_A(M,N)$ becomes a (B,C)-bimodule via

$$(b\,\varphi\,c)(m) := \varphi(m\,b)c$$

for all $m \in M$, $b \in B$, $c \in C$, $\varphi \in \operatorname{Hom}_A(M,N)$.

(ii) Similarly for ${}_AM_B, {}_CN_B$ bimodules $\operatorname{Hom}_B(M,N)$ becomes a (C,A)-bimodule via

$$(c\,\psi\,a)(m) := c\,\psi(a\,m)$$

for all $m \in M$, $a \in A$, $c \in C$, $\psi \in \operatorname{Hom}_B(M,N)$.

Proof. Checking that $(b\,\varphi\,c)$ in (i) is indeed well-defined and that the module axioms are satisfied are straight forward calculations. One should only convince oneself that the side from where one multiplies the ring elements with the morphisms is the correct one. As an example we will check (A.3) and (A.5) for (i).

For this let $b, b' \in B$, $m \in M$ and $\varphi \in \operatorname{Hom}_A(M,N)$. Then

$$((bb')\,\varphi)(m) = \varphi(m\,(bb')) = \varphi((m\,b)b')) = (b'\,\varphi)(m\,b) = (b\,(b'\,\varphi))(m)$$

which proves $(bb')\,\varphi = b\,(b'\,\varphi)$. Now let $c, c' \in C$. Then

$$(\varphi\,(cc'))(m) = \varphi(m)\,(cc') = (\varphi((m))\,c)\,c' = ((\varphi\,c)(m))\,c' = ((\varphi\,c)\,c')(m).$$

Thus also in this case associativity is proven. □

Whereas the construction of direct sums of (bi)modules over a common ring is obvious the construction of a tensor product between a left and a right module over a common ring is first of all not clear. So let A be a ring, M_A, ${}_AN$ modules and G an abelian group in the following.

Definition A.3. (i) A map $h : M \times N \to G$ is called *A-balanced* if it satisfies

$$\begin{aligned} h(m+m',n) &= h(m,n) + h(m',n), \\ h(m,n+n') &= h(m,n) + h(m,n'), \\ h(m\,a,n) &= h(m,a\,n). \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$ and $a \in A$.

(ii) A *tensor product* of M_A with ${}_AN$ over A is an abelian group $M \otimes_A N$ together with an A-balanced map $\tau : M \times N \to M \otimes_A N$ satisfying the following universal property: For all A-balanced maps $h : M \times N \to G$ into any abelian group G there exists an unique group homomorphism $h' : M \otimes_A N \to G$ such that $h' \circ \tau = h$.

Lemma A.1. *The tensor product of M_A and A_N over A exists and is unique up to isomorphism.*

Proof. The uniqueness is a straight forward consequence of the universal property.

Regarding the existence the construction of the tensor product of M_A and A_N will be sketched in the following. Let F be the free abelian group with basis $M \times N$ and U the subgroup generated by elements of the form

$$\begin{aligned} (m+m', n) - (m,n) - (m,n'), \\ (m, n+n') - (m,n) - (m,n'), \\ (ma, n) - (m, an). \end{aligned}$$

Then by construction the map

$$\tau : M \times N \xrightarrow{\iota} F \to F/U$$

is A-balanced. Now it remains to show that $(F/U, \tau)$ satisfies the universal property of the tensor product. So let G be an abelian group and $h : M \times N \to G$ an A-balanced map. Then because of the universal property of the free abelian group F there exists a unique group homomorphism $\tilde{h} : F \to G$ such that $h = \tilde{h} \circ \iota$. Since h is A-balanced $\ker(\tilde{h}) \subseteq U$ and therefore $\tilde{h}$ factors through F/U, ie. there exists a unique $h' : F/U \to G$

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[d, "h"] & F \arrow[r, two heads] \arrow[dl, "\tilde{h}"] & F/U \arrow[dll, dashed, bend left, "\exists! h'"] \\ G & & \end{tikzcd}$$

such that $h' \circ \tau = h$ which proves the universal property. $\square$

For $m \in M$ and $n \in N$ we will denote $\tau(m,n) =: m \otimes n$. Such elements are called simple tensors. An arbitrary element of $M \otimes_A N$ is the finite sum of those simple tensors.

Even though the tensor product of M_A with ${}_AN$ is first of all just an abelian group the following two examples show how we can define a module structure on it again in some cases.

Example A.2. (i) Let A, B be rings and ${}_BM_A$, ${}_AN$ modules. Then $M \otimes_A N$ becomes a left B-module via

$$b(m \otimes n) := (bm) \otimes n$$

for all $b \in B$, $m \in M$ and $n \in N$.

