EE140 Introduction to Communication Systems

Lecture 13

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Vector Space and Signal Space

Importance: Signal space (or vector) representation of signal is a very effective and useful tools in the analysis of digitally modulated signals.

- Many insights about signals depend not on time or frequency, but on the vector relationships
- Orthogonal expansions are best viewed in vector space terms
- Questions of limites and approximation are easily treated in vector space terms

$\{\phi_k(t)\}$

L_2 function $u(t) \longrightarrow$ orthogonal expansion $\longrightarrow \{\cdots u_k \cdots\}$

$\int |u(t)|^2 dt < \infty$ $\qquad \vec{u}$.

$\longrightarrow$ Vector space $\longrightarrow$ signal space

vector space + Inner product $\Rightarrow$ Inner product space

signal space + Inner product $\Rightarrow$ Inner product space of L_2 functions

Axioms and Basic Properties of Vector Space

Vector: An n-dimensional column vector $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T$

Vector space: A vector space $\mathcal{V}$ is a set of vectors $\boldsymbol{v} \in \mathcal{V}$, along with a set of rules for manipulating those objects and along with a set of ancillary elements $\alpha \in \mathbb{F}$.

- $\alpha \in \mathbb{F}$ is called scalar (real of complex)
- Real vector space if α is real
- Complex vector space if α is complex

Properties

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ Commutativity of addition
- for each scalar α, β, $\alpha(\beta \boldsymbol{v}) = (\alpha\beta)\boldsymbol{v}$ Scalar associativity
- $\left.\begin{array}{l} \alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v} \\ (\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v} \end{array}\right\}$ Distributive laws
- $\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^{n} |v_i|^2}$

Inner Product Space

Let $\boldsymbol{v}_1 = (v_{11}, v_{12}, \ldots, v_{1n})^t$ and $\boldsymbol{v}_2 = (v_{21}, v_{22}, \ldots, v_{2n})^t$

- Inner (dot) product:

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}^* = \boldsymbol{v}_2^H \boldsymbol{v}_1$$

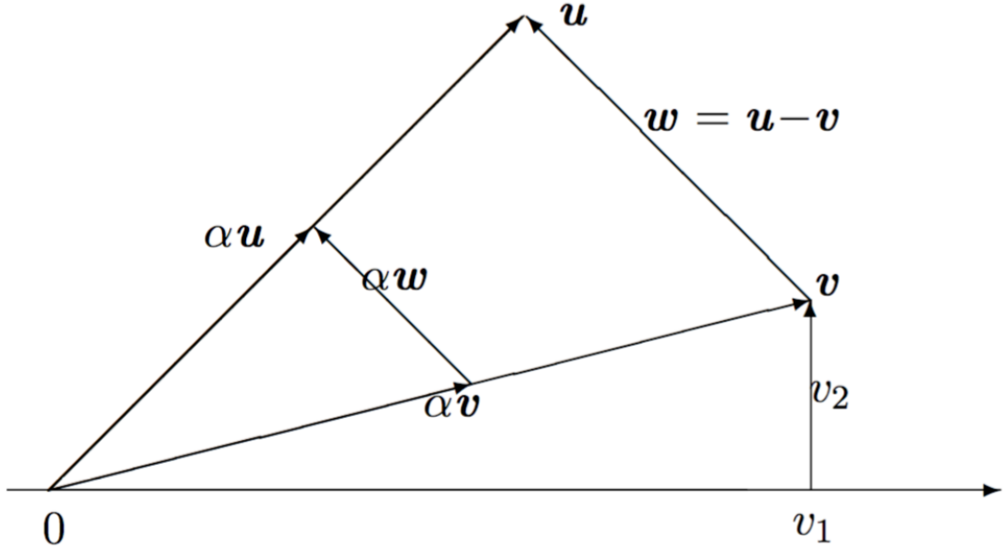
  where $\boldsymbol{A}^H$ denotes the Hermitian transpose of matrix $\boldsymbol{A}$

