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Necessary and Sufficient Graphical Conditions for Affine Formation Control

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***Abstract*—This paper introduces a new multi-agent control problem, called an affine formation control problem, with the objective of asymptotically reaching a configuration that preserves collinearity and ratios of distances with respect to a target configuration. Suppose each agent updates its own state using a weighted sum of its neighbor's relative states with possibly negative weights. Then the affine control problem can be solved for either undirected or directed interaction graphs. It is shown in this paper that an affine formation is stabilizable over an undirected graph if and only if the undirected graph is universally rigid, while an affine formation is stabilizable over a directed graph in the d-dimensional space if and only if the directed graph is $(d+1)$-rooted. Rigorous analysis is provided, mainly relying on Laplacian associated with the interaction graph, which contain both positive and negative weights.**

I. Introduction

This paper introduces a new type of collective pattern in multi-agent systems, called *an affine formation pattern*, and aims to uncover necessary and sufficient conditions so that an affine formation pattern can emerge from simple relative-state based local interactions. An affine formation in this paper refers to a collection of states for a network of agents, which associates with a target configuration via an affine transformation. That is, it preserves collinearity and ratios of distances (i.e., the agents lying on a line initially still lie on a line and maintain the ratios of distances with respect to the target configuration).

The set of affine formations is a superset of the set of rigid formations [1] (that preserve the distances between every pair of agents) and the set of translational formations (that preserve not only the distances between every pair of agents but also the orientation of the formation). The main motivation for the study of affine formations is to provide a new way for robotic formations as well as for cooperative localization in sensor networks. As we will show in this paper, a network of agents in an affine formation can be reshaped to form a rigid formation or a translational formation by controlling only a small number of agents in the network. This is advantageous for robotic formations to have better adaptivity to changing environments. Therefore, controlling a network of agents to achieve an affine formation is not only of its own interest, but also provides a new avenue towards reshapable robotic formations. Also, affine formation control is related to cooperative localization in sensor networks. With the presence of a number of anchor nodes knowing their absolute coordinates in a sensor network, every other sensor node updates its estimate about its own position according to affine formation control rules and it is then able to determine its absolute coordinate. A detailed study for cooperative localization in 2D can be found in [2].

A. Literature review

There has been a tremendous surge of interest among researchers from various disciplines of engineering and science on analysis of collective patterns in multi-agent systems by looking at local interaction rules using relative states. Suppose each agent updates its state according to a weighted sum of its neighbors's relative states. A collective pattern (namely, consensus) occurs provided that the graph (either undirected or directed) modeling the interaction topology has certain connectivity properties [3]–[6]. The notion of a *rooted graph*, meaning there is a node in the graph called root so that every other node is reachable from the root, unifies the necessary and sufficient connectivity properties for consensus in undirected and directed graphs. If a graph has multiple nodes playing the role of leaders, while the others update their states according to their neighbors' relative states, then the resulting collective pattern is either a straight line [5], [7] or a configuration within a convex hull spanned by the leaders [8]–[10].

The aforementioned works assume that the weights to generate the local interaction rules are all positive and real. Only few papers address collective behaviors using negative real weights. [11] shows that the use of negative weights may lead to faster convergence for distributed averaging consensus, while [12] shows that the use of negative weights can lead to a consensus value which is the same for all agents except for the sign. Moreover, the works by [13]–[15] consider negative weights as the inhibitory mechanism to desynchronize the interacting agents in different clusters. More recently, a novel idea using complex weights in the local interaction rules is proposed in [16]–[18], which achieve a collective pattern called a *similar formation* in the plane. That is, the achieved pattern is similar to the target configuration subject to rotations, translations and dilations. A necessary and sufficient graphical condition, called 2-rooted connectivity, is given in [17], [18] for the realizability of similar formations. There have also been similar works which consider control for a network of agents to achieve a formation that is invariant to rotations, permutations, and translations (see for example [19] and the references therein).

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B. Statement of contributions

This paper considers a local interaction law to update each agent's state using a weighted sum of its neighbor's relative states with both positive and negative weights and aims to understand how such a local interaction law leads to an affine formation pattern at the group level. In contrast to the consensus control law, the local interaction law in this paper takes both positive and negative weights to encode the target configuration. That is, the weights are designed to meet certain algebraic constraints related to the target configuration. Moreover, in contrast to the complex-weight based local interaction law in [16]–[18], which limits the state of each agent to the plane, the local interaction law in this paper can deal with pattern formation in any dimensional space, but the emergent pattern does not preserve angles between lines. Compared to our earlier work [20], this paper also extends to the setup with directed interaction graphs.

The main contribution of this paper is fourfold. First, the adoption of both positive and negative weights makes possible to achieve affine formation in any dimensional space. Second, necessary and sufficient graphical conditions are discovered to ensure from the structural viewpoint whether a simple local interaction law exists to steer the agents to an affine formation. On one hand, we show that, for the undirected graph setup, global rigidity is necessary and sufficient to make the affine image of a target configuration the equilibrium subspace. On the other hand, we reveal that global rigidity is not sufficient to guarantee the existence of a local interaction law to steer the agents to an affine formation. A stronger notion of rigidity, called universal rigidity, is necessary and sufficient for stabilizability. For the directed graph setup, we show that a $(d+1)$-rooted graph is necessary and sufficient to make the affine image of a target configuration the equilibrium subspace as well as to ensure stabilizability. Third, the local control law is developed for single-integrator agent models using relative distance measurements. This is then generalized to more general and realistic agent models by also incorporating velocity measurements. Moreover, a distributed control law without the need of centralized computation at the design stage is developed for multi-agent systems allowing communication between neighboring agents. Finally, it is revealed that when a network of agents are in an affine formation, they can be reshaped to form a globally rigid or translational formation by controlling only a small number of agents in the network.

C. Notation

Denote by $\mathbf{1}_n$ the n-dimensional vector of ones and denote by I_n the $n \times n$ identity matrix. Moreover, $\text{span}\{p_1, \dots, p_n\}$ is used to denote the linear span of vectors $p_1, \dots, p_n$ and $\text{diag}(A_1, \dots, A_n)$ represents the block diagonal matrix with each diagonal block A_i.

II. Preliminaries and Problem Formulation

In this section, we introduce preliminary results on graphs and formulate the affine formation control problems.

A. Basic notions and preliminary results for undirected graphs

An *undirected graph* (graph for short) is a set of n nodes $\mathcal{V}$ and m edges $\mathcal{E}$ consisting of unordered pairs of nodes, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A *configuration* in $\mathbb{R}^d$ (or simply called a configuration in this paper) of a set of n nodes $\mathcal{V}$ is defined by their coordinates in the Euclidean space $\mathbb{R}^d$, denoted as $p = [p_1^\intercal, \dots, p_n^\intercal]^\intercal$, where each $p_i \in \mathbb{R}^d$ for $1 \leq i \leq n$. A *framework* in $\mathbb{R}^d$ (or simply called a framework in this paper) in $\mathbb{R}^d$ is a graph $\mathcal{G}$ equipped with a configuration p in $\mathbb{R}^d$, denoted as $\mathcal{F} = (\mathcal{G}, p)$. An example of two frameworks in $\mathbb{R}^2$ is given in Fig. 1. The vertices of the framework are represented by little disks, while the edges are represented by straight lines.

Two frameworks $(\mathcal{G}, p)$ in $\mathbb{R}^{d_1}$ and $(\mathcal{G}, q)$ in $\mathbb{R}^{d_2}$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are said to be *equivalent*, written as $(\mathcal{G}, p) \cong (\mathcal{G}, q)$, if

$$\|p_i - p_j\| = \|q_i - q_j\|, \forall (i,j) \in \mathcal{E}.$$

That is, two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are equivalent as long as the distances are preserved for pairs of nodes with edges. On the other hand, two frameworks $(\mathcal{G}, p)$ in $\mathbb{R}^{d_1}$ and $(\mathcal{G}, q)$ in $\mathbb{R}^{d_2}$ are *congruent*, and we write $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$, if

$$\|p_i - p_j\| = \|q_i - q_j\|, \forall i, j \in \mathcal{V}.$$

That is, two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ in the same dimensional space are congruent if p and q can be obtained from each other by a rigid motion such as a rotation or translation.

A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called *globally rigid* if for any configuration q in $\mathbb{R}^d$, $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ implies $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$. A framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is called *universally rigid* if for any configuration q in $\mathbb{R}^s$ with s being any positive integer, $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ implies $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$. From the definitions of global rigidity and universal rigidity, it immediately follows that universal rigidity implies global rigidity but the converse is not true. As an example, the two frameworks in $\mathbb{R}^2$ in Fig. 1 are globally rigid while the framework in Fig. 1(a) is not universally rigid as there exists a framework in $\mathbb{R}^3$ equivalent to it but not congruent to it. That is, we keep points 1, 2, 3 and 4 in the plane while pushing them along the directions as shown in Fig. 1(a) and pulling node 5 out of the plane. Then a non-congruent configuration in $\mathbb{R}^3$ an be found to satisfy the distance constraints corresponding to the edges in the graph.

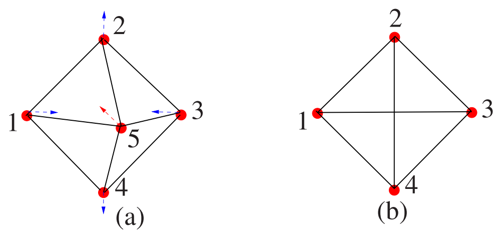

Fig. 1. (a) A framework in $\mathbb{R}^2$ that is globally rigid but not universally rigid; (b) A framework in $\mathbb{R}^2$ that is both globally rigid and universally rigid.

