35241 OQM

Lecture Note – Part 6

Sensitivity Analysis

1 Introduction

The real world is usually more complicated than the sorts of optimisation problems we formulate. Utilising mathematical optimisation modelling to describe the reality would need approximations. One of the significant approximations is the linearity assumption. Another important approximation comes from the data which we put into the LP model because we may be uncertain about them. Our knowledge of the relevant parameters such as the coefficients of objective function $\mathbf{c}$, the constraint matrix $\mathbf{A}$ and the rhs $\mathbf{b}$, in the considered problem may be imprecise, thus enforcing the approximations of their values. Furthermore, in many applications the values of an LP's parameters may change.

Sensitivity analysis is a systematic study of how sensitive the LP's optimal solution is to (small) changes in the LP's parameters, i.e. how changes in some parameters affect the optimal solution. If a parameter changes, sensitivity analysis often makes it unnecessary to solve the considered LP problem again.[1] A knowledge of sensitivity analysis often enables the analyst to determine from the original optimal solution how the changes in an LP's parameters

[1] Solving an LP with thousands of variables and constraints again would be a chore.

change the optimality. Sensitivity analysis in LP is presented to give answers to questions of the following forms:

1. If the objective function changes, how does the optimal solution change?

2. If the amount of resources available changes, how does the optimal solution change?

3. If an additional constraint is added to the LP, how does the optimal solution change?

Usually, sensitivity analysis is performed on "individual" variables, i.e.in each specific analysis only one parameter of the LP changes and all the other parameters remain fixed at their original values.
The types of sensitivity analysis to be introduced in this chapter include:

1. changes in the coefficients of the objective function, c_j, $\forall j = 1, \dots, n$;

2. changes in the rhs of constraints, b_i, $\forall i = 1, \dots, m$;

3. changes in the coefficients of the constraints, a_{ij}, $\forall i = 1, \dots, m$ and $j = 1, \dots, n$;

4. addition of a new constraint;

5. addition of a new variable.

To learn how to perform sensitivity analysis on an arbitrary LP, we recall the algebraic Simplex tableau. Assume that in an arbitrary LP in standard form:

$$\begin{aligned} \min \text{ (or max) } z = \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}, \end{aligned}$$

where $\mathbf{b} \geq \mathbf{0}$, we have the optimal bfs $\mathbf{x}^{*T} = (\mathbf{x}_\mathbf{N}^T \,|\, \mathbf{x}_\mathbf{B}^T)$ and the following final optimal Simplex tableau.

basis	$\mathbf{x}_\mathbf{N}$	$\mathbf{x}_\mathbf{B}$	rhs
z^*	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T$	$\mathbf{0}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

The optimal tableau gives that

$$\begin{aligned} \widehat{\mathbf{c}}_\mathbf{N}^T &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T \leq \text{ (or } \geq) \, \mathbf{0}^T; \\ \mathbf{x}_\mathbf{B} &= \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}; \\ \mathbf{x}_\mathbf{N} &= \mathbf{0}; \\ z^* &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}. \end{aligned}$$

The first inequality is the **optimality condition**.[2] The second inequality and the third equality give a basic solution the **feasibility conditions**.[3]

[2] If any reduced cost were positive (or negative), then the Simplex procedure would enter the corresponding nonbasic variable to the basis and improve the objective value.

[3] If any basic variable in a basic solution had been negative, this basic solution would not have been a bfs.

Later we will show type by type how to perform sensitivity analysis using the optimality condition and feasibility conditions. In the preceding chapter, we realised that the calculation of the basic matrix inverse $\mathbf{B}^{-1}$ and the Simplex multiplier $\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ is the kernel of the Simplex procedure. Actually, they are critical to sensitivity analysis as well. Assume that the initial bfs is $(\mathbf{x}_{\mathbf{N}_0}^T \mid \mathbf{x}_{\mathbf{B}_0}^T)$, and recall the derivation of the revised Simplex method. Then the above final optimal Simplex tableau can easily be revised as follows.

basis	$\mathbf{x}_{\mathbf{N}_0}$	$\mathbf{x}_{\mathbf{B}_0}$	rhs
z^*	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T$	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1} - \mathbf{c}_{\mathbf{B}_0}^T$	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}}$	$\mathbf{B}^{-1}\mathbf{N}_0$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Therefore, once the final optimal tableau is provided, we can obtain without calculation $\mathbf{B}^{-1}$ and $\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ from the columns corresponding to the initial basis $\mathbf{x}_{\mathbf{B}_0}$.