- $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle^*$ Hermitian Symmetry

$\|\vec{v}\| = \sqrt{<\vec{v}, \vec{v}>}$

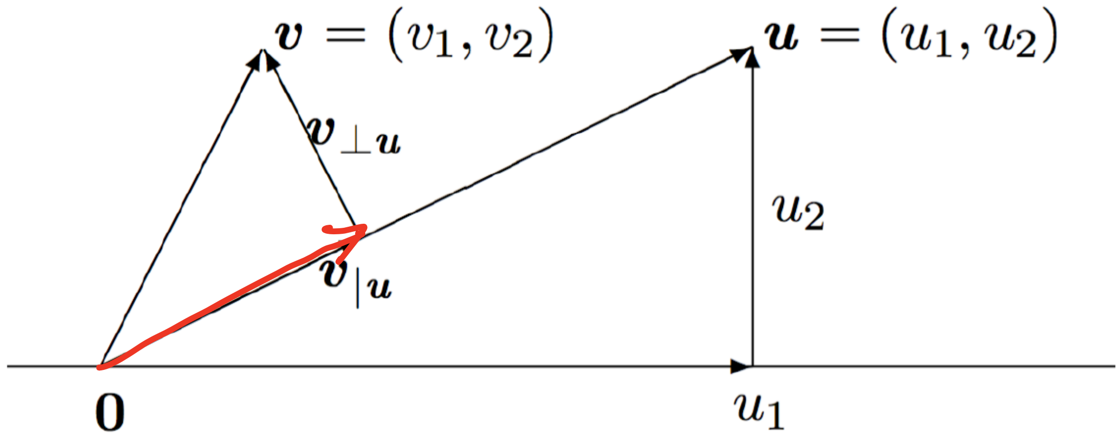
- $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle + \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle = 2\mathrm{Re}[\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle]$
- $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$
- A set of vectors $\boldsymbol{v}_k$ are orthogonal if $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$ for all $i \neq j$.
- A set of vectors are orthonormal if it is orthogonal and all $\|\boldsymbol{v}_i\| = 1$.

2-Dimensional Vector Space



- $\boldsymbol{v} = (v_1, v_2)$
- $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{u}$, $\boldsymbol{u} - \boldsymbol{w} = \boldsymbol{v}$
- $\cos(\angle(\boldsymbol{v}, \boldsymbol{u})) = \frac{v_1 u_1 + v_2 u_2}{\sqrt{v_1^2 + v_2^2}\sqrt{u_1^2 + u_2^2}} = \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{\|\boldsymbol{v}\| \|\boldsymbol{u}\|}$
- proof: hint $\cos A = (b^2 + c^2 - a^2)/2bc$

One-Dimensional Projection



- $\boldsymbol{v}_{\perp \boldsymbol{u}}$, $\boldsymbol{v}_{|\boldsymbol{u}}$: Projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$
- $\boldsymbol{v} = \boxed{\boldsymbol{v}_{\perp \boldsymbol{u}}} + \boxed{\boldsymbol{v}_{|\boldsymbol{u}}}$. ($\boldsymbol{v}_{\perp \boldsymbol{u}}$ and $\boldsymbol{v}_{|\boldsymbol{u}}$ are independent and orthogonal)
- $\cos(\angle(\boldsymbol{v}, \boldsymbol{u})) = \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{\|\boldsymbol{v}\| \|\boldsymbol{u}\|}$
- $\|\boldsymbol{v}_{|\boldsymbol{u}}\| = \|\boldsymbol{v}\| \cos(\angle(\boldsymbol{v}, \boldsymbol{u}))$; Unite vector: $\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$
- $\frac{\langle \vec{v}, \vec{u} \rangle}{\|u\|} \frac{\vec{u}}{\|\vec{u}\|}$ $\boxed{\boldsymbol{v}_{|\boldsymbol{u}} = \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|^2} \boldsymbol{u}}$

