GROMOV–WITTEN THEORY OF BICYCLIC PAIRS

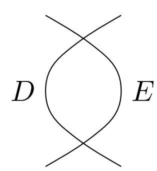
MICHEL VAN GARREL, NAVID NABIJOU, YANNIK SCHULER

ABSTRACT. A bicyclic pair consists of a smooth surface equipped with a pair of smooth divisors intersecting in two reduced points. Resolutions of self-nodal curves constitute an important special case. We investigate the logarithmic Gromov–Witten theory of bicyclic pairs. We establish all-genus correspondences with local Gromov–Witten theory and open Gromov–Witten theory, and a genus zero correspondence with orbifold Gromov–Witten theory. For self-nodal curves in $\mathbb{P}(1,1,r)$ we obtain a closed formula for the genus zero invariants and establish a conceptual relationship with the invariants of the smoothing. The technical heart of the paper is a detailed analysis of the degeneration formula for stable logarithmic maps, complemented by torus localisation and scattering techniques.

CONTENTS

INTRODUCTION

A **bicyclic pair** is a pair $(S \,|\, D+E)$ consisting of a smooth surface S and a pair of smooth divisors $D, E \subseteq S$ intersecting in two reduced points:



These arise in two distinct contexts:

(i) **Looijenga pairs** with two boundary components [Loo81]. In this case $D+E \in |-K_S|$ with D and E both rational. When D and E are nef there is a simple classification [BBvG20, Table 1]. The general classification proceeds via the minimal model program, see [FM83, Lemma 3.2] or [Fri15, Section 2].

(ii) **Self-nodal pairs.** Consider a surface containing an irreducible curve with a single nodal singularity. Let S be the blowup of the surface at the node, D the strict transform of the irreducible curve, and E the exceptional divisor. Then $(S \,|\, D+E)$ is a bicyclic pair. Note that E is not nef.

We study the Gromov–Witten theory of bicyclic pairs. We investigate connections to local, open and orbifold geometries, obtain closed formulae in important special cases, and probe the behaviour under smoothing of the divisor. The main theories we consider are:

$$\begin{tikzcd}
\boxed{(S \,|\, D+E)} \arrow[r, leftrightarrow, "\text{Section 1}"] & \boxed{(\mathcal{O}_S(-D) \,|\, E)} \arrow[r, leftrightarrow, "\text{Section 1.5}"] \arrow[d, leftrightarrow, "\text{Section 2}"] & \boxed{\mathcal{O}_S(-D) \oplus \mathcal{O}_S(-E)} \\
& \boxed{\mathcal{O}_S(-D)|_{S \setminus E}} \arrow[r, "\text{Section 3}"] & \boxed{\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-2)}
\end{tikzcd}$$

We work in several different settings. The precise hypotheses are stated clearly at the start of each section. Globally, the paper proceeds from most general to most specific.

0.1. **Logarithmic-local correspondence (Sections 1.1–1.4).** Our first main result establishes an all-genus correspondence between the Gromov–Witten theories of the following pairs:

$$(S \,|\, D+E) \leftrightarrow (\mathcal{O}_S(-D) \,|\, E).$$

This arises by considering the degeneration to the normal cone of the divisor $D \subseteq S$. This is a well-established technique [vGGR19, TY20, BFGW21] but additional complications occur due to the non-trivial logarithmic structure on the general fibre (see Remark 0.1).

Consider a bicyclic pair $(S \,|\, D+E)$ and fix a curve class $\beta \in A_1(S)$. We assume:

- $D \cong \mathbb{P}^1$ and $D^2 \geq 0$.
- $D \cdot \beta > 0$ and $E \cdot \beta \geq 0$.[1]

Fix a genus g and tangency data $\mathbf{c}$ with respect to E. From $\mathbf{c}$ we build tangency data $\hat{\mathbf{c}}$ with respect to $D+E$ by appending an additional marked point with tangency $D \cdot \beta$ along D.

Finally let γ be a collection of D-disjoint insertions (Definition 1.2) of the correct codimension. This permits arbitrary psi classes but requires evaluation classes to be disjoint from D.

Theorem A (Theorem 1.3). *We have the following equality of generating functions:*

$$\frac{(-1)^{D\cdot\beta-1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}\left(\sum_{g\geq 0}\mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle(-1)^g\lambda_g\gamma\rangle\cdot\hbar^{2g-1}\right)=\sum_{g\geq 0}\mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)\langle\gamma\rangle\cdot\hbar^{2g-2}.$$

Remark 0.1. The proof proceeds via the degeneration formula for stable logarithmic maps [ACGS20, Ran22] applied to the degeneration to the normal cone of $D \subseteq S$. The general fibre has non-trivial logarithmic structure corresponding to E, which greatly complicates the analysis. We develop combinatorial techniques for constraining the shapes of rigid tropical types, and intersection theoretic techniques for establishing the vanishing of certain contributions (see Section 1.4). We expect these to serve as a useful toolkit for future calculations.

Having established Theorem A, we explore two proximate results in genus zero.

0.2. **Nef pairs (Section 1.5).** Combining Theorem A with [vGGR19, Theorem 1.1] we obtain:

Theorem B (Theorem 1.26). *Suppose that $E^2 \geq 0$ and $E \cdot \beta > 0$ and let $\mathbf{c} = (E \cdot \beta)$ be maximal tangency contact data to E. Then we have:*

$$\mathsf{GW}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle = (-1)^{(D+E)\cdot\beta}(D\cdot\beta)(E\cdot\beta)\cdot\mathsf{GW}_{0,\mathbf{c},\beta}(\mathcal{O}_S(-D)\oplus\mathcal{O}_S(-E))\langle\gamma\rangle.$$

This gives another instance of the numerical logarithmic-local correspondence for normal crossings pairs [BBvG22, BBvG20, BBvG21, TY23]. See Remark 1.30 for the importance of D-disjoint insertions.

[1] The case $E \cdot \beta = 0$ can occur, e.g. for resolutions of self-nodal pairs. In this case, the spaces of stable logarithmic maps to $(S \,|\, D+E)$ and $(S|D)$ have the same virtual dimension, but their logarithmic Gromov–Witten invariants typically differ. See Remark 1.4.

0.3. **Root stacks and self-nodal pairs (Section 1.6).** We next impose that $E \cdot \beta = 0$. This includes the case of self-nodal pairs discussed above. In this setting we provide an alternative proof of Theorem A, by passing through the logarithmic-orbifold [BNR22] and orbifold-local [BNTY23] correspondences (Theorem 1.28). This allows us to remove the assumptions that D is rational and γ is D-disjoint. The key intermediate result is:

Theorem C (Proposition 1.29)**.** *Suppose that $E \cdot \beta = 0$. Then there is an equality between logarithmic and orbifold Gromov–Witten invariants:*

$$\mathsf{GW}^{\mathsf{log}}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle = \mathsf{GW}^{\mathsf{orb}}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle.$$

This holds without assuming that $D \cong \mathbb{P}^1$ or that γ is D-disjoint.

This result is proved using [BNR22, Theorem X]. The main technical step is to strongly constrain the shapes of tropical types of maps to $(S \,|\, D+E)$.

0.4. **Toric and open geometries (Section 2).** We next specialise to toric Calabi–Yau pairs. We assume:

- S is toric.
- E is a toric hypersurface.
- $D + E \in |-K_S|$.
- $E \cdot \beta = 0$.

We do not require that D is toric. In this setting, we obtain a logarithmic-open correspondence:

Theorem D (Theorem 2.3)**.** *For all $g \geq 0$ we have:*

$$\mathsf{GW}_{g,0,\beta}(\mathcal{O}_S(-D)|E) = \mathsf{GW}^T_{g,0,\iota^\star\beta}(\mathcal{O}_S(-D)|_{S\setminus E})$$

where the Gromov–Witten invariant on the right-hand side is defined by localising with respect to the action of the Calabi–Yau torus T (see Section 2.2).

The proof proceeds by localisation on both sides. The difficult step is to establish the vanishing of certain contributions, by isolating a weight zero piece of the obstruction bundle. For this it is crucial that we localise with respect to the Calabi–Yau torus T.

The target $\mathcal{O}_S(-D)|_{S\setminus E}$ is a toric Calabi–Yau threefold and hence its invariants can be computed using the topological vertex formalism [LLLZ09]. Theorem D combined with Theorem A thus provides a new means to efficiently compute the all-genus logarithmic invariants of $(S \,|\, D+E)$, otherwise a highly tedious endeavour.

0.5. **Self-nodal curves: calculations (Section 3.1).** We now specialise to our motivating example. Consider $S_r := \mathbb{P}(1,1,r)$ and let $D_r \in |-K_{S_r}|$ be an irreducible curve with a single nodal singularity at the singular point of S_r. We consider degree d curves in the pair $(S_r|D_r)$ which meet D_r in a single point of maximal tangency order $d(r+2)$.

Theorem E (Theorem 3.1)**.** *We have:*

$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) = \frac{r+2}{d^2}\binom{(r+1)^2 d - 1}{d-1}.$$

Remark 0.2. The numerator $(r+2)$ is the number of flex lines. The remaining factors strongly resemble the multiple cover formula of [GPS10, Proposition 6.1], however for a curve of tangency order $(r+1)^2+1$.

Theorem E is proved via computations on the corresponding local scattering diagram. Passing to the resolution $\mathbb{F}_r \to S_r$ and combining Theorems D and E we immediately obtain the following formula, already known in the physics literature [CGM+07, Equation (4.53)]:

Theorem F (Theorem 3.2)**.** *We have:*

$$\mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1}(-r-2)\big)=\frac{(-1)^{rd-1}}{d^3}\binom{(r+1)^2d-1}{d-1}.$$

0.6. **Self-nodal curves: smoothings (Section 3.2).** In the final section we focus on the case $r=1$. We compare the Gromov–Witten invariants of $(\mathbb{P}^2|D)$ and $(\mathbb{P}^2|E)$ where D and E are nodal and smooth plane cubics. Experimentally we observe that the former are always smaller than the latter. We provide a conceptual explanation for this defect, via the enumerative geometry of degenerating hypersurfaces. As in [BN22] we degenerate both D and E to the toric boundary $\Delta\subseteq\mathbb{P}^2$ and consider the invariants of the logarithmically singular central fibre.

Theorem G (Theorem 3.7)**.** *The invariants of $(\mathbb{P}^2|D)$ are precisely the central fibre contributions to the invariants of $(\mathbb{P}^2|E)$ arising from multiple covers of a single coordinate line.*

Finally (Theorem 3.10) we apply Theorem G to settle a conjecture in [BN22].

Remark 0.3. Taken together, Theorems E, F, G relate the Gromov–Witten invariants arising from two non-standard obstruction theories on the space of stable maps to $\mathbb{P}^1$: the local geometry and the degenerated hypersurface.

0.7. **Context.** The present paper fits into the broader body of work on logarithmic-local-open-quiver correspondences [Tak01, Gat03, vGWZ13, vGGR19, CvGKT20, CvGKT21a, CvGKT21b, BBvG22, BBvG20, BBvG21, NR22, BFGW21, LY22a, Yu23, FW21, Bou21, BW23, BS23, Wan22]. By [BNTY23] this is intimately connected to the logarithmic-orbifold correspondence, another area of intense study [Cad07, CC08, ACW17, TY20, BNR22]. This area enjoys close connections to Mirror Symmetry [Bou22, Bou24, LY22b, FTY19, TY23, You20, Grä22b, Grä22a, GRZ22, You22, Sha23].

In [NR22] it is shown that the naïve logarithmic-local correspondence fails for normal crossings pairs, and a corrected form is established. It is then observed that in many situations, the insertions cap trivially with the correction terms, collapsing the corrected correspondence to the naïve correspondence. The results of this paper provide another instance of this phenomenon.

Our techniques can be used to recover [BW23, Theorem 1.7], which relates the invariants of $(S|D)$ and $\mathcal{O}_S(-D)$ by applying the more general [BFGW21, Theorem 1.1]. Following [Bou21, Proposition 3.1] we pass to a deformation and identify the invariants of $(S|D)$ with the invariants of $(S\,|\,D+E)$ with tangency orders $(1,\ldots,1)$ along E. Similarly we identify the invariants of $\mathcal{O}_S(-D)$ with the invariants of $(\mathcal{O}_S(-D)\,|\,E)$. The correspondence [BW23, Theorem 1.7] then follows from Theorem A, with the following caveat: we must permit point insertions at markings with tangency along E. This can be achieved by suitable modifications of the statement and proof of Theorem 1.25.

0.8. **Prospects.**

0.8.1. *Quivers.* Two independent results in the literature suggest a relationship between Gromov–Witten invariants of bicyclic pairs and Donaldson–Thomas invariants of quivers:

(i) **Nef pairs.** In [BW23] the authors equate the Gromov–Witten invariants of $(S|D)$ and $\mathcal{O}_S(-D)$ to the Donaldson–Thomas invariants of certain quivers. The quivers in question differ by explicit framings, exhibiting one quiver moduli space as a projective bundle over the other, see [BW23, Equation (16)]. This produces a relationship between the Donaldson–Thomas invariants, involving the same correction factor as appears in Theorem A. The direct link to our results (see Section 0.7) suggests a similar identification for the invariants of the nef pair $(S\,|\,D+E)$.

(ii) **Self-nodal pairs.** The Gromov–Witten invariants of a self-nodal pair are governed by the wall-crossing functions attached to the central ray of a local scattering diagram (see Section 3.1 and [GPS10]). This is intimately related to quiver invariants by [GP10, Rei10].

It would be worthwhile to compare these two cases, and in the case of self-nodal pairs to obtain a relationship between local invariants and quiver invariants.

0.8.2. *Scattering.* Theorem E computes the genus zero Gromov–Witten invariants of $\mathbb{P}(1,1,r)$ relative to a self-nodal anticanonical divisor. We achieve this by equating these invariants with the wall-crossing function of the central ray of a local scattering diagram, with incoming ray directions ρ_1, ρ_2 satisfying $|\rho_1 \wedge \rho_2| = r+2$. We then use [GP10, Equation (1.4)] proven by Reineke [Rei11] for $\ell_1 = \ell_2$ to arrive at Theorem E, from which we deduce Theorem F.

It may be possible to reverse this logic, proving Theorem F first via a direct analysis of the local invariants (as in [CGM⁺07]) and using this to deduce Theorem E and hence [GP10, Equation (1.4)]. Speculatively, it may also be possible to study the case $\ell_1 \neq \ell_2$, by extending the arguments of Section 3 to weighted projective planes of the form $\mathbb{P}(1,a,b)$.

Parts of Section 3 extend readily to higher genus using Bousseau's quantum scattering [Bou20]. An analysis of the central ray of the quantum scattering diagram of the Kronecker quiver, via a correspondence with the all-genus Gromov–Witten generating function of local $\mathbb{P}^1$, is an attractive prospect.

0.8.3. *Logarithmic-open.* Combining Theorems A and D we obtain a logarithmic-open correspondence for two-component Looijenga pairs [BBvG20, Conjecture 1.3] in the setting where the curve class pairs trivially with one of the components. If the curve class pairs non-trivially with both components, we still expect Theorem A to allow us to make progress on the logarithmic-open correspondence. We intend to investigate this in future work.

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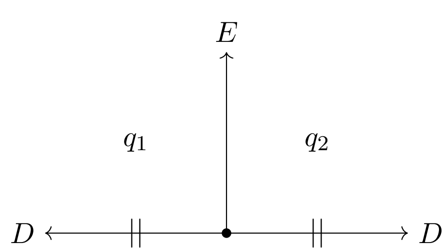
1. Bicyclic pairs

We introduce bicyclic pairs $(S \,|\, D+E)$. The main result of this section (Theorem 1.3) establishes a precise correspondence between the logarithmic Gromov–Witten theories of the pairs $(S \,|\, D+E)$ and $(\mathcal{O}_S(-D) \,|\, E)$. This occupies Sections 1.1–1.4. Applications and variations are discussed in Sections 1.5 and 1.6.