2 Change in the Objective Coefficient

How we approach a change in a coefficient of the objective function depends on whether the coefficient is associated with a basic or a nonbasic variable in the optimal bfs.

2.1 Nonbasic Variable

Consider a change in the objective coefficient of a nonbasic variable, $c_j \in \mathbf{c}_{\mathbf{N}}$ in the optimal bfs of a minimisation (or maximisation) LP. The only impact this change can have is that if the reduced cost of x_j becomes positive (or negative) we can improve the objective value by entering x_j into the basis. If the new reduced cost remains nonpositive (or nonnegative), the change has no impact on the optimal bfs. Denote by Δ the change in c_j, so the new objective coefficient for x_j is

$$c'_j = c_j + \Delta.$$

Notice that from the Simplex algebraic formulae this change only affects the current reduced cost $\widehat{c}_j$. Then we have its new reduced cost $\widehat{c}'_j$ as follows:

$$\begin{aligned}\widehat{c}'_j = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_j - c'_j &= \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_j - (c_j + \Delta) \\ &= (\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_j - c_j) - \Delta \\ &= \widehat{c}_j - \Delta.\end{aligned}$$

If this value is nonpositive (or nonnegative), then the current basis is still optimal. Thus, the optimal bfs will be unchanged if

$$\Delta \geq (\text{or } \leq)\ \widehat{c}_j.$$

Example 1.

Consider the following LP in standard form:

$$\begin{array}{rlcl}\min\ z = & -x_1 - 2x_2 & & \\ s.t. & -2x_1 + x_2 + x_3 & = & 2 \\ & -x_1 + 2x_2 + x_4 & = & 7 \\ & x_1 + x_5 & = & 3 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq & 0\end{array}$$

with the final optimal tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	-$\frac{1}{2}$	$\frac{3}{2}$	3

The final tableau shows that the optimal basis is $\mathbf{x_B} = (x_2, x_1, x_3)^T$. Besides, we can obtain the following information from the LP in standard form and the final tableau.

$$\widehat{\mathbf{c}}_{\mathbf{N}}^T = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T = (-1, -2),$$

$$\mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix},$$

$$\mathbf{B}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix},$$

$$\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} = (0, -1, -2).$$

Suppose now that in this cost minimisation problem there is an extra charge incurred by the nonbasic variable x_4, whose original price (objective coefficient) is zero. This incurred rate is \$$\Delta$/unit of x_4. So the new objective function is $-x_1 - 2x_2 + \Delta x_4$, i.e. the objective coefficient of x_4 is increased from $c_4 = 0$ to $c'_4 = (0+\Delta)$. Now we aim to perform sensitivity analysis to find the range of Δ so that the optimal basis is still $\mathbf{x_B} = (x_2, x_1, x_3)^T$. Since x_4 is a nonbasic variable in the minimisation LP, the value Δ must keep the corresponding new reduced cost $\widehat{c}'_4 \leq 0$, i.e.

$$\begin{aligned}
\widehat{c}'_4 &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_4 - c'_4 \\
&= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_4 - (c_4 + \Delta) \\
&= (\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_4 - c_4) - \Delta \\
&= \widehat{c}_4 - \Delta \\
&= -1 - \Delta \leq \ 0.
\end{aligned}$$

Hence, $\Delta \geq -1$.
In other words, if the new objective coefficient of x_4 satisfies $\Delta \geq -1$, i.e. $c'_4 = 0 + \Delta \geq -1$, the current basis is still optimal.

