EE140 Introduction to Communication Systems

Lecture 15

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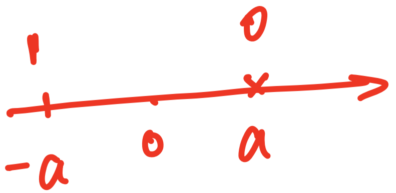
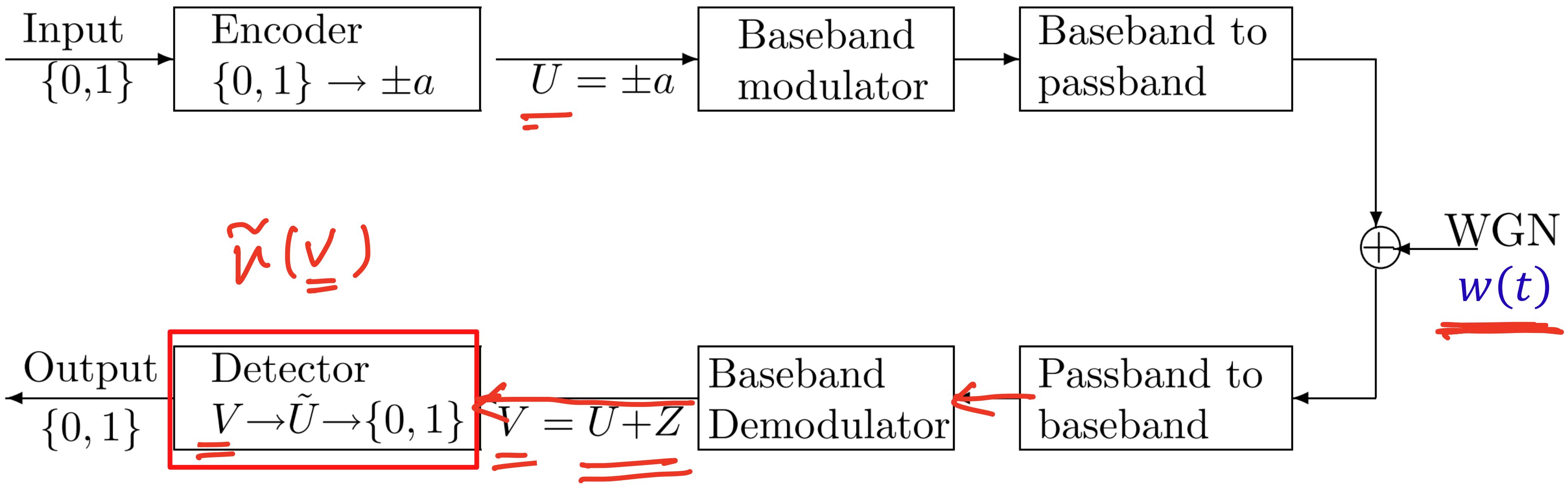
ShanghaiTech University, Fall 2025



>PAM

Detection



PAM antipodal modulation (*i.e.*, 2-PAM)

$V = U + Z$

- A detector observes a sample value v of a rv (vector, process or symbol) V, called observation, and on the basis of that observation, make a decision about the value of another rv U, with values in $\mathcal{A} = \{a_1, \ldots, a_M\}$, called M hypothesis.
- Synonyms: Decision making, hypothesis testing, decoding.

Detection

- Assume $U = a_m \in \mathcal{A}$ is sent, receiver observes $V = v$
  - Prior probability: $p_U(a_m) \triangleq p_m$
    - (the probability of the hypothesis before the observation of V)
  - Posteriori probability: $p_{U|V}(a_m|v)$
    - (the probability that a_m is the correct hypothesis conditional on the observation v)
  - Likelihood probability: $p_{V|U}(v|a_m)$
  - Criteria: maximize the correct probability, i.e., minimize the error probability
    - Maximum a posteriori probability (MAP) rule:

$$\boxed{\tilde{u}(v) = \arg\max_m [p_{U|V}(a_m \mid v)]}$$

$$\max \begin{cases} P_{U|V}(a_1|v) \\ P_{U|V}(a_2|v) \\ P_{U|V}(a_M|v) \end{cases}$$

$$\tilde{u}(v) = a_{m^*}$$

Detection

- Assume $U = a_m \in \mathcal{A}$ is sent, receiver observes $V = v$
  - Criteria: maximize the correct probability, i.e., minimize the error probability
    - Maximum a posteriori probability (MAP) rule:

$$\tilde{u}(v) = \arg\max_m [p_{U|V}(a_m \mid v)]$$

$P_{V|U}(v|a_m)P_U(a_m)$

$\Updownarrow$

$$\max_m P_{U|V}(a_m|v) = \frac{P_{U,V}(a_m,v)}{P_V(v)}$$

Theorem: The MAP rule maximizes the probability of correct decision, both for each observed sample value v and as an average over V. The MAP rule is determined solely by the joint distribution of U and V.

- Maximum likelihood (ML) rule:

$$\tilde{u}(v) = \arg\max_m \left[p_{V|U}(v|a_m)\right]$$

$$\left.\begin{matrix} P_{V|U}(v|a_1) \\ P_{V|U}(v|a_2) \\ \vdots \end{matrix}\right\} \max$$

- When prior probability is equal, MAP→ ML.

$\boxed{P_U(a_m) = \frac{1}{M}, \quad MAP \Leftrightarrow ML}$

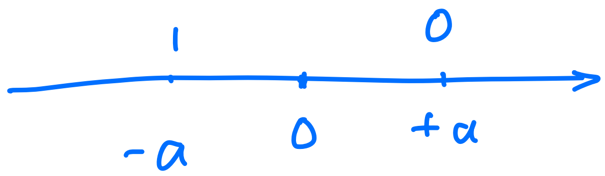
① How to MAP / ML Detection

② How to calculate Detection error probability (P_e)

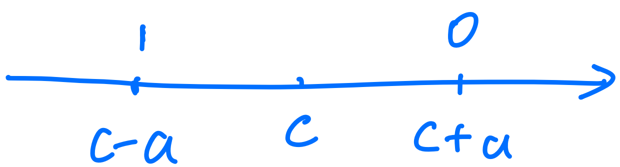
① Binary Detection in WGN

one bit $\{0, 1\}$ $M=2$

Standard 2PAM (Antipodal signal)

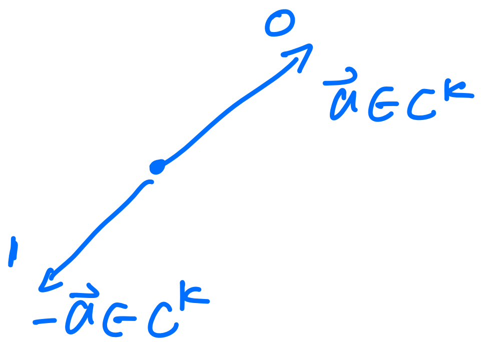


2PAM with offset (Non Antipodal signal)



Binary real vector Detection

Binary complex vector Detection



② M-ary Detection and sequence Detection

b-bit $\rightarrow \{\vec{a}_1, \vec{a}_2, \cdots \vec{a}_m\}$ $\quad \vec{a}_m \in R^n / C^n$

M-ary QAM : $\vec{a}_m \in R^2 / C^1$

M-ary Detection with arbitrary Modulation

$\Rightarrow$ Sequence Detection

Binary Detection

- Assume only one binary digit is being sent, rather than a sequence.
- Priori probability: $U = \begin{cases} 0, & p_0 \\ 1, & p_1 \end{cases}$

- constellation: $0 \to a_0, 1 \to a_1$.
- Likelihood: $f_{V|U}(v|a_0)$, $f_{V|U}(v|a_1)$
- Marginal density of V: $f_V(v) = p_0 f_{V|U}(v|a_0) + p_1 f_{V|U}(v|a_1)$
- Posteriori probability of U:

$$\boxed{p_{U|V}(a_m \mid v) = \frac{p_m f_{V|U}(v \mid a_m)}{f_V(v)}}$$

- MAP Rule:

$P(a_0|v)$ $P(a_1|v)$

$$\boxed{\frac{p_0 f_{V|U}(v \mid a_0)}{f_V(v)} \quad \overset{\tilde{U}=a_0}{\underset{\tilde{U}=a_1}{\gtrless}} \quad \frac{p_1 f_{V|U}(v \mid a_1)}{f_V(v)}}$$

Binary Detection: MAP

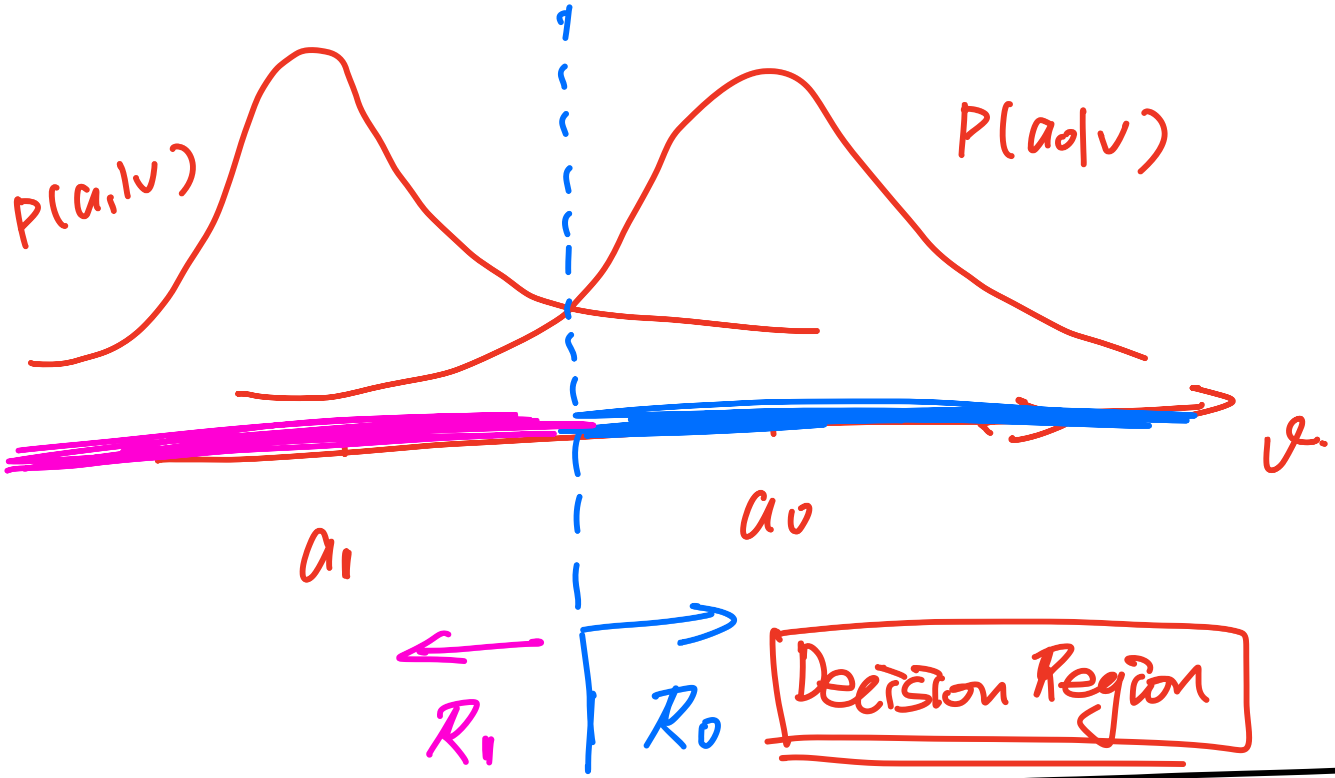
- MAP Rule:

$$\boxed{\frac{p_0 f_{V|U}(v \mid a_0)}{f_V(v)} \mathop{\gtrless}_{\tilde{U}=a_1}^{\tilde{U}=a_0} \frac{p_1 f_{V|U}(v \mid a_1)}{f_V(v)}}$$

- Likelihood ratio: $\Lambda(v) = f_{V|U}(v|a_0)/f_{V|U}(v|a_1)$, then MAP becomes

$$\boxed{\Lambda(v) = \frac{f_{V|U}(v \mid a_0)}{f_{V|U}(v \mid a_1)} \mathop{\gtrless}_{\tilde{U}=a_1}^{\tilde{U}=a_0} \frac{p_1}{p_0} = \eta}$$

- $\eta = p_1/p_0$: threshold, depends on priori probability



$$P(\hat{x}(v) = a_0)$$

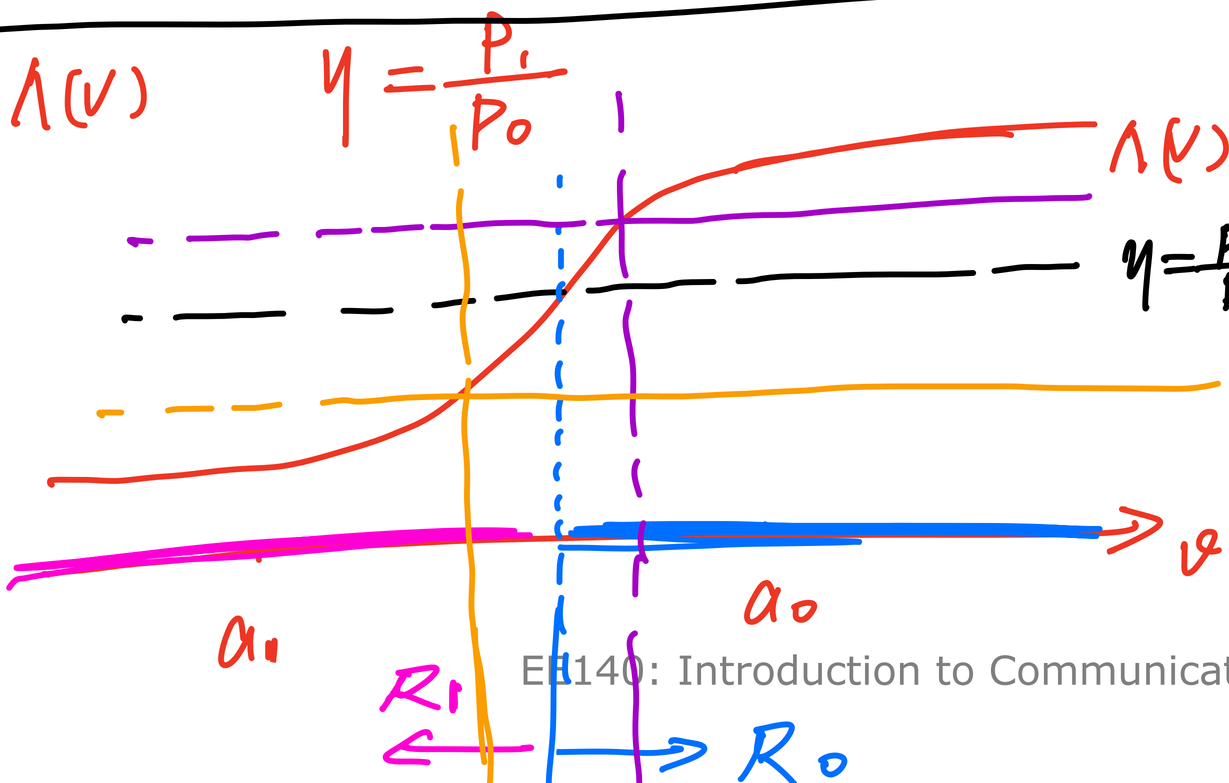
$$\Leftrightarrow P(v \in R_0)$$

$$\Leftrightarrow P\big(P(a_0|v) > P(a_1|v)\big)$$

$$P(\hat{x}(v) = a_1)$$

$$\Leftrightarrow P(v \in R_1)$$

$$\Leftrightarrow P\big(P(a_1|v) > P(a_0|v)\big)$$



$$\eta = \frac{P_1}{P_0} = 1$$

$\eta = 1$ $\quad P_1 = P_0$ (MAP $\Leftrightarrow$ ML)

