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Computationally efficient stochastic response determination of high-dimensional dynamical systems via a Wiener path integral variational formulation with free boundaries

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Outline

- Introduction
- Wiener path integral (WPI) technique - Standard formulation
- Wiener path integral (WPI) technique - Reformulation
- Numerical results
- Conclusions

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Introduction

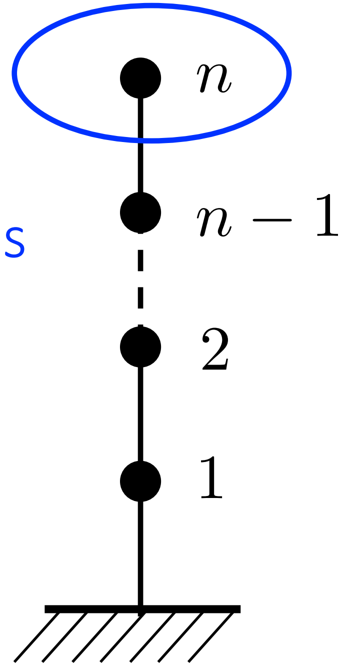
- Engineering Stochastic Dynamics
- Wiener Path Integral (WPI) techniques
- Variational formulation ⟶ **Joint response PDF**
- Recent work:
  - ➢ MDOF systems — Kougioumtzoglou et. al., J Appl. Mech. (2015); Psaros et. al., MSSP (2018)
  - ➢ Singular Diffusion Matrices (energy harvesting) — Petromichelakis et. al., Prob. Eng. Mech. (review); Petromichelakis et. al., Prob. Eng. Mech. (2018)
  - ➢ Realistic (non-white, non-Gaussian, etc.) excitations — Psaros et. al., J Sound Vibr. (2018)
  - ➢ Fractional derivative terms — Di Matteo et. al., Prob. Eng. Mech. (2014)

Introduction

- High-dimensional MDOF systems $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t)$
- Joint response PDF $\longrightarrow$ **intractable** (n-DOF system $\longrightarrow$ $2n$-dimensional PDF)
- Usually marginal or marginalized joint PDFs are sufficient
- Reformulation of the WPI with free boundaries
  - ➢ Modification of the variational problem to include free boundaries
  - ➢ Determination of any m-dimensional joint PDF directly

$$1 \leq m \leq 2n$$

  - ➢ Can be orders of magnitude faster than Monte Carlo simulation



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WPI technique - Standard formulation

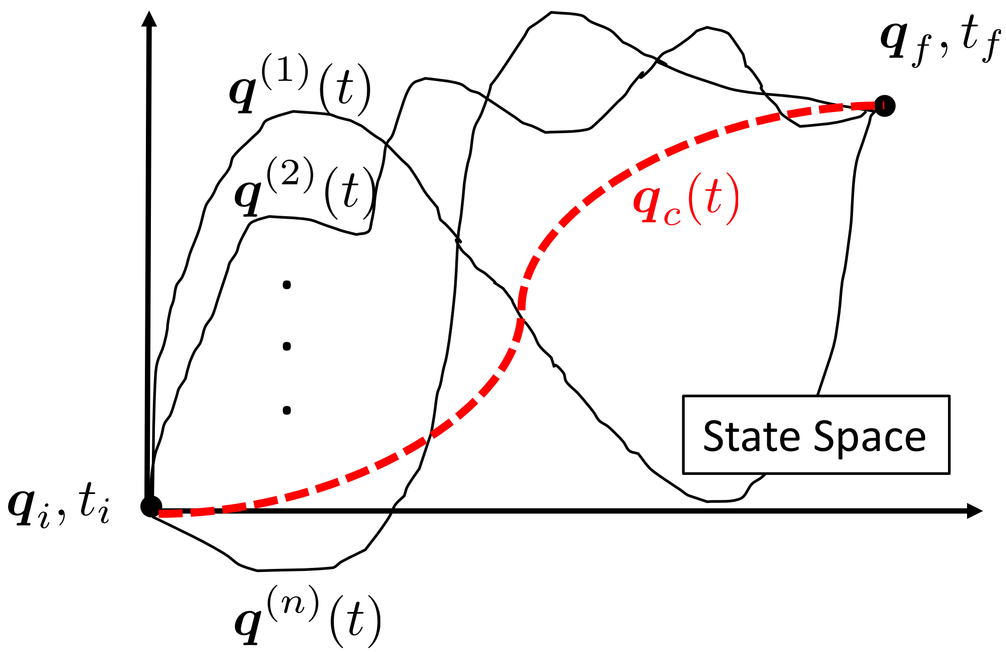
- Wiener path integral (WPI) $\longrightarrow$ Wiener (1921), Feynman (1948)

