Rutherford Scattering (Discussion 3)

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1   Derivations

The setup for the Rutherford scattering calculation is shown in Figure 1. We have an incoming particle, for example an α, which is going to deflect off the nucleus of an atom in the material. The impact parameter b is the perpendicular distance from the nucleus, and the scattering angle θ is the final angle at which the particle moves away after 'colliding' with the nucleus. We are interested in relating the scattering angle to the impact parameter. We will use a simplifying assumption that the nucleus is so heavy that it doesn't move: we fix it's position to be at the origin.
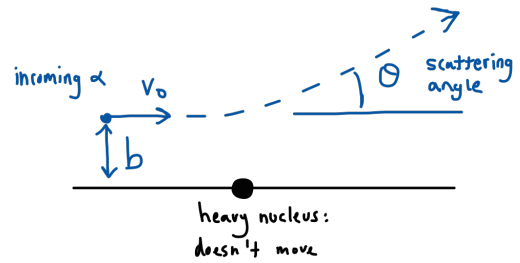
Figure 1: A diagram of the parameters in the scattering experiment

We can calculate the change in the y momentum of the incoming particle as

$$\Delta p_y = \int_{-\infty}^{\infty} dt \, F_y(t) \tag{1.1}$$

where $F_y(t)$ is the force between the nucleus and the particle. In this case, the force is the repulsive Coulomb force given by

$$\vec{F}(r) = \frac{qQ}{4\pi\epsilon_0 r^2}\hat{r} \Rightarrow F_y(r) = \frac{qQ}{4\pi\epsilon_0 r^2}\sin\phi \tag{1.2}$$

where q and Q are the charges of the incoming particle and the nucleus respectively and ϕ is the angular position of the particle. We can write also write these as $q = ze$ and $Q = Ze$ since the charges must be multiples of the fundamental electron charge e. Therefore, we have

$$p_y = \int_{t=-\infty}^{t=\infty} dt \, \frac{zZe^2}{4\pi\epsilon_0 r(t)^2}\sin\phi(t) \tag{1.3}$$

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where I emphasize that the distance and position are both functions of time. Lets invert the relation $\phi = \phi(t)$ so that we have $t = t(\phi)$. So, we are expressing the time as a function of angular position, rather than expressing the angle as a function of time. Carry out this change of variables in the integral: $dt(\phi) = \frac{dt}{d\phi}d\phi$ so

$$p_y = \int_\pi^\theta \frac{zZe^2}{4\pi\epsilon_0 r(\phi)^2} \sin\phi \frac{dt}{d\phi}d\phi \tag{1.4}$$

The new limits of integration reflect the fact that at time $t = -\infty$, the angular position of the particle was π, and at time $t = +\infty$ the particle has left at an angle θ. The factor of r^2 in the denominator is problematic since we can't integrate an unknown function of ϕ. We can eliminate it using conservation of angular momentum. The angular momentum is

$$I = m\omega r^2 = mr^2\frac{d\phi}{dt} \tag{1.5}$$

The statement that the angular momentum is conserved says the angular momentum at time t is equal to the angular momentum at time $-\infty$ which we can see is just $-mv_0 b$ (the sign just reflects that the ang. mom. is clockwise). Hence,

$$\frac{dt}{d\phi} = \left(\frac{d\phi}{dt}\right)^{-1} = -r^2/v_0 b \tag{1.6}$$

Plug this back into 1.4 to obtain an integral that we can do

$$
\begin{aligned}
p_y &= -\int_\pi^\theta \frac{zZe^2}{4\pi\epsilon_0 v_0 b} \sin\phi \, d\phi \\
&= \frac{zZe^2}{4\pi\epsilon_0 v_0 b} \cos\phi \Big|_\pi^\infty \\
p\sin\theta &= \frac{zZe^2}{4\pi\epsilon_0 v_0 b}(1 + \cos\theta)
\end{aligned}
\tag{1.7}
$$

Rearrange slightly to obtain:

$$b(\theta) = \frac{zZe^2}{4\pi\epsilon_0 mv_0^2}\frac{1+\cos\theta}{\sin\theta} \tag{1.8}$$

where we also used that $|p| = mv_0$ which we can get from conservation of energy. If you type $\frac{1+\cos\theta}{\sin\theta}$ into WolframAlpha, you will see that it is indeed equal to $\cot\frac{\theta}{2}$. So our (intermediate) final answer is:

$$b(\theta) = \frac{zZe^2}{4\pi\epsilon_0 mv_0^2}\cot\frac{\theta}{2} \tag{1.9}$$

Let's define a quantity called the cross section $d\sigma = 2\pi b \, db$. Evidently, this is the differential area element of a circular shell of radius b (hence, cross section). If we multiply by a *flux*
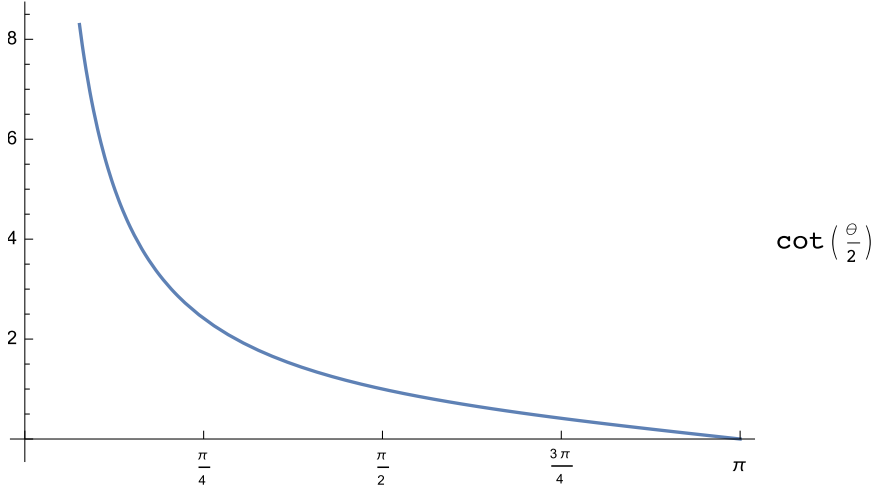
Figure 2: The shape of the function $b(\theta)$ from $\theta = 0$ to π.