(ii) Analogously if M_A and ${}_AN_B$ are modules, we can put a right B-module structure on $M \otimes_A N$ via

$$(m \otimes n)b := m \otimes (nb)$$

for all $b \in B$, $m \in M$ and $n \in N$.

The following properties of the tensor product are straight forward to show and are therefore stated without a proof.

Proposition A.2. *Let M_A, ${}_A(N_i)$ be modules over a ring A. Then there are isomorphisms*

(i) $A \otimes_A M \cong M$,

(ii) $M \otimes_A \left(\bigoplus_{i\in I} N_i\right) \cong \bigoplus_{i\in I} (M \otimes_A N_i)$. □

We close this chapter with the tensor-hom adjunction for modules which is often used throughout this thesis.

Lemma A.3. *Let A, B be rings and L_A, ${}_AM_B$ and N_B (bi)modules. Then there is an isomorphism of groups*

$$\mathrm{Hom}_B(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N)).$$

□

APPENDIX B

Division Rings

This chapter shall contain a brief introduction to division rings only covering results important for this thesis and some non-trivial examples of division rings.

Definition B.1. A ring with unity D is called a *division Ring* if every non-zero element is invertible.

The first example of division rings one can think of are fields. Another prominent non-trivial example of a division ring are the quaternions. The only difference between a field and a division ring is the commutativity of multiplication. Starting with a field k which admits a non-trivial automorphism one is able to construct a large range of examples following [13, 216ff.].

Definition B.2. Let k be a field and φ an automorphism of k. Then we define the *twisted ring of formal Laurent series* $k(\!(x;\varphi)\!)$ to contain the formal Laurent series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k x^k$$

as elements where there exists a $k_0 \in \mathbb{Z}$ such that $a_k = 0$ for all $k < k_0$. Summation is defined component-wise and the product of two elements $f(x) = \sum a_i x^i$, $g(x) = \sum b_j x^j$ is given by $f(x) \cdot g(x) = \sum c_k x^k$ where

$$c_k := \sum_{i+j=k} a_i \varphi^i(b_j).$$

Clearly $(k(\!(x;\varphi)\!), +, \cdot)$ is a ring.

Proposition B.1. *The twisted ring of formal Laurent series $k(\!(x;\varphi)\!)$ over a field k with automorphism φ is a division ring.*

Proof. Let $f(x) = \sum_{i=-\infty}^{\infty} a_i x^i$ and let $i_0 \in \mathbb{Z}$ such that $a_i = 0$ for all $i < i_0$ and $a_{i_0} \neq 0$. Then if we interpret $f(x) \cdot x^{-i_0} = \sum_{i=0}^{\infty} a_{i-i_0} x^i$ as a formal power series we can find an inverse $\tilde{g}(x) = \sum_{j=0}^{\infty} b_j x^j$ in $k[\![x]\!]$ as the lowest non-zero coefficient of $f(x) \cdot x^{-i_0}$ is a unit. If we now define

$$g(x) := \sum_{j=-i_0}^{\infty} \varphi^{-j-i_0}(b_j) x^j$$

this Laurent series satisfies

$$\begin{aligned} f(x)\cdot g(x) &= (f(x)\cdot x^{-i_0})\cdot(x^{i_0}\cdot g(x)) \\ &= \left(\sum_{i=0}^{\infty} a_{i-i_0}x^i\right)\cdot\left(\sum_{j=0}^{\infty}\varphi^{-j}(b_j)x^j\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{k=i+j} a_{i-i_0}b_j\right)x^k \\ &= 1 \end{aligned}$$

where in the last step we used that $\tilde{g}(x)$ is the inverse of $f(x)\cdot x^{-i_0}$ in $k[\![x]\!]$. Thus $g(x)$ is the inverse of $f(x)$ in $k(\!(x;\varphi)\!)$. □

Example B.1. As a concrete example consider $k = \mathbb{Q}(t)$ and $\varphi : k \to k$ defined via $\varphi(t) = 2t$. The elements of $k(\!(x;\varphi)\!)$ are then of the form

$$\sum_{k=k_0}^{\infty} a_k(t)x^k$$

with $a_k(t) \in \mathbb{Q}(t)$ and multiplication with x is given by $x \cdot a_k(t) = a_k(2t)x$. This example is due to Hilbert proving the existence of division rings which are infinite dimensional over their centers. The infinite dimensionality of this case is due to the following proposition.