A configuration p in $\mathbb{R}^d$ is said to be *generic* if all the coordinates $p_1, \dots, p_n$ are algebraically independent over the integers [21]. That is, p is generic if there does not exist a nonzero polynomial $f(x_1, \dots x_{nd})$ with integer coefficients such that $f(p_1^1, \dots, p_1^d, \dots, p_n^1, \dots, p_n^d) = 0$ where p_i^j is the jth element of the vector p_i. Intuitively speaking, a generic configuration has no degeneracy, i.e. no three points staying on the same line, no three lines go through the same point, etc. In nongeneric configurations, rigidity properties are more difficult to predict. Standard examples are those where a flexible framework becomes rigid, or where a framework loses global rigidity, when passing from a generic configuration to a nongeneric one. A framework $(\mathcal{G}, p)$ is *generically globally rigid* or *generically universally rigid* if it is globally rigid or universally rigid and p is generic. Generic rigidity is a property of the graph, not the configuration. Thus, with a slight abuse of notion, we also say a graph $\mathcal{G}$ is *globally rigid* or *universally rigid* if the framework $(\mathcal{G}, p)$ is generically globally rigid or generically universally rigid. We shall assume in this paper that the configuration considered is generic. The justification for this assumption is that in any dimensional space, the generic configurations form a set of full measure.

Finally, we introduce two results regarding global rigidity and universal rigidity in terms of the so-called stress matrix. For a configuration p in $\mathbb{R}^d$ of n points, a symmetric matrix L is called a *stress matrix* for the configuration p if it satisfies $L\mathbf{1}_n = 0$ and $(L \otimes I_d)p = 0$ ([22]). The first result below shows that a generic framework is globally rigid if and only if it has a stress matrix with its kernel of dimension $d+1$. The sufficiency is shown in [22]. The necessity is conjectured in [22] and proven in [21].

Theorem 2.1 ([21], [22]): Suppose an undirected graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then the framework $(\mathcal{G}, p)$ is globally rigid if and only if there exists a stress matrix L whose rank is $n-d-1$.

The second result below gives a necessary and sufficient condition for a generic framework to be universally rigid.

Theorem 2.2 ([23]): Suppose an undirected graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then the framework $(\mathcal{G}, p)$ is universally rigid if and only if there exists a stress matrix L that is of rank $n-d-1$ and positive semi-definite.

Remark 2.1: An immediate necessary condition for a d-dimensional framework $(\mathcal{G}, p)$ on $n \geq d+2$ nodes to be globally rigid or universally rigid is that the undirected graph $\mathcal{G}$ should be $(d+1)$-connected, that is, there does not exist a set of d nodes whose removal disconnects the graph. The necessary and sufficient connectivity condition is still not known yet. However, a recent work provides a construction based on nonconvex Grünbaum polygon to obtain generically universally rigid frameworks with the minimum number of edges in two or three dimensions [24].

B. Basic notation and preliminary results for directed graphs

A *directed graph* (digraph for short) is a set of n nodes $\mathcal{V}$ and m edges $\mathcal{E}$ consisting of ordered pairs of nodes, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. When (j, i) is an edge in $\mathcal{G}$, node j is called an *in-neighbor* of node i and node i is called an *out-neighbor* of node j.

Definition 2.1: For a digraph $\mathcal{G}$, a node v is said to be *k-reachable* from a non-singleton set $\mathcal{U}$ of nodes if there exists a path from a node in $\mathcal{U}$ to v after removing any $k-1$ nodes except node v (i.e., there are k disjoint paths from $\mathcal{U}$ to v).

Definition 2.2: A digraph is *k-rooted* if there exists a subset of k nodes called roots, from which every other node is k-reachable.

Definition 2.3: For a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a spanning k-tree of $\mathcal{G}$ rooted at $\mathcal{R} = \{r_1, r_2, \cdots, r_k\} \subset \mathcal{V}$ is a spanning subgraph $\mathcal{T} = (\mathcal{V}, \bar{\mathcal{E}})$ such that

(1) every node $r \in \mathcal{R}$ has no in-neighbor;
(2) every node $v \notin \mathcal{R}$ has k in-neighbors;
(3) every node $v \notin \mathcal{R}$ is k-reachable from $\mathcal{R}$.

Lemma 2.1: A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is k-rooted if and only if $\mathcal{G}$ has a spanning k-tree.

Proof. (Sufficiency) If $\mathcal{G}$ has a spanning k-tree, then by the definition of k-rooted graph, it is certain that $\mathcal{G}$ is k-rooted.

(Necessity) By the definition of a k-rooted graph, we know that there exists a subset of k nodes, called roots, such that every other node is k-reachable from them. Denote by $\mathcal{R} = \{r_1, r_2, \cdots, r_k\}$ the set composed of k roots.

First we remove all incoming edges to nodes in $\mathcal{R}$. By doing so, every node $v \notin \mathcal{R}$ is still k-reachable from $\mathcal{R}$. Second, we remove extra incoming edges for node $v \notin \mathcal{R}$ such that there remain k incoming edges for node v and is still k-reachable from $\mathcal{R}$. It is obvious that the removal of edges on node v does not affect the k-reachability from $\mathcal{R}$ to other nodes that do not have paths from $\mathcal{R}$ via node v. Moreover, for those nodes that have paths from $\mathcal{R}$ via node v, there must be another disjoint path not containing v connecting from $\mathcal{R}$ to u due to the k-reachability property. So the removal of the extra incoming edges on v also does not affect their k-reachability. Therefore, by Definition 2.3, a spanning k-tree is constructed. ■

C. Problem formulation

We consider a group of n agents, whose states are denoted by $z_1, \cdots, z_n \in \mathbb{R}^d$ (for example, mobile robots or unmanned aerial vehicles with their states being the positions in $\mathbb{R}^2$ or $\mathbb{R}^3$). Suppose each agent is governed by a single-integrator dynamics as follows.

$$\dot{z}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $u_i \in \mathbb{R}^d$ represents the control input of each agent. Define the aggregate state $z = [z_1^\intercal, \cdots, z_n^\intercal]^\intercal$, as a column vector in $\mathbb{R}^{nd}$.

Moreover, suppose that each agent i is able to access the relative states $(z_j - z_i)$ of its in-neighbors $j \in \mathcal{N}_i$ where $\mathcal{N}_i$ denotes the set of agent i's in-neighbors. If pairs of agents are mutually neighbors, i.e., $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$, then we use an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to model the information flow structure among the n agents. Otherwise, we use a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to model the information flow structure among the n agents. In the sequel, we use

$$z_{ij} = z_i - z_j, \quad j \in \mathcal{N}_i$$

to denote the relative state available to agent i.

Consider a target configuration $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ in $\mathbb{R}^d$, where each $p_i \in \mathbb{R}^d$ for $1 \leq i \leq n$. We denote the *affine image* of p as

$$\mathcal{A}(p) := \left\{ q = [q_1^\intercal, \ldots, q_n^\intercal]^\intercal \;\middle|\; \begin{array}{l} q_i = Ap_i + a, \ A \in \mathbb{R}^{d \times d} \\ a \in \mathbb{R}^d, \ \text{and } i = 1, \ldots, n \end{array} \right\}$$

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or equivalently,

$$\mathcal{A}(p) := \left\{ q = (I_n \otimes A)p + \mathbf{1}_n \otimes a \,\middle|\, A \in \mathbb{R}^{d\times d},\ a \in \mathbb{R}^d \right\}.$$

Notice that any real matrix A can be factorized by singular value decomposition as $A = U\Sigma V$ where U and V are unitary matrices, and Σ is a $d \times d$ diagonal matrix. This means that a configuration in $\mathcal{A}(p)$ is attained via an affine motion from p, namely, a rotation V, a scaling along different axes by Σ, and then another rotation U, followed by a translation a. So indeed, $\mathcal{A}(p)$ represents a set of patterns for the n agents in the state space. We say that the n agents, $z_1, z_2, \ldots, z_n$, form an *affine formation* if the state z belongs to $\mathcal{A}(p)$.

In this paper, we study the *affine formation control problem* with the objective of steering a group of agents to the affine image of a target configuration and uncovering necessary and sufficient topological conditions for the feasibility of affine formation control using the local interaction law

$$u_i = -\sum_{j\in\mathcal{N}_i} k_{ij} z_{ij},\ i = 1, \cdots, n. \tag{2}$$

where k_{ij} is a non-zero scalar weight on edge (j, i) of the graph $\mathcal{G}$ that models the information flow structure among the n agents. If $\mathcal{G}$ is an undirected graph, then we assume $k_{ij} = k_{ji}$. If $\mathcal{G}$ is a directed graph, then k_{ij} is not necessary to be equal to k_{ji}. Note that k_{ij} may be negative or positive, so it is different from the consensus control protocol [6].

With the local interaction law (2), the goal is then to find the necessary and sufficient graphical conditions for both undirected graphs and directed graphs and find control weights k_{ij}'s such that the trajectories of the closed-loop system satisfies

$$\lim_{t\to\infty} z(t) = z^* \text{ where } z^* \in \mathcal{A}(p).$$

Remark 2.2: If the matrix A in the definition of $\mathcal{A}(p)$ is an unitary matrix, the affine image $\mathcal{A}(p)$ is called a *rotation/translation image*, denoted as $\mathcal{R}(p)$. In other words, if $z \in \mathcal{R}(p)$, the agents form a configuration congruent to the target configuration. In particular, if the matrix A is an identity matrix, the affine image $\mathcal{A}(p)$ is called a *translation image*, denoted as $\mathcal{T}(p)$. That is, $z \in \mathcal{T}(p)$ implies the agents form a configuration that is congruent to the target configuration and has the same orientation as it.

As shown in [25], a generic framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ with $d + 1$ or fewer nodes is globally rigid if and only if $\mathcal{G}$ is a complete graph, a trivial case. Therefore, in this paper we assume that our graph has $d+2$ or more nodes, i.e., $n \geq d+2$. This also implies that a generic framework does not lie in any proper affine subspace of $\mathbb{R}^d$.

III. Affine formation over undirected networks

This section considers the undirected graph case, for which we will present necessary and sufficient graphical conditions for realizability as well as stabilizability of affine formation.

A. Realizability of affine formation

As the first step towards the goal of asymptotically achieving an affine formation, we explore a necessary and sufficient condition, under which the equilibrium set of the closed-loop system under the local interaction law (2) over an undirected graph $\mathcal{G}$ is the affine image $\mathcal{A}(p)$ for a generic target configuration p.