Inner Product Spaces

Let $\boldsymbol{v}_1 = (v_{11}, v_{12}, \ldots, v_{1n})^t$ and $\boldsymbol{v}_2 = (v_{21}, v_{22}, \ldots, v_{2n})^t$

- Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ satisfy the triangle inequality:

$$\|\boldsymbol{v}_1 + \boldsymbol{v}_2\| \leq \|\boldsymbol{v}_1\| + \|\boldsymbol{v}_2\|$$

- The Cauchy-Schwarz inequality:

$$|\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle| \leq \|\boldsymbol{v}_1\| \cdot \|\boldsymbol{v}_2\|$$

  with equality if $\boldsymbol{v}_1 = a\boldsymbol{v}_2$ for some complex scalar a.

- $$\|\boldsymbol{v}_1 + \boldsymbol{v}_2\|^2 = \|\boldsymbol{v}_1\|^2 + \|\boldsymbol{v}_2\|^2 + 2\mathrm{Re}[\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle]$$

  If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal, then $\|\boldsymbol{v}_1 + \boldsymbol{v}_2\|^2 = \|\boldsymbol{v}_1\|^2 + \|\boldsymbol{v}_2\|^2$

Linear Independent and Dependent

A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is

- linearly dependent if $\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j = 0$, where not all $\alpha_i = 0$. Thus, when $\alpha_k \neq 0$,

$$\boldsymbol{v}_k = \sum_{j \neq k} \frac{\alpha_j}{\alpha_k} \boldsymbol{v}_j$$

  (Each $\boldsymbol{v}_k$ for which $\alpha_k \neq 0$ is a a linear combination of the others)

- linearly independent if it's Not linearly dependent, i.e., if $\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j = 0$, it implies that each $\alpha_j = 0$.

  (No one vector can be represented as a linear combination of the remaining vectors.)

The word "linear " is often omitted.

Basis for Vector Space

- A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ spans a vector space $\mathcal{V}$ if every vector $\boldsymbol{v} \in \mathcal{V}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$

- For $\mathbb{R}^n$, let

$$\begin{cases} \boldsymbol{e}_1 = (1, 0, 0, \ldots, 0) \\ \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0) \\ \boldsymbol{e}_n = (0, 0, \ldots, 1) \end{cases}$$

  be the n unit vectors for $\mathbb{R}^n$. The unit vectors span $\mathbb{R}^n$ since every $\boldsymbol{v} \in \mathcal{V}$ is

$$\boldsymbol{v} = (\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} \alpha_j \boldsymbol{e}_j$$

- A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a basis for $\mathcal{V}$ if the set is linearly independent and spans $\mathcal{V}$, e.g., $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ form basis for $\mathbb{R}^n$ and $\mathbb{C}^n$

Finite-Dimensional Projections

Given

- a vector $\boldsymbol{v} \in \mathcal{V}$
- an orthonormal basis $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$ for vector space $\mathcal{S}$

$\boldsymbol{v}_{|\mathcal{S}}$: projection onto $\mathcal{S}$:

$$\boldsymbol{v}_{|\mathcal{S}} = \sum_{j=1}^{n} \langle \boldsymbol{v}, \boldsymbol{\phi}_j \rangle \boldsymbol{\phi}_j$$

When $n = 1$, then $\boxed{\boldsymbol{v}_{|\mathcal{S}} = \boldsymbol{v}_{|\boldsymbol{\phi}_1} = \langle \boldsymbol{v}, \boldsymbol{\phi}_1 \rangle \boldsymbol{\phi}_1}$

Gram-Schmidt Procedure

Given an arbitrary basis $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ spanning a space $\mathcal{S}$, generate an orthonormal basis $\{\phi_1, \phi_2, ..., \phi_n\}$ for $\mathcal{S}$. Let $\mathcal{S}_k$ be the subspace spanned by $\{\phi_1, \ldots, \phi_k\}$

