Engineering Mechanics Institute Conference (EMI)
Boston, MA
May 29 – June 1, 2018

Stochastic response analysis and optimum design of nonlinear electromechanical energy harvesters: A Wiener path integral approach

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Outline

- Introduction
- Vibration Energy Harvesting
  - Modelling aspects
  - Optimization
- Wiener path integral (WPI) technique
  - Standard formulation
  - Modification
- Numerical Results
- Conclusions

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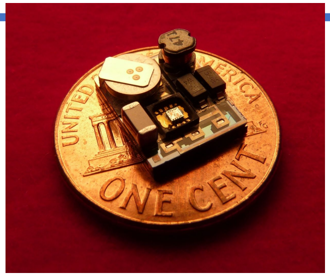
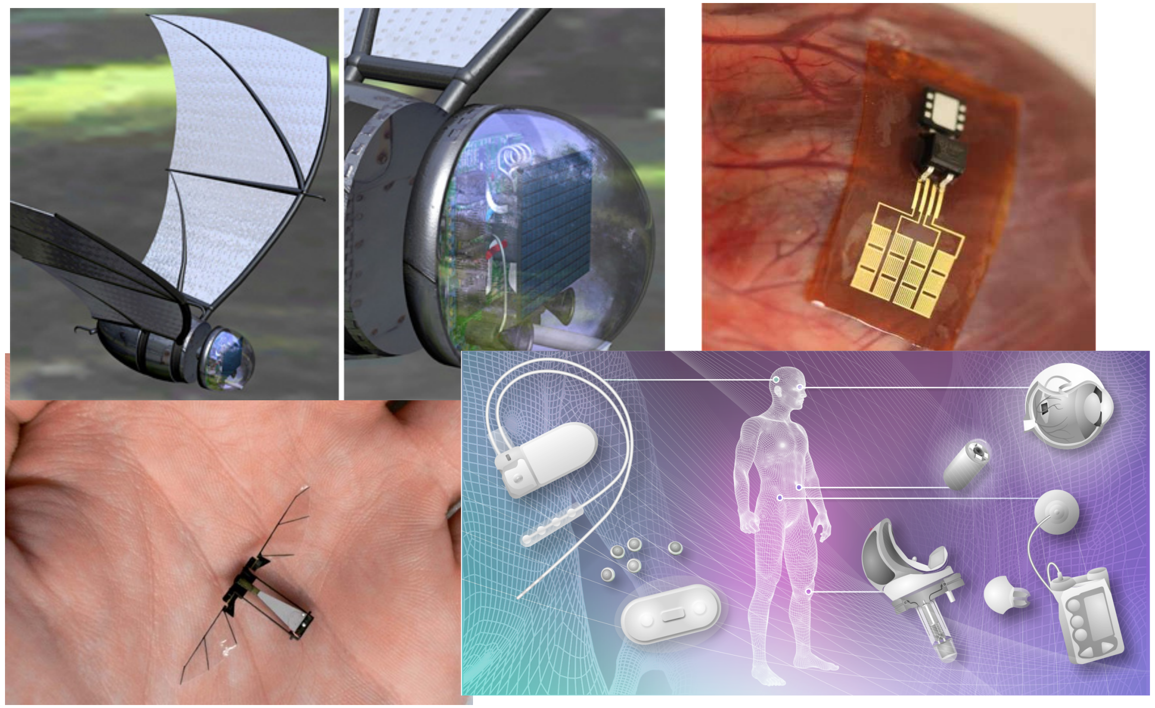
Introduction

- Vibration energy harvesting technology ---------→



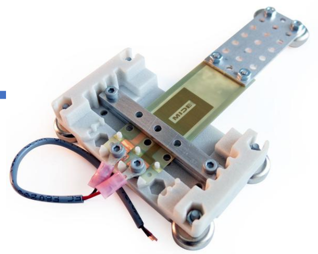
- Potential applications

Micro-robotic applications and
powering of biomedical implants

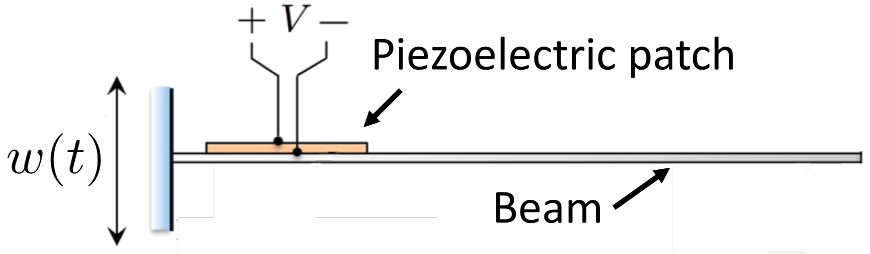


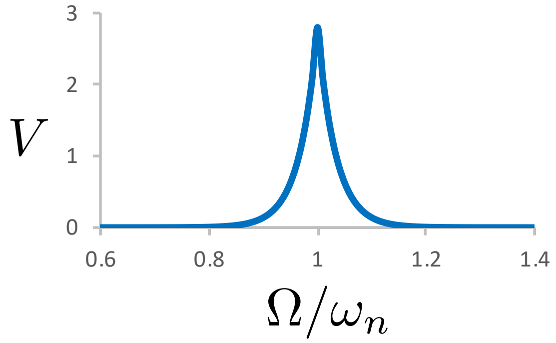
Wireless sensors networks for Structural Health
Monitoring