Under the local interaction law (2) over an undirected graph $\mathcal{G}$, the closed-loop system is of the following form:

$$\dot{z} = -(L \otimes I_d)z \tag{3}$$

where $L \in \mathbb{R}^{n\times n}$ is the matrix whose (i, j)th off-diagonal element is 0 if (j, i) is not an edge of $\mathcal{G}$ and the weight k_{ij} otherwise, and whose diagonal entry is the negative row sum of off-diagonal entries in the same row. In this paper, the Laplacian L associated with the undirected graph $\mathcal{G}$ may contain both positive and negative off-diagonal entries, which is different from the classic Laplacian matrix used in consensus study [6]. Since this section assumes that $\mathcal{G}$ is undirected, L is symmetric.

For a target configuration p, an affine formation of p is said to be *realizable* over the undirected graph $\mathcal{G}$ if there exists a symmetric matrix L associated with $\mathcal{G}$ such that the equilibrium set of system (3) equals to $\mathcal{A}(p)$ (i.e., $(L \otimes I_d)z = 0$ if and only if $z \in \mathcal{A}(p)$).

Next we present our main result. But before that, we give a preliminary result regarding the affine image $\mathcal{A}(p)$.

Lemma 3.1: Consider $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ with every $p_i \in \mathbb{R}^d$. If $\text{span}\{p_1, \ldots, p_n\} = \mathbb{R}^d$, then $\mathcal{A}(p)$ is a linear subspace of dimension $d^2 + d$.

The proof of Lemma 3.1 is given in the appendix.

Theorem 3.1: Suppose an undirected graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then an affine formation of p is realizable over $\mathcal{G}$ if and only if $\mathcal{G}$ is globally rigid.

Proof: (Sufficiency) From Theorem 2.1 we know that if the undirected graph $\mathcal{G}$ is globally rigid, then there exists a stress matrix L whose rank is $n - d - 1$. As a result, for a generic configuration p in $\mathbb{R}^d$, the null space of $L \otimes I_d$ is of dimension $(d + 1)d$. On the other hand, since $(L \otimes I_d)p = 0$, it turns out that for any $A \in \mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$,

$$\begin{aligned}(L \otimes I_d)[(I_n \otimes A)p + \mathbf{1}_n \otimes a] &= (L \otimes A)p \\ &= (I_n \otimes A)(L \otimes I_d)p = 0,\end{aligned}$$

which means the affine image $\mathcal{A}(p)$ is a subset of the equilibrium set. Moreover, from Lemma 3.1 we know that $\mathcal{A}(p)$ is a linear subspace of dimension $(d + 1)d$, which equals to the dimension of null space of $L \otimes I_d$. Therefore, it is certain that the equilibrium set of system (3) equals to $\mathcal{A}(p)$, that means an affine formation of p is realizable over $\mathcal{G}$.

(Necessity) If an affine formation of p is realizable over $\mathcal{G}$, then there exists a symmetric matrix L associated with $\mathcal{G}$ such that the equilibrium set of system (3) equals to $\mathcal{A}(p)$. Thus, it can be inferred that $(L \otimes I_d)p = 0$ due to $p \in \mathcal{A}(p)$. Moreover, since the dimension of $\mathcal{A}(p)$ is $d^2 + d$ by Lemma 3.1, it then follows that $\text{rank}(L) = n - d - 1$. Therefore, by Theorem 2.1, it is concluded that $\mathcal{G}$ is globally rigid. ∎

B. Stabilizability of affine formation

The preceding subsection shows that in order to make an affine formation an equilibrium of the system under the local

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interaction law (2), the undirected graph $\mathcal{G}$ needs to be globally rigid. The result in this subsection will show that global rigidity is not sufficient to ensure the existence of L making the system asymptotically stable, while universal rigidity is the necessary and sufficient condition.

For a target configuration p, an affine formation of p is said to be *stabilizable* over the undirected graph $\mathcal{G}$ if there exists a symmetric matrix L associated with $\mathcal{G}$ such that the state of the closed-loop system (3) converges to a point in $\mathcal{A}(p)$. The theorem below provides a necessary and sufficient condition for stabilizability of an affine formation.

Theorem 3.2: Suppose an undirected graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then an affine formation of p is stabilizable over $\mathcal{G}$ if and only if $\mathcal{G}$ is universally rigid.

Proof: (Sufficiency) If $\mathcal{G}$ is universally rigid, then by Theorem 2.2, for a generic configuration p, there exists a stress matrix L (satisfying $L\mathbf{1}_n = 0$ and $(L \otimes I_d)p = 0$) that is of rank $n-d-1$ and positive semi-definite. Thus, we construct such a matrix L for system (3), for which the eigenvalues of L are all positive other than $d+1$ zero eigenvalues with $d+1$ linearly independent associated eigenvectors. Then it follows from the properties of Kronecker product that system (3) is asymptotically stable, namely, converging to $\mathcal{A}(p)$.

(Necessity) If an affine formation of p is stabilizable over $\mathcal{G}$, then there exists a symmetric matrix L associated with $\mathcal{G}$ such that the equilibrium set of system (3) equals to $\mathcal{A}(p)$ and the state of system (3) converges to a point in $\mathcal{A}(p)$. This implies that other than $d+1$ zero eigenvalues, the eigenvalues of L are positive. Therefore, it is positive semi-definite. Thus, by Theorem 2.2 it follows that $\mathcal{G}$ is universally rigid. ∎

C. Design of control weights for stabilization

Next, we come to design k_{ij}'s when the undirected graph $\mathcal{G}$ is universally rigid, that is, solving the affine formation control problem over undirected graphs with the objective to steer a group of agents to the affine image $\mathcal{A}(p)$.

Denote q_1 the n-dimensional vector by aggregating the first components of $p_1, \ldots, p_n$. Similarly, we denote $q_2, \ldots, q_d$ the corresponding aggregate vectors. Since for a generic configuration p, $\mathbf{1}_n, q_1, \ldots, q_d$ are linearly independent, we can find an $(n-d-1) \times n$ matrix Q with orthonormal rows that are each orthogonal to $\mathbf{1}_n, q_1, \ldots, q_d$; that is

$$Q\mathbf{1}_n = 0,\ Qq_1 = 0,\ \ldots,\ Qq_d = 0,\ QQ^\intercal = I_{n-d-1}.$$

Then we are ready to present a stability criteria that will be used in the control weight design.

Theorem 3.3: Suppose $L\mathbf{1}_n = 0$ and $(L \otimes I_d)p = 0$. Then an affine formation of p is stabilizable over $\mathcal{G}$ if and only if $\lambda_{min}(QLQ^\intercal) > 0$ where $\lambda_{min}(\cdot)$ represents the smallest eigenvalue of a symmetric matrix.

Proof: When $L\mathbf{1}_n = 0$ and $(L \otimes I_d)p = 0$, then $\lambda_{min}(QLQ^\intercal) > 0$ is equivalent to the fact that the eigenvalues of L are all positive except the ones at the origin. Thus the conclusion follows. ∎

Based on Theorem 3.3, we formulate the control weight design problem as an optimization problem in the following.

We suppose the undirected graph $\mathcal{G}$ has m edges, with labels $1, \ldots, m$. We arbitrarily assign an orientation for each edge. The choice of orientation does not change the analysis. The *incidence matrix* $B \in \mathbb{R}^{n\times m}$ is defined as

$$B_{il} = \begin{cases} 1 & \text{if edge } l \text{ starts from node } i, \\ -1 & \text{if edge } l \text{ ends at node } i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each edge l of the undirected graph is associated with a single weight $w_l = k_{ij} = k_{ji}$, where edge l is incident to nodes i and j. We let $w \in \mathbb{R}^m$ denote the vector of weights on the m edges. Using this notation, the matrix L can be written as

$$L(w) = B\text{diag}(w)B^\intercal$$

where $\text{diag}(w)$ stands for the $m \times m$ diagonal matrix with the lth diagonal entry w_l. Thus, the control design problem turns out to be the design problem of the weight vector w subject to certain equality or inequality constraints, to meet the stability requirement of the multi-agent formation. That is,

$$\begin{array}{ll} \underset{w}{\textbf{maximize}} & \lambda \\ \textbf{subject to} & 0 < \lambda \leq \bar{\lambda}, \\ & QL(w)Q^\intercal \succ \lambda I_{n-d-1}, \\ & L(w)q_i = 0,\ i = 1, \ldots, d \end{array}$$

where $A \succ B$ refers that $A - B$ is positive definite and $\bar{\lambda}$ can be any positive constant to make the optimization problem have a bounded solution. From Theorem 3.2, we know that if $\mathcal{G}$ is universally rigid, the optimization always has a solution. The constraints in the above optimization problem are represented by linear matrix inequalities. The optimization is a semi-definite programming that can be efficiently and globally solved by using a polynomial-time interior point method [26].

IV. Affine formation over directed networks

This section considers the directed graph case, for which we will present necessary and sufficient graphical conditions for realizability as well as stabilizability of affine formation.

A. Realizability of affine formation

Similar to the undirected graph case, we firstly aim to explore a necessary and sufficient condition, under which the equilibrium set of the closed-loop system under the local interaction law (2) over a directed graph $\mathcal{G}$ is exactly the affine image $\mathcal{A}(p)$ of a target configuration p.

Under the local interaction law (2), the closed-loop system is of the following form:

$$\dot{z} = -(L \otimes I_d)z \tag{4}$$

where $L \in \mathbb{R}^{n\times n}$ is the matrix whose (i,j)th off-diagonal element is 0 if (j,i) is not an edge of $\mathcal{G}$ and k_{ij} otherwise, and whose diagonal entry is the negative row sum of off-diagonal entries in the same row. We call L the *Laplacian* associated with a directed graph $\mathcal{G}$.