- Transition probability density

$$p(\boldsymbol{q}_f, t_f | \boldsymbol{q}_i, t_i) = \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} W[\boldsymbol{q}(t)][\mathrm{d}\boldsymbol{q}(t)]$$

$$= \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} \Phi \exp\left(- \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q})\, \mathrm{d}t \right) [\mathrm{d}\boldsymbol{q}(t)]$$

$$\approx \Phi \exp\left(- \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q_c})\, \mathrm{d}t \right)$$



$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t) \longrightarrow \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \frac{1}{2}[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]^T \boldsymbol{B}^{-1} [\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]$$

WPI technique - Standard formulation

- Determine $\boldsymbol{q}_c = (\boldsymbol{x}_c, \dot{\boldsymbol{x}}_c, \ddot{\boldsymbol{x}}_c)$ by solving:

  Extremality condition: $\delta \mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = 0$

Variational problem

minimize $\mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})\,\mathrm{d}t$

Single Degree of Freedom (SDOF)

$$\cancel{\left[\left(\mathcal{L}_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}}\right)\delta x\right]_{t_i}^{t_f} + \left[\mathcal{L}_{\ddot{x}}\delta\dot{x}\right]_{t_i}^{t_f}} + \int_{t_i}^{t_f}\left(\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}}\right)\delta x\mathrm{d}t = 0$$