2.2 Basic Variable

Suppose now that

$$\mathbf{c}'_\mathbf{B} = \mathbf{c}_\mathbf{B} + \Delta\mathbf{c}_\mathbf{B},$$

i.e. there is a change in the objective coefficient of some basic variable. It is obvious that this change won't alter the reduced costs of the basis, which will remain to be zeros. However, we must check all reduced costs of the nonbasic variables to observe whether any would be eligible to enter the basis.
The new reduced cost for any nonbasic variable x_j is

$$\widehat{c}'_j = (\mathbf{c}_\mathbf{B} + \Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{A}_j - c_j$$

$$= (\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_j - c_j) + (\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{A}_j$$
$$= \widehat{c}_j + (\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{A}_j.$$

If in the minimisation (or maximisation) LP these values are all nonpositive (or nonnegative), i.e.

$$\widehat{c}_j + (\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{A}_j \leq (\text{or } \geq)\ 0,$$

$$(\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{A}_j \leq (\text{or } \geq)\ -\widehat{c}_j$$

for any nonbasic variable $x_j \in \mathbf{x}_\mathbf{N}$, the current basis is still optimal. The above inequalities can be presented by a vector inequality shown below.

$$\widehat{\mathbf{c}}_\mathbf{N}'^T = \widehat{\mathbf{c}}_\mathbf{N}^T + (\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{N} \leq (\text{or } \geq)\ \mathbf{0}^T \quad \Leftrightarrow \quad (\Delta\mathbf{c}_\mathbf{B})^T\mathbf{B}^{-1}\mathbf{N} \leq (\text{or } \geq)\ -\widehat{\mathbf{c}}_\mathbf{N}^T$$

Example 2.

Consider the LP in Example 1. Now we aim to find by how much the objective coefficient of x_3 can be changed without altering the optimal basis $\mathbf{x}_\mathbf{B}$.

Denote the new objective coefficient of x_3 by $c_3' = c_3 + \Delta$. Then the new objective coefficient vector of basis is $\mathbf{c}_\mathbf{B}^T + (0, 0, \Delta)$, i.e. $\Delta\mathbf{c}_\mathbf{B} = (0, 0, \Delta)^T$. Since x_3 is a basic variable in the minimisation LP, the value Δ must keep the reduced cost of all nonbasic variables nonpositive, i.e.

$$
\begin{aligned}
\widehat{\mathbf{c}}'^T_{\mathbf{N}} &= \mathbf{c}'^T_{\mathbf{B}}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}^T_{\mathbf{N}} \\
&= \big(\mathbf{c}^T_{\mathbf{B}} + (0,0,\Delta)\big)\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}^T_{\mathbf{N}} \\
&= (\mathbf{c}^T_{\mathbf{B}}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}^T_{\mathbf{N}}) + (0,0,\Delta)\mathbf{B}^{-1}\mathbf{N} \\
&= \widehat{\mathbf{c}}^T_{\mathbf{N}} + (0,\ 0,\ \Delta)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \\
&= (-1,\ -2) + (-\tfrac{\Delta}{2},\ \tfrac{3\Delta}{2}) \\
&= (-1 - \tfrac{\Delta}{2}\ ,\ -2 + \tfrac{3\Delta}{2}) \leq \mathbf{0}
\end{aligned}
$$

Hence, $-2 \leq \Delta \leq \dfrac{4}{3}$.

3 Change in the Right Hand Side

Suppose now that there is a change in the rhs of a constraint. This will affect the values of the basic variables, and hence also the optimal objective value. That is, the rhs column in the Simplex tableau is affected. Hence, the feasibility rather than optimality is affected.[4]
Denote the new rhs by

$$\mathbf{b}' = \mathbf{b} + \Delta\mathbf{b}.$$

The value $\Delta\mathbf{b}$ must keep the corresponding new basic solution feasible, i.e. $\mathbf{x}'_{\mathbf{B}} \geq \mathbf{0}$. Hence, whether a minimisation or maximisation LP is considered, the value $\Delta\mathbf{b}$ must satisfy the inequality

[4]You can notice this from the aforementioned Simplex algebraic formulae.

$$\begin{aligned}
\mathbf{x}'_{\mathbf{B}} &= \mathbf{B}^{-1}\mathbf{b}' \\
&= \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \\
&= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}\Delta\mathbf{b} \geq \mathbf{0},
\end{aligned}$$

that is,

$$\mathbf{B}^{-1}\mathbf{b} \geq -\mathbf{B}^{-1}\Delta\mathbf{b}.$$

Example 3.

