

Exploration in Interactive Personalized Music Recommendation: A Reinforcement Learning Approach

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Current music recommender systems typically act in a greedy fashion by recommending songs with the highest user ratings. Greedy recommendation, however, is suboptimal over the long term: it does not actively gather information on user preferences and fails to recommend *novel* songs that are potentially interesting. A successful recommender system must balance the needs to *explore* user preferences and to *exploit* this information for recommendation. This paper presents a new approach to music recommendation by formulating this exploration-exploitation trade-off as a reinforcement learning task called the multi-armed bandit. To learn user preferences, it uses a Bayesian model, which accounts for both audio content and the novelty of recommendations. A piecewise-linear approximation to the model and a variational inference algorithm are employed to speed up Bayesian inference. One additional benefit of our approach is a single unified model for both music recommendation and playlist generation. Both simulation results and a user study indicate strong potential for the new approach.

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1. INTRODUCTION

A music recommendation system recommends songs from a large database by matching songs with a user's preferences. An *interactive* recommender system infers the user's preferences by incorporating

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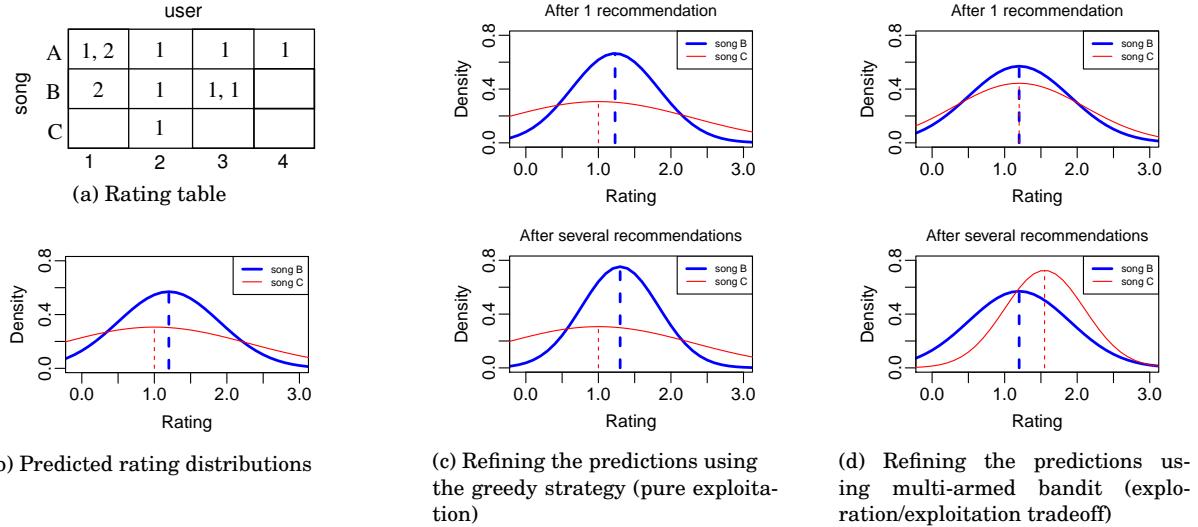


Fig. 1: Uncertainty in recommendation

user feedback into recommendations. Each recommendation thus serves two objectives: (i) satisfy the user’s current musical need, and (ii) elicit user feedback in order to improve future recommendations.

Current recommender systems typically focus on the first objective, while completely ignoring the other. They recommend songs with the highest user ratings. Such a greedy strategy, which does not actively seek user feedback, often results in suboptimal recommendations over the long term.

Consider the simple example in Figure 1. The table contains the ratings for three songs by four users (Figure 1a). For simplicity, let us assume that the recommender chooses between two songs B and C only. The target user is 4, whose true ratings for B and C are 1.3 and 1.6, respectively. The true ratings are real numbers, because a user may give the same song different ratings as a result of external factors. The true rating is the expected rating of a song by the user. In this case, a good recommender should choose C .

Since the true user ratings are unknown to the recommender, it may approximate the rating distributions for B and C as Gaussians, P_B and P_C (Figure 1b), respectively, using the data in Figure 1a. The distribution P_B has mean 1.2. The distribution P_C has mean 1. P_B has much lower variance than P_C , because B has more rating data. A greedy recommender, including, e.g., the highly successful collaborative filtering (CF) approach, recommends the song with the highest mean rating and thus B . In response to this recommendation, user 4 gives a rating, whose expected value is 1.3. The net effect is that the mean of P_B likely shifts towards 1.3 and its variance further reduces (Figure 1c). Consequently the greedy recommender is even more convinced that user 4 favors B and will always choose B for all future recommendations. It will never choose C and find out its true rating, resulting in clearly suboptimal performance.

To overcome this difficulty, the recommender must take into account *uncertainty* in the mean ratings. If it considers both the mean and the variance of the rating distribution, the recommendation will change. Consider again Figure 1b. Although P_C has slightly lower mean than P_B , it has very high variance. It may be worthwhile to recommend it and gather additional user feedback in order to reduce the variance. User 4’s rating on C has expected value 1.6. Therefore, after one recommendation, the mean of P_C will likely shift towards 1.6 (Figure 1d). By recommending C several times and gathering user feedback, we will then find out user 4’s true preference C .

The gist here is that a good interactive music recommender system must *explore* user preferences actively rather than merely *exploit* rating information available. Balancing exploration and exploitation is critical, especially when the system is faced with a *cold start*, i.e., when a new user or a new song appears.

Another crucial issue for music recommendation is playlist generation. People often listen to a group of related songs together and may repeat the same song multiple times. This is unique to music recommendation and does not occur often for other recommendation domains, such as newspaper articles or movies. A playlist is a group of songs arranged in a suitable order. The songs in a playlist have strong interdependencies. For example, they share the same genre [Chen et al. 2012], but are diversified at the same time [Zhang et al. 2012]. They have a consistent mood [Logan 2002]. They may repeat, but are not repetitive. Existing recommender systems based on CF or audio content analysis typically recommend one song at a time and do not consider their interdependencies during the recommendation process. They divide playlist generation into two distinct steps [Chen et al. 2012]. First, choose a set of favored songs through CF or content analysis. Next, arrange the songs into a suitable order in a process called automatic playlist generation (APG).

In this work, we formulate interactive, personalized music recommendation as a reinforcement learning task called the *multi-armed bandit* [Sutton and Barto 1998] and address both exploration-exploitation trade-off and that of playlist generation with a single unified model:

- Our bandit approach systematically balances exploration and exploitation, a central issue well studied in reinforcement learning. Experimental results show that our recommender system mitigates the difficulty of cold start and improves recommendation performance, compared with the traditional greedy approach.
- We build a single rating model that captures both user preference over audio content and the novelty of recommendations. It seamlessly integrates music recommendation and playlist generation.
- We also present an approximation to the rating model and new probabilistic inference algorithms in order to achieve real-time recommendation performance.

Although our approach is designed specifically for music recommendation, it is possible to be generalized to other media types as well. The detailed discussion will be presented in Section 6.

In the following, Section 2 describes related work. Section 3 formulates the rating model and our bandit approach to music recommendation. Section 4 presents the approximate Bayesian models and inference algorithms. Section 5 describes the evaluations of our models and algorithms. Section 6 discusses the possible generalization directions of the approach and future research directions. Section 7 concludes this work.

2. RELATED WORK

2.1 Music recommendation

Since [Song et al. 2012] have given a very recent and comprehensive review of existing music recommendation works, we will first provide a brief overview of the status quo and discuss highly relevant work in detail later. Currently, music recommender systems can be classified according to their methodologies into four categories: *collaborative filtering* (CF), *content-based* methods, *context-based* methods, and hybrid methods. Collaborative filtering recommends songs by considering those preferred by other like-minded users. The state-of-the-art method for performing CF is non-negative matrix factorization, which is well summarized by [Koren et al. 2009]. Although CF is one of the most widely used methods, it suffers from the notorious cold-start problem since it cannot recommend songs to new users whose preference are unknown (the *new-user* problem) or recommend new songs to users (the *new-song* prob-

lem). Unlike CF, content-based method recommends songs which have similar audio content to the user's preferred songs. The recommendation quality of content-based systems is largely determined by acoustic features, the most useful ones of which, timbre and rhythm, are incorporated into our proposed system [Song et al. 2012]. Content-based systems remedies the new-song problem but not the new-user problem. Recently, context-based music recommender systems have become popular. They recommend songs to match various aspects of the user context, e.g., activities, environment, mood, physiological states [Wang et al. 2012]. Hybrid methods combine two or more of the above methods.

Relatively few works have attempted to combine music recommendation with playlist generation. In [Chen et al. 2012], a playlist is modeled as a Markov process whose transition probability models both user preferences and playlist coherence. In [Zheleva et al. 2010], a model similar to Latent Dirichlet Allocation is used to capture user latent taste and mood of songs. In [Aizenberg et al. 2012], a new CF model is developed to model playlists in Internet radio stations. While the three works also combine recommendation with playlist generation, our model differs in three aspects: (1) it is based on audio content while the previous three depend only on usage data; (2) our model is highly efficient so allowing easy online updates; (3) our model is crafted and evaluated based on real-life user interaction data, not data crawled from the web. Zhang *et al.* tries to recommend using a linear combination of CF's results with the results from an existing novelty model [Zhang et al. 2012], which ranks songs by CF before generating the playlists according to novelty. The parameters for the linear combination are adjusted manually, not optimized simultaneously. Moreover, they provide only system-wise control of novelty while our method provides user-wise control. Other works like [Hu and Ogihara 2011] generate music playlists within a user's own music library, in which case his/her preference is already known and need not to be inferred.

2.2 Reinforcement learning

Unlike supervised learning (e.g. classification), which considers only prescribed training data, a reinforcement learning (RL) algorithm *actively* explores its environment to gather information and exploits the learnt knowledge to make decision or prediction.