Introduction

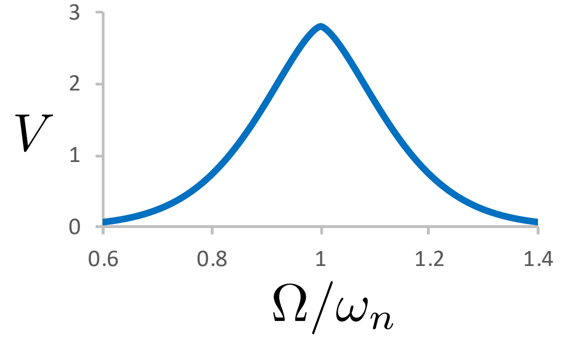


- Linear energy harvester

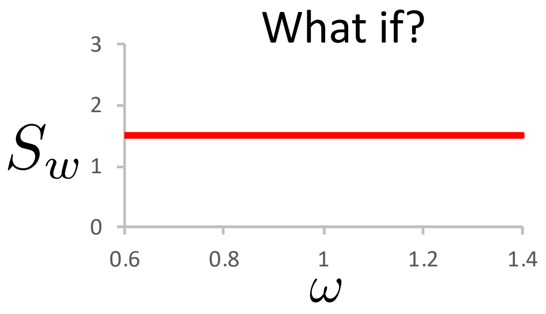




- Nonlinear energy harvester



- More realistic approach: $w(t)$ → stochastic process



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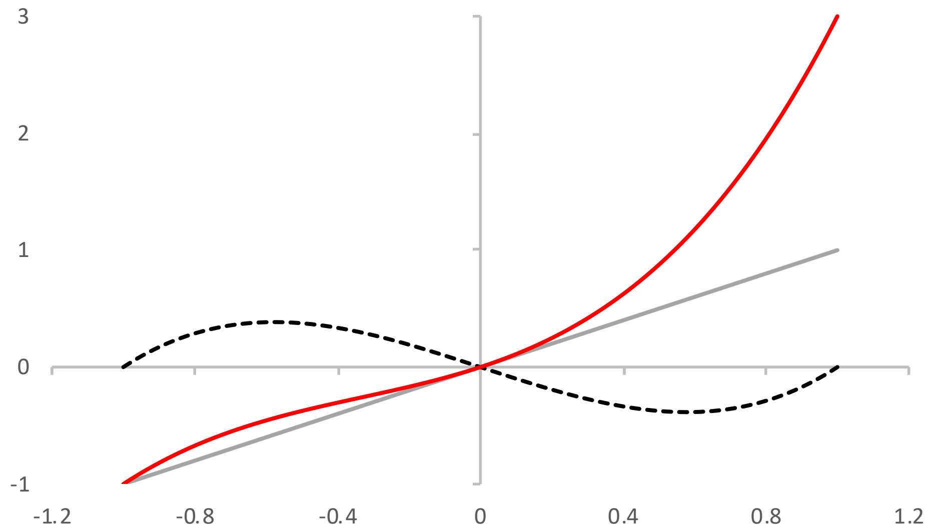
Energy Harvester – Modeling aspects

$$\left.\begin{aligned} &\ddot{x} + 2\zeta\dot{x} + \frac{dU(x)}{dx} + \kappa^2 y = w(t) \\ &\dot{y} + \alpha y - \dot{x} = 0 \end{aligned}\right\}$$

Coupled electromechanical system
(non-dimensional)

Potential $U(x)$	Restoring force	$\frac{dU(x)}{dx}$
monostable	Linear	x
bistable	Nonlinear	$-x + \delta x^3$
monostable if $0 < \lambda \leq 2\sqrt{\delta}$	Nonlinear	$x + \lambda x^2 + \delta x^3$



- Nonlinear (Duffing) harvester with asymmetric monostable potential

- Maximum output for $\lambda = 2\sqrt{\delta}$

He & Daqaq (2016) *Meccanica*

Energy Harvester – Optimization

Electromechanical system

$$\ddot{x} + 2\zeta\dot{x} + x + 2\sqrt{\delta}x^2 + \delta x^3 + \kappa^2 y = w(t)$$

$$\dot{y} + \alpha y - \dot{x} = 0$$

Mean harvested power

$$P_h = \alpha E[y^2]$$

- Ultimate goal: Maximize output → P_h
- Optimum design problem

  Find (δ_*, α_*) that maximize P_h

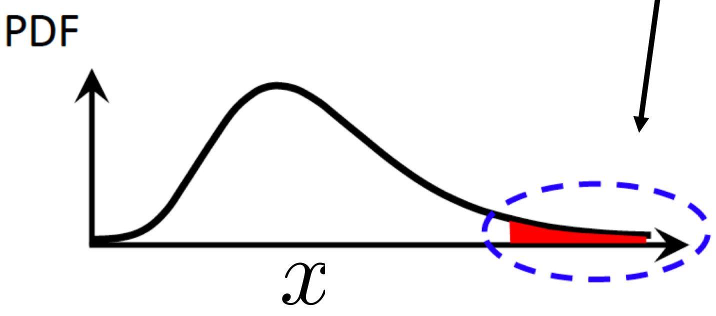
  Subject to: $P_f \leq \epsilon$

- Additional design criteria
  - Prevent mechanical failure
  - Safeguard electrical equipment
  - Space limitations

Constraint on failure probability P_f

- Response PDF → accuracy at the tails

  Commonly used **statistical linearization** treatment → PDF inaccurate at the tails



