EE140 Introduction to Communication Systems

Lecture 11

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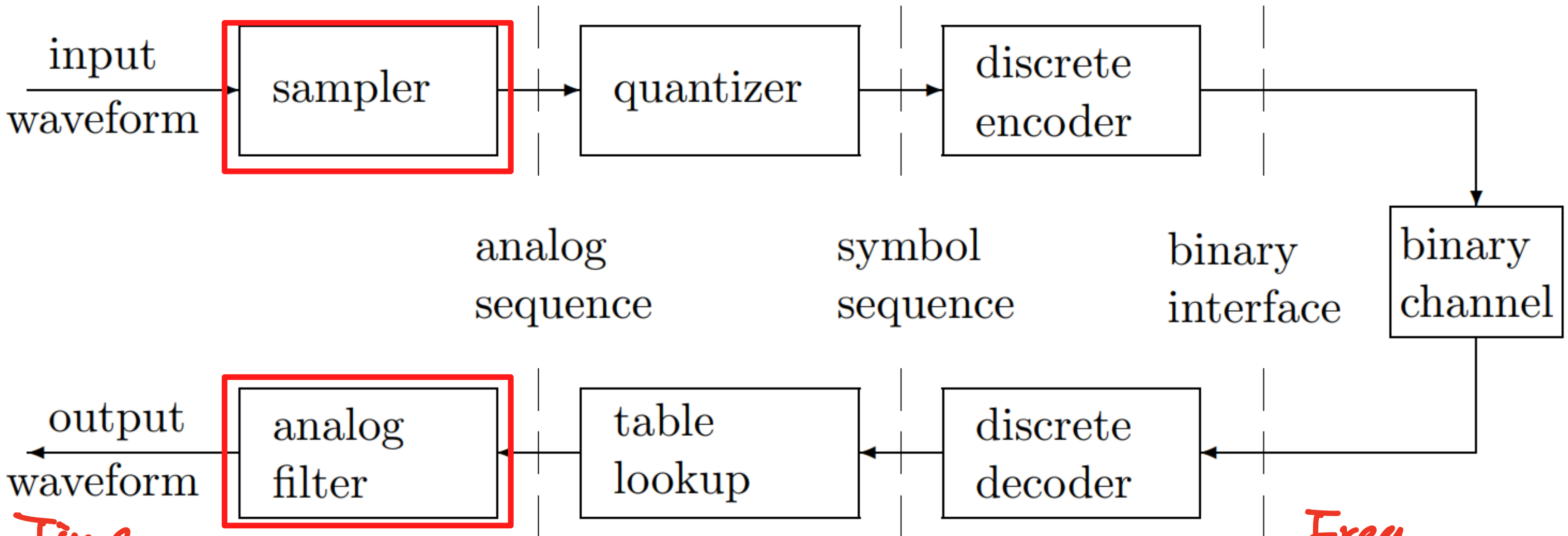
ShanghaiTech University, Fall 2025

Sampling

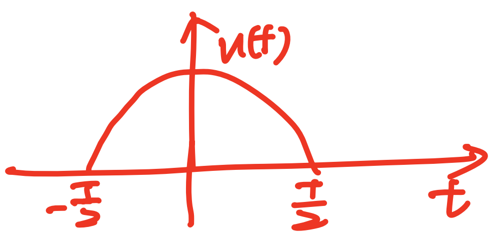
$u(t) = \{u_k\}$



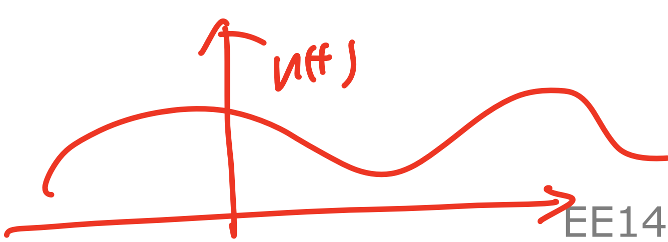
Time

Freq

① Time-limited



② Time-unlimited



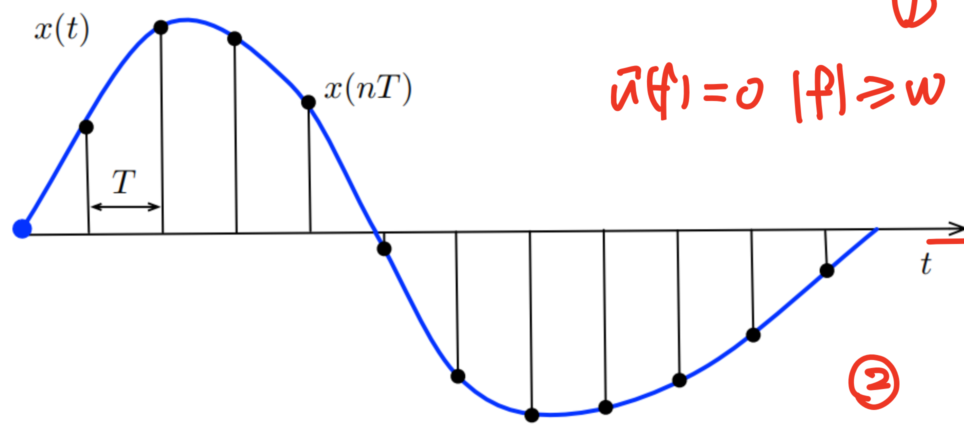
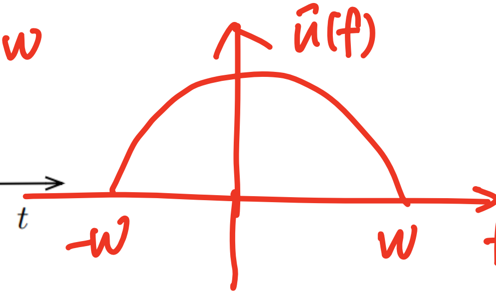


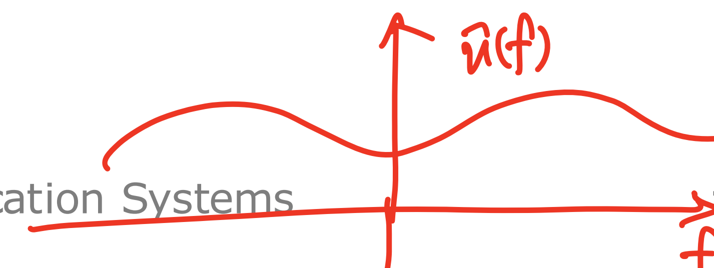
Figure: Signal sampling

① Band limited

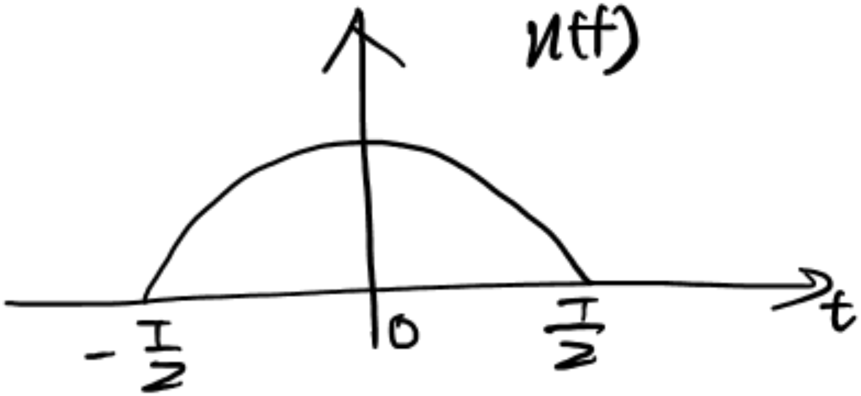
$\hat{u}(f) = 0 \quad |f| \geq W$



② Band unlimited



1. Time Limited signal



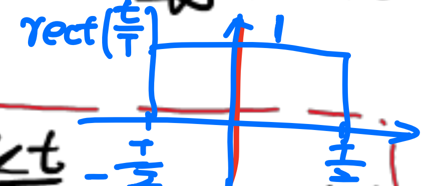
Fourier Series

$$x(t)=\begin{cases}\sum_{k=-\infty}^{+\infty}\hat{x}_k e^{j2\pi k\frac{t}{T}} & -\frac{T}{2}\le t\le\frac{T}{2}\\ 0 & \text{, otherwise}\end{cases}$$