1.1. Setup and statement of correspondence.

Definition 1.1. A **bicyclic pair** $(S \,|\, D+E)$ consists of a smooth projective surface S and smooth divisors $D, E \subseteq S$ such that D and E intersect in two reduced points, denoted q_1 and q_2. The normal crossings pair $(S \,|\, D+E)$ has tropicalisation:

(1)



Fix a bicyclic pair $(S \mid D+E)$ and a curve class $\beta \in A_1(S)$ such that:

- $D \cong \mathbb{P}^1$ and $D^2 \geq 0$.
- $D \cdot \beta > 0$ and $E \cdot \beta \geq 0$.

We consider stable logarithmic maps to $(S \mid D+E)$ with genus g, class β and markings $x, y_1, \ldots, y_b$, $z_1, \ldots, z_c$ carrying the following tangency conditions:

- x has maximal tangency $D \cdot \beta$ with respect to D.
- y_i has tangency $\alpha_i > 0$ with respect to E (with $\Sigma_{i=1}^b \alpha_i = E \cdot \beta$).
- z_i has no tangency (interior marking).

We let $\hat{\mathbf{c}}$ denote the matrix of tangency data. We restrict to the following class of insertions.

Definition 1.2. A class $\gamma_i \in A^\star(S)$ is **D-disjoint** if there exists a regularly embedded subvariety $Z_i \subseteq S$ such that $[Z_i] = \gamma_i$ and $Z_i \cap D = \emptyset$. An assembly of insertions at the interior markings

$$\gamma = \prod_{i=1}^{c} \mathrm{ev}_{z_i}^\star(\gamma_i)\psi_{z_i}^{k_i}.$$

is **D-disjoint** if each γ_i is D-disjoint. This provides an enlargement of the stationary sector. We only permit insertions at interior markings.

With the above setup, we use the moduli space of stable logarithmic maps [Che14, AC14, GS13] to define logarithmic Gromov–Witten invariants with a lambda class insertion

$$\mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)\langle(-1)^g\lambda_g\gamma\rangle := (-1)^g\lambda_g\gamma \cap [\overline{M}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)]^{\mathrm{virt}} \in \mathbb{Q}. \tag{2}$$

We next consider the pair $(\mathcal{O}_S(-D) \mid E)$.[2] We obtain tangency data $\mathbf{c}$ from $\hat{\mathbf{c}}$ by deleting the marking x, and define logarithmic Gromov–Witten invariants of the local target

$$\mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle := \gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)]^{\mathrm{virt}} \in \mathbb{Q}. \tag{3}$$

Since $D^2 \geq 0$ it follows that D is nef, and then $D \cdot \beta > 0$ implies that $H^0(C, f^\star\mathcal{O}_S(-D)) = 0$ for any stable map $f\colon C \to S$ of class β. The local theory of $\mathcal{O}_S(-D)$ is thus well-defined. The main result of this section is a correspondence between the invariants (2) and (3).

Theorem 1.3 (Theorem A)**.** *We have the following equality of generating functions:*

$$\frac{(-1)^{D\cdot\beta-1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}\sum_{g\geq 0}\mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)\langle(-1)^g\lambda_g\gamma\rangle\hbar^{2g-1} = \sum_{g\geq 0}\mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle\hbar^{2g-2}.$$

Remark 1.4. The case $E \cdot \beta = 0$ is possible, and in fact one of the most interesting (see Section 3). In this case $b = 0$, but we emphasise that the spaces

$$\overline{M}_{g,\mathbf{c},\beta}(S \mid D+E) \qquad \text{and} \qquad \overline{M}_{g,\mathbf{c},\beta}(S|D)$$

are *not* the same. The difference is clearly visible for $E \subseteq \mathbb{F}_1$ the (-1)-curve and β a curve class pulled back along the morphism $\mathbb{F}_1 \to \mathbb{P}^2$ contracting E. The moduli space

$$\overline{M}_{0,n,\beta}(\mathbb{F}_1)$$

has a large number of excess components. On the other hand the moduli space

$$\overline{M}_{0,(0,\ldots,0),\beta}(\mathbb{F}_1|E)$$

[2] For $\pi\colon V \to S$ a vector bundle and $E \subseteq S$ a divisor, we abuse notation and write $(V|E)$ for the pair $(V|\pi^{-1}(E))$.

is irreducible of the expected dimension. Although the markings carry no tangency, the logarithmic structure imposes tangency conditions at the nodes, which cut down the excess components to produce a space of the expected dimension. The Gromov–Witten invariants also differ, with the theory of $(\mathbb{F}_1|E)$ being essentially equivalent to the theory of $\mathbb{P}^2$.

1.2. **Target degeneration.** The proof of Theorem 1.3 follows the degeneration argument of [vGGR19]. Because our degeneration has logarithmic structure on the general fibre, much greater care is required when enumerating rigid tropical types and performing gluing. This accounts for the significantly more involved proof. Despite this, the shape of the final formula is relatively simple (Theorem 1.3), as we strongly constrain the rigid tropical types which contribute.

Consider the degeneration of S to the normal cone of D as illustrated in Figure 1. This is a family

$$\mathcal{S} \to \mathbb{A}^1 \tag{4}$$

with general fibre S. Let P denote the projective completion of the normal bundle of $D \subseteq S$. The central fibre $\mathcal{S}_0$ of (4) is obtained by gluing S and P along the divisors $D \subseteq S$ and the zero section $D_0 \subseteq P$. We write $D_0 \subseteq \mathcal{S}_0$ for the gluing divisor.

Let $\mathcal{E} \subseteq \mathcal{S}$ denote the strict transform of $E \times \mathbb{A}^1$. This intersects the component S of the central fibre in $E \subseteq S$, and the component P of the central fibre in the union of the two fibres $E_1, E_2 \subseteq P$ of the $\mathbb{P}^1$-bundle $P \to D$ over the points $\{q_1, q_2\} = D \cap E$. We equip the total space $\mathcal{S}$ with the divisorial logarithmic structure corresponding to $S + P + \mathcal{E}$.

Now take the strict transform of $D \times \mathbb{A}^1$ under the blowup $\mathcal{S} \to S \times \mathbb{A}^1$ and let $\mathcal{L}$ be the inverse of the corresponding line bundle on $\mathcal{S}$. There is a flat morphism

$$\mathcal{L} \to \mathcal{S}$$

which we use to pull back the logarithmic structure on $\mathcal{S}$. On the general fibre, the resulting logarithmic scheme is

$$(\mathcal{O}_S(-D) \,|\, E).$$

On the central fibre we have $\mathcal{L}|_S = \mathcal{O}_S$ and $\mathcal{L}|_P = \mathcal{O}_P(-D_\infty)$. The central fibre $\mathcal{L}_0$ is therefore obtained by gluing

$$(S \times \mathbb{A}^1 \,|\, D + E) \qquad \text{and} \qquad (\mathcal{O}_P(-D_\infty) \,|\, D_0 + E_1 + E_2)$$

along the divisors $D \times \mathbb{A}^1 \subseteq S \times \mathbb{A}^1$ and $D_0 \times \mathbb{A}^1 \subseteq \mathcal{O}_P(-D_\infty)$.

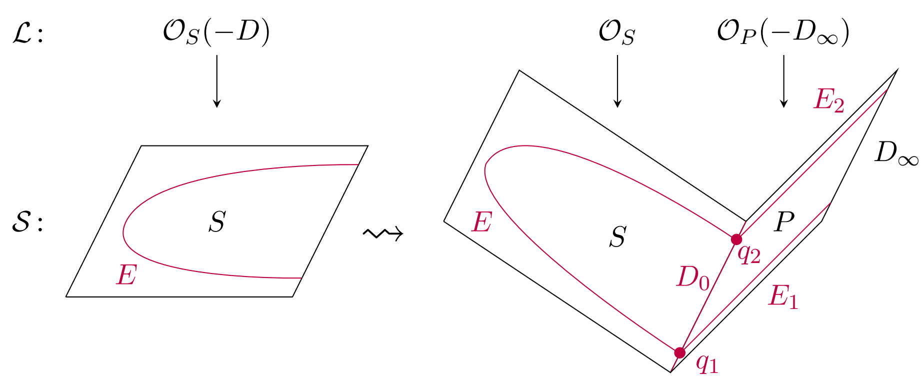
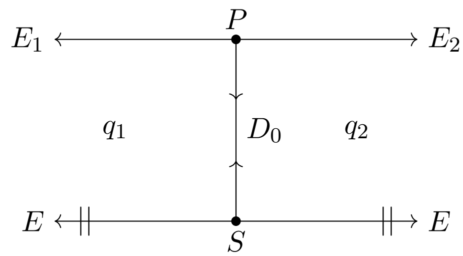


FIGURE 1. The degeneration $\mathcal{S} \to \mathbb{A}^1$ and the line bundle $\mathcal{L}$ on $\mathcal{S}$. The logarithmic structure is indicated in purple.

1.3. **Tropical types and balancing.** The logarithmic morphism $\mathcal{S} \to \mathbb{A}^1$ induces a morphism of tropicalisations

$$\Sigma(\mathcal{S}) \to \Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}.$$

The fibre over $1 \in \mathbb{R}_{\geq 0}$ is a polyhedral complex which we denote Σ:



Notation 1.5. We use the notation $S, P, D_0, E, E_1, E_2, q_1, q_2$ to refer both to the strata of $\mathcal{S}_0$ and to the corresponding polyhedra in Σ.

The moduli space of stable logarithmic maps to the central fibre $\mathcal{L}_0$ decomposes into virtual irreducible components indexed by rigid tropical types of maps to Σ.

Definition 1.6 ([ACGS20, Definition 2.23])**.** A **tropical type of map to** Σ consists of:

(i) **Source graph.** A finite graph Γ with vertices $V(\Gamma)$, finite edges $E(\Gamma)$, unbounded legs $L(\Gamma)$, and a genus assignment $g\colon V(\Gamma) \to \mathbb{N}$.

(ii) **Polyhedra assignments.** An inclusion-preserving function $\sigma\colon V(\Gamma) \sqcup E(\Gamma) \sqcup L(\Gamma) \to \Sigma$. We often write $\mathsf{f}(w) \in \sigma$ instead of $\sigma_w = \sigma$.

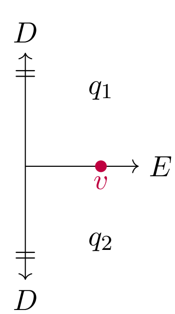
(iii) **Curve classes.** A curve class $\beta_v \in A_1(\mathcal{S}_0(\sigma_v))$ for every $v \in V(\Gamma)$. Here $\mathcal{S}_0(\sigma_v) \subseteq \mathcal{S}_0$ is the closed stratum corresponding to the polyhedron $\sigma_v \in \Sigma$.

(iv) **Slopes.** Vectors $m_{\vec{e}} \in N(\sigma_e)$ for every oriented edge $\vec{e} \in \vec{E}(\Gamma) \sqcup L(\Gamma)$, satisfying $m_{\vec{e}} = -m_{\overleftarrow{e}}$. Here $N(\sigma_e)$ is the lattice associated to the polyhedron σ_e.

A tropical type has an associated tropical moduli space, parametrising choices of edge lengths and vertex positions. A tropical type is **rigid** if its tropical moduli space is a point. See [ACGS20, Sections 2.5 and 3.2] for details.

The above data is required to be balanced at each vertex. For vertices in q_1 and q_2 this means that the sum of outgoing slopes is zero. For vertices in S and P it is the usual balancing condition for logarithmic maps to the normal crossings pairs $(S \,|\, D + E)$ and $(P \,|\, D_0 + E_1 + E_2)$ as in [GS13, Proposition 1.15]. We now explain balancing for vertices in E, D_0, E_1, and E_2.

1.3.1. *Vertices on E.* Local to v the tropical target Σ has the following structure:



For $i \in \{1,2\}$ let $m_D^i \in \mathbb{N}$ denote the sum of vertical slopes of outgoing edges which enter q_i. Letting $m_D \in \mathbb{Z}$ denote the sum of vertical slopes of all outgoing edges, we have

$$m_D = m_D^1 + m_D^2.$$

On the other hand, let $m_E \in \mathbb{Z}$ denote the total sum of horizontal slopes of all outgoing edges. The curve class $\beta_v \in A_1(E)$ must be of the form

$$\beta_v = kE$$

for some $k \in \mathbb{N}$. We have $D \cdot \beta_v = kDE = 2k$ and $E \cdot \beta_v = kE^2$ where $E^2 \in \mathbb{Z}$ is the self-intersection inside S. The balancing condition therefore gives

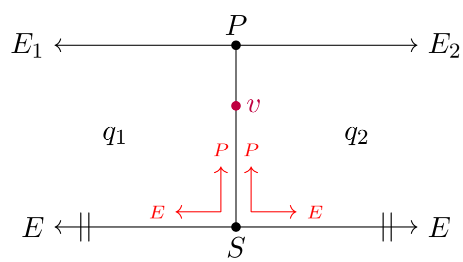
$$m_D = 2k, \qquad m_E = kE^2.$$

However, there is a stronger constraint. Since $f_v \colon C_v \to E$ is a degree k cover, the tangency orders m_D^1, m_D^2 give the ramification profiles of f_v over the two intersection points q_1, q_2. Therefore we must have

$$m_D^1 = m_D^2 = k.$$

Geometrically, this means that when we lay the tropicalisation flat along the "spine" E, the sum of outgoing vertical slopes is zero.

1.3.2. *Vertices on D_0.* This is similar to the previous case. Local to v the tropical target Σ has the following structure:



We choose coordinates for the adjacent polyhedra q_1, q_2 as indicated in red. For $i \in \{1,2\}$ we let $m_E^i \in \mathbb{N}$ denote the sum of horizontal slopes of outgoing edges which enter q_i. Letting $m_E \in \mathbb{Z}$ denote the sum of horizontal slopes of all outgoing edges, we have

$$m_E = m_E^1 + m_E^2.$$

On the other hand, let $m_P \in \mathbb{Z}$ denote the sum of vertical slopes of all outgoing edges (note that if instead we choose coordinates corresponding to S, E then we have $m_S = -m_P$). The curve class $\beta_v \in A_1(D_0)$ necessarily takes the form

$$\beta_v = kD_0$$

for some $k \in \mathbb{N}$. We have $E \cdot \beta_v = 2k$ and $\deg f_v^\star N_{D_0|P}^\vee = kD^2$ where $D^2 \geq 0$ is the self-intersection inside S. The balancing condition is therefore

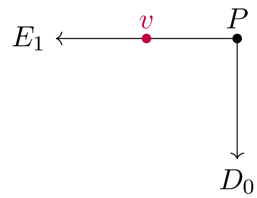
$$m_E = 2k, \qquad m_P = kD^2.$$

As in the previous case, we have the stronger constraint

$$m_E^1 = m_E^2 = k.$$

This means that when we lay the tropicalisation flat along the spine D_0, the sum of outgoing horizontal slopes is zero.

1.3.3. *Vertices on E_1, E_2.* This is the simplest case. We restrict to E_1 without loss of generality. Local to v there is a single maximal polyhedron, coordinatised by E_1, D_0:



We let $m_{E_1} \in \mathbb{Z}$ and $m_{D_0} \in \mathbb{N}$ denote, respectively, the sums of horizontal and vertical outgoing slopes. The curve class is $\beta_v = kE_1$ giving $E_1 \cdot \beta_v = 0$ and $D_0 \cdot \beta_v = k$. The balancing condition gives:

$$m_{E_1} = 0, \qquad m_{D_0} = k.$$

Unlike the previous cases there is no stronger constraint, as there is only one adjacent polyhedron.

