2-5-A: Rutherford Scattering

Rutherford analyzed Coulomb scattering by using **the classical mechanics (Newtonian mechanics)** (1911). Let us explain this below.

As shown in **Fig. 1**, an α particle of the mass M and the charge $+2e$ is supposed to come in with a speed v_0 from the distant left along the path of the **impact parameter** b. It moves under an interaction by the Coulomb repulsive force

$$\frac{2Ze^2}{4\pi\varepsilon_0 r^2}$$

exerted by the charge of the nucleus, $+Ze$, which is assumed to stay at rest at the origin of the coordinate frame.

The trajectory of the α particle

We use the polar coordinate (r, φ) instead of the Cartesian coordinate (x, y).

Separating Newtonian equation of motion into the radial part (r part) and the angular part (φ part), we have

$$M\left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right] = \frac{2Ze^2}{4\pi\varepsilon_0 r^2}, \tag{1}$$

$$M\left[2\frac{dr}{dt}\frac{d\varphi}{dt} + r\frac{d^2\varphi}{dt^2}\right] = 0. \tag{2}$$

We can rewrite Eq. (2) as

$$M\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) = 0,$$

then we have

$$Mr^2\frac{d\varphi}{dt} = \text{const.} = L. \tag{3}$$

This is nothing other than the **law of angular momentum conservation.**

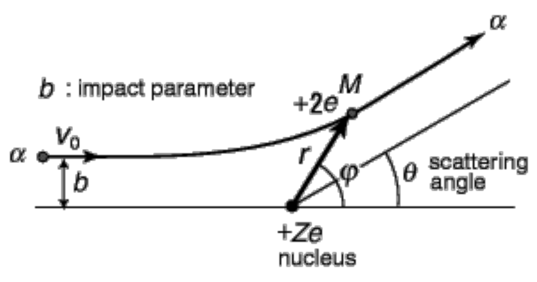

Fig. 1: Trajectory of the α particle in Rutherford scattering

Using the speed v_0 at the long distance and the impact parameter b, we can write the angular momentum L as

$$L = -Mv_0 b. \tag{4}$$

From Eqs. (4) and (3), we get

$$r^2\frac{d\varphi}{dt} = -v_0 b. \tag{5}$$

Putting this in Eq. (1), we obtain

$$\frac{d^2r}{dt^2} - \frac{v_0^2 b^2}{r^3} = \frac{2Ze^2}{4\pi\varepsilon_0 M r^2}. \tag{6}$$

We can obtain the solution $r(t)$ of the differential equation (6) as a function of time t. Inserting this $r(t)$ into Eq. (5), we have a differential equation which φ satisfies. Solving this, we can get the solution $\varphi(t)$. Thereby, we obtain the position of the α particle, (r, φ), as a function of time t. Consequently, we can obtain the trajectory of the α particle.

Instead of taking such a roundabout way, we can directly get a relation between r and φ without using the variable t explicitly. Considering r as a function of φ, and φ as a function t, we have

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = -\frac{v_0 b}{r^2}\frac{dr}{d\varphi}. \tag{7}$$

Putting $r = 1/u$, we think of u instead of r. From Eq. (7), we get

$$\frac{d}{dt} = -v_0 b u^2 \frac{d}{d\varphi}.$$

Therefore we have

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-v_0 b u^2 \frac{d}{d\varphi}\frac{1}{u}\right) = -v_0^2 b^2 u^2 \frac{d^2u}{d\varphi^2}.$$

Inserting this into Eq. (6), we obtain

$$\frac{d^2u}{d\varphi^2} = -\left(u + \frac{2Ze^2}{4\pi\varepsilon_0 M v_0^2 b^2}\right). \tag{8}$$

In this differential equation (8), let the contents in the parentheses on the right-hand side be w. Then we have a simplest oscillator equation as

$$\frac{d^2w}{d\varphi^2} = -w.$$

Because its general solution is written $w = A\cos(\varphi + \delta)$, we obtain a solution

$$\frac{1}{r} = A\cos(\varphi + \delta) - \frac{2Ze^2}{4\pi\varepsilon_0 M v_0^2 b^2}. \tag{9}$$

Two constants A and δ in the solution (9) are determined by the initial conditions. Differentiating the both sides of Eq. (9) with t and using Eq. (5), we get

$$\frac{dr}{dt} = -Av_0 b\sin(\varphi + \delta). \tag{10}$$

Since $dr/dt = -v_0$ at the distant left, i.e., at $r = \infty$, $\varphi = \pi$, we put these into Eqs. (9) and (10), and we have

$$A\cos\delta = -\frac{2Ze^2}{4\pi\varepsilon_0 M v_0^2 b^2}, \tag{11}$$

$$Ab\sin\delta = -1. \tag{12}$$

Accordingly,

$$\tan\delta = \frac{4\pi\varepsilon_0 M v_0^2 b}{2Ze^2}. \tag{13}$$

Thus we can determine the constant δ, and using the result and considering Eq. (12), we can determine the constant A. Consequently, the **trajectory of the α particle**, (9), is completely determined.

The closest approach

As seen in **Fig. 2**, as the incident α particle moves forward from the distant left in the right direction, the angle φ decreases from π. When $\varphi = \pi - \delta$, the α particle reaches the closest to the origin or the nucleus. We call this point P the **closest point** and the distance between P and the nucleus, $r_{\min}$, the **closest approach.** The closest point is the point where the radial component of the velocity of the particle becomes zero, i.e., $dr/dt = 0$.