Consider the LP in Example 1. Now we aim to find by how much the rhs of the second constraint can be changed without altering the optimal basis $\mathbf{x_B}$.

Let the new rhs be

$$\mathbf{b}' = \mathbf{b} + \Delta\mathbf{b} = \mathbf{b} + (0, \Delta, 0)^T.$$

The value Δ must keep the corresponding new basic solution $\mathbf{x}'_{\mathbf{B}} \geq \mathbf{0}$, i.e. it must satisfy the inequality

$$\begin{aligned}
\mathbf{x}'_{\mathbf{B}} &= \mathbf{B}^{-1}\mathbf{b}' \\
&= \mathbf{B}^{-1}\big(\mathbf{b} + (0, \Delta, 0)^T\big) \\
&= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}(0, \Delta, 0)^T \\
&= (5,\, 3,\, 3)^T + \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta \\ 0 \end{pmatrix} \\
&= (5,\, 3,\, 3)^T + (\tfrac{\Delta}{2},\, 0,\, -\tfrac{\Delta}{2})^T \\
&= (5 + \tfrac{\Delta}{2},\, 3,\, 3 - \tfrac{\Delta}{2})^T \;\geq\; \mathbf{0}
\end{aligned}$$

Hence, $-10 \leq \Delta \leq 6$.

4 Change in a Column of the Constraint Matrix

There is no simple way to perform sensitivity analysis for discrete changes in the coefficients of the constraint equations if the constraint coefficients to be changed are associated with a basic variable. This is because we would need to recalculate $\mathbf{B}^{-1}$. However, a change in the constraint coefficient associated with a nonbasic variable is relatively easy to deal with.

A change in the constraint coefficient associated with a nonbasic variable x_j will affect the row-operated j^{th} column of the Simplex tableau and the corresponding reduced cost. Suppose that the column $\mathbf{A}_j$ is replaced with $\mathbf{A}'_j = \mathbf{A}_j + \Delta\mathbf{A}_j$, and the corresponding variable x_j is nonbasic in the optimal bfs. The optimal basis will remain unchanged if the new reduced cost $\widehat{c}'_j$ is kept nonpositive (or nonnegative) for a minimisation (or maximisation) LP, i.e. the value $\Delta\mathbf{A}_j$ satisfies the inequality

$$\begin{aligned}
\widehat{c}'_j = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}'_j - c_j &= \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}(\mathbf{A}_j + \Delta\mathbf{A}_j) - c_j \\
&= \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_j + \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\Delta\mathbf{A}_j - c_j \\
&= (\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_j - c_j) + \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\Delta\mathbf{A}_j \\
&= \widehat{c}_j + \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\Delta\mathbf{A}_j \ \leq (\text{or } \geq)\ 0,
\end{aligned}$$

that is,

$$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\Delta\mathbf{A}_j \ \leq (\text{or } \geq) - \widehat{c}_j.$$

Example 4.

Consider the LP in Example 1. We want to find by how much the coefficient of x_5 in the third constraint can be changed without altering the optimal basis $\mathbf{x_B}$.

Denote the new column vector corresponding to x_5 in $\mathbf{A}$ by

$$\mathbf{A}_5' = \mathbf{A}_5 + (0, 0, \Delta)^T.$$

Since x_5 is a nonbasic variable in the optimal bfs of the considered minimisation LP, the value Δ must keep the new reduced cost of x_5 nonpositive, i.e.

$$\begin{aligned}
\widehat{c}_5' &= \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{A}_5' - c_5 \\
&= \mathbf{c_B}^T\mathbf{B}^{-1}\big(\mathbf{A}_5 + (0,0,\Delta)^T\big) - c_5 \\
&= (\mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{A}_5 - c_5) + \mathbf{c_B}^T\mathbf{B}^{-1}(0,0,\Delta)^T \\
&= \widehat{c}_5 + (0,-1,-2)(0,0,\Delta)^T \\
&= -2 - 2\Delta \ \leq \ 0.
\end{aligned}$$

Hence, $\quad \Delta \ \geq \ -1.$

Instead of standard form, we will consider for the next two sections an LP in general form:

$$\begin{array}{rrcl}
\min \text{ (or max) } z = & \mathbf{c}^T\mathbf{x} & & \\
s.t. & \mathbf{Ax} & \geq \text{(or} \leq) & \mathbf{b} \\
& \mathbf{x} & \geq & \mathbf{0}.
\end{array}$$

and let $\mathbf{x_B}$ be an optimal basis for its corresponding standard form.