$\eta > 1$ $\quad P_1 > P_0$

$\eta < 1$ $\quad P_1 < P_0$

$\mathcal{R}_1$ $\mathcal{R}_0$

$\mathcal{R}_1$ $\mathcal{R}_0$

Binary Detection: ML

- When $p_0 = p_1$, the MAP decision rule becomes ML Rule:

$$\frac{f_{V|U}(v|a_0)}{f_{V|U}(v|a_1)} \underset{\tilde{U}=a_1}{\overset{\tilde{U}=a_0}{\gtrless}} 1$$

- Error probability:

$$\Pr\{e\} = p_0 \Pr\{e \,|\, U{=}a_0\} + p_1 \Pr\{e \,|\, U{=}a_1\}$$

In radar (image detection) field

- $\Pr[e|U=0]$: probability of false alarm (no signal, but detect signal)
- $\Pr[e|U=1]$: probability of a miss (there is signal, but detect nothing)

In statistics field

- $\Pr[e|U=0]$ probability of error of the first kind
- $\Pr[e|U=1]$ probability of error of the second kind

Binary Detection: Error Probability

- Calculation of Error probability:

The MAP decision rule

$$\wedge(v) = \frac{f_{V|U}(v|0)}{f_{V|U}(v|1)} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} \frac{p_1}{p_0} = \eta$$

Partition the space of observed sample values into 2 regions:

$$R_0 = \{v : \wedge(v) \geq \eta\}, \quad R_1 = \{v : \wedge(v) < \eta\}$$

$P(e|u=a_0)$

$$\Pr[e|U=0] = \int_{R_1} f_{V|U}(v|0)dv, \quad \Pr[e|U=1] = \int_{R_0} f_{V|U}(v|1)dv$$

$= P(\hat{u}=a_1|u=a_0) = P(V \in R_1 | u=a_0)$

Or calculate by:

$$\Pr[e|U=0] = \Pr[\wedge(V) < \eta | U=0], \quad \Pr[e|U=1] = \Pr[\wedge(V) \geq \eta | U=1]$$

Binary Detection: Sufficient Statistic

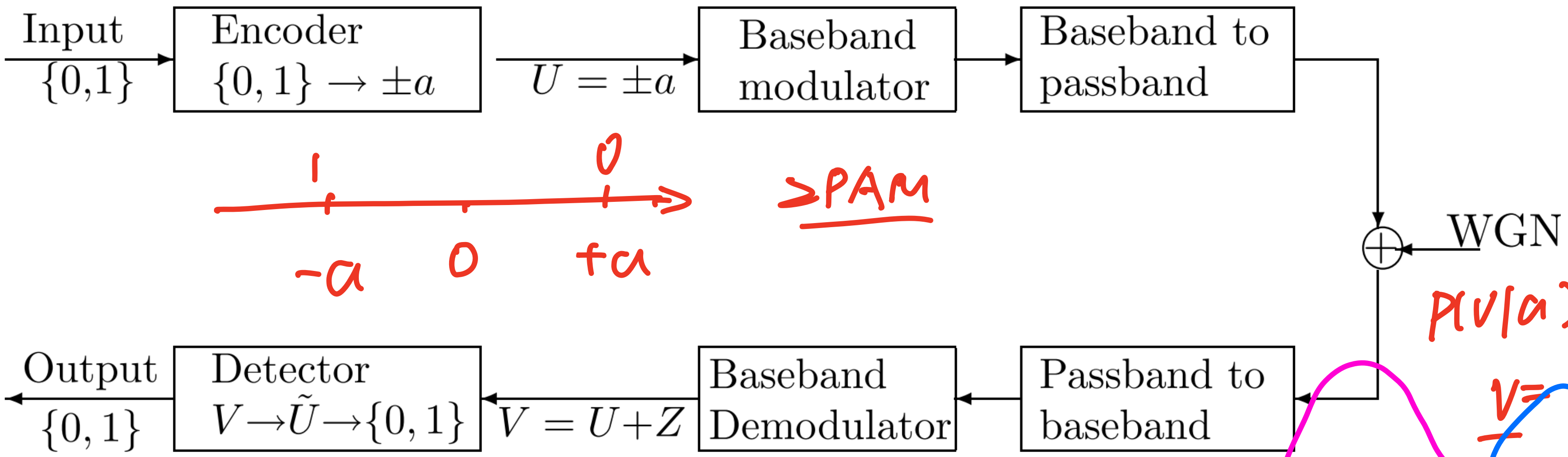
Sufficient Statistic: Given a set $\mathbf{X}$ of independent identically distributed data conditioned on an unknown parameter θ , a sufficient statistic is a function $T(\mathbf{X})$ whose value contains all the information needed to compute any estimate of the parameter.

Example: $\wedge(V)$ is sufficient statistic when doing ML estimate:

- Any function of the observation v from which the likelihood ratio can be calculated.
- $\wedge(V)$ and $\boxed{LLR(V) = \ln[\wedge(V)]}$ are sufficient statistic
- Usefulness: When calculating $\Pr[e]$, it is often simpler to work with $\wedge(V)$ than v.

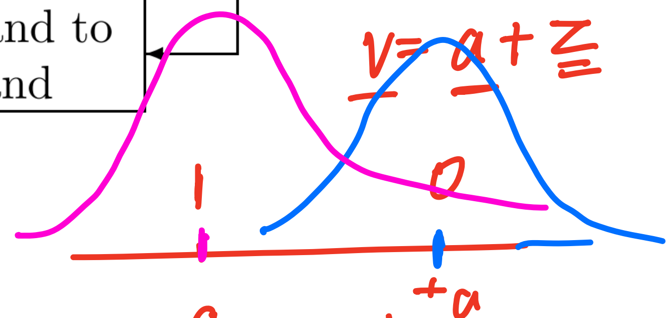
Binary Detection in WGN: Antipodal Sig

- Detection for 2PAM antipodal signals (send only a single binary symbol rather than a sequence)



$P(V|a) \sim N(a, \frac{N_0}{2})$

$V = a + Z$



$P(V|-a) \sim N(-a, \frac{N_0}{2})$

- Modulator input: $U = \pm a$ $(0 \rightarrow +a, 1 \rightarrow -a)$,
  - $p(U = +a) = p_0$, $p(U = -a) = p_1$
- Demodulator output: $V = a + Z$ *if* $U = a$ *or* $V = -a + Z$ *if* $U = -a$.
- Noise $Z \sim N(0, \frac{N_0}{2})$, independent of U.

Binary Detection in WGN: Antipodal Sig

$V = a + Z$

$V = -a + Z \quad Z \sim N(0, \frac{N_0}{2})$

- Likelihood function:

$$f_{V|U}(v \mid a) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(v-a)^2}{N_0}\right] \qquad f_{V|U}(v \mid -a) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(v+a)^2}{N_0}\right]$$

- Likelihood ratio:

$$\Lambda(v) = \exp\left[\frac{-(v-a)^2 + (v+a)^2}{N_0}\right] = \exp\left[\frac{4av}{N_0}\right]$$

- MAP rule:

$$\boxed{\exp\left[\frac{4av}{N_0}\right] \quad \substack{\geq^{\tilde{U}=a} \\ <_{\tilde{U}=-a}} \quad \frac{p_1}{p_0} = \eta}$$

- Logarithm likelihood ratio (LLR):

$$\boxed{\mathrm{LLR}(v) = \left[\frac{4av}{N_0}\right] \quad \substack{\geq^{\tilde{U}=a} \\ <_{\tilde{U}=-a}} \quad \ln(\eta)} \Longrightarrow \boxed{v \quad \substack{\geq^{\tilde{U}=a} \\ <_{\tilde{U}=-a}} \quad \frac{N_0 \ln(\eta)}{4a}}$$

Binary Detection in WGN: Antipodal Sig

$V = \frac{N_0 \ln(\eta)}{4a}$

- MAP rule:

$$\boxed{v \underset{\tilde{U}=-a}{\overset{\tilde{U}=a}{\gtrless}} \frac{N_0 \ln(\eta)}{4a}}$$

$\eta = \frac{P_1}{P_0} = 1 \quad P_1 = P_0 \quad \text{MAP=ML}$

minimum Distance Det.

$P_1 > P_0 \quad \eta > 1$

$P_1 < P_0 \quad \eta < 1$

$$\eta = \frac{p_1}{p_0} = \frac{p_U(-a)}{p_U(a)}$$

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$P(e \mid u=-a) = P(\tilde{u}(v) = a \mid u=-a)$

$= P(v \in R_0 \mid u=-a)$

$= P(v \geq \frac{N_0 \ln \eta}{4a} \mid u=-a)$

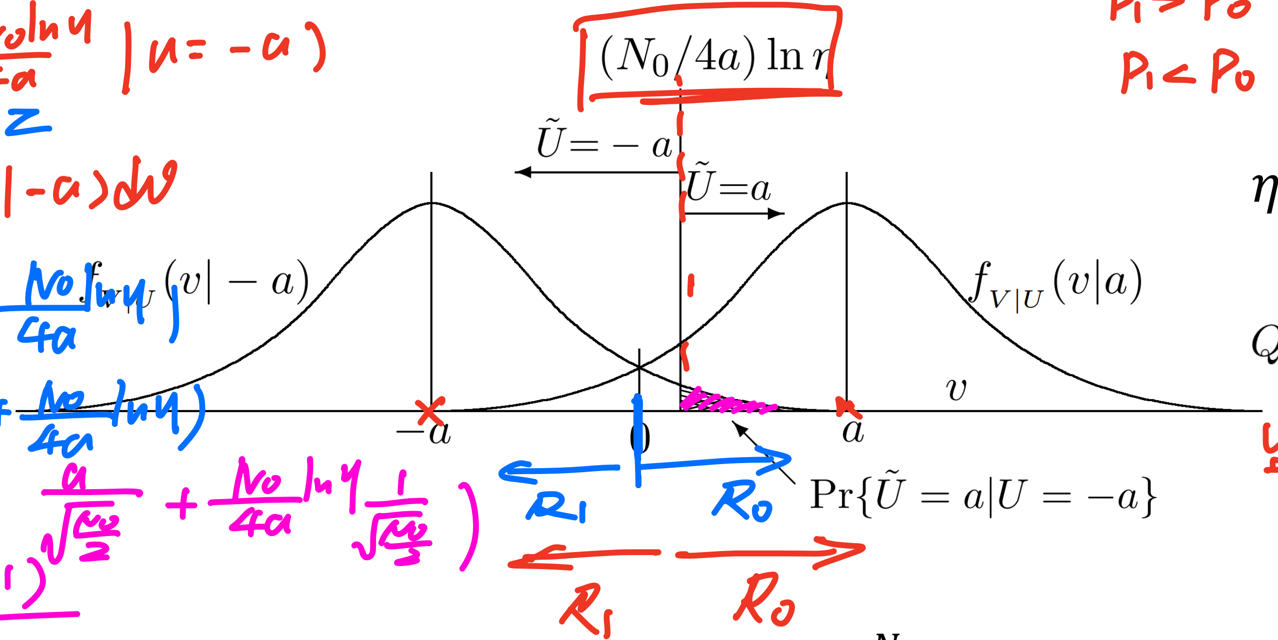
$v = -a+z$

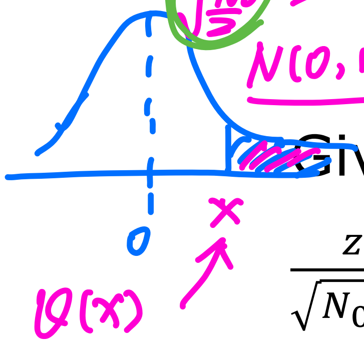
$= \int_{R_0} f(v \mid -a) dv$

$= P(-a+z \geq \frac{N_0 \ln \eta}{4a})$

$= P(z \geq a + \frac{N_0 \ln \eta}{4a})$

$= P\left(\frac{z}{\sqrt{\frac{N_0}{2}}} \geq \frac{a}{\sqrt{\frac{N_0}{2}}} + \frac{N_0 \ln \eta}{4a} \frac{1}{\sqrt{\frac{N_0}{2}}}\right)$





Given $U = -a$, error occurs if $v \geq \frac{N_0}{4a} \ln \eta \Leftrightarrow z \geq a + \frac{N_0}{4a} \ln \eta \Leftrightarrow$

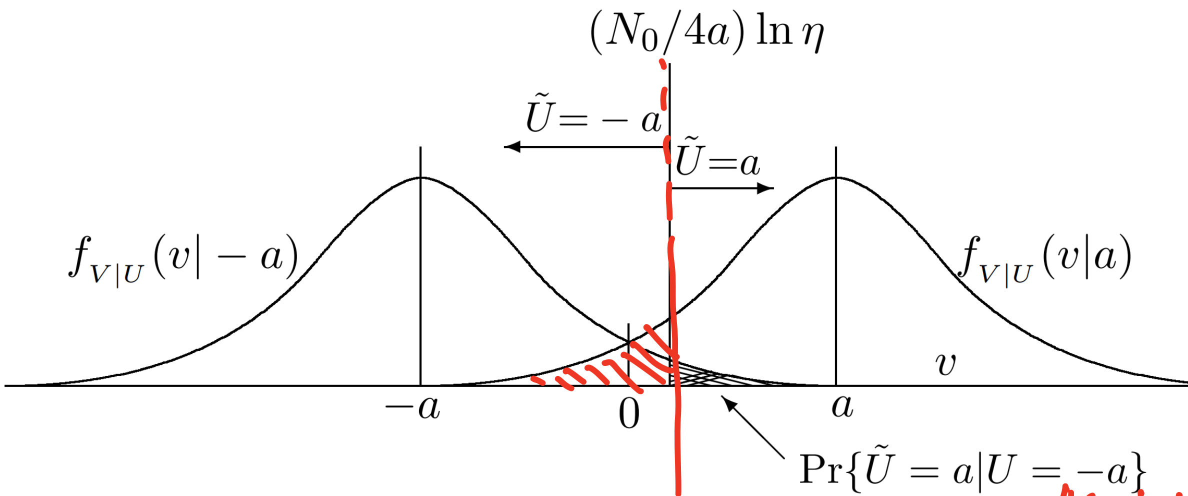
$\frac{z}{\sqrt{N_0/2}} \geq \frac{a}{\sqrt{N_0/2}} + \frac{\sqrt{N_0/2}}{2a} \ln \eta$, thus $\Pr\{e \mid U = -a\} = Q\left(\frac{a}{\sqrt{N_0/2}} + \frac{\sqrt{N_0/2}\ln\eta}{2a}\right)$

$\left(\sqrt{\frac{2a^2}{N_0}} + \_\_\_\_ \right)$

Binary Detection in WGN: Antipodal Sig

- MAP rule:

$$\boxed{v \;\underset{\tilde{U}=-a}{\overset{\tilde{U}=a}{\gtrless}}\; \frac{N_0 \ln(\eta)}{4a}}$$



$$\eta = \frac{p_1}{p_0} = \frac{p_U(-a)}{p_U(a)}$$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$V = a+z$ $\qquad z \le \frac{N_0}{4a}\ln\eta - a$

Given $U=a$, error occurs if $v \le \frac{N_0}{4a}\ln\eta \Leftrightarrow -z \ge a - \frac{N_0}{4a}\ln\eta \Leftrightarrow$ $\frac{-z}{\sqrt{N_0/2}} \ge \frac{a}{\sqrt{\frac{N_0}{2}}} - \frac{\sqrt{N_0/2}}{2a}\ln\eta$, thus $\Pr\{e\,|\,U=a\} = Q\left(\frac{a}{\sqrt{N_0/2}} - \frac{\sqrt{N_0/2}\ln\eta}{2a}\right)$

$Q\left(\sqrt{\frac{2a^2}{N_0}} - \_\_\_\_\right)$

Binary Detection in WGN: Antipodal Sig

$= P_0 a^2 + P_1 a^2$

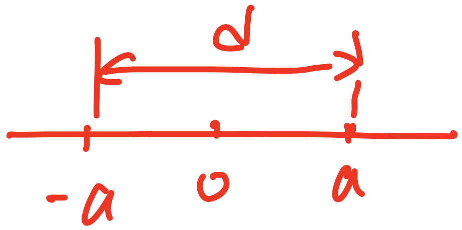
- The energy per bit is $E_b = a^2$
- For communication, usually assume $p_0 = p_1$, so $\eta = 1$
- ML=MAP, we have

$$\boxed{\Pr\{e\} = \Pr\{e \,|\, U{=}-a\} = \Pr\{e \,|\, U{=}a\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

$Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$

- Only ratio E_b/N_0 can be relevant, since both can be scaled together.

