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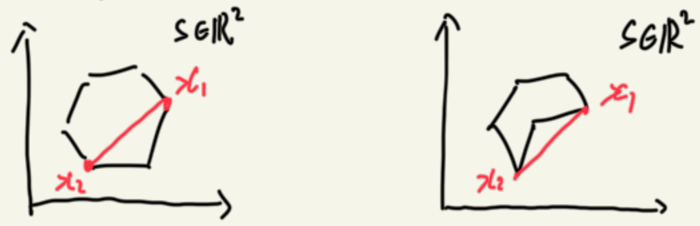
① Some Fundamental Law for LP

i. Definitions

Lemma 2:
The intersection of any collection of convex sets is convex

Lemma 1:
Every closed half-space is a convex set.

Definition 1 Convex set: A set S in n-dimensional space is said convex if whenever any two points x_1 and x_2 belong to S so does every point of the line segment between x_1 and x_2



Definition 2 Closed half-space: Given a n-dimensional row vector a and a costant b, the set of all vectors x in n-dimensional space satisfying $ax \le b$ is called a closed half-space. The set of vectors for which $ax = b$ is called the boundary of the closed half-space.

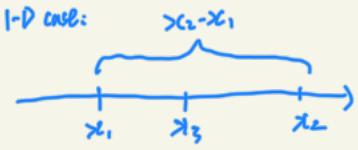
⇐ Defining spaces through inequalities

Definition 3 Extreme point: Given a convex set S of n-dimentional vectors, a point x^* is called an extreme point of S if there do not exist any two points x_1 and x_2 in S and any value $\alpha \in (0,1)$, such that

$$x^* = \alpha x_1 + (1-\alpha) x_2$$

endpoint of each line segment in S

Represent a point between two poi



⇐ $x_3 = x_2 - \alpha(x_2 - x_1)$
$= \alpha x_1 + (1-\alpha) x_2$
so does higher dimentional case

ii. Inference

Premise 1: Each constraint form as " $a_i^T x \le b_i$ " is a closed half-space. ⇐ recall the lemma 1. each closed half-space is convex set

Premise 2: The feasible set of an LP, if it exists, consists of all vectors that simultaneously satisfy a finite number of linear constraints.

$\Downarrow$ premise 1 + premise 2

Feasible set is the intersection of a finite number of closed half-spaces.

Theorem 1
The feasible set of an LP is a convex set

Theorem 2
The feasible region for any LP has a finite number of extreme points

☆ **Theorem 3**
If the feasible set is non-empty and one optimal solution exists to the LP, then there is an optimal solution at one of the extreme points

iii. Aim of solving LP: Find the extreme point that are optimal solution.

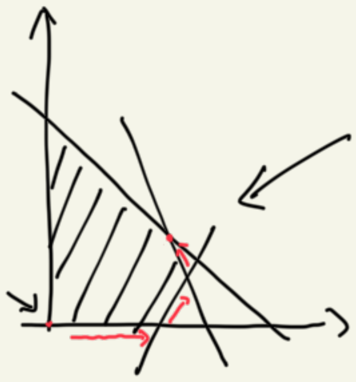
② Simplex Method Prereview

"The fundamental theorem of linear programming reduces to a finite value the number of feasible solutions that need to be evaluated"

← That's why we need Simplex Method

A diamond has many extreme point

⇒ "In practice, identifying the coordinates of every extreme point and evaluate the objective function at each is not efficient because the number of extreme points can be very large."



Simplex method begins by identifying an initial extreme point of the feasible set.

"origin"

Then the simplex method looks along each edge intersecting the extreme point and compute the net effect on the objective function if we were to move along the edge.

i. Preparing for LP.

Aim: Changing LP to standard form

① Nonnegtive RHS.

if not, multiplying through the entire constraint by "-1"

e.g.

$$3x_1 + 4x_2 - 2x_3 \le -6$$
$$\Downarrow$$
$$-3x_1 - 4x_2 + 2x_3 \ge 6$$

② All Constraints Must Be Equalities

if not, add slack / surplus variable.