For a target configuration p, an affine formation of p is said to be *realizable* over a directed graph $\mathcal{G}$ if there exists a Laplacian L associated with $\mathcal{G}$ such that the equilibrium set

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of system (4) equals to $\mathcal{A}(p)$ (i.e., $(L \otimes I_d)z = 0$ if and only if $z \in \mathcal{A}(p)$).

Next we present our main result. Before that, we give a result that will be used in the proof of our main result.

Lemma 4.1: For a generic Laplacian L of a digraph $\mathcal{G}$, if $\mathcal{G}$ is k-rooted with the root set $\mathcal{R} = \{r_1, \cdots, r_k\}$, then

(a) all the principal minors of $L_{\mathcal{R}}$ are distinct from zero, where $L_{\mathcal{R}}$ is the sub-matrix of L with the rows and columns corresponding to nodes in $\mathcal{R}$ crossed out;

(b) $\det(M) \neq 0$ where M is a sub-matrix of L by deleting the k rows corresponding to the k roots and any k columns.

The proof of Lemma 4.1 is given in the appendix.

Theorem 4.1: Suppose a directed graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then an affine formation of p is realizable over $\mathcal{G}$ if and only if $\mathcal{G}$ is $(d+1)$-rooted.

Proof: (Sufficiency) If $\mathcal{G}$ is $(d+1)$-rooted, we show in the following that there exists a Laplacian L satisfying $(L \otimes I_d)p = 0$ and $\text{rank}(L) = n - d - 1$.

Since $\mathcal{G}$ is $(d+1)$-rooted, by Lemma 2.3 we know that $\mathcal{G}$ has a spanning $(d+1)$-tree. Denote by $\mathcal{T}$ the spanning $(d+1)$-tree. Let T be a generic Laplacian associated with $\mathcal{T}$. It is known that the rows of T corresponding to the $d+1$ roots are all zero vectors. Moreover, by Lemma 4.1, $\text{rank}(T) \geq n - d - 1$. So the kernel of T is a $(d+1)$-dimensional subspace, for which one basis is $\mathbf{1}_n$. Denote $\eta_0 = \mathbf{1}_n$ and the other linearly independent bases as $\eta_1, \cdots, \eta_d$.

Next we show that for the tall matrix $[\eta_0, \eta_1, \ldots, \eta_d]$, by removing any $n-d-1$ rows, the remaining square matrix is of full rank. To see this, suppose by contradiction that it is not. That is, by row switching, $[\eta_0, \eta_1, \ldots, \eta_d]$ transforms to the form $\begin{bmatrix} M \\ N \end{bmatrix}$ where $M \in \mathbb{R}^{(d+1)\times(d+1)}$ is not of full rank. In other words, there is a nonzero vector ξ such that $M\xi = 0$. By corresponding column switching for T followed by removing the rows in T corresponding to the $d+1$ roots, denote the resulting transformed sub-matrix as $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$ where T_1 is $(n-d-1)$-by-$(d+1)$ and T_2 is $(n-d-1)$-by-$(n-d-1)$. Then we have

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = 0. \tag{5}$$

Plugging $M\xi = 0$ into eq. (5) leads to $T_2 N\xi = 0$. On the other hand, from Lemma 4.1, we know that for a generic Laplacian T, T_2 is of full rank. Therefore, $N\xi = 0$, which together with $M\xi = 0$ imply $\eta_0, \eta_1, \ldots, \eta_d$ are not linearly independent, a contradiction.

Since $\mathcal{T}$ is a $(d+1)$-tree, each non-root node has exactly $(d+1)$ in-neighbors, which implies the corresponding row of T has at most $(d+2)$ nonzero entries. Denote by μ the sub-vector of a row of T corresponding to a non-root node, which aggregates only the $(d+2)$ nonzero entries. Since we just showed that for the tall matrix $[\eta_0, \eta_1, \ldots, \eta_d]$, by removing any $n-d-1$ rows the remaining $(d+1)$-by-$(d+1)$ square matrix is of full rank, it means μ lies in a one-dimensional subspace. So for a generic p and a Laplacian T' associated with $\mathcal{T}$ and satisfying $(T' \otimes I_d)p = 0$, the corresponding μ' of T' is in a subspace of at least one dimension. Therefore, T' has the same zero/nonzero pattern as T and $\text{rank}(T') = \text{rank}(T) = n - d - 1$ in a generic sense.

For a $(d+1)$-rooted graph $\mathcal{G}$ and a Laplacian L satisfying $(L \otimes I_d)p = 0$, T' can be considered as a Laplacian of $\mathcal{G}$ for a special choice of weights with edges not in $\mathcal{T}$ being 0. Thus, by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that $\text{rank}(L) = n - d - 1$, too.

On the other hand, since $(L \otimes I_d)p = 0$, it turns out that for any $A \in \mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$,

$$\begin{aligned} (L \otimes I_d)[(I_n \otimes A)p + \mathbf{1}_n \otimes a] &= (L \otimes A)p \\ &= (I_n \otimes A)(L \otimes I_d)p \\ &= 0, \end{aligned}$$

which means the affine image $\mathcal{A}(p)$ is a subset of the equilibrium set. Moreover, from Lemma 3.1 we know that $\mathcal{A}(p)$ is a linear subspace of dimension $(d+1)d$, which equals to the dimension of null space of $L \otimes I_d$. Therefore, it is certain that the equilibrium subspace of system (4) equals to $\mathcal{A}(p)$, that means an affine formation of p is realizable over $\mathcal{G}$.

(Necessity) If an affine formation of p is realizable over a directed graph $\mathcal{G}$, then there exists a Laplacian L associated with $\mathcal{G}$ such that the equilibrium set of system (4) equals to $\mathcal{A}(p)$. Thus, it can be inferred that $(L \otimes I_d)p = 0$ due to $p \in \mathcal{A}(p)$. Moreover, since the dimension of $\mathcal{A}(p)$ is $d^2 + d$ as shown in Lemma 3.1, it then follows that $\text{rank}(L) = n - d - 1$. Thus, there exist $d+1$ rows of L, which can be transformed to zero vectors by elementary row operations. Denote by $\mathcal{R}$ the set of nodes corresponding to the indices of these $d+1$ rows.

Now suppose by contradiction that $\mathcal{G}$ is not $(d+1)$-rooted. Then there exists a node $i \notin \mathcal{R}$ such that after deleting d nodes, without loss of generality say $\{1, 2, \ldots, d\}$, i is not reachable from $\mathcal{R}$. Let $\mathcal{U}$ be the set of nodes not in $\mathcal{R}$ (including node i) such that all nodes in $\mathcal{U}$ are not reachable from $\mathcal{R}$ after removing $\{1, 2, \ldots, d\}$. Let

$$\bar{\mathcal{U}} = \mathcal{V} - \mathcal{U} - \{1, 2, \ldots, d\}.$$

Then it is clear that there is no edge from any node in $\bar{\mathcal{U}}$ to any node in $\mathcal{U}$. So by relabeling the nodes in $\mathcal{U}$ and $\bar{\mathcal{U}}$ in a consecutive manner respectively, the matrix L transforms to the following form by a permutation matrix P, i.e.,

$$PLP^\intercal = L' := \left[\begin{array}{c|c|c} L_{11} & L_{12} & L_{13} \\ \hline L_{21} & L_{22} & 0 \\ \hline L_{31} & L_{32} & L_{33} \end{array}\right].$$

where the rows and columns in L_{11} correspond to nodes $1, 2, \ldots, d$, the rows and columns in L_{22} correspond to the nodes in $\mathcal{U}$, and the rows and columns in L_{33} correspond to the nodes in $\bar{\mathcal{U}}$. Thus, $(L \otimes I_d)p = 0$ is equivalent to

$$(L' \otimes I_d)(P \otimes I_d)p = 0,$$

from which we have

$$\left(\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix} \otimes I_d\right)(P \otimes I_d)p = 0.$$

Moreover, we have $\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix}\mathbf{1} = 0$. Therefore,

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$\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix}$ is not of full row rank, which together with the fact that $d+1$ rows corresponding nodes not in $\mathcal{U}$ can be transformed to zero vectors by elementary row operations, imply rank$(L) \leq n-d-2$. It contradicts to the above conclusion rank$(L) = n-d-1$. Therefore, $\mathcal{G}$ is $(d+1)$-rooted. ■

B. Stabilizability of affine formation

This subsection will show that a $(d+1)$-rooted graph is also necessary and sufficient to ensure the existence of a Laplacian L making the system asymptotically stable.

For a target configuration p, an affine formation of p is said to be *stabilizable* over a directed graph $\mathcal{G}$ if there exists a Laplacian L associated with $\mathcal{G}$ such that the state of the closed-loop system (4) converges to a point in $\mathcal{A}(p)$. The theorem below provides a necessary and sufficient condition for stabilizability of an affine formation.

Theorem 4.2: Suppose a directed graph $\mathcal{G}$ has n nodes with $n \geq d+2$ and $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. Then an affine formation of p is stabilizable over $\mathcal{G}$ if and only if $\mathcal{G}$ is $(d+1)$-rooted.

The proof requires the following result related to the multiplicative inverse eigenvalue problem (MIEP) [27] over the real field.

Lemma 4.2 ([27]): Let A be an $n \times n$ real matrix with all of its leading principal minors being nonzero. Then there is an $n \times n$ diagonal matrix D such that all the roots of DA are positive and simple.

Proof of Theorem 4.2. (Sufficiency) If $\mathcal{G}$ is $(d+1)$-rooted, then by Lemma 4.1 it follows that for a generic Laplacian L, there is a permutation operation (multiplying with a permutation matrix P, equivalent to relabeling the nodes) such that PLP^T has the form

$$L' = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

with $B_1 \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$ and B_2, B_3 and B_4 of appropriate dimension, and all principal minors of B_1 are nonzero. The property also holds for L satisfying $(L \otimes I_d)p = 0$ for a generic p, which can be shown by the same argument as to show rank$(L) = n-d-1$ in Theorem 4.1. Thus, by Lemma 4.2, there exists a diagonal matrix $D_1 = \text{diag}(d_1, \cdots, d_{n-d-1})$ such that all the eigenvalues of D_1B_1 are in the right half plane.