Multi-armed bandit is the most thoroughly studied reinforcement learning problem. For a bandit (slot) machine with M arms, pulling arm i will result in a random payoff r , sampled from an unknown and arm-specific distribution p_i . The objective is to maximize the *total payoff* given a number of interactions. Namely, the set of arms is $\mathcal{A} = \{1 \dots M\}$, known to the player; each arm $i \in \mathcal{A}$ has a probability distribution p_i , unknown to the player. The player also knows he has n rounds of pulls. At the l -th round, he can pull an arm $I_l \in \mathcal{A}$, and receive a random payoff r_{I_l} , sampled from the distribution p_{I_l} . The objective is to wisely choose the n arms, i.e., $(I_1, I_2, \dots, I_n) \in \mathcal{A}^n$ to maximize

$$\text{Total payoff} = \sum_{l=1}^n r_{I_l}$$

A naive solution to the problem could be: the player first randomly pulls arms to gather information to learn p_i (exploration) and then always pulls the arm that yields the maximum predicted payoff (exploitation). However, both too much exploration, i.e., the learnt information is not used much, or too much exploitation, i.e., the player lacks information to make accurate predictions, result in suboptimal total payoff. Thus, how to balance the amount of the two is important.

Multi-armed bandit approach provides a principled solution to this problem. The simplest multi-armed bandit approach, namely ϵ -greedy, chooses the arm with the highest predicted payoff with probability $1 - \epsilon$ or chooses arms uniformly at random with probability ϵ . An approach better than ϵ -greedy is based on a simple and elegant idea called upper confidence bound (UCB) [Auer 2003]. Let U_i be

the true expected payoff for arm i , i.e., the expectation of p_i ; UCB-based algorithms estimate both its expected payoff \hat{U}_i and a confidence bound c_i from history payoffs, so that U_i lies in $(\hat{U}_i - c_i, \hat{U}_i + c_i)$ with high probability. Intuitively, selecting an arm with large \hat{U}_i corresponds to exploitation, while selecting one with large c_i corresponds to exploration. To balance exploration and exploitation, UCB-based algorithms always select the arm that maximizes $\hat{U}_i + c_i$, the principle of which is called “optimism in the face of uncertainty”.

Bayes-UCB [Kaufmann et al. 2012] is one of the state-of-the-art Bayesian counterparts of the UCB approach. In Bayes-UCB, the expected payoff U_i is regarded as a random variable, and the posterior distribution of U_i given the history payoffs \mathcal{D} , denoted as $p(U_i|\mathcal{D})$, is maintained, and the fixed-level *quantile* of $p(U_i|\mathcal{D})$ is used to mimic the upper confidence bound. Similar to UCB, every time Bayes-UCB selects the arm with the maximum quantile. More interestingly, UCB-based algorithms require an explicit form of the confidence bound, which is difficult to derive in our case, but in Bayes-UCB, the quantiles of the posterior distributions of U_i can be easily obtained using *Bayesian inference*. We therefore choose Bayes-UCB.

There are more sophisticated RL methods such as Markov Decision Process (MDP) [Szepesvári 2010], which generalizes the bandit problem by assuming that the states of the system can change following a Markov process. Although MDP can model a broader range of problems than bandit, it requires much more data to train and is usually computationally expensive.

2.3 Reinforcement learning in recommender systems

Previous works have used reinforcement learning to recommend web pages, travel information, books, and news etc. For example, Joachims *et al.* use Q-learning to guide users through web pages [Joachims et al. 1997]. Golovin *et al.* propose a general framework for web recommendation, as well as user implicit feedback to update the system [Golovin and Rahm 2004]. Zhang *et al.* propose a personalized web-document recommender, where user profile is represented as vector of terms. The weight of the terms are updated based on the temporal difference method using both implicit and explicit feedback [Zhang and Seo 2001]. In [Srivihok and Sukonmanee 2005], a Q-learning based travel recommender is proposed, where trips are ranked using a linear function of several attributes including trip duration, price and country, and the weights are updated using user feedback. Shani *et al.* use a MDP to model the dynamics of user preference in book recommendation [Shani et al. 2005], where purchase history is used as the states, and the generated profit is used as the payoffs. Similarly, in a web recommender [Taghipour and Kardan 2008], history web pages are used as the states; web content similarity and user behavior are combined as the payoffs.

In a seminal work done by [Li et al. 2012], news are represented as feature vectors; the click-through rates of news are treated as the payoffs and assumed to be a linear function of news feature vectors. A bandit model called LinUCB is proposed to learn the weights of the linear function. Our work differs from this work in two aspects. Fundamentally, music recommendation is different from news recommendation due to the sequential relationship between songs. Technically, the additional novelty factor of our rating model makes the reward function nonlinear and the confidence bound difficult to obtain. Therefore we need the Bayes-UCB approach and the sophisticated Bayesian inference algorithms developed in Section 4. Moreover, we cannot apply the offline evaluation techniques developed in [Li et al. 2011] because we assume that ratings change dynamically over time. As a result, we must conduct online evaluation with real human subjects.

Although we believe reinforcement learning has great potential in improving music recommendation, it has received relatively little attention and found only limited application. Liu *et al.* use MDP to recommend music based on a user’s heart rate to help the user maintain it within the normal range [Liu et al. 2009]. States are defined as different levels of heart rate, and biofeedback is used

as payoffs. However, (1) parameters of the model are not learnt from exploration, and thus exploration/exploitation tradeoff is not needed; (2) the work does not disclose much information about the evaluation of the approach. Chi *et al.* uses MDP to automatically generate playlist [Chi et al. 2010]. Both SARSA and Q-learning are used to learn user preference, and, similar to [Shani et al. 2005], states are defined as mood categories of the recent listening history. However, in this work, (1) exploration/exploitation tradeoff is not considered; (2) mood or emotion, while useful, can only contribute so much to effective music recommendation; and (3) the MDP model cannot handle long listening history, as the state space grows exponentially with history length; as a result, too much exploration and computation will be required to learn the model. Independent of and concurrent with our work, Liebman *et al.* build a DJ agent to recommend playlists based on reinforcement learning [Liebman and Stone 2013]. Their work differs from ours in that: (1) exploration/exploitation tradeoff is not considered; (2) the reward function does not consider the novelty of recommendations; (3) their approach is based on a simple tree-search heuristic, ours the thoroughly studied multi-armed bandit; (4) not much information about the simulation study is disclosed, and no user study is conducted.

The active learning approach developed by [Karimi et al. 2011] *only explores* songs in order to optimize the predictive performance on a pre-determined test dataset. Our approach, on the other hand, requires no test dataset and balances *both* exploration and exploitation to optimize the entire interactive recommendation process between the system and users. Since many recommender systems in reality do not have test data or at least have no data for new users, our bandit approach is more realistic compared with the active learning approach.

Our work is, to the best of our knowledge, the first to balance exploration and exploitation based on reinforcement learning and particularly multi-armed bandit in order to improve recommendation performance and mitigate the cold-start problem in music recommendation.

3. A BANDIT APPROACH TO MUSIC RECOMMENDATION

3.1 Personalized user rating model

Music preference is a combined effect of many factors including music audio content, novelty, diversity, moods and genres of the songs, user emotional states, and user context information [Wang et al. 2012]. As it is unrealistic to cover all the factors in this paper, we focus on audio content and novelty.

Music Audio Content - Whether a user likes or dislikes a song is highly related to its audio content. We assume that the music audio content of a song can be described as a feature vector \mathbf{x} . Without considering other factors, a user's preference can be represented as a linear function of \mathbf{x} as:

$$U_c = \theta' \mathbf{x} \quad (1)$$

where the parameter vector θ represents user preference of different music features. Users may have different preference and thus different values of θ . To keep the problem simple, we assume a user's preference is invariant, i.e. θ remains a constant, and leave modeling changing θ as future work.

Although the idea of exploration/exploitation tradeoff can be applied on collaborative filtering (CF) as long as the rating distribution can be estimated as shown in Figure 1, we choose the content-based approach instead of the popular CF-based methods for a number of reasons. First, we need a posterior distribution of U_c in order to use Bayes-UCB as introduced in Section 2.2, so non-Bayesian methods cannot be used. Second, existing Bayesian matrix factorization methods [Salakhutdinov and Mnih 2008; Silva and Carin 2012] are much more complicated than the linear model and also require large amount of training data; these render the user study unwieldy and expensive. Third, our bandit approach requires the model to be updated whenever a new rating is obtained, but existing Bayesian matrix factorization methods are too slow [Salakhutdinov and Mnih 2008; Silva and Carin 2012]. Fourth, CF suffers from the new song problem while the content-based method does not. Fifth, CF

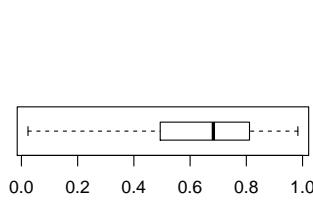


Fig. 2: Proportion of repetitions in users' listen history

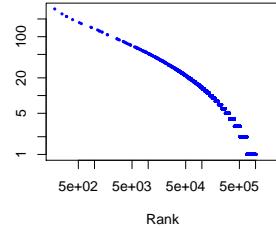


Fig. 3: Zipf's law of song repetition frequency

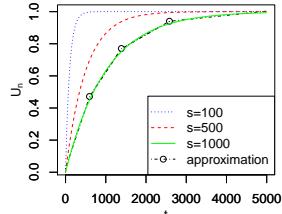


Fig. 4: Examples of $U_n = 1 - e^{-t/s}$. The line marked with circles is a 4-segment piecewise linear approximation.

captures correlation instead of causality and thus does not explain why a user likes a song. However, as science usually pursues causal models, the content-based approach captures one important aspect of the causality, i.e. music content.

Novelty - We define that novelty is about repeating songs at proper frequencies, which is in essence consistent with the definition in [Gunawardana and Shani 2009]. We examined the repetition distributions of 1000 users' listening histories collected from Last.fm¹. The box plot in Figure 2 shows the proportion of repetitions, which is defined as: $1 - \frac{\text{number of unique songs}}{\text{listening history length}}$. Note that since Last.fm does not record users' listening histories outside of Last.fm, the actual proportion should be even larger than the 68.3% shown here. Thus, most of the songs the user listens to are repeats. We also studied the song repetition frequency distribution of every individual user's listening history: the frequencies of songs were first computed for every user; then all users' frequencies were ranked in decreasing order; finally the frequencies versus ranks were plotted on a log-log scale (Figure 3). The distribution approximately follows the Zipf's law [Newman 2005]—only a small set of songs are repeated for most of the time while all the rest are repeated much less often. Most other types of recommenders, however, do not follow Zipf's law. Recommending books that have been bought or movies that have been watched makes little sense. In music recommendation, however, it is critically important to repeat songs appropriately.

