Exercise 9

1 Nonlinear MPC

The following is a dynamic model of a nonlinear double integrator:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x, u) = \begin{bmatrix} u \\ \alpha(x_1) + x_1 \end{bmatrix}$$

subject to constraints $\|x\|_\infty \leq 1$ and $|u| \leq 1$. Treat α as a symbolic function.

1. Discretize the system $x^+ = \phi(x, u)$ with a sample period of $h = 1$ seconds using Runge-Kutta 2. Assume that during the sample period, the input is constant, i.e., $u(t) = u_i$ for all $t \in [h(i), h(i + 1)]$ for $i \in \{0, 1, 2, \ldots\}$.

2. Linearize the discrete-time system around the point $x_s = (0, 0)$ and $u_s = 0$ and for $\alpha(x_1) = \sin(x_1)$.

3. Design a stabilizing and recursively feasible linear MPC controller for the linear discrete-time model computed in the previous part.

4. If the input from your linear MPC controller is applied to the nonlinear system, will the closed-loop system necessarily satisfy the constraints?

   - If yes, prove that this is the case.

   - If no, give suggestions for how you can modify your controller to ensure that it does.

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1. Discretize the system $x^+ = \phi(x, u)$ with a sample period of $h = 1$ seconds using Runge-Kutta 2. Assume that during the sample period, the input is constant, i.e., $u(t) = u_i$ for all $t \in [h(i), h(i + 1)]$ for $i \in \{0, 1, 2, \ldots\}$.

   The RK2 integration scheme needs two intermediate variables k_1, k_2

   $$k_1 = f(x) = \begin{bmatrix} u \\ \alpha(x_1) + x_1 \end{bmatrix} \quad x + hk_1 = \begin{bmatrix} x_1 + hu \\ x_2 + h\alpha(x_1) + hx_1 \end{bmatrix}$$

   $$k_2 = f(x + hk_1) = \begin{bmatrix} u \\ \alpha(x_1 + hu) + x_1 + hu \end{bmatrix}$$

   $$x^+ \approx x + \frac{h}{2}k_1 + \frac{h}{2}k_2$$

   $$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{h}{2} \begin{bmatrix} u \\ \alpha(x_1) + x_1 \end{bmatrix} + \frac{h}{2} \begin{bmatrix} u \\ \alpha(x_1 + hu) + x_1 + hu \end{bmatrix}$$

   $$= \begin{bmatrix} x_1 + hu \\ x_2 + hx_1 + 0.5h^2u + 0.5h\alpha(x_1) + 0.5h\alpha(x_1 + hu) \end{bmatrix} = \phi(x, u)$$

2. Linearize the discrete-time system around the point $x_s = (0, 0)$ and $u_s = 0$ and for $\alpha(x_1) = \sin(x_1)$.

   Compute the Jacobian of ϕ with respect to x, u:

   $$J_x = \begin{bmatrix} \frac{\partial \phi_1}{\partial \phi_{x_1}} & \frac{\partial \phi_1}{\partial \phi_{x_2}} \\ \frac{\partial \phi_2}{\partial \phi_{x_1}} & \frac{\partial \phi_2}{\partial \phi_{x_2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ h + 0.5h\cos(x_1) + 0.5h\cos(x_1 + hu) & 1 \end{bmatrix}$$

   $$J_x|_{x=0, u=0} = \begin{bmatrix} 1 & 0 \\ 2h & 1 \end{bmatrix}$$

   $$J_u = \begin{bmatrix} \frac{\partial \phi_1}{\partial \phi_u} \\ \frac{\partial \phi_2}{\partial \phi_u} \end{bmatrix} = \begin{bmatrix} h \\ 0.5h^2 + 0.5h\cos(x_1 + hu)h \end{bmatrix}$$

   $$J_u|_{x=0, u=0} = \begin{bmatrix} h \\ h^2 \end{bmatrix}$$

3. Design a stabilizing and recursively feasible linear MPC controller for the linear discrete-time model computed in the previous part.

   Any standard MPC setup will do the job. Here the simplest one (terminal constraint is zero) is used:

   $$\min_u \quad \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k$$

   $$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k \quad \forall k \in [N]$$

   $$\|x_k\|_\infty \le 1 \quad \forall k \in [N]$$

   $$-1 \le u_k \le 1 \quad \forall k \in [N]$$

   $$x_N = 0$$

4. If the input from your linear MPC controller is applied to the nonlinear system, will the closed-loop system necessarily satisfy the constraints?

- If yes, prove that this is the case.

- If no, give suggestions for how you can modify your controller to ensure that it does.

No, it will not because the linearization and discretization are approximate.

One way to ensure that it does is to bound the difference between the linear discrete time system and the continuous time one (in our case, $\|\phi - f\|$), and then use robust MPC to compensate for this error.