EE140 Introduction to Communication Systems

Lecture 4

Instructor: Prof. Lixiang Lian

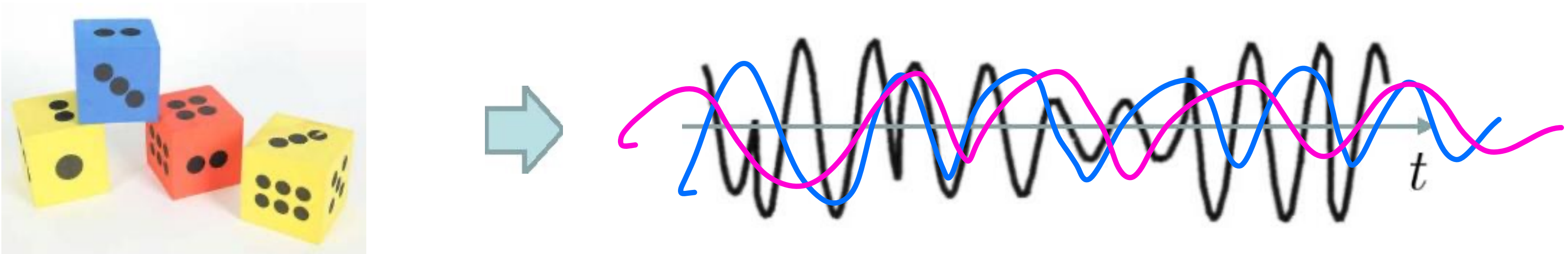
ShanghaiTech University, Fall 2025

Contents

- Random signals
  - Review of probability and random variables
  - Random processes: basic concepts
  - Gaussian white processes

Random Process

- A random process (stochastic process, or random signal) is the evolution of random variables over time.