5 Addition of a New Constraint

Example 5.

Consider the following maximisation LP

$$
\begin{array}{rrcl}
\max \ z = & 5x_1 + 3x_2 & & \\
s.t. & x_1 + \ x_2 & \leq & 10 \\
& x_1 \qquad & \leq & 4 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

with the following final optimal tableau, where s_1 and s_2 are slack variables in the first and second constraints, respectively.

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	3	2	38
x_2	0	1	1	-1	6
x_1	1	0	0	1	4

The optimal basis is $\mathbf{x_B} = (x_2, x_1) = (6, 4)$.
Again from the LP in standard form and the final optimal tableau, we can have the following information:

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{c}_\mathbf{B}^T \mathbf{B}^{-1} = (3, 2).$$

Suppose now that a new constraint $x_1 + 3x_2 \leq 15$ is added to give the following new LP.

$$
\begin{aligned}
\max\ z = \quad & 5x_1 + 3x_2 & & \\
s.t. \quad & x_1 + \ x_2 & \leq & \ 10 \\
& x_1 & \leq & \ 4 \\
& x_1 + 3x_2 & \leq & \ 15 \\
& x_1,\, x_2 & \geq & \ 0
\end{aligned}
$$

It is easy to check whether the optimal basis for the original LP is still optimal to the new LP.[5] Since the original optimal basis $\mathbf{x_B} = (x_2, x_1) = (6, 4)$ does not satisfy the additional constraint, the optimal solution to the new LP will change. Then we aim to find an optimal solution for the new LP based on the original $\mathbf{x_B}$ and the related information obtained from the final tableau of the original LP. This may need to apply the dual Simplex method which will be introduced in the following chapter.

After rewriting the additional constraint by adding a slack variable as follows

$$x_1 + 3x_2 + s_3 = 15,$$

we have the new constraint row vector

$$(1,\ 3,\ 0,\ 0,\ 1)$$

with a rhs constant term equal to 15. However, in order to create a Simplex tableau for the new LP by adding this new constraint row to the final tableau of the original LP we need to rearrange it to satisfy the canonical form, i.e. the requirement that the coefficient of the newly introduced basic variable is 1 and the coefficients of all other basic variables are 0s. To obtain the desired new constraint

[5] Obviously, the optimal objective value of the new LP will not be better than that of the original LP.

row, we use the first two constraint rows of the final tableau and perform EROs as we have seen in the Simplex procedure:

$$x_2 + s_1 - s_2 = 6 \quad \Leftrightarrow \quad x_2 = 6 - s_1 + s_2,$$

$$x_1 + s_2 = 4 \quad \Leftrightarrow \quad x_1 = 4 - s_2.$$

Then the new constraint becomes

$$x_1 + 3x_2 + s_3 = (4 - s_2) + 3(6 - s_1 + s_2) + s_3 = 15$$

$$\Leftrightarrow \quad -3s_1 + 2s_2 + s_3 = -7.$$

Now we add this derived new constraint row into the final tableau of the original LP. Notice that there is a "negative rhs" and the dual Simplex method must be used to recover feasibility.[6] Since this basic solution is infeasible, s_3 must leave the basis and s_1 will enter the basis as shown below.

basis	x_1	x_2	s_1	s_2	s_3	rhs	
z	0	0	3	2	0	38	
x_2	0	1	1	-1	0	6	
x_1	1	0	0	1	0	4	
s_3	0	0	$\boxed{-3}$	2	1	-7	
z	0	0	0	4	1	31	$R_0' \leftarrow R_0 + R_3$
x_2	0	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{11}{3}$	$R_1' \leftarrow R_1 - R_3'$
x_1	1	0	0	1	0	4	
s_1	0	0	1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{7}{3}$	$R_3' \leftarrow -\frac{1}{3}R_3$

[6]We skip the detailed dual Simplex procedure at this stage.

