Bayesian community detection[1]

Random graphs and network statistics

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[1] https://github.com/yannmclatchie/karate

Community detection

Suppose we observe an adjacency matrix $A = (A_{ij})$ of a graph, and task to infer the community memberships of each node (z_i), $i = 1, \dots, n$. One way to do this is to model the structure of the graph, and specifically model $A \overset{d}{=} \mathsf{SBM}(z, P)$ with the link probability matrix P also unobserved.

Aim

Pas and Vaart (2018) task to produce a consistent Bayesian estimator of the community structure for an SBM given a fixed number of communities.

Consistency

An estimator $\bar{X}_n$ of a random variable X is deemed *weakly consistent* (partial recovery) if it converges *in probability* to the true value of the variable X^*,

$$\lim_{n\to\infty} \mathbb{P}(|\bar{X}_n - X^*| > \epsilon) = 0, \ \forall \epsilon > 0.$$

An estimator $\bar{X}_n$ of a random variable X is deemed *strongly consistent* (exact recovery) if it converges *almost surely* to the true value of the variable X^*,

$$\mathbb{P}(\lim_{n\to\infty} \bar{X}_n = X^*) = 1.$$

Community structure in the SBM

We have an undirected random graph G on n nodes, each belonging to one of $K \in \mathbb{N}$ classes. Each node is randomly labelled according to i.i.d. $Z_1, \ldots, Z_n$ random variables with probability $\pi_1, \ldots, \pi_K$. Given this set of labels, edges between nodes are independently sampled from a Bernoulli random variable dependent on the label, $\mathbb{P}(A_{ij} = 1 \mid Z) = P_{Z_i, Z_j}$.

The likelihood of our SBM is then defined as

$$\prod_{1 \leq i < j \leq n} P_{Z_i, Z_j}^{A_{ij}} (1 - P_{Z_i, Z_j})^{1 - A_{ij}} \prod_{1 \leq i \leq n} \pi_{Z_i}.$$

Bayesian inference

We wish to infer parameter $\theta \in \Theta$ over which we have prior information $p(\theta)$. We achieve a *posterior* belief $p(\theta \mid x_{1:n})$ by combining our prior with a likelihood $p(x_{1:n} \mid \theta)$ and by performing the belief update (Bernardo and Smith 2009).

$$p(\theta \mid x_{1:n}) \propto p(x_{1:n} \mid \theta)p(\theta).$$

Prior choices

$$\begin{aligned} \pi &\sim \text{Dirichlet}(\alpha, \dots, \alpha) \\ P_{i,j} &\overset{\text{i.i.d.}}{\sim} \text{Beta}(\beta_1, \beta_2) \\ e_i \mid \pi, P &\sim \pi \\ A_{ij} \mid \pi, P, e &\sim \text{Bernoulli}(P_{e_i, e_j}) \end{aligned}$$

(Hyper-priors over α, β_1, β_2 also available, and not very sensitive, can use for instance $\alpha = 0.5$ and $\beta_1 = \beta_2 = 0.5$).

The posterior

Pas and Vaart (2018) call the posterior class distribution $p(e \mid A)$ the *Bayesian modularity*, denoted $Q_B(e)$, and we then assign class labels according to

$$\hat{e} = \arg\max_{e} Q_B(e).$$

As such, we classify nodes into community based on the maximum *a posteriori* (MAP) estimate of e. The Bayesian modularity is connected to the *likelihood modularity* of Bickel and Chen (2009), in that the latter exists as a special case of the former.

Why be Bayesian?

- Computationally efficiency of Gibbs sampler compared to maximum likelihood for large n
- Complete posterior predictive distribution
  - Uncertainty quantification
  - Decision analysis
- Ability to encode prior beliefs
- "A Bayesian version will usually make things better." (Gelman 2022)

The main result

Theorem

Denote $\rho_n = \sum_{i,j} \pi_i \pi_j P_{i,j}$, then:
1. if (P, π) is fixed and identifiable (π has strictly positive coordinates, and rows of P are distinct) then the MAP estimator $\hat{e}$ is strongly consistent;
2. if $P = \rho_n S$ with (S, π) is fixed and identifiable then the MAP estimator $\hat{e}$ is strongly consistent if $(n-1)\rho_n \gg (\log n)^2$, where $\mathbb{E}[deg_G(i)] = (n-1)\rho_n$.

How much data is enough data?

Weakly consistent community detection for a given number of classes is possible under the assumption $(n-1)\rho_n \gg \log n$ (Lei and Rinaldo 2015), and the likelihood modularity is strongly consistent in the case $K = 2$.

Strong consistency of the likelihood modularity for arbitrary K was claimed by Bickel and Chen (2009) to hold under $(n-1)\rho_n \gg \log n$. **However**, this was shown under the assumption that the modularity is globally Lipschitz.