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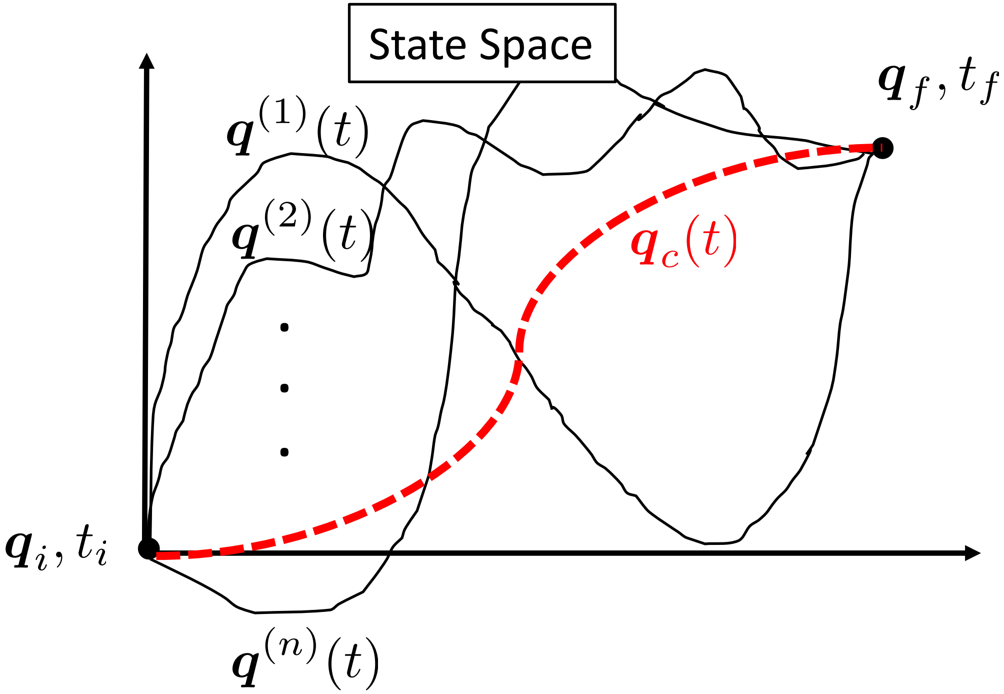
WPI technique - Standard formulation

- Wiener path integral (WPI) ⟶ Wiener (1921), Feynman (1948)

- Transition probability density $p(\boldsymbol{q}_f, t_f | \boldsymbol{q}_i, t_i)$

$$= \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} W[\boldsymbol{q}(t)][\mathrm{d}\boldsymbol{q}(t)] = \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} \Phi \exp\left(-\int_{t_i}^{t_f} \underset{\substack{\uparrow \\ \text{Lagrangian functional}}}{\mathcal{L}}(\boldsymbol{q})\, \mathrm{d}t\right) [\mathrm{d}\boldsymbol{q}(t)]$$

$$\approx \Phi \exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q}_c)\, \mathrm{d}t\right)$$



- Determine $\boldsymbol{q}_c$ by solving:

Variational problem

minimize $\mathcal{J}(\boldsymbol{q}) = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q})\, \mathrm{d}t$

→ Euler-Lagrange equations

→ Rayleigh-Ritz direct method

WPI technique - Extension

Governing equations

$$\boldsymbol{D}[\boldsymbol{q}(t)] = \boldsymbol{w}(t) \longrightarrow \cancel{\mathcal{L}(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{D}[\boldsymbol{q}]^T \boldsymbol{B}^{-1}\boldsymbol{D}[\boldsymbol{q}]}$$

$\boldsymbol{w}(t)$ white noise process

$$E[\boldsymbol{w}(t_1)\boldsymbol{w}(t_2)] = \underbrace{\begin{bmatrix} 2\pi S_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 2\pi S_0 \end{bmatrix}}_{\boldsymbol{B}} \delta(t_2 - t_1)$$

Mechanical oscillator - Underdetermined SDE

$$\ddot{x} + 2\zeta\dot{x} + x + 2\sqrt{\delta}x^2 + \delta x^3 + \kappa^2 y = w(t)$$

Electrical circuit - Constraint

$$\dot{y} + \alpha y - \dot{x} = 0$$

$$\boldsymbol{D}[\boldsymbol{q}(t)] = \begin{bmatrix} w(t) \\ 0 \end{bmatrix}$$

$\boldsymbol{B}$ singular

$$E\{\begin{bmatrix} w(t_1) & 0 \end{bmatrix} \begin{bmatrix} w(t_2) \\ 0 \end{bmatrix}\} = \overbrace{\begin{bmatrix} 2\pi S_0 & 0 \\ 0 & 0 \end{bmatrix}}^{\boldsymbol{B}} \delta(t_2 - t_1)$$

$$\mathcal{L}(x,y) = \frac{1}{4\pi S_0}\left[\ddot{x} + 2\zeta\dot{x} + x + 2\sqrt{\delta}x^2 + \delta x^3 + \kappa^2 y\right]^2$$