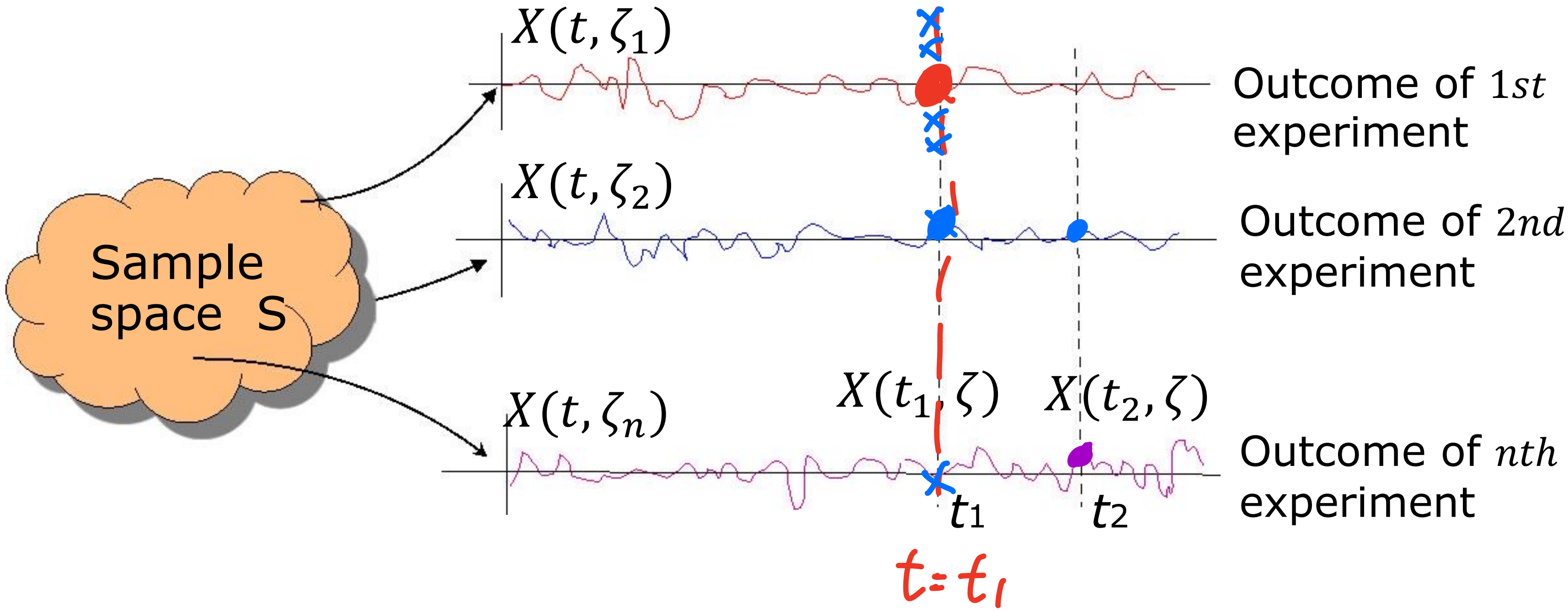


Description of Random Process

ζ: random variable

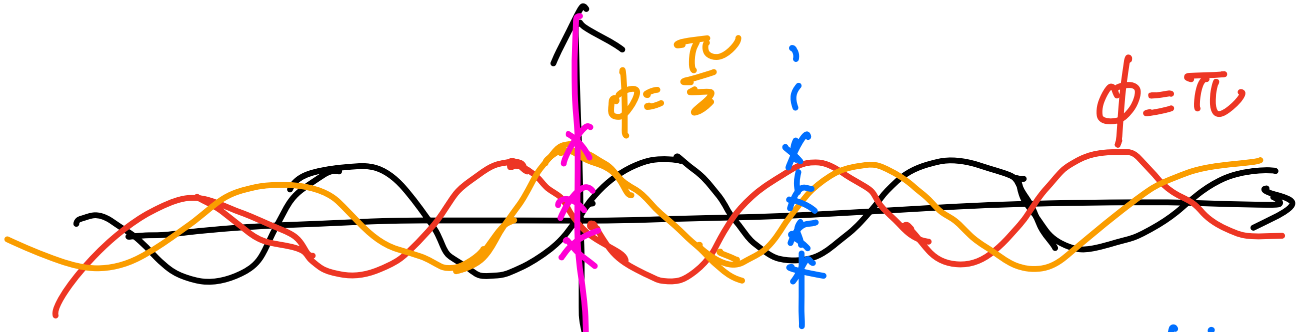
- $X(t, \zeta)$: random process
- $X(t, \zeta_i)$: sample function of the random process, ζ_i is a member of a sample space $\mathcal{S}$.
- $X(t_j, \zeta)$: a random variable
- $X(t_j, \zeta_i)$: a number