1.4. **Degeneration formula analysis.** The blowup morphism $p\colon \mathcal{S} \to S \times \mathbb{A}^1$ satisfies $p^{-1}(E \times \mathbb{A}^1) = \mathcal{E}$. Restricting to the central fibre, we obtain a logarithmic morphism $\mathcal{S}_0 \to (S|E)$. Combined with the line bundle projection $\mathcal{L}_0 \to \mathcal{S}_0$ this induces a pushforward morphism

$$\rho\colon \overline{M}_{g,\mathbf{c},\beta}(\mathcal{L}_0) \to \overline{M}_{g,\mathbf{c},\beta}(S|E).$$

The conservation of number principle [ACGS20, Theorem 1.1] and the decomposition theorem [ACGS20, Theorem 1.2] give the following identity in the Chow homology of $\overline{M}_{g,\mathbf{c},\beta}(S|E)$:

$$[\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}} = \sum_{\tau} \frac{m_\tau}{|\mathrm{Aut}(\tau)|} \rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}}.$$

The sum runs over rigid tropical types τ of tropical stable maps to Σ of type $(g, \mathbf{c}, \beta)$. Here m_τ is the smallest integer such that scaling Σ by m_τ produces a tropical stable map with integral vertices and edge lengths, and ι is the inclusion of the virtual irreducible component $\overline{M}_\tau \hookrightarrow \overline{M}_{g,\mathbf{c},\beta}(\mathcal{L}_0)$. Capping with the D-disjoint insertions γ we obtain:

$$\gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}} = \sum_{\tau} \frac{m_\tau}{|\mathrm{Aut}(\tau)|} \rho_\star \iota_\star \left(\iota^\star \rho^\star \gamma \cap [\overline{M}_\tau]^{\mathrm{virt}} \right). \tag{5}$$

We now use the degeneration formula of [Ran22, Section 6] to describe the terms in the sum. Let Γ denote the source graph of the tropical type τ. There exists a subdivision of the product

$$\bigtimes_{v \in V(\Gamma)} \overline{M}_v \to \prod_{v \in V(\Gamma)} \overline{M}_v$$

and for each edge $e \in E(\Gamma)$ a smooth universal divisor $\mathcal{D}_e \to \bigtimes_v \overline{M}_v$ supporting an evaluation section for each flag $(v \in e)$. We consider the universal doubled divisor

$$\mathcal{D}_e^{\{2\}} := \mathcal{D}_e \times_{\bigtimes_v \overline{M}_v} \mathcal{D}_e.$$

There is a universal diagonal $\Delta\colon \mathcal{D}_e \hookrightarrow \mathcal{D}_e^{\{2\}}$ which is a regular embedding since $\mathcal{D}_e \to \bigtimes_v \overline{M}_v$ is smooth. We obtain a diagram

$$\begin{tikzcd} \overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bigtimes_v \overline{M}_v \arrow[d] \\ & \prod_e \mathcal{D}_e \arrow[r, hook, "\Delta"] & \prod_e \mathcal{D}_e^{\{2\}} \end{tikzcd} \tag{6}$$

with an equality of classes

$$\nu_\star [\overline{M}_\tau]^{\mathrm{virt}} = \Delta^! [\bigtimes_v \overline{M}_v]^{\mathrm{virt}}$$

in the Chow homology of $\overline{N}_\tau$.

1.4.1. *Reduction outline.* We will prove Theorem 1.3 by analysing the contributions of rigid tropical types τ to the degeneration formula (5). For the rest of Section 1.4 we fix a rigid tropical type τ whose contribution to (5) is nontrivial:

$$\rho_\star\iota_\star\left(\iota^\star\rho^\star\gamma\cap[\overline{M}_\tau]^{\mathrm{virt}}\right)\neq 0.$$

We will show that the shape of τ is tightly constrained. We proceed via three reductions:

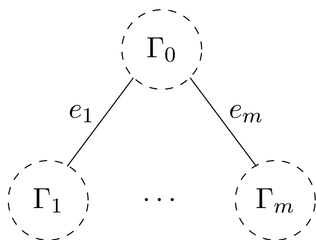
- In Section 1.4.2 we show that τ is weakly star-shaped (Proposition 1.8).
- In Section 1.4.3 we show that τ is star-shaped (Proposition 1.9).
- In Section 1.4.4 we show that τ has a single edge (Proposition 1.19).

Finally in Section 1.4.5 we calculate the contributions of the remaining τ, arriving at Theorem 1.3.

Remark 1.7. As in [BFGW21] we obtain a reduction to star-shaped graphs (Proposition 1.9). In fact we reduce further to single edge graphs (Proposition 1.19), using crucially the fact that D is rational.

1.4.2. *First reduction: weakly star-shaped graphs.*

Proposition 1.8 (First reduction)**.** *The tropical type τ is* weakly star-shaped *in the following sense. The source graph Γ decomposes into subgraphs*



with the vertices of Γ_0 mapping to P, E_1, E_2 and the vertices of $\Gamma_1,\ldots,\Gamma_m$ mapping to S, E, D, q_1, q_2.

Proof. Let $\Gamma'\subseteq\Gamma$ be a maximal connected subgraph contained in the union of the following strata

$$S, E, D_0, q_1, q_2. \tag{7}$$

Let $E'\subseteq E(\Gamma)$ denote the set of edges connecting Γ' to the rest of Γ. It is sufficient to show that $|E'|=1$. The following argument parallels [vGGR19, Lemma 3.1].

The line bundle $\mathcal{L}_0$ is trivial when restricted to each of the strata in (7). It follows from the definition of Γ' that for each $e\in E'$ the corresponding universal divisor $\mathcal{D}_e$ decomposes as

$$\mathcal{D}_e=\overline{\mathcal{D}}_e\times\mathbb{A}^1$$

where $\overline{\mathcal{D}}_e$ is the universal divisor for the compact degeneration $\mathcal{S}_0$ (since $\mathcal{L}_0\to\mathcal{S}_0$ is flat and strict, all expansions of $\mathcal{L}_0$ are pulled back from the corresponding expansions of $\mathcal{S}_0$). We define

$$\overline{\mathcal{D}}:=\prod_{e\in E(\Gamma)\setminus E'}\mathcal{D}_e\times\prod_{e\in E'}\overline{\mathcal{D}}_e$$

and note that the product of universal divisors appearing in (6) decomposes as $\Pi_{e\in E(\Gamma)}\mathcal{D}_e=\overline{\mathcal{D}}\times\mathbb{A}^{E'}$. We now examine evaluations at the nodes corresponding to edges $e\in E'$.

Given $e\in E'$ the corresponding node has an adjacent irreducible component C_e which is mapped to P and has positive intersection with the divisor D_0. Since $D^2\geq 0$ in S it follows that the pullback of the line bundle $\mathcal{L}_0|_P=\mathcal{O}_P(-D_\infty)$ to this component has negative degree. We conclude that evaluation of $f|_{C_e}$ at the given node factors through the zero section of $\mathcal{L}_0$.

On the other hand, the subcurve C' corresponding to Γ' is connected and maps to S. Since $\mathcal{L}_0|_S=\mathcal{O}_S$ it follows that this subcurve is mapped to a constant section of the bundle $S\times\mathbb{A}^1\to S$. Therefore the evaluations of $f|_{C'}$ at the nodes corresponding to $e\in E'$ all coincide.

Taken together, we obtain the following cartesian diagram extending (6):

$$\begin{tikzcd}
\overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bigtimes_v \overline{M}_v \arrow[d] \\
\overline{\mathcal{D}} \times \mathbb{A}^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{D}}^{\{2\}} \times \mathbb{A}^1 \arrow[d] \\
\overline{\mathcal{D}} \times \mathbb{A}^{E'} \arrow[r, hook, "\Delta"] & \overline{\mathcal{D}}^{\{2\}} \times \mathbb{A}^{2E'}.
\end{tikzcd}$$

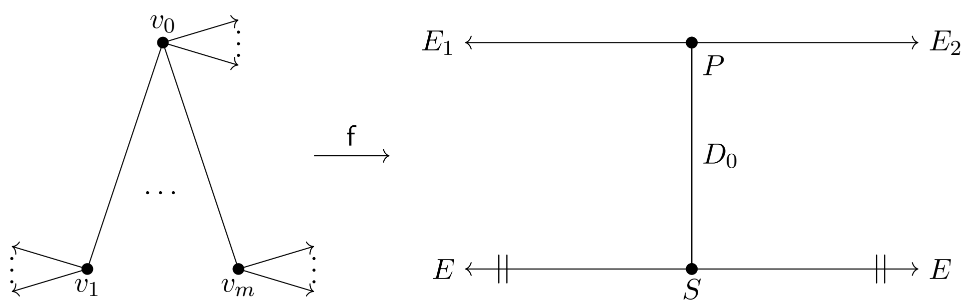
If $|E'| \geq 2$ the excess bundle is a trivial vector bundle of positive rank. Consequently the excess class [Ful98, Theorem 6.3] vanishes, and we obtain

$$\nu_\star[\overline{M}_\tau]^{\mathrm{virt}} = \Delta^![\bigtimes_v \overline{M}_v]^{\mathrm{virt}} = 0.$$

We conclude that $|E'| = 1$. $\square$

1.4.3. *Second reduction: star-shaped graphs.* The next step is to constrain the shape of the subgraphs Γ_i appearing in Proposition 1.8.

Proposition 1.9 (Second reduction)**.** *For $i \in \{0, 1, \ldots, m\}$ the subgraph Γ_i consists of a single vertex v_i. We have $\mathsf{f}(v_0) \in P$ and $\mathsf{f}(v_1), \ldots, \mathsf{f}(v_m) \in S$. The tropical type τ therefore takes the following form:*



The proof proceeds by a sequence of intermediate reductions. We fix a weakly star-shaped tropical type τ as in Proposition 1.8.

Notation 1.10. For each $i \in \{1, \ldots, m\}$ we let $v_i \in V(\Gamma_i)$ and $w_i \in V(\Gamma_0)$ denote the endpoints of e_i.

Proposition 1.11. *Each of the graphs $\Gamma_1, \ldots, \Gamma_m$ is a tree.*

Proof. Recall that an edge of a connected graph is **separating** if deleting it produces a graph with two connected components, and that a graph is a tree if and only if every edge is separating.

Fix $i \in \{1, \ldots, m\}$. We will prove that every edge of Γ_i is separating by inducting on the vertices. At each step we prove that the edges adjacent to the given vertex are separating. We pass to the next step by traversing along all adjacent edges, excluding the vertex we arrived by. The starting point is the vertex v_i which is connected to w_i along the separating edge e_i. When we arrive at a new vertex, there is by induction a path of separating edges connecting it to w_i.

Suppose that we arrive at a vertex $v \in V(\Gamma)$ via a separating edge $\tilde{e}$. Let $\tilde{e}_1, \ldots, \tilde{e}_r$ be the other adjacent edges and suppose for a contradiction that $\tilde{e}_1$ is not separating. Since $\tilde{e}$ is separating, the subgraph behind $\tilde{e}_1$ must coincide (without loss generality) with the subgraph behind $\tilde{e}_2$.

Split the graph Γ at the edges $\tilde{e}_1, \tilde{e}_2$. This produces a new combinatorial type τ_{12} with four open half-edges corresponding to the previously closed edges $\tilde{e}_1, \tilde{e}_2$. For $i \in \{1, 2\}$ let $\mathcal{D}_i$ denote the corresponding universal divisor. As in Section 1.4.2 this decomposes as $\mathcal{D}_i = \overline{\mathcal{D}}_i \times \mathbb{A}^1$ and we have a diagram

$$\begin{tikzcd}
\overline{M}_{\tau} \arrow[r, "\nu"] & \overline{N}_{\tau} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_{12}} \arrow[d] \\
& \prod_{i=1}^{2}(\overline{\mathcal{D}}_i \times \mathbb{A}^1) \arrow[r, hook, "\Delta"] & \prod_{i=1}^{2}(\overline{\mathcal{D}}_i^{\{2\}} \times \mathbb{A}^2).
\end{tikzcd}$$

We now show that the composite $\overline{M}_{\tau_{12}} \to \Pi_{i=1}^{2}(\overline{\mathcal{D}}_i^{\{2\}} \times \mathbb{A}^2) \to \mathbb{A}^4$ factors through the linear subspace:

$$\begin{aligned}\epsilon\colon \mathbb{A}^1 &\hookrightarrow \mathbb{A}^4\\ t &\mapsto (0,t,0,t).\end{aligned}$$

The inductive argument connects the current vertex v to the vertex w_i via a path of separating edges, which are hence distinct from $\tilde{e}_1$ and $\tilde{e}_2$. The line bundle $f^\star\mathcal{L}_0|_{C_{w_i}}$ is negative and so $f|_{C_{w_i}}$ factors through the zero section of $\mathcal{L}_0$. Since C_{w_i} is connected to C_v through nodes which are not split in τ_{12} it follows that $f|_{C_v}$ also factors through the zero section of $\mathcal{L}_0$. This explains the two zero entries. On the other hand, the two t entries occur because the subgraphs behind $\tilde{e}_1$ and $\tilde{e}_2$ coincide. We thus obtain

$$\begin{tikzcd}
\overline{M}_{\tau} \arrow[r, "\nu"] & \overline{N}_{\tau} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tilde{\tau}} \arrow[d] \\
& \left(\prod_{i=1}^{2}\overline{\mathcal{D}}_i\right) \times \mathbb{A}^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \left(\prod_{i=1}^{2}\overline{\mathcal{D}}_i^{\{2\}}\right) \times \mathbb{A}^1 \arrow[d, hook, "\mathrm{Id}\times\epsilon"] \\
& \left(\prod_{i=1}^{2}\overline{\mathcal{D}}_i\right) \times \mathbb{A}^2 \arrow[r, hook, "\Delta"] & \left(\prod_{i=1}^{2}\overline{\mathcal{D}}_i^{\{2\}}\right) \times \mathbb{A}^4.
\end{tikzcd}$$

Since the codimensions of the lower two horizontal arrows differ, the excess bundle is a trivial bundle of rank one, and hence the excess class vanishes. As in the proof of Proposition 1.8, it follows that the contribution of τ vanishes. We conclude that the edges $\tilde{e}_1,\ldots,\tilde{e}_r$ adjacent to v are all separating, and this completes the induction step. □

The arguments now shift from intersection theory to tropical geometry. The background developed in Section 1.3 is essential.

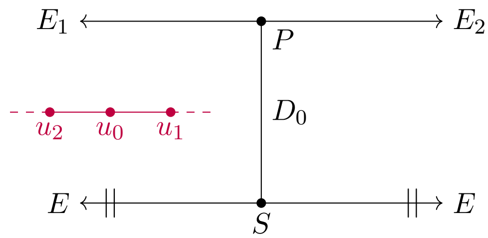
Convention 1.12. By Proposition 1.11, each Γ_i is a tree equipped with a root vertex v_i. This defines a canonical flow starting at v_i. From now on we orient each edge of Γ_i according to this flow. The edge e_i is also oriented from w_i to v_i. In this way, every vertex of Γ_i has a unique incoming edge.

Lemma 1.13. *Let $e \in E(\Gamma_i) \cup \{e_i\}$, oriented as above. Suppose that $\mathsf{f}(e)$ is contained in q_1,q_2, or D_0. Then the vertical slope of e is negative.*

Proof. First we show that the vertical slope of e cannot be positive. Suppose for a contradiction that e has positive vertical slope. Since Γ_i has only a single incoming edge and no outgoing edges, the edge e leads to another vertex of Γ_i. This vertex is also contained in q_1, q_2, or D_0. The balancing condition (Section 1.3.2) ensures that it has an outgoing edge with positive vertical slope. Continuing in this way, we produce a path in Γ_i consistent with the flow and with positive vertical slope along each edge. This path continues indefinitely, a contradiction.

It remains to consider the case where e has zero vertical slope. Let Γ_e denote the subgraph of Γ_i behind the oriented edge e. By the previous paragraph, no edge of Γ_e has positive vertical slope. By balancing, it follows that no edge of Γ_e has negative vertical slope. Therefore every edge of Γ_e has zero vertical slope, and every vertex and edge is mapped to q_1, q_2, or D_0.