Constrained variational problem

minimize $\mathcal{J}(x,y) = \int_{t_i}^{t_f} \mathcal{L}(x,y)\,\mathrm{d}t$

subject to $\dot{y} + \alpha y - \dot{x} = 0$

WPI technique - Extension

Constrained variational problem

minimize $\mathcal{J}(x,y) = \int_{t_i}^{t_f} \mathcal{L}(x,y)\,\mathrm{d}t$

subject to $\dot{y} + \alpha y - \dot{x} = 0$

- Rayleigh-Ritz method and nullspace approach

Expansion in a polynomial basis: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx \boldsymbol{\psi}(t) + \boldsymbol{c}^T \boldsymbol{h}(t)$

Functional $\mathcal{J}(x,y)$: function $J(\boldsymbol{c})$ of $\boldsymbol{c}$ → minimize $J(\boldsymbol{c})$

Constraint: polynomial function of t, linear in $\boldsymbol{c}$ → $\boldsymbol{A}\boldsymbol{c} - \boldsymbol{b} = 0$

$$\frac{\partial J(\hat{\boldsymbol{c}})}{\partial \hat{\boldsymbol{c}}} = 0$$

$$\hat{\boldsymbol{c}} \in \mathrm{Null}(\boldsymbol{A})$$

- Euler-Lagrange equations and Lagrange multipliers

Unconstrained variational problem

Lagrange multiplier

minimize $\mathcal{J}^*(x,y) = \int_{t_i}^{t_f} \mathcal{L}^*(x,y)\,\mathrm{d}t$ where $\mathcal{L}^*(x,y) = \mathcal{L}(x,y) + \lambda(t)(\dot{y} + \alpha y - \dot{x})$

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Numerical results

- Transition PDF

$$p(x_f, y_f, \dot{x}_f, t_f | x_i, y_i, \dot{x}_i, t_i) \approx \Phi \exp\left(-\int_{t_i}^{t_f} \mathcal{L}\left(x_c, y_c\right) \mathrm{d}t \right) \longrightarrow$$