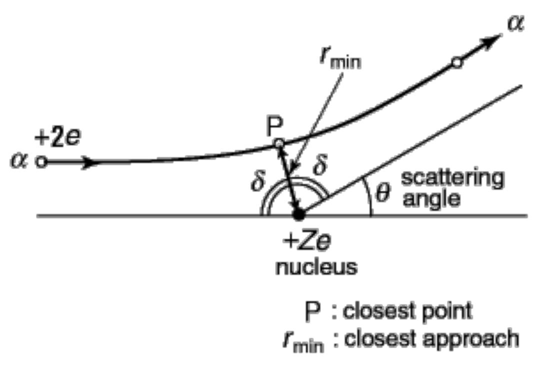

Fig. 2: The closest approach in Rutherford scattering

At the closest point, $\varphi = \pi - \delta$. If we use this relation in the equation of the trajectory, Eq. (9), we can obtain the closest approach $r_{\min}$. However, let us show another method here.

The **energy conservation law** in the present case is written

$$\frac{1}{2}M\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] + \frac{2Ze^2}{4\pi\varepsilon_0 r} = \frac{1}{2}Mv_0^2. \tag{14}$$

The first term on the left-hand side is the kinetic energy, in which the first term in $[\cdots]$ is 0 at the closest point and the second term becomes $v_0^2 b^2/r_{\min}^2$ because of Eq. (5). We therefore have

$$r_{\min}^2 - \left(\frac{4C}{Mv_0^2}\right)r_{\min} - b^2 = 0, \qquad C = \frac{Ze^2}{4\pi\varepsilon_0}. \tag{15}$$

Solving Eq. (15) we can get the closest approach as

$$r_{\min} = \frac{C}{E}\left[1 + \sqrt{1 + \left(\frac{Eb}{C}\right)^2}\,\right], \quad C = \frac{Ze^2}{4\pi\varepsilon_0}, \quad E = \frac{1}{2}Mv_0^2. \tag{16}$$

As seen in Eq. (16), the α particle can come the closest to the nucleus when $b = 0$ which means a head-on collision. Therefore the minimum value of the closest approach is

$$(\text{The minimum value of } r_{\min}) = \frac{2C}{E} = \frac{2Ze^2}{4\pi\varepsilon_0 E}, \quad E = \frac{1}{2}Mv_0^2. \tag{17}$$

Relation between impact parameter and scattering angle

We shall obtain the relation between the impact parameter b and the scattering angle θ. Suppose that the α particle is scattered in the direction of an angle θ. Setting $r = \infty$ in Eq. (9), and comparing the results for $\varphi = \theta$ and $\varphi = \pi$, we get

$$\cos(\theta + \delta) = \cos(\pi + \delta).$$

For a scattering angle $\theta (\neq \pi)$, we therefore have $2\delta = \pi - \theta$, and

$$\delta = \frac{\pi}{2} - \frac{\theta}{2}, \quad \text{or} \quad \tan\delta = \cot\frac{\theta}{2}.$$

$$b = \frac{2Ze^2}{4\pi\varepsilon_0 M v_0^2}\cot\frac{\theta}{2}. \tag{18}$$

The impact parameter b of the α particle scattered in the direction of an angle θ is uniquely obtained by the above equation (18). Consequently we can calculate at what probability the α particle is scattered in any specified scattering angle.

The differential scattering cross section, Rutherford's formula

Inserting this into Eq. (13), we obtain

Equation (18) gives the relation between the impact parameter and the scattering angle. Let us write this equation simply as

$$b = b(\theta). \tag{19}$$

Here the right-hand side of Eq. (18) is expressed by $b(\theta)$.

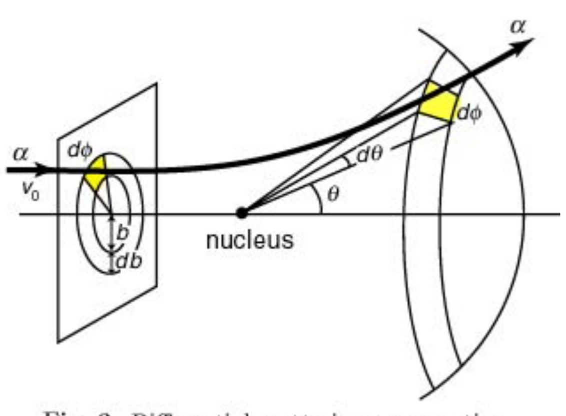

Fig. 3: Differential scattering cross section

Let the number of α particles entering into a unit area in unit time be N.

The number of the particles that pass through the yellow-colored element of area $b\,db\,d\phi$, where the impact parameter is between b and $b + db$ and the angle between ϕ and $\phi + d\phi$, is given by $dN = Nb\,db\,d\phi$. As seen in **Fig. 3**, these α particles would enter into the yellow-colored area on the unit sphere, i.e., the solid angle $\sin\theta\, d\theta\, d\phi$, in which the angles θ and $\theta + d\theta$ are related to the impact parameters b and $b + db$, respectively, through Eq. (18).

Let the number of α particles scattered into a unit solid angle around the angle θ in a unit time be $\sigma(\theta)$. The above dN is written

$$dN = Nb\,db\,d\phi = N\sigma(\theta)\sin\theta\,d\theta\,d\phi.$$

Therefore we have

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \tag{20}$$

Calculating $db/d\theta$ by differentiating Eq. (18) with θ, we obtain

$$\sigma(\theta) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{Z^2 e^4}{M^2 v_0^4} \times \frac{1}{\sin^4(\theta/2)}. \tag{21}$$

This is the famous **Rutherford's formula.** The quantity $\sigma(\theta)$ is the **differential cross section** of Rutherford scattering.