Since all the reduced costs are nonnegative, the optimal bfs to the new LP is

$$(x_1, x_2) = (4, \frac{11}{3}) \quad \text{with} \quad z_{\max} = 31.$$

6 Addition of a New Variable

Suppose that added to the LP is a new variable x_j with the objective coefficient c_j and a column in the constraint matrix $\mathbf{A}_j$. If we consider a minimisation (or maximisation) LP and

$$\widehat{c}_j = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_j - c_j \leq (\text{or } \geq)\ 0,$$

then x_j is put in the nonbasis and $\mathbf{x_B}$ remains the optimal basis. Otherwise, we need to enter the new variable x_j into the basis, and the optimal bfs will change.

Example 6.

Consider the LP in Example 5. Suppose now that a new variable x_3 is added to yield the following new LP:

$$\begin{array}{rlcl}
\max\ z = & 5x_1 + 3x_2 + 8x_3 & & \\
s.t. & x_1 + x_2 + x_3 & \leq & 10 \\
& x_1 \qquad + 2x_3 & \leq & 4 \\
& x_1,\ x_2,\ x_3 & \geq & 0
\end{array}$$

Hence, we have $c_3 = 8$ and $\mathbf{A}_3 = (1, 2)^T$.

Since the corresponding reduced cost

$$\widehat{c}_3 = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_3 - c_3 = (3, 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 8 = -1$$

is negative, the current basis $\mathbf{x_B}$ is no longer optimal and x_3 shall be entered into the basis. To create a Simplex tableau for the new LP, the corresponding constraint column $\widehat{\mathbf{A}}_3$ to be added into the original final tableau is

$$\widehat{\mathbf{A}}_3 = \mathbf{B}^{-1}\mathbf{A}_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The Simplex procedure is now continued as follows.

basis	x_1	x_2	x_3	s_1	s_2	rhs	
z	0	0	-1	3	2	38	
x_2	0	1	-1	1	-1	6	
x_1	1	0	$\boxed{2}$	0	1	4	
z	$\frac{1}{2}$	0	0	3	$\frac{5}{2}$	40	$R_0' \leftarrow R_0 + R_2'$
x_2	$\frac{1}{2}$	1	0	1	-$\frac{1}{2}$	8	$R_1' \leftarrow R_1 + R_2'$
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	2	$R_2' \leftarrow \frac{1}{2}R_2$

All the reduced costs are nonnegative, so this is the final optimal tableau. The optimal bfs occurs at

$$\mathbf{x} = (x_1, x_2, x_3)^T = (0, 8, 2)^T \quad \text{with} \quad z_{\max} = 40.$$

7 A Comprehensive Example

Now consider the following LP:

$$
\begin{array}{llrcl}
\max\ z = & 10x_1 + 7x_2 + 6x_3 & & & \\
s.t. & 3x_1 + 2x_2 + \ x_3 & \leq & 36 \\
 & x_1 + \ x_2 + 2x_3 & \leq & 32 \\
 & 2x_1 + \ x_2 + \ x_3 & \leq & 22 \\
 & x_1,\, x_2,\, x_3 & \geq & 0
\end{array}
$$

with the final optimal Simplex tableau for the corresponding LP in standard form:

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

Again, we can obtain the following information from the LP in standard form and the final tableau.

$$\widehat{\mathbf{c}}_{\mathbf{N}}^T = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T = (\widehat{c}_1, \widehat{c}_4, \widehat{c}_6) = (3, 1, 5),$$

$$\mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix}, \quad \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 14 \\ 2 \\ 8 \end{pmatrix},$$

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix},$$

$$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1} = (1, 0, 5).$$

7.1 By how much can the objective coefficient of x_1 be changed without altering the optimal basis?

Note first that x_1 is a nonbasic variable. Suppose that we change its objective coefficient to $c_1' = c_1 + \Delta$. Then the new reduced cost is

$$\begin{aligned} \widehat{c}_1' &= \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_1 - c_1' \\ &= \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_1 - (c_1 + \Delta) \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A}_1 - c_1) - \Delta \\ &= \widehat{c}_1 - \Delta \\ &= 3 - \Delta. \end{aligned}$$