Stationary joint response PDF

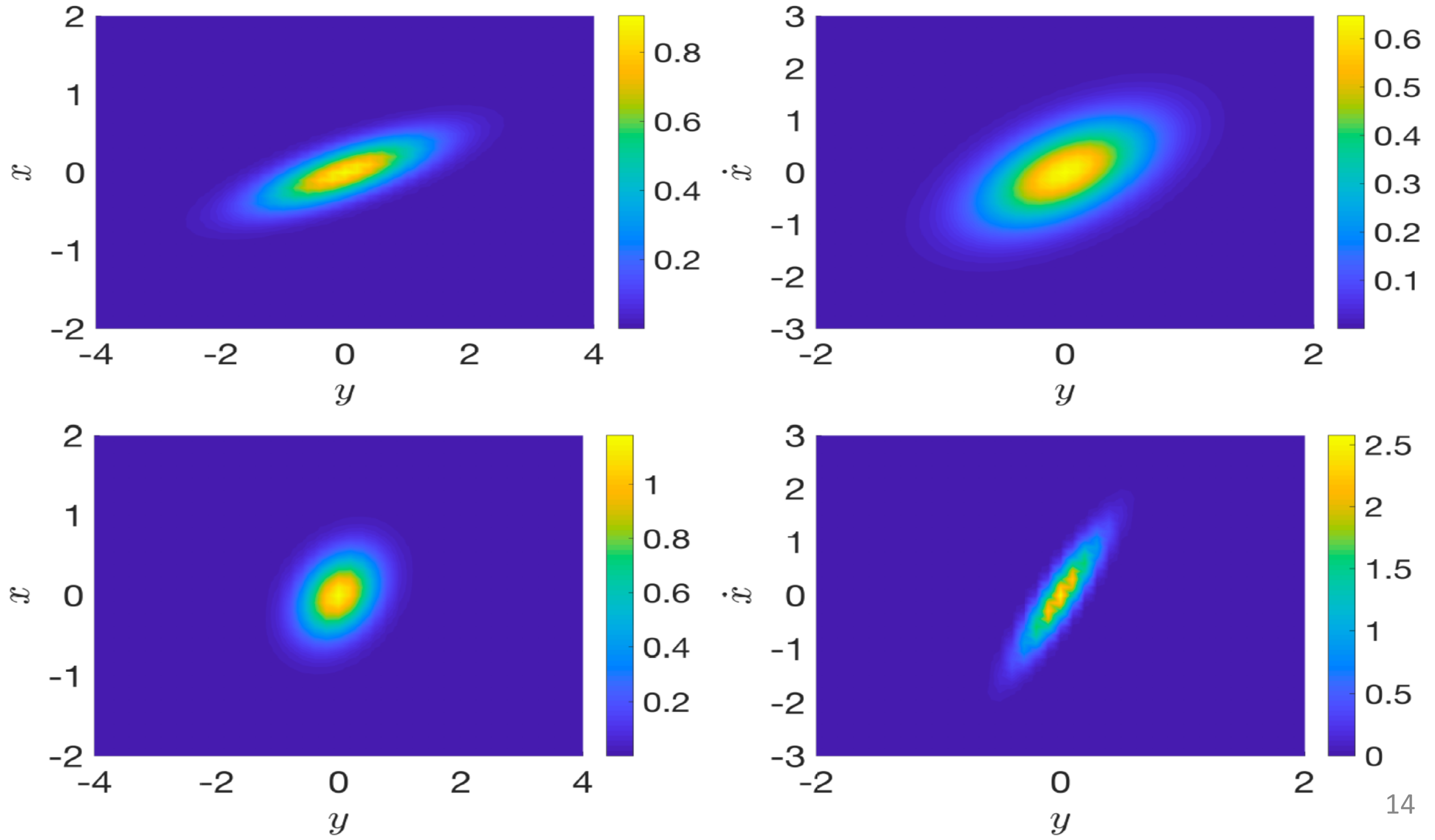
$$p(x, y, \dot{x})$$

- Linear Harvester

$$\dot{y} + \alpha y - \dot{x} = 0$$

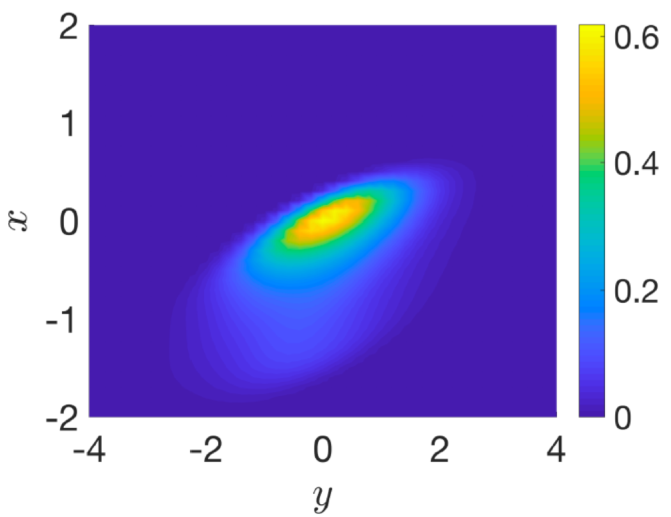
$$\alpha = 0.8$$

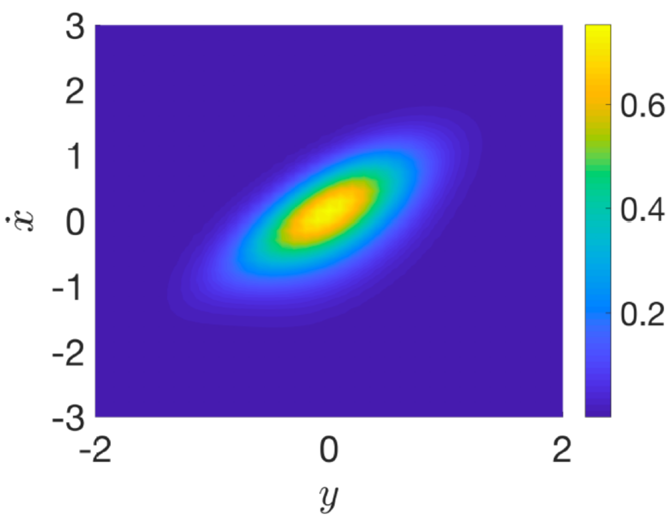
$$\alpha = 3$$

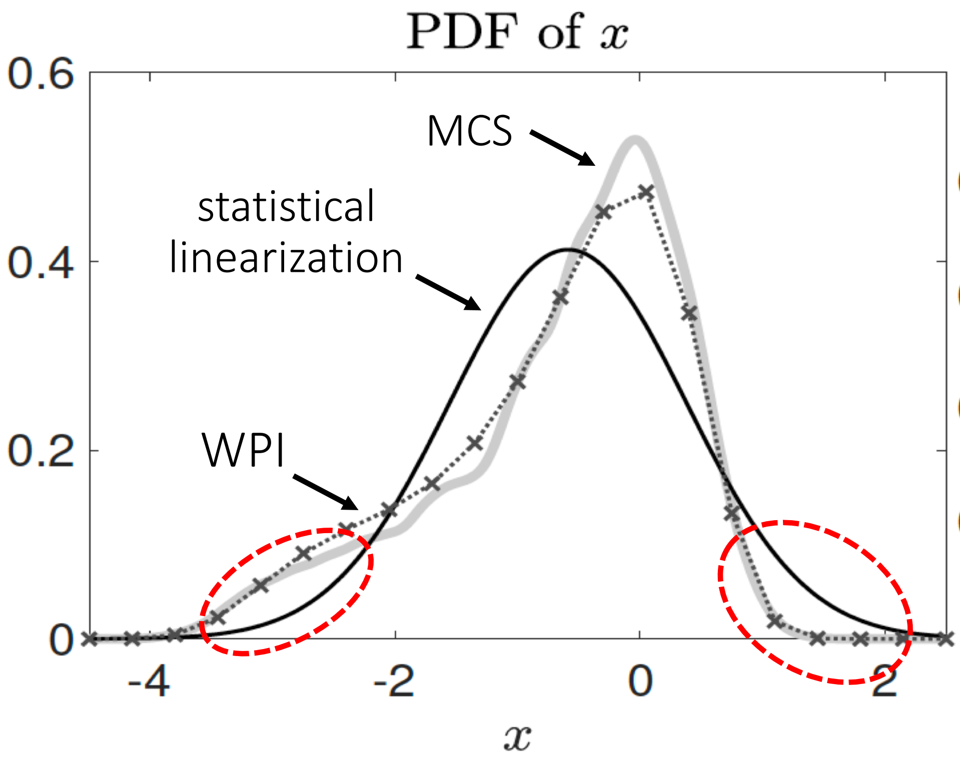


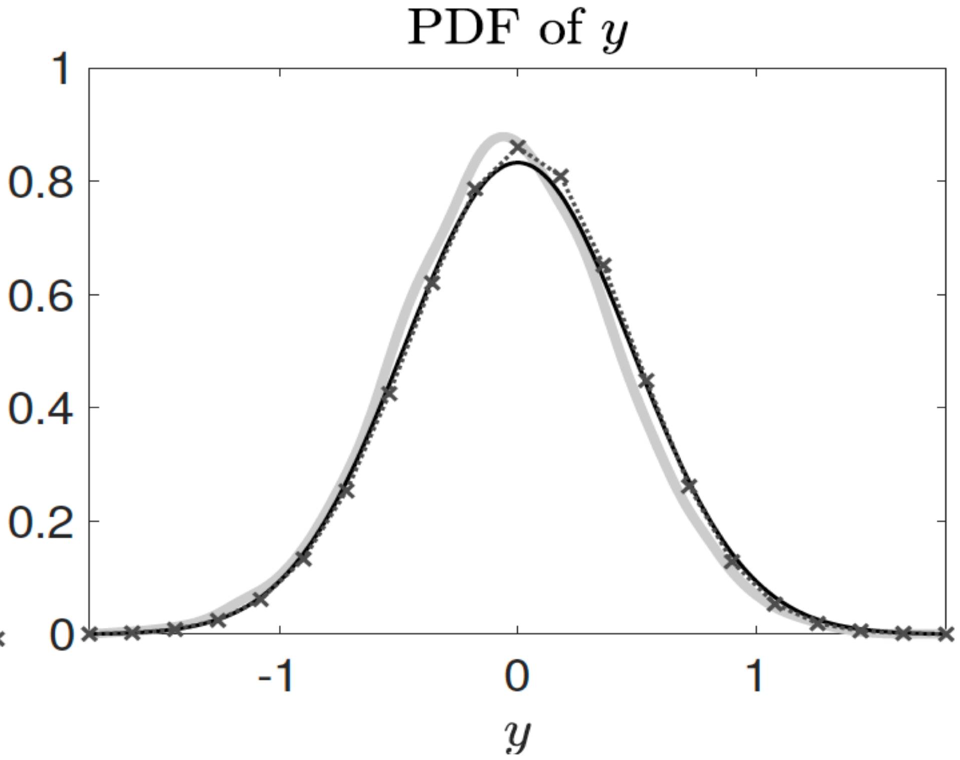