1. $\phi_1 = \boldsymbol{s}_1 / \|\boldsymbol{s}_1\|$. $\phi_1 \to S_1$
2. $(\boldsymbol{s}_2)_{\perp \mathcal{S}_1} = \boldsymbol{s}_2 - \boxed{(\boldsymbol{s}_2)_{|\mathcal{S}_1}}$
3. $\phi_2 = (\boldsymbol{s}_2)_{\perp \mathcal{S}_1} / \|(\boldsymbol{s}_2)_{\perp \mathcal{S}_1}\|$. ϕ_2
4. Using induction, $\boxed{\{\phi_1, \ldots \phi_k\} \to S_k}$
5. $\boxed{(\boldsymbol{s}_{k+1})_{|\mathcal{S}_k}} = \sum_{j=1}^{k} \langle \boxed{\boldsymbol{s}_{k+1}}, \phi_j \rangle \phi_j$ $S_{k+1}|_{S_k}$
6. $(\boldsymbol{s}_{k+1})_{\perp \mathcal{S}_k} = \boldsymbol{s}_{k+1} - \sum_{j=1}^{k} \langle \boldsymbol{s}_{k+1}, \phi_j \rangle \phi_j$
7.

$$\phi_{k+1} = \frac{(\boldsymbol{s}_{k+1})_{\perp \mathcal{S}_k}}{\|(\boldsymbol{s}_{k+1})_{\perp \mathcal{S}_k}\|}$$

Signal Space

- Norm $|x(t)|$. Energy $\mathcal{E}_x$

$$\|x(t)\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 dt\right)^{1/2} = \sqrt{\mathcal{E}_x}$$

- Inner produce:

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

- $x_1(t), x_2(t)$ are orthogonal if their inner product is 0.
- Signals are orthonormal if they are orthogonal and their norm are unity
- Signals are linearly independent if no signal can be represented as a linear combination of remaining signals
- $\|x_1(t) + x_2(t)\| \leq \|x_1(t)\| + \|x_2(t)\|$
- $\|\langle x_1(t), x_2(t)\rangle\| \leq \|x_1(t)\|\|x_2(t)\|$

Gram-Schmidt for Signals

Given an finite energy signal waveforms $\{s_m(t)\}_{m=1}^{M}$, generate an orthonormal waveforms $\{\phi_1(t), \phi_2(t), ..., \phi_M(t)\}$

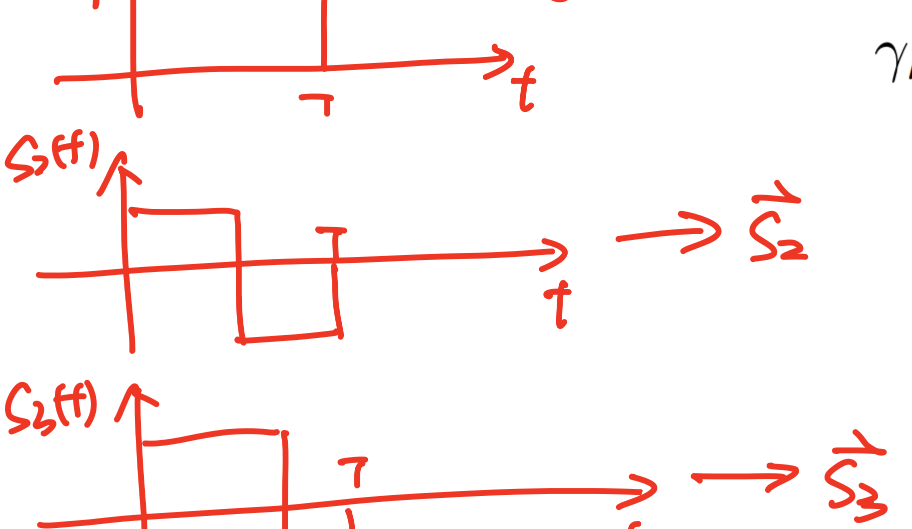
1. $\phi_1(t) = s_1(t)/\sqrt{E_1}$.
2. $c_{21} = \langle s_2(t), \phi_1(t)\rangle$
3. $\gamma_2(t) = s_2(t) - c_{21}\phi_1(t)$, and let $E_2 = \int_{-\infty}^{\infty} \gamma_2^2(t)dt$
4. $\phi_2(t) = \gamma_2(t)/\sqrt{E_2}$
5. $\phi_k(t) = \gamma_k(t)/\sqrt{E_k}$