Existing novelty models do not take time into consideration [Lathia et al. 2010; Castells et al. 2011; Zhang et al. 2012], and as a result songs heard year ago and just now have the same impact on the current recommendation. Inspired by [Hu and Ogihara 2011], we assume that the novelty of a particular song decays immediately after listening to it and then gradually recovers. Let t be the time elapsed since the last listening of the song, the novelty recovers following the function:

$$U_n = 1 - e^{-t/s} \quad (2)$$

where s is a parameter indicating the recovery speed, with slower recovery having a higher s . Figure 4 shows examples of U_n with different values of s .

Different users can have different recovery rates s . As can be seen from the broad distribution in Figure 2, some may repeatedly listen to their favorite songs more often, while the others would be keen to exploring new songs. Therefore we assume s to be a personalized value to be learnt through the user interactive process.

Combined Model - A user's preference of a recommendation can be represented as a rating; the higher the rating is, the more the user likes the recommendation. Unlike traditional recommenders which assume ratings are static, we assume that a rating is the combined effect of the user's preference

¹<http://ocelma.net/MusicRecommendationDataset/lastfm-1K.html>

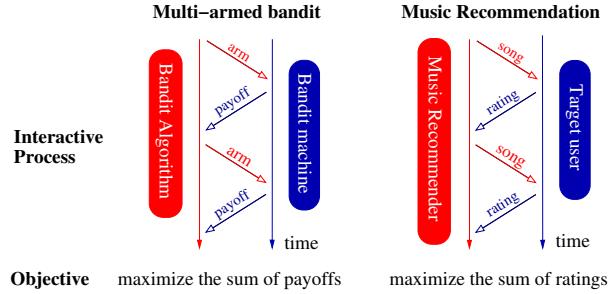


Fig. 5: Relationship between the multi-armed bandit problem and music recommendation

of the song's content and the dynamically changing novelty. Therefore, a song rated as 5 last time could be rated as 2 this time because the novelty has decreased. Finally, we define the complete user rating model as:

$$U = U_c U_n = \theta' \mathbf{x} \left(1 - e^{-t/s}\right). \quad (3)$$

In this model, the more the user likes a particular song the more likely it will be repeated—a song with larger U_C requires less time (t) to recover U and becomes eligible for repeat. Also, given that the user's favorites comprise a small subset of his/her library, the U model behaves in accordance with Zipf's Law and ensures that only a small proportion of songs will be repeated often. This property of the model will be verified in Section 5.3.2.

In Section 5.3.1, we will show that the product form of Equation (3) leads to significantly better performance than the alternative linear combination $U = aU_c + bU_n$.

3.2 Interactive music recommendation

Under our rating model, each user is represented by a set of parameters $\Omega = \{\theta, s\}$. If we know the values of Ω , we can simply recommend the songs with the highest rating according to Equation (3). However, Ω needs to be estimated from historical data, and thus uncertainty always exists. In this case, the greedy strategy used by traditional systems is suboptimal, and it is necessary to take the uncertainty into account and balance exploration and exploitation, as explained in Section 1.

The multi-armed bandit approach introduced in Section 2.2 offers a way for balancing exploration and exploitation for the interactive music recommendation process between the target user and the recommender. As illustrated in Figure 5, we treat songs as arms, and user ratings as payoffs². The music recommendation problem is then transformed into a multi-armed bandit problem, and the objective of a music recommender is also changed to maximizing the sum of the ratings given by the target user *over the long term*. We argue that the cumulative rating is a more realistic objective than the *myopic* predictive accuracy used by traditional music recommenders, because users usually listen to songs for a long time instead of focusing on one individual song.

We adopt the Bayes-UCB algorithm introduced in Section 2.2 for our recommendation task. First we denote the rating given by the target user to recommendation i as a random variable R_i , and the

²Although in reality users usually do not give explicit feedback (i.e. ratings) to every recommended song, implicit feedback (e.g. skipping a song, listening to a song fully) can be obtained much more easily. In this paper, we focus on explicit feedback to keep the problem simple.

expectation of R_i is U given the feature vector (\mathbf{x}_i, t_i) :

$$\mathbb{E}[R_i] = U_i = \boldsymbol{\theta}' \mathbf{x}_i \left(1 - e^{-t_i/s}\right) \quad (4)$$

Then, we develop Bayesian models to estimate the posterior distribution of U given the history recommendation information. We sketch the framework here and explain it in greater detail in Section 4. We assume that the prior distribution of Ω is $p(\Omega)$ and that, at the $(l+1)$ -th recommendation, we have accumulated l history recommendations $\mathcal{D}_l = \{(\mathbf{x}_i, t_i, r_i)\}_{i=1}^l$ as training samples, where r_i is the rating given by the user to the i -th recommendation. The posterior distribution of Ω can then be obtained based on Bayes' rule:

$$p(\Omega|\mathcal{D}_l) \propto p(\Omega)p(\mathcal{D}_l|\Omega) \quad (5)$$

and then the expected rating of song k , denoted as U_k can be predicted as:

$$p(U_k|\mathcal{D}_l) = \int p(U_k|\Omega)p(\Omega|\mathcal{D}_l)d\Omega \quad (6)$$

Later, we use λ_k^l to denote $p(U_k|\mathcal{D}_l)$ for simplicity.

Finally, to balance exploration and exploitation, Bayes-UCB recommends song k^* , which maximizes the quantile function:

$$k^* = \arg \max_{k=1\dots|\mathcal{S}|} Q(\alpha, \lambda_k^l)$$

where Q satisfies $\mathbb{P}[U_k \leq Q(\alpha, \lambda_k^l)] = \alpha$ and \mathcal{S} is all songs in the database. We set $\alpha = 1 - \frac{1}{l+1}$. The detail of the recommendation algorithm is listed in Algorithm 1.

The cold-start problem is caused by the lack of information required for making good recommendations. There are many ways for mitigating the cold-start problem, most of which rely on additional information about the users or songs, e.g., popularity/metadata information about the songs [Hariri et al. 2012], context/demographic information about the users [Wang et al. 2012]. Although music audio content is required by U_c , it is usually easy to obtain in industry. Our bandit addresses the cold-start problem without relying on additional information about users and songs. Instead, it wisely explores and exploits information during the whole interactive process. Thus, the bandit approach presents a fundamentally different method to tackle the cold-start problem, yet it can be used in conjunction with existing methods.

There are other Bayesian multi-arm bandit approaches such as Thompson sampling [Agrawal and Goyal 2012] and optimistic Bayesian sampling [May et al. 2012]. Theoretical performance comparisons between them are interesting research problems. Empirical results of them are, however, usually comparable. These comparisons are not the focus of this work. Moreover, since all of them are based on the Bayesian approach, it is very easy to replace Bayes-UCB with other approaches even if Bayes-UCB is shown to be inferior in the future.

ALGORITHM 1: Recommendation using Bayes-UCB

```

for  $l = 1$  to  $n$  do
  for all song  $k = 1, \dots, |\mathcal{S}|$  do
    compute  $q_k^l = Q(1 - 1/l, \lambda_k^{l-1})$ 
  end for
  recommend song  $k^* = \arg \max_{k=1\dots|\mathcal{S}|} q_k^l$ 
  gather rating  $r_l$ ; update  $p(\Omega|\mathcal{D}_l)$  and  $\lambda_k^l$ 
end for

```



Fig. 6: Graphical representation of the Bayesian models. Shaded nodes represent observable random variables, while white nodes represent hidden ones. The rectangle (plate) indicates that the nodes and arcs inside are replicated for N times.

4. BAYESIAN MODELS AND INFERENCE

4.1 Exact Bayesian model

To compute Equations (5) and (6), we develop the following Bayesian model with its graphical representation shown in Figure 6a.

$$\begin{aligned} R|\mathbf{x}, t, \theta, s, \sigma^2 &\sim \mathcal{N}\left(\theta' \mathbf{x} \left(1 - e^{-t/s}\right), \sigma^2\right) \\ \theta|\sigma^2 &\sim \mathcal{N}(\mathbf{0}, a_0 \sigma^2 \mathbf{I}) \\ s &\sim \mathcal{G}(b_0, c_0) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(f_0, h_0) \end{aligned}$$

Every line of the model indicates a probability dependency and the corresponding distribution, e.g., $\theta|\sigma^2 \sim \mathcal{N}(\mathbf{0}, a_0 \sigma^2 \mathbf{I})$ suggests $p(\theta|\sigma^2) = \mathcal{N}(\mathbf{0}, a_0 \sigma^2 \mathbf{I})$. $\mathcal{N}(\cdot, \cdot)$ is a (multivariate) Gaussian distribution with the mean and (co)variance parameters, and $\mathcal{G}(\cdot, \cdot)$ is a Gamma distribution with the shape and rate parameters. The rating R is assumed to be normally distributed following the convention of recommender systems. A gamma prior is put on s because s is positive. Following the conventions of Bayesian regression models, A normal prior is put on θ and a gamma one for τ . We depend on σ^2 for θ because it shows better convergence in the simulation study.

Since there is no closed form solution to Equation (5) for this model, Markov Chain Monte Carlo (MCMC) is used as the approximate inference procedure. Directly evaluating Equation (6) is also impossible. Thus we use Monte Carlo simulation to obtain λ_k^l : for every sample obtained from the MCMC procedure, we substitute it into Equation (4) to obtain a sample of U_i , and then use the histogram of the samples of U_i as an approximation of λ_k^l .

This approach is easy to understand and implement. However, it is very slow and users can hardly wait for a recommendation for tens of seconds until the Markov chain converges. To make the algorithm more responsive, we will develop an approximate Bayesian model and a highly efficient variational inference algorithm in the following sections.

4.2 Approximate Bayesian model

4.2.1 Piecewise Linear approximation. It is very difficult to develop better inference algorithms for the exact Bayesian model because of the irregular form of function $U_n(t)$. Fortunately, we find U_n can be approximated by a piecewise linear function (as shown in Figure 4), which enables us to develop an efficient model.