Numerical results

- Nonlinear harvester
- Statistical linearization: inaccurate probability of failure P_f







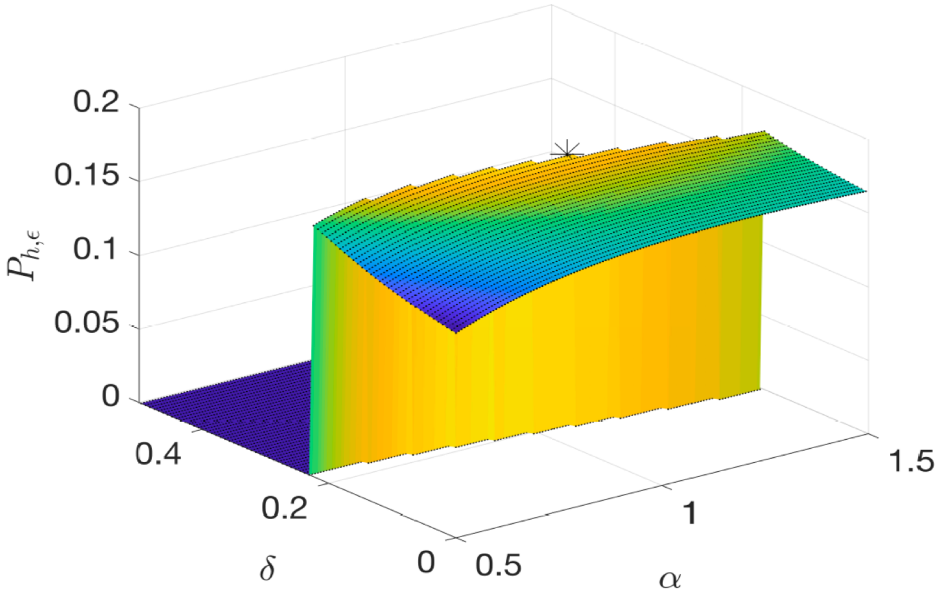


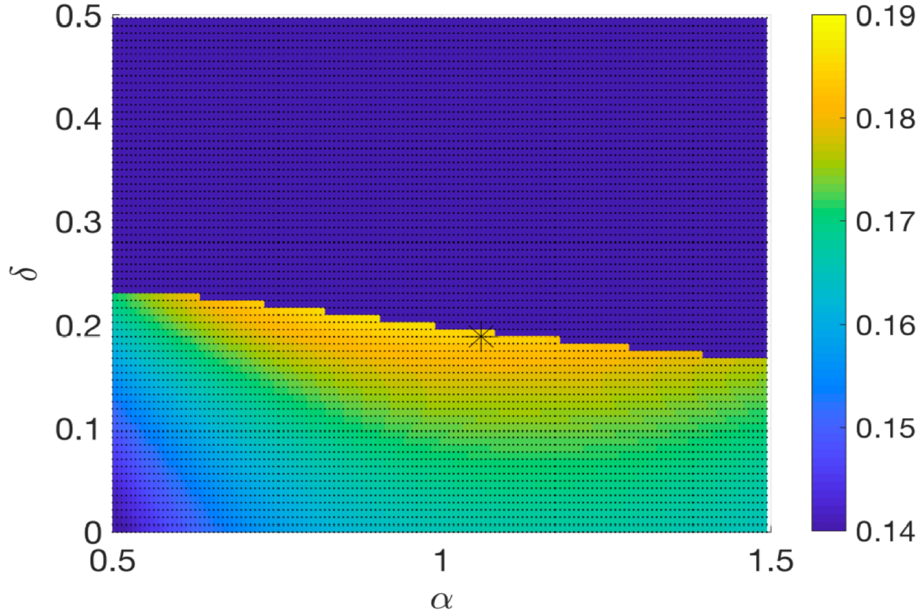
Numerical results – Optimum design

- Design of nonlinear energy harvester for <u>maximum mean harvested power</u> **and** <u>constrained probability of failure.</u>