$\{\hat{x}_k\}$

$$\hat{x}_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-j2\pi k\frac{t}{T}}$$

$-\infty<k<+\infty$



$\{\theta_k(t)\}$ orthogonal

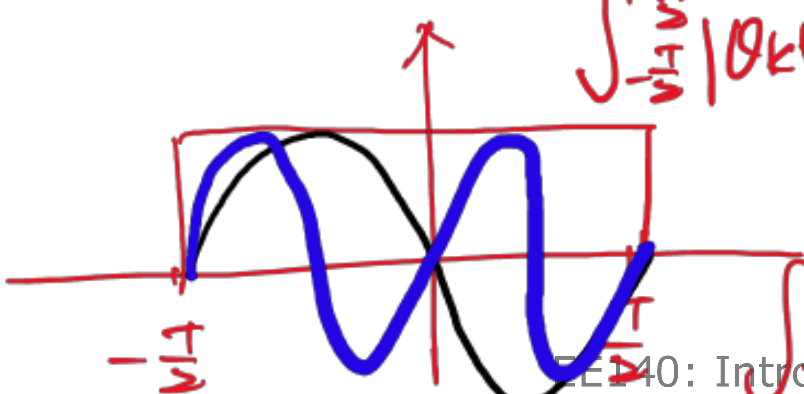
Energy of $\theta_k(t)=T$

$$x(t)=\sum_{k=-\infty}^{+\infty}\hat{x}_k\boxed{e^{j2\pi k\frac{t}{T}}\,\mathrm{rect}\left(\frac{t}{T}\right)}$$

$$=\sum_{k\in Z}\hat{x}_k\,\underline{\theta_k(t)}$$ (orthogonal expansion)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}|\theta_k(t)|^2dt=T$$

$$\Leftarrow\ \boxed{\theta_k(t)=e^{j2\pi k\frac{t}{T}}\,\mathrm{rect}\left(\frac{t}{T}\right)}$$



$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\theta_k(t)\,\theta_m^*(t)\,dt=0$$

$f_k=k\cdot\frac{1}{T}$

$$u(t) \rightleftarrows \{\hat{u}_k\}_{k=-\infty}^{+\infty}$$

$$u(t) = \sum_{k'} \hat{u}_{k'}\,\theta_{k'}(t)$$

① Fourier Series Expansion

② $$\frac{1}{T}\int_{-\infty}^{+\infty} u(t)\,\theta_k^*(t)\,dt = \hat{u}_k$$

$$\frac{1}{T}\int u(t)\,\theta_k^*(t)\,dt = \frac{1}{T}\hat{u}_k \int |\theta_k(t)|^2 dt = \frac{1}{T}T\,\hat{u}_k = \hat{u}_k$$

$$u(t) = \sum_{k=\infty}^{+\infty} \hat{u}_k\,\theta_k(t)$$

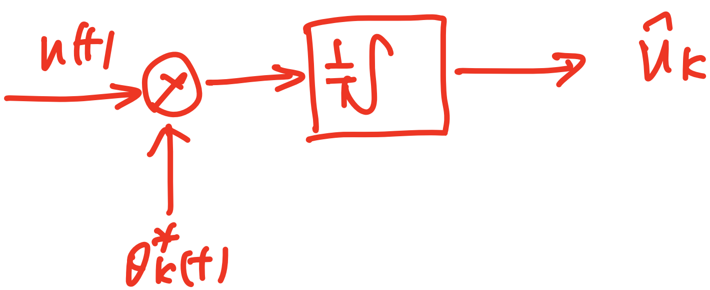
$$\lim_{\ell\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \Big|u(t) - \sum_{k=-\ell}^{+\ell} \hat{u}_k\,\theta_k(t)\Big|^2 = 0$$

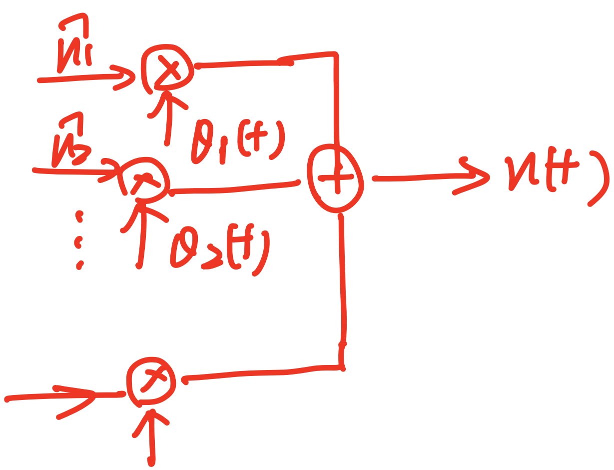
converge in $\mathcal{L}_2$.

or $\mathcal{L}_2$ convergence

or l.i.m. limit in mean-square

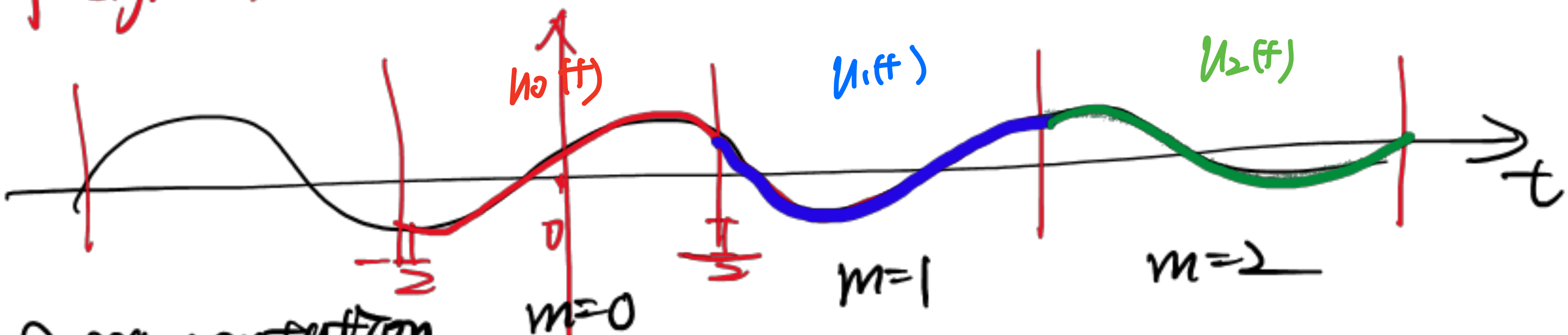
$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k e^{j2\pi \frac{kt}{T}}\,\text{rect}\left(\frac{t}{T}\right)$$





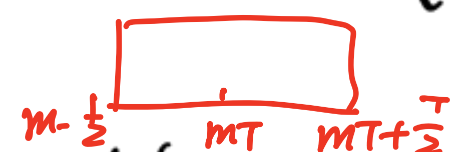
If signal is not time limited

$u(t) \Longleftrightarrow \{\hat{u}_{k,m}\}_{k,m}$



① segmentation

$$u(t) = \sum_m u_m(t) \qquad \underline{u_m(t)} = u(t)\,\mathrm{rect}\left(\frac{t}{T} - m\right)$$



$u_m(t)$ duration is T

$$\underline{u_m(t)} = \sum_k \underline{\hat{u}_{k,m}}\, e^{j2\pi k \frac{t}{T}}\,\mathrm{rect}\left(\frac{t}{T} - m\right)$$

$$\hat{u}_{k,m} = \frac{1}{T}\int_{mT-\frac{T}{2}}^{mT+\frac{T}{2}} u_m(t)\, e^{-j2\pi k\frac{t}{T}}\,dt$$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} u(t)\,\mathrm{rect}\left(\frac{t}{T} - m\right) e^{-j2\pi k \frac{t}{T}}\,dt$$

orthogonal expansion

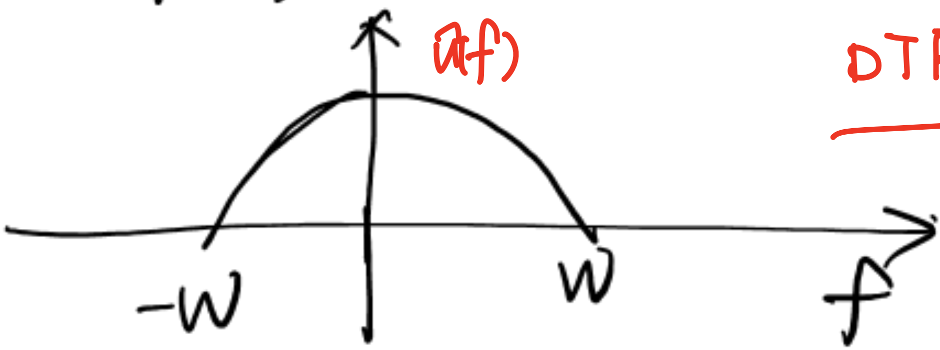
$$\underline{u(t)} = \sum_m \sum_k \hat{u}_{k,m}\, \underbrace{\boxed{e^{j2\pi k\frac{t}{T}}\,\mathrm{rect}\left(\frac{t}{T} - m\right)}}_{\theta_{k,m}(t)}$$