ensemble



Example. $x(t) = A\sin(2\pi f t + \phi) \triangleq X(t, \zeta)$

① A, f fixed. $\phi \sim U[0, 2\pi)$ $X(t, \phi)$



$t=0$

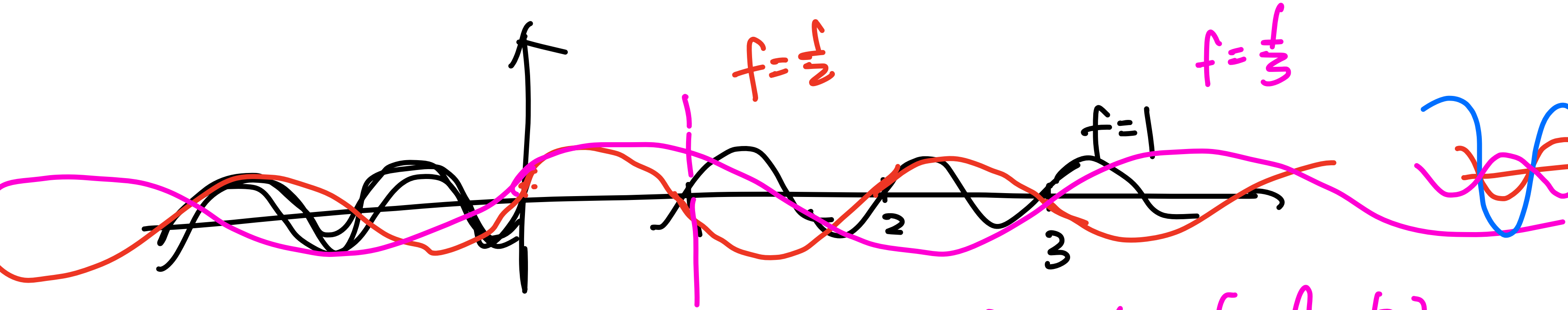
$A\sin(\phi)$

$[-A, A]$

$t=1$ $X(t=1, \phi) = A\sin(2\pi f + \underline{\phi})$

$[-A, A]$

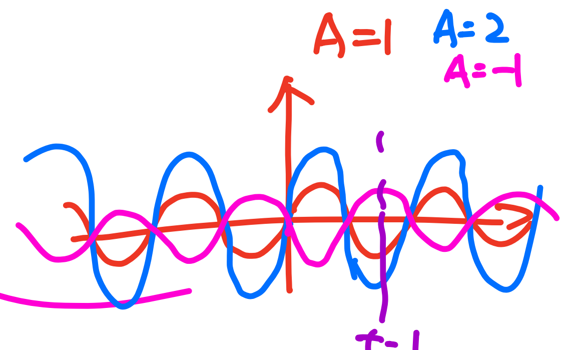
② A, ϕ fixed. $f \sim U[0,1]$ $X(t, f)$



$t=1$ $X(t=1, f) = A\sin(2\pi \underline{f} + \phi)$

③ f, ϕ fixed

$A \sim N(0,1)$



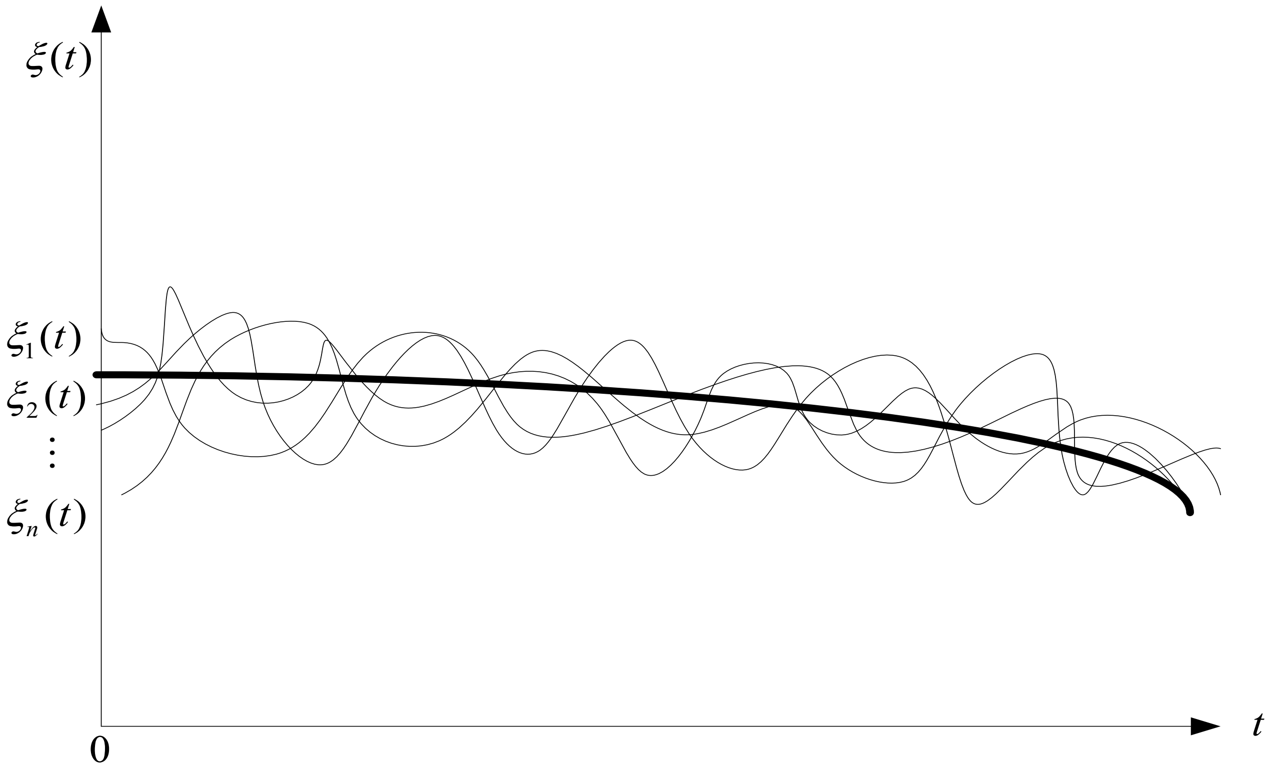
$X(t=1, A) = A\sin(2\pi f + \phi)$

$\sim N(0, \sin^2(\ ))$

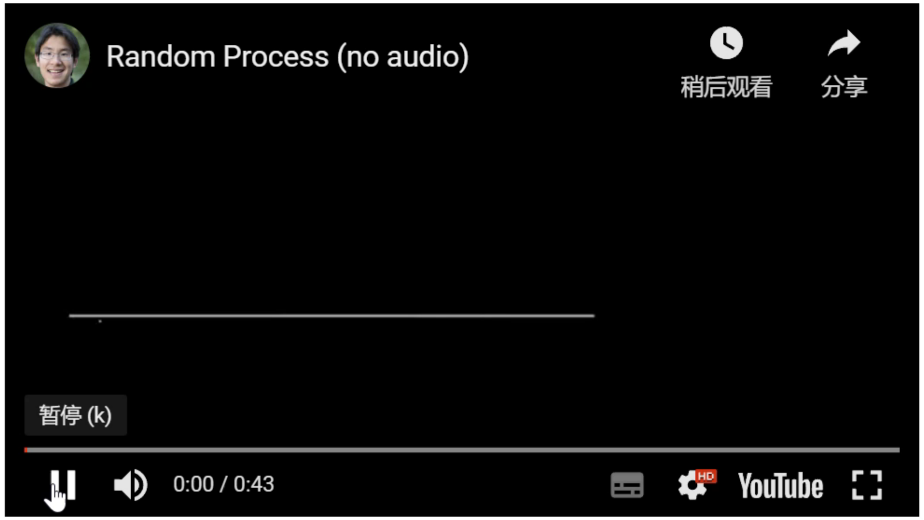
$t=5$ $X(t=5, A) = N(0, \sin^2(\ ))$

Example

- Observation of noise
  - $\xi_i(t)$, one realization, deterministic
  - $\xi(t) = \{\xi_1(t), \xi_2(t), \ldots, \xi_n(t)\}$, random process, the set of all realizations.



