35241 OQM

Lecture Note – Part 2

Linear Programming and Simplex Method

1 Introduction to Linear Programming

In the end of Chapter 1, we saw how to solve two-variable LP problems graphically. Unfortunately, most real-life LPs have many variables, so a technique is needed to solve LPs with more than two variables.
Consider an optimisation problem

$$\begin{array}{rrll} \text{min (or max)} & z = f(\mathbf{x}) & & \\ \text{s.t.} & \mathbf{g}(\mathbf{x}) & \geq \text{(or mixed with } \leq, =) & \mathbf{0} \\ & \mathbf{x} & \geq \text{(or mixed with } \leq, \text{ urs)} & \mathbf{0} \end{array}$$

If the objective function $f(\mathbf{x})$ and all of the constraints $\mathbf{g}(\mathbf{x})$ are linear, then the considered optimisation problem is an LP, which can be widely found and used in industry. The formulation shown above is called the general form of an LP. It can have both inequality and equality constraints, and also can have variables that are required to be nonnegative as well as those allowed to be *unrestricted in sign* (urs).

It is possible that there will be a set of constraints that never can be simultaneously satisfied, in which case *no feasible solution* exists to the LP. Hence, there is no optimal solution to the LP.[1]

It is also possible that the feasible set/region is open or unrestricted in some direction. If this happens, an optimal solution may not be yielded to the LP, which is called *unbounded* (possible but not guaranteed).[2] For example, consider the following LP

$$\begin{array}{rrcl} \min z = & -2x_1 - 3x_2 & & \\ \text{s.t.} & x_1 + x_2 & \geq & 3 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

[1]cf. Case 1 at p.10 in Lecture Note Part 1

[2]cf. Case 4 at p.10 in Lecture Note Part 1

This LP is unbounded. Both decision variables x_1 and x_2 can be made as large as we wish, giving an objective function value below any bound (loosely, the objective function value heads to $-\infty$).

However, consider another LP with the same constraints

$$\begin{array}{rrcl} \min z = & 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + x_2 & \geq & 3 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

This LP is not unbounded, and does have an optimal solution. This is because the feasible region is unrestricted in some direction but not in the moving direction of the iso-cost line.

Before introducing another LP format called "standard" form, we first describe two conversion processes.

1.1 Nonnegativity of Decision Variables

Although in most practical situations the decision variables, e.g. the number of cars produced, quantity of raw material used, etc, are usually required to be nonnegative, some urs variables could exist in an LP. In solving LPs with the Simplex method, we will need to perform a value examination called "ratio test", which depends on the condition that any feasible solution requires all variables to be nonnegative. Thus, if some variables are allowed to be urs, the ratio test and therefore the Simplex algorithm are no longer valid.

Here the conversion of urs variables to nonnegative variables is introduced. Any variable not already constrained to be nonnegative (by the nature of the problem) can be converted to the difference of two new nonnegative variables. For example, decision variables x_1 and x_2 in the following LP are urs:

$$\begin{array}{rrcl} \min z = & 25x_1 + 30x_2 & & \\ \text{s.t.} & 4x_1 + 7x_2 & \geq & 1 \\ & 8x_1 + 5x_2 & \geq & 3 \\ & 6x_1 + 9x_2 & \geq & -2 \\ & x_1,\, x_2 & \text{urs} & \end{array}$$

Since a real number can be written as the difference between two nonnegative numbers, we can convert this problem to an equivalent one with nonnegative decision variables by introducing four new nonnegative decision variables p_1, q_1, p_2 and q_2, such that

$$\begin{aligned} x_1 &= p_1 - q_1 \\ x_2 &= p_2 - q_2 \end{aligned}$$

Then we have an equivalent LP as follows:

$$\begin{array}{rrcl} \min z = & 25p_1 - 25q_1 + 30p_2 - 30q_2 & & \\ \text{s.t.} & 4p_1 - 4q_1 + 7p_2 - 7q_2 & \geq & 1 \\ & 8p_1 - 8q_1 + 5p_2 - 5q_2 & \geq & 3 \\ & 6p_1 - 6q_1 + 9p_2 - 9q_2 & \geq & -2 \\ & p_1,\, q_1,\, p_2,\, q_2 & \geq & 0 \end{array}$$

1.2 Slack and Surplus Variables and the Matrix Form for Linear Constraints

In order to use the techniques for the solution of the system of equations, it is necessary to convert the inequality constraints of an LP into equality constraints.