Fixed Boundary Conditions

$x(t_i) = 0 \qquad x(t_f) = x_f$

$\dot{x}(t_i) = 0 \qquad \dot{x}(t_f) = \dot{x}_f$

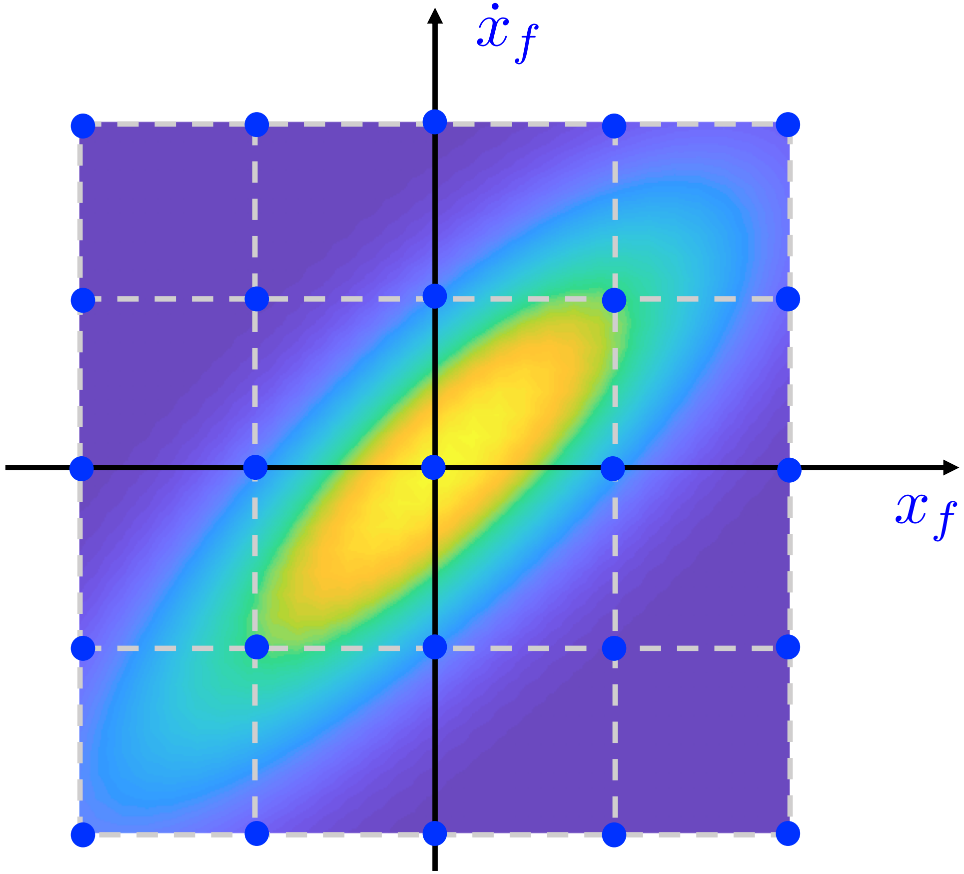
Euler-Lagrange equation

$$\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}} = 0$$

BVP

WPI technique - Standard formulation

$$p(x_f, \dot{x}_f, t_f | 0, 0, t_i) \approx \Phi \exp\left(- \int_{t_i}^{t_f} \mathcal{L}\left(x_c, \dot{x}_c, \ddot{x}_c\right) \mathrm{d}t \right)$$



<u>SDOF system</u>

N^2 BVP solutions

<u>n - DOF system</u>

N^{2n} BVP solutions

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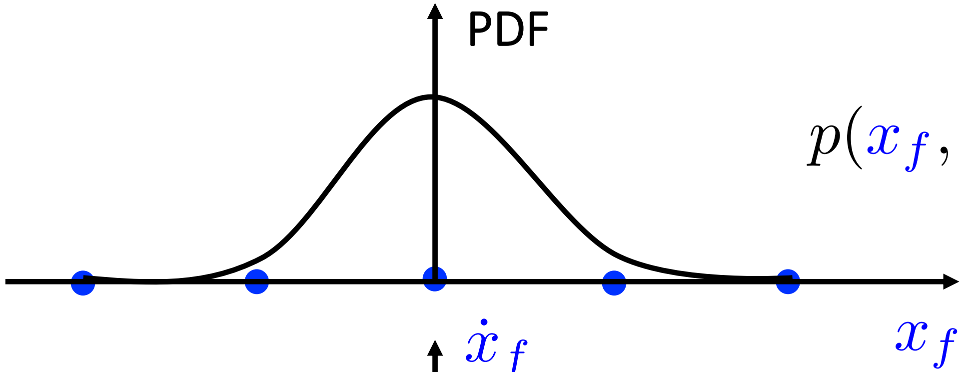
WPI technique - Reformulation