For simplicity, we discretize time t into K predetermined intervals: $[0, \xi_1], [\xi_1, \xi_2], \dots, [\xi_{K-1}, +\infty)$, and only consider the class of piecewise linear functions whose consecutive line segments intersect at the boundaries of the intervals. It is not difficult to see that this class of functions can be compactly represented as a linear function [Hastie et al. 2009]. We first map t into a vector $\mathbf{t} = [(t -$

$\xi_1)_+, \dots (t - \xi_{K-1}), t, 1]$, where $(t - \xi)_+ = \max(t - \xi, 0)$, and then approximate $U_n(t)$ as $U_n(t) \approx \beta' t$, where $\beta = [\beta_1, \dots, \beta_{K+1}]'$ is a vector of parameters to be learnt from training data. Now, we can represent U as the product of two linear functions: $U = U_c U_n \approx \theta' x \beta' t$.

Based on this approximation, we revise the distributions of R and the parameters of the exact Bayesian model as follows:

$$\begin{aligned} R|x, t, \theta, \beta, \sigma^2 &\sim \mathcal{N}(\theta' x \beta' t, \sigma^2) \\ \theta|\sigma^2 &\sim \mathcal{N}(\mu_{\theta 0}, \sigma^2 D_0) \\ \beta|\sigma^2 &\sim \mathcal{N}(\mu_{\beta 0}, \sigma^2 E_0) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(a_0, b_0) \end{aligned} \quad (7)$$

where θ, β, τ are parameters. $D_0, E_0, \mu_{\theta 0}, \mu_{\beta 0}, a_0, b_0$ are hyperparameters of the priors to be specified beforehand. D_0 and E_0 are positive definite matrices. The graphical representation of the model is shown in Figure 6b. We use conjugate priors for θ, β, τ , which make the variational inference algorithm described later very efficient.

4.2.2 Variational inference. Recall that our objective is to compute the posterior distribution of parameters Ω (now it is $\{\theta, \beta, \tau\}$) given the history data $\mathcal{D} = \{(x_i, t_i, r_i)\}_{i=1}^N$, i.e., $p(\theta, \beta, \tau | \mathcal{D})$. Using piecewise linear approximation, we can now develop an efficient variational inference algorithm.

Following the convention of mean-field approximation [Friedman and Koller 2009], we assume that the joint posterior distribution can be approximated by a restricted distribution $q(\theta, \beta, \tau)$, which consists of three independent factors [Friedman and Koller 2009]:

$$p(\Omega | \mathcal{D}) = p(\theta, \beta, \tau | \mathcal{D}) \approx q(\theta, \beta, \tau) = q(\theta)q(\beta)q(\tau).$$

Because of the choice of the conjugate priors, it is easy to show that the restricted distributions $q(\theta)$, $q(\beta)$, and $q(\tau)$ take the same parametric forms as the prior distributions. Specifically,

$$q(\theta) \propto \exp\left(-\frac{1}{2}\theta' \Lambda_{\theta N} \theta + \eta'_{\theta N} \theta\right), \quad q(\beta) \propto \exp\left(-\frac{1}{2}\beta' \Lambda_{\beta N} \beta + \eta'_{\beta N} \beta\right), \quad q(\tau) \propto \tau^{a_N - 1} \exp(-b_N \tau).$$

To find the values that minimize the KL-divergence between $q(\theta, \beta, \tau)$ and the true posterior $p(\theta, \beta, \tau | \mathcal{D})$ for parameters $\Lambda_{\theta N}, \eta_{\theta N}, \Lambda_{\beta N}, \eta_{\beta N}, a_N$, and b_N , we use the coordinate descent method. Specifically, we first initialize the parameters of $q(\theta)$, $q(\beta)$, and $q(\tau)$, and then iteratively update $q(\theta)$, $q(\beta)$, and $q(\tau)$ until the variational lower bound \mathcal{L} (elaborated in the Appendix) converges. Further explanation about the principle can be found in [Friedman and Koller 2009]. The detailed steps are in Algorithm (2), where p, K are the dimensionalities of x and t , respectively; the moments of θ, β, τ are in the Appendix.

4.2.3 Predict the posterior distribution $p(U | \mathcal{D})$. Because $q(\theta)$ and $q(\beta)$ are normal distributions, $\theta' x$ and $\beta' t$ are also normally distributed:

$$p(\theta' x | x, t, \mathcal{D}) \approx \mathcal{N}(x' \Lambda_{\theta N}^{-1} \eta_{\theta N}, x' \Lambda_{\theta N}^{-1} x), \quad p(\beta' t | x, t, \mathcal{D}) \approx \mathcal{N}(t' \Lambda_{\beta N}^{-1} \eta_{\beta N}, t' \Lambda_{\beta N}^{-1} t)$$

and the posterior distribution of U in Equation (6) can be computed as:

$$p(U | x, t, \mathcal{D}) = p(\theta' x \beta' t | x, t, \mathcal{D}) = \int p(\theta' x = a | x, t, \mathcal{D}) p(\beta' t = \frac{U}{a} | x, t, \mathcal{D}) da.$$

Since there is no closed-form solution to the above integration, we use Monte Carlo simulation: we first obtain one set of samples for each of $\theta' x$ and $\beta' t$, and then use the element-wise products of the two group of samples to approximate the distribution of U . Because $\theta' x$ and $\beta' t$ are normally distributed univariate random variables, the sampling can be done very efficiently. Moreover, prediction for different songs is trivially parallelizable and is thus scalable.

ALGORITHM 2: Variational inference

input: $\mathcal{D}, \mathbf{D}_0, \mathbf{E}_0, \boldsymbol{\mu}_{\theta_0}, \boldsymbol{\mu}_{\beta_0}, a_0, b_0$
 initialize $\boldsymbol{\Lambda}_{\theta N}, \boldsymbol{\eta}_{\theta N}, \boldsymbol{\Lambda}_{\beta N}, \boldsymbol{\eta}_{\beta N}, a_N, b_N$

repeat

- update $q(\theta)$: $\boldsymbol{\Lambda}_{\theta N} \leftarrow \mathbb{E}[\tau] \left(\mathbf{D}_0^{-1} + \sum_{i=1}^N \mathbf{x}_i \mathbf{t}_i' \mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'] \mathbf{t}_i \mathbf{x}_i' \right), \quad \boldsymbol{\eta}_{\theta N} \leftarrow \mathbb{E}[\tau] \left(\mathbf{D}_0^{-1} \boldsymbol{\mu}_{\theta_0} + \sum_{i=1}^N r_i \mathbf{x}_i \mathbf{t}_i' \mathbb{E}[\boldsymbol{\beta}] \right)$
- update $q(\beta)$: $\boldsymbol{\Lambda}_{\beta N} \leftarrow \mathbb{E}[\tau] \left(\mathbf{E}_0^{-1} + \sum_{i=1}^N \mathbf{t}_i \mathbf{x}_i' \mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'] \mathbf{x}_i \mathbf{t}_i' \right), \quad \boldsymbol{\eta}_{\beta N} \leftarrow \mathbb{E}[\tau] \left(\mathbf{E}_0^{-1} \boldsymbol{\mu}_{\beta_0} + \sum_{i=1}^N r_i \mathbf{t}_i \mathbf{x}_i' \mathbb{E}[\boldsymbol{\theta}] \right)$
- update $q(\tau)$: $a_N \leftarrow \frac{p+K+N}{2} + a_0,$

$$b_N \leftarrow \frac{1}{2} [\text{tr} [D_0^{-1} (\mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'])] + (\boldsymbol{\mu}'_{\theta_0} - 2\mathbb{E}[\boldsymbol{\theta}']) \mathbf{D}_0^{-1} \boldsymbol{\mu}_{\theta_0}] + \frac{1}{2} [\text{tr} [E_0^{-1} (\mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'])] + (\boldsymbol{\mu}'_{\beta_0} - 2\mathbb{E}[\boldsymbol{\beta}']) \mathbf{E}_0^{-1} \boldsymbol{\mu}_{\beta_0}]$$

$$+ \frac{1}{2} \sum_{i=1}^N \left(r_i^2 + \mathbf{x}_i' \mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}^T] \mathbf{x}_i \mathbf{t}_i' \mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}^T] \mathbf{t}_i \right) - \sum_{i=1}^N r_i \mathbf{x}_i' \mathbb{E}[\boldsymbol{\theta}] \mathbf{t}_i' \mathbb{E}[\boldsymbol{\beta}] + b_0$$

until \mathcal{L} converges
return $\boldsymbol{\Lambda}_{\theta N}, \boldsymbol{\eta}_{\theta N}, \boldsymbol{\Lambda}_{\beta N}, \boldsymbol{\eta}_{\beta N}, a_N, b_N$

4.2.4 *Integration of other factors.* Although the approximate model considers music audio content and novelty only, it is easy to integrate other factors as long as they can be approximated by linear functions. For instance, diversity is another important factor for a playlist. If we measure the diversity that a song contributes to a playlist as d , and user preference of d follows a function that can be approximated by a piecewise linear function. Following the method in Section 4.2.1, we can map d into a vector \mathbf{d} and modify the approximate Bayesian model in Section (4.2.1) by extending Equation (7) with an additional term $\gamma' \mathbf{d}$ and put a prior on γ as following:

$$R | \mathbf{x}, \mathbf{t}, \mathbf{d}, \sigma^2, \boldsymbol{\theta}, \boldsymbol{\beta}, \gamma \sim \mathcal{N}(\boldsymbol{\theta}' \mathbf{x} \boldsymbol{\beta}' \mathbf{t} \gamma' \mathbf{d}, \sigma^2), \quad \gamma | \sigma^2 \sim \mathcal{N}(\boldsymbol{\mu}_{\gamma_0}, \sigma^2 \mathbf{F}_0).$$

Following the symmetry between \mathbf{x} , \mathbf{t} , and \mathbf{d} , we can easily modify Algorithm 2 accordingly without further derivation.

Similarly, we could incorporate in the model more factors such as coherence of mood and genre. Moreover, although the model is designed for music recommendation, it can also be applied for other regression as long as the regression function can be factorized into the product of a few linear functions.

5. EXPERIMENTS

We compare the results from our evaluations of 6 recommendation algorithms in this section. Extensive experimental evaluations of both efficiency and effectiveness of the algorithms and models have been conducted, and the results show significant promise from both aspects.

5.1 Experiment setup

5.1.1 *Comparison recommendation algorithms.* To study the effectiveness of the exploration/exploitation tradeoff, we introduced the Random and Greedy baselines. The Random approach represents pure exploration and recommends songs uniformly at random. The Greedy approach represents pure exploitation and always recommends the song with the highest predicted rating. Therefore, the Greedy approach simulates the strategy used by the traditional recommenders. For Greedy, minimum mean square error approach was used to estimate the parameters $\{\boldsymbol{\theta}, s\}$, which were optimized by the L-BFGS-B algorithm [Byrd et al. 1995].