We can convert any inequality constraint into an equality constraint by adding slack or surplus variables as appropriate. These variables are defined to be nonnegative. For example, the constraints

$$\begin{array}{rcl} x_1 - 2x_2 & \leq & 3 \\ x_1,\, x_2 & \geq & 0 \end{array}$$

is equivalent to

$$\begin{array}{rcl} x_1 - 2x_2 + s_1 & = & 3 \\ x_1,\, x_2,\, s_1 & \geq & 0 \end{array}$$

The variable s_1 is referred to as a *slack* variable. It results from the fact that, in order for the smaller left-hand-side (lhs) to equal the right-hand-side (rhs), some nonnegative value must be added to the lhs. Similarly, the constraints

$$\begin{array}{rcl} 2x_1 + x_2 & \geq & 3 \\ x_1,\, x_2 & \geq & 0 \end{array}$$

is equivalent to

$$\begin{array}{rcl} 2x_1 + x_2 - e_1 & = & 3 \\ x_1,\, x_2,\, e_1 & \geq & 0 \end{array}$$

The variable e_1 is referred to as a *surplus* (or *excess*) variable. As the lhs is bigger, some nonnegative value must be subtracted from it to achieve equality.

Hence, the system of constraints in any LP, regardless of the direction of the inequality, can be rearranged in the form:

$$\begin{array}{lcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

or, equivalently

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or even more briefly as $\mathbf{Ax} = \mathbf{b}$.

It is usual that we have more variables than constraints, i.e. $n > m$. Besides, the built constraints normally shall be *consistent* (i.e. a row of the form $[0\ 0 \cdots 0 \mid c]$ with $c \neq 0$ does not exist after applying Gauss-Jordan elimination); otherwise, there exists no feasible solution. In fact, if we have $n > m$ and consistent constraints, then there will be an infinite number of feasible solutions. In other words, the system of linear equations has $n - m$ degrees of freedom. This means that it is possible to find the "best" solution from those feasible ones with regard to the objective function, i.e. an optimal solution. And it makes sense to consider "optimisation" for the problems of this type.

1.3 The Standard Form of an LP

Before the Simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative. An LP in this form is said to be in *standard form.* Any LP in general form can be transformed into an equivalent LP in standard form using the two fore-introduced conversion techniques, as shown below:

$$\begin{array}{rlcl}
\max\ (\text{or}\ \min)\ z = & c_1x_1 + c_2x_2 + \cdots + c_nx_n & & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\
& & \vdots & \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \\
& x_1,\ x_2,\ \cdots,\ x_n\ \geq 0 & &
\end{array}$$

where $b_i \geq 0$ for $i = 1, 2, \cdots, m$.

Converting to the standard form can be done as follows:

- You can choose whether you want to maximise or minimise the objective function, or leave the objective as it is. If the problem is in minimisation (or maximisation) sense, multiply it by -1 to convert the objective to a maximisation (or minimisation) one.
- Convert any inequality to an equality constraint by the addition of slack or surplus variables (as appropriate).
- If any rhs b_i is negative, multiply the whole constraint by -1.
- Any urs x_j can be replaced by two nonnegative variables: $x_j = x'_j - x''_j$.

In matrix format, an LP in standard form can be written as

$$\begin{array}{rlcl}
\max\ (\text{or}\ \min)\ z = & \mathbf{c}^T\mathbf{x} & & \\
\text{s.t.} & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\
& \mathbf{x} & \geq & \mathbf{0}
\end{array}$$

where $\mathbf{x}$ and $\mathbf{c}$ are n-dimensional vectors, $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{b}$ is an m-dimensional vector. Note that $\mathbf{b} \geq \mathbf{0}$.

1.4 Fundamental Law of LP

Convex set: A set $\mathbf{S}$ in n-dimensional space is said *convex* if whenever any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to $\mathbf{S}$ so does every point of the line segment between $\mathbf{x}_1$ and $\mathbf{x}_2$. In other words, a set $\mathbf{S}$ is a convex set if the line segment joining any pair of points in $\mathbf{S}$ is wholly contained in $\mathbf{S}$.