   where

$S_{k-1} = \{\phi_1(t) \cdots \phi_{k-1}(t)\}$

$E_1 = \int |s_1(t)|^2 dt \Rightarrow \vec{s}_1$

$s_k(t) \,|\, S_{k-1}$



$$\gamma_k(t) = s_k(t) - \boxed{\sum_{i=1}^{k-1} c_{ki}\phi_i(t)}$$

$$c_{ki} = \langle s_k(t), \phi_i(t)\rangle$$

$$E_k = \int_{-\infty}^{\infty} \gamma_k^2(t)dt$$

Orthonormal Expansions in $\mathcal{L}_2$

- Given an orthonormal basis $\{\phi_1(t), \phi_2(t), \cdots\}$ of $\mathcal{L}_2$, any $\mathcal{L}_2$ signal x(t) can be represented as $x(t) = \sum_j x_j \phi_j(t)$.
  - $x_j = \langle x(\mathrm{t}), \phi_j(t) \rangle = \int x(t)\phi_j^*(t)dt$.
  - $\boldsymbol{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots)$
  - Equality holds in the sense of MSE=0.
    - $\int_{-\infty}^{\infty} \left| x(t) - \sum_j x_j \phi_j(t) \right|^2 dt = 0$
  - $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \sum_j |x_j|^2 = \|\boldsymbol{x}\|^2$
  - $\langle x(t), s(t) \rangle = \langle \boldsymbol{x}, \boldsymbol{s} \rangle$
- Example
  - Fourier Series
  - Sampling Theorem

$x(t) \longleftrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x \end{bmatrix}$

orthonormal basis for some subspace of $\mathcal{L}_2$

Time-limited $\mathcal{L}_2$ functions $\rightarrow$

Band-limited $\mathcal{L}_2$ functions $\rightarrow$

Orthonormal Expansions in $\mathcal{L}_2$

- Fourier Series

Given a function $v(t)$, with duration $[-T/2, T/2]$

- Define $\theta_k(t) = e^{2\pi ikt/T}\text{rect}(t/T)$
- $\{\theta_k(t)\}_k$ are orthogonal functions, with $||\theta_k(t)||^2 = T$
- Let $\phi_k(t) = \theta_k(t)/\sqrt{T}$:

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{2\pi ikt/T}\text{rect}(t/T)$$

- By FSE, $v(t)$ can be written as

$$v(t) = \sum_k \alpha_k \phi_k(t)$$

where $\alpha_k =< v(t), \phi_k(t) >$.