$\theta_{k,m}(t)$ are orthogonal

$\text{Energy}[\theta_{k,m}(t)] = T$

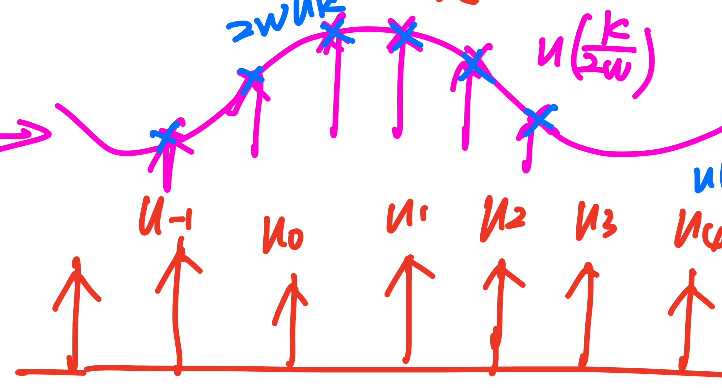
T-spaced Truncated sinusoid expansion

2. If signal is Bandlimited.



IFT

DTFT



$T = \frac{1}{2W}$

$u(\frac{k}{2W}) = 2W u_k$

$u(kT) = \frac{u_k}{T}$

$u_k = T u(kT)$

DTFT $\quad \hat{u}(f) = \sum_k u_k e^{-j2\pi k \frac{f}{2W}} \operatorname{rect}\left(\frac{f}{2W}\right)$

DTFT coefficients. $\quad u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{j2\pi k \frac{f}{2W}} df$

FT $\quad \hat{u}(f) \xrightarrow{\text{IFT}} u(t)$

$u(t) \overset{?}{\longleftrightarrow} \{u_k\} \Rightarrow$ sampling theorem

Proof: Sampling Theorem

$\{\phi_k(f)\}$ orthogonal

- DTFT of $\hat{x}(f)$ over [-W, W]:

DTFT coefficients

$$\hat{x}(f) = \sum_{k=-\infty}^{\infty} x_k e^{-2\pi i k f/2W} \text{rect}(\frac{f}{2W}) = \sum_{k=-\infty}^{\infty} x_k \hat{\phi}_k(f)$$

$x(t) = \sum_k x_k \phi_k(t)$

where:

$$\hat{\phi}_k(f) = e^{-2\pi i k f/2W} \text{rect}(\frac{f}{2W})$$

$$x_k = \frac{1}{2\pi} \int_{-W}^{W} \hat{x}(f) e^{2\pi i k f/2W}\, df$$

- IFT of $\hat{x}(f)$:

Since $\text{rect}(f/2W) \leftrightarrow 2W\text{sinc}(2Wt)$,

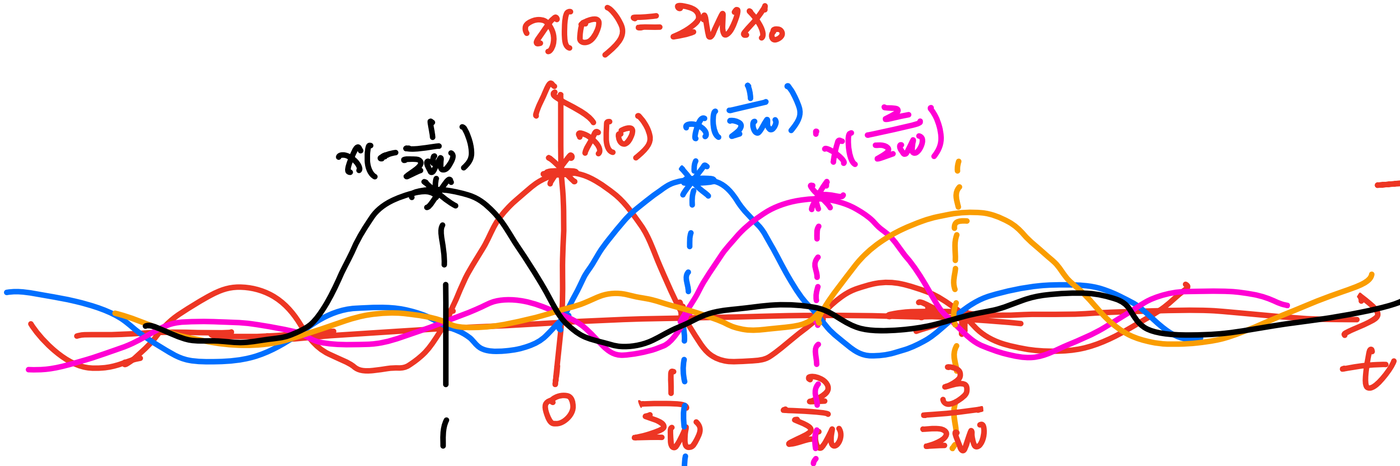
$$\phi_k(t) = 2W\text{sinc}(2Wt - k) \Leftarrow \hat{\phi}_k(f) = e^{-2\pi i k f/2W} \text{rect}(\frac{f}{2W})$$

Thus, the inverse of $\hat{x}(f)$ is

DTFT coefficient

$= x(\frac{k}{2W})$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \phi_k(t) = \sum_{k=-\infty}^{\infty} \boxed{2W x_k} \text{sinc}(2Wt - k)$$



let $t=\frac{k}{2W}$ $\quad T=\frac{1}{2W}$

$$x\left(\frac{k}{2W}\right) = \sum_{k'} 2W x_{k'} \, \mathrm{sinc}\left(2W\frac{k}{2W}-k'\right)$$

$$= \sum_{k'} 2W x_{k'} \, \mathrm{sinc}(k-k')$$

$$= 2W x_k$$

Sampling theorem $T=\frac{1}{2W}$

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} x\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt-k)}$$

orthogonal expansion

$\boxed{|f| \geq W \quad X(f)=0}$

$$x(t) = \sum_{k=-\infty}^{+\infty} x(kT) \, \mathrm{sinc}\left(\frac{t}{T}-k\right)$$

$\{\mathrm{sinc}(2Wt-k)\}_k$ orthogonal

Energy of $[\mathrm{sinc}(2Wt-k)]$

$$= \frac{1}{2W} = T$$

Proof: Sampling Theorem

- Rewrite:

$$x(t) = \sum_{k=-\infty}^{\infty} 2W x_k \text{sinc}(2Wt - k) \qquad (5)$$

Let $t = k/2W$, since $\text{sinc}(0) = 1, \text{sinc}(j) = 0$, for $j = \pm 1, \pm 2, \ldots,$, we have

$$x(k/2W) = 2W x_k$$

Substitute this to (5), we obtain sampling equation

$$x(t) = \sum_{k=-\infty}^{\infty} x(k/2W)\text{sinc}(2Wt - k)$$

Same as the equation in Sampling theorem:

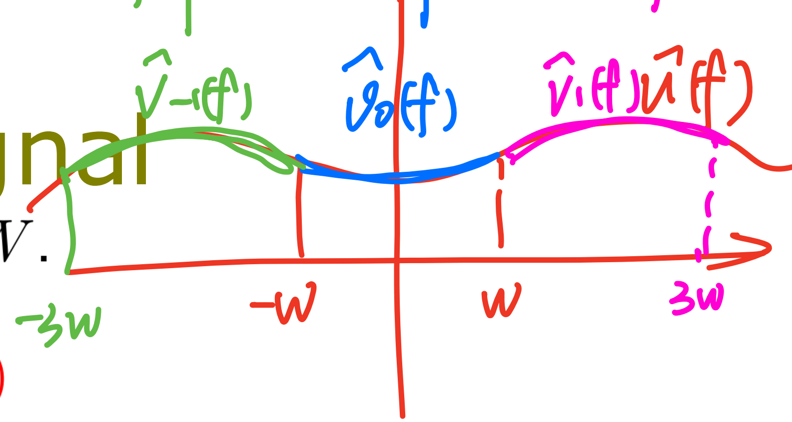
$\{v_{-1}(kT)\}_k$ $\{v_0(kT)\}_k$ $\{v_1(kT)\}_k$

$$x(t) = \sum_{k=-\infty}^{\infty} x(\frac{k}{2W})\text{sinc}\left[2W(t - \frac{k}{2W})\right]$$

$T = \frac{1}{2W}$ $v_{-1}(f)$ $v_0(f)$ $v_1(f)$

IFT IFT IFT

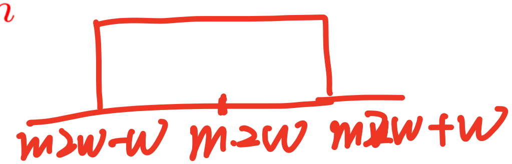
Segment Signal



- Break $\hat{x}(f)$ into segments, each of duration $2W$.

$$\hat{x}(f) = \lim \sum_m \hat{v}_m(f)$$

where the m-th segment is



$$\hat{v}_m(f) = \hat{x}(f)\text{rect}(f/2W - m) \quad (6)$$

Then by Sampling Theorem (4), $v_m(t) \leftrightarrow \hat{v}_m(f)$, with $T = 1/2W$,

$$v_m(t) = \sum_k v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\boxed{e^{2\pi i m \frac{t}{T}}} = \sum_k v_m(kT)\psi_{m,k}(t)$$

$\{\psi_{m,k}(t)\}_{m,k}$ orthogonal

Thus,

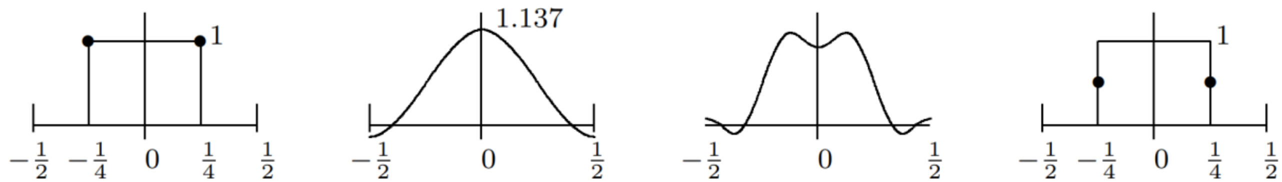
$$x(t) = \lim \sum_{m,k} v_m(kT)\underbrace{\boxed{\text{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}}}_{\text{T-spaced sinc-weighted sinusoids}}$$

- m and k is the index of segments and Fourier coefficients
- $\psi_{m,k}(t) = \text{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}$ is orthogonal (orthogonal expansion)

Encoding Source Waveform

1. Expand the waveform into an orthogonal expansion.
   - Time limited signal: $u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi i k t/T} \operatorname{rect}(\frac{t}{T}) = \sum_{k \in \mathbb{Z}} \hat{u}_k \theta_k(t)$

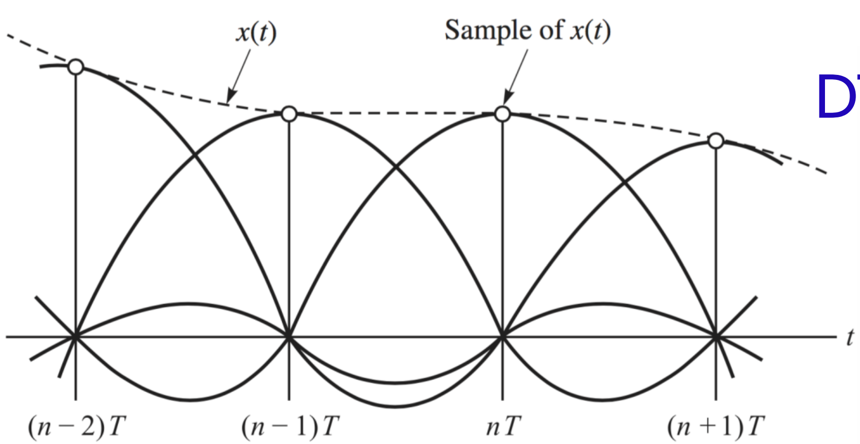
Fourier series expansion



$u(t) = \operatorname{rect}(2t)$ | $\frac{1}{2} + \frac{2}{\pi}\cos(2\pi t)$ | $\frac{1}{2} + \frac{2}{\pi}\cos(2\pi t) - \frac{2}{3\pi}\cos(6\pi t)$ | $\sum_k u_k e^{2\pi i k t} \operatorname{rect}(t)$

$= 2W u_k = \frac{1}{T} u_k$

   - Baseband limited signal: $u(t) = \sum_{k=-\infty}^{\infty} u(kT) \operatorname{sinc}(\frac{t}{T} - k)$, $T = 1/(2W)$



DTFT and FT

Sampling Theorem

Encoding Source Waveform

1. Expand the waveform into an orthogonal expansion.

   – Time unlimited signal:

     Segmentation -> Shifted Fourier series expansion

$$u(t) = \text{l.i.m.} \sum_m \sum_k \hat{u}_{k,m} \theta_{k,m}(t) \qquad \text{where} \quad \theta_{k,m}(t) = e^{2\pi i k t/T} \text{rect}\left(\frac{t}{T} - m\right).$$

T-spaced truncated sinusoid expansion

T-spaced truncated sinusoids

   – Band unlimited signal:

     Segmentation -> Shifted sampling theorem $T = 1/(2W)$

$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT) \psi_{m,k}(t) \qquad \psi_{m,k}(t) = \text{sinc}\left(\frac{t}{T} - k\right) e^{2\pi i m t/T}$$

T-spaced sinc-weighted sinusoid expansion

T-spaced sinc-weighted sinusoids

ɔmr

Encoding Source Waveform

1. Expand the waveform into an orthogonal expansion.

$$u(t) = \sum_k u_k \theta_k(t) \qquad u(t) \Leftrightarrow \{u_k\}$$

2. Quantize the coefficients in that expansion.
3. Use discrete source coding on the quantizer output.

- Q: why orthogonal expansion?

Why orthogonal?

$$u(t) = \sum_k u_k \phi_k(t) \xrightarrow{\text{Sampling}} \{u_k\} \xrightarrow{\text{Quantization}} \{v_k\} \xrightarrow{\text{Source Encode}} 01001$$

$$v(t) = \sum_k v_k \phi_k(t) \longleftarrow \{v_k\} \longleftarrow \{v_k\} \longleftarrow 01001$$

if $\{\phi_k(t)\}$ orthogonal waveforms

Energy of $u(t)$:

$$\int u^2(t)\,dt = \int \left|\sum u_k \phi_k(t)\right|^2 dt$$

$$= \sum_k u_k^2 \int |\phi_k(t)|^2 dt$$

$$= T \sum |u_k|^2$$

Reconstruction error: $\int |u(t) - v(t)|^2 dt$

$$= \int \left|\sum (u_k - v_k)\phi_k(t)\right|^2 dt$$

$$= \sum_k T |u_k - v_k|^2$$

Quantization error

$$\boxed{MSE = E\,|V - U|^2}$$

minimize

Sampling

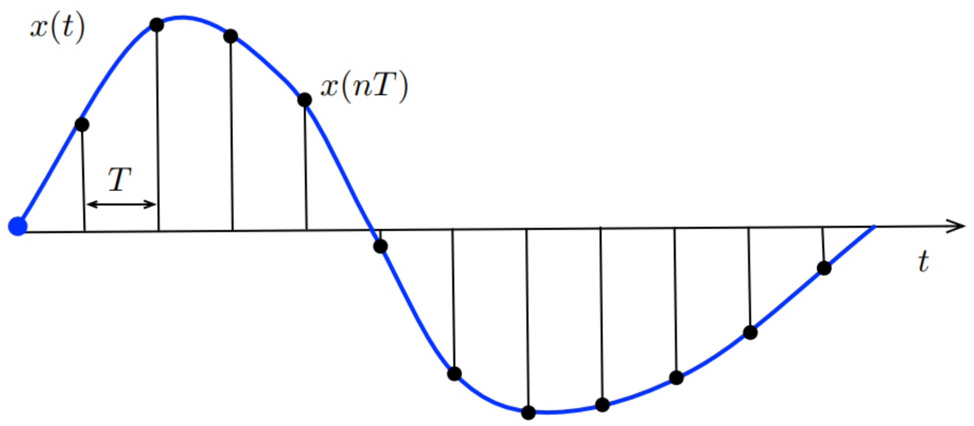


Figure: Signal sampling

- $x(nT)$ are samples of $x(t)$
- ideal sampled signal $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$
- T sampling interval (period), $f_s = 1/T$ sampling frequency (rate)

Question: Whether/ how we can reconstruct $x(t)$ from $x(nT)$?