To study the effectiveness of the rating model, the LinUCB baseline was introduced. LinUCB is a bandit algorithm which assumes that the expected rating is a linear function of the feature vector [Li

et al. 2012]. In LinUCB, ridge regression is used as the regression method, and upper confidence bound is used to balance exploration and exploitation.

Two combinations of the factors U_c , U_n were evaluated: U_c and $U_c U_n$. We write them as C and CN for short, where C and N indicate content and novelty respectively, e.g., Bayes-UCB-CN contains both content and novelty.

For the Bayes-UCB algorithm, the exact Bayesian model with the MCMC inference algorithm (Section 4.1) is indicated by Bayes-UCB-CN, and the approximate model with the variational inference algorithm (Section 4.2) is indicated by Bayes-UCB-CN-V.

We evaluated 6 recommendation algorithms, which were combinations of the four approaches and three factors: Random, LinUCB-C, LinUCB-CN, Bayes-UCB-CN, Bayes-UCB-CN-V, and Greedy-CN. Because LinUCB-CN cannot handle nonlinearity and thus cannot directly model $U_c U_n$, we combined the feature vector \mathbf{x} in U_c and the time variable t in U_n as one vector, and assumed the expected rating is a linear function of the combined vector. Greedy-C was not included because it was not related to our objective. As discussed in Section 3.2, the bandit approach can also combine with existing methods to solve the cold-start problem. We plan to study the effectiveness of such combinations in future works.

5.1.2 Songs and Features. Ten thousand songs from different genres were used in the experiments. Videos of the songs were first crawled from YouTube and converted by ffmpeg³ into mono channel WAV files with a 16KHz sampling rate. For every song, a 30-second audio clip was used [Wang et al. 2012]. Feature vectors were then extracted using a program developed based on the MARSYAS library⁴, in which a window size of 512 was used without overlapping. The features we used and their dimensionalities are ZeroCrossing (1), Centroid (1), Rolloff (1), Flux (1), MFCC (13), Chroma (14), SCF (24) and SFM (24). These features are well accepted in the music retrieval/recommendation domain. To represent a 30-second clip in one feature vector, we used the mean and standard deviation of all feature vectors from the clip. Next, we added the 1-dimensional feature *tempo* to the summarized feature vectors, and the resulting feature dimensionality is $79 \times 2 + 1 = 159$. Directly using the 159-dimensional features requires a large amount of data to train the models and makes user studies very expensive and time-consuming. To reduce the dimensionality, we conducted Principal Component Analysis (PCA) with 90% of variance reserved. The final feature dimensionality is thus reduced to 91.

The performance of these features in music recommendation was checked based on a dataset that we built. We did not use existing music recommendation datasets because they lack explicit ratings, and dealing with implicit feedbacks is not our focus. Fifty-two undergraduate students with different cultural backgrounds contributed to the dataset, with each student annotating 400 songs with a 5-point Likert scale from “very bad” (1) to “very good” (5). We first computed the 10-fold cross-validation RMSE of U_c for each user. We then averaged the accuracy over all users. The resulting RMSE is 1.10, significantly lower than the RMSE (1.61) of the random baseline with the same distribution as the data. Therefore these audio features indeed provide useful information for recommendation. Feature engineering can improve the accuracy, but it is not our focus and we leave it as future work.

5.1.3 Evaluation protocol. In [Li et al. 2011], an offline approach is proposed for evaluating contextual-bandit approaches with the assumption that the context (including the audio features and the elapsed time of songs) at different iterations are identically independently distributed. Unfortunately, this is not true in our case because when a song is not recommended, its elapsed time t keeps increasing and is thus strongly correlated. Therefore, online user study is the most reliable way of evaluation.

³<http://ffmpeg.org>

⁴<http://marsyas.sourceforge.net>

To reduce the cost of the user study, we first conducted comprehensive simulation study to verify the approaches. Only if they passed the simulations, we then proceeded to user study for further verification. The whole process underwent for a few iterations, during which the models and algorithms were continually refined. The results hereby presented are from the final iteration, and intermediate results are either referred to as preliminary study whenever necessary or omitted due to page limitation.

5.2 Simulations

5.2.1 Effectiveness study . $U = U_c U_n$ was used as the true model because the preliminary user studies showed that this resulted in better performance, which will be verified in Section 5.3 again. During the simulation, songs were recommended and rated about every 50 seconds. After 20 songs, the simulation paused for about 4 minutes to simulate the gap between two recommendation sessions.

Priors for the Bayesian models were set as uninformative ones or chosen based on preliminary simulation and user studies. For the exact Bayesian model, they are: $a_0 = 10$, $b_0 = 3$, $c_0 = 10^{-2}$, $f_0 = 10^{-3}$, $h_0 = 10^{-3}$, where f_0, h_0 are uninformative and a_0, b_0, c_0 are based on preliminary studies. For the approximate Bayesian model, they are: $D_0 = E_0 = 10^{-2}I$, $\mu_{\theta 0} = \mu_{\beta 0} = 0$, $a_0 = 2$, $b_0 = 2 \times 10^{-8}$, where $\mu_{\theta 0}, \mu_{\beta 0}, a_0, b_0$ are uninformative and D_0, E_0 are based on preliminary studies; I is the identity matrix.

U_n was discretized into the following intervals (in minutes) according to the exponentially decaying characteristics of human memory [Ebbinghaus et al. 1913]: $[0, 2^{-3}], [2^{-3}, 2^{-2}], \dots, [2^{10}, 2^{11}], [2^{11}, +\infty)$. We defined the smallest interval as $[0, 2^{-3}]$ because people usually don't listen to a song for less than 2^{-3} minute. The largest interval was defined as $[2^{11}, +\infty)$ because our preliminary user study showed that evaluating one algorithm takes no more than 1.4 day, i.e., about 2^{11} minutes. Further discretization of $[2^{11}, +\infty)$ should be easy. For songs that had not been listened to by the target user, the elapsed time t was set as one month to ensure the U_n is close to 1.

We compared the recommendation performance of the 6 recommendation algorithms in terms of regret, which is a widely used metric in RL literatures. First we define that for the l -th recommendation, the difference between the maximum expected rating $\mathbb{E}[\hat{R}^l] = \max_{k=1\dots|\mathcal{S}|} U_k$ and the expected rating of the recommended song is $\Delta_l = \mathbb{E}[\hat{R}^l] - \mathbb{E}[R^l]$. Then, the regret till the n -th recommendation can be written as Equation 8, where a smaller \mathfrak{R}_n indicates better performance.

$$\mathfrak{R}_n = \sum_{l=1\dots n} \Delta_l = \sum_{l=1\dots n} \mathbb{E}[\hat{R}^l] - \mathbb{E}[R^l] \quad (8)$$

Different values of parameters $\{\theta, s\}$ were tested. Elements of θ were sampled from standard normal distribution and s was sampled from uniform(100, 1000), where the range (100, 1000) was determined based on preliminary user study. We conducted 10 runs of the simulation study. Figure 7 shows the means and standard errors of the regret of different algorithms at different number of recommendations n . From the figure, we see that the algorithm Random (pure exploration) performs the worst. The two LinUCB-based algorithms are worse than Greedy-CN because LinUCB-C does not capture the novelty and LinUCB-CN does not capture the nonlinearity within U_c and U_n although both LinUCB-C and LinUCB-CN balance exploration and exploitation.

Bayes-UCB-based algorithms performed better than Greedy-CN because Bayes-UCB balances exploration and exploitation. In addition, the difference between Bayes-UCB and Greedy increases very fast when n is small. This is because small n means small number of training samples and thus high uncertainty, i.e., the cold-start stage. Greedy algorithms, which are used by most existing recommendation systems, do not handle the uncertainty well, while Bayes-UCB can reduce the uncertainty quickly and thus improves the recommendation performance. The good performance of Bayes-UCB-CN-V also indicates that the piecewise linear approximation and variational inference is accurate.

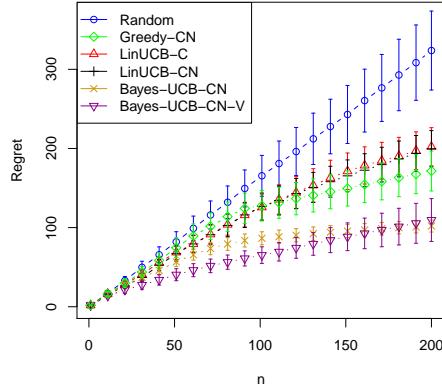


Fig. 7: Performance comparison in simulation

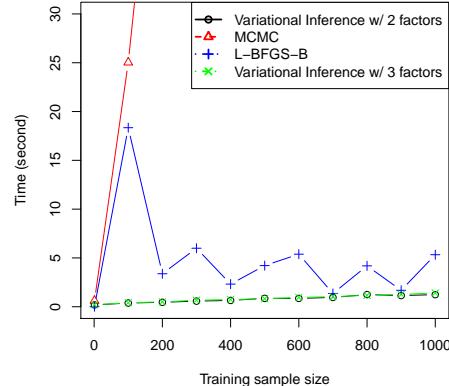


Fig. 8: Efficiency comparison

5.2.2 Efficiency study. Theoretical efficiency study of MCMC and variational inference algorithms are difficult to analyze due to their iterative nature and deserve future work. Instead, we conducted empirical efficiency study of the training algorithms for Bayes-UCB-CN (MCMC), Bayes-UCB-CN-V (variational inference), Greedy-CN (L-BFGS-B). In addition, the variational inference algorithm for the 3-factor model describe in Section 4.2.4 was also studied. LinUCB and Random were not included because the algorithms are much simpler and thus faster (but also perform much worse). Experiments were conducted on a computer with 16 cores (Intel Xeon CPU L5520 @ 2.27GHz) and 32GB main memory. No multi-threading or GP-GPU were used in the comparisons. The programming language R is used to implement all the six algorithms.