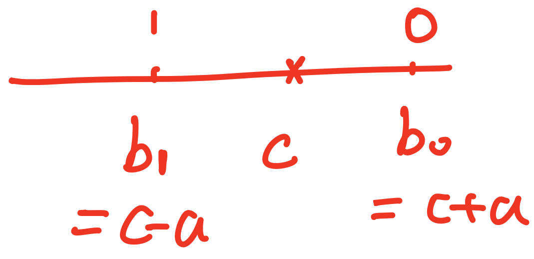
SNR $\uparrow$ Pe $\downarrow$



$P_0 = P_1 \Rightarrow a = \frac{d}{2}$ $\quad$ $Pe = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

$d \uparrow \begin{cases} E_s \uparrow \\ Pe \downarrow \end{cases}$

Binary Detection in WGN: Non-Antipodal



- Detection for 2PAM non-antipodal signals (send only a single binary symbol rather than a sequence)
- Modulator input: $U = \{b_0, b_1\}\ (0 \to b_0, 1 \to b_1)$,
  - $p(U = b_0) = p_0,\ p(U = b_1) = p_1$
  - Center point $c = \frac{b_0 + b_1}{2}$
  - $a = b_0 - c = c - b_1 (b_1 < b_0)$
- Demodulator output: $V = c + a + Z\ if\ U = b_0\ \ or\ V = c - a + Z\ if\ U = b_1$.

  $c + a$
- Noise $Z \sim N(0, \frac{N_0}{2})$, independent of U.
- Define $\tilde{V} = V - c = \pm a + Z$. Estimate $\tilde{U} = a\ or - a$ based on $\tilde{V}$. Map $a \to b_0, -a \to b_1$.
- Compared to antipodal case, shift the result by constant c.

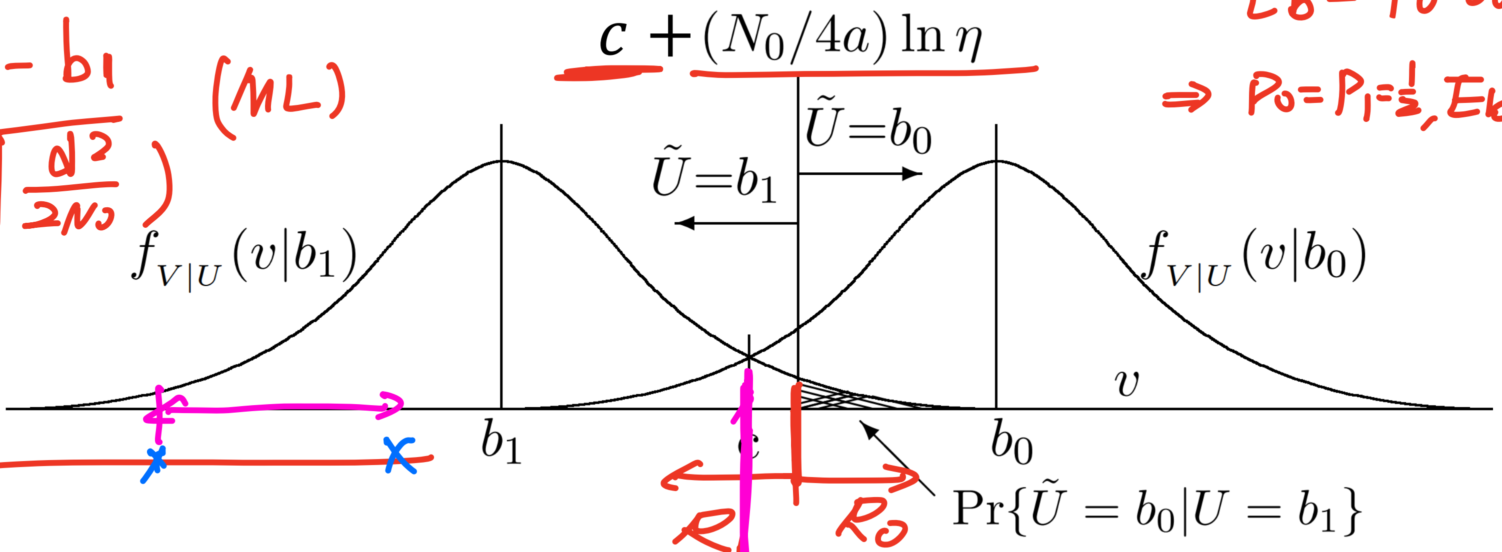
Binary Detection in WGN: Non-Antipodal

- The error probability

$d = b_0 - b_1$

(ML)

$P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$



$E_b = P_0 b_0^2 + P_1 b_1^2$

$\Rightarrow P_0 = P_1 = \frac{1}{2},\ E_b = \frac{1}{2}(a+c)^2 + \frac{1}{2}(c-a)^2 = a^2 + c^2$

$\gamma = \frac{a^2}{a^2+c^2}$

$a^2 = \gamma \cdot E_b$

$$\Pr\{e\,|\,U=-a\} = Q\left(\frac{a}{\sqrt{N_0/2}} + \frac{\sqrt{N_0/2}\ln\eta}{2a}\right)$$

$$\Pr\{e\,|\,U=a\} = Q\left(\frac{a}{\sqrt{N_0/2}} - \frac{\sqrt{N_0/2}\ln\eta}{2a}\right)$$

$P_e = Q\left(\sqrt{\frac{2a^2}{N_0}}\right) = Q\left(\sqrt{\frac{2\gamma E_b}{N_0}}\right)$

Binary Detection in WGN: Non-Antipodal

$P_1 = P_0 = \frac{1}{2}$

- The energy per bit: $E_b = \frac{b_0^2+b_1^2}{2} = a^2 + c^2$. Let $\gamma = \frac{a^2}{a^2+c^2}$. We have

$$\Pr\{e \,|\, U{=}b_1\} = Q\left(\sqrt{\frac{2\gamma E_b}{N_0}} + \frac{\ln \eta}{2\sqrt{2\gamma E_b/N_0}}\right)$$

$$\Pr\{e \,|\, U{=}b_0\} = Q\left(\sqrt{\frac{2\gamma E_b}{N_0}} - \frac{\ln \eta}{2\sqrt{2\gamma E_b/N_0}}\right)$$

- For ML, we have

$$\Pr[e|U=1] = \Pr[e|U=0] = Q\left(\sqrt{\frac{2\gamma E_b}{N_0}}\right)$$

- γ is the fraction of energy E_b used for signal

$P_0 = P_1 = \frac{1}{2}$

$E_b = P_0 4a^2 + P_1 0$

$= 2a^2$

$\gamma = \frac{a^2}{2a^2} = \frac{1}{2}$

1 0

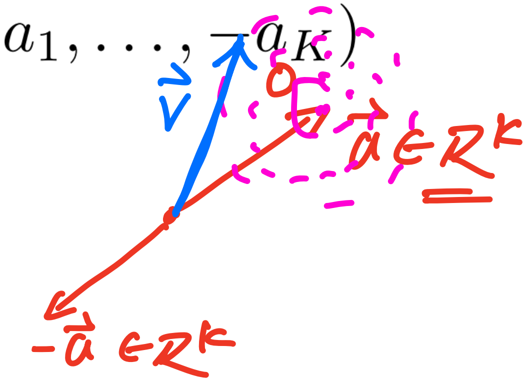
0 2a

- Q: $\Pr[e]$ for on-off keying scheme: $0 \to 2a, 1 \to 0$.
- Ans: In this case, $\gamma = ½$. For ML, the probability of error then becomes $Q(\sqrt{E_b/N_0})$.

$Q\sqrt{\frac{2\gamma E_b}{N_0}} \overset{\gamma=\frac{1}{2}}{=\!=} Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

Binary Vector Detection in WGN

- $U = 0 \rightarrow \boldsymbol{a} = (a_1, \ldots, a_K)$ and $U = 1 \rightarrow -\boldsymbol{a} = (-a_1, \ldots, -a_K)$



$$\boldsymbol{V} = \pm\boldsymbol{a} + \boldsymbol{Z}$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_K)$, i.i.d, $Z_i \sim \mathcal{N}(0, N_0/2)$.

$$f_{V|U}(\boldsymbol{v}|\boldsymbol{a}) = \frac{1}{\pi N_0^{k/2}} e^{\frac{-\|\boldsymbol{v}-\boldsymbol{a}\|^2}{N_0}}, \quad f_{V|U}(\boldsymbol{v}|-\boldsymbol{a}) = \frac{1}{\pi N_0^{k/2}} e^{\frac{-\|\boldsymbol{v}+\boldsymbol{a}\|^2}{N_0}}$$

$$= \prod_{i=1}^{k} f(v_i|a_i)$$

$$= \prod_{i=1}^{k} N(a_i, \tfrac{N_0}{2})$$

$$\text{LLR}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v}-\boldsymbol{a}\|^2 + \|\boldsymbol{v}+\boldsymbol{a}\|^2}{N_0} = \frac{4\langle \boldsymbol{v}, \boldsymbol{a}\rangle}{N_0}$$

- MAP test:

$$\text{LLR}(\boldsymbol{v}) = \frac{4\langle \boldsymbol{v}, \boldsymbol{a}\rangle}{N_0} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} \ln\frac{p_1}{p_0} = \ln\eta$$

- In other words, $\langle \boldsymbol{v}, \boldsymbol{a}\rangle$ is a sufficient statistic.

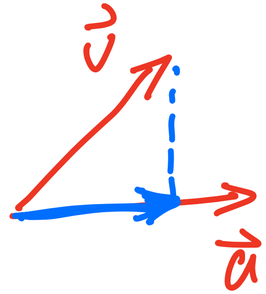
Binary Vector Detection in WGN

MAP test:

$$\mathsf{LLR}(\boldsymbol{v}) = \frac{4\langle \boldsymbol{v}, \boldsymbol{a}\rangle}{N_0} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} \ln\frac{p_1}{p_0} = \ln\eta$$

Equivalent to:

$$\frac{\langle \boldsymbol{v}, \boldsymbol{a}\rangle}{\|\boldsymbol{a}\|} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}$$



ML test:

$$\langle \boldsymbol{v}, \boldsymbol{a}\rangle \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} 0$$

$$\vec{v}|_{\vec{a}} = \frac{\langle \vec{v}, \vec{a}\rangle}{\|\vec{a}\|} \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

- The projection of $\boldsymbol{v}$ onto $\boldsymbol{a}$ is $\frac{\langle \boldsymbol{v},\boldsymbol{a}\rangle}{\|\boldsymbol{a}\|}\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$
- Decision only depends on the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{a}$
- $\langle \boldsymbol{v}, \boldsymbol{a}\rangle = ||\boldsymbol{a}||^2 + \langle \boldsymbol{z}, \boldsymbol{a}\rangle$
- Only the noise in the direction of the signal should be relevant in detecting the signal.

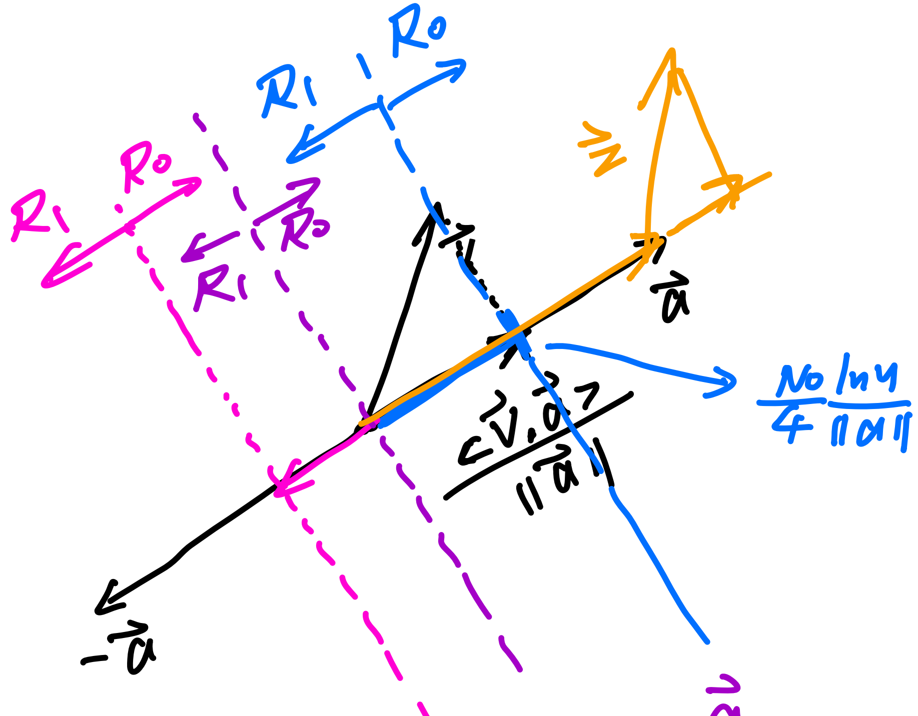
$$\underbrace{\frac{\langle \vec{v},\vec{a}\rangle}{\|\vec{a}\|}}_{V'} \underset{\check{u}=-\vec{a}}{\overset{\check{u}=\vec{a}}{\gtrless}} \frac{N_0 \ln \gamma}{4\|a\|} \qquad \gamma=\frac{P_1}{P_0}$$

If $\vec{u}=\vec{a}$

$$\frac{\langle \vec{v},\vec{a}\rangle}{\|\vec{a}\|} = \frac{\langle \vec{a}+\vec{z},\vec{a}\rangle}{\|\vec{a}\|} = \|\vec{a}\| + \frac{\langle \vec{z},\vec{a}\rangle}{\|\vec{a}\|}$$

$$\Downarrow$$

$$\|Z|\vec{a}\|$$



① $P_1>P_0$ $\ln\gamma>0$ $\quad$ $P_1=P_0$ $\ln\gamma=0$ $\quad$ $\langle \vec{v},\vec{a}\rangle \underset{-\vec{a}}{\overset{\vec{a}}{\gtrless}} 0$ (minimum Distance Detection)

$P_1<P_0$ $\ln\gamma<0$

$$\|\vec{v}-\vec{a}\| \underset{-\vec{a}}{\overset{\vec{a}}{\lessgtr}} \|v+\vec{a}\|$$