Bandlimited Signal

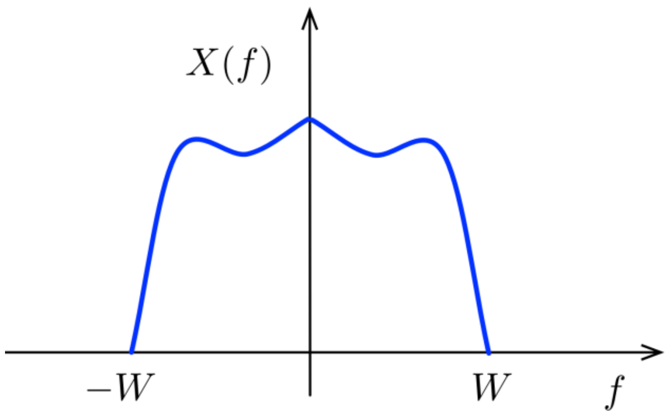


Figure: Bandlimited signal

- Given a signal $x(t)$, it is bandlimited if its Fourier Transform $\hat{x}(f)$: $\hat{x}(f) = 0$, for $|f| > W$.

- Sampling Theorem:

Let $x(t)$ be a continuous signal baseband-limited to W. Then

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt - k)$$

where $\mathrm{sinc}(\mathrm{x}) = \frac{\sin(\pi x)}{\pi x}$.

Sampling Theorem

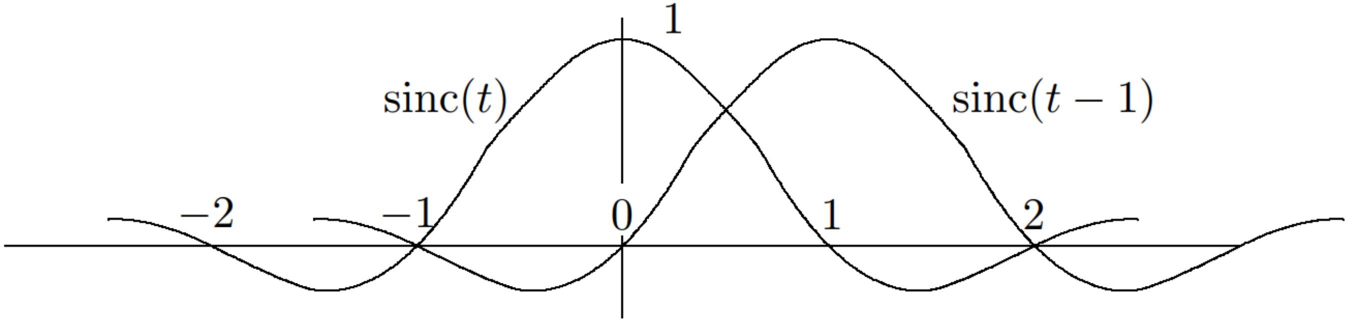
- Sampling Theorem:

Let x(t) be a continuous signal baseband-limited to W. Then

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt - k)$$

where $\mathrm{sinc(x)} = \frac{\sin(\pi x)}{\pi x}$.

- $\{\mathrm{sinc}(2Wt - k)\}$ orthogonal. $2\mathrm{W}\mathrm{sinc}(2\mathrm{W}t - k) \longleftrightarrow e^{-2\pi i k f/(2\mathrm{W})}\mathrm{rect}(\frac{f}{2\mathrm{W}})$



- $\mathrm{T} = 1/2\mathrm{W} \Longrightarrow f_s = 2W$, Nyquist rate/frequency.
  - Sampling equation: $x(t) = \sum_{k=-\infty}^{\infty} x(kT)\mathrm{sinc}(t/T - k)$

Sampling and Reconstruction: Time Domain

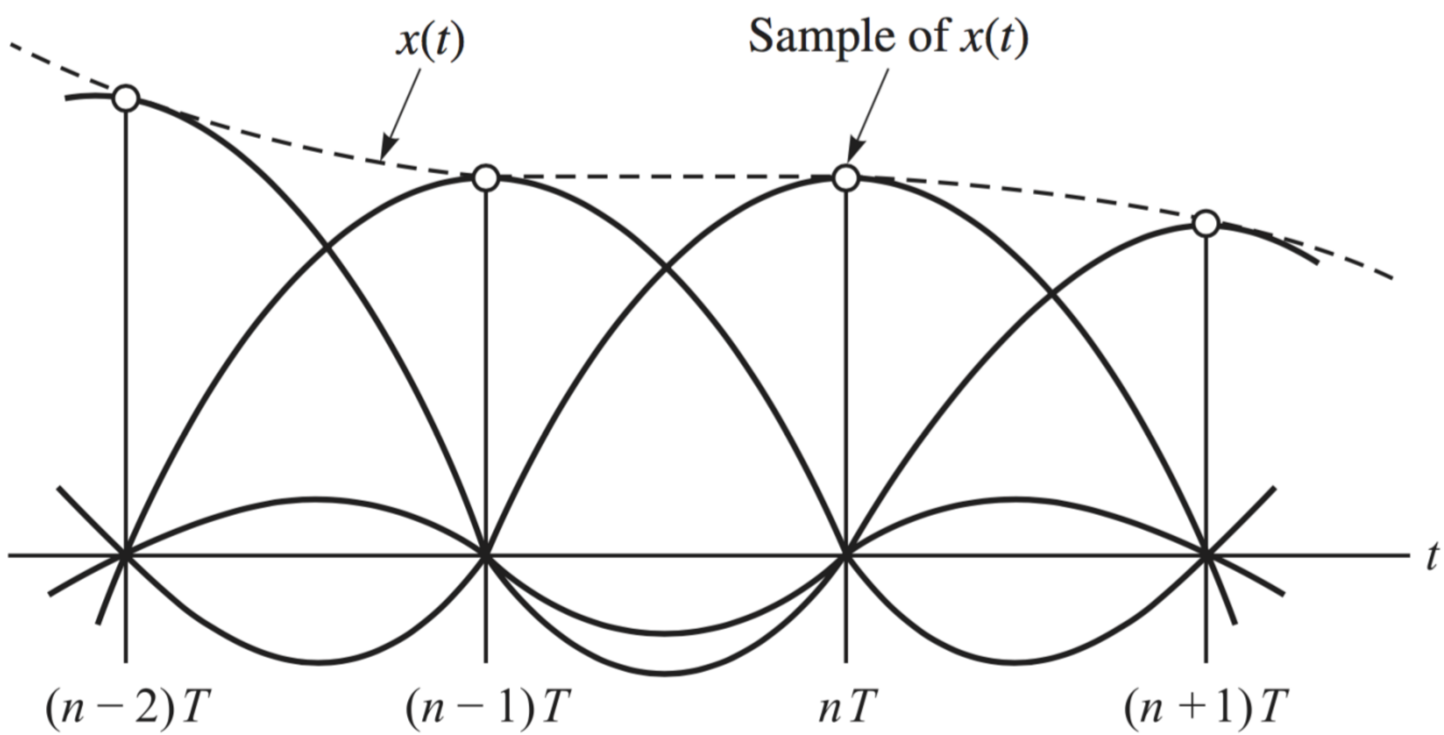


Figure: Reconstruction from samples (time domain) [Proakis'B]