It's not just a simple transformation of the formula, but more like re-modeling from the perspective of equations to re-understand the problem. ⇒

case1: $3x_1 + 4x_2 \le 12$
$$\Downarrow$$
$$3x_1 + 4x_2 + s_1 = 12$$

case2: $2x_1 + 4x_2 \ge 10$
$$\Downarrow$$
$$2x_1 + 4x_2 - s_2 = 10$$

slack / surplus variables all have physical meaning.

③ All Variables Must Be Nonnegative

Any variable not already constrained to be nonnegative can be converted to the difference of two new nonnegative variables

e.g.
$$\min z = 25x_1 + 30x_2$$
$$\text{s.t.} \quad 4x_1 + 7x_2 \ge 1$$
$$x_1, x_2 \text{ urs}$$
$$\Downarrow$$

replace x_1, x_2 by
$$x_1 = a_1 - a_2$$
$$x_2 = a_3 - a_4$$
$$a_1, a_2, a_3, a_4 \ge 0$$

$\Downarrow$

$$\min Z = 25(a_1-a_2)+20(a_3-a_4)$$
$$4(a_1-a_2)+7(a_3-a_4)\geq 1$$
$$a_1,a_2,a_3,a_4\geq 0$$

④ Each Constraint Must Have a Unique Variable with a "+1" Coefficient. If not, adding artifical variables a_i.

ii. Maximum number of Extreme point.

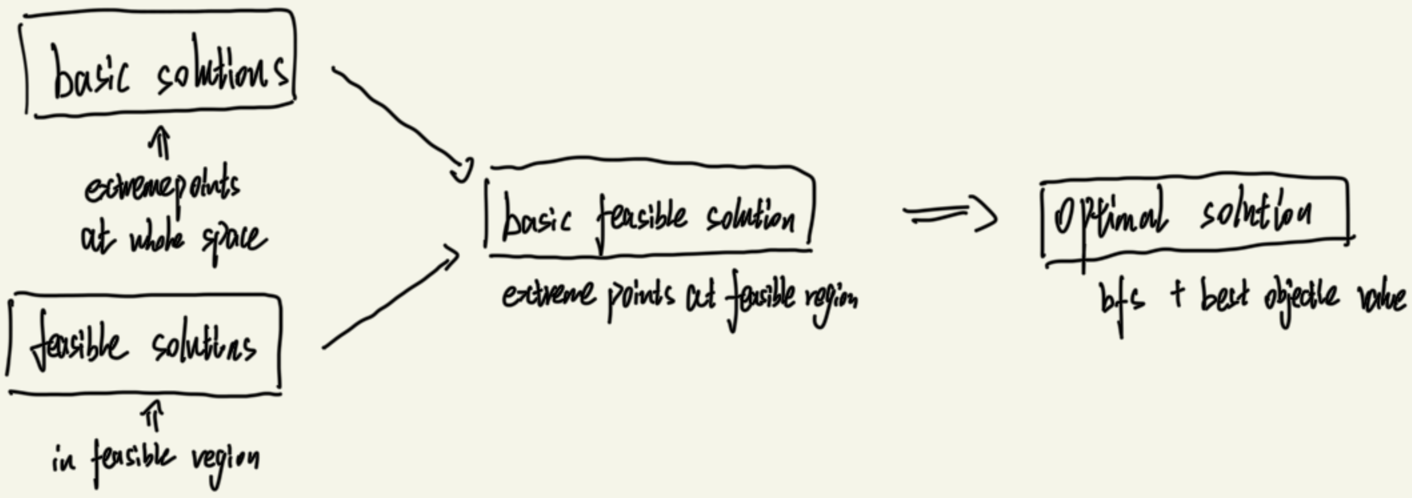
The standard form LP has m simultaneous linear equations (constraints) and n nonnegative variables, with $m<n$.