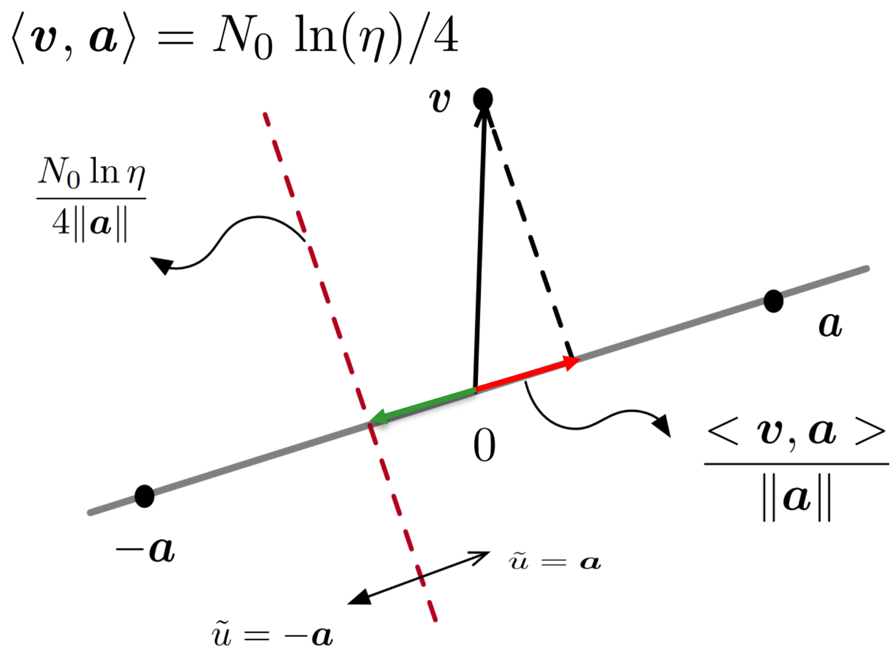
② $\vec{v} = \underbrace{\vec{v}|\vec{a}}_{V} + \underbrace{\vec{v}\perp\vec{a}}_{X}$ (Theorem of irrelevance)

$\vec{z} = \underbrace{\vec{z}|\vec{a}}_{V} + \underbrace{\vec{z}\perp\vec{a}}_{X}$

③ Vector Detection $\Leftrightarrow$ $\frac{\text{Scalar Detection}}{V'}$

Binary Vector Detection in WGN

- MAP test

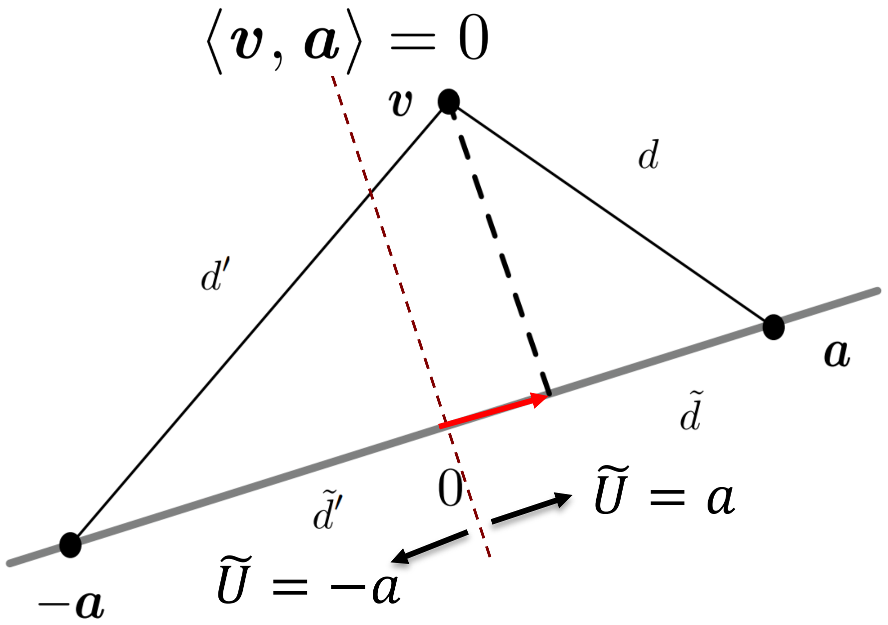


$$\frac{\langle \boldsymbol{v}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}$$

  - The noise is spherically symmetric around the origin

Binary Vector Detection in WGN

- ML test

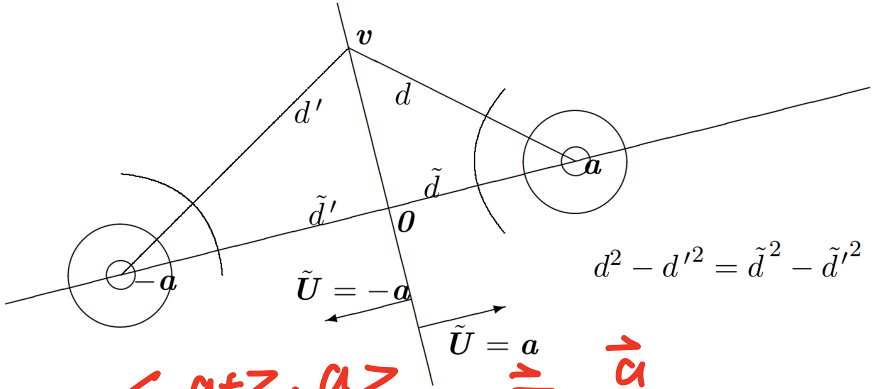


$$\langle \boldsymbol{v}, \boldsymbol{a} \rangle \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} 0$$

$$\mathrm{LLR}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v}-\boldsymbol{a}\|^2 + \|\boldsymbol{v}+\boldsymbol{a}\|^2}{N_0} \underset{\hat{U}=1}{\overset{\hat{U}=0}{\gtrless}} 0$$

  - The likelihoods depend only on the distance from the origin.
  - $d = \big||\boldsymbol{v} - \boldsymbol{a}|\big|, d' = \big||\boldsymbol{v} + \boldsymbol{a}|\big|$
  - If d>d', then $\widetilde{U} = -a$, vice versa. Choose estimate which is closer to $\boldsymbol{v}$. (minimum distance rule)

Binary Vector Detection in WGN



- Error Probability

$$\frac{\langle \boldsymbol{v}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|} \quad \underset{<_{\tilde{U}=-\boldsymbol{a}}}{\ge^{\tilde{U}=\boldsymbol{a}}} \quad \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}$$

  – Given $\boldsymbol{U} = -\boldsymbol{a}$, $\boldsymbol{V} = -\boldsymbol{a} + \boldsymbol{Z}$. Thus $\frac{<a+z,\, a>}{\|a\|}$

$$\frac{<\boldsymbol{V},\boldsymbol{a}>}{\|\boldsymbol{a}\|} = -\|\boldsymbol{a}\| + \boxed{< \boldsymbol{Z}, \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} >}$$

$<\vec{Z}, \frac{\vec{a}}{\|a\|}>$ $\quad Z_i \sim N(0, \frac{N_0}{2})$

$= \frac{\sum_{i=1}^{k} Z_i a_i}{\|a\|}$ $\quad \frac{\sum_{i=1}^{k} a_i^2 \cdot \frac{N_0}{2}}{\|a\|^2}$ Variance

  – Z_k is $\mathcal{N}(0, N_0/2)$. $\frac{<\boldsymbol{V},\boldsymbol{a}>}{\|\boldsymbol{a}\|}$ is $\mathcal{N}(-\|\boldsymbol{a}\|, N_0/2)$. $Z \sim N(0, \frac{N_0}{2})$

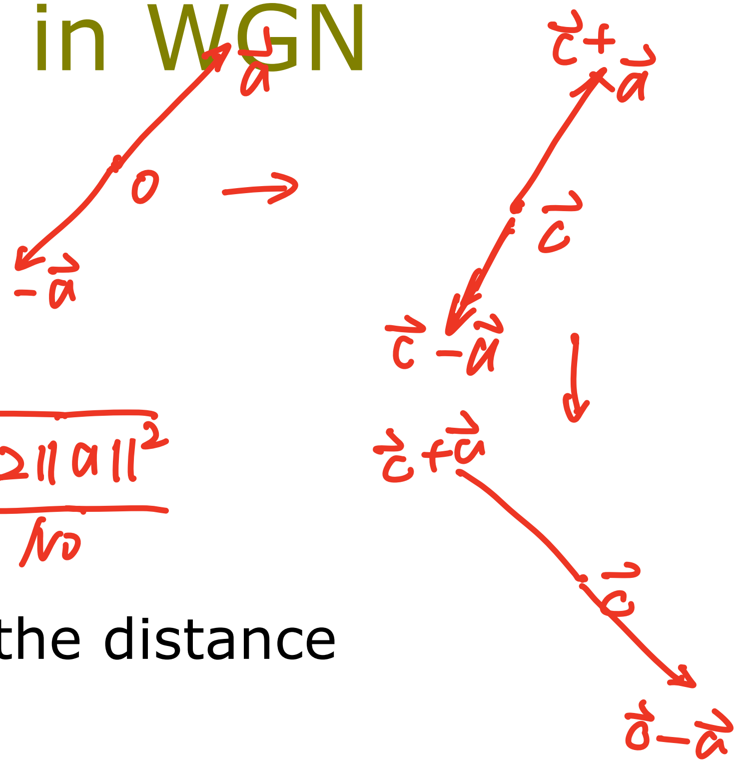
$V = -a + Z$

  – $\Pr\{e|\boldsymbol{U} = -\boldsymbol{a}\} \Leftrightarrow \Pr\left\{\frac{<\boldsymbol{V},\boldsymbol{a}>}{\|\boldsymbol{a}\|} \ge \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}\right\} \Leftrightarrow \Pr\left\{Z \ge \|\boldsymbol{a}\| + \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}\right\}$

$$\Pr\{e \mid \boldsymbol{U} = -\boldsymbol{a}\} = Q\left(\sqrt{\frac{2\|\boldsymbol{a}\|^2}{N_0}} + \frac{\ln \eta}{2\sqrt{2\|\boldsymbol{a}\|^2/N_0}}\right)$$

$$\Pr\{e \mid \boldsymbol{U} = \boldsymbol{a}\} = Q\left(\sqrt{\frac{2\|\boldsymbol{a}\|^2}{N_0}} - \frac{\ln \eta}{2\sqrt{2\|\boldsymbol{a}\|^2/N_0}}\right)$$

Binary Vector Detection in WGN



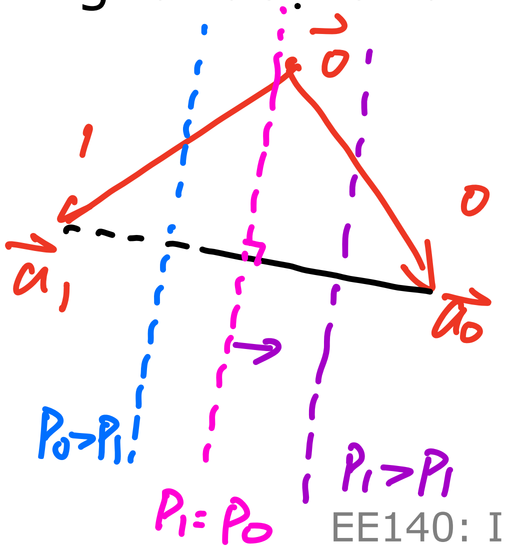
- Error Probability
  - The energy per bit: $\mathrm{E_b} = \|\boldsymbol{a}\|^2$
  - If $\eta = 1$, we have

$P_0 = P_1$, ML

$$\Pr\{e\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$Q\sqrt{\frac{2\|a\|^2}{N_0}}$$

  - The error probability only depends on the distance $2\|\boldsymbol{a}\|$ between the signals.
  - The component of $\boldsymbol{v}$ in directions orthogonal to the signal do not affect the LLR (theorem of irrelevance)



$P_1 = P_0$ $\quad P_e = Q\left(\sqrt{\frac{2\left\|\vec{a}_0 - \frac{\vec{a}_0+\vec{a}_1}{2}\right\|^2}{N_0}}\right)$

$d = \|\vec{a}_0 - \vec{a}_1\|$

$\Downarrow$

$P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

Binary Complex Vector Detection in WGN

- Input $\boldsymbol{U} = \boldsymbol{a}$ with p_0, $\boldsymbol{U} = -\boldsymbol{a}$ with p_1, $\boldsymbol{a}$ is a complex random n-vector.
- Observation is a complex random n-vector: $\boldsymbol{V} = \boldsymbol{U} + \boldsymbol{Z}$
- $\boldsymbol{Z}$ is complex random vectors with $\mathcal{R}(Z_k) \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$, $\mathcal{I}(Z_k) \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$, thus $Z_k \sim \mathcal{CN}(0, N_0)$.

$Z_k = \mathcal{R}(Z_k) + j\,\mathcal{I}(Z_k)$

- Likelihood:

$$f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}) = f_{\boldsymbol{V}'|\boldsymbol{U}'}(\boldsymbol{v}'|\boldsymbol{a}') = \frac{1}{(\pi N_0)^n} \exp \sum_{k=1}^{n} \frac{-\Re(v_k - a_k)^2 - \Im(v_k - a_k)^2}{N_0}$$

$-\dfrac{\|\vec{v} - \vec{a}\|^2}{N_0}$

$$f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|-\boldsymbol{a}) = f_{\boldsymbol{V}'|\boldsymbol{U}'}(\boldsymbol{v}'|-\boldsymbol{a}') = \frac{1}{(\pi N_0)^n} \exp \sum_{k=1}^{n} \frac{-\Re(v_k + a_k)^2 - \Im(v_k + a_k)^2}{N_0}.$$

$-\dfrac{\|\vec{v} + \vec{a}\|^2}{N_0}$

$$\text{LLR}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v} - \boldsymbol{a}\|^2 + \|\boldsymbol{v} + \boldsymbol{a}\|^2}{N_0}$$

$\|\vec{v} - \vec{a}\|^2$
$= \|\vec{v}\|^2 + \|\vec{a}\|^2 - 2\,\mathcal{R}(\langle \vec{v}, \vec{a} \rangle)$

- MAP test: $$\boxed{\frac{\Re[\langle \boldsymbol{v}, \boldsymbol{a} \rangle]}{\|\boldsymbol{a}\|}} \;\underset{<_{\tilde{U}=-\boldsymbol{a}}}{\geq^{\tilde{U}=\boldsymbol{a}}}\; \frac{N_0 \ln(\eta)}{4\|\boldsymbol{a}\|}$$
  - Error probabilities are the same.