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT)\mathrm{sinc}(t/T - k)\,, T = \frac{1}{2W}.$$

Sampling and Reconstruction: Frequency Domain

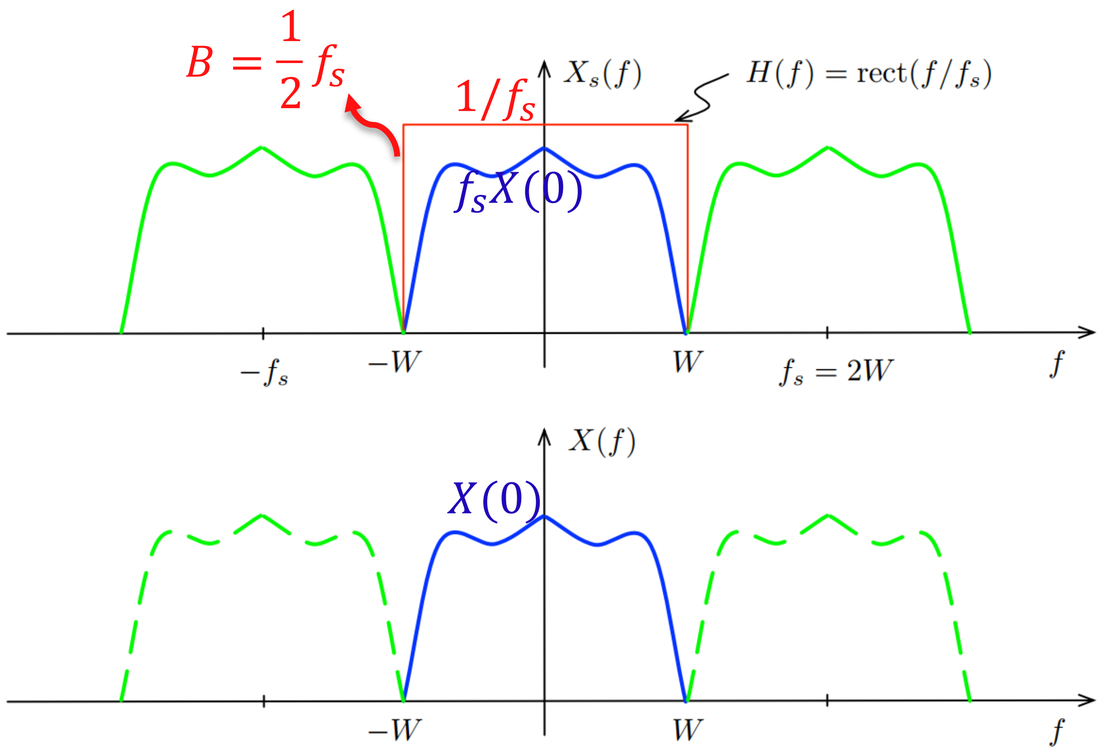


Figure: Recover $X(t)$ from $X(n/2W)$ (frequency domain)

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\mathrm{sinc}\left(\frac{t}{T_s} - k\right), f_s \geq 2W.$$

Segment Signal

- Break $\hat{x}(f)$ into segments, each of duration $2W$.

$$\hat{x}(f) = \lim \sum_m \hat{v}_m(f)$$

where the m-th segment is

$$\hat{v}_m(f) = \hat{x}(f)\text{rect}(f/2W - m) \tag{6}$$

Then by Sampling Theorem (4), $v_m(t) \leftrightarrow \hat{v}_m(f)$, with $T = 1/2W$,

$$v_m(t) = \sum_k v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}} = \sum_k v_m(kT)\psi_{m,k}(t)$$

Thus,

$$x(t) = \lim \sum_{m,k} v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}$$

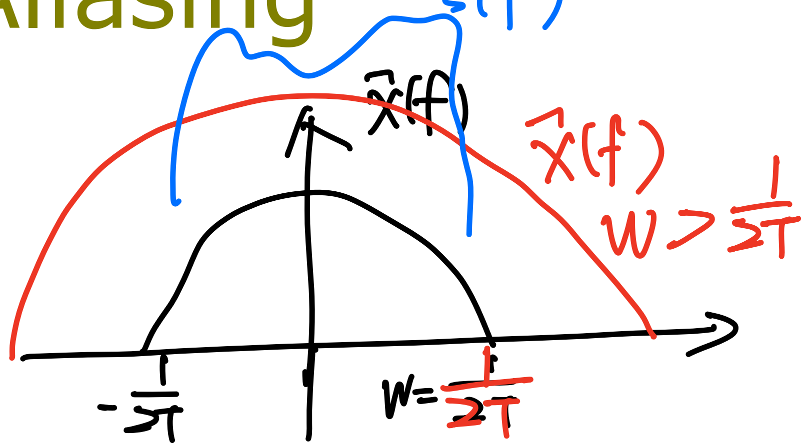
- m and k is the index of segments and Fourier coefficients
- $\psi_{m,k}(t) = \mathrm{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}$ is orthogonal (orthogonal expansion)

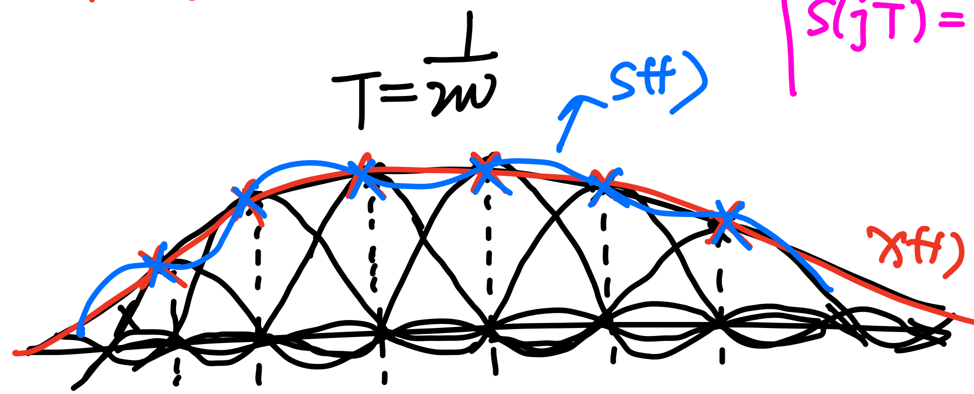
$\hat{x}(f)$ Bandwidth : $\underline{W}$

T: sampling period $f_s = \frac{1}{T}$

$$\begin{cases} s(t) \neq x(t) \\ s(jT) = x(jT) \quad \forall j \end{cases}$$

Aliasing





$$\sum_k x\left(\frac{k}{2W}\right) \operatorname{sinc}(2Wt-k)$$

$$\hat{s}(f) \longleftarrow s(t) = \sum x(kT) \operatorname{sinc}\left(\frac{t}{T}-k\right)$$

Sampling approximation of $x(t)$

$$\begin{cases} \text{Time} : & x(t), \quad s(t) \\ \text{freq} : & \hat{x}(f) \quad \hat{s}(f) \end{cases}$$

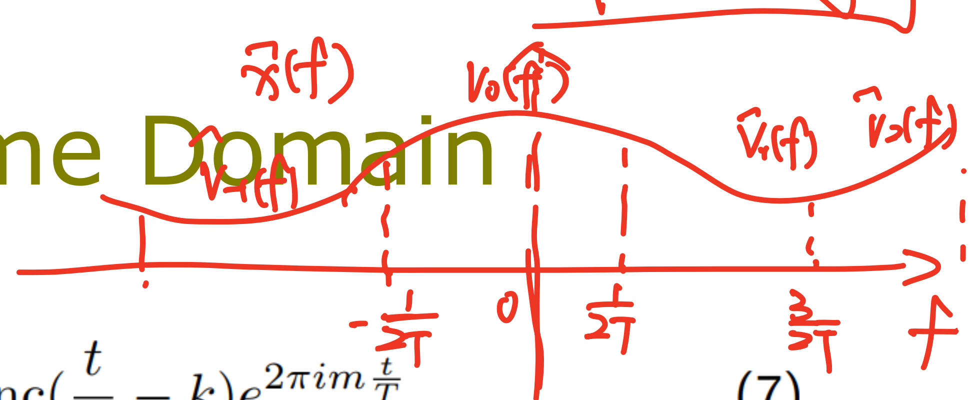
If $\underline{W \le \frac{1}{2T}}$ $\left(T \le \frac{1}{2W},\ W \le \frac{f_s}{2},\ f_s \ge 2W \right)$