Closed half-space: Given a n-dimensional row vector $\mathbf{a}$ and a constant b, the set of all vectors (i.e. points) $\mathbf{x}$ in n-dimensional space satisfying $\mathbf{ax} \leq b$ is called a *closed half-space*. The set of vectors for which $\mathbf{ax} = b$ is called the boundary of the closed half-space.

Extreme point: Given a convex set $\mathbf{S}$ of n-dimensional vectors, a point $\mathbf{x}^*$ is called an *extreme point* of $\mathbf{S}$ if there do not exist any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbf{S}$ and any value $\alpha \in (0, 1)$, such that

$$\mathbf{x}^* = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2.$$

In other words, for any convex set $\mathbf{S}$, a point $\mathbf{x}^*$ in $\mathbf{S}$ is an extreme point if each line segment that lies completely in $\mathbf{S}$ and contains the point $\mathbf{x}^*$ has $\mathbf{x}^*$ as an endpoint of the line segment.

Lemma 1 *Every closed half-space is a convex set.*

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in a closed half-space consisting of points which satisfy $\mathbf{ax} \leq b$. Let $\mathbf{x}_3$ be any point on the line segment between $\mathbf{x}_1$ and $\mathbf{x}_2$. Then we have $\mathbf{ax}_1 \leq b$, $\mathbf{ax}_2 \leq b$, and $\mathbf{x}_3 = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$ for some $\alpha \in [0, 1]$.
So it can be shown that

$$\mathbf{ax}_3 = \mathbf{a}\big(\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2\big) = \alpha\mathbf{ax}_1 + (1 - \alpha)\mathbf{ax}_2 \leq \alpha b + (1 - \alpha)b = b,$$

which means $\mathbf{x}_3$ also lies in the considered closed half-space.

Lemma 2 *The intersection of any collection of convex sets is convex.*

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points in the intersection. Then $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to each convex set of the collection. Each convex set contains the line segment between $\mathbf{x}_1$ and $\mathbf{x}_2$. Thus, the line segment belongs to each convex set in the collection so that it belongs to the intersection.

Each feasible set of an LP, if it exists, consists of all vectors that simultaneously satisfy a finite number of linear constraints. Each constraint defines a closed half-space. Thus the feasible set is the intersection of a finite number of closed half-spaces, each of which is convex as per Lemma 1. Lemma 2 then gives us the following theorem.

Theorem 1 *The feasible set of an LP is a convex set.*

Since the LP feasible region is a convex set constructed by a finite number of closed half-space boundaries, it is a polyhedron with a finite number of sides (and therefore vertices). Then we have the following theorem.

Theorem 2 *The feasible region for any LP has a finite number of extreme points.*

Theorem 3 *If the feasible set is non-empty and one optimal solution exists to the LP, then there is an optimal solution at one of the extreme points.*

To see this, suppose that $\mathbf{x}^*$ is an optimal solution to an minimisation LP, but not an extreme point. Then, from the definition of extreme point, there must be two other points $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbf{S}$ and a value α, $0 < \alpha < 1$, such that $\mathbf{x}^* = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$.

Since $\mathbf{x}^*$ is an optimal solution, so $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}_1$ and $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}_2$. Hence

$$\begin{aligned}\mathbf{c}^T\mathbf{x}^* &= \mathbf{c}^T(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) = \alpha\mathbf{c}^T\mathbf{x}_1 + (1-\alpha)\mathbf{c}^T\mathbf{x}_2 \\ &\geq \alpha\mathbf{c}^T\mathbf{x}^* + (1-\alpha)\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T\mathbf{x}^*\end{aligned}$$

The only situation that the above derivation can be satisfied is $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}^*$, which contradicts the assumption that $\mathbf{x}^*$ is not an extreme point.

1.5 Basic Feasible Solutions

1.5.1 Definition

Considering an LP in standard form, we have the following system of constraints

$$\begin{aligned}\mathbf{Ax} &= \mathbf{b} \qquad (1)\\ \mathbf{x} &\geq \mathbf{0} \qquad (2)\end{aligned}$$

with n variables and m constraints, where $n > m$. Suppose that the feasible region exists and there are an infinite number of solutions. How can we find a solution?