M-ary Detection in WGN

- Input: n-vector $\boldsymbol{U} \in (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_M)$ with a priori proba. $p_1, \ldots, p_M$
- Observation: n-vector $\boldsymbol{V}$
- Posteriori probability: $p_{\boldsymbol{U}|\boldsymbol{V}}(\boldsymbol{a}_m|\boldsymbol{v})$
- Likehood: $f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}_m)$

MAP rule:

$$\begin{aligned}\hat{\boldsymbol{U}}(v) &= \arg\max_{\boldsymbol{a}_m} \; p_{\boldsymbol{U}|\boldsymbol{V}}\{\boldsymbol{a}_m|\boldsymbol{v}\} \\ &= \arg\max_{\boldsymbol{a}_m} \; p_m f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}_m)\end{aligned}$$

M-ary rule is multiple binary hypothesis testing problems: $\hat{\boldsymbol{U}}(v) = \boldsymbol{a}_m$ if for all m'

$$\wedge_{m,m'}(\boldsymbol{v}) = \frac{f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}_m)}{f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}_{m'})} \geq \frac{p_{m'}}{p_m}$$

$a_{m'} \in A / a_m$

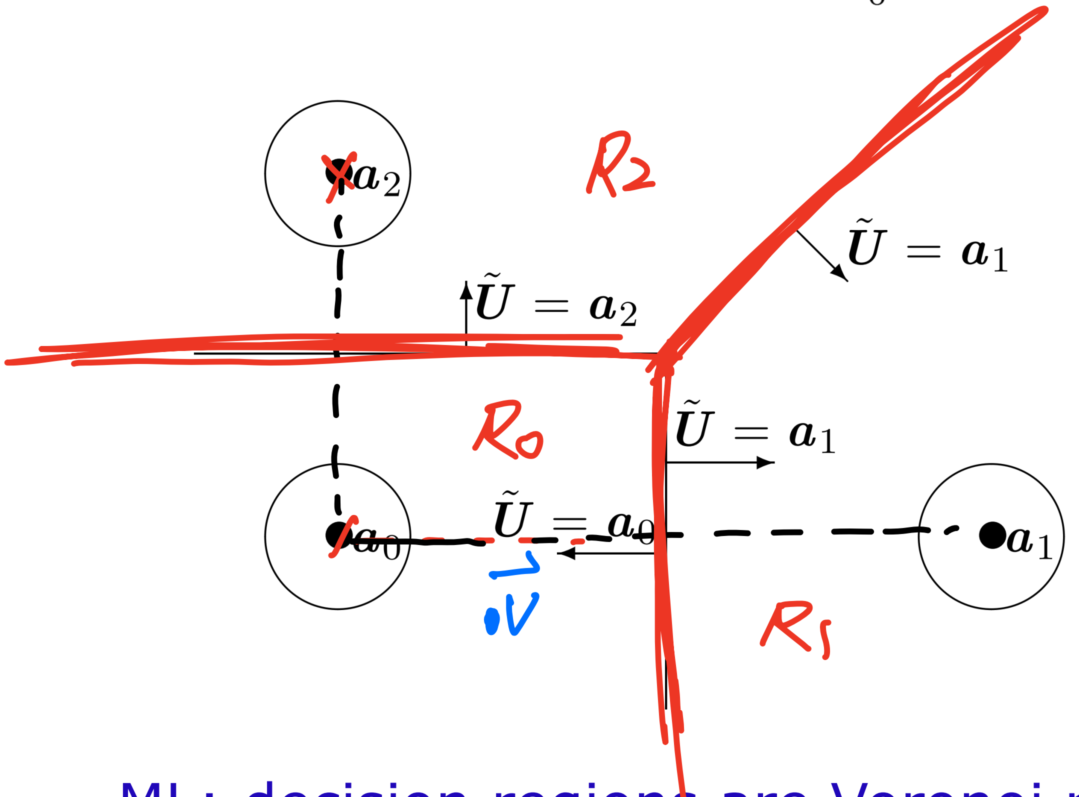
M-ary Complex Vector Detection in WGN

$$\boldsymbol{V} = \boldsymbol{U} + \boldsymbol{Z}$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_K)$, i.i.d, $\mathrm{Re}[Z_i], \mathrm{Im}[Z_i] \sim \mathcal{N}(0, N_0/2)$.

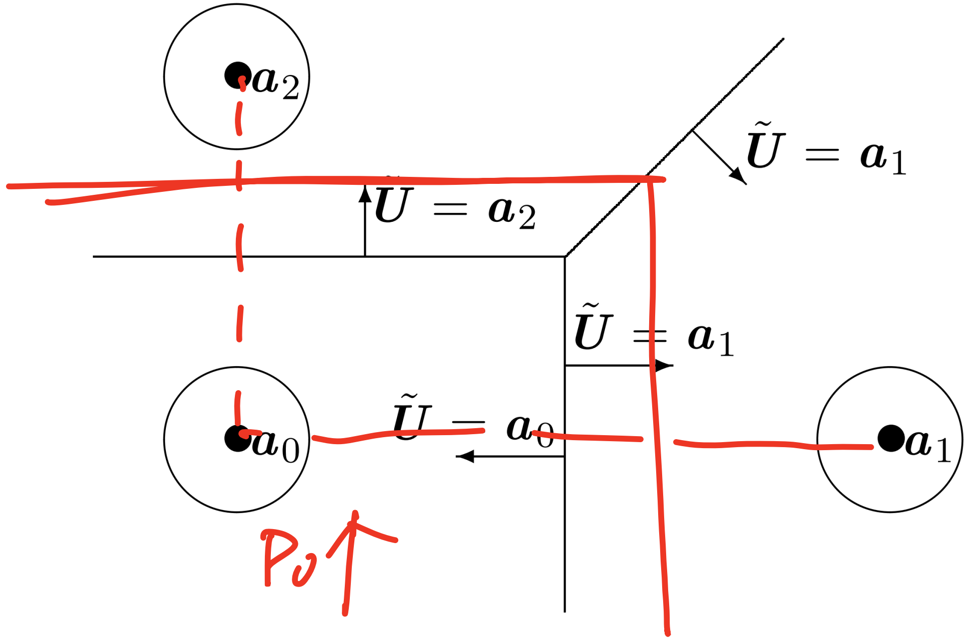
$$\mathrm{LLR}_{m,m'}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v} - \boldsymbol{a}_m\|^2 + \|\boldsymbol{v} - \boldsymbol{a}_{m'}\|^2}{N_0}$$

$M=3$



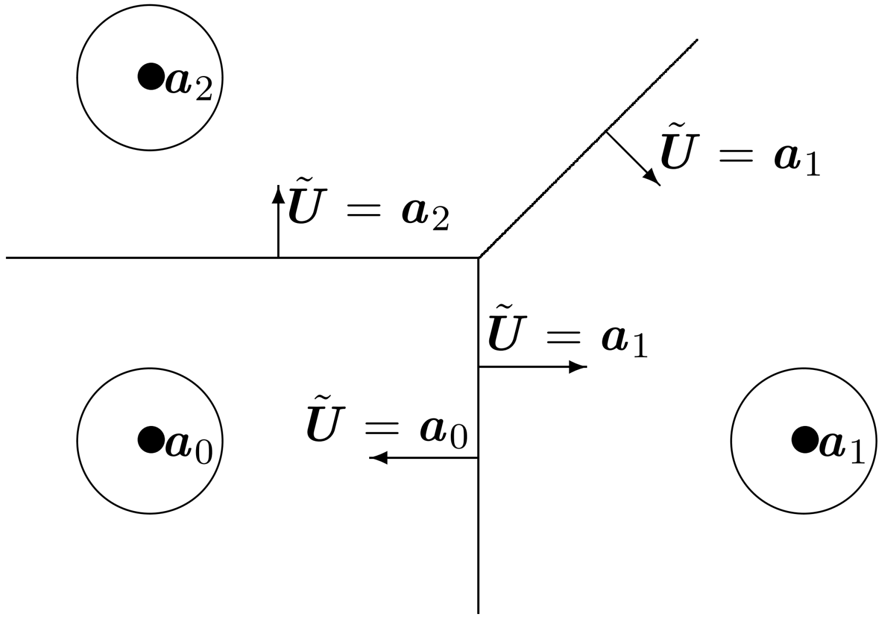
ML: decision regions are Voronoi regions

M-ary Complex Vector Detection in WGN



- For ML, each binary test separates the observation space into two regions separated by the perpendicular bisector between the two points.
- If $\{p_1, \ldots, p_M\}$ are unequal, perpendicular bisectors are shifted.

M-ary Complex Vector Detection in WGN



Union bound:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

- Error probability:

Given $\boldsymbol{U} = \boldsymbol{a}_m$,

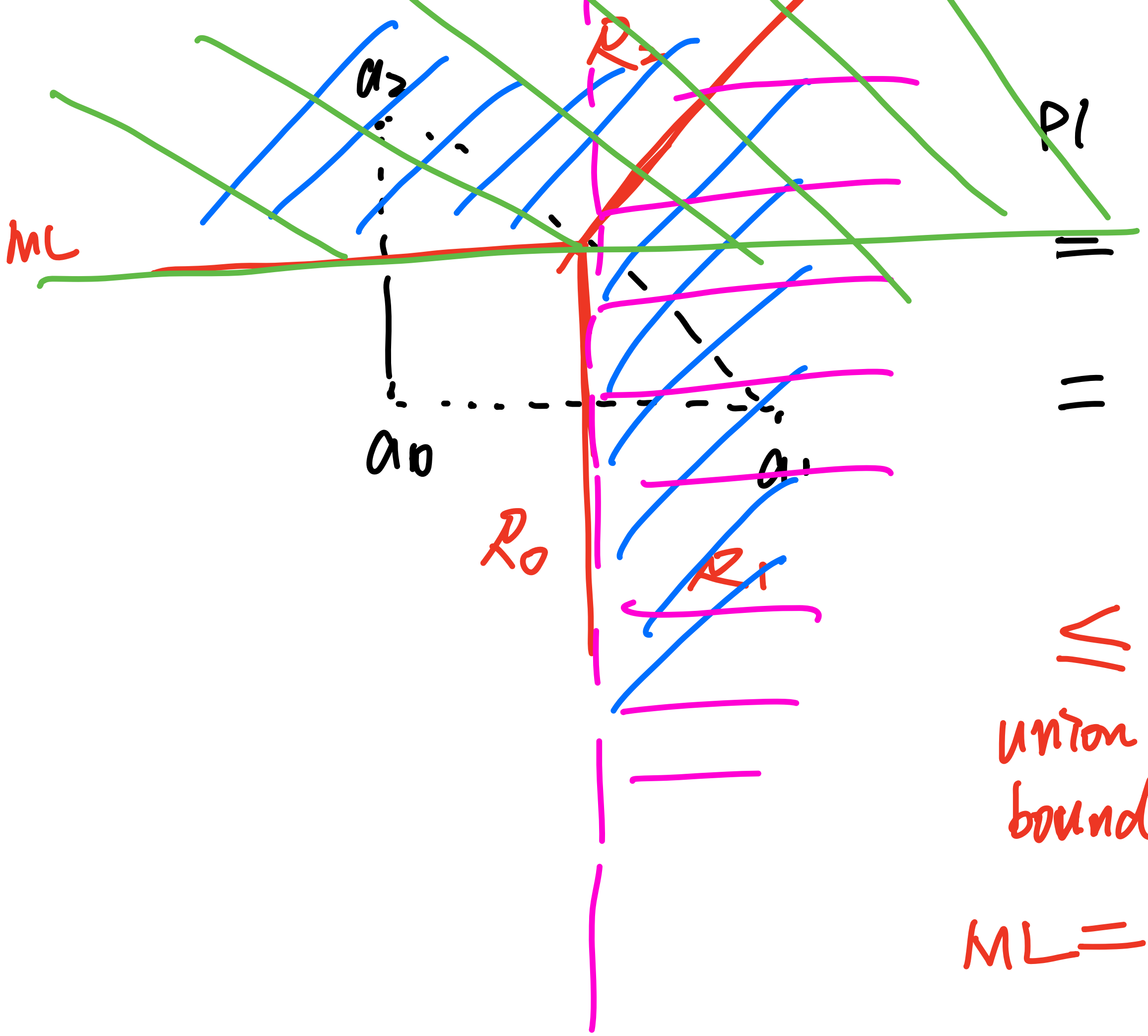
$$\Pr[e|\boldsymbol{U} = \boldsymbol{a}_m] = \Pr\left[\cup_{m' \neq m} \left(\tilde{\boldsymbol{U}}(\boldsymbol{v}) = \boldsymbol{a}_{m'} | \boldsymbol{U} = \boldsymbol{a}_m\right)\right]$$

$$\leq \sum_{m' \neq m} \Pr\left(\tilde{u}(v) = a_{m'} | U = a_m\right)$$

$$Q\left(\sqrt{\frac{\|a_m - a_{m'}\|^2}{2N_0}} + 0\right)$$

Thus,

$$\Pr[e] = \sum_m p_m \Pr[e|\boldsymbol{U} = \boldsymbol{a}_m]$$



$$P(e \mid u=a_0)$$

$$= P(v \in R_1 \cup R_2 \mid u=a_0)$$

$$= \int_{R_1 \cup R_2} f(v \mid u=a_0)\, dv$$

$$\le \underline{P(\hat{u}=a_1 \mid u=a_0)} + P(\hat{u}=a_2 \mid u=a_0)$$

union bound

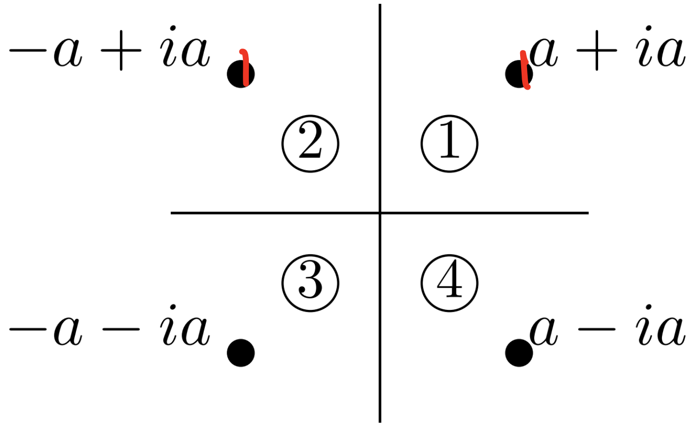
$$\text{ML} = Q\left(\sqrt{\frac{\|\vec{a}_0-\vec{a}_1\|^2}{2N_0}}\right) + Q\left(\sqrt{\frac{\|\vec{a}_0-\vec{a}_2\|^2}{2N_0}}\right)$$

$$Q\left(\sqrt{\frac{2\left\|\vec{a}_0-\frac{\vec{a}_0+\vec{a}_1}{2}\right\|^2}{N_0}}\right) + Q \cdots$$

M-ary QAM Detection in WGN

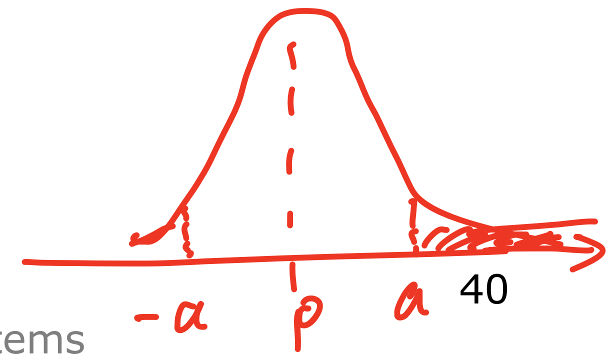
- Example:
- Consider 4-QAM with four signal points $u = \pm a \pm ia$. Assume Gaussian noise with $\mathcal{N}\left(0, \frac{N_0}{2}\right)$ per dimension. Sketch the signal set and ML decision regions. Find the exact probability of error (in terms of Q function) for this signal set using ML detection.
- Sol:

2a



$P(Z > a)$

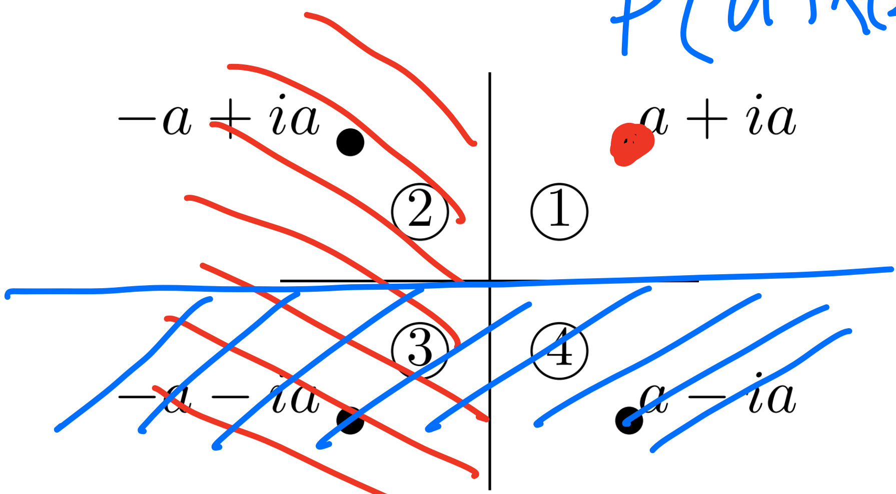
$= Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$