Denote $D_2 = \text{diag}(d_{n-d}, \cdots, d_n)$ and $D = \text{diag}(D_1, D_2)$. Moreover, define $M(D_1, D_2) = DL'$. Then it is clear that

$$M(D_1, 0) = \begin{bmatrix} D_1B_1 & D_1B_2 \\ 0 & 0 \end{bmatrix}.$$

So the eigenvalues of $M(D_1, 0)$ consist of $d+1$ zero eigenvalues and the eigenvalues of D_1B_1. Then by the continuity property of eigenvalues, for sufficiently small entries in D_2, $M(D_1, D_2)$ also has eigenvalues in the right half plane except the fixed $d+1$ zero eigenvalues. This means, $P^\intercal DPL$ can be used to replace the original L and stabilize the closed-loop system to an affine formation, where $P^\intercal DPL$ is another Laplacian associated with $\mathcal{G}$ because $P^\intercal DP$ is a diagonal matrix just scaling each row of L. Thus, the conclusion follows.

(Necessity) If an affine formation is stabilizable over $\mathcal{G}$, then it can be inferred that an affine formation must be realizable over $\mathcal{G}$. So by Theorem 4.1, $\mathcal{G}$ is $(d+1)$-rooted. ■

Remark 4.1: From Remark 2.1, we know that global rigidity or universal rigidity of an undirected graph implies $(d+1)$-connectedness, which further implies that the undirected graph is $(d+1)$-rooted according to its definition. The reason that the undirected graph case requires a stronger graphical condition (global rigidity for realizability and universal rigidity for stabilizability) is that a single weight is considered on each edge in the undirected graph case while two different weights are allowed to be associated with the edge with different orientations when a directed graph becomes bidirectional, equivalent to an undirected graph in terms of topology.

C. Design of control weights for stabilization

In this subsection, we come to solve the affine formation control problem over directed graphs with the objective to steer a group of agents to the affine image $\mathcal{A}(p)$, namely, to find proper control weights k_{ij}'s such that $L\mathbf{1}_n = 0$, $(L \otimes I_d)p = 0$, rank$(L) = n-d-1$ and the nonzero eigenvalues of L all have positive real parts. Theorem 4.2 shows that such L exists if $\mathcal{G}$ is $(d+1)$-rooted. From the proof of Theorem 4.2, finding proper weights k_{ij} for this purpose can be decomposed into two steps.

First, find weights k'_{ij} so that $L'\mathbf{1}_n = 0$, $(L' \otimes I_d)p = 0$, and rank$(L') = n-d-1$, where L' is the Laplacian associated to $\mathcal{G}$ with weights k'_{ij}. This step can be done in a distributed manner. For a given target configuration $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$, each agent i is supposed to know p_i and p_j for $j \in \mathcal{N}_i$ as otherwise the problem is not solvable. Then agent i computes k'_{ij}'s according to the following formula

$$\sum_{j \in \mathcal{N}_i} k'_{ij}(p_j - p_i) = 0. \tag{6}$$

Since $\mathcal{G}$ is $(d+1)$-rooted, by the definition of a rooted graph, it is known that each node has at least $(d+1)$ in-neighbors, which means that (6) must have a solution and the solution space is of at least one dimension. Eq. (6) is a linear equation and can be solved easily. For each agent i, picking any one solution to (6) gives a choice of k'_{ij} for $j \in \mathcal{N}_i$, which certainly ensures $L'\mathbf{1}_n = 0$ and $(L' \otimes I_d)p = 0$. Moreover, from the proof of Theorem 4.1, we know that for arbitrarily picking k'_{ij}'s as described above, rank$(L') = n-d-1$ holds in the generic sense since $\mathcal{G}$ is $(d+1)$-rooted.

Secondly, design a diagonal matrix D such that the nonzero eigenvalues of DL' all have positive real parts. The existence of D is assured by Theorem 4.2. Usually, this step requires a centralized computation.

V. Extension to other agent models

Suppose the motion agent i is governed by a second-order dynamic model, i.e.,

$$M_i\ddot{z}_i = f_i(\dot{z}_i) + \tau_i \tag{7}$$

where $M_i \in \mathbb{R}^{d\times d}$ represents the inertia matrix, $f_i(\dot{z}_i)$ is the damping term, which does not depend on the absolute position of the agent, and $\tau_i \in \mathbb{R}^d$ is the actuation force.

As we will show in the following, the local control law (2) designed for the single-integrator case and its corresponding results serve as the starting point for formation control with more complex and realistic agent dynamics. Define u_i to be an auxiliary variable for each agent (7):

$$u_i = -\sum_{j\in\mathcal{N}_i} k_{ij} z_{ij},\ i = 1,\cdots,n, \tag{8}$$

which is exactly the same as (2). Consistent with the back-stepping philosophy, we adopt an objective of designing τ_i to ensure that

$$\dot{z}_i \to u_i,\ \text{as } t\to\infty.$$

By virtue of the definition of u_i in (8), it can be seen that if it were replaced by $\dot{z}_i$, we would have the same closed-loop system formula for z_i as the single-integrator case and the desired formation could be achieved with $\dot{z}_i$ going to zero for each i. It is then intuitively reasonable that if $\dot{z}_i$ exponentially converges to u_i, we should still achieve the desired formation.

Now we come to design τ_i for this objective. Define an error

$$e_i = \dot{z}_i - u_i. \tag{9}$$

Towards the goal of making e_i converge to zero exponentially, we shall design τ_i such that the error obeys

$$\dot{e}_i = -K_i e_i$$

for some positive definite K_i. By considering (7) and (9), it can be obtained that

$$\begin{aligned}\dot{e}_i &= \ddot{z}_i - \dot{u}_i\\ &= M_i^{-1} f_i(\dot{z}_i) + M_i^{-1}\tau_i + \textstyle\sum_{j\in\mathcal{N}_i} k_{ij}\dot{z}_{ij}.\end{aligned}$$

Thus, we can choose

$$\tau_i = M_i\left[-K_i e_i - \sum_{j\in\mathcal{N}_i} k_{ij}\dot{z}_{ij}\right] - f_i(\dot{z}_i), \tag{10}$$

which makes $\dot{e}_i = -K_i e_i$. The convergence to a desired formation is stated in the following theorem.

Theorem 5.1: Suppose $\mathcal{G}$ is universally rigid the undirected graph case or $(d+1)$-rooted for the directed graph case. If each agent i can access its own velocity $\dot{z}_i$ and the relative velocity of its in-neighbors, i.e., $\dot{z}_{ij} = \dot{z}_i - \dot{z}_j$, then the local control law (10), with k_{ij}'s in (8) designed exactly the same as for the single-integrator case, solves the affine formation control problem for the agents modelled in (7).

Proof: Consider the multi-agent system with each agent modelled as (7) and the local control law (10). It can be obtained that

$$\begin{cases}\dot{e}_i = -K_i e_i,\\ \dot{z}_i = -\sum_{j\in\mathcal{N}_i} k_{ij} z_{ij} + e_i,\end{cases}$$

or in the matrix form

$$\begin{cases}\dot{e} = -Ke,\\ \dot{z} = -(L\otimes I_d)z + e,\end{cases}$$

where $e = [e_1^\intercal,\dots,e_n^\intercal]^\intercal$, $z = [z_1^\intercal,\dots,z_n^\intercal]^\intercal$, and $K = \operatorname{diag}(K_1,\dots,K_n)$. Observe from this matrix form that the closed-loop system is a cascade linear system with e converging to zero exponentially. Therefore, with a stable H as designed in Section III and Section IV for the undirected graph and directed graph case respectively, z converge to an affine formation of p. ■

The control law (10) indicates that the relative velocity of in-neighbor j has to be available to agent i as stated in Theorem 5.1. Therefore, in the scenario when the relative velocity information is not accessible by agent i, the control law (10) is not implementable. In the following, we will develop an alternative control law to solve the affine formation control problem without using the relative velocity information but each agent's own velocity information. The latter can be measured by onboard tachometers. The idea is based on building an approximate of u_i and seeking to have $\dot{z}_i$ approach the approximate rather than u_i itself. Then the requirement of knowing the relative velocity falls away.

Consider an approximate $\hat{u}_i$ governed by the following dynamics

$$\dot{\hat{u}}_i = -\gamma\hat{u}_i + u_i, \tag{11}$$

where $\gamma > 0$ is a parameter to be designed. Next, we define an error

$$\eta_i = \dot{z}_i - \hat{u}_i.$$

Then considering the agent dynamics (7), we obtain that

$$\begin{aligned}\dot{\eta}_i &= \ddot{z}_i - \dot{\hat{u}}_i\\ &= M_i^{-1} f_i(\dot{z}_i) + M_i^{-1}\tau_i + \gamma\hat{u}_i - u_i.\end{aligned}$$

Thus, we can choose

$$\tau_i = M_i\left[-K_i\eta_i - \gamma\hat{u}_i + u_i\right] - f_i(\dot{z}_i), \tag{12}$$

for some positive definite K_i, which makes

$$\dot{\eta}_i = -K_i\eta_i.$$

The following result states the convergence property.

Theorem 5.2: Suppose $\mathcal{G}$ is universally rigid for the undirected graph case or $(d+1)$-rooted for the directed graph case. If each agent i can access its own velocity $\dot{z}_i$, then the local control law (11)-(12), with k_{ij}'s in (8) designed exactly the same as for the single-integrator case and γ satisfying

$$\gamma^2 > \frac{\mathrm{Im}^2(\lambda_i)}{\mathrm{Re}(\lambda_i)},$$

where λ_i are the non-zero eigenvalues of L, solves the affine formation control problem for the agents modelled in (7).