- This is $\max_{(\alpha,\delta)} P_h \quad \text{s.t.} \quad P(x < -3) \leq 10^{-2}$

- Full grid using **statistical linearization**: 10296 objective function evaluations

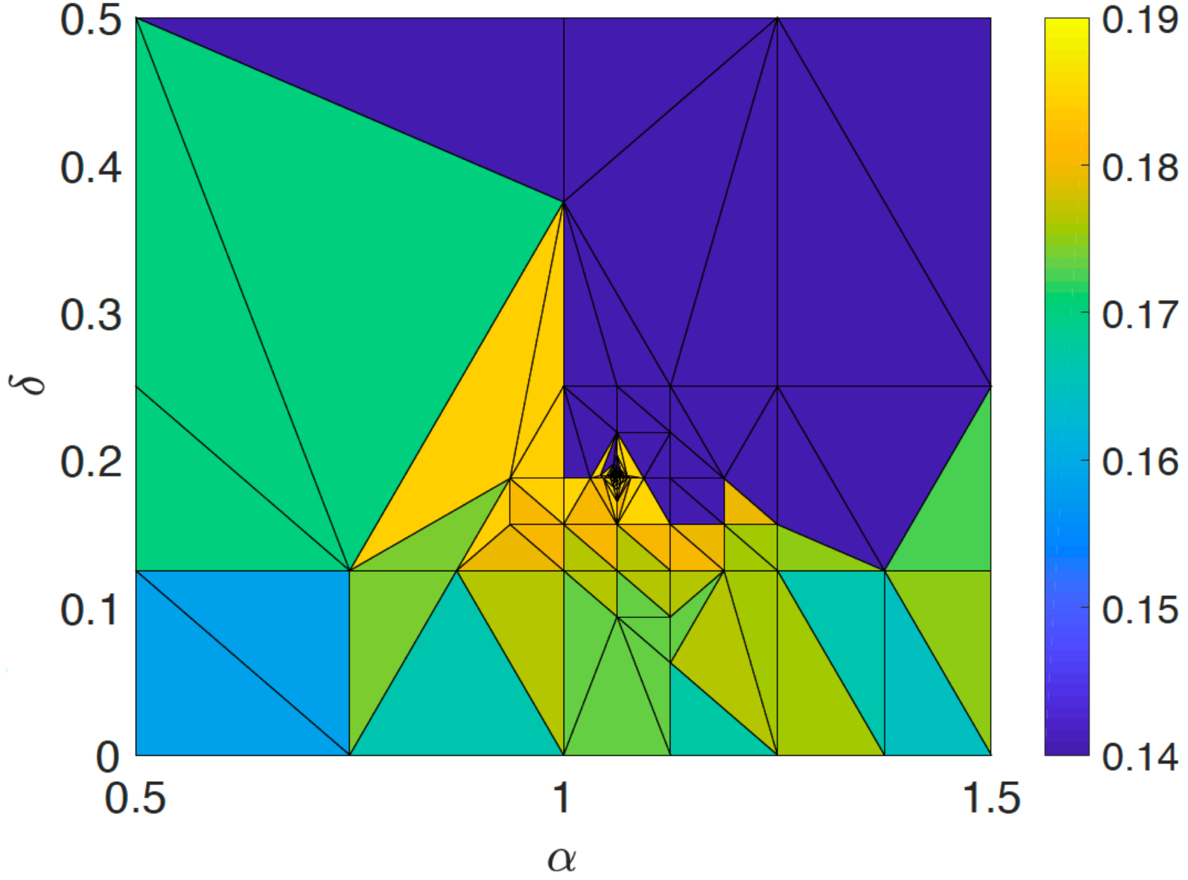




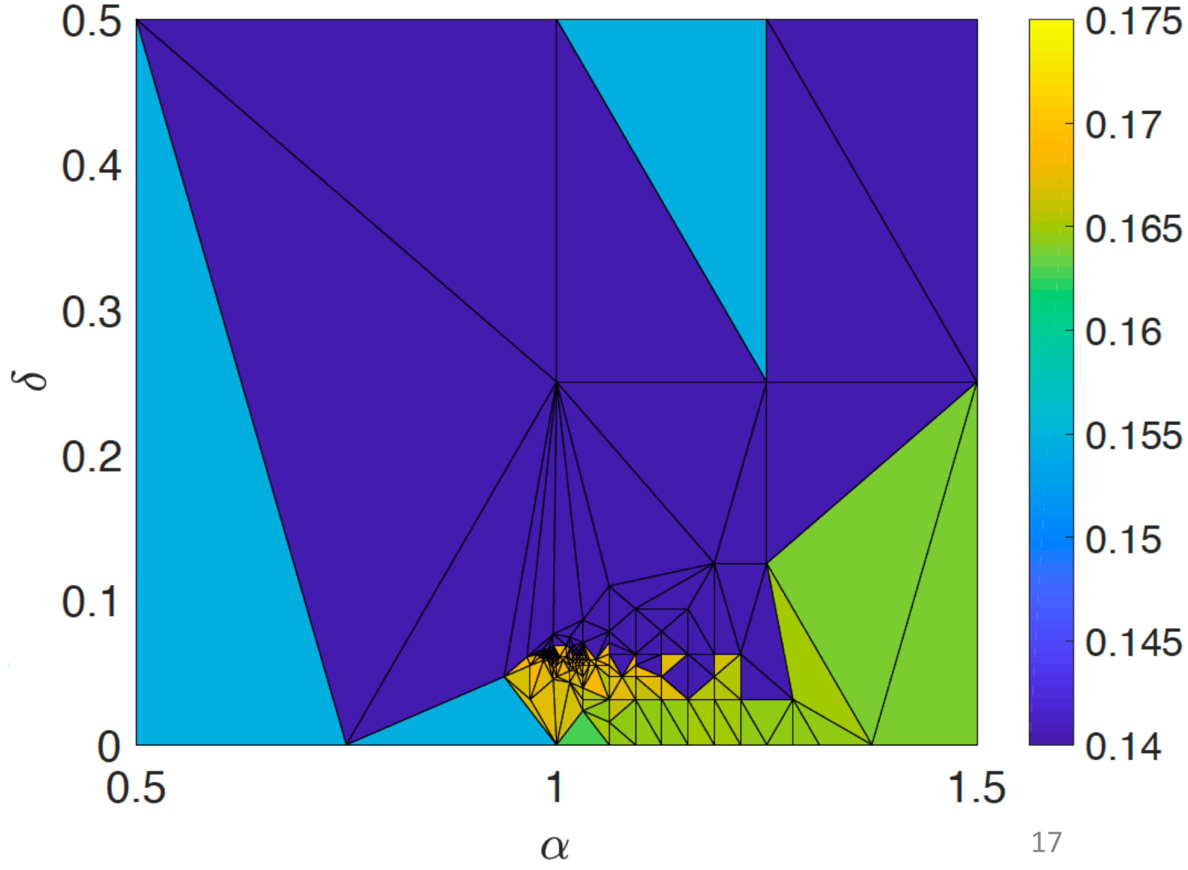
Numerical results – Optimum design

- Gradient-free optimization algorithm: **Generalized Pattern Search (GPS)**

statistical linearization: 164 evaluations



WPI technique: 144 evaluations



Numerical results – Optimum design

- Assessment of optimum designs using Monte-Carlo Simulation (50000 realizations)

Constraint: $P_f = P(x < -3.0) \leq 10^{-2}$		
	WPI optimum $(\alpha, \delta) = (0.9874{,}0.0625)$	Stat. Lin. optimum $(\alpha, \delta) = (1.0580{,}0.1907)$
P_h	0.16886	0.18530
P_f	0.00998	0.00999

$P_f > 10^{-2}$

constraint violation