The maximum number of the extreme points $= C_n^m = \dfrac{n!}{m!(n-m)!}$

Each time setting m variable to be the basic variables and $(n-m)$ variable to be 0. With no sequence.

As with extreme points, the basic feasible solutions completely define the candidates for the optimum solution in the algebraic solution space.

iii. variables and solutions



③ Simplex Method Algebra

After we get the standard LP as follow:

$$Z - C^T x = 0$$
$$\text{s.t.}\quad A x = b$$
$$x \geq 0$$

$\Leftarrow$ As we can write in in canonical form and write as Simplex tableau.

We can apply the simplex method.

Step1: find the entering variable.

Simplex method select m variables (the number of constraints) to be the basic variables and $(n-m)$ variables to be the non-basic variables.

thus for the coefficients we have $x=[x_N^T \mid x_B^T]^T$

$c=[c_N^T \mid c_B^T]^T$

$$C = [C_N \mid C_B]$$
$$A = [N \mid B]$$

We cares about how the changes in the values of non-basic variables affect the value of the objective function

$$z - C^T x = 0$$
$$z - C_N^T x_N - C_B^T x_B = 0$$

⇐ At this time, both x_B, x_N is in the same row. we need to eliminate x_B

Rewrite

$$A x = b$$
$$N x_N + B x_B = b$$
$$x_B = B^{-1}(b - N x_N)$$
$$x_B = B^{-1} b - B^{-1} N x_N$$

⇐ Now we know the mapping relationship between x_N and x_B

substitute $x_B = B^{-1}b - B^{-1}N x_N$ to the z-row.

Finilly, We established the relationship between the objective value z and the non-basic variable x_N, taking into account the mutual influence between x_N and x_B ⇒

$$z - C_N^T x_N - C_B^T (B^{-1} b - B^{-1} N x_N) = 0$$
$$z - C_N^T x_N + C_B^T B^{-1} N x_N = C_B^T B^{-1} b$$
$$z + (C_B^T B^{-1} N - C_N^T) x_N = C_B^T B^{-1} b$$

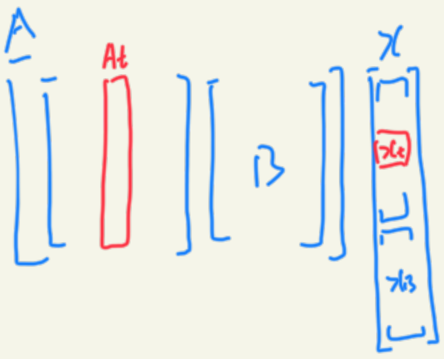
⇐ At this step, we make the coefficients of x_B to 0. In Simplex tableau, we use EROs to do pivoting.

set $\hat{C}_N^T = C_B^T B^{-1} N - C_N^T$, and $\hat{C}_N^T$ named cost value.

Because In the simplex method, we only change one non-basic variable at one iteration. So we need to pick the most significant one.

entering variable x_t $\begin{cases} \text{choose the most large positive } \hat{C}_{N,t}^T, & \text{if LP is Minimum problem} \\ \text{choose the most small negative } \hat{C}_{N,t}, & \text{if LP is Maximum problem} \end{cases}$

When we choose $x_t \in x_N$ to join in the basis, we need to "choose one x_B to leave the basis"



⇒

Recall $x_B = B^{-1} b - B^{-1} N x_N$

we only have $x_t \neq 0$ in x_N, thus we have

$$x_B = B^{-1} b - B^{-1} A_t x_t$$

for each row i, we have.

$$x_{B,i} = (B^{-1} b)_i - (B^{-1} A_t)_i x_t$$

We need to find, when x_t increasing, which element in x_B turns to zero first

$$s = \underset{i}{\arg\min} \left\{ \frac{(B^{-1} b)_s}{(B^{-1} A_t)_s} \; ; \; (B^{-1} A_t)_s > 0 \right\}$$

⇐ In simplex tableau, this is called "ratio-test".

and $x_s \in x_B$ is th leaving variable

At last, we use x_t to replace x_s, set x_t to pivot and do pivoting.