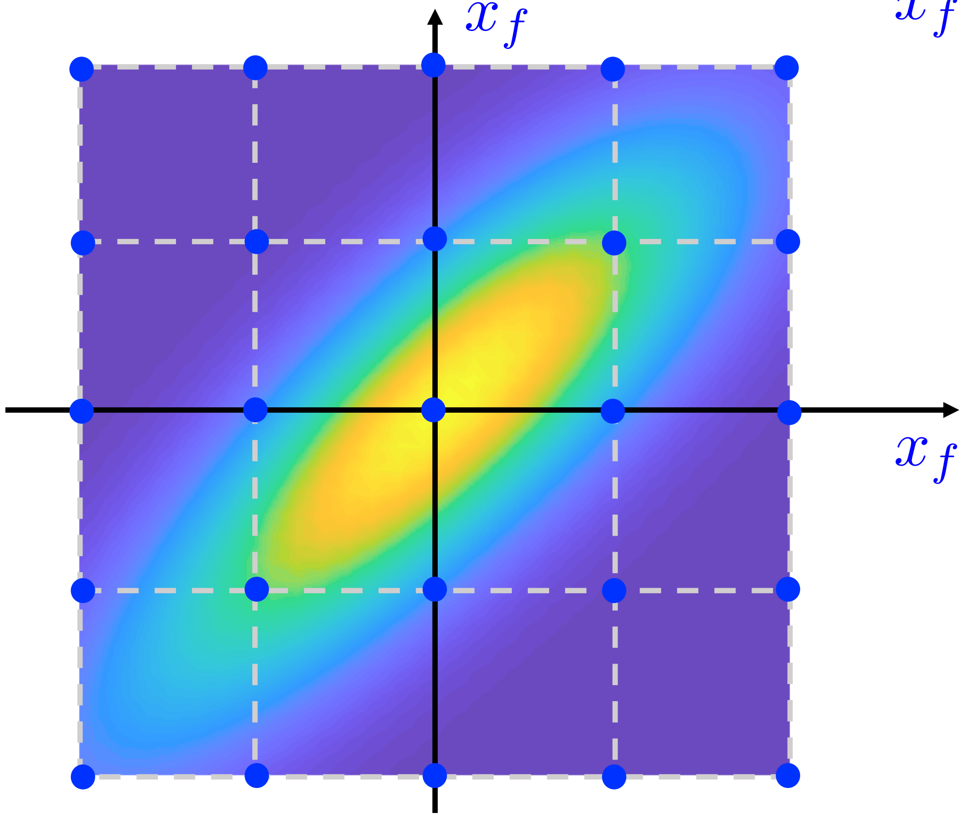
N

points

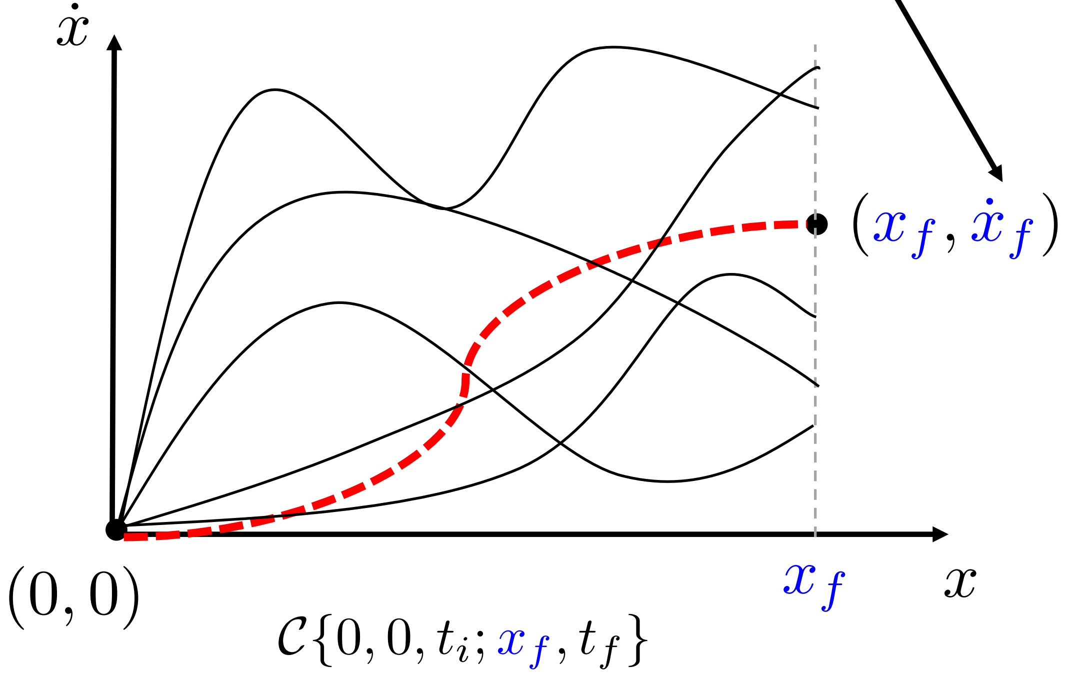


N^2

points



FREE endpoint velocity $\dot{x}_f$



WPI technique - Reformulation

Variational problem: minimize $\mathcal{J}\left(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}\right)=\int_{t_i}^{t_f}\mathcal{L}\left(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}\right)\mathrm{d}t$

Extremality condition:

$\delta\mathcal{J}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})=0$

$$\left[\left(\mathcal{L}_{\dot{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}}\right)\delta x\right]_{t_i}^{t_f}+\left[\mathcal{L}_{\ddot{x}}\delta\dot{x}\right]_{t_i}^{t_f}+\int_{t_i}^{t_f}\left(\mathcal{L}_x-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}}\right)\delta x\mathrm{d}t=0$$

Free endpoint velocity

$$p(x_f)\begin{cases} x(t_i)=0 & x(t_f)=x_f \\ \dot{x}(t_i)=0 & \left[\mathcal{L}_{\ddot{x}}\right]_{t=t_f}=0 \end{cases}$$

Free endpoint displacement

$$p(\dot{x}_f)\begin{cases} x(t_i)=0 & \left[\mathcal{L}_{\dot{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}}\right]_{t=t_f}=0 \\ \dot{x}(t_i)=0 & \dot{x}(t_f)=\dot{x}_f \end{cases}$$