Proof: Consider the multi-agent system with each agent modelled as (7) and the local control law (11)-(12). It can be obtained that

$$\begin{cases}\dot{\eta}_i = -K_i\eta_i,\\ \dot{z}_i = \hat{u}_i + \eta_i,\\ \dot{\hat{u}}_i = -\gamma\hat{u}_i - \sum_{j\in\mathcal{N}_i} k_{ij} z_{ij}\end{cases}$$

or in the matrix form

$$\begin{cases}\dot{\eta} = -K\eta,\\ \dot{z} = \hat{u} + \eta,\\ \dot{\hat{u}} = -\gamma\hat{u} - (L\otimes I_d)z,\end{cases}$$

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where $\eta = [\eta_1^\intercal, \ldots, \eta_n^\intercal]^\intercal$, $z = [z_1^\intercal, \ldots, z_n^\intercal]^\intercal$, $\hat{u} = [\hat{u}_1^\intercal, \ldots, \hat{u}_n^\intercal]^\intercal$, and $K = \text{diag}(K_1, \ldots, K_n)$. Notice that this closed-loop system can be considered as a cascade linear system of subsystem

$$\dot{\eta} = -K\eta$$

and subsystem

$$\begin{bmatrix} \dot{z} \\ \dot{\hat{u}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -(L \otimes I_d) & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ \hat{u} \end{bmatrix} + \begin{bmatrix} \eta \\ 0 \end{bmatrix},$$

for which η is known to be convergent to zero exponentially. Thus, it remains to show that the eigenvalues of the matrix

$$\begin{bmatrix} 0 & I \\ -(L \otimes I_d) & -\gamma I \end{bmatrix}$$

are all in the open left complex plane except $d(d+1)$ fixed eigenvalues at 0. According to the condition given in the proof of Theorem 3.6 in [16], the above conclusion holds if

$$\gamma^2 > \frac{\text{Im}^2(\lambda_i)}{\text{Re}(\lambda_i)}$$

where λ_i are the non-zero eigenvalues of L. ∎

Remark 5.1: The formula given in Theorem 5.2 provides a lower bound for a valid γ. To compute this lower bound, it needs to know the entire graph Laplacian.

Remark 5.2: Based on the local control law for the single-integrator case, extensions can also be made to a class of more general nonlinear agent models with the same technique developed in this section. Suppose each agent i has a model of the form

$$\begin{aligned} \dot{\zeta}_i &= f_i(\zeta_i) + G_i(\zeta_i)u_i \\ z_i &= h_i(\zeta_i) \end{aligned}$$

where $z_i \in \mathbb{R}^d$ is the position, $u_i \in \mathbb{R}^d$ is the control input, and $\zeta_i \in \mathbb{R}^{p_i}$ for some $p_i \geq d$. Further suppose that each agent model has vector relative degree $[m_i, \ldots, m_i]$ with respect to z_i (see e.g. [28]) and can be transformed into the normal form

$$\left\{ \begin{aligned} \dot{\xi}_i^1 &= \xi_i^2 \\ \dot{\xi}_i^2 &= \xi_i^3 \\ &\vdots \\ \dot{\xi}_i^{m_i} &= g_i(\xi_i, \eta_i) + B_i(\xi_i, \eta_i)u_i \\ \dot{\eta}_i &= q(\eta_i, \xi_i) \\ z_i &= \xi_i \end{aligned} \right. \tag{13}$$

by a diffeomorphism $[\xi_i^\intercal, \eta_i^\intercal]^\intercal = T_i(\zeta_i)$, where $\xi_i = [\xi_i^1, \ldots, \xi_i^{m_i}]^\intercal$. The same feedback linearization technique considered in this section can be applied iteratively to find a feasible local control law solving the affine formation control problem as long as the zero dynamics $\dot{\eta}_i = q(\eta_i, \xi_i)$ holds certain property such as input-to-state stability with respect to ξ_i. The model (7) is a simple example of (13) where $m_i = 2$ and the dimension of η_i is 0. However, feedback linearization is based on exact mathematical cancellation of the nonlinear terms g_i and B_i. For practical reasons such as model simplification, parameter uncertainty, and computation errors, accurate models may not be available, for which robust state feedback stabilization techniques such as sliding mode control, Lyapunov redesign, and high-gain feedback can be applied (see [29] for details).

Remark 5.3: In the same way, extensions can be made to multi-agent systems with the Lagrange agent model, expressed as

$$M_i(z_i)\ddot{z}_i + C_i(z_i, \dot{z}_i)\dot{z}_i + g_i(z_i) = Y_i(z_i, \dot{z}_i, \ddot{z}_i)\theta_i = \tau_i$$

where $z_i, \dot{z}_i, \ddot{z}_i$ are vectors of $\mathbb{R}^d$. $M_i(z_i)$ is the $d \times d$ positive definite inertia matrix. $C_i(z_i, \dot{z}_i)\dot{z}_i$ represents the Coriolis and centripetal forces. $g_i(z_i)$ is the gravity vector and τ_i is the applied motor forces. The parametrization $\tau_i = Y_i\theta_i$ simply means the system parameter vector θ_i enters linearly in the system. If the model contains uncertainty (i.e., the system parameter vector θ_i is unknown), then an adaptive control law (see for example [30]) can be adopted to combine with the formation control law extended from the one for the single-integrator case as done in this section.

VI. Extension to multi-agent networks allowing communication

In Sections III, IV and V, the distributed control laws for affine formation control do not require communication but only onboard sensing for their implementation. The control parameters in such distributed control laws need a centralized computation. However, if local communication between neighboring agents is allowed, then an alternative control law for affine formation control can be developed such that both its design and implementation can be done in a distributed manner without relying on centralized computation and the knowledge of the entire graph. This issue is addressed in this section.

As the undirected graph case is a special one of the directed graph case and the more complex dynamic agent models can be tackled based on the results for the single-integrator case as done in Section V, we will restrict our focus to the case with directed sensing graphs and single-integrator agent models in this section.

Consider a directed sensing graph $\mathcal{G}$ characterizing that agent i can measure the relative position of agent j when (j, i) is an edge of $\mathcal{G}$. We assume in this section that agent i and j can communicate to each other if either (j, i) or (i, j) is an edge of the sensing graph $\mathcal{G}$. This assumption is practically reasonable as usually communication is bidirectional and the communication range is larger than the sensing range.

Consider a target configuration $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$. Suppose each agent i knows its own p_i in the target configuration. Then by communication with its neighbors, agent i can know p_j, $j \in \mathcal{N}_i$, as well. Thus, agent i is able to select its own control parameters $k_{ij}, j \in \mathcal{N}_i$, to satisfy

$$\sum_{j \in \mathcal{N}_i} k_{ij}(p_j - p_i) = 0,$$

which can be carried out without a centralized computation. As discussed in Subsection IV-C, such a selection process makes the Laplacian L associated to the directed sensing graph $\mathcal{G}$ with weights k_{ij} have rank $n - d - 1$ and satisfy $(L \otimes I_d)p = 0$ if $\mathcal{G}$ is $(d+1)$-rooted.

The following control law is then proposed to solve the affine formation control problem.

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$$\begin{cases} \dot{z}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}\zeta_i + \sum_{i\in\mathcal{N}_j} k_{ji}\zeta_j, \\ \dot{\zeta}_i = -a\zeta_i + \sum_{j\in\mathcal{N}_i} k_{ij}z_{ij}, \end{cases} \tag{14}$$

where $\zeta_i \in \mathbb{R}^d$ is an auxiliary state and $a > 0$ is any constant parameter.

Remark 6.1: The control law (14) requires the following relative position information by agent i:

- $(z_j - z_i)$ of its in-neighbors in the sensing graph $\mathcal{G}$,

and requires the following information via communication:

- the auxiliary information $k_{ji}\zeta_j$ from its out-neighbors in $\mathcal{G}$.

By our assumption that communication is bidirectional, the whole piece of information $k_{ji}\zeta_j$ that is known by agent j can be sent to agent i. In other words, the control law (14) is locally implementable in a distributed manner by allowing communication between neighboring agents.

Denote $z = [z_1, \ldots, z_n]^\intercal$ and $\zeta = [\zeta_1, \ldots, \zeta_n]^\intercal$. The closed-loop system under the control law (14), can be described as

$$\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & -L^\intercal \otimes I_d \\ L \otimes I_d & -aI \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix}. \tag{15}$$

The following result shows the convergence to an affine formation of p by using the control law (14).

Theorem 6.1: Suppose $\mathcal{G}$ is $(d+1)$-rooted. Then the local control law (14) solves the affine formation control problem.

Proof. Denote

$$A = \begin{bmatrix} 0 & -L^\intercal \\ L & -aI_n \end{bmatrix}.$$

It can be observed that the system matrix in (15) has d duplicate copies of the eigenvalues of A. Thus, to show that the local control law (14) solves the affine formation control problem, it remains to show that the eigenvalues of A are in the open left complex plane except $d+1$ zero eigenvalues.

Let λ be an eigenvalue of A and let $\begin{bmatrix} \omega \\ \varpi \end{bmatrix}$ be its associated eigenvector. Then we have

$$\left(\lambda I_{2n} - \begin{bmatrix} 0 & -L^\intercal \\ L & -aI_n \end{bmatrix}\right) \begin{bmatrix} \omega \\ \varpi \end{bmatrix} = 0.$$

By several steps of mathematical manipulation, it can then be obtained that $-L^\intercal L\omega = \lambda(\lambda + a)\omega$, which means, $\lambda(\lambda + a)$ is an eigenvalue of $-L^\intercal L$ with ω being its associated eigenvector. In other words, let σ_i be an eigenvalue of the matrix $L^\intercal L$. Then the roots of the polynomial equation

$$\lambda^2 + a\lambda + \sigma_i = 0, \; i = 1, \ldots, n \tag{16}$$

are the eigenvalues of A. Note that the roots of the polynomial equation (16) have the following explicit formula.

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4\sigma_i}}{2}, \; i = 1, \ldots, n. \tag{17}$$

The matrix $L^\intercal L$ is positive semi-definite with its rank the same as L (namely, $n-d-1$). So it has $d+1$ zero eigenvalues and all other eigenvalues are positive and real. Thus, it follows from (17) and the condition $a > 0$ that the eigenvalues of A lies in the open left complex plane except $d+1$ zero eigenvalues. ■

VII. Extra conditions for rigid formation

In this section, we show how a group of agents in an affine formation transforms to a globally rigid formation or a translational formation with a few extra constraints. Necessary and sufficient conditions are provided below.