From the results in Figure 8, we can see that time consumed by both MCMC and variational inference grows linearly with the training set size. However, variational inference is more than 100 times faster than the MCMC, and significantly faster than the L-BFGS-B algorithm. Comparing the variational inference algorithm with two factors and it with three factors, we find that adding another factor to the approximate Bayesian model only slightly slows down the variational inference algorithm. Moreover, when the sample size is less than 1000, the variational inference algorithm can finish in 2 seconds, which makes online updating practical and meets the user requirement well. Implementing the algorithms in more efficient languages like C/C++ can result in even better efficiency.

Time consumed in the prediction phase of the Bayesian methods is larger than that of Greedy and LinUCB-based methods because of the sampling process. However, for the two factors model Bayes-UCB-CN-V, prediction can be accelerated significantly by the PRODCLIN algorithm without sacrificing the accuracy [MacKinnon et al. 2007]. In addition, since prediction for different songs is trivially parallelizable, scaling variational inference to large music databases should be easy.

5.3 User study

Fifteen subjects (9 females and 6 males) participated in the evaluation process. All are undergraduate students with different majors and cultural backgrounds including Chinese, Malay, Indian and Indonesian. All listen to music regularly (at least 3 hours per week). Every subject was rewarded with a small token payment for their effort and time. For each of the 6 algorithms, a subject evaluated 200 recommendations, a number more than sufficient to cover the cold-start stage. Every recommended song was listened to for at least 30 seconds (except when the subject was very familiar with the song *a priori*) and rated based on a 5-point Likert-scale as before. Subjects were required to rest for at least 4 minutes after listening to 20 songs to ensure the quality of the ratings and simulate recommenda-

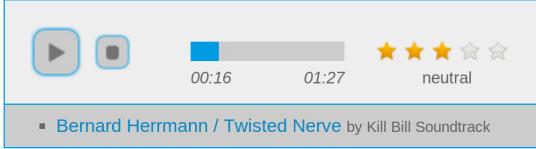


Fig. 9: User evaluation interface

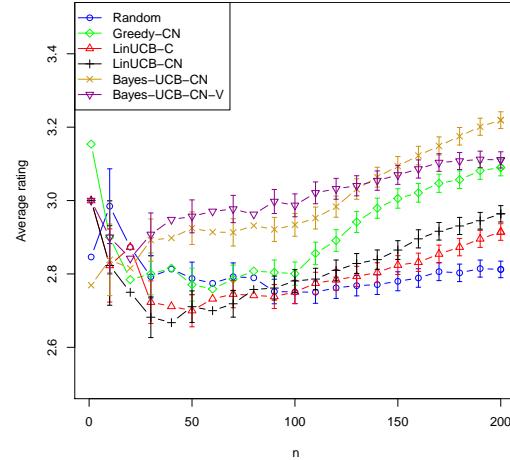


Fig. 10: Performance comparison in user study

tion sessions. To minimize the carryover effect, subjects were not allowed to evaluate more than two algorithms within one day, and there must be a gap of more than 6 hours between two algorithms. The user study lasted for one week. Every subject spent more than 14 hours in total. The dataset will be released after the publication of this paper. During the evaluation, the recommendation models were updated immediately whenever a new rating was obtained. The main interface used for evaluation is shown as Figure 9.

5.3.1 The overall recommendation performance. Because the true model is not known in user study, the regret used in simulations cannot be used here. We thus choose average rating as the evaluation metric, which is also popular in evaluations of RL algorithms. Figure 10 shows the average ratings and standard errors of every algorithm from the beginning to the n -th recommendation.

T-tests at different iterations show Bayes-UCB-CN outperforms Greedy-CN since the 45th iteration with p -values < 0.039 . Bayes-UCB-CN-V outperforms Greedy-CN from the 42th to the 141th iteration with p -values < 0.05 , and afterwards with p -values < 0.1 . Bayes-UCB-CN and Greedy-CN share the same rating model and the only difference between them is that Bayes-UCB-CN balances exploration/exploitation while Greedy-CN only exploits. Therefore, the improvement of Bayes-UCB-CN against Greedy-CN is solely contributed by the exploration/exploitation tradeoff, affirming its effectiveness.

More interestingly, when $n \leq 100$ (cold-start stage) the differences between Bayes-UCB-CN and Greedy-CN are even more significant. This is because during the cold-start stage, the uncertainty is very high; Bayes-UCB explores and thus reduces the uncertainty quickly while Greedy-CN always exploits and thus cannot reduce the uncertainty as efficiently as Bayes-UCB-CN. To verify this point, we first define a metric for uncertainty as

$$\text{uncertainty} = \frac{1}{|\mathcal{S}|} \sum_{k=1}^{|\mathcal{S}|} \text{SD}[p(U_k | \mathcal{D}_n)]$$

which is the mean of the standard deviations of all song's posterior distributions $p(U_k | \mathcal{D}_n)$ estimated using the exact Bayesian model. Larger standard deviation means larger uncertainty as illustrated in Figure 1. Given the iteration n , we calculate an uncertainty measure based on each user's recommendation history. The means and standard errors of the uncertainties among all users at different

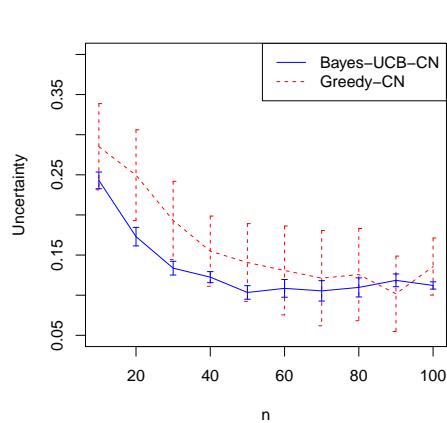


Fig. 11: Uncertainty

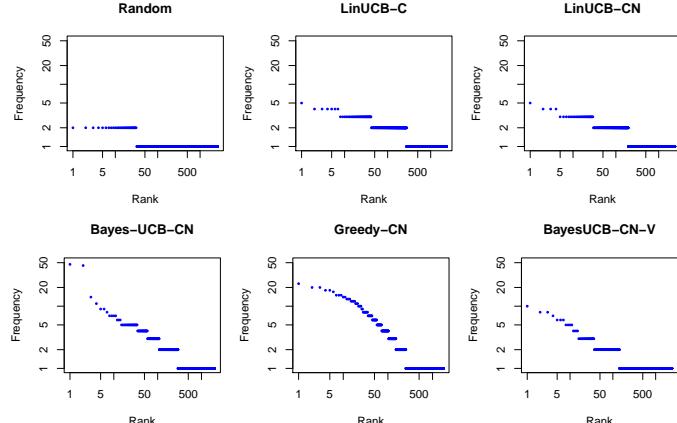


Fig. 12: Distributions of song repetition frequency

iterations are shown in Figure 11. When the number of training data points increases, the uncertainty decreases. Also as expected, the uncertainty of Bayes-UCB-CN decreases faster than Greedy-CN when n is small, and later the two remain comparable because both have obtained enough training samples to fully train the models. Therefore, this verifies that our bandit approach handles uncertainty better during the initial stage, and thus mitigate the cold-start problem.

Results in Figure 10 also show that all algorithms involving CN outperforms LinUCB-C, indicating that the novelty factor of the rating model improves recommendation performance. In addition, Bayes-UCB-CN outperforms LinUCB-CN significantly, suggesting that multiplying U_c and U_n together works better than linearly combining them.

5.3.2 Playlist generation. As discussed in Section 3.1, repeating songs following the Zipf's law is important for playlist generation. Therefore, we evaluated the playlists generated during the recommendation process by examining the distribution of songs repetition frequencies for every user. We generated the plots of the distributions in the same way we generated Figure 3 for the six algorithms. Ideal algorithms should reproduce repetition distributions of Figure 3.

The results of the six algorithms are shown in Figure 12. As we can see all algorithms with U_c and U_n multiplied together (i.e. Bayes-UCB-CN, Greedy-CN, BayesUCB-CN-V) reproduce the Zipf's law pattern well, while the algorithms without U_c (Random, LinUCB-C) or with U_c and U_n added together (LinUCB-CN) do not. This confirms that our model $U = U_c U_n$ can effectively reproduce the Zipf's law distribution. Thus, we successfully modeled an important part for combining music recommendation and playlist generation.

5.3.3 Piecewise linear approximation. In addition to the studies detailed above, the piecewise linear approximation of the novelty model is tested again by randomly selecting four users and showing in Figure 13 their novelty models learnt by Bayes-UCB-CN-V. Specifically, the posterior distributions of β^t for $t \in (0, 2^{11})$ are presented. Black lines represent the mean values of β^t and the red regions the confidence bands of one standard deviation. The scale of β^t is not important because β^t is multiplied together with the content factor, and any constant scaling of one factor can be compensated by the scaling of the other one. Comparing Figure 13 and Figure 4, we can see that the learnt piecewise linear novelty factors match our analytic form U_n well. This again confirms the accuracy of the piecewise linear approximation.

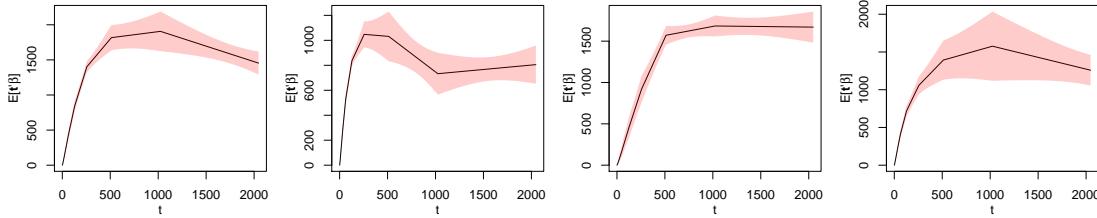


Fig. 13: Four users' diversity factors learnt from the approximate Bayesian model

6. DISCUSSION

Exploring user preferences is a central issue for recommendation systems, regardless of the specific media types. Under uncertainty, the greedy approach usually results in suboptimal results, and balancing exploration/exploitation is important. One successful example of exploration/exploitation tradeoff is the news recommender [Li et al. 2012]. Our work in this paper has shown its effectiveness in music recommendation. Given that uncertainty exists universally in all kinds of recommenders, it will be interesting to examine its effectiveness in recommenders for other media types such as video and image.