If e has zero horizontal slope then it is contracted by the tropical map f, in which case the tropical type is not rigid. If it has non-zero horizontal slope, we may traverse Γ_e using the balancing condition. It is easy to see that we arrive at a configuration of the following form:



Fixing u_1 and u_2 and varying the position of u_0 we see that the tropical type is not rigid. □

Lemma 1.14. *There is no vertex $v \in V(\Gamma_i)$ with $\mathsf{f}(v) \in E$.*

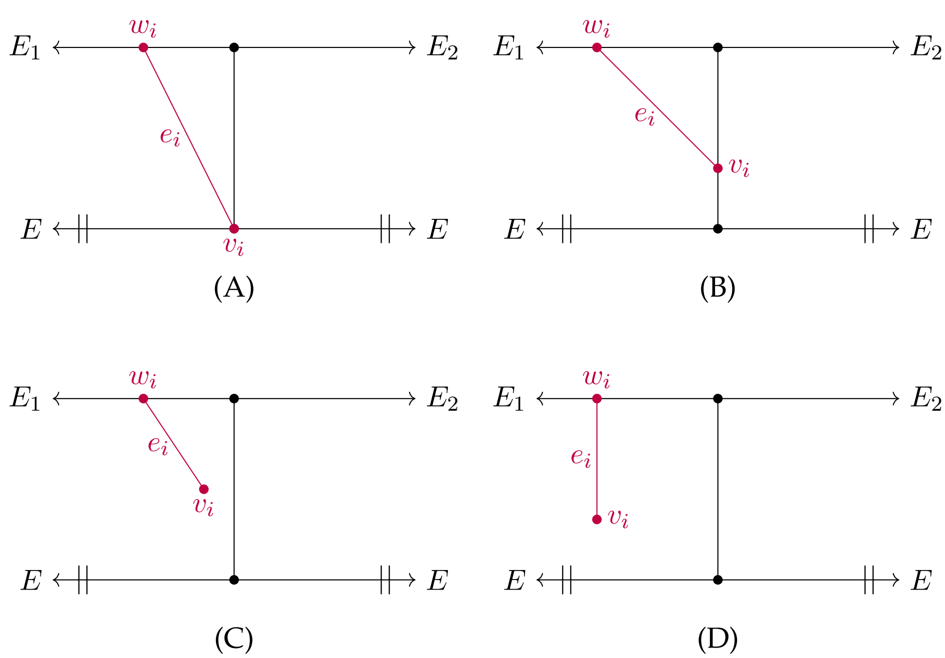
Proof. Suppose for a contradiction that such a vertex v exists. If it has an adjacent edge with non-zero vertical slope, then by balancing (Section 1.3.1) it has at least two adjacent edges with non-zero vertical slope. At most one of these can be the incoming edge, and so at least one is outgoing, i.e. an edge oriented according to the flow with positive vertical slope. This contradicts Lemma 1.13. We conclude that all adjacent edges to v have zero vertical slope. It follows immediately that the tropical type is not rigid. □

Lemma 1.15. *Let $v \in V(\Gamma_i)$ be such that $\mathsf{f}(v)$ belongs to q_1 or q_2. Then the incoming edge at v cannot have positive horizontal slope.*

Proof. Suppose without loss of generality that $\mathsf{f}(v) \in q_1$. By balancing, there must exist an edge outgoing from v with positive horizontal slope. By Lemma 1.13 this edge has negative vertical slope. Follow the edge to the next vertex. If the vertex also belongs to q_1 then we repeat the argument. Eventually, we obtain a vertex w with $\mathsf{f}(w) \in E$. This contradicts Lemma 1.14. □

Lemma 1.16. *We have $\mathsf{f}(w_i) \in P$.*

Proof. It is equivalent to show that $\mathsf{f}(w_i)$ does not belong to E_1 or E_2. Lemmas 1.14 and 1.15 eliminate many cases. The only ones which remain are



along with the matching cases for E_2. We deal with cases (A), (B), (C) together. In each case balancing at w_i (Section 1.3.3) ensures that there exists an outgoing edge with positive horizontal slope. If this edge has zero vertical slope, we follow it to the next vertex and repeat the argument. Eventually, we arrive at a vertex on E_1 supporting an outgoing edge with positive horizontal slope and negative vertical slope. This leaves Γ_0 and enters one of the other subgraphs Γ_j. But Lemmas 1.14 and 1.15 preclude this.

It remains to consider (D). By Lemma 1.13, Lemma 1.15, and balancing, all edges outgoing from v_i must have zero horizontal slope and negative vertical slope. Inducting along the path, we eventually arrive at a vertex on E, contradicting Lemma 1.14. □

Lemma 1.17. *For $i \in \{1, \ldots, m\}$ the graph Γ_i consists only of the vertex v_i, with $\mathsf{f}(v_i) \in S$.*

Proof. We first show $\mathsf{f}(v_i) \in S$. By Lemma 1.14 we have $\mathsf{f}(v_i) \notin E$. On the other hand Lemma 1.15 and 1.16 together imply that $\mathsf{f}(v_i) \notin q_1$ or q_2. It remains to consider the case $\mathsf{f}(v_i) \in D_0$. By Lemma 1.16 we have $\mathsf{f}(w_i) \in P$. Since Γ_i is a tree, it follows in this case that the corresponding tropical type is not rigid. We conclude that $\mathsf{f}(v_i) \in S$.

By Lemma 1.13 the vertex v_i has no outgoing edges with positive vertical slope. By Lemma 1.14 it also has no outgoing edges with positive horizontal slope. It follows that the only outgoing edges have slope zero in both directions. These do not exist because the tropical type is rigid. □

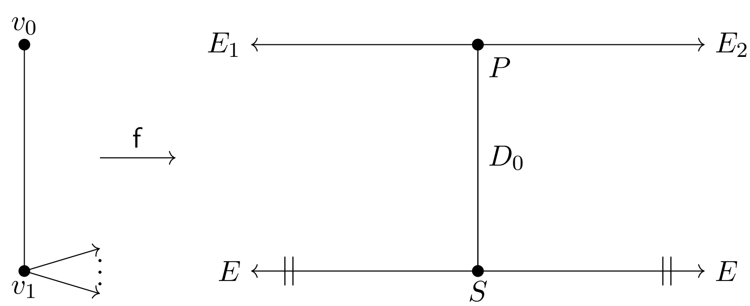
Lemma 1.18. *Γ_0 consists only of a single vertex w_0, with $\mathsf{f}(w_0) \in P$.*

Proof. By Lemma 1.17 we know that each Γ_i consists of a single vertex v_i with $\mathsf{f}(v_i) \in S$ and that v_i is connected to Γ_0 by an edge e_i with $\mathsf{f}(e_i) \subseteq D_0$. From this we see that if Γ_0 contains any vertices along E_1 or E_2 then the tropical type is not rigid. On the other hand if Γ_0 contains more than one vertex over P then it must contain contracted edges, and again the tropical type is not rigid. □

Proof of Proposition 1.9. Combine Lemmas 1.17 and 1.18. □

1.4.4. *Third reduction: single-edge graphs.* By Proposition 1.9 we know that the tropical type τ is star-shaped. We now establish the third and final reduction:

Proposition 1.19 (Third reduction). *Let τ be a star-shaped tropical type, as in Proposition 1.9. Then $m = 1$ and v_1 contains all of the markings:*



Proof. Combine Propositions 1.23 and 1.24 below. □

The proof necessitates a careful analysis of the gluing appearing in the degeneration formula. We begin (Lemma 1.22) by simplifying the contribution of the tropical type τ.

The remarks in [Ran22, Section 6.5.3] apply to our setting, since S is a surface and the source graph Γ has genus zero with each vertex (respectively edge) mapping to a zero-dimensional (respectively one-dimensional) polyhedron of Σ. The following is an immediate consequence.

Lemma 1.20. *There is a map $\nu\colon \overline{M}_\tau \to \overline{N}_\tau$ with target the fibre product over the unexpanded diagonal*

$$\begin{tikzcd} \overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{i=0}^m \overline{M}_{v_i} \arrow[d] \\ & \prod_{i=1}^m (D_0 \times \mathbb{A}^1) \arrow[r, hook, "\Delta"] & \prod_{i=1}^m (D_0 \times \mathbb{A}^1)^2 \end{tikzcd}$$

and an equality of virtual classes

$$\nu_\star[\overline{M}_\tau]^{\mathrm{virt}} = \Delta^! \left(\Pi_{i=0}^m [\overline{M}_{v_i}]^{\mathrm{virt}}\right)$$

in the Chow homology of $\overline{N}_\tau$.

Lemma 1.21. *There exists a gluing morphism $\theta\colon \overline{N}_\tau \to \overline{M}_{g,\mathbf{c},\beta}(S|E)$ such that the following square commutes:*

$$\begin{tikzcd} \overline{M}_\tau \arrow[r, "\nu"] \arrow[d, "\iota"] & \overline{N}_\tau \arrow[d, "\theta"] \\ \overline{M}_{g,\mathbf{c},\beta}(\mathcal{S}_0) \arrow[r, "\rho"] & \overline{M}_{g,\mathbf{c},\beta}(S|E). \end{tikzcd} \tag{8}$$

Proof. There are morphisms of pairs $(S \mid D+E) \to (S|E)$ and $(P \mid E_1 + E_2 + D_0) \to (S|E)$ which induce pushforward morphisms $\overline{M}_{v_i} \to \overline{M}_{g_i,\mathbf{c}_i,\beta_i}(S|E)$ for $i \in \{0,1,\ldots,m\}$. We wish to glue the resulting stable logarithmic maps to $(S|E)$ along the nodes corresponding to $e_1,\ldots,e_m$.

The key observation is that these nodes have tangency order 0 with respect to E. Consequently, when the $m+1$ tropical maps are glued, the new finite edges are contracted by the tropical map. As such their edge lengths constitute free parameters. We conclude that there is a unique choice of logarithmic enhancement of the glued curve, and a unique logarithmic map to $(S|E)$. $\square$

Combining Lemma 1.20 and Lemma 1.21 we obtain

$$\rho_\star\iota_\star[\overline{M}_\tau]^{\mathrm{virt}} = \theta_\star\Delta^! \left(\Pi_{i=0}^m [\overline{M}_{v_i}]^{\mathrm{virt}}\right). \tag{9}$$

Recall from Proposition 1.9 that $v_0 \in V(\Gamma)$ is the unique vertex mapping to P. Compose the corresponding stable map $C_{v_0} \to \mathcal{L}_0$ with the line bundle projection $\mathcal{L}_0 \to P$ to obtain a map $C_{v_0} \to P$. Since $D^2 \geq 0$ (Section 1.1) this has positive intersection with the bundle $\mathcal{O}_P(D_\infty)$. Since $\mathcal{L}_0|_P = \mathcal{O}_P(-D_\infty)$ (Section 1.2) it follows that $C_{v_0} \to \mathcal{L}_0$ factors through the zero section, so the evaluation map

$$\overline{M}_{v_0} \xrightarrow{\mathrm{ev}_0} \prod_{i=1}^m (D_0 \times \mathbb{A}^1)$$

admits a factorisation through a codimension-m subvariety:

$$\begin{tikzcd} \overline{M}_{v_0} \arrow[r, "\overline{\mathrm{ev}}_0"] \arrow[rr, bend left, "\mathrm{ev}_0"] & \prod_{i=1}^m D_0 \arrow[r, hook, "\epsilon"] & \prod_{i=1}^m (D_0 \times \mathbb{A}^1). \end{tikzcd}$$

For $i \in \{1,\ldots,m\}$ the moduli space $\overline{M}_{v_i}$ parametrises logarithmic maps to $\mathcal{L}_0|_S$ with tangency along the divisors $D+E$. If τ_i denotes the star of v_i in τ then there is a closed embedding

$$\overline{M}_{\tau_i}(S \mid D+E) \hookrightarrow \overline{M}_{v_i}$$

parametrising logarithmic maps which factor through the zero section. Letting δ denote the diagonal $\Pi_{i=1}^m D_0 \hookrightarrow \Pi_{i=1}^m D_0^2$ we obtain the following diagram

$$
\begin{tikzcd}
\overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{v_0} \times \prod_{i=1}^m \overline{M}_{\tau_i}(S\,|\,D+E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{v_0} \times \prod_{i=1}^m \overline{M}_{v_i} \arrow[d, "\overline{\mathrm{ev}}_0 \times \Pi_{i=1}^m \mathrm{ev}_i"] \\
\prod_{i=1}^m D_0 \arrow[r, "\delta"] \arrow[d] \arrow[drr, phantom, "\square"] & \prod_{i=1}^m D_0 \times \prod_{i=1}^m D_0 \arrow[r, "\mathrm{Id}\times\epsilon"] & \prod_{i=1}^m D_0 \times \prod_{i=1}^m (D_0 \times \mathbb{A}^1) \arrow[d, "\epsilon\times\mathrm{Id}"] \\
\prod_{i=1}^m (D_0 \times \mathbb{A}^1) \arrow[rr, "\Delta"] & & \prod_{i=1}^m (D_0 \times \mathbb{A}^1)^2.
\end{tikzcd} \tag{10}
$$

The following explains the appearance of lambda classes in Theorem 1.3.

Lemma 1.22. *Given τ be a star-shaped tropical type as in Proposition 1.9 we have*

$$\rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}} = \theta_\star \delta^! \left([\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}} \right).$$

Proof. From (10) we obtain

$$\Delta^! \left(\Pi_{i=0}^m [\overline{M}_{v_i}]^{\mathrm{virt}} \right) = \delta^! (\mathrm{Id} \times \epsilon)^! \left(\Pi_{i=0}^m [\overline{M}_{v_i}]^{\mathrm{virt}} \right) = \delta^! \left([\overline{M}_{v_0}]^{\mathrm{virt}} \times \epsilon^! \left(\Pi_{i=1}^m [\overline{M}_{v_i}]^{\mathrm{virt}} \right) \right)$$

where the first equality follows from [Ful98, Theorems 6.2(c) and 6.5] and the second equality follows from [Ful98, Example 6.5.2]. Since $\mathcal{L}_0$ trivialises over the irreducible component $S \subseteq \mathcal{S}_0$ we have

$$\overline{M}_{v_i} = \overline{M}_{\tau_i}(S\,|\,D+E) \times \mathbb{A}^1$$

for $i \in \{1, \ldots, m\}$. A direct comparison of obstruction theories then produces

$$\epsilon^! \left(\Pi_{i=1}^m [\overline{M}_{v_i}]^{\mathrm{virt}} \right) = \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}$$

which we combine with (9) to obtain the result. □

The proof of Proposition 1.19 proceeds in two steps: we show first that v_0 contains no markings (Proposition 1.23), and then that $m = 1$ (Proposition 1.24). Recall that the insertions

$$\gamma = \prod_{i=1}^c \mathrm{ev}_{z_i}^\star(\gamma_i) \psi_{z_i}^{k_i}$$

are D-disjoint. The following vanishing result involves a dimension count, for which it is crucial that the codimension of γ coincides with the virtual dimension of $\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)$.

For $i \in \{0, 1, \ldots, m\}$ let $I(i) \subseteq \{1, \ldots, c\}$ denote the set of interior markings at the vertex v_i and define the class

$$\gamma_i \coloneqq \prod_{j \in I(i)} \mathrm{ev}_{z_j}^\star(\gamma_j) \psi_{z_j}^{k_j}$$

so that $\gamma = \Pi_{i=0}^m \gamma_i$.