of incoming particles I_0 (flux = particles/area) then this gives the number of particles that pass between radius b and $b + db$. Schematically,

$$\text{cross section} = \frac{\text{number of scattered particles per unit area per time}}{\text{incident flux}} \qquad (1.10)$$

We can use our formula 1.9 to make this into a statement about the number of particles that scatter through the range of angles θ to $\theta + d\theta$ by using the chain rule: $db = \frac{db}{d\theta}d\theta$. So we need to take the derivative $db/d\theta$, and multiply by $2\pi b$. This, as well as some manipulations involving trig id's are left to the reader. The final result is

$$d\sigma = \left(\frac{zZe^2}{8\pi\epsilon_0 mv_0^2}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \, d\Omega \qquad (1.11)$$

where $d\Omega$ is the *differential solid angle* $2\pi \sin\theta \, d\theta$. Now let's try to understand what all of this means.

2   What have we learned?

Now we need to interpret the information contained in these results. Equation 1.9 relates the impact parameter to the scattering angle. However in an experiment, there is not really a way to know what the impact parameter is going to be. So as it stands, the equation doesn't tell us a whole lot. However there is still something we can learn from it: by looking at the behavior of the function (2) we see that a large scattering angle corresponds to a small impact parameter, and vice versa. In fact to get to a scattering angle of zero requires that the impact parameter b be infinity. This will help us interpret a puzzling issue we will soon encounter.

Differential solid angle d Ω

Differential cross section d σ

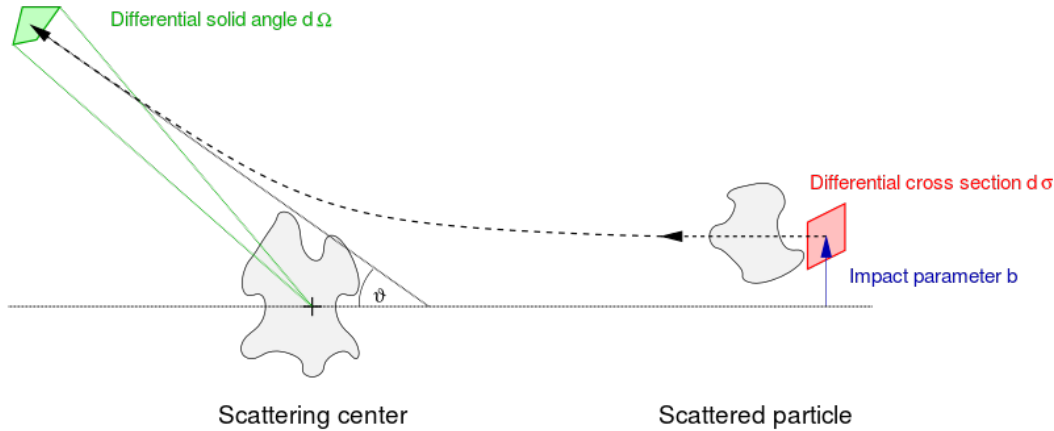**Impact parameter b**

ϑ

Scattering center

Scattered particle

Figure 3: Illustration of the differential cross section

Moving on to the cross section: what is the quantity $d\sigma/d\Omega$ physically? It is the number of particles scattered into direction (θ, ϕ) per unit time per unit solid angle divided by incident flux. See Figure 3 for a helpful diagram which I stole from wikipedia [1]. Let's try to break this down step by step in terms of how we got there. We started from $d\sigma = 2\pi b \, db$ which is the area available to scatter at impact parameters between b and $b + db$. We then expressed it in terms of the scattering angle θ, which gave us the area available to scatter between angles θ and $\theta + d\theta$. Finally we wrote it in terms of the differential solid angle $d\Omega$. So if we multiply by an incident flux I_0, it gives us the number of particles that scatter through that little solid angle. In other words, the number of scattered particles per solid angle per time as a function of the scattering angle $I_s(\theta)$ is given by $I_0 d\sigma/d\Omega$.

Let's confront an apparent issue in the equation we derived 1.11. As the angle becomes small, the cross section blows up and tends toward infinity. However we can resolve this paradox by appealing to the relation between scattering angle and impact parameter. In an experimental setup, we might be shining alpha particles at a sheet of gold, for instance. In that case, there is a maximum value that the impact parameter can take, which is half the spacing between the nuclei of the atoms. So we are protected from the problem due to our inability to look at very small angles. A more general resolution of this issue will require significantly more complex physics, so we leave it for the future.

If you want to read more, another good discussion of scattering and cross section can be found in the first several pages of http://www.tcm.phy.cam.ac.uk/~bds10/aqp/lec20-21_compressed.pdf

References

[1] "Differential cross section". Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Differential_cross_section.svg#/media/File:Differential_cross_section.svg