$\Re(Z), \Im(Z) \sim N(0, \frac{N_0}{2})$

M-ary QAM Detection in WGN

- Example:
- Sol:



$P(a+\Re(Z)<0 \cap a+\Im(Z)<0)$

$= P(a+\Re(Z)<0) \cdot P(a+\Im(Z)<0)$

$= P(\Re(Z)<-a) \times P(\Im(Z)<-a)$

- $P(e|U = a+ia) = P(a+ia+\mathcal{R}(Z)+i\mathcal{I}(Z) \in \{R_2 \cup R_3 \cup R_4\})$

$$= P(\{a+\mathcal{R}(Z)<0\} \cup \{a+\mathcal{I}(Z)<0\})$$

$$= P(\{a+\mathcal{R}(Z)<0\}) + P(\{a+\mathcal{I}(Z)<0\}) - P(\{a+\mathcal{R}(Z)<0\} \cap \{a+\mathcal{I}(Z)<0\})$$

$$= 2Q\left(\sqrt{\frac{2a^2}{N_0}}\right) - Q\left(\sqrt{\frac{2a^2}{N_0}}\right)^2$$

$= Q^2(\quad)$

* We focus on signal (symbol) error probability (constellation)
Not bit error probability (mapping from bits to symbol)

$(M^2-1)(2a)^2$

MPAM: $\frac{(M^2-1)(\ldots)}{12} = 5a^2$ $M=4$

M-ary QAM Detection in WGN

- Example:

4QAM

- For QPSK, $E_b = \frac{2a^2}{2} = a^2$, $P(e) = 2Q\left(\sqrt{\frac{2a^2}{N_0}}\right) - Q\left(\sqrt{\frac{2a^2}{N_0}}\right)^2$

$P(e \mid u=3a) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$

- For 4PAM, $E_b = \frac{5a^2}{2}$, $\mathcal{A} = \{-3a, -a, a, 3a\}$

$-3a \quad -a \quad 0 \quad a \quad 3a$

$$P(e|U=a) = P(a+Z<0) + P(a+Z>2a)$$
$$= P(-Z>a) + P(Z>a)$$
$$= 2Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$$
$$P(e) = \frac{1}{2} * 2Q\left(\sqrt{\frac{2a^2}{N_0}}\right) + \frac{1}{2} * Q\left(\sqrt{\frac{2a^2}{N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$$

- Comparing 4PAM and QPSK, for the same error probability, approximately 2.5 times more transmit energy is needed. This loss is even more significant for larger constellations.
- It is much more energy-efficient to use both I and Q channels.

QPSK

QPSK

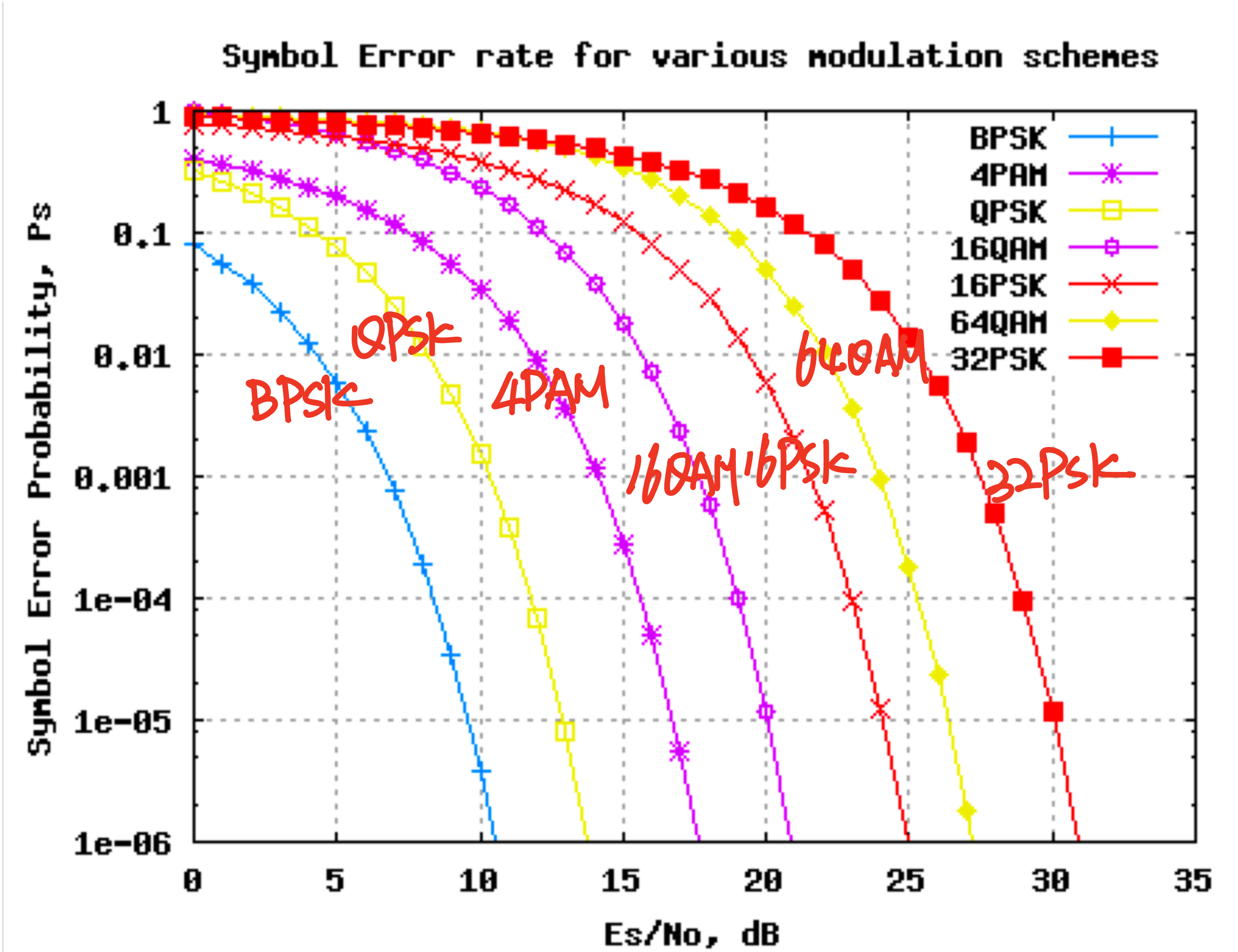
4QAM. $P_e \approx 2Q\sqrt{\frac{2E_b}{N_0}}$

4PAM $P_e \approx \frac{3}{2}Q\sqrt{\frac{4E_b}{5N_0}}$

$\Longrightarrow$

$2E_b^{4QAM} = \frac{4}{5}E_b^{4PAM}$

$2.5\,E_b^{4QAM} = E_b^{4PAM}$



same P_e,

$M\uparrow$ $E_s\uparrow$

M-ary QAM Detection in WGN

- Consider a QAM modulation using pulse $p(t)$
  - $\{p(t-kT)\}$ are orthonormal functions.
  - $\mathcal{A} = \{a_1, \dots, a_M\}$
  - Baseband waveform: $u(t) = \sum_{k=1}^{n} u_k p(t-kT)$.
  - Let $\{\phi_k(t); k \geq 1\}$ be an orthonormal basis of complex $\mathcal{L}_2$ waveforms such that the first n basis are given by $\phi_k(t) = p(t-kT), 1 \leq k \leq n$.
  - Baseband noise: $z(t) = \sum_k Z_k \phi_k(t)$
  - The received baseband signal:

$$V(t) = \sum_{k=1}^{\infty} V_k \phi_k(t) = \sum_{k=1}^{n} (u_k + Z_k) p(t-kT) + \sum_{k>n} Z_k \phi_k(t)$$

- Detection:
  - M^n hypotheses for $\{U_1, \dots, U_n\}$
  - $\boldsymbol{u} = (u_1, \dots, u_n)^T, \boldsymbol{v} = (v_1, \dots, v_n, \dots, v_l)^T, l > n,$

$$\{u_k\}_{k=1}^{n} \xrightarrow{\{p(t-kT)\}_{k=1}^{n}} u(t) \longrightarrow$$

B.B.

$z(t)$ B.B. WGN

$$\{u_k\}_{k=1}^{n} \xleftarrow{\text{Detection.}} \{v_k\} \longleftarrow v(t) \longleftarrow$$

B.B

$$\phi_k(t) = \begin{cases} p(t-kT) & 1 \le k \le n \\ \phi_k(t) & k > n \end{cases}$$

complex $\mathcal{L}_2$. B.B space

$$u(t) = \sum_{k=1}^{n} u_k\, p(t-kT) \iff \vec{u} = [u_1, \cdots u_k] \qquad u_k \in \mathcal{A}$$

$$z(t) = \sum_{k=1}^{\infty} z_k \phi_k(t) = \sum_{k=1}^{n} z_k\, p(t-kT) + \sum_{k>n} z_k \phi_k(t)$$

$$\iff [\ \vec{z} = [z_1, \cdots z_k],\ \vec{z}'\ ]$$

$$v(t) = u(t) + z(t) = \sum_{k=1}^{n} (u_k + z_k)\, p(t-kT) + \sum_{k'>n} z_k \phi_k(t)$$

$$\iff [\ \vec{v},\ \vec{v}'\ ] \qquad \vec{v} = \vec{u} + \vec{z}$$

$$\vec{v}' = \vec{z}'$$

$$[\vec{v}, \vec{v}'] \longrightarrow \vec{u} \qquad \boxed{M^n}$$

$$f_{\vec{V}|\vec{u}}([\underline{\vec{V}}, \underline{\vec{V}'}] \mid \vec{u}) = f(\vec{u}+\vec{z}, \vec{z}' \mid \vec{u})$$

$\vec{z}'$, is independent to $\vec{u}, \vec{z}$.

$$= f(\vec{u}+\vec{z} \mid \vec{u}) f(\vec{z}')$$

$$= f_z(\vec{v}-\vec{u}) f_{z'}(\vec{z}')$$

$$f_{\vec{v}|\vec{u}'}([\vec{v}, \vec{v}'] \mid \vec{u}') = f_z(\vec{v}-\vec{u}') f_{z'}(\vec{z}')$$

$$LR = \frac{f_z(\vec{v}-\vec{u})}{f_z(\vec{v}-\vec{u}')}$$

Theorem of Irrelavence

$$\eta = \frac{p(\vec{u}')}{p(\vec{u})}$$

$$LLR = \frac{-\|\vec{v}-\vec{u}\|_2^2 + \|\vec{v}-\vec{u}'\|_2^2}{N_0} \underset{\vec{u}'}{\overset{\vec{u}}{\gtrless}} \ln \eta$$

$$ML: \quad \ln \eta = 0$$

$$\hat{\vec{u}}(\vec{v}) = \arg\min_{\vec{u}_i \in A^n} \|\underline{\underline{\vec{v}}} - \vec{u}_i\|_2^2$$

M^n

$$= \arg\min_{\{u_{i,k}\} \in A} \sum_{k=1}^{n} |v_k - u_{ik}|^2$$

$$\hat{u}_k(v_k) = \arg\min_{u_{ik} \in A} |v_k - u_{ik}|^2 \quad \forall k \in \{1, \cdots n\}$$

M-ary QAM Detection in WGN

$$V(t) = \sum_{k=1}^{\infty} V_k \phi_k(t) = \sum_{k=1}^{n} (u_k + Z_k) p(t - kT) + \sum_{k>n} Z_k \phi_k(t)$$

- Likelihood of $\boldsymbol{v}$ conditioned on $\boldsymbol{u}$ is

$$f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{u}) = \prod_{k=1}^{n} f_Z(v_k - u_k) \prod_{k=n+1}^{\ell} f_Z(v_k)$$

- Likelihood ratio:

$$\Lambda_{\boldsymbol{u},\boldsymbol{u}'}(\boldsymbol{v}) = \prod_{k=1}^{n} \frac{f_Z(v_k - u_k)}{f_Z(v_k - u'_k)}$$

$$\mathrm{LLR}_{\boldsymbol{u},\boldsymbol{u}'}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v} - \boldsymbol{u}\|^2 + \|\boldsymbol{v} - \boldsymbol{u}'\|^2}{N_0}$$

  - The only relevant terms in the decision are $\boldsymbol{v} = (v_1, \dots, v_n)^T$

- ML detection:
  - $\tilde{u}(\boldsymbol{v}) = \arg\min_{i \in M^n} \|\boldsymbol{v} - \boldsymbol{u_i}\|$ (minimum distance detection)

M-ary QAM Detection in WGN

- ML detection:
  - $\tilde{u}(\boldsymbol{v}) = \arg\min_{\boldsymbol{u_i} \in \mathcal{A}^n} \|\boldsymbol{v} - \boldsymbol{u_i}\|$ (minimum distance detection)
  - $\tilde{u}(v_k) = \arg\min_{u_k \in \mathcal{A}} \|v_k - u_k\|$
  - ML sequence detector with M^n hypothesis detects each U_k by minimizing $(v_k - u_k)^2$ over M hypothesis for that U_k.
  - Theorem: Let $U(t) = \sum_{k=1}^{n} U_k p(t - kT)$ be a QAM or PAM baseband input to a WGN channel, assume $\{p(t - kT)\}$ are orthonormal functions. Then the M^n-ary ML decision on $\boldsymbol{U} = (U_1, \dots, U_n)^T$ is equivalent to making separate M-ary ML decision on each U_k.
  - Detection Error Probability:
    - $\Pr(e) = 1 - (1 - \mathrm{P})^{\mathrm{n}}$, P is the error probability for single symbol.