Orthonormal Expansions in $\mathcal{L}_2$

- Sampling Theorem

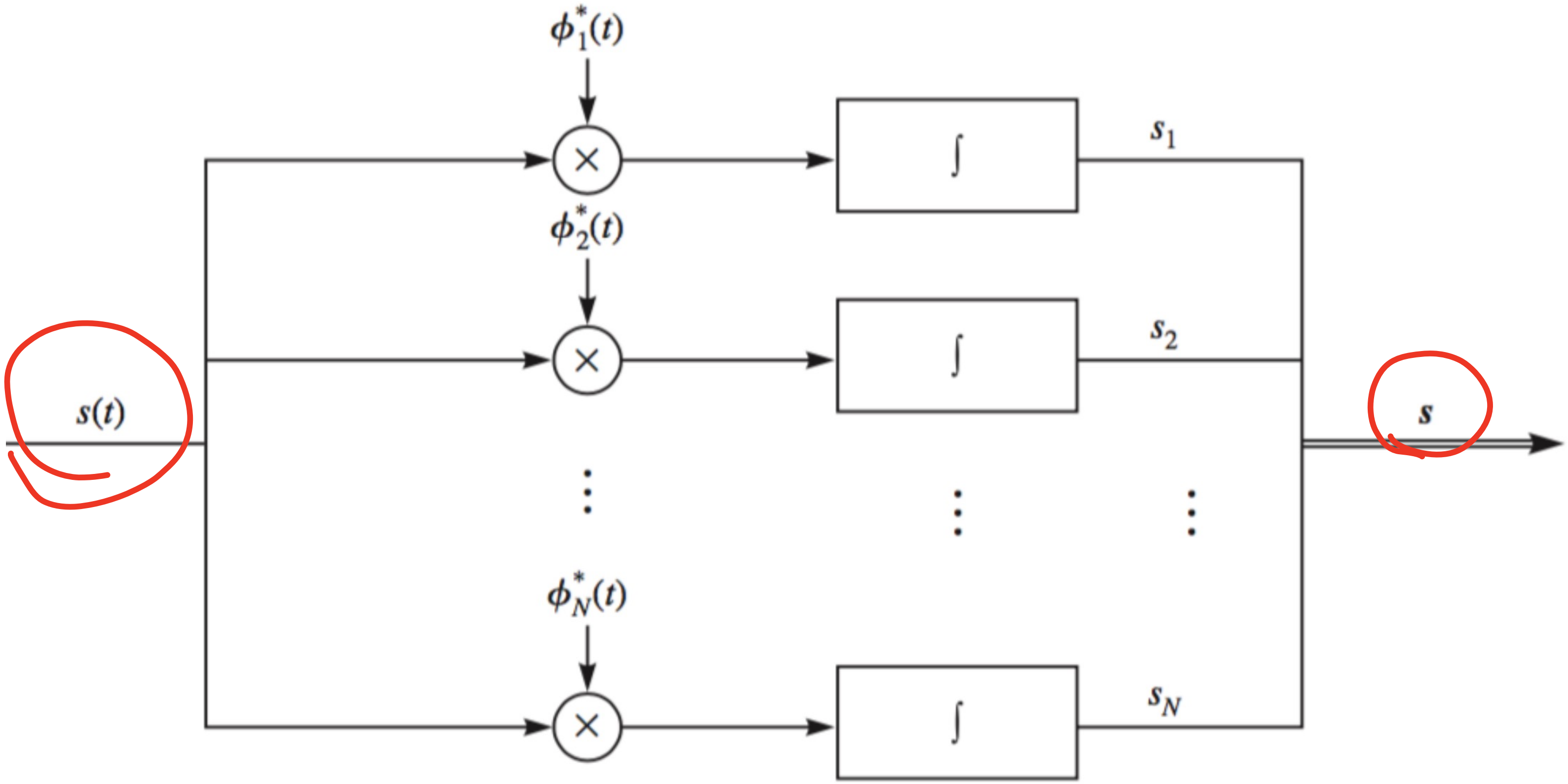
Sampling Theorem: If $x(t)$ is a band-limited in W, then

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left[\left(\frac{t}{T} - n\right)\right]$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, $T = 1/2W$.