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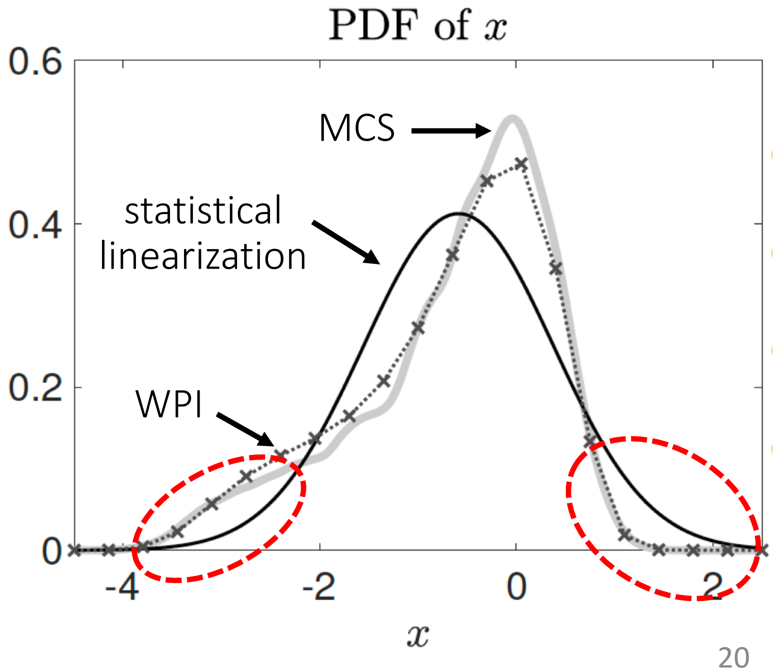
Conclusions

- Path integral technique modification to enable the analysis of vibratory energy harvesters
- Formulation of constrained variational problem
  - E-L equations and Lagrange multipliers ⟶ Calculus of Variations
  - Rayleigh-Ritz and nullspace ⟶ computationally efficienrt
- Efficient computational scheme for accurate determination of response PDF
- Design optimization for maximum output and constrained probability of failure P_f
- Path integral significantly outperforms a commonly utilized statistical linearization solution treatment
- Statistical linearization based (constrained P_f) optimum design yields either sub-optimal solutions, or solutions that violate the constraint.



Thank you!