Since this is not the case in general, it is unclear if $(n-1)\rho_n \gg \log n$ is sufficient for strong consistency. However, it is known that $(n-1)\rho_n \gg (\log n)^2$ is sufficient for the likelihood (and thus also the Bayesian) modularity to be strongly consistent.

An application in Stan: Zachary's karate club

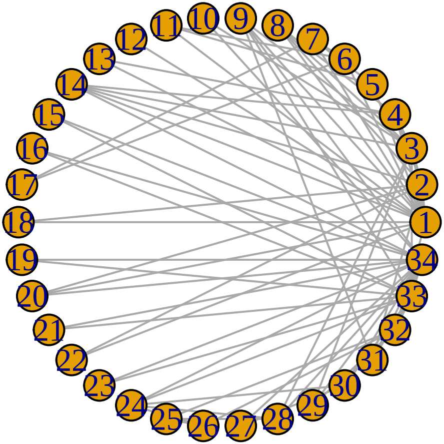

Figure 1: The karate club graph

The model in Stan

This code is adapted from Sarkar (2018).

```
model {
  for(i in 1:K){
    for(j in 1:K){
      // prior over kernel matrix
      phi[i][j] ~ beta(beta[1], beta[2]);
    }
  }
  // prior over mixture distribution
  pi ~ dirichlet(alpha);
  for(i in 1:N){
    for(j in i+1:N){ //symmetry and ignore diagonals
      // likelihood
      graph[i][j] ~ bernoulli(pi' * phi * pi);
    }
  }
}
```

The fitted model

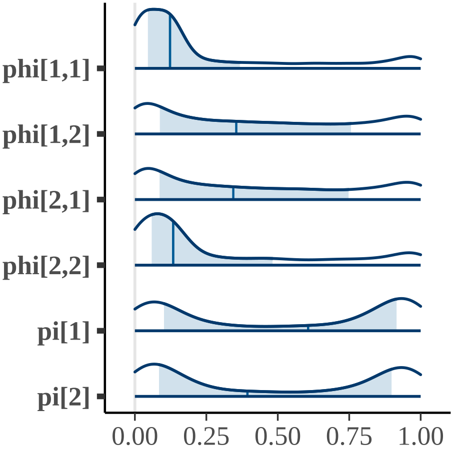

Figure 2: Posterior SBM parameters.

Decision analysis

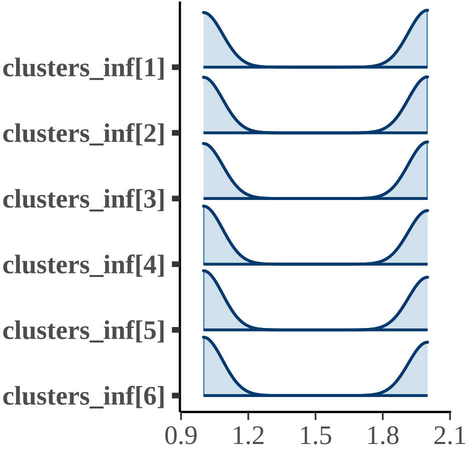

Figure 3: Posterior predictive distribution over six individual node communities.

Prediction

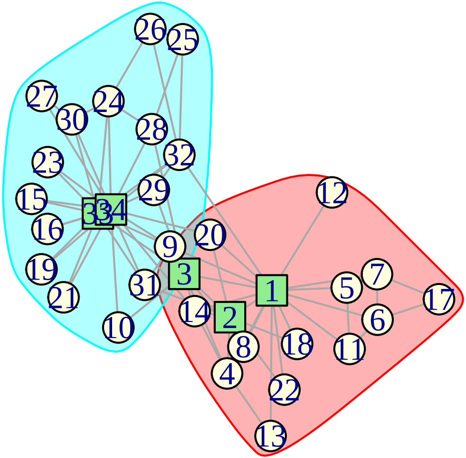

Figure 4: MAP cluster predictions from the Bayesian modularity. Shape and colour of node show predicted class, while coloured clouds indicate true communities.

Prior and likelihood sensitivity

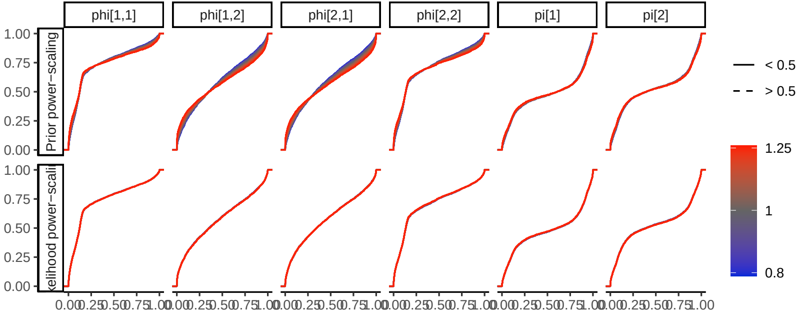

Figure 5: Prior and likelihood sensitivity of posterior.

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References II

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