Proposition B.2. *The Division ring $D := k(\!(x;\varphi)\!)$ is finite dimensional over its center $Z(D)$ if and only if φ is of finite order.*

Proof. See [13, Prop. 14.2]. □

For more examples of division rings consult [13, 216ff.]. We close this chapter with an important property of division rings regularly used throughout this thesis.

Lemma B.3. *Let D be a division ring. Then every right D-module is free.*

Proof. The proof follows [15]. Let M be a right D-module and choose an $m \in M$ such that $m \neq 0$. This element induces an homomorphism of right D-modules $f_m : D \to M, a \mapsto ma$. Now since $f_m(1) = m \neq 0$ we have $\ker(f_m) \neq D$ and hence $\ker(f_m) = 0$ as the only right ideals of D are 0 and D. This implies that $\{m\}$ is a set of linearly independent elements in M.

Now let $S := \{E \subset M | E \text{ is linearly indep.}\}$. We already know that $\{m\} \in S \neq \varnothing$. We can apply Zorn's lemma on $(S, \subseteq)$ with $\subseteq$ the inclusion of sets since every chain is bounded. We can now show that a maximal element $\tilde{E}$ in S is a basis of M. So take an arbitrary $m \in M$ then either $m \in \tilde{E}$ which means that m lies in the span of $\tilde{E}$ or $m \notin M$ and $\{m\} \cup \tilde{E}$ is linearly dependent as $\tilde{E}$ is maximal. Therefore there exist $m_1, \dots, m_n \in \tilde{E}$ and $a_0, a_1, \dots, a_n \in D$ such that

$$a_0 m + a_1 m_1 + \dots + a_n m_n = 0$$

with especially $a_0 \neq 0$. But this means that we can invert a_0 and see that

$$m = -a_0^{-1}(a_1 m_1 + \dots + a_n m_n)$$

lies in the span of $\tilde{E}$ which makes $\tilde{E}$ a basis of M and thus M a free D-module. □

Remark B.1. One can even prove the converse of this lemma. Namely, if R is a unital ring with the property that all right R-modules are free then R is a division ring. See [15] for a proof of this statement.

APPENDIX C

Dynkin and extended Dynkin diagrams

$$\mathbb{A}_n \qquad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n \end{tikzcd}$$

$$\mathbb{B}_n \qquad \begin{tikzcd} 1 \arrow[r, dash, "{(1,2)}"] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n \end{tikzcd}$$

$$\mathbb{C}_n \qquad \begin{tikzcd} 1 \arrow[r, dash, "{(2,1)}"] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n \end{tikzcd}$$

$$\mathbb{D}_n \qquad \begin{tikzcd} 1 \arrow[dr, dash] & & & & & \\ & 3 \arrow[r, dash] & 4 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n \\ 2 \arrow[ur, dash] & & & & & \end{tikzcd}$$

$$\mathbb{E}_6 \qquad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \\ & & 6 & & \end{tikzcd}$$

$$\mathbb{E}_7 \qquad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \\ & & 7 & & & \end{tikzcd}$$

$$\mathbb{E}_8 \qquad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \\ & & 8 & & & & \end{tikzcd}$$

$$\mathbb{F}_4 \qquad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash, "{(1,2)}"] & 3 \arrow[r, dash] & 4 \end{tikzcd}$$

$$\mathbb{G}_2 \qquad \begin{tikzcd} 1 \arrow[r, dash, "{(1,3)}"] & 2 \end{tikzcd}$$

Figure C.1: Dynkin diagrams.