One intuitive approach is to set $n - m$ components of $\mathbf{x}$ equal to zeros. If the columns in $\mathbf{A}$ corresponding to the remaining m variables are linearly independent, solving for the values of these m variables will yield a unique solution. A solution produced by the unique values for the m variables coupled with the zeros for the other $n - m$ variables is called a *basic solution.*

Assume without loss of generality that we take the first m components[3] and call them *basic variables.* The vector constructed with them is named *basis* and denoted by $\mathbf{x_B}$ ($\mathbf{B}$ is for basic). Then denote the vector of the remaining $n - m$ components, i.e. *nonbasic variables*, by $\mathbf{x_N}$, which is called *nonbasis* ($\mathbf{N}$ is for nonbasic). By setting $\mathbf{x_N} = \mathbf{0}$, we ensure that we have the same number of unknowns as that of equations, and open up the possibility of yielding a unique solution

$$(x_1, x_2, \cdots, x_n) = (\mathbf{x}_\mathbf{B}^T | \mathbf{x}_\mathbf{N}^T).$$

The first m columns, associated with the basic variables, of $\mathbf{A}$ can be labeled $\mathbf{B}$ and the last $n - m$ columns $\mathbf{N}$. Since $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$, Eq. (1) can be written as

$$\mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b}.$$

By setting $\mathbf{x_N} = \mathbf{0}$, we get

$$\mathbf{B}\mathbf{x_B} = \mathbf{b}.$$

If it can be assumed that the columns of $\mathbf{B}$ are linearly independent, then we have a unique solution

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}.$$

Then we say that the variables in $\mathbf{x_B}$ are in the basis, and that $\mathbf{x}^T = (\mathbf{x}_\mathbf{B}^T|\mathbf{0})$ is a basic solution. If Eq. (2) is satisfied, i.e. $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then we call it a *basic feasible solution* (bfs). A bfs exactly represents an extreme point of the considered LP feasible region.

And the objective value of the bfs is

$$\begin{aligned} z &= \mathbf{c}^T\mathbf{x} &&= \mathbf{c}_\mathbf{B}^T\mathbf{x_B} + \mathbf{c}_\mathbf{N}^T\mathbf{x_N} \\ &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}. \end{aligned}$$

Obviously, we could swap the order of the variables (and correspondingly, the columns of matrix $\mathbf{A}$), and choose any m variables to be in the basis.

[3]The numbering in the subscripts of the variables is entirely arbitrary so that basic variables may not always be the "first" m components.

Thus, there can be a huge number of possible choices, so we have many possible bases.

We can rewrite Theorem 3 as:

Theorem 4 *If the feasible set is non-empty and one optimal solution exists to the LP, then there is a basic feasible solution giving the optimal value.*

Consider the following LP for example,

$$\begin{array}{rrcl} \min z = & x_1 + x_2 & & \\ \text{s.t.} & 2x_1 + 5x_2 & \leq & 7 \\ & x_1 + 8x_2 & \leq & 4 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

Its standard form is

$$\begin{array}{rlcl} \min z = & x_1 + x_2 & & \\ \text{s.t.} & 2x_1 + 5x_2 + s_1 & = & 7 \\ & \quad x_1 + 8x_2 \qquad + s_2 & = & 4 \\ & \qquad\quad x_1,\, x_2\ s_1,\, s_2 & \geq & 0 \end{array}$$

A bfs is $(x_1, x_2, s_1, s_2) = (0, 0, 7, 4)$ with $z = 0$.

1.5.2 Degeneracy

It is possible that more than one bfs represents the same extreme point of the feasible set. This is known as *degeneracy*, which would cause the inefficiency of the Simplex method for solving an LP.

Another feature of a degenerate bfs is that some of the elements of $\mathbf{x_B}$ are equal to 0s (along with $\mathbf{x_N} = \mathbf{0}$).

We will look at degeneracy again later on.

2 Introduction to the Simplex Method

2.1 Background

The Simplex Method was devised in 1947 by George Dantzig. It was a remarkably successful technique for solving LPs, and no other solution approach was really considered until the 1980s when interior point methods were devised. This sparked new interest in improving implementations of the Simplex Method and today, on most problems, its modified variant is approximately as quick as the best interior point methods.

However, the Simplex Method does not have good theoretical bounds on its performance. It has been shown that most variations of the Simplex method (in the worst case) can actually search every single extreme point before finding the solution. In practice, though, it seems this sort of problems do not arise.