Proposition 1.23. *Let τ be a star-shaped tropical type as in Proposition 1.9. The contribution*

$$\gamma \cap \rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}}$$

to the Gromov–Witten invariant vanishes unless v_0 carries no markings. In this case, we have the following equality in $A_0(\overline{M}_{g,\mathbf{c},\beta}(S|E))$:

$$\gamma \cap \rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}} = \theta_\star \delta^! \left([\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}} \right). \tag{11}$$

Proof. It is sufficient to prove the vanishing result. We consider separately the case of an interior marking and a marking with tangency along E. First suppose that v_0 carries an interior marking. Choose regularly embedded subvarieties $Z_1, \ldots, Z_c \subseteq S$ with $[Z_i] = \gamma_i$ and $Z_i \cap D = \emptyset$. Let

$$\mathcal{Z}_i \hookrightarrow \mathcal{S} \tag{12}$$

denote the strict transform of $Z_i \times \mathbb{A}^1 \hookrightarrow S \times \mathbb{A}^1$. Since $Z_i \cap D = \emptyset$ it follows that on the central fibre the inclusion (12) factors through the irreducible component S:

$$Z_i \hookrightarrow S \hookrightarrow \mathcal{S}_0.$$

Set $Z = \Pi_{i=1}^c Z_i$ and combine the above morphisms into a regular embedding

$$\eta_Z \colon Z \hookrightarrow S^c.$$

Similarly, for $i \in \{0, \ldots, m\}$ we denote the corresponding inclusion

$$\eta_i \colon Z^{(i)} = \Pi_{j \in I(i)} Z_j \hookrightarrow S^{I(i)}.$$

Recall the gluing morphism θ constructed in Lemma 1.21. We perform the pullback along ι_Z to produce closed substacks with constrained evaluation:

$$\begin{tikzcd}
\overline{N}_\tau|_Z \arrow[r, hook] \arrow[d, "\theta_Z"] \arrow[dr, phantom, "\square"] & \overline{N}_\tau \arrow[d, "\theta"] \\
\overline{M}_{g,\mathbf{c},\beta}(S|E)|_Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,\mathbf{c},\beta}(S|E) \arrow[d, "\mathrm{ev}"] \\
Z \arrow[r, hook, "\eta_Z"] & S^c.
\end{tikzcd}$$

Applying $\eta_Z^!$ to Lemma 1.22 we obtain

$$\begin{aligned}
\eta_Z^! \rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}} &= \theta_{Z\star} \delta^! \eta_Z^! \left([\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}(S \,|\, D+E)]^{\mathrm{virt}} \right) \\
&= \theta_{Z\star} \delta^! \left(\eta_0^! [\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \cap \eta_i^! [\overline{M}_{\tau_i}(S \,|\, D+E)]^{\mathrm{virt}} \right)
\end{aligned} \tag{13}$$

where the first equality follows from [Ful98, Theorem 6.2(a) and 6.4] and the second from [Ful98, Example 6.5.2 and Proposition 6.3].

If v_0 carries an interior marking z_i then the associated evaluation map factors through $D \subseteq S$. Since $Z_i \cap D = \emptyset$ it follows that the fibre product

$$\overline{M}_{v_0} \times_{S^{I(0)}} Z^{(0)}$$

is empty. Therefore $\eta_0^! [\overline{M}_{v_0}]^{\mathrm{virt}} = 0$ and by (13) the contribution vanishes. (This is the only point in the paper where we use the assumption of D-disjoint insertions.) We conclude that v_0 contains no interior markings.

Now suppose that v_0 contains a marking with tangency along E. Pushing forward (13) to $\overline{M}_{g,\mathbf{c},\beta}(S|E)$ and capping with the psi classes appearing in the insertions γ produces:

$$\gamma \cap \rho_\star \iota_\star [\overline{M}_\tau]^{\mathrm{virt}} = \theta_\star \delta^! \left([\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \gamma_i \cap [\overline{M}_{\tau_i}(S \,|\, D+E)]^{\mathrm{virt}} \right).$$

We now perform a dimension count. For $i \in \{0, 1, \ldots, m\}$ let $c_i := |I(i)|$ denote the number of interior markings contained in v_i, and let b_i denote the number of markings y_j with tangency along E

contained in v_i. We have $c_0 = 0$ by the previous arguments. The virtual dimension of $\overline{M}_{\tau_i}(S\,|\,D+E)$ is $-\beta_i \cdot (K_S + D + E) + g_i + b_i + c_i$. It follows that the class

$$\prod_{i=1}^{m} (-1)^{g_i} \lambda_{g_i} \gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}} \tag{14}$$

has dimension:

$$\begin{aligned}
&\sum_{i=1}^{m} \left(-\beta_i \cdot (K_S + D + E) + b_i + c_i - \Sigma_{j \in I(i)} (k_i + \mathrm{codim}(Z_i, S)) \right) \\
&= \pi_\star \beta_0 \cdot (K_S + D + E) - b_0 + (-\beta \cdot (K_S + D + E) + b + c - \Sigma_{i=1}^{c} (k_i + \mathrm{codim}(Z_i, S))) \\
&= \pi_\star \beta_0 \cdot (K_S + D + E) - b_0. \qquad (15)
\end{aligned}$$

Here $\pi\colon P \to S$ is the bundle projection and $\beta_0 \in A_1(P)$ is the curve class attached to v_0. The first equality follows from

$$\beta = \pi_\star \beta_0 + \sum_{i=1}^{m} \beta_i, \qquad b = \sum_{i=0}^{m} b_i\,, \qquad c = \sum_{i=1}^{m} c_i$$

while the second equality holds because the assumption that the codimension of γ is equal to the virtual dimension of $\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)$.

The pushforward $\pi_\star \beta_0 \in A_1(D)$ is necessarily a multiple of the fundamental class, say $\pi_\star \beta_0 = kD$ for some $k \geq 0$. By adjunction and $D \cdot E = 2$ we obtain

$$\pi_\star \beta_0 \cdot (K_S + D + E) = kD \cdot (K_S + D) + 2k = 2kg(D) = 0.$$

(This is the first of two points in the argument where we use the assumption that D is rational.) By (15) this implies that the cycle (14) has dimension $-b_0$. The contribution therefore vanishes unless $b_0 = 0$. $\square$

Since v_0 carries no markings and the nodes $q_1, \ldots, q_m$ have no tangency with respect to E_1 and E_2, the balancing condition determines the curve class β_0 as

$$\beta_0 = (D \cdot \beta) F$$

where $F \in A_1(P)$ is the class of a fibre. Consequently the evaluation morphism $\overline{M}_{v_0} \to \prod_{i=1}^{m} D_0$ factors through the small diagonal $D_0 \hookrightarrow \Pi_{i=1}^{m} D_0$. Let

$$\psi : \overline{M}_{v_0} \to \overline{M}_{g_0,m} \times D_0$$

denote the morphism remembering only the stabilised source curve and the evaluation. We obtain a diagram

$$\begin{tikzcd}
\overline{M}_{g,\mathbf{c},\beta}(S|E) & \overline{N}_\tau \arrow[l, "\theta"'] \arrow[r] \arrow[d] & \overline{M}_{v_0} \times \prod_{i=1}^{m} \overline{M}_{\tau_i}(S\,|\,D+E) \arrow[d, "\psi \times \mathrm{Id}"] \\
& \overline{M}_{g_0,m} \times \overline{L}_\tau \arrow[ul, "\phi"] \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & \overline{M}_{g_0,m} \times D_0 \times \prod_{i=1}^{m} \overline{M}_{\tau_i}(S\,|\,D+E) \arrow[d] \\
& \overline{L}_\tau \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & D_0 \times \prod_{i=1}^{m} \overline{M}_{\tau_i}(S\,|\,D+E) \arrow[d] \\
& \prod_{i=1}^{m} D_0 \arrow[r, "\delta"] \arrow[ur, phantom, "\square"] & \prod_{i=1}^{m} D_0^2
\end{tikzcd}$$

in which $\overline{L}_\tau$ is defined via the bottom cartesian square.[3] It parametrises logarithmic maps $f_i\colon C_i \to (S\,|\,D+E)$ for $i \in \{1,\ldots,m\}$, such that $f_1(q_1) = \ldots = f_m(q_m)$. The gluing morphism ϕ is defined similarly to θ (Lemma 1.21).

For $g, d \geq 0$ consider the moduli space

$$\overline{M}_{g,(d),d}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)$$

of stable logarithmic maps with maximal tangency at a single marking, and consider the pushforward of its virtual fundamental class to the moduli space of curves:

$$C_{g,d} \in A_{2g-2}(\overline{M}_{g,1}).$$

We obtain the following simplified expression of the contribution of τ.

Proposition 1.24. *Let τ be a star-shaped tropical type as in Proposition 1.9. The contribution $\gamma \cap \rho_\star\iota_\star[\overline{M}_\tau]^{\mathrm{virt}}$ vanishes unless $m=1$. In this case $\overline{L}_\tau = \overline{M}_{\tau_1}(S\,|\,D+E)$ and we have*

$$\gamma \cap \rho_\star\iota_\star[\overline{M}_\tau]^{\mathrm{virt}} = (-1)^g \phi_\star\left(\left(\lambda_{g_{v_0}} \cap C_{g_0,D\cdot\beta}\right) \times \left(\lambda_{g_1}\gamma \cap [\overline{M}_{\tau_1}(S\,|\,D+E)]^{\mathrm{virt}}\right)\right). \tag{16}$$

Proof. Using the compatibility of proper push forward and outer product [Ful98, Proposition 1.10], the identity (11) simplifies to

$$\gamma \cap \rho_\star\iota_\star[\overline{M}_\tau]^{\mathrm{virt}} = \phi_\star\delta^!\left(\psi_\star[\overline{M}_{v_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i}\lambda_{g_i}\gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right). \tag{17}$$

Since D_0 is a rational curve, [Ful98, Example 1.10.2] ensures that the outer product morphism

$$A_\star(\overline{M}_{g_0,m}) \otimes A_\star(D_0) \to A_\star(\overline{M}_{g_0,m} \times D_0)$$

is surjective (this is the second and final point in the argument where we use the assumption that D is rational). We may thus write

$$\psi_\star[\overline{M}_{v_0}]^{\mathrm{virt}} = (A \times [D_0]) + (B \times [\mathrm{pt}]) \tag{18}$$

where $A, B \in A_\star(\overline{M}_{g_0,m})$ and $[\mathrm{pt}]$ is the generator of $A_0(D_0)$. Starting with the second term, we have by [Ful98, Example 6.5.2]:

$$\delta^!\left(B \times [\mathrm{pt}] \times \prod_{i=1}^m (-1)^{g_i}\lambda_{g_i}\gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right) = B \times \delta^!\left([\mathrm{pt}] \times \prod_{i=1}^m (-1)^{g_i}\lambda_{g_i}\gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right).$$

A dimension count similar to (15) then shows that

$$\delta^!\left([\mathrm{pt}] \times \prod_{i=1}^m (-1)^{g_i}\lambda_{g_i}\gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right) \in A_{-m}(\overline{L}_\tau).$$

We always have $m \geq 1$ and hence this contribution always vanishes. Considering the second term of (18), a similar analysis shows that

$$\delta^!\left([D_0] \times \prod_{i=1}^m (-1)^{g_i}\lambda_{g_i}\gamma_i \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right) \in A_{1-m}(\overline{L}_\tau)$$

and so the contribution vanishes unless $m=1$. This proves the vanishing statement.

Now suppose that $m=1$. We have:

$$\gamma \cap \rho_\star\iota_\star[\overline{M}_\tau]^{\mathrm{virt}} = \phi_\star\left(A \times \delta^!\left([D_0] \times (-1)^{g_1}\lambda_{g_1}\gamma \cap [\overline{M}_{\tau_1}(S\,|\,D+E)]^{\mathrm{virt}}\right)\right). \tag{19}$$

[3] If $g_0 = 0$ and $m < 3$ we adopt the convention $\overline{M}_{0,1} = \overline{M}_{0,2} = \operatorname{Spec}\Bbbk$.

It remains to determine the cycle A. Recall that $\psi_\star[\overline{M}_{v_0}]^{\mathrm{virt}} = A \times [D_0] + (B \times [\mathrm{pt}])$. Choose a closed point $\xi\colon \operatorname{Spec}\Bbbk \hookrightarrow D_0$ and form the fibre product:

$$\begin{tikzcd}
\overline{M}_{g_{v_0},(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \overline{M}_{v_0} \arrow[d, "\psi"] \\
\overline{M}_{g_{v_0},1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_{v_0},1} \times D_0 \arrow[d] \\
\operatorname{Spec}\Bbbk \arrow[r, hook, "\xi"] & D_0.
\end{tikzcd}$$

We then obtain

$$A = \xi^! \psi_\star [\overline{M}_{v_0}]^{\mathrm{virt}} = \pi_\star \xi^! [\overline{M}_{v_0}]^{\mathrm{virt}} = \pi_\star\left((-1)^{g_0}\lambda_{g_0} \cap [\overline{M}_{g_0,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)]^{\mathrm{virt}}\right) = (-1)^{g_0}\lambda_{g_0} \cap C_{g_0,D\cdot\beta}$$

where in the final step we use the fact that lambda classes are preserved by pullback along stabilisation morphisms. The identity (19) then immediately implies (16). ☐

As an immediate consequence of Proposition 1.24 we obtain:

Theorem 1.25. *The following identity holds in* $A_0(\overline{M}_{g,\mathbf{c},\beta}(S|E))$

$$\gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}} = (-1)^g (D\cdot\beta) \sum_{g_1+g_2=g} \phi_\star\left((\lambda_{g_1} \cap C_{g_1,D\cdot\beta}) \times (\lambda_{g_2}\gamma \cap [\overline{M}_{g_2,\hat{\mathbf{c}},\beta}(S\,|\,D+E)]^{\mathrm{virt}})\right)$$

where the tangency data $\hat{\mathbf{c}}$ *is obtained from* $\mathbf{c}$ *by introducing one additional marked point with maximal tangency along* D *(see Section 1.1).*

1.4.5. *Evaluating the integrals.* Finally, we show that the cycle-theoretic Theorem 1.25 implies the numerical Theorem 1.3.

Proof of Theorem 1.3. Push the formula in Theorem 1.25 forward to a point, and form the generating function by summing over g. We obtain:

$$\sum_{g\geq 0} \mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)\langle\gamma\rangle \cdot \hbar^{2g-2} = (D\cdot\beta)\left(\sum_{g_1\geq 0}\mathsf{GW}_{g_1,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)\langle(-1)^{g_1}\lambda_{g_1}\rangle \cdot \hbar^{2g_1-1}\right)$$
$$\left(\sum_{g_2\geq 0}\mathsf{GW}_{g_2,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle(-1)^{g_2}\lambda_{g_2}\gamma\rangle \cdot \hbar^{2g_2-1}\right).$$

The theorem follows immediately from [BP08, Lemma 6.3] which gives:

$$\sum_{g_1\geq 0}\mathsf{GW}_{g_1,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)\langle(-1)^{g_1}\lambda_{g_1}\rangle \cdot \hbar^{2g_1-1} = \left(\frac{1}{D\cdot\beta}\right)\frac{(-1)^{D\cdot\beta+1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}.$$ ☐

1.5. **Nef pairs.** In this section, we adopt the setup of Section 1.1 and impose the following additional assumptions:

- $E\cdot\beta > 0$.
- $E^2 \geq 0$.

This includes nef Looijenga pairs as studied in [BBvG20]. The assumptions ensure that the local theory of $\mathcal{O}_S(-E)$ is well-defined, see Section 1.1 for the analogous argument for D.

We consider stable logarithmic maps with two markings of maximal tangency to D and E and possibly additional interior markings. Denote this contact data by $\hat{\mathbf{c}}$ and by $\mathbf{c}$ the result of deleting the two tangency markings. We obtain the following correspondence in genus zero:

Theorem 1.26 (Theorem B)**.** *The following identity holds between the genus zero Gromov–Witten invariants with insertions γ avoiding D:*

$$\mathsf{GW}_{0,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle\gamma\rangle = (-1)^{(D+E)\cdot\beta}(D\cdot\beta)(E\cdot\beta)\cdot\mathsf{GW}_{0,c,\beta}(\mathcal{O}_S(-D)\oplus\mathcal{O}_S(-E))\langle\gamma\rangle.$$

Proof. Take Theorem 1.3 in genus zero and apply the logarithmic-local correspondence for smooth pairs [vGGR19, Theorem 1.1]. □

This extends [BBvG20, Theorem 5.2], which gives the above result when $(S\,|\,D+E)$ is logarithmically Calabi–Yau and has stationary insertions supported at a single interior marking.

Remark 1.27. A higher genus analogue of Theorem 1.26 may be obtained by combining Theorem 1.3 with [BFGW21, Theorem 2.7]. The graph sum simplifies, since the logarithmic invariants carry an insertion of λ_g^2 which vanishes unless $g=0$. The resulting correspondence will thus have limited use, as it compares the genus zero invariants of $(S\,|\,D+E)$ to the higher genus invariants of the local geometry. We leave a detailed analysis to future work.

1.6. **Root stacks and self-nodal pairs.** In this section we adopt the setup of Section 1.1 with the following modifications, which apply only to this section:

- **Stricter:** $g=0$ and $E\cdot\beta=0$.
- **Looser:** D is no longer required to be rational, insertions γ are no longer required to avoid D.