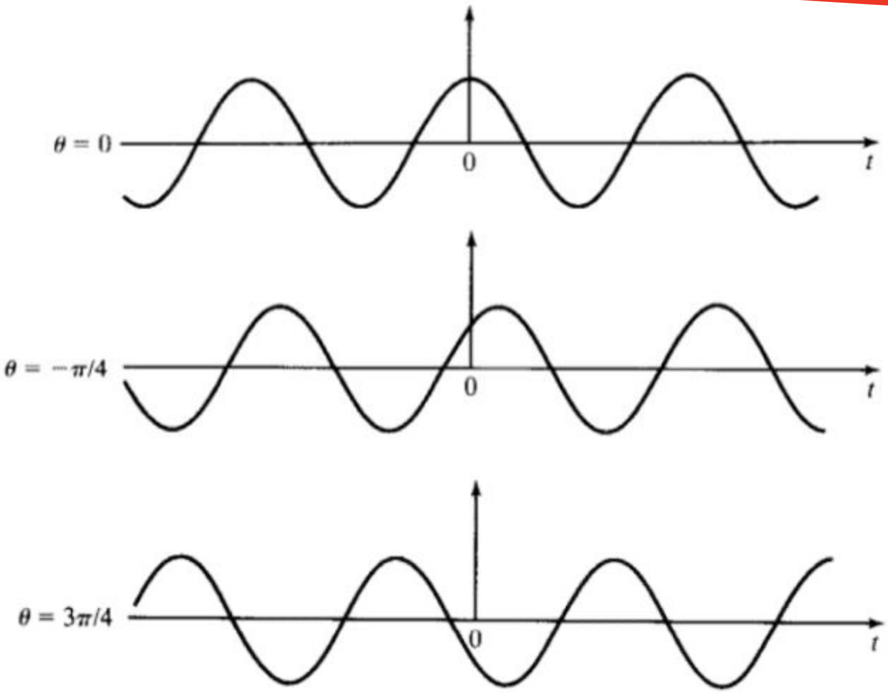
Examples

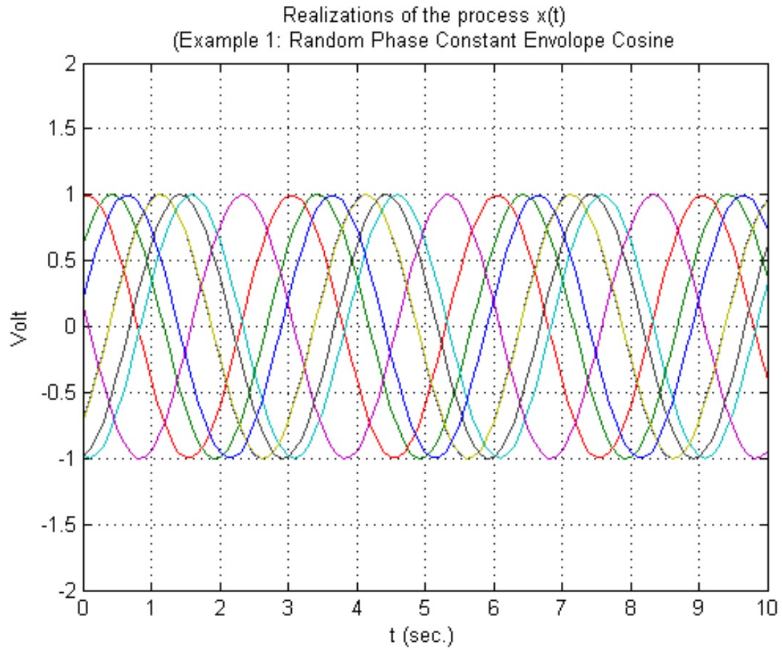


Example

- Uniformly choose a phase θ between $[0, 2\pi]$ and generate a sinusoid with a fixed amplitude and frequency but with a random phase θ.
- In this case, the random process is

$$X(t) = A\cos(2\pi f_0 t + \theta)$$





Statistics of Random Processes

- An infinite collection of random variables specified at time $t_1, t_2, \cdots, t_n, \forall n$

$X(t, \xi) = X(t)$

$$\{X(t_1), X(t_2), \ldots, X(t_n)\}$$

- Joint pdf (different notations)