2.2 Simplex Algorithm Procedure

Recall that the Simplex method uses the optimality criterion (cf. Step 2 below) to perform an efficient search of the extreme points (i.e. bfs) of the feasible region. The method usually starts where all original decision variables are zero. The method "greedily" moves from one extreme (or corner) point of the feasible region to an adjacent one by changing one basic variable at a time. The feasibility criterion (cf. Step 3 below), which is a ratio test, ensures that each basic solution remains feasible (i.e. satisfies all constraints). The method ceases when no further improvement in the value of the objective function can be obtained. The method is outlined below:

Step 1: (Preprocessing – Initial bfs) Convert the LP to standard form. Check whether the system of linear equations is in *canonical form* where each equation has a variable with a coefficient of 1 in that equation and zero coefficients in all other equations. If this is the case and the rhs of each equation in the canonical form is nonnegative, an initial bfs can be obtained by inspection, i.e. a basis which is feasible can be constructed by those variables with a coefficient of 1 in one equation and a zero coefficient in any other equation. Then generate an initial Simplex tableau from the objective and equations written in canonical form. Following the notation in the text, the objective function row, also called the reduced cost row or row 0, is constructed by moving the rhs of the objective to the lhs (so the constant term, usually 0, will be on the rhs).[4]

Step 2: (Entering) For a maximisation LP, if there is no negative number in the row 0 (the objective function row), then STOP – the current bfs is optimal. Otherwise, select a nonbasic variable with the most negative number (which is called reduced cost) in row 0 to be the *entering variable* x_t, which will become basic variable after iteration. For minimisation, if there is no positive entry in row 0, then STOP – the current bfs is optimal. Otherwise, select a nonbasic variable with the largest positive reduced cost as the entering variable.

Step 3: (Leaving – Ratio Test) Let $\widehat{\mathbf{b}} = (\widehat{b}_1, \widehat{b}_2, \cdots, \widehat{b}_m)^T$ be the right-most column, which is called "column rhs", and let $\widehat{\mathbf{A}}_t = (\widehat{a}_{1t}, \widehat{a}_{2t}, \cdots, \widehat{a}_{mt})^T$ be the column vector corresponding to the entering variable x_t. Find an index

$$s = \arg \min_{1 \le i \le m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\}$$

i.e. find the smallest "positive" ratio formed by the divisor of column rhs and the entering-variable column. The basic variable with the smallest positive ratio is the *leaving variable*, i.e. if

$$\min_{1 \le i \le m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\} = \frac{\widehat{b}_s}{\widehat{a}_{st}}$$

then x_s is the leaving variable.
If $\widehat{a}_{it} \le 0$ for all $i = 1, 2, \cdots, m$, then STOP – the problem is unbounded.

Step 4: (Pivoting) Update the tableau by pivoting on a_{st}, i.e. perform EROs on the tableau to get a 1 in the pivot position, and 0s above and below it. Recall that the pivot is the intersection of the entering-variable column

[4] In the representation of the Simplex Tableau in the text, an additional column is added to the tableau. This column has the same entries regardless of the row operations, a 1 followed by 0s. Hence, in most representations this column can be omitted. This will be the case in our lecture notes.

and the leaving-variable row. By doing this, you are solving the system of linear equations with the updated nonbasic variables being set zeros. This tableau then yields the new bfs. The process then returns to **Step 2** to commence the next iteration, if necessary.

Important Point: The pivot element a_{st} will always be a "positive" number since **Step 3** ignores any negative elements in the entering-variable column. All the elements in the rhs will remain nonnegative (At this stage of the subject, notice that if you find any negative rhs entry, it's a sure sign that you've made a mistake somewhere during the Simplex procedure! More complicated situations will be addressed in our later lectures.)