Since $E\cdot\beta=0$ there are no markings with tangency along E. A case of particular interest is resolutions of irreducible self-nodal curves, where β is a curve class pulled back along the blowup.

In this setting, we establish the analogue of Theorem 1.3.

Theorem 1.28. *The following identity holds between the genus zero Gromov–Witten invariants:*

$$\mathsf{GW}_{0,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle\gamma\rangle = (-1)^{D\cdot\beta+1}(D\cdot\beta)\cdot\mathsf{GW}_{0,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)\langle\gamma\rangle.$$

Instead of using the degeneration formula, the result is proved via the enumerative geometry of root stacks. We pass through the following correspondences of genus zero Gromov–Witten theories:

$$\mathsf{Log}(S\,|\,D+E) \overset{[\text{BNR22}]}{\dashleftarrow\dashrightarrow} \mathsf{Orb}(S\,|\,D+E) \overset{[\text{BNTY23}]}{\longleftrightarrow} \mathsf{Orb}(\mathcal{O}_S(-D)\,|\,E) \overset{[\text{ACW17}]}{\longleftrightarrow} \mathsf{Log}(\mathcal{O}_S(-D)\,|\,E).$$

The second correspondence follows from [BNTY23, Theorem 1.2] (the result is stated for D nef, but in fact β-nef is all that is required in the proof). The third correspondence follows from [ACW17, Theorem 1.1]. Only the first correspondence requires further justification. Theorem 1.28 thus reduces to the following:

Proposition 1.29 (Theorem C)**.** *There is an equality of logarithmic and orbifold Gromov–Witten invariants:*

$$\mathsf{GW}^{\mathsf{log}}_{0,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle\gamma\rangle = \mathsf{GW}^{\mathsf{orb}}_{0,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle\gamma\rangle.$$

Proof. Let Σ be the tropicalisation of $(S\,|\,D+E)$. A kaleidoscopic double cover is depicted in Figure 2. Given a tropical type of map to Σ, the balancing conditions of Sections 1.3.1 and 1.3.2 apply verbatim.

By [BNR22, Theorem X] it is sufficient to show that $(S\,|\,D+E)$ is slope-sensitive with respect to the given numerical data [BNR22, Section 4.1]. Fix a naive type of tropical map to Σ as in [BNR22, Section 3]. We must show that there is no oriented edge $\vec{e}$ of the source graph Γ such that the associated cone $\sigma_e\in\Sigma$ is maximal and the slope $m_{\vec{e}}\in N_{\sigma_e}$ belongs to the positive quadrant.

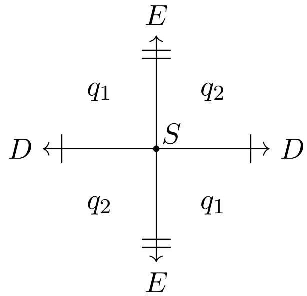


FIGURE 2. Representing Σ as a quotient of a double cover.

We begin with a useful construction. Given an oriented edge $\vec{e} \in \vec{E}(\Gamma)$ terminating at a vertex $v \in V(\Gamma)$ with $\sigma_e \in \{q_1, q_2\}$ and $\sigma_v \in \{q_1, q_2, D\}$, we let

$$\Gamma(\vec{e}) \subseteq \Gamma$$

denote the maximal connected subgraph which contains $\overleftarrow{e}$ as an outgoing half-edge and is such that all vertices and half-edges have associated cones q_1, q_2, or D. This means that all outgoing half-edges besides $\overleftarrow{e}$ are either unbounded marking legs, or finite edges terminating at E or S (the vertical dividing line in Figure 2).

Now suppose for a contradiction that there exists an oriented edge $\vec{e}_1 \in \vec{E}(\Gamma)$ such that $\sigma_{e_1} = q_1$ and $m_{\vec{e}_1} \in N_{q_1}$ belongs to the positive quadrant. The assumption on the slope ensures that the subgraph $\Gamma(\vec{e}_1)$ is well-defined. We claim that $\Gamma(\vec{e}_1)$ contains the (unique) marking leg with positive tangency to D. Since $D^2 \geq 0$ it follows by balancing (Section 1.3.2) that at every vertex of $\Gamma(\vec{e}_1)$ the sum of the outgoing slopes in the D-direction is non-negative. Summing over all the vertices, we see that the sum of outgoing slopes from $\Gamma(\vec{e}_1)$ in the D-direction is non-negative. The slope of $\overleftarrow{e}_1$ in the D-direction is negative, hence there exists an outgoing edge whose slope in the D-direction is positive. Such an edge cannot terminate at E or S and so, by the definition of $\Gamma(\vec{e}_1)$, it must be the marking leg with positive tangency to D.

A similar argument shows that $\Gamma(\vec{e}_1)$ also has an outgoing edge with positive slope in the E-direction. Since $E \cdot \beta = 0$ there are no marking legs with tangency to E, and so this must be a finite edge terminating at a vertex v_0 with $\sigma_{v_0} = E$. By balancing (Section 1.3.1) we see that v_0 supports an outgoing edge $\vec{e}_2$ with $\sigma_{e_2} = q_2$.

The same argument as above now shows that $\Gamma(\vec{e}_2)$ also contains the marking leg with positive tangency to D. However, $\Gamma(\vec{e}_1)$ and $\Gamma(\vec{e}_2)$ are disjoint: deleting the vertex v_0 separates them, since Γ has genus zero. $\square$

Remark 1.30. While restricted to genus zero, Theorem 1.28 is strong in that it establishes an equality of virtual fundamental classes. This contrasts with Theorem 1.3 which also establishes an equality of Chow classes, but only after capping with suitable insertions. In the latter case, we expect that even in genus zero, the counterexamples of [NR22, Sections 1 and 3.7] can be adapted to produce pathological insertions (such as naked psi classes) violating the correspondence.

2. TORIC AND OPEN GEOMETRIES

In this section we restrict to toric targets. The main result (Theorem 2.3) equates the Gromov–Witten theories of the local and open geometries:

$$(\mathcal{O}_S(-D) \mid E) \leftrightarrow \mathcal{O}_S(-D)|_{S \setminus E}.$$

The proof proceeds via torus localisation. For the open geometry, the computation is controlled by the topological vertex [LLLZ09]. The difficult step is to show that the contributions of certain localisation graphs vanish.

2.1. **Setup.** We retain the setup of Section 1.1 and introduce the following additional assumptions:

- S is a toric surface.
- E is a toric hypersurface.
- $D + E \in |-K_S|$.
- $E \cdot \beta = 0$.

We do not require that D is toric. An important example is the resolution of an irreducible self-nodal cubic in the plane (Section 3).

With these assumptions, the enumerative setup of Section 1.1 specialises. There is a single marked point x with tangency $D \cdot \beta$ along D, and no marked points with tangency along E. The space of stable logarithmic maps has virtual dimension g and we consider the Gromov–Witten invariant with a lambda class and no additional insertions:

$$\mathsf{GW}_{g,(D\cdot\beta,0),\beta}(S\,|\,D+E)\langle(-1)^g\lambda_g\rangle := (-1)^g\lambda_g \cap [\overline{M}_{g,(D\cdot\beta,0),\beta}(S\,|\,D+E)]^{\mathrm{virt}} \in \mathbb{Q}.$$

The space of stable logarithmic maps to the local target $(\mathcal{O}_S(-D)\,|\,E)$ has virtual dimension zero and we consider the Gromov–Witten invariant with no insertions:

$$\mathsf{GW}_{g,0,\beta}(\mathcal{O}_S(-D)\,|\,E) := [\overline{M}_{g,0,\beta}(\mathcal{O}_S(-D)\,|\,E)]^{\mathrm{virt}} \in \mathbb{Q}.$$

Theorem 1.3 furnishes a correspondence between these invariants. In this section we relate the latter to the invariants of the open target $\mathcal{O}_S(-D)|_{S\setminus E}$.

2.2. **Open invariants.** We establish conventions for toric geometry. We consider fans Σ not contained in a proper linear subspace of the ambient lattice; these correspond to toric varieties with no torus factors. We write $\Sigma(k)$ for the set of k-dimensional cones. Letting n denote the dimension of the ambient lattice, we introduce notation for closed toric strata appearing in critical dimensions:

- For $\rho \in \Sigma(1)$ we denote the corresponding toric hypersurface D_ρ.
- For $\tau \in \Sigma(n-1)$ we denote the corresponding toric curve L_τ.
- For $\sigma \in \Sigma(n)$ we denote the corresponding torus-fixed point P_σ.

Write Σ_S for the fan of S and $\rho_E \in \Sigma_S(1)$ for the cone corresponding to E. Since $D + E \in |-K_S|$ we have the following identity in the class group of S:

$$D = \sum_{\substack{\rho\in\Sigma_S(1)\\ \rho\neq\rho_E}} D_\rho. \tag{20}$$

Set $X := \mathcal{O}_S(-D)$ and consider the open subvariety:

$$X^\circ := \mathcal{O}_S(-D)|_{S\setminus E}.$$

Equip X° with the trivial logarithmic structure and X with the logarithmic structure induced by E. The open embedding $\iota\colon X^\circ \hookrightarrow X$ is strict.

Following the formalism of the topological vertex [LLLZ09] we define Gromov–Witten invariants of the open manifold X° by localising with respect to an appropriate torus.

Definition 2.1 ([LLLZ09, Section 3.1])**.** Let $P \in X^\circ$ be a torus-fixed point. Consider the action of the three-dimensional dense torus on $\bigwedge^3 T_P X^\circ$ and let χ_P denote the associated character. The **Calabi–Yau torus** is denoted and defined

$$T := \operatorname{Ker} \chi_P.$$

It is a two-dimensional subtorus of the dense torus. The definition is independent of the choice of P because X° is Calabi–Yau.

While the moduli space of stable maps to X° is non-proper, its T-fixed locus is proper. This is used to define Gromov–Witten invariants, via localisation. Let Q_T denote the localisation of $A^\star_T(\mathrm{pt})$ at the set of homogeneous elements of non-zero degree, and let $Q_{T,k}$ denote its kth graded piece.

Definition 2.2. The **T-localised Gromov–Witten invariant** of X° is denoted and defined:

$$\mathsf{GW}^T_{g,0,\iota^\star\beta}(X^\circ) \coloneqq \int_{[\overline{M}_{g,0,\iota^\star\beta}(X^\circ)^T]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}})} \in Q_{T,0}.$$

A priori this is a rational function in the equivariant weights, with numerator and denominator homogeneous polynomials of the same degree. However [LLLZ09, Theorem 4.8] shows that the numerator and denominator are in fact constant, so that:

$$\mathsf{GW}^T_{g,0,\iota^\star\beta}(X^\circ) \in \mathbb{Q}.$$

For this it is crucial to restrict to the Calabi–Yau torus.

The main result of this section is the following:

Theorem 2.3 (Theorem D)**.** *For all $g \geq 0$ we have:*

$$\mathsf{GW}_{g,0,\beta}(X|E) = \mathsf{GW}^T_{g,0,\iota^\star\beta}(X^\circ). \tag{21}$$

2.3. **Localisation calculation.** To prove Theorem 2.3, we compute the left-hand side of (21) by localising with respect to the Calabi–Yau torus. We show that most contributions vanish, and identify the remaining contributions with the right-hand side.

2.3.1. *Fixed loci.* The action $T \curvearrowright X$ lifts to actions on $\overline{M}_{g,0,\beta}(X)$ and $\overline{M}_{g,0,\beta}(X|E)$ (in the latter case, this is because T sends E to itself and hence lifts to a logarithmic action $T \curvearrowright (X|E)$).

Restricting from the dense torus of X to the subtorus T does not change the zero- and one-dimensional orbits in X. It follows that it also does not change the fixed locus in the moduli space of stable maps. Since X is a toric variety, this fixed locus is well-understood, see e.g. [Spi00, Section 6], [Liu13, Section 5.2], [GP99, Section 4], [CK99, Section 9.2], or [Beh02, Section 4].

Briefly, the fixed locus decomposes into a union of connected components indexed by **localisation graphs**. A localisation graph Γ is a graph equipped with marking legs, degree labelings $d_e > 0$ for every edge $e \in E(\Gamma)$ and genus labelings $g_v \geq 0$ for every vertex $v \in V(\Gamma)$. Furthermore every vertex $v \in V(\Gamma)$ is assigned a cone $\sigma(v) \in \Sigma_X(3)$ and every edge $e \in E(\Gamma)$ is assigned a cone $\sigma(e) \in \Sigma_X(2)$. The corresponding connected component of the fixed locus is denoted

$$F_\Gamma(X)$$

and generically parametrises stable maps with components C_v contracted to torus-fixed points and components C_e forming degree d_e covers of toric curves, totally ramified over the torus-fixed points.

We let $\Omega_{g,0,\beta}(X)$ denote the set of localisation graphs, so that:

$$\overline{M}_{g,0,\beta}(X)^T = \bigsqcup_{\Gamma \in \Omega_{g,0,\beta}(X)} F_\Gamma(X).$$

2.3.2. *Comparison of fixed loci.* Consider the morphism forgetting the logarithmic structures:

$$\overline{M}_{g,0,\beta}(X|E) \to \overline{M}_{g,0,\beta}(X).$$

This is T-equivariant, and hence restricts to a morphism between T-fixed loci. For each $\Gamma \in \Omega_{g,0,\beta}(X)$ we define $F_\Gamma(X|E)$ via the fibre product

$$\begin{tikzcd} F_\Gamma(X|E) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,0,\beta}(X|E)^T \arrow[d] \\ F_\Gamma(X) \arrow[r, hook] & \overline{M}_{g,0,\beta}(X)^T \end{tikzcd}$$

and this produces a decomposition of $\overline{M}_{g,0,\beta}(X|E)^T$ into clopen substacks:

$$\overline{M}_{g,0,\beta}(X|E)^T = \bigsqcup_{\Gamma \in \Omega_{g,0,\beta}(X)} F_\Gamma(X|E).$$

Note that we do not claim that each $F_\Gamma(X|E)$ is connected, nor that $F_\Gamma(X|E) \to F_\Gamma(X)$ is virtually birational. Virtual localisation [GP99] gives:[4]

$$\mathsf{GW}_{g,0,\beta}(X|E) = \sum_{\Gamma \in \Omega_{g,0,\beta}(X)} \int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})}. \tag{22}$$

Turning to X° we note that there is an inclusion

$$\Omega_{g,0,\iota^\star\beta}(X^\circ) \subseteq \Omega_{g,0,\beta}(X)$$

consisting of localisation graphs which do not interact with cones in $\Sigma_X \setminus \Sigma_{X^\circ}$. Since the logarithmic structure on $(X|E)$ is trivial when restricted to X° it follows that for $\Gamma \in \Omega_{g,0,\iota^\star\beta}(X^\circ)$ we have

$$F_\Gamma(X|E) = F_\Gamma(X) = F_\Gamma(X^\circ).$$

The perfect obstruction theories coincide when restricted to these loci, producing an identification of the induced virtual fundamental classes and virtual normal bundles. We conclude:

Proposition 2.4. *We have:*

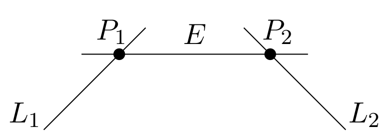
$$\mathsf{GW}_{g,0,\beta}(X|E) = \mathsf{GW}^T_{g,0,\iota^\star\beta}(X^\circ) + \sum_{\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^\star\beta}(X^\circ)} \int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})}.$$

2.3.3. *Vanishing of remaining contributions.* Fix a localisation graph $\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^\star\beta}(X^\circ)$. To prove Theorem 2.3 it remains to show

$$\int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})} = 0.$$

This requires a detailed analysis of the shape of the localisation graph and its contribution.

Notation 2.5. Local to $E \subseteq S$ the toric boundary takes the following form



The zero section gives a closed embedding $S \hookrightarrow X$ as a union of toric boundary strata. Let

$$\tau_E, \tau_1, \tau_2 \in \Sigma_X(2)$$

[4]Virtual localisation for spaces of stable logarithmic maps presents conceptual difficulties, as the obstruction theory is defined over the Artin fan which is typically singular. Since the divisor $E \subseteq X$ is smooth, we circumvent these issues by passing to Kim's space of expanded logarithmic maps [Kim10], which has an absolute obstruction theory and arises as a logarithmic modification of the Abramovich–Chen–Gross–Siebert space (see e.g. [BNR21, Section 2.1]). The arguments of this section are insensitive to the choice of birational model of the moduli space. See [MR19] for a treatment of localisation in this setting.