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n), \forall n$$

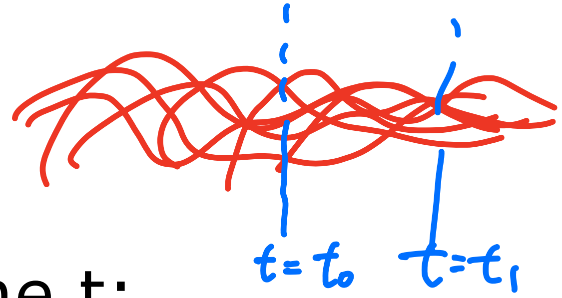
$$f_{X_1,X_2,\cdots,X_n}(x_1, x_2, \cdots, x_n), \forall n$$

lectures

$$f_{X_1,X_2,\cdots,X_n}(x_1, t_1; x_2, t_2; \cdots; x_n, t_n), \forall n$$

textbook

First Order Statistics



- Probability density function of $X(t)$ at time t:

$$f_{X(t)}(x)$$

- Mean $E[X(t_0)] = E[X(t = t_0)] = \int_{-\infty}^{\infty} x f_{X(t_0)}(x)dx$

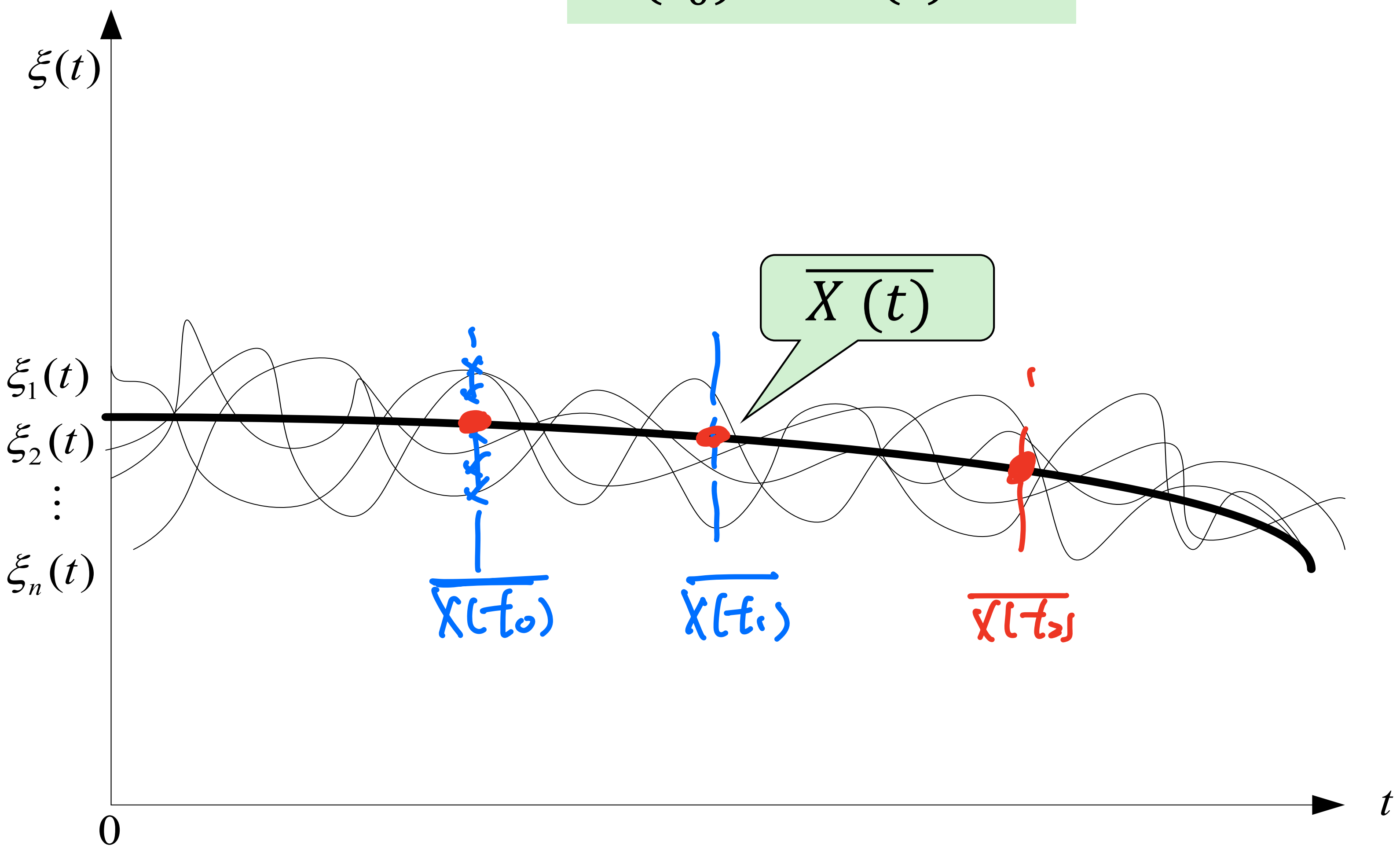
  t.