2.3 Example of the Simplex Method in Tabular Form

Consider the following LP:

$$\begin{array}{rrcr} \max z = & x_1 + x_2 & & \\ \text{s.t.} & x_1 + \frac{1}{2}x_2 & \leq & 2 \\ & 3x_1 + 2x_2 & \leq & 12 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

The first step is to convert the general form to its standard form, giving

$$\begin{array}{rrrrcr} \max z = & x_1 + x_2 & +0s_1 & +0s_2 & & \\ \text{s.t.} & x_1 + \frac{1}{2}x_2 & +s_1 & & = & 2 \\ & 3x_1 + 2x_2 & & +s_2 & = & 12 \\ & x_1,\, x_2, & s_1, & s_2 & \geq & 0 \end{array}$$

Since it is in canonical form as well, it is obvious to select the initial basis $\mathbf{x_B} = (s_1, s_2)^T$. Thus our initial Simplex tableau is

basis	x_1	x_2	s_1	s_2	rhs
z	-1	-1	0	0	0
s_1	1	$\frac{1}{2}$	1	0	2
s_2	3	2	0	1	12

Here

- the current basis is listed on left-most side;
- the values of these basic variables at this bfs are in the right-most column, i.e. $(s_1, s_2) = (2, 12)$. Recall that nonbasic variables are zeros, i.e. $(x_1, x_2) = (0, 0)$;
- the 0 in row 0 and the column rhs indicates that the objective value at this bfs is 0;
- row 0 give us the reduced costs of all decision variables;
- the rest of the tableau gives the constraint matrix.

As we are maximising, we look for the most negative reduced cost. In this case, there is a tie (between the reduced cost for x_1 and x_2). When this happens, we can arbitrarily choose one, for example x_1, to be the entering variable. Then we need to find a basic variable to leave the basis. This is done by dividing the numbers in column rhs by the numbers in the entering-variable column. We look for the minimum of $\{\frac{2}{1}, \frac{12}{3}\}$. Thus the first row gives the leaving variable, which is s_1. We now pivot on the element in the entering-variable column and the leaving-variable row, which is marked with a box. The pivot element must become a "1". Since this is already the case, no row operation is necessary on the pivot row.

basis	x_1	x_2	s_1	s_2	rhs
z	-1	-1	0	0	0
s_1	$\boxed{1}$	$\frac{1}{2}$	1	0	2
s_2	3	2	0	1	12

The other two entries in the pivot column, currently "-1" and "3", must become zeros. For row 0, this is done by adding the pivot row to it. For row 2, the entry is modified to zero by adding minus three times the pivot row to it. (These row operations are indicated on the rhs of the tableau in the usual way). These row operations yield the tableau below.

basis	x_1	x_2	s_1	s_2	rhs	
z	0	-$\frac{1}{2}$	1	0	2	$R_0' \leftarrow R_0 + R_1$
x_1	1	$\frac{1}{2}$	1	0	2	
s_2	0	$\frac{1}{2}$	-3	1	6	$R_2' \leftarrow R_2 - 3R_1$

To interpret this tableau,

- the current basis is now $\mathbf{x_B} = (x_1, s_2)^T$;
- the values of these basic variables at this current bfs are on the right, i.e. $(x_1, s_2) = (2, 6)$, and nonbasic variables are zeros, i.e. $(x_2, s_1) = \mathbf{0}$;
- the objective value at this bfs is 2.
- row 0 gives us the new reduced costs.
- The rest of the tableau gives us $\mathbf{B}^{-1}$ times the original constraint matrix $\mathbf{A}$.

Then we start the next iteration with this updated Simplex tableau. The most negative reduced cost now corresponds to x_2, so it is our entering variable. To find a leaving variable, the ratio test $\min\left\{\frac{2}{\frac{1}{2}}, \frac{6}{\frac{1}{2}}\right\}$ decides that the first basic variable, i.e. x_1, leaves. Thus the marked number is our pivot.

basis	x_1	x_2	s_1	s_2	rhs
z	0	-$\frac{1}{2}$	1	0	2
x_1	1	$\boxed{\frac{1}{2}}$	1	0	2
s_2	0	$\frac{1}{2}$	-3	1	6

The EROs yield the tableau below.

basis	x_1	x_2	s_1	s_2	rhs	
z	1	0	2	0	4	$R_0'' \leftarrow R_0' + R_1'$
x_2	2	1	2	0	4	$R_1'' \leftarrow 2R_1'$
s_2	-1	0	-4	1	4	$R_2'' \leftarrow R_2' - R_1'$

Now all the reduced costs are nonnegative, so this is an optimal solution. The solution occurs when $x_2 = 4$, $s_2 = 4$, $x_1 = 0$ and $s_1 = 0$. And the optimal objective value is 4.

Further reading: Section 4.1 – 4.6, and 4.14 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)