Theorem 7.1: Suppose $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. A configuration $z \in \mathcal{A}(p)$ is congruent to p (i.e., $z \in \mathcal{R}(p)$) if and only if there exist at least $d+1$ agents, labelled $m_1, \ldots, m_{d+1}$, such that the dimension of the convex hull of $z_{m_1}, \ldots, z_{m_{d+1}}$ is d and

$$\|z_{m_i} - z_{m_j}\| = \|p_{m_i} - p_{m_j}\| \text{ for any } i, j. \tag{18}$$

Proof: (Sufficiency) For $z \in \mathcal{A}(p)$, there exist $A \in \mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$ such that $z_i = Ap_i + a$ for all i. Moreover, the condition (18) implies that there are at least $d+1$ agents in a globally rigid formation, from which we can infer that A is a unitary matrix, i.e.,

$$A^\intercal A = I.$$

As a result, for any i and j,

$$\|z_i - z_j\|^2 = (p_i - p_j)^\intercal A^\intercal A(p_i - p_j) = \|p_i - p_j\|^2, \tag{19}$$

meaning that the distance between any pair of nodes is preserved, i.e., z is congruent to p.

(Necessity) If z is congruent to p, then eq. (19) holds for any i and j. Thus, the conclusion follows. ■

Theorem 7.2: Suppose $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ is a generic configuration in $\mathbb{R}^d$. A configuration $z \in \mathcal{A}(p)$ is a translation of p (i.e., $z \in \mathcal{T}(p)$) if and only if there exist at least d pairs of agents such that the dimension of the convex hull of d pairs of agents is d and

$$z_k - z_j = p_k - p_j. \tag{20}$$

Proof: (Sufficiency) For $z \in \mathcal{A}(p)$, there exist $A \in \mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$ such that $z_i = Ap_i + a$ for all i. Moreover, if there exist d pairs of agents such that eq. (20) holds, then the following holds for these d pairs of agents,

$$p_k - p_j = z_k - z_j = A(p_k - p_j)$$

and

$$(I - A)(p_k - p_j) = 0.$$

By the condition that the dimension of the convex hull of d pairs of agents position is d, we know that the linear span of $(p_k - p_j)$ of d pairs, equals to $\mathbb{R}^d$. Thus,

$$(I - A)(p_k - p_j) = 0 \text{ for all } j \in \mathcal{N}_k$$

for these d pairs implies

$$A = I.$$

As a result, for any i and j, we have

$$z_i - z_j = p_i - p_j \tag{21}$$

meaning that z is a translation of p.

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(Necessity) If z is a translation of p, then eq. (21) holds for any i and j and thus the conclusion follows. ∎

Remark 7.1: From Theorem 7.1, we know that if we choose a network of agents to be in the affine image $\mathcal{A}(p)$ and make $d+1$ agents maintain as a globally rigid formation, then a globally rigid formation of the entire network is achieved. Thus, it is flexible to reshape the formation by controlling only a small number of agents in the group for better adaptivity to a possibly changing environment.

Remark 7.2: Similarly, in addition to make the agents converge to the affine image $\mathcal{A}(p)$, if we can control d pairs of agents to attain their desired relative positions in a common reference frame, then the whole group of agents can achieve a globally rigid formation subject to only translations. This can also be interpreted from the dimension of the affine image $\mathcal{A}(p)$ and the number of constraints. In a generic sense, the preservation of relative positions for d edges in $\mathbb{R}^d$ results in d^2 linearly independent constraints, which reduces the (d^2+d)-dimensional equilibrium subspace $\mathcal{A}(p)$ to a subspace of dimension d, corresponding to the translation motions.

VIII. Simulations

In this section, we present several simulations to illustrate our results in $\mathbb{R}^3$.

Consider a system consisting of 27 agents. Its interaction topology is modelled by an undirected graph with 27 nodes and 128 edges. The target configuration p is shown in Fig. 2. The goal is to reach an affine formation of the target configuration by distributed control laws. By solving the optimization problem described in Section III-C, a positive semi-definite $L \in \mathbb{R}^{27\times 27}$ of rank 23 is obtained.

First, we consider the single integrator case. From Theorem 3.3, we know that an affine formation of p is achievable under the control law (3) with the obtained L above. Fig. 3 shows the simulated trajectories starting from a random initial condition, from which it is observed that the 27 agents exponentially converge to form an affine formation of p in $\mathbb{R}^3$.

Second, we consider the double integrator case. We carry out two simulation for the agents with the double integrator model (7) using two different control laws (namely, (10) and (12) respectively). Without loss of generality, we assume that M_i is the identity matrix and $f_i(\dot{z}_i) = -\dot{z}_i$. To demonstrate the effectiveness of the generalized control laws for double integrator agents, we use the same interaction graph, the same target configuration, the same positive semi-definite L, and the same initial condition as for the single integrator case. For both control laws,

$$K_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is used, while additionally $\gamma_i = 5$ is used in (12) to satisfy the condition in Theorem 5.2. With these control parameters, the simulated trajectories by executing the control law (10) and (12) are plotted in Fig. 4 and Fig. 5. From the simulation results, we can see that the 27 agents exponentially converge to form desired affine formations in $\mathbb{R}^3$, which validate Theorem 5.1 and Theorem 5.2.

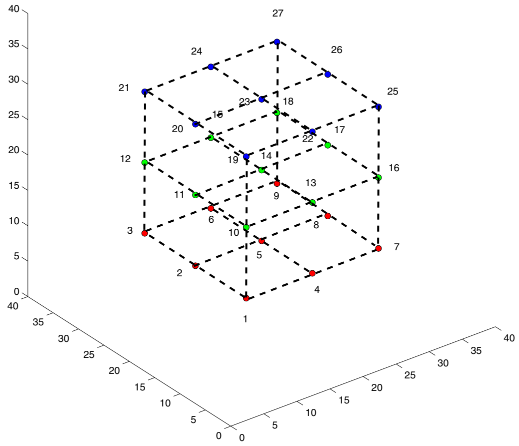

Fig. 2. The target configuration p for the simulation example in $\mathbb{R}^3$. The dashed straight lines connecting the points are used only to outline the geometric shape and do not have any meaning. There are totally 128 edges in the interaction graph, which are not drawn in the figure.

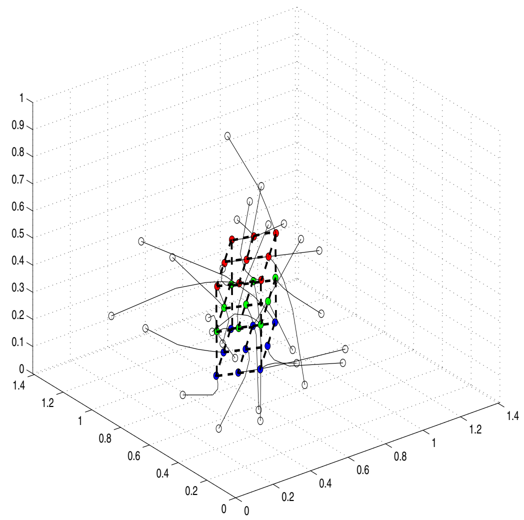

Fig. 3. The simulated trajectories of 27 agents governed by (3) in achieving an affine formation. The hollow circles indicate the initial states of the agents while the colored dots represent the states the agents converge to. The dashed straight lines connecting the final states are plotted only for the purpose of outlining the formation shape.

IX. Conclusions

This paper generalizes the well-known consensus control protocol by allowing both positive and negative weights to analyze a new emergent collective pattern. The connection of the generic rank property of Laplacian with both positive

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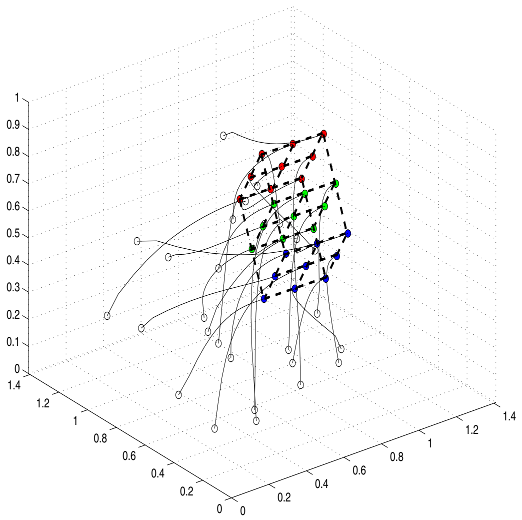

Fig. 4. The simulated trajectories of 27 agents governed by (7) and (10) in achieving an affine formation.

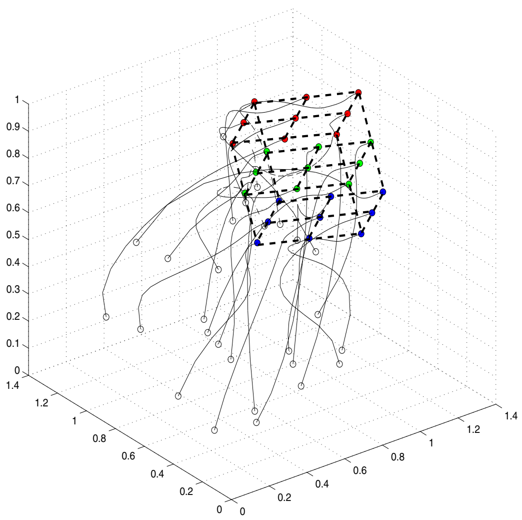

Fig. 5. The simulated trajectories of 27 agents governed by (7) and (12) in achieving an affine formation.

and negative weights and the connectivity of a graph is established, with which necessary and sufficient conditions are obtained for a multi-agent system to ensure the emergence of collective pattern by local interactions for both undirected and directed networks. It is then shown how extension of the local interaction law for the single-integrator case can be made to deal with more general and realistic dynamic agent models. Moreover, with the allowance of communication between neighboring agents, an alternative control law is developed to solve the formation control problem, which is distributed in both design stage and implementation stage. Finally, we show how a network of agents in an affine formation can be reshaped to form a globally rigid formation or translational formation with a few extra constraints on a small number of agents in the network. Along the line, many interesting problems can be explored, such as rigid formation control of multi-agent systems and localization of sensor networks in high dimensional spaces, the coordination of heterogeneous dynamic systems, etc.