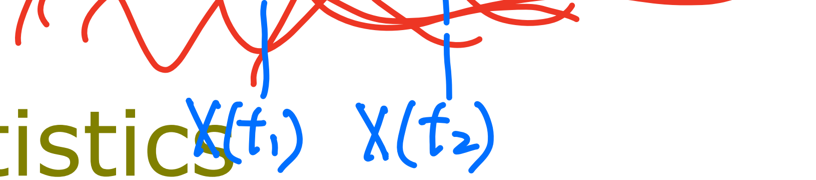
  $= \overline{X\,(t_0)}$

- Variance $E\left[\left|X(t_0) - \overline{X\,(t_0)}\right|^2\right] = \sigma_X^2(t_0)$

Example

$$\overline{X\ (t_0)} \Rightarrow \overline{X\ (t)}$$





Second-Order Statistics

- Joint pdf of the random variables $X(t_1), X(t_2)$

$$f_{X(t_1),X(t_2)}(x_1, x_2) \triangleq f_{X_1,X_2}(x_1, x_2),$$
$$X_1 = X(t_1), X_2 = X(t_2).$$

- Autocorrelation function of the process $X(t)$ (correlation within a process):

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, x_2)dx_1 dx_2$$

- Autocovariance function

$$\mu_X(t_1, t_2) = E\{[X(t_1) - \overline{X(t_1)}][X(t_2) - \overline{X(t_2)}]\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_1 - \overline{X(t_1)}][x_2 - \overline{X(t_2)}] f_{X_1,X_2}(x_1, x_2)dx_1 dx_2$$
$$= R_X(t_1, t_2) - \overline{X(t_1)}\,\overline{X(t_2)}$$

$t_1 = t_2$, variance of X(t)

$\sigma_X^2(t_1) = \mu_X(t_1, t_1)$
$= R_X(t_1, t_1) - \overline{X(t_1)}^2$

Example 1

- Consider $X(t) = A\cos(2\pi f t + \theta)$, where θ is uniform in $[-\pi, \pi]$
- Mean

$$E[X(t)] = \int_{-\pi}^{\pi} A\cos(2\pi f t + \theta)\frac{1}{2\pi}\,d\theta = 0$$

- Autocorrelation

Let $t_1 = t, t_2 = t + \tau$

$$E[X(t_1)X(t_2)] = E[A\cos(2\pi f t + \theta)A\cos(2\pi f(t+\tau) + \theta)]$$

$$= \frac{A^2}{2}E[\cos(4\pi f t + 2\pi f \tau + 2\theta) + \cos(2\pi f \tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f t + 2\pi f\tau + 2\theta)d\theta + \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(2\pi f\tau)d\theta$$

$$= 0 + \frac{A^2}{2}\cos(2\pi f\tau)$$

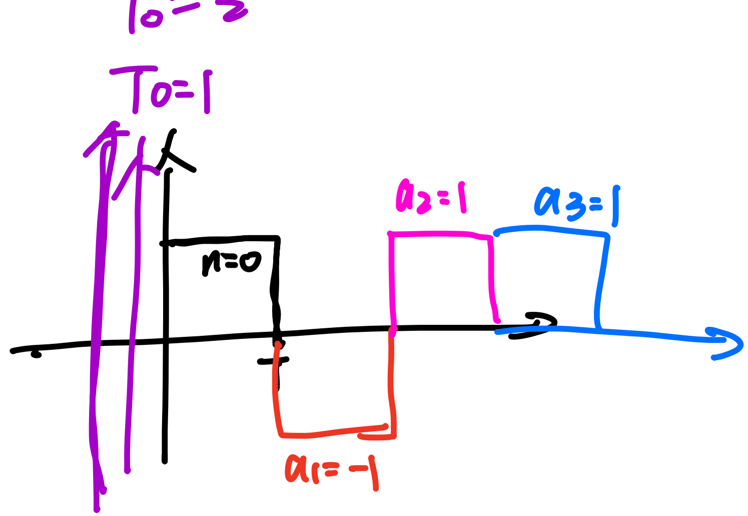
$$\Rightarrow R_X(t, t+\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$$

Example 2

- Consider $Y(t) = B\cos\omega_c t$, where $B \sim N(0$,! $b^2)$
- Find its mean and autocorrelation function

$$E[Y(t)] = 0$$

$$E[Y(t)Y(t+\tau)] = E[B^2]\cos\omega_c t\cos\omega_c(t+\tau)$$
$$= b^2 \cos\omega_c t\cos\omega_c(t+\tau)$$