M-ary Detection with Arbitrary Modulation in WGN

- PAM
  - Real hypotheses $\mathcal{A} = \{a_0, \dots, a_{M-1}\}$ P.B.
  - $\{u_k; k \in \mathbb{Z}\} \rightarrow u(t) = \sum_k u_k p(t - kT) \rightarrow x(t) = 2\cos 2\pi f_c t \left(\sum_k u_k p(t - kT)\right)$
- QAM
  - Complex hypotheses $\mathcal{A} = \{a_0, \dots, a_{M-1}\}$ DSB-SC
  - $\{u_k; k \in \mathbb{Z}\} \rightarrow u(t) = \sum_k u_k p(t - kT) \rightarrow x(t) = 2\cos 2\pi f_c t \sum_k u'_k p(t - kT) - 2\sin 2\pi f_c t \sum_k u''_k p(t - kT)$
- Arbitrary modulation scheme
  - Signal set $\mathcal{A} = \{\boldsymbol{a_0}, \dots, \boldsymbol{a_{M-1}}\}$, $\boldsymbol{a_m} = \left(a_{m,1}, \dots, a_{m,n}\right)^T \in \mathbb{R}^n$
  - Choose orthonormal functions $\{\phi_k(t); k \geq 1\}$, span the space of real $\mathcal{L}_2$ waveforms
  - $\boldsymbol{a_m} \rightarrow \boldsymbol{b_m}(\boldsymbol{t}) = \sum_{k=1}^{n} a_{m,k}\phi_k(t)$
  - $\{s_1, s_2, \dots\} \rightarrow \sum_l \boldsymbol{b_{s_l}}(t - lT)$

$\mathbb{R}^n$

$\vec{a}_0 \longrightarrow b_0(t)$

$\vec{a}_1 \longrightarrow b_1(t)$

$\vdots$

$\vec{a}_M \longrightarrow b_M(t)$

M-ary Detection with Arbitrary Modulation in WGN

- Arbitrary modulation scheme
  - Signal set $\mathcal{A} = \{\boldsymbol{a_0}, \dots, \boldsymbol{a_{M-1}}\}$, $\boldsymbol{a_m} = \left(a_{m,1}, \dots, a_{m,n}\right)^T \in \mathbb{R}^n$
  - Choose orthonormal functions $\{\phi_k(t); k \geq 1\}$, span the space of real $\mathcal{L}_2$ waveforms
  - $\boldsymbol{a_m} \to \boldsymbol{b_m}(\boldsymbol{t}) = \sum_{k=1}^{n} a_{m,k}\phi_k(t)$
  - $\{s_1, s_2, \dots\} \to \sum_l \boldsymbol{b}_{\boldsymbol{s_l}}(t - lT)$ $= \sum_l \sum_k a_{s_l,k}\phi_k(t-lT)$
  - the successive transmitted signal waveforms are all orthogonal to each other
    - $\{\phi_k(t - lT), k, l \in \mathbb{Z}\}$ are all orthonormal
  - PAM: $n = 1$, $\phi_1(t) = p(t)$ (BB), $\phi_1(t) = \sqrt{2}p(t)\cos 2\pi f_c t$ (PB)
  - QAM: $n = 2$, $\phi_1(t) = \sqrt{2}p(t)\cos 2\pi f_c t$, $\phi_2(t) = -\sqrt{2}p(t)\sin 2\pi f_c t$ (PB)

$$y(t) \longrightarrow \begin{cases} b_1(t) \leftrightarrow \vec{a}_1 \\ b_2(t) \leftrightarrow \vec{a}_2 \\ \vdots \\ b_M(t) \leftrightarrow \vec{a}_M \end{cases}$$

M-ary Detection with Arbitrary Modulation in WGN

- Arbitrary modulation scheme
  - $X(t)$ is a choice from M waveforms $\{\boldsymbol{b_m}(t)\}_{m=1}^{M}$
  - $X(t) = \sum_{k=1}^{n} X_k \phi_k(t)$
  - Received random waveform
    - $Y(t) = \sum_{k=1}^{l} Y_k \phi_k(t) = \sum_{k=1}^{n} (X_k + Z_k)\, \phi_k(t) + \underline{\sum_{k=n+1}^{l} Y_k \phi_k(t)}$

Signal Noise in signal space

Noise perpendicular to the signal space and Contribution of signal waveforms other than X (successive signals, signals from other users)

M-ary Detection with Arbitrary Modulation in WGN

- Arbitrary modulation scheme
  - $X(t)$ is a choice from M waveforms $\{\boldsymbol{b_m}(t)\}_{m=1}^{M}$
  - $X(t) = \sum_{k=1}^{n} X_k \phi_k(t)$
  - Received random waveform $\quad Y \quad Y'$
    - $Y(t) = \sum_{k=1}^{l} Y_k \phi_k(t) = \underbrace{\sum_{k=1}^{n} (X_k + Z_k)\,\phi_k(t)}_{Y} + \underbrace{\sum_{k=n+1}^{l} Y_k \phi_k(t)}_{Y'}$
  - $Y(t) \rightarrow \boldsymbol{Y} = (Y_1, \dots, Y_n)^T$ and $\boldsymbol{Y}' = (Y_{n+1}, \dots, Y_l)^T$
  - $\mathrm{X(t)} \rightarrow \boldsymbol{X} = (X_1, \dots, X_n)^T$ $\quad \vec{X} \in \mathcal{A} = \{\vec{a}_1, \cdots \vec{a}_M\}$
  - WGN $Z(t) \rightarrow \boldsymbol{Z} = (Z_1, \dots, Z_n)^T$ and $\boldsymbol{Z}' = (Z_{n+1}, \dots, Z_l)^T$
  - $V(t)$ (contributions from other signals)$\rightarrow \boldsymbol{V}' = (V_{n+1}, \dots, V_l)^T$
  - $\vec{\boldsymbol{Y}} = \vec{\boldsymbol{X}} + \vec{\boldsymbol{Z}}, \boldsymbol{Y}' = \boldsymbol{Z}' + \boldsymbol{V}'$
  - Assumption:
    - $\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Z}', \boldsymbol{V}'$ are statistically independent.
    - $\boldsymbol{Z}'$ and $\boldsymbol{V}'$ are orthogonal to $\boldsymbol{X}$ and $\boldsymbol{Z}$

M-ary Detection with Arbitrary Modulation in WGN

- Arbitrary modulation scheme
  - $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z}, \boldsymbol{Y}' = \boldsymbol{Z}' + \boldsymbol{V}'$
  - Likelihood:

$$f_{\boldsymbol{Y}\boldsymbol{Y}'|\boldsymbol{X}}(\boldsymbol{y}\boldsymbol{y}'|\boldsymbol{a}_m) = f_{\boldsymbol{Z}}(\boldsymbol{y} - \boldsymbol{a}_m) f_{\boldsymbol{Y}'}(\boldsymbol{y}') \text{ (independence)}$$

  - Likelihood ratio:

$$\Lambda_{m,m'}(\boldsymbol{y}) = \frac{f_{\boldsymbol{Z}}(\boldsymbol{y} - \boldsymbol{a}_m)}{f_{\boldsymbol{Z}}(\boldsymbol{y} - \boldsymbol{a}_{m'})}$$

  - Y is sufficient statistic for MAP on X, Y′ is irrelevant
  - Theorem of irrelevance: Assume that $\boldsymbol{Y}'$ is statistically independent of the pair $\boldsymbol{X}, \boldsymbol{Z}$. Then the MAP detection of $\boldsymbol{X}$ from the observation of $(\boldsymbol{Y}, \boldsymbol{Y}')$ depends only on $\boldsymbol{Y}$. That is, the observed sample value of $\boldsymbol{Y}'$ is irrelevant.
  - Reduce the problem to a finite dimensional problem. The other signals can be view as part of $\boldsymbol{Y}'$, assuming they are orthogonal and independent of $\boldsymbol{X}$.

Example: $\mathcal{A} = \{a_1, a_2 \cdots a_M\}$. $\vec{a}_m \in R^n$

$$\vec{a}_m \in R^n \xrightarrow{\{\phi_k(t)\}_{k=1}^{n}} S_m(t) = \sum_{k=1}^{n} a_{m,k}\,\phi_k(t) \rightarrow \text{AWGN}$$

$$\vec{a}_m \leftarrow S_m(t) \leftarrow \boxed{\text{Waveform Det.}} \leftarrow y(t) \leftarrow \text{AWGN}$$

$\downarrow$

$\boxed{\text{Symbol Det.}}$ "vectorization"

given $S_m(t)$, $m = \{1, \cdots, M\}$ and $y(t)$, How to perform Detection

① $\{S_1(t), S_2(t) \cdots S_M(t)\} \xrightarrow{\text{Gram-Schmidt orthonormalization}} \{\phi_1(t), \phi_2(t) \cdots \phi_K(t)\}$.

② $S_1(t) \longleftrightarrow \vec{S}_1$ $\quad S_2(t) \longleftrightarrow \vec{S}_2$ $\quad \cdots\cdots$ $\quad S_M(t) \longleftrightarrow \vec{S}_M$

③ $y(t) \longleftrightarrow \vec{Y}$ in signal space $\quad y(t)|_S = \sum_k y_k \phi_k(t) \quad y_k = \langle \quad \rangle$

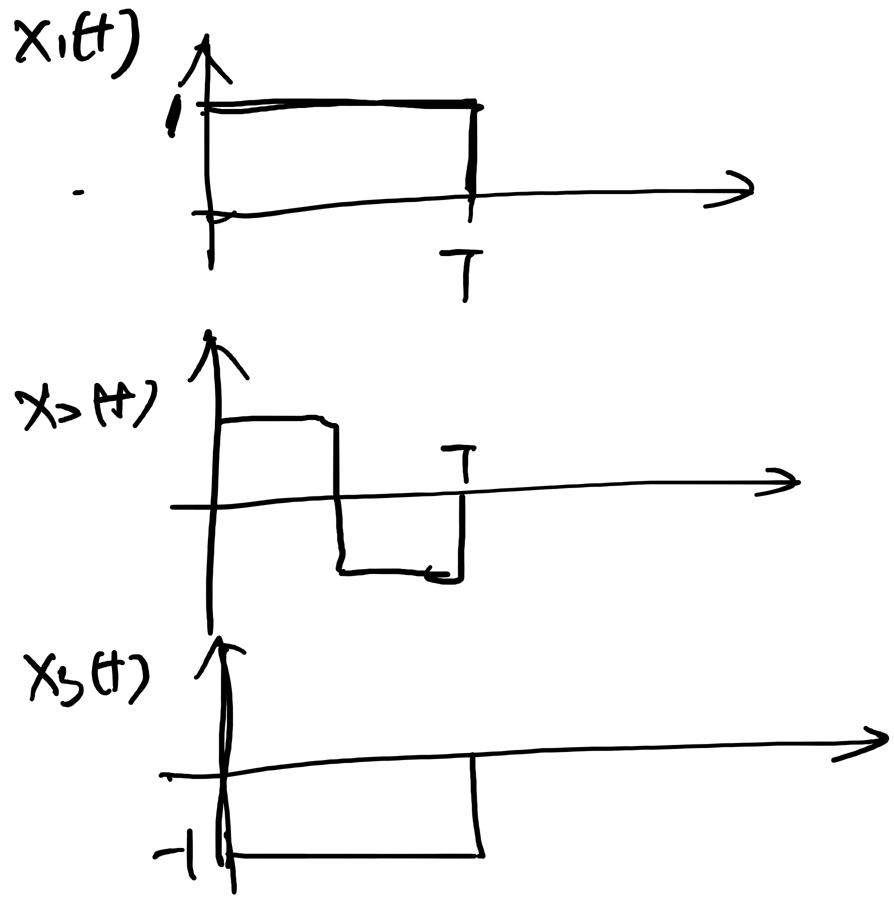
④ $\vec{Y} \longrightarrow \{\vec{S}_1, \vec{S}_2, \cdots \vec{S}_M\}$ M-ary vector Detection.

Example:

$$a_m \in \mathbb{R}^n \xrightarrow{\{\phi_k(t)\}_{k=1}^{n}} x_m(t) = \sum_{k=1}^{n} a_{m,k}\phi_k(t)$$ waveform Detection

$x_m(t) \Updownarrow a_m$ symbol Detection

$x_m(t) \longrightarrow a_m \longrightarrow$ symbol Detection



Q: MAP Detection & Pe

① Find the orthonormal Basis of signal space $\{\phi_k(t)\}$

Gram-schmidt orthonormalization

$$x_1(t) \longrightarrow \frac{x_1(t)}{\|x_1(t)\|} = \phi_1(t)$$

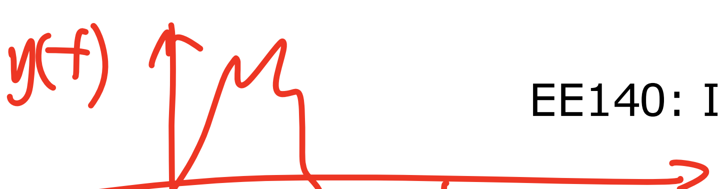
$$\frac{x_2(t) - \langle x_2(t), \phi_1(t)\rangle \phi_1(t)}{\| \quad \|} = \phi_2(t)$$

② $\{\phi_1(t), \phi_2(t)\}$

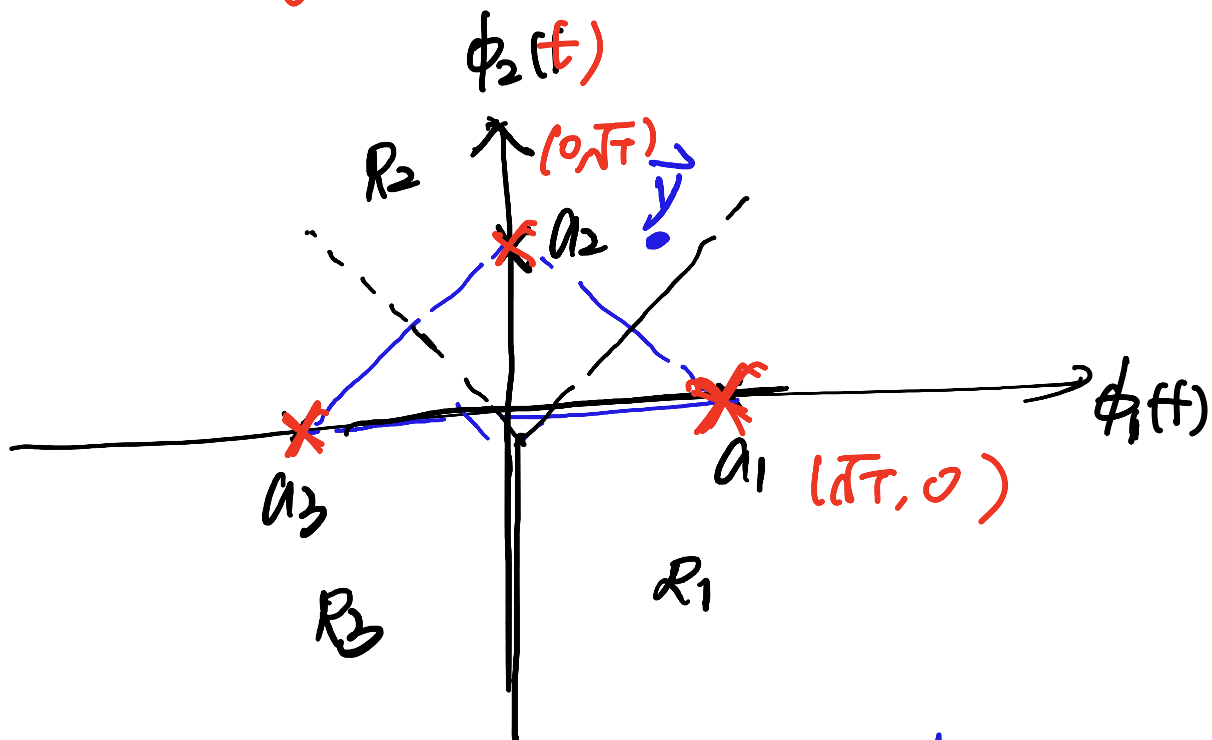
$y(t) \longrightarrow \vec{y} = [y_1, y_2]$

$x_1(t) \longrightarrow a_1 = [a_{1,1}, a_{1,2}]$

$x_2(t) \longrightarrow a_2 = [a_{2,1}, a_{2,2}]$



$x_3(t) \xrightarrow{\{\phi_1(t),\, \phi_2(t)\}} \vec{a}_3 = [a_{3,1},\ a_{3,2}]$



① Draw the decision boundary

or Binary Detection

or If ML, minimum Distance Detection $= \arg\min_m \|\vec{y} - \underline{a_m}\|$

② Calculation Detection error Probability

$P_e = P_1 P(e \mid u = a_1) + P_2 P(e \mid u = a_2) + P_3 P(e \mid u = a_3)$

$\leq$ ______ (union bound) depends on symbol Distance.