Euler-Lagrange equation

$$\mathcal{L}_x-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}}=0$$

WPI technique - Reformulation

- n-DOF systems $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t)$
- m-dimensional marginalized joint PDF $p(\boldsymbol{x}_{U,f}, \dot{\boldsymbol{x}}_{V,f}, t_f)$

$|U| + |V| = m$

$1 \leq m \leq 2n$

$$\mathcal{L}_{x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}_i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}_i} = 0$$ (Euler-Lagrange equations)

If $i \in U$ $x_i(t_f) = x_{i,f}$ (fixed displacement)

Else $\left[\mathcal{L}_{\dot{x}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}_i}\right]_{t=t_f} = 0$ (free displacement)

If $i \in V$ $\dot{x}_i(t_f) = \dot{x}_{i,f}$ (fixed velocity)

Else $[\mathcal{L}_{\ddot{x}_i}]_{t=t_f} = 0$ (free velocity)

for all $i = 1, \ldots, n$

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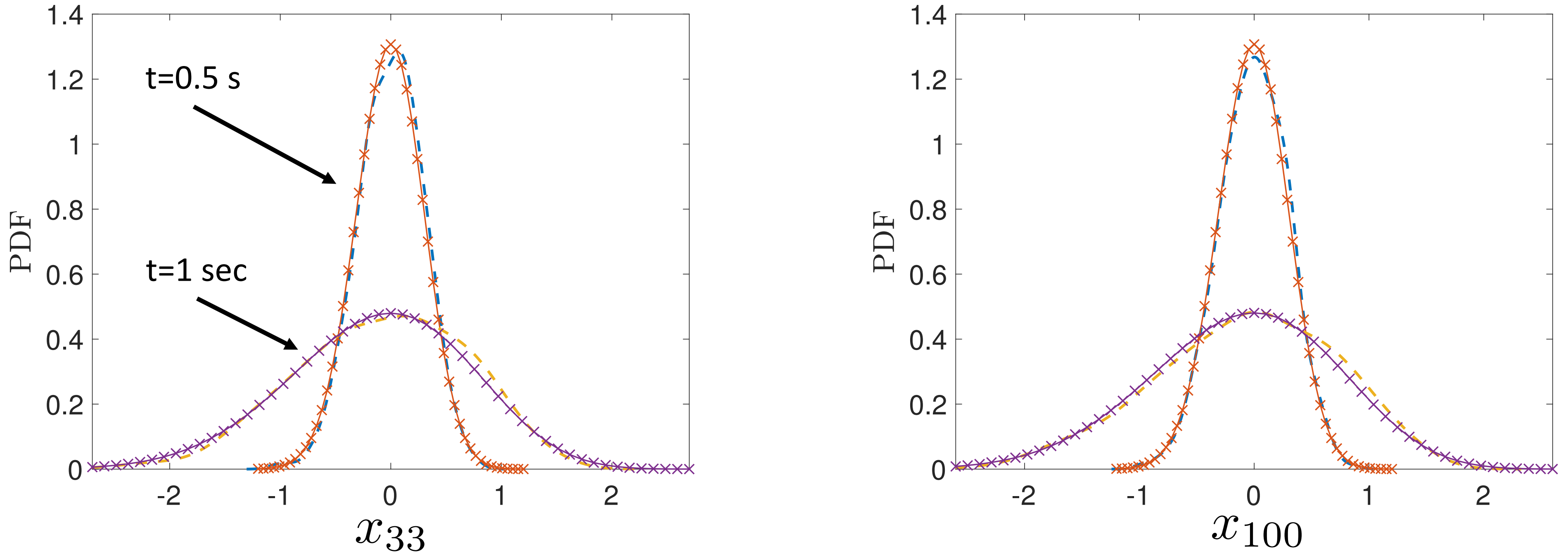
1st Example: 100-DOF under white noise

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + g(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t)$$

$$\boldsymbol{M} = \begin{bmatrix} m_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_0 \end{bmatrix}$$