Example 3

- Given a binary random signal

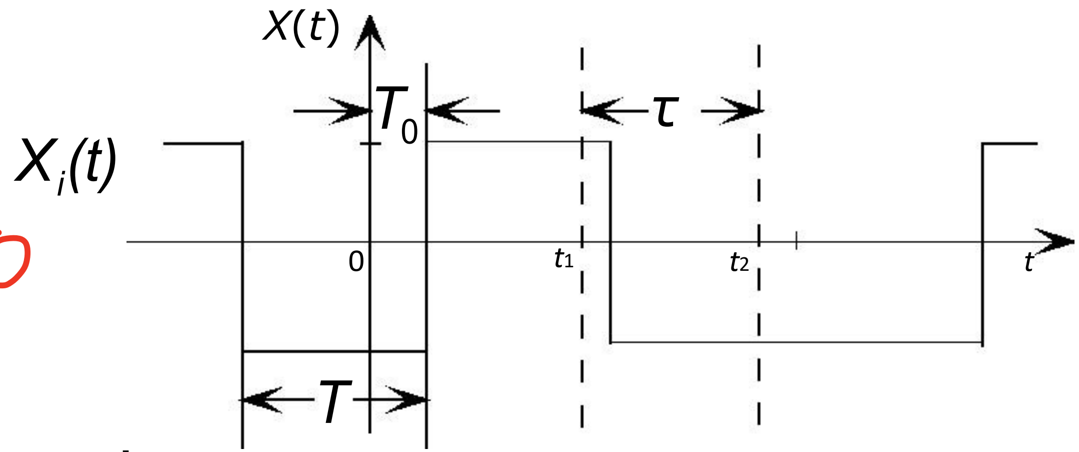
$$X(t) = \sum_n a_n p(t - nT - T_0)$$

  - $p(t)$ is a rectangular pulse shaping function with width T
  - a_n is a random variable that takes +1 or -1 with equal probability, and it is independent for different n
  - T_0 is a random time delay uniformly distributed within $[0, T]$
  - A typical sample function of X(t) is

$|\tau| > T$

$E[X(t) X(t+\tau)] = 0$

$E[X(t)] E[X(t+\tau)] = 0$



Find its autocorrelation function

Example 3

- Given a binary random signal

$$X(t) = \sum_n a_n p(t - nT - T_0)$$

$E[a_n^2] = 1$

$E[a_n a_m] = 0$

- Solution

$$R_X(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E\left[\sum_n a_n p(t - nT - T_0) \sum_m a_m p(t + \tau - mT - T_0)\right]$$

Independence of a_n

$$= \sum_n E[a_n^2] E[p(t - nT - T_0)\, p(t + \tau - nT - T_0)]$$

$$= \sum_n \int_0^T \frac{1}{T} p(t - nT - T_0)\, p(t + \tau - nT - T_0) \mathrm{d}T_0$$

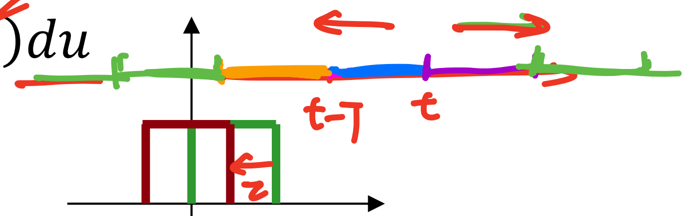
Let $t - nT - T_0 = u$

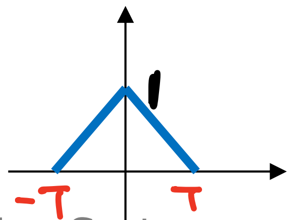
$T_0 \in [0, T)$

$$= \sum_n \int_{t-nT-T}^{t-nT} \frac{1}{T} p(u)\, p(u + \tau) du$$

$$= \int_{-\infty}^{\infty} \frac{1}{T} p(u) p(u + \tau) du$$

$$= \begin{cases} \frac{T - |\tau|}{T}, |\tau| < T \\ 0, |\tau| \geq T \end{cases}$$





$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \qquad \sigma_{XY} = E[XY] - E[X]E[Y]$$

$$\rho_{X(t)X(t+\tau)} = \frac{\mu_X(t, t+\tau)}{\sigma_X(t)\,\sigma_X(t+\tau)} = \frac{R_X(t, t+\tau) - \overline{X(t)}\,\overline{X(t+\tau)}}{\sigma_X(t)\,\sigma_X(t+\tau)}$$

$$X(t) = 0$$

$$R_X(\tau) = \begin{cases} \frac{T-|\tau|}{T} & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

$$\mu_X(t,t) = \sigma_X^2(t) = R_X(t,t) - 0 \cdot 0 = R_X(0) = 1$$

$$\Rightarrow \rho_{X(t)X(t+\tau)} = R_X(\tau)$$

Stationary Processes

in strict sense

- A stochastic process is said to be stationary if for any *n* and *τ*:

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1,x_2,\ldots,x_n) = f_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_n+\tau)}(x_1,x_2,\ldots,x_n), \quad \forall n,\tau$$