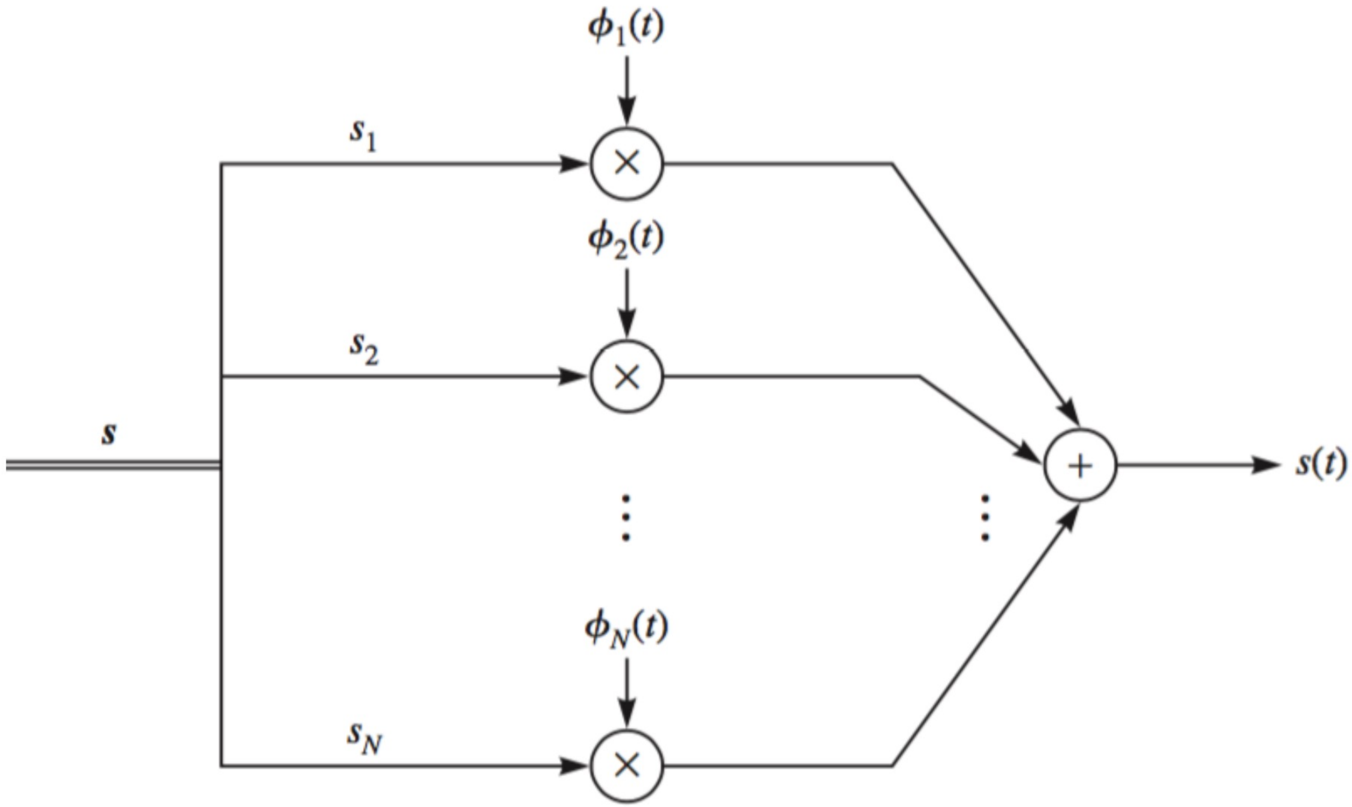
- $x(nT)$ is coefficient (x_k)
- $\{\text{sinc}\left[\left(\frac{t}{T} - n\right)\right]\}_{n=-\infty}^{\infty}$ are orthogonal function
- Equality holds in sense of $E_e = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 = 0$
- View signal $x(t)$ as vectors $x(nT)$

Map Signal to Vector



$$s_k = \int_{-\infty}^{\infty} s(t)\phi_k^*(t)dt = \langle s(t), \phi_k(t)\rangle$$

Map Vector to Signal



$$s(t) = \sum_{k=1}^{N} s_k \phi_k(t)$$

Thanks for your kind attention!

Questions?