Also, our models and algorithms could be generalized to other recommenders. First, the mathematical form of the approximate Bayesian model is general enough to cover a family of rating functions that can be factorized as the product of a few linear functions (Section 4.2.4). Moreover, we can often approximate nonlinear functions with linear ones. For instance, we can use a feature mapping function $\phi(x)$ and make $U_c = \theta'\phi(x)$ to capture the non-linearity in our content model. Therefore, it will be interesting to explore our approximate Bayesian model and the variational inference algorithm in other recommendation systems. Second, the proposed novelty model may not be suitable for movie recommendation due to different consumption patterns in music and movie—users may listen to their favorites songs for many times, but repetitions are relatively rare for movies. However, the novelty model may suit recommenders which repeat items (e.g. food or makeup recommenders [Liu et al. 2013]). If their repetition patterns also follow the Zipf's law, both the exact and approximate Bayesian models can be used; otherwise, the approximate Bayesian model can be used at least.

As for extensions of this work, the first interesting direction is to model the correlations between different users to further reducing the amount of exploration. This could be achieved by extending the Bayesian models to hierarchical Bayesian models. Another interesting direction is to consider more factors such as diversity, mood, and genres to generate even better playlists, for the integration of which, our approximate Bayesian model could be a good starting point.

7. CONCLUSION

In this paper, we have described a bandit approach to interactive music recommendation that balances exploration and exploitation, mitigates the cold-start problem, and improves recommendation performance. We have also described a rating model including music audio content and novelty to integrate music recommendation and playlist generation. To jointly learn the parameters of the rating model, a Bayesian regression model together with a MCMC inference procedure were developed. To make the Bayesian inference efficient enough for online updating and generalize the model for more factors such as diversity, a piecewise linear approximate Bayesian regression model and a variational inference algorithm were built. The results from simulation demonstrate that our models and algorithms are accurate and highly efficient. User study results show that (1) the bandit approach mitigates the cold-

start problem and improves recommendation performance, and (2) the novelty model together with the content model capture the Zipf's law of repetitions in recommendations.

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APPENDIX

The following is the variational lower bound, where $\psi(\cdot)$ is the digamma function.

$$\begin{aligned}
 \mathcal{L} &= \mathbb{E}[\ln(\mathcal{D}, \tau, \boldsymbol{\theta}, \boldsymbol{\beta})] - \mathbb{E}[\ln q(\boldsymbol{\theta}, \tau, \boldsymbol{\beta})] \\
 &= \mathbb{E}[\ln p(\tau)] + \mathbb{E}[\ln p(\boldsymbol{\theta}|\tau)] + \mathbb{E}[\ln p(\boldsymbol{\beta}|\tau)] + \sum_{i=1}^N \mathbb{E}[\ln p(r_i|\mathbf{x}_i, \mathbf{t}_i, \boldsymbol{\theta}, \boldsymbol{\beta}, \tau)] - \mathbb{E}[\ln q(\boldsymbol{\theta})] - \mathbb{E}[\ln q(\boldsymbol{\beta})] - \mathbb{E}[\ln q(\tau)] \\
 &= a_0 \ln b_0 + (a_0 - 1)[\psi(a_N) - \ln b_N] - b_0 \frac{a_N}{b_N} - \frac{p}{2} \ln(2\pi) - \frac{1}{2} \ln |\mathbf{D}_0| + \frac{p}{2} (\psi(a_N) - \ln(b_N)) \\
 &\quad - \frac{a_N}{2b_N} [\text{tr}(\mathbf{D}_0 \boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \mathbb{E}[\boldsymbol{\theta}])' \mathbf{D}_0^{-1} (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \mathbb{E}[\boldsymbol{\theta}])] - \frac{K}{2} \ln(2\pi) - \frac{1}{2} \ln |\mathbf{E}_0| + \frac{K}{2} (\psi(a_N) - \ln(b_N)) \\
 &\quad - \frac{a_N}{2b_N} [\text{tr}(\mathbf{E}_0 \boldsymbol{\Lambda}_{\boldsymbol{\beta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\beta}0} - \mathbb{E}[\boldsymbol{\beta}])' \mathbf{E}_0^{-1} (\boldsymbol{\mu}_{\boldsymbol{\beta}0} - \mathbb{E}[\boldsymbol{\beta}])] - \frac{1}{2} \ln(2\pi) + \frac{1}{2} (\psi(a_N) - \ln b_N) \\
 &\quad - \frac{a_N}{2b_N} \sum_{i=1}^N (r_i^2 + \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta}\boldsymbol{\theta}'] \mathbf{x}_i \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta}\boldsymbol{\beta}'] \mathbf{t}_i) + \frac{a_N}{b_N} \sum_{i=1}^N r_i \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta}] \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta}] + \frac{K}{2} [1 + \ln(2\pi)] + \frac{1}{2} \ln |\boldsymbol{\Lambda}_{\boldsymbol{\beta}N}^{-1}| \\
 &\quad + \frac{p}{2} [1 + \ln(2\pi)] + \frac{1}{2} \ln |\boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1}| - (a_N - 1)\psi(a_N) - \ln b_N + a_N
 \end{aligned}$$

The moments of $\boldsymbol{\theta}$, $\boldsymbol{\beta}$, and τ :

$$\mathbb{E}[\boldsymbol{\beta}\boldsymbol{\beta}'] = \boldsymbol{\Lambda}_{\boldsymbol{\beta}N}^{-1} + \mathbb{E}[\boldsymbol{\beta}]\mathbb{E}[\boldsymbol{\beta}'], \quad \mathbb{E}[\boldsymbol{\beta}] = \boldsymbol{\Lambda}_{\boldsymbol{\beta}N}^{-1} \boldsymbol{\eta}_{\boldsymbol{\beta}N}, \quad \mathbb{E}[\boldsymbol{\theta}\boldsymbol{\theta}'] = \boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1} + \mathbb{E}[\boldsymbol{\theta}]\mathbb{E}[\boldsymbol{\theta}'], \quad \mathbb{E}[\boldsymbol{\theta}] = \boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1} \boldsymbol{\eta}_{\boldsymbol{\theta}N}, \quad \mathbb{E}[\tau] = \frac{a_N}{b_N}$$

Online Appendix to: Exploration in Interactive Personalized Music Recommendation: A Reinforcement Learning Approach

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A. CONDITIONAL DISTRIBUTIONS FOR THE APPROXIMATE BAYESIAN MODEL

Given N training samples $\mathcal{D} = \{r_i, \mathbf{x}_i, \mathbf{t}_i\}_{i=1}^N$, the conditional distribution $p(\boldsymbol{\theta}|\mathcal{D}, \tau, \boldsymbol{\beta})$ remains a normal distribution as:

$$\begin{aligned}
p(\boldsymbol{\theta}|\mathcal{D}, \tau, \boldsymbol{\beta}) &\propto p(\tau)p(\boldsymbol{\theta}|\tau)p(\boldsymbol{\beta}|\tau) \prod_{i=1}^N p(r_i|\mathbf{x}_i, \mathbf{t}_i, \boldsymbol{\theta}, \boldsymbol{\beta}, \tau) \\
&\propto p(\boldsymbol{\theta}|\tau) \prod_{i=1}^N p(r_i|\mathbf{x}_i, \mathbf{t}_i, \boldsymbol{\theta}, \boldsymbol{\beta}, \tau) \\
&\propto \exp\left(-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}0})' (\sigma^2 \mathbf{D}_0)^{-1} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}0})\right) \exp\left(\sum_{i=1}^N -\frac{1}{2} (r_i - \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta})^2 (\sigma^2)^{-1}\right) \\
&\propto \exp\left[-\frac{1}{2}\boldsymbol{\theta}' \left(\tau \mathbf{D}_0^{-1} + \tau \sum_{i=1}^N \mathbf{x}_i \mathbf{t}'_i \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{t}_i \mathbf{x}'_i\right) \boldsymbol{\theta} + \tau \left(\boldsymbol{\mu}'_{\boldsymbol{\theta}0} \mathbf{D}_0^{-1} + \sum_{i=1}^N r_i \boldsymbol{\beta}' \mathbf{t}_i \mathbf{x}'_i\right) \boldsymbol{\theta}\right] \\
&\propto \exp\left(-\frac{1}{2}\boldsymbol{\theta}' \boldsymbol{\Lambda}_{\boldsymbol{\theta}N} \boldsymbol{\theta} + \boldsymbol{\eta}'_{\boldsymbol{\theta}N} \boldsymbol{\theta}\right)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\Lambda}_{\boldsymbol{\theta}N} &= \tau \left(\mathbf{D}_0^{-1} + \sum_{i=1}^N \mathbf{x}_i \mathbf{t}'_i \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{t}_i \mathbf{x}'_i \right) \\
\boldsymbol{\eta}'_{\boldsymbol{\theta}N} &= \tau \left(\boldsymbol{\mu}'_{\boldsymbol{\theta}0} \mathbf{D}_0^{-1} + \sum_{i=1}^N r_i \boldsymbol{\beta}' \mathbf{t}_i \mathbf{x}'_i \right)
\end{aligned}$$

Due to the symmetry between $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$, we can easily obtain

$$p(\boldsymbol{\beta}|\mathcal{D}, \tau, \boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2}\boldsymbol{\beta}' \boldsymbol{\Lambda}_{\boldsymbol{\beta}N} \boldsymbol{\beta} + \boldsymbol{\eta}'_{\boldsymbol{\beta}N} \boldsymbol{\beta}\right),$$

where

$$\begin{aligned}\Lambda_{\beta N} &= \tau \left(\mathbf{E}_0^{-1} + \sum_{i=1}^N \mathbf{t}_i \mathbf{x}_i' \boldsymbol{\theta} \boldsymbol{\theta}' \mathbf{x}_i \mathbf{t}_i' \right) \\ \boldsymbol{\eta}'_{\beta N} &= \tau \left(\boldsymbol{\mu}'_{\beta 0} \mathbf{E}_0^{-1} + \sum_{i=1}^N r_i \boldsymbol{\theta}' \mathbf{x}_i \mathbf{t}_i' \right).\end{aligned}$$