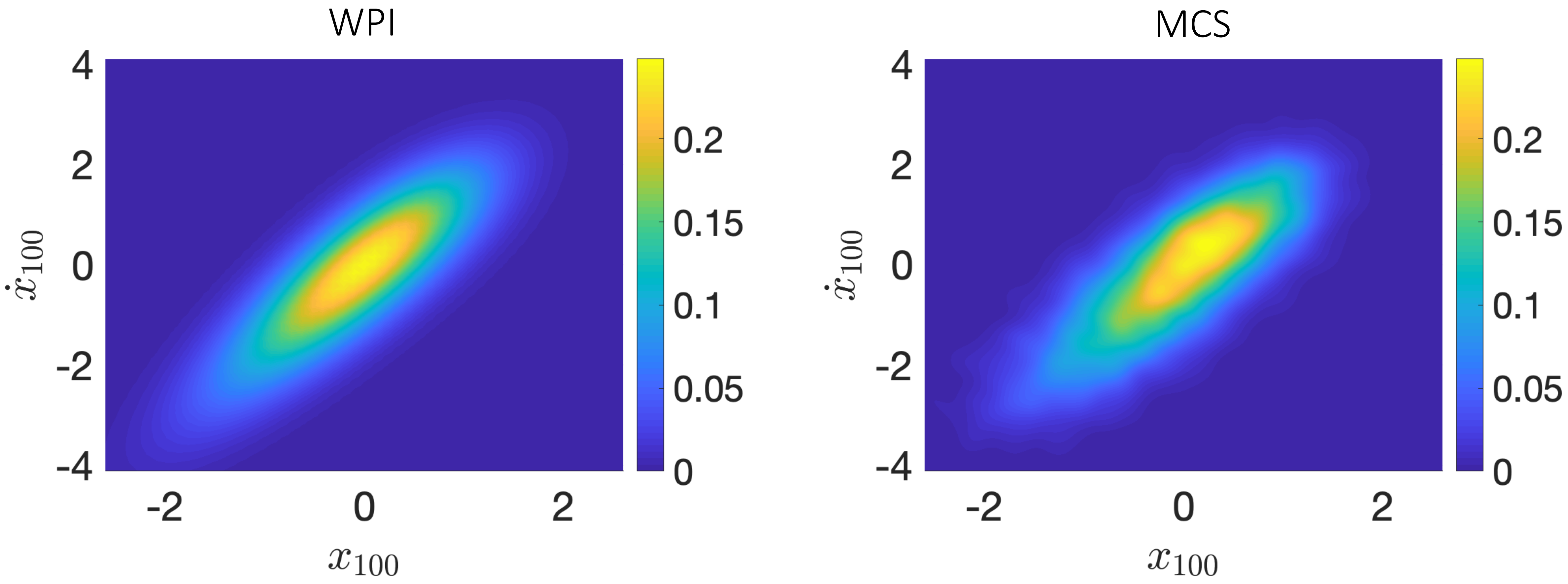
$$\boldsymbol{C} = \begin{bmatrix} 2c_0 & -c_0 & \cdots & 0 \\ -c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -c_0 \\ 0 & \cdots & -c_0 & 2c_0 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 2k_0 & -k_0 & \cdots & 0 \\ -k_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -k_0 \\ 0 & \cdots & -k_0 & 2k_0 \end{bmatrix}$$

$$\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \begin{bmatrix} \epsilon_k k_0 x_1^3 + 2\sqrt{\epsilon_k k_0} x_1^2 + \epsilon_c c_0 \dot{x}_1^3 + 2\sqrt{\epsilon_c c_0} \dot{x}_1^2 \\ \vdots \\ \epsilon_k k_0 x_n^3 + 2\sqrt{\epsilon_k k_0} x_n^2 + \epsilon_c c_0 \dot{x}_n^3 + 2\sqrt{\epsilon_c c_0} \dot{x}_n^2 \end{bmatrix}$$



1st Example: 100-DOF under white noise

- Marginalized joint PDF $p(x_{100}, \dot{x}_{100})$



2nd Example: 10-DOF under white noise

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + g(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t)$$

$$\boldsymbol{M} = \begin{bmatrix} m_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 2c_0 & -c_0 & \cdots & 0 \\ -c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -c_0 \\ 0 & \cdots & -c_0 & 2c_0 \end{bmatrix} \qquad \boldsymbol{K} = \begin{bmatrix} 2k_0 & -k_0 & \cdots & 0 \\ -k_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -k_0 \\ 0 & \cdots & -k_0 & 2k_0 \end{bmatrix}$$