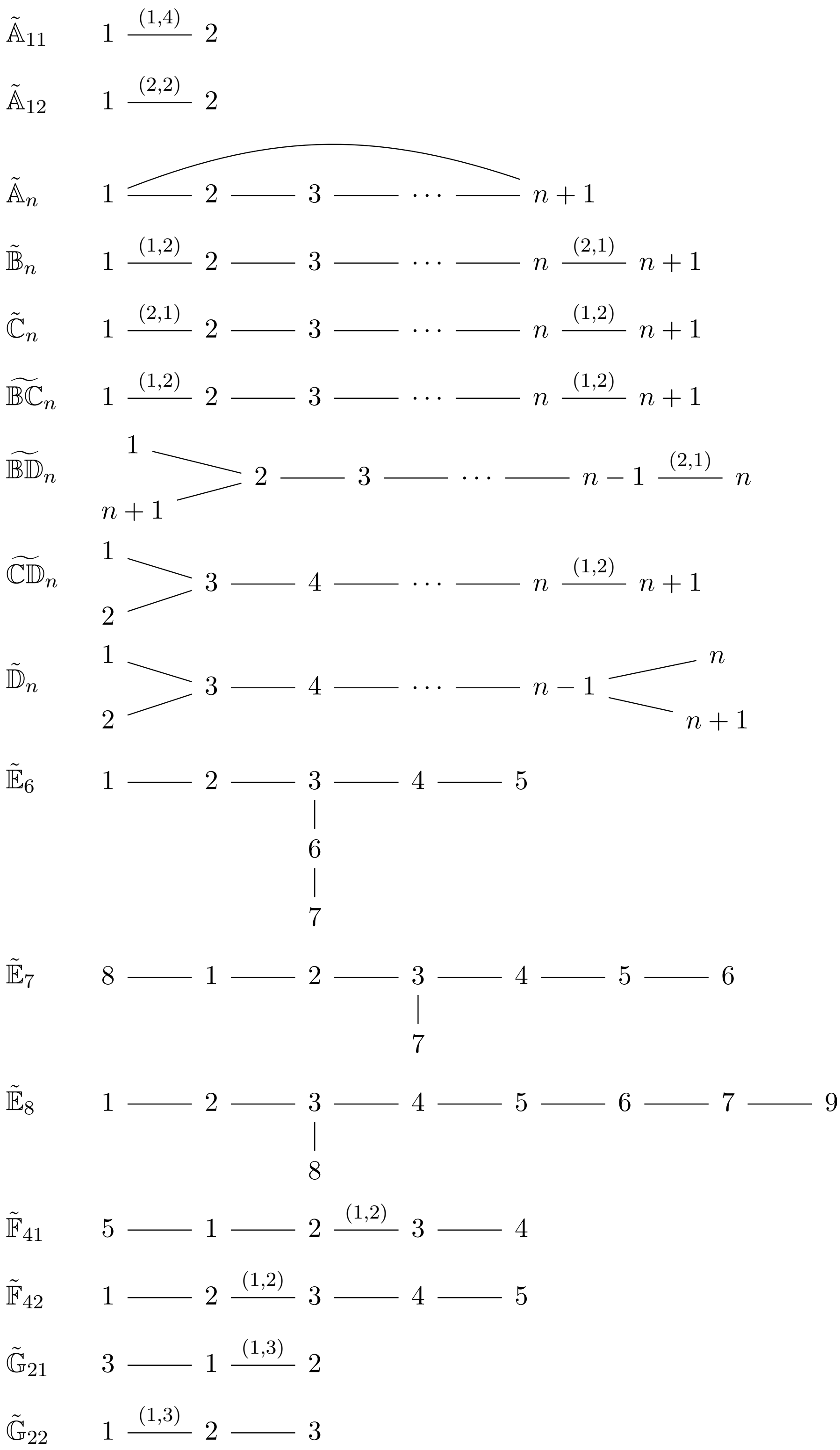

Figure C.2: Extended Dynkin diagrams.