If the cost value shows that can be further optimized, iterate again.

cost value is generated after pivoting is done.

basis	x_N	x_B	rhs
z	$C_B^T B^{-1} N - C_N^T$	0^T	$C_B^T B^{-1} b$

⇐ canonical form tableau

z	$c_B B^{-1}N - c_N$	0	$c_B B^{-1} b$
x_B	$B^{-1}N$	I	$B^{-1}b$

after pivoting

(4) M-method and Two-phase method

i. LP problem

- Simple case: "$\leq$" constraints, $b \geq 0$
  initial bfs: set all slack variable as basic variable, start from the origin.
- Complex case: "$\geq$, $=$" constraints
  initial bfs: slack variable is not enough for basis, need artifical variable to fill in.

Not like slack/surplus ⇒ variable, artifical variable don't have physical significance. If $a_i > 0$, LP changes to another LP problem. The optimal solution is also changed

ii. Aim for M-method and Two-phase method: Restrict the artifical variable to 0

M-method ⇑ add restrict to original objective function

Two-phase method ⇑ add restrict to a new objective function

iii. M-method

Step 1: Convert LP to standard form (Satisfy the above three conditions) ⇐
- ① nonnegative RHS
- ② all variable non-negative
- ③ all constraints are "=" constraints (add slack and surplus variable)

Step 2: For each constraint without a slack variable (surplus variables always negative), add an artifical variable a_i in LHS. also add nonnegative constraints $a_i \geq 0$

Step 3: Define a sufficiently large positive value M, as a penalty coefficients for a_i

- case 1: $\min z = C^T x \Rightarrow \min z = C^T x + M_i a_i$, if $a_i > 0$, $z \to \infty$
- case 2: $\max z = C^T x \Rightarrow \max z = C^T x - M_i a_i$, if $a_i > 0$, $z \to -\infty$

Step 4: Solving LP using simplex method.
If all artifical variables are equal to 0 in the optimal solution, then we find the solution for original LP. If any artifical variables are positive in optimal solution, then the original problem is infeasible.

iv. Two-phase method

The introduction of large number M in original objective function will cause "round off error"

Step 1 ~ Step 2 is same as M-method.

step 3: Ignore the original LP's objective function.
solving the new one form as:

$$\min w = \sum^{N} a_i$$ ⇐ Phase I (force the artificial variable to zero)

$\min w = \sum_{i=1} a_i$ ← Phase I (force the artificial variable to zero)

Using the original LP's constraints $Ax = b$. $x, a \geq 0$

As for $a_i \geq 0$, there are three cases.

case 1: $W_{min} > 0$, LP has no feasible solution

case 2: $W_{min} = 0$, and all a_i leaves the basis.

I. In this case, drop all columns in the Phase-I tableau that correspond to the artificial variable

II. Combine the original objective function with the new constraints from the modified optimal Phase-I tableau

no artificial variable in it.

turn to phase II

Case 3: $W_{min} = 0$, but at least one a_i remaining in the basis

I. In this case, drop non-basic artificial variables ① and non-artificial variables (also non-basic) that have negative coefficients in row 0

II. combine the original objective function with the new constraints from the modified optimal phase I tableau

step 4: solving Phase II using simplex method.

⑤ Special Cases in Simplex Method

i. Degeneracy: the objective value stay the same after one iteration

In some cases, the degeneracy just causes more iterations and the inefficiency of the Simplex method. The simplex method might still reach an optimal solution even though a degeneracy.

ii. Alternative Optima: infinite optimal solution

exists some coefficients for non-basic variable are 0

iii. Unbounded Solution: The model is poorly constructed

the objective value can still improve shown by the cost value, but don't know the direction

iv. Infeasible Solution: At least one artificial variable is positive.

The model may not be formulated correctly and constraints mutual contradiction.