Mathematical Preliminaries

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Group

A group $(G, *)$ consists of the following:

(i) A set G

(ii) A rule or binary operation '*' (set G is closed under this operation) which associates with each pair of elements x and y in G, an element $(x * y)$ in G such that

a) This binary operation is associative:

i.e., $x *(y * z)=(x * y) * z \qquad \forall\ x, y, z \in G$

b) There exists an element 'e' called identity of group G

s.t. $\quad e * x = x * e = x \qquad \forall\ x \in G$

c) For every $x \in G$, there exists an element x^{-1} such that

$x * x^{-1} = e = x^{-1} * x \qquad \forall\ x \in G$

d) If $x * y = y * x \quad \forall\, x, y \in G$

$\Longrightarrow$ Commutative group or Abelian group

Examples:

1. Set of n x n invertible matrices with matrix multiplication

2. Set of periodic signals with time period T under binary operation ‘*’ = ‘+’

Field

Group : A field $(F, +, \cdot)$ consists of the following:

(i) A set F

(ii) Two binary operations '+' and '·' such that

a) $(F, +)$ is an Abelian group

b) Define $F^*=F-\{0\}$.

$(F^*, \cdot)$ is an Abelian group

(c) The multiplication operation distributes over addition:

Left distributive:

$x.(y+z)=x.y + x.z \qquad \forall\, x, y, z \in F$

Right distributive:

$(x+y).\, z=x.z + y.z \qquad \forall\, x, y, z \in F$

Examples: Check for $F = \mathbf{R}$; $F = \mathbf{C}$; $F= \mathbf{Z}$; $F=\mathbf{Q}$

Vector Space

A set V together with a map

'+': $V \times V \rightarrow V$

$(v_1,v_2) \rightarrow (v_1+v_2)$ called vector addition

and

'.' : $F \times V \rightarrow V$

$(\alpha,v) \rightarrow (\alpha . v)$ called scalar multiplication

is called a F-vector space or vector space over the field F if the following are satisfied:

(i) $(V, +)$ is an Abelian group

(ii) $\alpha . (v_1+v_2) = \alpha . v_1 + \alpha . v_2$ $\quad \forall\, v_1,v_2 \in V$ and $\forall\, \alpha \in F$

(iii) $(\alpha_1 + \alpha_2) . v = \alpha_1 . v + \alpha_2 . v$ $\quad \forall\, v \in V$ and $\forall\, \alpha_1, \alpha_2 \in F$

(iv) $(\alpha\beta) \,.\, v = \alpha(\beta v) = \beta(\alpha v)$ $\quad \forall\, v \in V$ and $\forall\, \alpha, \beta \in F$

(v) $1.v = v$ $\quad \forall\, v \in V$

(vi) $\alpha.\boldsymbol{0} = \boldsymbol{0}$ $\quad \forall\, \alpha \in F$

(vii) $0.v=\boldsymbol{0}$ $\quad \forall\, v \in V$

(viii)If $v \neq \boldsymbol{0}$, then $\alpha.v = \boldsymbol{0}$ implies that $\alpha = 0$.

(ix) If V is a vector space over a field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V.

Examples:

1. Set of periodic signals with the same time period T over the field $F = \mathbf{R}$ form a vector space
2. Set of finite energy signals over the field $F = \mathbf{R}$ form a vector space
3. $V_n(F)$ = n-dimensional vector space over the field F

$F = \mathbf{R}$ or $\mathbf{C}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} \;\middle|\; x_i \in F \right\}$$

Metric Space

Definition: Metric is a map

$$d: X \times X \to R$$

that satisfies the following:

(i) $d(x, y) \geq 0$ and $d(x, y)=0$ iff $x = y$ $\quad \forall\, x, y \in X$

(ii) $d(x, y) = d(y, x)$ $\quad \forall\, x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ $\quad \forall\, x, y, z \in X$

This map is called a **metric** and a set equipped with this map is called a **metric space** and is denoted by (X, d).

Note that metric is a generalization of the notion of distance.

Examples ????

Norm

Definition:

Let V be a F-vector space. A map

$\|\cdot\|: \mathrm{V} \rightarrow R$ is called a **norm** if it satisfies the following:

(i) $\|v\| \geq 0$ and $\|v\| = 0$ iff v=0 $\quad \forall\, v \in V$

(ii) $\|\alpha v\| = |\alpha|\, \|v\|$ $\quad \forall\, v \in V$ and $\forall\, \alpha \in F$

(iii) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ $\quad \forall\, v_1, v_2 \in V$

A vector space equipped with a norm is called a **normed vector space**.

Example: Let V be a F-vector space equipped with a norm. Prove that $d(v_1, v_2) = \|v_1 - v_2\|$ is a proper metric.

Inner Product

Definition:

Let V be a F-vector space. A map

$$< , > : V \text{ x } V \longrightarrow F$$

is called an inner product if it satisfies the following:

(i) $< v, v > \;\geq\; 0$ and $< v, v > \; = 0$ iff v=0 $\qquad \forall\, v \in V$

(ii) $< v_1, v_2 > = < v_2, v_1 >^*$ $\qquad \forall\, v_1,v_2 \in V$

'*' denotes complex conjugate

(iii) It is linear in the first coordinate.

$$< \alpha_1 v_1 + \alpha_2 v_2, w> = \alpha_1 < v_1, w> + \alpha_2 <v_2, w>$$

$$\forall\, v_1,v_2, w \in V \text{ and } \forall\, \alpha_1, \alpha_2 \in F$$

(iv) It is conjugate linear in the second coordinate.

$$<v,\ \alpha_1 w_1 + \alpha_2 w_2> = \alpha_1{}^* < v,\ w_1> + \alpha_2{}^* <v,\ w_2>$$

$$\forall\ w_1, w_2,\ v \in V \text{ and } \forall\ \alpha_1,\ \alpha_2 \in F$$

Example 1: Define $\|v\| = < v,\ v>^{1/2}$. If it is an inner product space, we can define norm and then define metric.

Example 2: Consider $V_n(F)$ where vectors are n-tuples of scalars, i.e., $\mathbf{x} \in V$ and $x_i \in F$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

Define inner product as: $< \mathbf{x}, \mathbf{y} > = \sum_{i=1}^{n} x_i \bar{y}_i$

Show that it satisfies all the properties of an inner product.

Linear Independence of Vectors

A set of vectors $\{v_1, v_2, \ldots.., v_k\}$ in $V_n(F)$ is called linearly independent (LI) if $\sum_{i=1}^{k} \alpha_i v_i = 0$ implies $\alpha_i = 0$ for all i.

Definition: The number of maximal LI vectors in a vector space V is called the **dimension** of the vector space and the maximal LI vectors is called a **basis** for V.

If $\{e_1, e_2, \ldots.., e_n\}$ is a basis for $V_n(F)$, then for any $v \in V$

$$v = \sum_{i=1}^{n} \alpha_i e_i \qquad \text{for some } \alpha_i \in F$$

Orthonormal Basis

Definition: A set of vectors $\{v_1, v_2, \ldots.., v_n\}$ belonging to an inner product space V is called **orthonormal** if

(i) $v_i \neq 0 \qquad \forall\ i.$

(ii) $< v_i\ , v_j > = 0 \qquad \forall\ \ i \neq j$

(iii) $< v_i\ , v_j > = 1 \qquad \forall\ \ i = j$

We can also write $< v_i\ , v_j > = \delta_{ij}$.

References

- Linear Algebra by Hoffman and Kunze