The conditional distribution $p(\tau|\mathcal{D}, \boldsymbol{\theta}, \boldsymbol{\beta})$ also remains a Gamma distribution:

$$\begin{aligned}p(\tau|\mathcal{D}, \boldsymbol{\theta}, \boldsymbol{\beta}) &\propto \tau^{a_N - 1} \exp(-b_N \tau) \\ p(\tau|\mathcal{D}, \boldsymbol{\theta}, \boldsymbol{\beta}) &\propto p(\tau) p(\boldsymbol{\theta}|\tau) p(\boldsymbol{\beta}|\tau) \prod_{i=1}^N p(r_i|\mathbf{x}_i, \mathbf{t}_i, \boldsymbol{\theta}, \boldsymbol{\beta}, \tau) \\ &= b_0^{a_0} \frac{1}{\Gamma(a_0)} \tau^{a_0 - 1} \exp(-b_0 \tau) \times \\ &\quad \frac{1}{(2\pi)^{p/2} |\sigma^2 \mathbf{D}_0|^{1/2}} \exp \left(-\frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta 0})' (\sigma^2 \mathbf{D}_0)^{-1} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta 0}) \right) \\ &\quad \times \frac{1}{(2\pi)^{K/2} |\sigma^2 \mathbf{E}_0|^{1/2}} \exp \left(-\frac{1}{2} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta 0})' (\sigma^2 \mathbf{E}_0)^{-1} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta 0}) \right) \\ &\quad \times \left(\frac{1}{(2\pi)^{1/2} |\sigma^2|^{1/2}} \right)^N \exp \left(\sum_{i=1}^N -\frac{1}{2} (r_i - \mathbf{x}_i' \boldsymbol{\theta} \mathbf{t}_i' \boldsymbol{\beta})^2 (\sigma^2)^{-1} \right) \\ &\propto \tau^{a_N - 1} \exp(-b_N \tau)\end{aligned}$$

where a_N and b_N are the parameters of the Gamma distribution, and they are

$$\begin{aligned}a_N &= \frac{p + K + N}{2} + a_0 \\ b_N &= b_0 + \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta 0})' \mathbf{D}_0^{-1} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta 0}) + \frac{1}{2} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta 0})' \mathbf{E}_0^{-1} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta 0}) \\ &\quad + \frac{1}{2} \sum_{i=1}^N (r_i - \mathbf{x}_i' \boldsymbol{\theta} \mathbf{t}_i' \boldsymbol{\beta})^2\end{aligned}$$

B. VARIATIONAL INFERENCE

To calculate the joint posterior distribution $p(\boldsymbol{\theta}, \tau, \boldsymbol{\beta}|\mathcal{D})$, we can use Gibbs sampling based on the conditional distributions. However, this is slow too, and therefore, we resort to variational inference (mean field approximation specifically). We assume that $p(\boldsymbol{\theta}, \tau, \boldsymbol{\beta}|\mathcal{D}) \approx q(\boldsymbol{\theta}, \boldsymbol{\beta}, \tau) = q(\boldsymbol{\theta})q(\boldsymbol{\beta})q(\tau)$. In the restricted distribution $q(\boldsymbol{\theta}, \boldsymbol{\beta}, \tau)$, every variable is assumed independent from the other variables. Because all the conditional distributions $p(\boldsymbol{\theta}|\mathcal{D}, \tau, \boldsymbol{\beta})$, $p(\tau|\mathcal{D}, \boldsymbol{\theta}, \boldsymbol{\beta})$, and $p(\boldsymbol{\beta}|\mathcal{D}, \boldsymbol{\theta}, \boldsymbol{\beta})$ are in the exponential families, their restricted distributions $q(\boldsymbol{\theta})$, $q(\boldsymbol{\beta})$, $q(\tau)$ lie in the same exponential families as their conditional distributions. We then obtain the restricted distributions and update rules as in Section 4.2.2.

The expectation of b_N with respect to $q(\boldsymbol{\theta})$ and $q(\boldsymbol{\beta})$ might be a bit tricky to derive. We thus show it as the following:

$$\begin{aligned} b_N &= b_0 + \frac{1}{2}\mathbb{E}[(\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}0})' \mathbf{D}_0^{-1} (\boldsymbol{\theta} - \boldsymbol{\mu}_{\boldsymbol{\theta}0})] + \frac{1}{2}\mathbb{E}[(\boldsymbol{\beta} - \boldsymbol{\mu}_{\boldsymbol{\beta}0})' \mathbf{E}_0^{-1} (\boldsymbol{\beta} - \boldsymbol{\mu}_{\boldsymbol{\beta}0})] + \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^N (r_i - \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta})^2\right] \\ &= b_0 + \frac{1}{2} [\text{tr}[\mathbf{D}_0^{-1} (\mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'])] + (\boldsymbol{\mu}'_{\boldsymbol{\theta}0} - 2\mathbb{E}[\boldsymbol{\theta}']) \mathbf{D}_0^{-1} \boldsymbol{\mu}_{\boldsymbol{\theta}0}] \\ &\quad + \frac{1}{2} [\text{tr}[\mathbf{E}_0^{-1} (\mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'])] + (\boldsymbol{\mu}'_{\boldsymbol{\beta}0} - 2\mathbb{E}[\boldsymbol{\beta}']) \mathbf{E}_0^{-1} \boldsymbol{\mu}_{\boldsymbol{\beta}0}] + \frac{1}{2} \sum_{i=1}^N \mathbb{E}[(r_i - \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta})^2]. \end{aligned}$$

Since $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ are assumed independent, we have

$$\begin{aligned} \mathbb{E}[(r_i - \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta})^2] &= \mathbb{E}[r_i^2 - 2r_i \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta} + \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta} \mathbf{x}'_i \boldsymbol{\theta} \mathbf{t}'_i \boldsymbol{\beta}] \\ &= r_i^2 - 2r_i \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta}] \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta}] + \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'] \mathbf{x}'_i \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'] \mathbf{t}_i. \end{aligned}$$

Therefore b_N can be calculated as

$$\begin{aligned} b_N &= b_0 + \frac{1}{2} [\text{tr}[\mathbf{D}_0^{-1} (\mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'])] + (\boldsymbol{\mu}'_{\boldsymbol{\theta}0} - 2\mathbb{E}[\boldsymbol{\theta}']) \mathbf{D}_0^{-1} \boldsymbol{\mu}_{\boldsymbol{\theta}0}] \\ &\quad + \frac{1}{2} [\text{tr}[\mathbf{E}_0^{-1} (\mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'])] + (\boldsymbol{\mu}'_{\boldsymbol{\beta}0} - 2\mathbb{E}[\boldsymbol{\beta}']) \mathbf{E}_0^{-1} \boldsymbol{\mu}_{\boldsymbol{\beta}0}] \\ &\quad + \frac{1}{2} \left[\sum_{i=1}^N r_i^2 - 2r_i \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta}] \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta}] + \mathbf{x}'_i \mathbb{E}[\boldsymbol{\theta} \boldsymbol{\theta}'] \mathbf{x}'_i \mathbf{t}'_i \mathbb{E}[\boldsymbol{\beta} \boldsymbol{\beta}'] \mathbf{t}_i \right]. \end{aligned}$$

C. VARIATIONAL LOWER BOUND

It might be a bit tricky to derive

$$\mathbb{E}[\ln p(\boldsymbol{\theta}|\tau)] = \iint p(\boldsymbol{\theta}|\tau) q(\boldsymbol{\theta}) d\boldsymbol{\theta} q(\tau) d\tau$$

which is part of the lower bound \mathcal{L} . We assume that $P = p(\boldsymbol{\theta}|\tau)$, and $Q = q(\boldsymbol{\theta})$, and we have $\int p(\boldsymbol{\theta}|\tau) q(\boldsymbol{\theta}) d\boldsymbol{\theta} = -H(Q, P)$, where $H(Q, P)$ is the cross entropy between Q and P . Given Q and P are multivariate normal distributions, the KL-divergence between Q and P and the entropy of Q are

$$\begin{aligned} D_{KL}(Q||P) &= \frac{1}{2} \left[\text{tr}(\Sigma_P^{-1} \Sigma_Q) + (\boldsymbol{\mu}_P - \boldsymbol{\mu}_Q)' \Sigma_P^{-1} (\boldsymbol{\mu}_P - \boldsymbol{\mu}_Q) - \ln \frac{|\Sigma_Q|}{|\Sigma_P|} - p \right] \\ &= \frac{1}{2} \left[\text{tr}(\tau \mathbf{D}_0 \Lambda_{\boldsymbol{\theta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})' \tau \mathbf{D}_0^{-1} (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N}) - \ln |\Lambda_{\boldsymbol{\theta}N}^{-1}| + \ln \left| \frac{1}{\tau} \mathbf{D}_0 \right| - p \right] \\ H(Q) &= \frac{1}{2}(p + p \ln(2\pi) + \ln |\Lambda_{\boldsymbol{\theta}N}^{-1}|). \end{aligned}$$

Therefore

$$\begin{aligned} \int p(\boldsymbol{\theta}|\tau) q(\boldsymbol{\theta}) d\boldsymbol{\theta} &= -H(Q, P) \\ &= -H(Q) - D_{KL}(Q||P) \\ &= -\frac{p}{2} \ln(2\pi) - \frac{1}{2} \ln |\mathbf{D}_0| + \frac{p}{2} \ln \tau - \frac{1}{2} [\text{tr}(\tau \mathbf{D}_0 \Lambda_{\boldsymbol{\theta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})' \tau \mathbf{D}_0^{-1} (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})], \end{aligned}$$

and

$$\begin{aligned}
 \mathbb{E}[\ln p(\boldsymbol{\theta}|\tau)] &= \iint p(\boldsymbol{\theta}|\tau)q(\boldsymbol{\theta})d\boldsymbol{\theta}q(\tau)d\tau \\
 &= -\frac{p}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{D}_0| + \frac{p}{2}\mathbb{E}[\ln\tau] - \frac{1}{2}\left[\text{tr}(\mathbf{D}_0\boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})' \mathbf{D}_0^{-1}(\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})\right]\mathbb{E}[\tau] \\
 &= -\frac{p}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{D}_0| + \frac{p}{2}(\psi(a_N) - \ln(b_N)) \\
 &\quad - \frac{a_N}{2b_N}\left[\text{tr}(\mathbf{D}_0\boldsymbol{\Lambda}_{\boldsymbol{\theta}N}^{-1}) + (\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})' \mathbf{D}_0^{-1}(\boldsymbol{\mu}_{\boldsymbol{\theta}0} - \boldsymbol{\mu}_{\boldsymbol{\theta}N})\right]
 \end{aligned}$$