Since the considered problem is a maximisation LP, the current basis $\mathbf{x_B}$ remains optimal in the case of

$$3 - \Delta \geq 0 \quad \Leftrightarrow \quad \Delta \leq 3.$$

7.2 By how much can the objective coefficient of x_2 be changed without altering the optimal basis?

Since x_2 is a basic variable, all the reduced costs will be affected by changing its objective coefficient. So we need to check the following reduced costs[7]:

$$\widehat{\mathbf{c}}_{\mathbf{N}}'^T = (\mathbf{c_B} + \Delta\mathbf{c_B})^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_{\mathbf{N}}^T,$$

where $\Delta\mathbf{c_B} = (\Delta, 0, 0)^T$. This gives us

[7] Note that x_2 is the first component in the basis $\mathbf{x_B} = (x_2, s_2, x_3)$.

$$
\begin{aligned}
\widehat{\mathbf{c}}'^T_{\mathbf{N}} &= \big(\mathbf{c}_{\mathbf{B}}^T + (\Delta, 0, 0)\big)\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T \\
&= (\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T) + (\Delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} \\
&= \widehat{\mathbf{c}}^T_{\mathbf{N}} - (\Delta,\ 0,\ 0)\begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix} \\
&= (3, 1, 5) + (\Delta, \Delta, -\Delta) \\
&= (3 + \Delta, 1 + \Delta, 5 - \Delta).
\end{aligned}
$$

Since the considered problem is a maximisation LP, these reduced costs must be nonnegative to keep the optimal basis, i.e. $3+\Delta \geq 0$, $1 + \Delta \geq 0$ and $5 - \Delta \geq 0$. Hence we have

$$-1 \leq \Delta \leq 5.$$

Note that within this range the optimal basis stays the same, but the optimal objective value would change. In the optimal bfs, we have $x_2 = 14$, which is not affected by increasing its objective coefficient by Δ within the range $-1 \leq \Delta \leq 5$. Thus the optimal objective value will increase by 14Δ.

7.3 By how much can the rhs of the first constraint be changed without altering the optimal basis?

Let the new rhs be

$$\mathbf{b}' = \mathbf{b} + (\Delta, 0, 0)^T$$

The value Δ must keep the corresponding new basic solution $\mathbf{x}'_{\mathbf{B}} \geq \mathbf{0}$, i.e. it must satisfy the inequality

$$
\begin{aligned}
\mathbf{x}'_{\mathbf{B}} &= \mathbf{B}^{-1}\mathbf{b}' \\
&= \mathbf{B}^{-1}\big(\mathbf{b} + (\Delta, 0, 0)^T\big) \\
&= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}(\Delta, 0, 0)^T \\
&= \mathbf{B}^{-1}\mathbf{b} + \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} \\
&= (14, 2, 8)^T + (\Delta, \Delta, -\Delta)^T \\
&= (14 + \Delta,\ 2 + \Delta,\ 8 - \Delta)^T \ \geq \ \mathbf{0}
\end{aligned}
$$

Hence, we have $-2 \leq \Delta \leq 8$.

7.4 By how much can the coefficient of x_1 in the first constraint be changed without altering the optimal basis?

Let the new column vector corresponding to x_1 in $\mathbf{A}$ be

$$\mathbf{A}'_1 = \mathbf{A}_1 + (\Delta, 0, 0)^T.$$

Since x_1 is a nonbasic variable for the maximisation LP, the value Δ must keep the new reduced cost of x_1 nonnegative, i.e.

$$
\begin{aligned}
\widehat{c}'_1 &= \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}'_1 - c_1 \\
&= \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\big(\mathbf{A}_1 + (\Delta, 0, 0)^T\big) - c_1 \\
&= (\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_1 - c_1) + \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}(\Delta, 0, 0)^T \\
&= \widehat{c}_1 + (1, 0, 5)(\Delta, 0, 0)^T \\
&= 3 + \Delta \geq 0.
\end{aligned}
$$

Hence, we have $\Delta \geq -3$.

Further reading: Section 6.1–6.4 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)