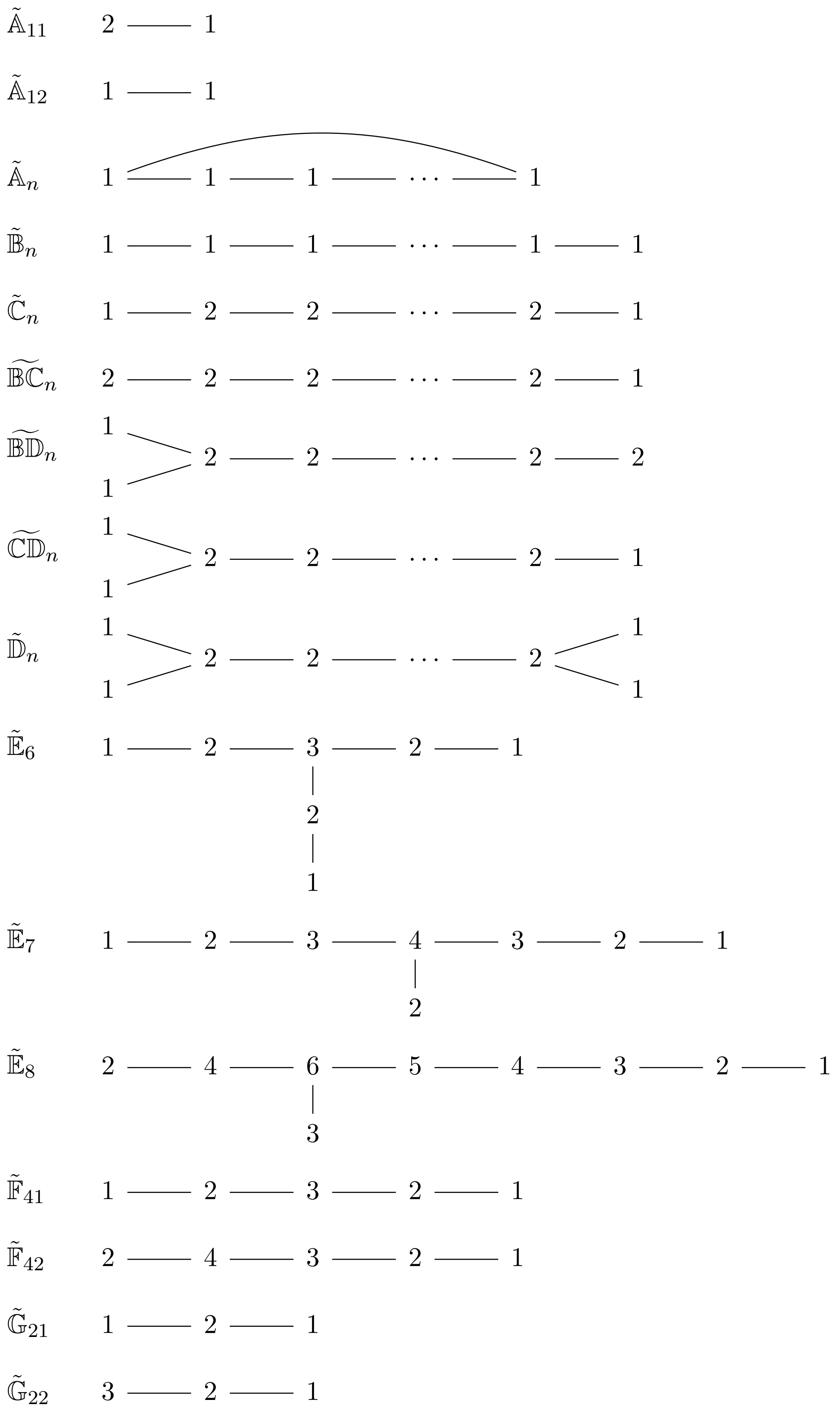

Figure C.3: The vectors δ satisfying $(\delta, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}$ for each extended Dynkin diagram [4, pp. 39-53].

Bibliography

[1] I. Assem, D. Simson and A. Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006.

[2] D. Benson. *Representations and Cohomology: Volume 1, Basic Representation Theory of Finite Groups and Associative Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[3] I. N. Bernstein, I. M. Gel'fand and V. A. Ponomarev. 'Coxeter Functors and Gabriel's Theorem'. *Russian Mathematical Surveys* 28.2 (1973), pp. 17–32.

[4] V. Dlab and C. Ringel. *Indecomposable Representations of Graphs and Algebras*. American Mathematical Society: Memoirs of the American Mathematical Society Nr. 173. American Mathematical Society, 1976.

[5] V. Dlab and C. M. Ringel. 'On algebras of finite representation type'. *Journal of Algebra* 33.2 (1975), pp. 306 –394.

[6] V. Dlab and C. M. Ringel. 'Représentations des graphes valués'. *C. R. Acad. Sci. Paris Sér. A* 278 (1974), pp. 537–540.

[7] Y. Drozd et al. *Finite Dimensional Algebras*. Springer-Verlag, 1994.

[8] P. Gabriel. 'Indecomposable representations. II.' *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)* (1973), pp. 81–104.

[9] P. Gabriel. 'Unzerlegbare Darstellungen I'. *manuscripta mathematica* 6.1 (1972), pp. 71–103.

[10] C. Geiss, B. Leclerc and J. Schröer. 'Quivers with relations for symmetrizable Cartan matrices I: Foundations'. *Invent. Math.* 209.1 (2017), pp. 61–158.

[11] N. Jacobson. *Basic Algebra II: Second Edition*. Dover Books on Mathematics. Dover Publications, 2012.

[12] H. Krause. *Representations of quivers via reflection functors*. 2008. eprint: arXiv:0804.1428.

[13] T. Lam. *A First Course in Noncommutative Rings*. Graduate Texts in Mathematics. Springer New York, 2013.

[14] J. Lemay. *Valued Graphs and the Representation Theory of Lie Algebras*. 2011. eprint: arXiv:1109.2107.

[15] *Rings whose every module is free*. 2013. URL: https://planetmath.org/ringswhoseeverymoduleisfree (visited on 30/09/2010).

[16] R. Schiffler. *Quiver Representations*. CMS Books in Mathematics. Springer International Publishing, 2014.