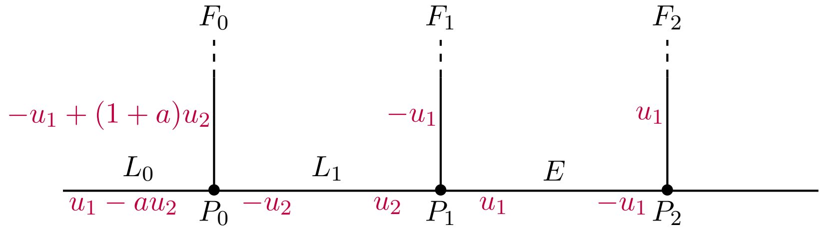


FIGURE 3. The toric skeleton of X locally around $L_1 \cup E$, used in the proof of Lemma 2.7. Edges represent boundary curves and vertices represent torus-fixed points. The purple label at a flag (P, L) records the weight $c_1^T(T_P L)$.

denote the cones corresponding to the toric curves $E, L_1, L_2 \hookrightarrow S \hookrightarrow X$. Similarly let

$$\sigma_1, \sigma_2 \in \Sigma_X(3)$$

denote the cones corresponding to the torus-fixed points $P_1, P_2 \in S \hookrightarrow X$.

Lemma 2.6. *There exists an edge $\tilde{e} \in E(\Gamma)$ with $\sigma(\tilde{e}) = \tau_E$.*

Proof. Suppose for a contradiction that $\sigma(e) \neq \tau_E$ for all $e \in E(\Gamma)$. Since $\Gamma \notin \Omega_{g,0,\iota^\star\beta}(X^\circ)$ there exists a vertex $v \in V(\Gamma)$ with $\sigma(v) \in \{\sigma_1, \sigma_2\}$. This vertex is adjacent to an edge $\tilde{e} \in E(\Gamma)$, and since $\sigma(\tilde{e}) \neq \tau_E$ we must have $\sigma(\tilde{e}) \in \{\tau_1, \tau_2\}$. We then find

$$E \cdot \beta = \sum_{e \in E(\Gamma)} d_e \left(E \cdot L_{\sigma(e)}\right) = \sum_{\substack{e \in E(\Gamma) \\ \sigma(e) \in \{\tau_1, \tau_2\}}} d_e \geq d_{\tilde{e}} > 0$$

which contradicts $E \cdot \beta = 0$. □

Lemma 2.7. *The following relation holds in $A^1_T(\mathrm{pt})$:*

$$c_1^T(T_{P_1}E) + c_1^T\left(\mathcal{O}_S(-D)|_{P_1}\right) = 0.$$

Proof. Figure 3 illustrates the toric skeleton of X in a neighbourhood of $L_1 \cup E$. The horizontal edges index boundary curves contained in the zero section $S \hookrightarrow X$ while the vertical edges index fibres of the projection $X \to S$ over torus-fixed points. We define:

$$u_1 := c_1^T(T_{P_1}E), \qquad u_2 := c_1^T(T_{P_1}L_1).$$

We now calculate the weights of the T-action on $T_{P_0}L_0$ and $T_{P_0}L_1$. The standard theory of torus actions on projective lines gives:

$$c_1^T(T_{P_0}L_1) = -c_1^T(T_{P_1}L_1) = -u_2. \tag{23}$$

Turning to $T_{P_0}L_0$ we have natural identifications:

$$T_{P_0}L_0 = N_{L_1|S}|_{P_0}, \qquad T_{P_1}E = N_{L_1|S}|_{P_1}.$$

Let $a_1 := \deg N_{L_1|S}$ denote the self-intersection of the divisor $L_1 \subseteq S$. We then have:

$$c_1^T(T_{P_0}L_0) = c_1^T(N_{L_1|S}|_{P_0}) = c_1^T(N_{L_1|S}|_{P_1}) - a_1 c_1^T(T_{P_1}L_1) = u_1 - a_1 u_2. \tag{24}$$

From the definition of the Calabi–Yau torus and (23), (24) we obtain

$$0 = c_1^T\left(\bigwedge^3 T_{P_0}X\right) = c_1^T(T_{P_0}F_0) + (-u_2) + (u_1 - a_1 u_2)$$

from which we deduce:

$$c_1^T(T_{P_0}F_0) = -u_1 + (a_1+1)u_2.$$

By (20) we have $\mathcal{O}_S(-D)|_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(-a_1 - 1)$ from which we conclude

$$c_1^T(T_{P_1}F_1) = c_1^T(T_{P_0}F_0) + (a_1+1)c_1^T(T_{P_0}L_1) = -u_1 = -c_1^T(T_{P_1}E)$$

which completes the proof. □

Proof of Theorem 2.3. Since $D^2 \geq 0$ and $D \cdot \beta > 0$ it follows that
$$\overline{M}_{g,0,\beta}(X|E) = \overline{M}_{g,0,\beta}(S|E).$$
Since $(S|E) \hookrightarrow (X|E)$ is strict, there is a short exact sequence
$$0 \to T^{\log}_{S|E} \to T^{\log}_{X|E}\big|_S \to N_{S|X} \to 0$$
and using $N_{S|X} = \mathcal{O}_S(-D)$ we obtain:
$$[\overline{M}_{g,0,\beta}(X|E)]^{\mathrm{virt}}_T = e^T(\mathbf{R}^1\pi_\star f^\star \mathcal{O}_S(-D)) \cap [\overline{M}_{g,0,\beta}(S|E)]^{\mathrm{virt}}_T.$$
Fix a graph $\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^\star\beta}(X^\circ)$. By Proposition 2.4 it suffices to show that the contribution of Γ vanishes. We will prove that the T-equivariant vector bundle
$$\mathbf{R}^1\pi_\star f^\star \mathcal{O}_S(-D)|_{F_\Gamma(X|E)}$$
has a weight zero summand in K-theory. This ensures that the T-equivariant Euler class vanishes, and the claim follows.

Perform the partial normalisation of the source curve at the nodes forced by the localisation graph Γ (such nodes correspond to flags based at either a vertex of valency at least three, or a vertex of valency two which is the intersection of two bounded edges). The normalisation sequence produces a surjection:
$$H^1(C, f^\star\mathcal{O}_S(-D)) \twoheadrightarrow \bigoplus_{e\in E(\Gamma)} H^1(C_e, f^\star\mathcal{O}_S(-D)).$$
It suffices to show that one summand of the codomain has vanishing equivariant Euler class. By Lemma 2.6 there exists an edge $\tilde{e} \in E(\Gamma)$ with $\sigma(\tilde{e}) = \tau_E$. Every point of $F_\Gamma(X|E)$ parametrises a stable logarithmic map whose underlying curve contains an irreducible component $C_{\tilde{e}}$ which maps to E with positive degree $d_{\tilde{e}}$ and is totally ramified over the torus-fixed points. Using Lemma 2.7 we write
$$u_1 = -c_1^T(\mathcal{O}_S(-D)|_{P_1}) = c_1^T(T_{P_1}E).$$
A Riemann–Roch calculation (see e.g. [Liu13, Example 19]) then shows that:
$$\mathrm{ch}_T\left(H^1(C_{\tilde{e}}, f^\star\mathcal{O}_S(-D))\right) = \sum_{j=1}^{2d_{\tilde{e}}-1} \exp(-u_1 + j\tfrac{u_1}{d_{\tilde{e}}}).$$
Taking $j = d_{\tilde{e}}$ we see that $H^1(C_{\tilde{e}}, f^\star\mathcal{O}_S(-D)$ has a vanishing Chern root, and hence its equivariant Euler class vanishes as claimed. □

3. Self-nodal plane curves

In this section we focus on an important special case. Fix $r \geq 1$ and consider the toric variety
$$S_r := \mathbb{P}(1,1,r).^5$$
Let $D_r \in |-K_{S_r}|$ be an irreducible curve with a single nodal toric singularity at the singular point of S_r (or at one of the torus-fixed points if $r = 1$). The pair $(S_r|D_r)$ is logarithmically smooth. By a curve in S_r of degree d we mean a curve whose class is d times the class of the toric hypersurface with self-intersection r. Given a curve in S_r of degree d, its intersection number with D_r is $d(r+2)$. We consider the genus zero maximal tangency Gromov–Witten invariants:
$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) \in \mathbb{Q}.$$

[5] We can also take $r = 0$ in which case we have $S_0 := \mathbb{P}^1 \times \mathbb{P}^1$ and D_0 the union of a $(1,0)$ curve and a smooth $(1,2)$ curve. In this case, the results of this section follow from a direct calculation, using [GPS10, Proposition 6.1]. Similarly we can take $r = -1$ which results in the local geometry of a (-1)-curve in a surface.

We begin by deriving an explicit formula for these invariants (Theorem 3.1) and applying it to deduce a formula for the invariants of local $\mathbb{P}^1$ (Theorem 3.2).

We then specialise to $r = 1$ and establish a relationship between the invariants of $(\mathbb{P}^2|D_1)$ and $(\mathbb{P}^2|E)$ for E a smooth cubic (Theorem 3.7). We apply this to prove a conjecture of Barrott and the second-named author (Theorem 3.10).

3.1. **Scattering calculation.** The main result of this section is:

Theorem 3.1 (Theorem E)**.** *We have:*

$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) = \frac{r+2}{d^2}\binom{(r+1)^2d-1}{d-1}.$$

The following result, already known in the physics literature [CGM+07, Equation (4.53)], is a direct consequence of Theorem 1.28 and Theorem 2.3.

Theorem 3.2 (Theorem F)**.** *We have*

$$\mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1}(-r-2)\big) = \frac{(-1)^{rd-1}}{d^3}\binom{(r+1)^2d-1}{d-1}$$

where the left-hand side is defined via localisation with respect to the Calabi–Yau torus, as in Section 2.2.

Proof. There is a toric resolution of singularities $\mathbb{F}_r \to \mathbb{P}(1,1,r)$. Consider $D_r \subseteq \mathbb{F}_r$ the strict transform and $E \subseteq \mathbb{F}_r$ the exceptional divisor. Realise the Hirzebruch surface as a $\mathbb{P}^1$-bundle

$$\mathbb{F}_r \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1})$$

with $E \subseteq \mathbb{F}_r$ the zero section and $E_\infty \subseteq \mathbb{F}_r$ the infinity section. By [AW18] the Gromov–Witten invariants of $(S_r|D_r)$ are identified with the Gromov–Witten invariants of the bicyclic pair $(\mathbb{F}_r \mid D_r + E)$. We then have

$$\begin{aligned}\mathsf{GW}_{0,(d(r+2),0),d}(S_r|D_r) &= \mathsf{GW}_{0,(d(r+2),0),dE_\infty}(\mathbb{F}_r \mid D_r+E)\\ &= (-1)^{d(r+2)+1}d(r+2)\cdot\mathsf{GW}^T_{0,0,d\,\iota^\star E_\infty}(\mathcal{O}_{\mathbb{F}_r}(-D_r)|_{\mathbb{F}_r\setminus E})\end{aligned}\tag{25}$$

where the second equality follows by combining Theorems 1.28 and 2.3. Now note that $\mathbb{F}_r \setminus E$ is the total space of $N_{E_\infty|\mathbb{F}_r}$ which is a degree r line bundle on $E_\infty \cong \mathbb{P}^1$. Similarly, we have $\deg \mathcal{O}_{\mathbb{F}_r}(-D_r)|_{E_\infty} = -r-2$. This establishes the identity

$$\mathsf{GW}^T_{0,0,d\,\iota^\star E_\infty}(\mathcal{O}_{\mathbb{F}_r}(-D_r)|_{\mathbb{F}_r\setminus E}) = \mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1}(-r-2)\big)$$

and the result follows by combining (25) and Theorem 3.1. □

It remains to prove Theorem 3.1. This proceeds via the connection between the logarithmic Gromov Witten invariants of $(S_r|D_r)$ and local scattering diagrams, following the analyses in [BBvG20, BS23]. Since the combinatorics of scattering diagrams play only a secondary role, we economise on detail and refer to [BBvG20, Section 4] for a full account.

We work on the lattice $M = \mathbb{Z}^2$ equipped with the standard skew-symmetric form $\omega(v,w) = v \wedge w$ and define

$$\rho_1 := (-1,0), \qquad \rho_2 := (1, r+2).$$

Let $\mathfrak{D}^{\mathrm{in}}_{r+2}$ be the initial scattering diagram with walls $\mathfrak{d}_i := \rho_i\mathbb{R}$ decorated with wall-crossing functions

$$f_{\rho_i} := 1 + t_i z^{-\rho_i} \in \mathbb{Q}[\![t_1,t_2]\!][M].$$

Write $\mathfrak{D}_{r+2}$ for the consistent scattering diagram obtained by completing $\mathfrak{D}^{\mathrm{in}}_{r+2}$ using the algorithm of [GPS10]. For us, the relevant information is encoded in the wall-crossing function $f_{(0,-1)}$ of the central ray $(0,-1)\mathbb{R}_{\geqslant 0}$.

Proposition 3.3. *We have:*

$$f_{(0,-1)} = \exp\left(\sum_{d>0}(r+2)d\,\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r)\,\left(t_1t_2z^{(0,-(r+2))}\right)^d\right).$$

Proof. We proceed as in [BS23, Section 2.2.1] and start by constructing a toric model for $(S_r|D_r)$. In the following we will identify a divisor with its strict transform (respectively image) under a blowup (respectively blowdown).

As in the proof of Theorem 3.2 we consider the resolution of singularities $\mathbb{F}_r \to S_r$, writing $D_r \subseteq \mathbb{F}_r$ for the strict transform of the self-nodal curve and $E \subseteq \mathbb{F}_r$ for the exceptional divisor. We now further blowup at the two intersection points $\{q_1, q_2\} = D_r \cap E$ and let F_1, F_2 denote the resulting exceptional divisors. We write

$$\phi\,:\,(\widetilde{S}_r, \widetilde{D}_r) \coloneqq (\mathrm{Bl}_{q_1,q_2}\mathbb{F}_r \,|\, D_r + F_1 + E + F_2) \longrightarrow (S_r|D_r)$$

for the composition of these blowups. Let $L_1, L_2 \subseteq \widetilde{S}_r$ denote the strict transforms of the tangent lines at the singularity of $D_r \subseteq S_r$. These are (-1)-curves which we blow down to produce:

$$\pi\,:\,(\widetilde{S}_r|\widetilde{D}_r) \to (\overline{S}_r|D_r + F_1 + E + F_2).$$

A quick calculation shows that the self-intersection numbers of $D_r, F_1, E, F_2 \subseteq \overline{S}_r$ are $(r+2), 0, -(r+2), 0$. By [Fri15, Lemma 2.10] this identifies the pair $(\overline{S}_r|D_r + F_1 + E + F_2)$ with $(\mathbb{F}_{r+2}\,|\,\partial\mathbb{F}_{r+2})$.

We observe that $\mathfrak{D}_{r+2}$ as defined above is the scattering diagram associated to the toric model π. By [GPS10, Theorem 5.6] (see also [Bou20, Theorem 3.2]) we have

$$f_{(0,-1)} = \exp\left(\sum_{d>0} d(r+2)\,\mathsf{GW}_{0,(d(r+2),0,0,0),d(E_\infty - L_1 - L_2)}(\widetilde{S}_r|\widetilde{D}_r)\,(t_1t_2)^d\,z^{(0,-d(r+2))}\right)$$

where we write E_∞ for the toric hypersurface in $\mathbb{F}_{r+2}$ with self-intersection $(r+2)$. The result now follows from birational invariance [AW18], which gives:

$$\mathsf{GW}_{0,(d(r+2),0,0,0),d(E_\infty - L_1 - L_2)}(\widetilde{S}_r|\widetilde{D}_r) = \mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r).$$ □

It remains to compute $f_{(0,-1)}$. For this, we first identify $\mathfrak{D}_{r+2}$ with the scattering diagram associated to the $(r+2)$-Kronecker quiver. Consider the skew-symmetric form $\widetilde{\omega}(v,w) = (r+2)\,(v \wedge w)$ on $\mathbb{Z}^2$ and denote the resulting symplectic lattice:

$$(\widetilde{M}, \widetilde{\omega}).$$

Set $v_1 = (-1,0), v_2 = (0,1)$, and let $\widetilde{\mathfrak{D}}_{r+2}$ denote the consistent scattering diagram for $(\widetilde{M}, \widetilde{\omega})$ with initial walls $(v_i\mathbb{R}, 1 + t_iz^{-v_i})$.