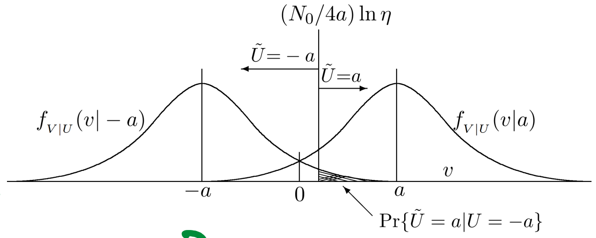
Summary:

1. prior distribution $P(U=a_m)=P_m$,
posteriori probability $P_{U|V}(a_m|v)$
Likelihood probability $P_{V|U}(v|a_m)$
MAP: $\tilde{u}(v)=\arg\max_m P_{U|V}(a_m|v)$
ML: $\tilde{u}(v)=\arg\max_m P_{V|U}(v|a_m)$

equal prior

2. Detection (MAP/ML, P_e)

① Binary detection
- scalar
  - Antipodal

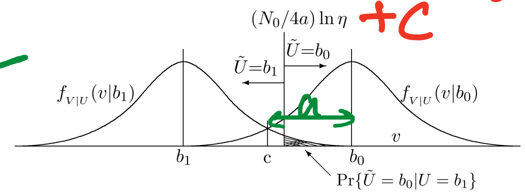


$E_b=a^2$

$$\boxed{\Pr\{e\}=\Pr\{e\,|\,U{=}-a\}=\Pr\{e\,|\,U{=}a\}=Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$ (ML)

+C

  - NonAntipodal



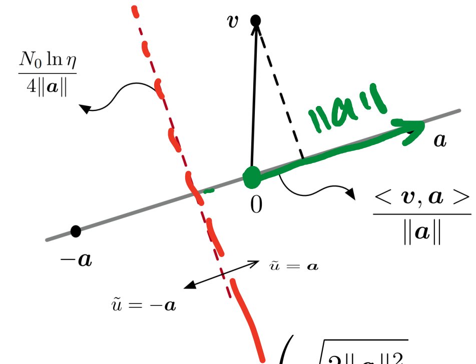
$$\Pr\{e\,|\,U{=}-a\}=Q\left(\frac{a}{\sqrt{N_0/2}}+\frac{\sqrt{N_0/2}\ln\eta}{2a}\right)$$

- vector
  - real vector
  - complex vector

real vector

vector

complex vector



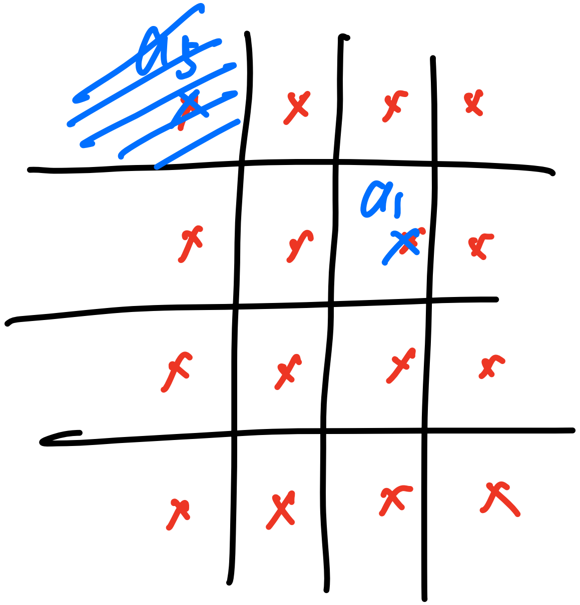
$P(\tilde{u}=a_5 \mid a_1)$

$P(e \mid a_1)$

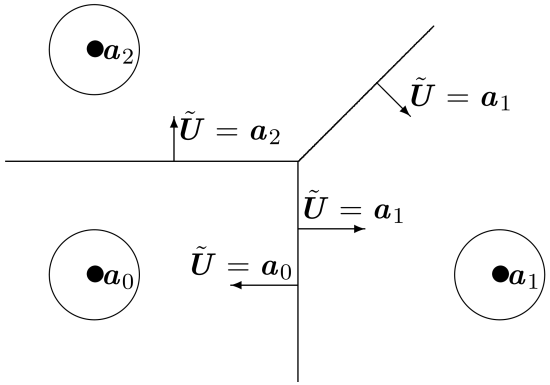
$$\Pr\{e \mid \boldsymbol{U}=-\boldsymbol{a}\} = Q\left(\sqrt{\frac{2\|\boldsymbol{a}\|^2}{N_0}} + \frac{\ln\eta}{2\sqrt{2\|\boldsymbol{a}\|^2/N_0}}\right)$$

$$\Pr\{e\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\frac{\Re[\langle \boldsymbol{v}, \boldsymbol{a}\rangle]}{\|\boldsymbol{a}\|} \underset{<_{\tilde{U}=-\boldsymbol{a}}}{\geq^{\tilde{U}=\boldsymbol{a}}} \frac{N_0\ln(\eta)}{4\|\boldsymbol{a}\|}$$



② M-ary Detection



$$\Pr[e|\boldsymbol{U}=\boldsymbol{a}_m]$$
$$= \Pr\left[\cup_{m'\neq m}\left(\tilde{\boldsymbol{U}}(\boldsymbol{v}) = \boldsymbol{a}_{m'} \middle| \boldsymbol{U}=\boldsymbol{a}_m\right)\right]$$
$$\leq \sum_{m'\neq m} \Pr\left(\tilde{U}(v)=a_{m'} \mid U=a_m\right)$$

③ M-ary QAM =

$$V(t)=\sum_{k=1}^{\infty} V_k\phi_k(t)=\sum_{k=1}^{n}(u_k+Z_k)p(t-kT)+\sum_{k>n} Z_k\phi_k(t)$$

$$\mathrm{LLR}_{\boldsymbol{u},\boldsymbol{u}'}(\boldsymbol{v}) = \frac{-\|\boldsymbol{v}-\boldsymbol{u}\|^2+\|\boldsymbol{v}-\boldsymbol{u}'\|^2}{N_0}$$

ML sequence detector with M^n hypothesis detects each U_k by minimizing $(v_k-u_k)^2$ over M hypothesis for that U_k.

④ M-ary Arbitrary Modulation:

$$\boldsymbol{a_m} \to \boldsymbol{b_m(t)} = \sum_{k=1}^{n} a_{m,k}\phi_k(t)$$

$$\{s_1, s_2, \ldots\} \to \sum_{l} \boldsymbol{b}_{\boldsymbol{s_l}}(t-lT)$$

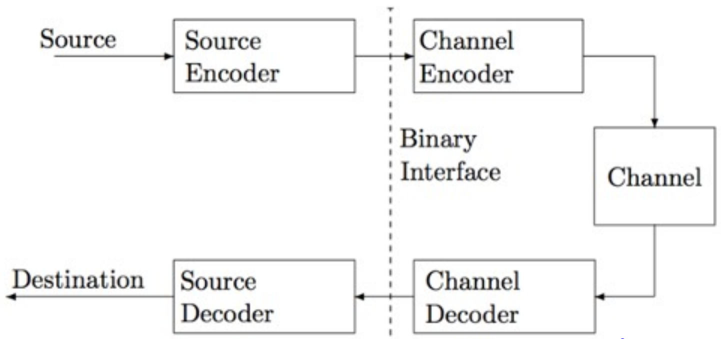
$$Y(t)=\sum_{k=1}^{l} Y_k\phi_k(t)$$

$$=\sum_{k=1}^{n}(X_k+Z_k)\,\phi_k(t)+\sum_{k=n+1}^{l} Y_k\phi_k(t)$$

$$\Lambda_{m,m'}(\boldsymbol{y})=\frac{f_Z(\boldsymbol{y}-\boldsymbol{a}_m)}{f_Z(\boldsymbol{y}-\boldsymbol{a}_{m'})}$$

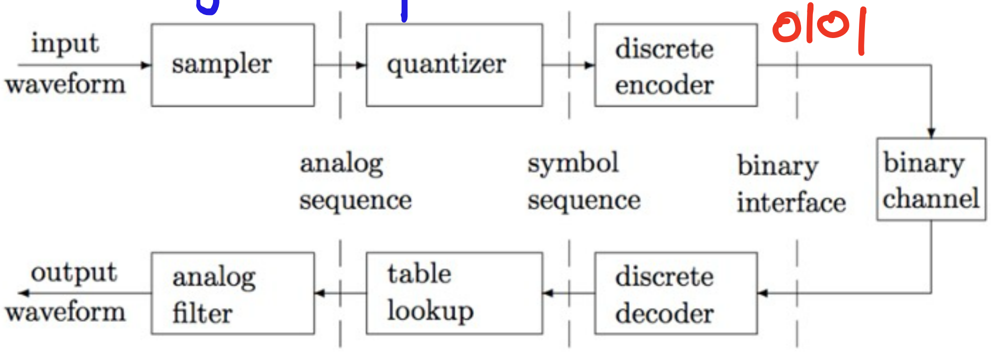
waveform Detection ⟺ symbol Detection

Digital Communication

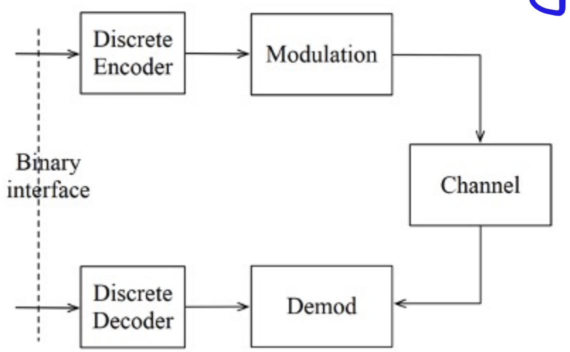


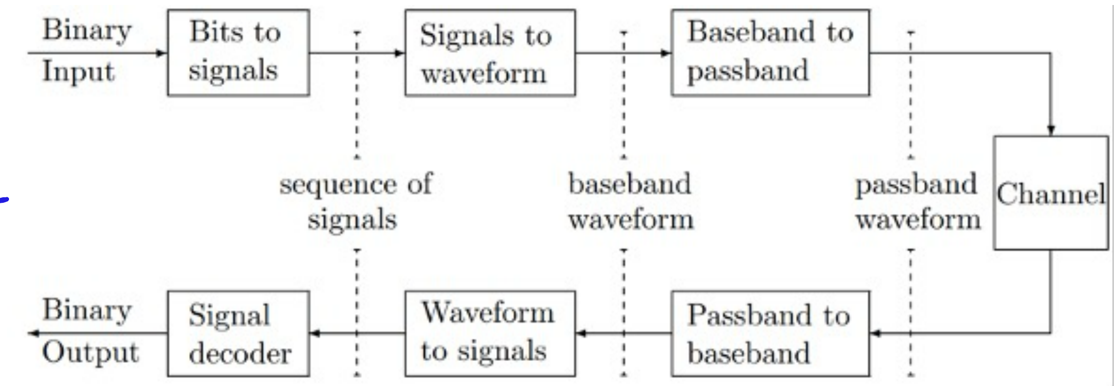
general framework

0101



source coding



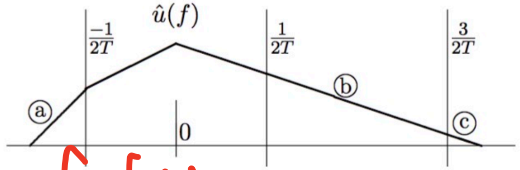


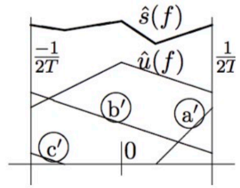
channel coding

Source Coding:

① Sampling: {

- $u(t) \to \{u_k\}$, orthogonal Expansion
- sampling theorem: $x(t) = \sum_{k=-\infty}^{+\infty} x\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt-k)$
- Aliasing:





$\begin{cases} W > \frac{1}{2T} \\ f_s < 2W \end{cases}$

② Quantization {

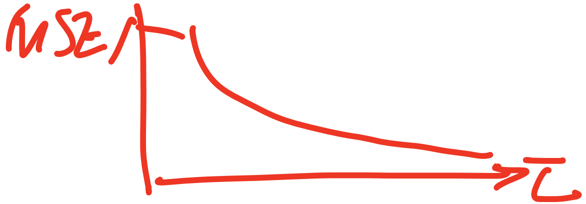
- Analog sequences $\{u_k\}$ $f_U(u)$ → $\boxed{Q}$ → Discrete sequence $\{v_k\}$
- MSE: $\begin{cases} E[|U-V|^2] \\ E_V E_{U|V}[|U-V|^2] \end{cases}$

  MSE_j $f_j(u)$ $E[f_j(u)]$

- $\min_{\{R_j\}\{a_j\}} MSE = E[|U-V|^2]$ s.t. M → **Lloyd-Max Algorithm**
- $\min$ MSE s.t. $H(V) = R$ → Entropy-coded quantization (High rate, uniform)

MSE vs. $\bar{L}$ Tradeoff ←

$\begin{cases} MSE = \frac{\Delta^2}{12} \\ H[V] \approx h(U) - \log_2 \Delta \end{cases}$



③ Discrete Encoder:

- Lossless (uniquely decodable)
- $\bar{L}_{min}$

Fixed-length code: $\log_2 M \le L < \log_2 M + 1$

fixed to fixed length code: $\log_2 M \le \bar{L} < \log_2 M + \frac{1}{n}$

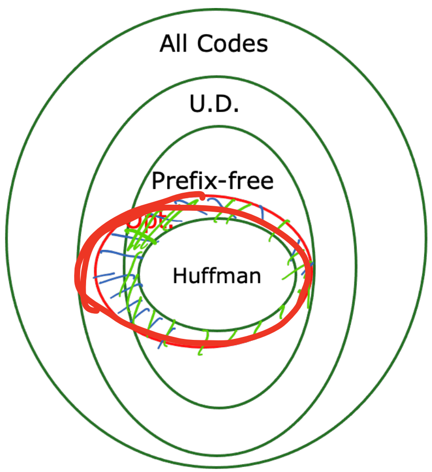
variable-length code:
- prefix-free code
  ↓
- Kraft-Inequality $\{l(x)\}$
  ↓
- Huffman-code

$H(X) \le \bar{L}_{min} < H(X) + 1$

(entropy bound of P.F. code)

~~variable~~ fixed to variable length code:

$H(X) \le \bar{L}_{min,n} < H(X) + \frac{1}{n}$



$E_s = E\left[\int |u_k(t)|^2 dt\right] = E\left[|u_k|^2\right] E_p = \sum P_m |a_m|^2 E_p$

Channel Encoder

↓

$\boxed{\text{Detection}}$

PAM

(Constellation, Es, Mod, DeM.)

QAM $\boxed{\text{DSB-SC}}$

(Constellation, Es, Mod, DeM.)

Nyquist Criterion (Zero ISI)

Time

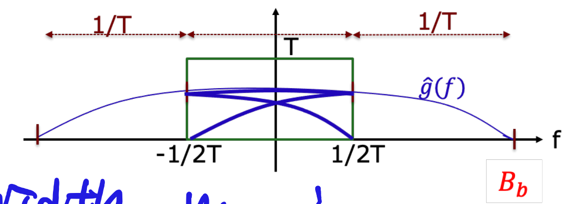
$g(kT)=1, k=0$

$g(kT)=0, k\neq 0,$

freq:

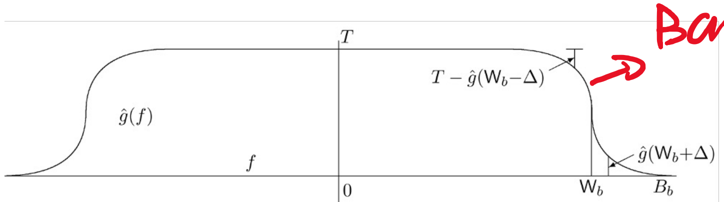
$\sum \hat{g}(f+\frac{k}{T})\,\mathrm{rect}(fT)$

$= T\,\mathrm{rect}(fT)$



Nyquist Bandwidth: $W_b=\frac{1}{2T}$

$2W_b > B_b \geq W_b$



→ Band edge symmetry

Matched Filter: $g(f)=P^*(f) \Leftrightarrow \{p(t-kT)\}$ orthonormal functions

bits to signal mapping

↓

signal to BB. waveform

$u(t)=\sum u_k p(t-kT)$

↓

BB → PB Mapping

↓ channel

P.B. → BB

↓

BB → signals

(Nyquist criterion)

↓

Signal → bits

(Detection)

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Thanks for your kind attention!

Questions?