Appendix

Proof of Lemma 3.1. First of all, it can be verified that $\mathcal{A}(p)$ is closed under linear combinations. So it is a linear subspace. Moreover, notice that

$$\{\mathbf{1}_n \otimes a : a \in \mathbb{R}^d\}$$

is a linear subspace of dimension d. Hence, it remains to show that

$$\left\{(I_n \otimes A)p : A \in \mathbb{R}^{d\times d}\right\}$$

is a linear subspace of dimension d^2. Denote $E_{ij} \in \mathbb{R}^{d\times d}$ ($i = 1,\ldots,d$ and $j = 1,\ldots,d$) the matrix with only the (i,j)th entry being 1 and others being 0. Then it is clear that $\left\{(I_n \otimes A)p : A \in \mathbb{R}^{d\times d}\right\}$ is the linear span of vectors

$$(I_n \otimes E_{11})p,\ \ldots,\ (I_n \otimes E_{dd})p.$$

Thus, in order to show that $\left\{(I_n \otimes A)p : A \in \mathbb{R}^{d\times d}\right\}$ is a linear subspace of dimension d^2, we just need to show the vectors

$$(I_n \otimes E_{11})p,\ \ldots,\ (I_n \otimes E_{dd})p$$

are linearly independent. To see this, we let

$$\alpha_{11}(I_n \otimes E_{11})p + \cdots + \alpha_{dd}(I_n \otimes E_{dd})p = 0,$$

from which we get

$$\begin{aligned}(\alpha_{11}E_{11} + \cdots + \alpha_{dd}E_{dd})p_1 &= 0,\\ &\vdots\\ (\alpha_{11}E_{11} + \cdots + \alpha_{dd}E_{dd})p_n &= 0.\end{aligned}$$

Since $\mathrm{span}\{p_1,\ldots,p_n\} = \mathbb{R}^d$, it follows that

$$\alpha_{11}E_{11} + \cdots + \alpha_{dd}E_{dd} = 0$$

in order to make the above inequalities hold. Recall that $E_{11},\ldots,E_{dd}$ are linearly independent. Therefore, $\alpha_{11} = \cdots = \alpha_{dd} = 0$, which implies

$$(I_n \otimes E_{11})p,\ \ldots,\ (I_n \otimes E_{dd})p$$

are linearly independent. ■

Proof of Lemma 4.1. We prove this lemma by induction.

Step 1: Consider $k = 1$.

Part (a): By Lemma 2.1, a rooted graph $\mathcal{G}$ has a spanning tree, denoted as $\mathcal{T}$, with a root set $\mathcal{R}_1 = \{i\}$ without loss of generality. For a generic Laplacian L' of $\mathcal{T}$, $\det(L'_{\mathcal{R}_1}) \neq 0$. Adding any one other node in $\mathcal{R}_1$ to make a new set $\mathcal{R}_2$, all the remaining nodes are reachable from $\mathcal{R}_2$ then. So $\det(L'_{\mathcal{R}_2}) \neq 0$ for a generic L'. Repeating this argument, it can be concluded that all principal minors of $L'_{\mathcal{R}_1}$ are distinct from zero for a generic L' associated with $\mathcal{T}$. For a rooted graph $\mathcal{G}$ and a generic Laplacian L associated with $\mathcal{G}$, L has possibly more

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nonzero entries compared with L'. So by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that all principal minors of L are distinct from zero.

Part (b): Next we show that for a generic L' corresponding to $\mathcal{T}$, $\det(M') \neq 0$ where M' is the sub-matrix of L' by deleting the row corresponding to the root i and the column corresponding to any one node, say $j \in \mathcal{V} - \mathcal{R}_1$. Denote l'_i and l'_j the row vectors of L' corresponding to node i and j respectively. Clearly, $l'_i = 0$ since i is the root of $\mathcal{T}$. We take the following elementary row transformation,

$$L' = \begin{bmatrix} \vdots \\ l'_i \\ \vdots \\ l'_j \\ \vdots \end{bmatrix} \Rightarrow \bar{L}' = \begin{bmatrix} \vdots \\ l'_i + l'_j \\ \vdots \\ l'_j \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ l'_j \\ \vdots \\ l'_j \\ \vdots \end{bmatrix}.$$

Denote by $\bar{\mathcal{T}}$ the graph associated with $\bar{L}'$. It is certain that node i is also a root of $\bar{\mathcal{T}}$. Moreover, notice that $\bar{L}'(i,j) = L'(j,j) \neq 0$ (where $\bar{L}'(i,j)$ and $L'(j,j)$ stand for the corresponding entry of $\bar{L}'$ and L'). So there is an edge from j to i, which means j is also a root of $\bar{\mathcal{T}}$. Thus, according to (1a), $\det(\bar{L}'_{\{j\}}) \neq 0$. Notice that M' can be obtained from $\bar{L}'_{\{j\}}$ via elementary row transformations, so $\det(M') \neq 0$, too. By the same argument as given in the end of (1a), then it is known that the conclusion also holds for a generic Laplacian L associated with a rooted graph $\mathcal{G}$.

Step 2: Suppose $k = d$ and for a generic Laplacian L associated with a d-rooted graph $\mathcal{G}$ with the root set $\mathcal{R} = \{1, \cdots, d\}$, all principal minors of $L_{\mathcal{R}}$ are nonzero and $\det(M) \neq 0$. Next we show that the statement also holds for $k = d + 1$.

Part (a): Without loss of generality, consider a $(d+1)$-rooted graph $\mathcal{G}$ with the root set $\mathcal{R} = \{1, \cdots, d+1\}$ and let L be a generic Laplacian of $\mathcal{G}$. By removing an arbitrary node in $\mathcal{R}$, say $d+1$ without loss of generality, and its incident edges, we denote the resulting sub-graph by $\mathcal{G}_1$. Clearly, $\mathcal{G}_1$ is d-rooted with the root set $\mathcal{R}_1 = \{1, \cdots, d\}$. Denote by L_1 the corresponding Laplacian of $\mathcal{G}_1$. Then it is known by assumption that all principal minors of $(L_1)_{\mathcal{R}_1}$ are nonzero, and $\det(M_1) \neq 0$ where M_1 is the sub-matrix of L_1 by deleting the d rows corresponding to nodes in $\mathcal{R}_1$ and any d columns. This implies, all principal minors of $L_{\mathcal{R}}$ are nonzero and $\det(M) \neq 0$ for M being the sub-matrix of L by deleting the $(d+1)$ rows corresponding to nodes $\mathcal{R}$ and $(d+1)$ columns corresponding to at least one node in $\mathcal{R}$ and others not in $\mathcal{R}$.

Part (b): Then it remains to show that $\det(M) \neq 0$ for M being the sub-matrix of L by deleting the $(d+1)$ rows corresponding to nodes in $\mathcal{R}$ and any $(d+1)$ columns corresponding to nodes all not in $\mathcal{R}$. Consider any node not in $\mathcal{R}$, say j. Denote by l_i the row vector of L corresponding to node i. We make the following elementary row transformation for L, i.e.,

$$L = \begin{bmatrix} l_1 \\ \vdots \\ l_{d+1} \\ \vdots \\ l_j \\ \vdots \end{bmatrix} \Rightarrow \bar{L} = \begin{bmatrix} \sum_{i=1}^n k_i l_i \\ \vdots \\ l_{d+1} \\ \vdots \\ l_j \\ \vdots \end{bmatrix}$$

Since $\mathcal{G}$ is $(d+1)$-rooted, every root in $\mathcal{R}$ has at least one out-going edge. Thus, with a proper choice of k_i's, $\bar{L}(1,1), \cdots, \bar{L}(1,d+1)$ and $\bar{L}(1,j)$ can all become nonzero. Denote by $\bar{\mathcal{G}}$ the graph associated with $\bar{L}$. It is certain that $\bar{\mathcal{G}}$ is also $(d+1)$-rooted with a root set $\mathcal{R}$. Moreover, there are edges from nodes $2, \ldots, d+1$ and node j to node 1. So node 1 is $(d+1)$-reachable from the set $\bar{\mathcal{R}} = \{2, \cdots, d+1, j\}$. Now we consider any node v in $\mathcal{V} - \{\mathcal{R}, j\}$ and show that v is $(d+1)$-reachable from $\bar{\mathcal{R}}$, too. Certainly, if $d+1$ disjoint paths from $\mathcal{R}$ to v does not pass through node j, then there exist $d+1$ disjoint paths from $\bar{\mathcal{R}}$ to v with one path goes from node j to 1 and then the original path from 1 to v. If for $d+1$ disjoint paths from $\mathcal{R}$ to v, there exists a path, say from node m to v, passing through node j, then there still exist $d+1$ disjoint paths from $\bar{\mathcal{R}}$ to v with two new paths: one is the path from j to v taken from the path m to v and the other is the path connecting from m to 1 and 1 to v. This means, $\bar{\mathcal{G}}$ is $(d+1)$-rooted with the root set $\bar{\mathcal{R}}$. According to the argument in case 1, we can know that $\det(\bar{L}_{\bar{\mathcal{R}}}) \neq 0$. Notice that M can be obtained from $\bar{L}_{\bar{\mathcal{R}}}$ via elementary row transformations, so $\det(M) \neq 0$. Thus, the conclusion follows by induction. ■

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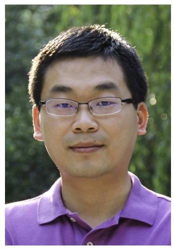

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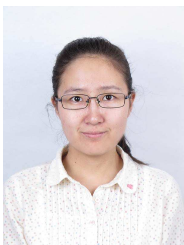

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