$s(t) = x(t)$ $\hat{s}(f) = \hat{x}(f)$

If $W > \frac{1}{2T}$ $\left(T > \frac{1}{2W},\ W > \frac{f_s}{2},\ f_s < 2W \right)$

$s(t) \neq x(t)$, $\hat{s}(f) \neq \hat{x}(f)$ $\Downarrow$ $\boxed{\text{Aliasing}}$

Aliasing: Time Domain



Rewrite:

$$x(t) = \lim \sum_{m,k} v_m(kT)\,\text{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}} \qquad (7)$$

$t = jT$

$x(jT) = \sum_m \sum_k v_m(kT)\,\text{sinc}(j-k) = \sum_m v_m(jT)$

- Here, $v_m(kT) = \mathcal{F}^{-1}[\hat{v}_m(f)]|_{t=kT}$
- $x(t)$ above is the true signal described by $\{v_m(kT)\}$

Let $s(t)$ denotes the sampling approximation of $x(t)$:

$t = jT$

$s(jT) = \sum_k x(kT)\,\text{sinc}(j-k)$

$= x(jT)$

$$s(t) = \sum\nolimits_k x(kT)\,\text{sinc}(\frac{t}{T} - k) \qquad (8)$$

- If $x(t)$ is baseband-limited to 1/2T, then $s(t) = x(t)$ (only one segment $m = 0$, proved by DTFT before)
- If $x(t)$ is not baseband-limited to 1/2T, when the samples of $s(t)$, $x(t)$ are same, we have $\boxed{s(jT) = x(jT) = \sum_m v_m(jT)}$, for all j. Thus,

$$s(t) = \sum_{k,m} v_m(kT)\,\text{sinc}(\frac{t}{T} - k) \qquad (9)$$

- By (7) and (9), we find $\int_{-\infty}^{\infty} |x(t) - s(t)|^2 dt \neq 0$, i.e., not $\mathcal{L}_2$ equivalent

Aliasing: Frequency Domain

Rewrite:w

$$x(t) = \lim \sum_m v_m(t) = \lim \sum_{m,k} v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}$$

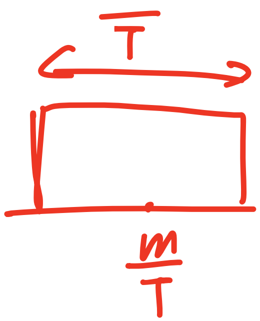
$$s(t) = \sum_{k,m} v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k) \triangleq \sum_m s_m(t)$$

where $v_m(t) = \sum_k v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k)e^{2\pi i m \frac{t}{T}}$ and with $s_m(t) = \sum_k v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k)$.

- compare $s_m(t)$ with $v_m(t)$, we have

$$\boxed{v_m(t) = s_m(t)e^{2\pi i m \frac{t}{T}}}$$

$\hat{v}_m(f)$:



  thus $\hat{v}_m(f) = \hat{s}_m(f - \frac{m}{T})$, and $\hat{s}_m(f) = \hat{v}_m(f + \frac{m}{T})$

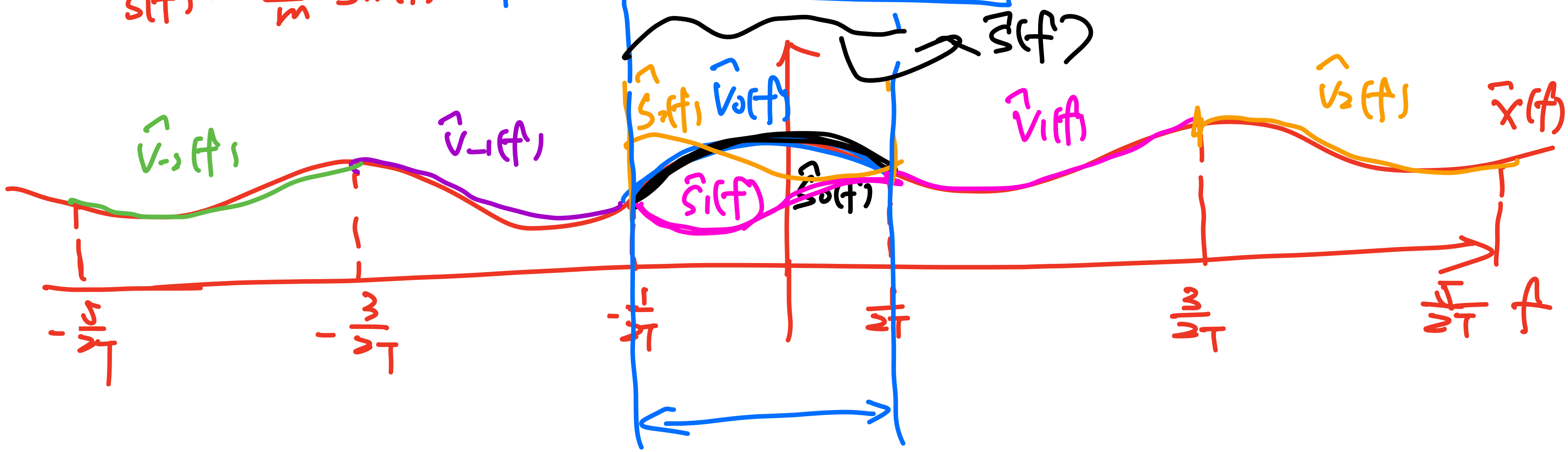
- Since $\hat{v}_m(f) = \hat{x}(f)\mathrm{rect}(fT - m)$ by (6), we have $\hat{v}_m(f + \frac{m}{T}) = \hat{x}(f + \frac{m}{T})\mathrm{rect}(fT)$, and thus

$$\hat{s}(f) = \sum_m \hat{s}_m(f) = \sum_m \hat{x}(f + \frac{m}{T})\mathrm{rect}(fT)$$

$$\hat{x}(f) = \sum_m \hat{v}_m(f)$$

$$\hat{s}(f) = \sum_m \hat{s}_m(f)$$

$$\boxed{\hat{s}_m(f) = \hat{v}_m\left(f + \frac{m}{T}\right)}$$



Aliasing: Frequency Domain

Given a signal $x(t)$, its sampling approximation

$$s(t) = \sum_k x(kT)\,\mathrm{sinc}(\frac{t}{T} - k)$$

converges to a function $s(t)$, and the Fourier transform of $s(t)$ satisfies:

$$\hat{s}(f) = \sum_m \hat{x}(f + \frac{m}{T})\mathrm{rect}(fT)$$

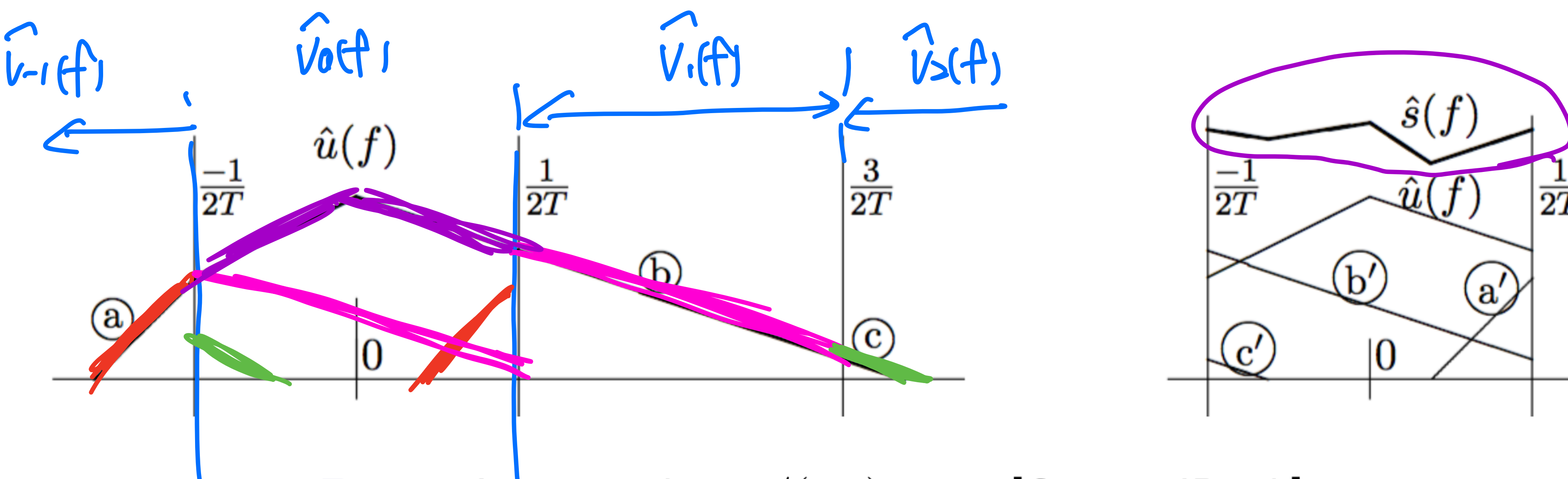


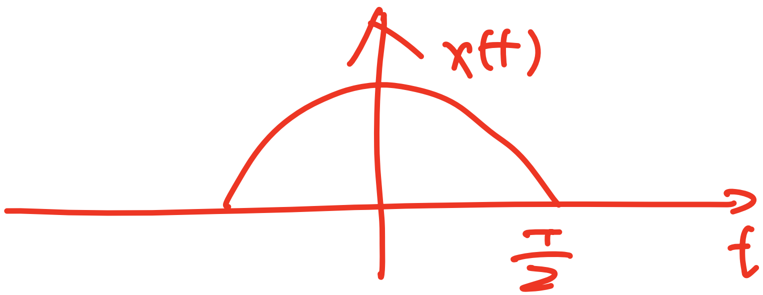
Figure: Aliasing when $1/(2T) < W$ [Gallagar'Book]

Summary

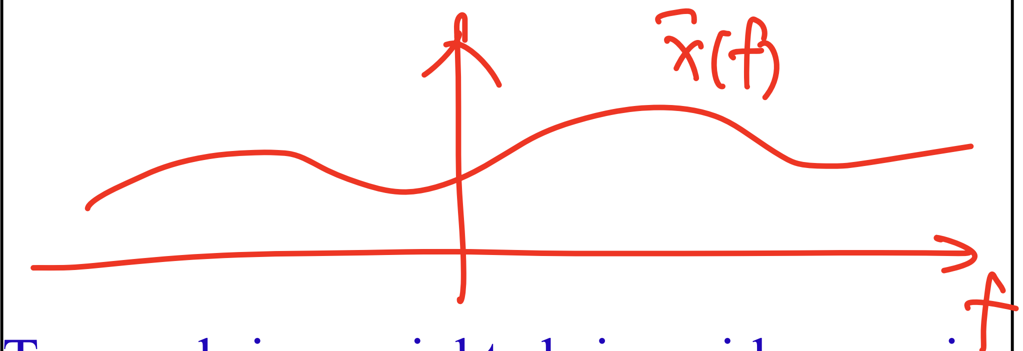
$$x_k = \frac{1}{T}\int x(t)\theta_k^*(t)dt$$

Orthogonal expansion: $x(t) = \sum_k x_k \theta_k(t) \quad x(t) \leftrightarrow \{x_k\}$

Time | **Frequency**

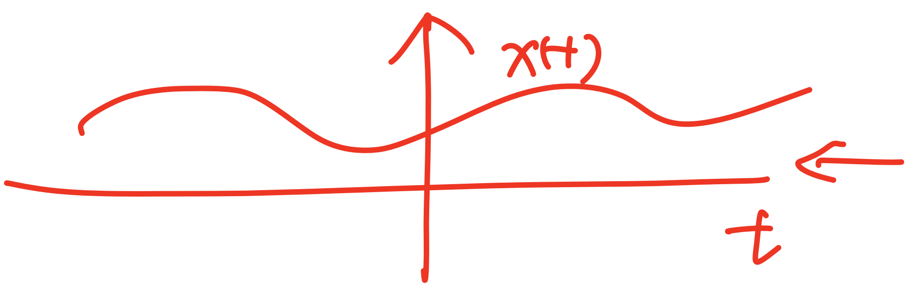


FS: $x(t) = \sum_k \hat{u}_k e^{\frac{2\pi i k t}{T}} rect(\frac{t}{T})$



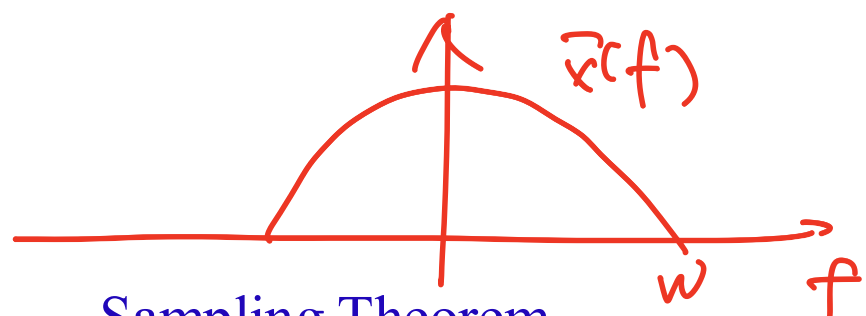
T-spaced sinc-weighted sinusoid expansion

$$x(t) = \lim \sum_{m,k} v_m(kT)\,\mathrm{sinc}(\frac{t}{T} - k) e^{2\pi i m \frac{t}{T}}$$



T-spaced truncated sinusoid expansion

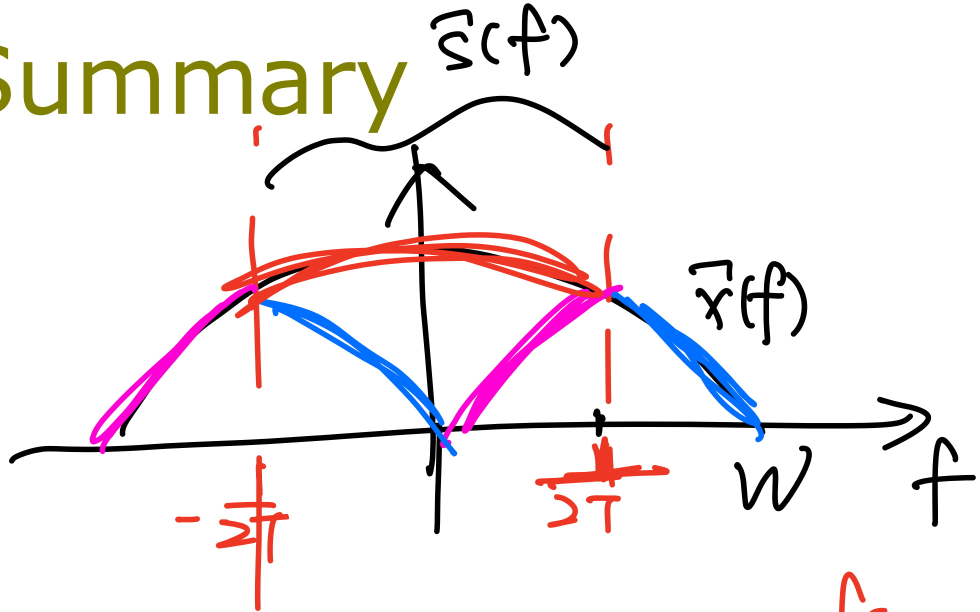
$$x(t) = \sum_m \sum_k \hat{u}_{k,m} e^{\frac{2\pi i k t}{T}} rect(\frac{t}{T} - m)$$



Sampling Theorem

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt - k)$$

Summary



$\underline{W}$, $\underline{T}$, $\underline{f_s}$

① $W > \frac{1}{2T}$ $\quad W > \frac{f_s}{2}$ $\quad \Rightarrow$ Aliasing

$T > \frac{1}{2W}$ $\quad f_s < 2W$

② $s(f) \neq x(f)$, $\quad s(kT) = x(kT), \forall k$

③ $\underline{\tilde{s}(f)}$

Thanks for your kind attention!

Questions?