The resulting scattering pattern is intricate, featuring a subcone in the lower-right quadrant where the support of the walls is dense. However, Reineke has found an explicit description of the wall-crossing function attached to the central wall $(1,-1)\mathbb{R}_{\geq 0}$.

Lemma 3.4 ([Rei11, Theorem 6.4])**.** *Let μ be the Möbius function and for $k \geq 1$ set:*

$$N_{k,r} \coloneqq \frac{1}{rk^2}\sum_{i|k}\mu(k/i)\,(-1)^{(r+2)i+1}\binom{(r+1)^2 i - 1}{i}.$$

Then $N_{k,r} \in \mathbb{Z}$ and

$$\widetilde{f}_{(1,-1)} = \prod_{k>0}\left(1 - (-1)^{(r+2)k^2}\left(t_1t_2z^{(-1,1)}\right)^k\right)^{k(r+2)\,N_{k,r}}.$$

Remark 3.5. Strictly speaking, [Rei11, Theorem 6.4] concerns the scattering diagram with initial wall-crossing functions $(1+t_i z^{-v_i})^{r+2}$ on a lattice equipped with the standard skew-symmetric form $(r+2)^{-1}\widetilde{\omega}$. However, the "change of lattice" trick [GHKK18, Section C.3, Step IV] translates Reineke's theorem into Lemma 3.4. See also [GP10] for an account using this scattering diagram.

Proof of Theorem 3.1. Consider the morphism of lattices $\pi : \widetilde{M} \to M$ given by $v_i \mapsto \rho_i$. The construction of [DM21, Lemma 2.11] produces a consistent scattering diagram

$$\pi_\star \widetilde{\mathfrak{D}}_{r+2}$$

on (M, ω). Under π the initial wall $(v_i\mathbb{R},\, 1+t_i z^{-v_i})$ is taken to $(\rho_i\mathbb{R},\, 1+t_i z^{-\rho_i})$. We conclude that up to equivalence $\pi_\star\widetilde{\mathfrak{D}}_{r+2} \cong \mathfrak{D}_{r+2}$. Hence, under π the central wall $((1,-1)\mathbb{R}_{\geq 0}, \widetilde{f}_{(1,-1)}) \in \widetilde{\mathfrak{D}}_{r+2}$ is taken to $((0,-1)\mathbb{R}_{\geq 0}, f_{(0,-1)}) \in \mathfrak{D}_{r+2}$. By Lemma 3.4 we find

$$\begin{aligned} f_{(0,-1)} &= \prod_{k>0} \left(1-(-1)^{(r+2)k^2}\left(t_1 t_2 z^{(0,-(r+2))}\right)^k\right)^{k(r+2)\,N_{k,r}} \\ &= \exp\left(-\sum_{l>0}\frac{r+2}{l}\sum_{k>0}(-1)^{(r+2)k^2 l} k\, N_{k,r}\left(t_1t_2z^{(0,-(r+2))}\right)^{kl}\right) \\ &= \exp\left(\sum_{d>0}\frac{r+2}{rd}\binom{(r+1)^2 d-1}{d}\left(t_1t_2z^{(0,-(r+2))}\right)^d\right) \end{aligned}$$

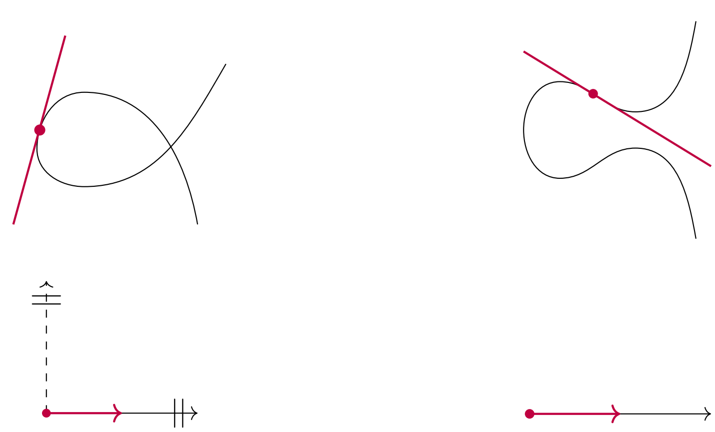
where the last equality is proven using the Möbius inversion formula. The statement of Theorem 3.1 now follows from Proposition 3.3 and the identity:

$$\binom{(r+1)^2d-1}{d} = r(r+2)\binom{(r+1)^2d-1}{d-1}. \qquad \square$$

3.2. **Nodal cubics versus smooth cubics.** In this final section we set $r=1$. We have

$$D_1 = D \subseteq \mathbb{P}^2$$

an irreducible cubic with a single nodal singularity. Let $E \subseteq \mathbb{P}^2$ be a smooth cubic. For each of the two pairs we consider the moduli space of genus zero stable logarithmic maps, with maximal tangency at a single marking (see Figure 4).



(a) $(\mathbb{P}^2|D)$ (b) $(\mathbb{P}^2|E)$

FIGURE 4. Tangent curves to plane cubics, nodal and smooth.

In genus zero, each moduli space has virtual dimension zero and produces a system of enumerative invariants indexed by d. Both theories are completely solved: the case of $(\mathbb{P}^2|D)$ is solved in Theorem 3.1, while the case of $(\mathbb{P}^2|E)$ is solved in [Gat03, Example 2.2] with inspiration from [Tak01]. The numbers do not agree, as the following table demonstrates:

d	$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2\|D)$	$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2\|E)$
1	3	9
2	$21/4$	$135/4$
3	$55/3$	244
4	$1,365/16$	$36,999/16$
5	$11,628/25$	$635,634/25$
6	$33,649/12$	$307,095$

Experimentally, we always have

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) < \mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|E). \tag{26}$$

In this section we provide a conceptual explanation for this defect, via the geometry of degenerating hypersurfaces (Theorem 3.7). We then settle a conjecture posed in [BN22] (Theorem 3.10).

The paper [BN22] degenerates a smooth cubic to the toric boundary, and studies the resulting logarithmic Gromov–Witten theory on the central fibre. The following construction is similar, except that our starting point is a nodal cubic. We explain how the arguments adapt to this setting, assuming familiarity with [BN22]. Let $\Delta \subseteq \mathbb{P}^2$ denote the toric boundary and consider a degeneration $D \rightsquigarrow \Delta$, i.e. a divisor

$$\mathcal{D} \subseteq \mathbb{P}^2 \times \mathbb{A}^1$$

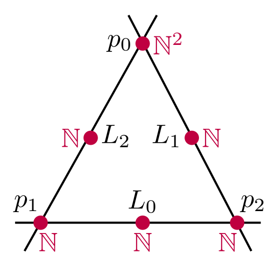
whose general fibre is an irreducible nodal cubic and whose central fibre is Δ. We can choose $\mathcal{D}$ to be irreducible with normal crossings singularities, and such that for $\pi^{-1}(t) \cap \mathcal{D}^{\text{sing}}$ is the nodal point of $\mathcal{D}_t$ for $t \neq 0$, and

$$\pi^{-1}(0) \cap \mathcal{D}^{\text{sing}} = p_0$$

where $p_0 = [1,0,0]$. Consider the logarithmically regular logarithmic scheme

$$\mathcal{Y} = (\mathbb{P}^2 \times \mathbb{A}^1 \,|\, \mathcal{D}).$$

Equip $\mathbb{A}^1$ with the trivial logarithmic structure and consider the logarithmic morphism $\mathcal{Y} \to \mathbb{A}^1$. This is not logarithmically smooth, but is logarithmically flat; the proof is similar to [BN22, Lemma 3.7]. The general fibre $\mathcal{Y}_t$ is the logarithmic scheme associated to the smooth pair $(\mathbb{P}^2|\mathcal{D}_t)$. The central fibre, on the other hand, is logarithmically singular. The stalks of the ghost sheaf of $\mathcal{Y}_0$ are



where $L_0, L_1, L_2 \subseteq \mathbb{P}^2$ are the coordinate lines and $p_0, p_1, p_2 \in \mathbb{P}^2$ the coordinate points.

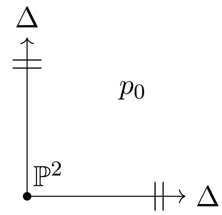
Following [BN22, Section 2] we construct, in genus zero, a virtual fundamental class on the space of stable logarithmic maps to the central fibre $\mathcal{Y}_0$. Integrals against this class recover the invariants of $(\mathbb{P}^2|D)$ by the conservation of number principle. We now study the moduli space

$$\overline{M}_{0,\mathbf{c},d}(\mathcal{Y}_0).$$

As in [BN22, Lemma 4.5] we find that every logarithmic map to $\mathcal{Y}_0$ must factor through Δ. In fact, in our new setting we obtain a stronger constraint.

Lemma 3.6. *Given a logarithmic map to $\mathcal{Y}_0$ the underlying schematic map to $\mathbb{P}^2$ factors through L_0.*

Proof. Let $C \to \mathcal{Y}_0$ be a logarithmic map and consider its tropicalisation $\mathsf{f}\colon \Sigma C \to \Sigma\mathcal{Y}_0$. The target $\Sigma\mathcal{Y}_0$ is the cone complex:



Following [BN22, proof of Lemma 4.5] it suffices to show that the image of f does not intersect the interior of the maximal cone p_0.

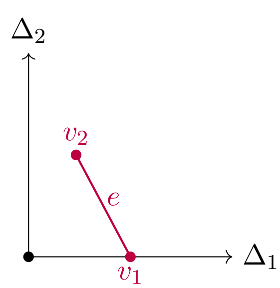
We first describe the balancing condition. For $v \in V(\Gamma)$ let $C_v \subseteq C$ denote the corresponding irreducible component. Since $f(C_v) \subseteq \Delta$ we have that $\mathsf{f}(v)$ belongs to either Δ or p_0. In the latter case, the balancing condition states that the sum of the outgoing slope vectors is zero. The interesting case is $\mathsf{f}(v) \in \Delta$. We distinguish two possibilities:

- If $f(C_v) \subseteq L_0$ then all outgoing edges from v are contained in Δ and the sum of their slopes is equal to $3d_v$.
- If $f(C_v) \subseteq L_1$ or L_2 then the slope of each outgoing edge from v can be broken into components tangent to Δ and normal to Δ. The sum of the slopes tangent to Δ is equal to $2d_v$, while the sum of the slopes normal to Δ is equal to d_v. Moreover, edges with positive slope normal to Δ must all enter the same neighbourhood of v in the above chart. Which neighbourhood they enter depends on which of L_1 or L_2 the component C_v maps to.

Due to the balancing condition, every tropical map $\Sigma C \to \Sigma\mathcal{Y}_0$ admits a lift to the standard cover of the target:

$$\begin{tikzcd} & \mathbb{R}^2_{\geqslant 0} \arrow[d] \\ \Sigma C \arrow[ur] \arrow[r] & \Sigma\mathcal{Y}_0. \end{tikzcd}$$

Choose such a lift, and suppose for a contradiction that the image of f intersects the interior of p_0. Then there exists an edge $e \in E(\Gamma)$ such that $\mathsf{f}(e)$ intersects the interior of p_0 and such that one of the end vertices of e is mapped to Δ. On the lift we have:



Define $\Gamma_i \subseteq \Gamma$ by cutting Γ at e and taking the subgraph containing v_i. Start with Γ_1. By balancing we have $d_{v_1} > 0$ and so v_1 supports an outgoing edge with positive slope in the Δ_1-direction. Traversing along this edge to the next vertex, we again conclude by balancing that there exists an outgoing edge with positive slope in the Δ_1-direction. Continuing in this way, we eventually arrive at the marking leg. It follows that the marking leg Γ_1.

Now consider Γ_2. By balancing the vertex v_2 supports an outgoing edge with positive slope in the Δ_2-direction (this occurs both if $\mathsf{f}(v_2) \in p_0$ and if $\mathsf{f}(v_2) \in \Delta_2$). As above, we inductively traverse the graph and produce a path consisting of edges with positive slope in the Δ_2-direction. Eventually we arrive at the marking leg. It follows that the marking leg is contained in Γ_2.

The marking leg is thus contained in both Γ_1 and Γ_2. But these subgraphs are disjoint: since Γ has genus zero, the edge e is separating. □

Using Lemma 3.6 we can show as in [BN22, Proposition 4.11] that

$$\overline{M}_{0,\mathbf{c},d}(\mathcal{Y}_0) = \overline{M}_{0,1,d}(L_0) \cong \overline{M}_{0,1,d}(\mathbb{P}^1).$$

This space carries a virtual fundamental class of dimension zero, arising from logarithmic deformation theory. In [BN22, Section 4] this is expressed as an obstruction bundle integral, and in [BN22, Section 5] it is computed via localisation. These calculations apply directly to our new setting, because the logarithmic scheme $\mathcal{Y}_0$ agrees with the logarithmic scheme $\mathcal{X}_0$ of [BN22, Section 3.1] in a neighbourhood of L_0. We conclude:

Theorem 3.7 (Theorem G)**.** *The invariant of $(\mathbb{P}^2|D)$ is precisely the central fibre contribution to the invariant of $(\mathbb{P}^2|E)$ arising from multiple covers of a single line of Δ. In the notation of [BN22, Section 5.5]:*

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) = \mathrm{C}_{\mathrm{ord}}(d,0,0).$$

Thus the invariants of $(\mathbb{P}^2|D)$ constitute one contribution, amongst many, to the invariants of $(\mathbb{P}^2|E)$. Experimentally, all contributions are positive: this explains the inequality (26).

Theorem 3.7 also allows us to compute $\mathrm{C}_{\mathrm{ord}}(d,0,0)$. In [BN22, Section 5] these are computed up to $d=8$ by computer-assisted torus localisation. Based on these numerics, the following formula is proposed:

Conjecture 3.8 ([BN22, Conjecture 5.9(39)])**.** *We have the following hypergeometric expression for the contribution of degree d covers of L_0:*

$$\mathrm{C}_{\mathrm{ord}}(d,0,0) = \frac{1}{d^2}\binom{4d-1}{d}.$$

It is then shown [BN22, Proposition 5.13] that Conjecture 3.8 is equivalent to the following conjecture in pure combinatorics, which is verified by computer up to $d=50$:

Conjecture 3.9 ([BN22, Conjecture 5.12])**.** *Fix an integer $d \geq 1$. Then we have*

$$\sum_{(d_1,\ldots,d_r)\vdash d} \frac{2^{r-1}\cdot d^{r-2}}{\#\mathrm{Aut}(d_1,\ldots,d_r)} \prod_{i=1}^{r} \frac{(-1)^{d_i-1}}{d_i}\binom{3d_i}{d_i} = \frac{1}{d^2}\binom{4d-1}{d}$$

where the sum is over strictly positive unordered partitions of d (of any length).

Using Theorem 3.7 we can now prove both these conjectures.

Theorem 3.10. *Conjecture 3.8, and hence also Conjecture 3.9, holds.*

Proof. By Theorem 3.7 it is equivalent to show that

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) = \frac{1}{d^2}\binom{4d-1}{d}.$$

This follows from Theorem 3.1 in the case $r=1$. See also [BN22, Remark 5.10]. □

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Michel van Garrel, University of Birminghan, m.vangarrel@bham.ac.uk

Navid Nabijou, Queen Mary University of London, n.nabijou@qmul.ac.uk

Yannik Schuler, University of Sheffield, yschuler1@sheffield.ac.uk