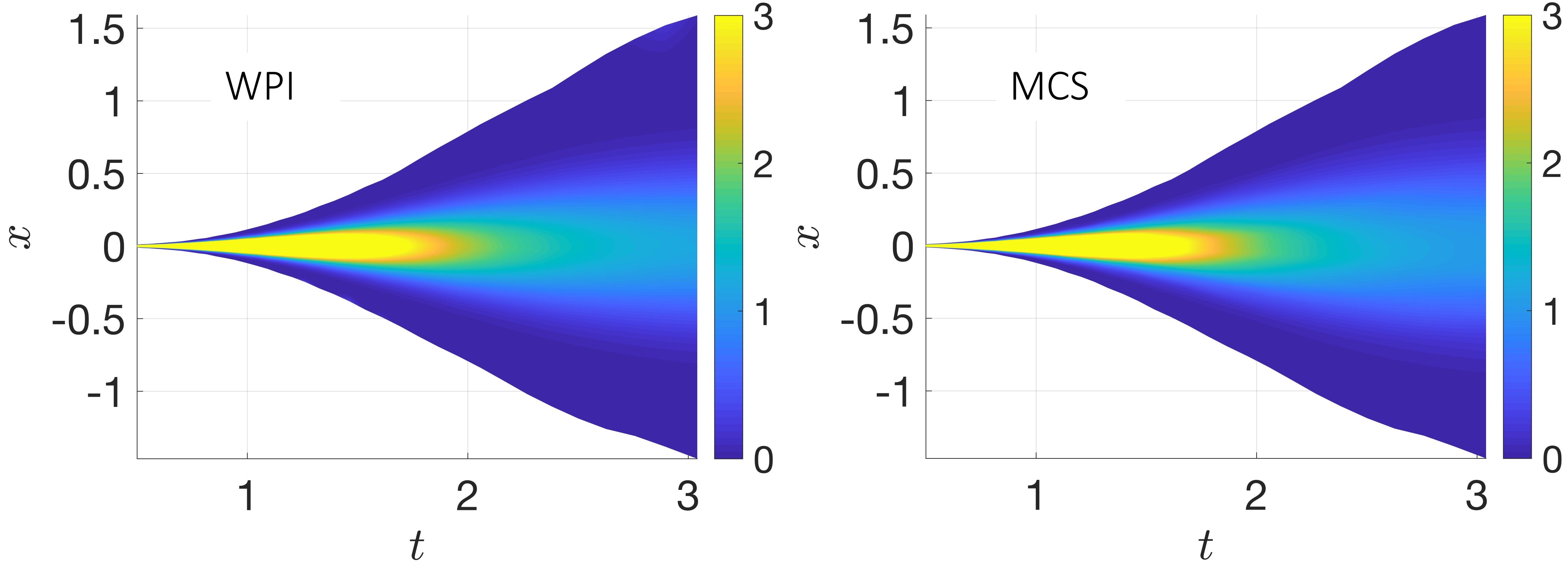
$$\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \begin{bmatrix} \epsilon_k k_0 x_1^3 + 2\sqrt{\epsilon_k k_0} x_1^2 + \epsilon_c c_0 \dot{x}_1^3 + 2\sqrt{\epsilon_c c_0} \dot{x}_1^2 \\ \vdots \\ \epsilon_k k_0 x_n^3 + 2\sqrt{\epsilon_k k_0} x_n^2 + \epsilon_c c_0 \dot{x}_n^3 + 2\sqrt{\epsilon_c c_0} \dot{x}_n^2 \end{bmatrix}$$

Live calculation...

3rd Example: Morison oscillator under colored noise

$$\left.\begin{aligned} p\ddot{y} + q\dot{y} + ry &= w(t) \\ \ddot{x} + 2\omega\xi\dot{x} + \omega^2 x + \frac{1}{2}\frac{C_D \rho D}{M_0}|V+\dot{x}|(V+\dot{x}) &= y(t) \end{aligned}\right\}$$

4-th order equation of motion:
Extra free boundary conditions



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Conclusions

- Direct determination of marginalized PDFs
- Path integral representation with free boundaries
- Modification of the core variational problem – free boundary conditions
- Depending on the dimensionality of the target PDF can be orders of magnitude more efficient than MCS
- Examples: 100-DOF under white noise, Morison oscillator under colored noise

Thank you!