$$\Downarrow$$

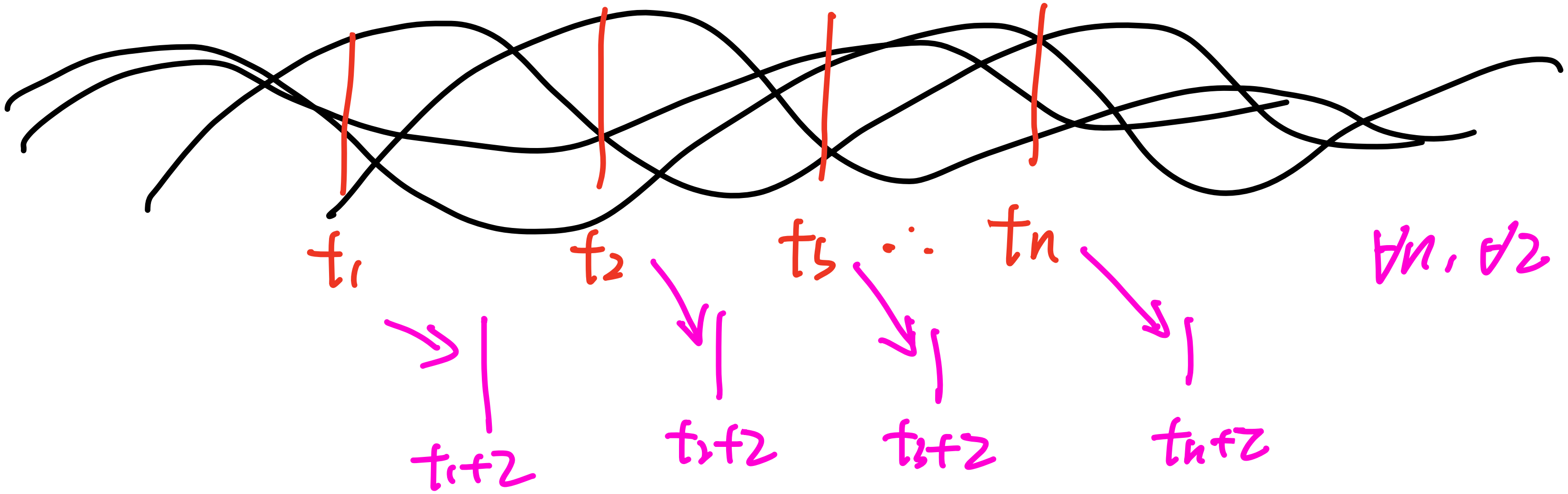
- First-order statistics is independent of t

$$E\{X(\mathrm{t})\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx = m_X, E\left\{X(\mathrm{t}) - \overline{X(t)}\right\}^2 = \sigma_X^2.$$

- Second-order statistics only depends on the gap

$\tau = t_2 - t_1$

$$R_X(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1dx_2$$
$$= R_X(t_2 - t_1) = R_X(\tau), \text{where } \tau = t_2 - t_1$$



if $n=1$ $\quad f_{X(t)}(x) = f_{X(t+\tau)}(x) \quad \forall \tau,$

$E[X(t)] = m_X$ constant $\qquad Var[X(t)] = \sigma_X^2$ constant

if $n=2$ $\quad f_{X(t_1)X(t_2)}(x_1,x_2) = f_{X(t_1+\tau)X(t_2+\tau)}(x_1,x_2) \quad \forall \tau$

$$R_X(t_1,t_2) = \int x_1 x_2 \boxed{f_{X(t_1)X(t_2)}(x_1,x_2)}\, dx_1\, dx_2$$

$$= R_X(t_2-t_1) = R_X(\tau)$$

Wide-Sense Stationary (WSS)

- A random process is said to be WSS when

$$E\{X(\mathrm{t})\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx = m_X,\ E\{X(\mathrm{t}) - \overline{X(t)}\}^2 = \sigma_X^2.$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$
$$= R_X(t_2 - t_1) = R_X(\tau)$$

  - Defined with the first order and second order statistics only

- Compare the strictly stationary

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n)$$
$$= f_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_n+\tau)}(x_1, x_2, \ldots, x_n),\quad \forall n, \tau$$

Examples

- Example 1: Determine if $X(t)$ is WSS

$$X(t) = A\cos(2\pi f t + \theta), \text{ where } \theta \sim U(-\pi, \pi)$$

  – Check the first order and second order statistics

$$E[X(\mathrm{t})] = 0$$

$$R_X(t, t+\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

$\Rightarrow$ $X(t)$ is WSS

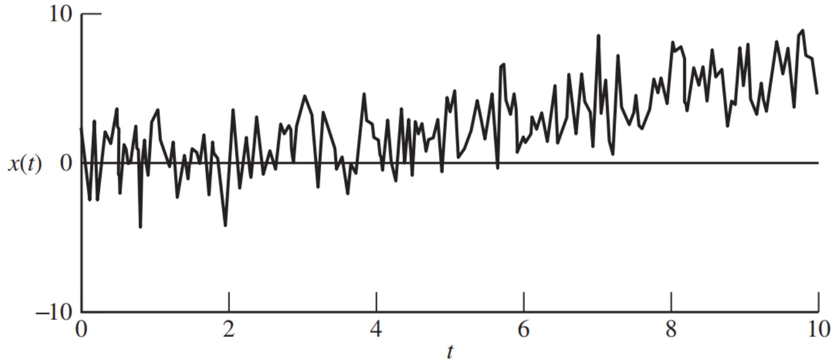
- Example 2: Determine if $Y(t)$ is WSS

$$Y(t) = B\cos\omega_c t\text{, where } B \sim N(0\ !\,\mathrm{b}^2)$$

$$E[Y(t)] = 0$$

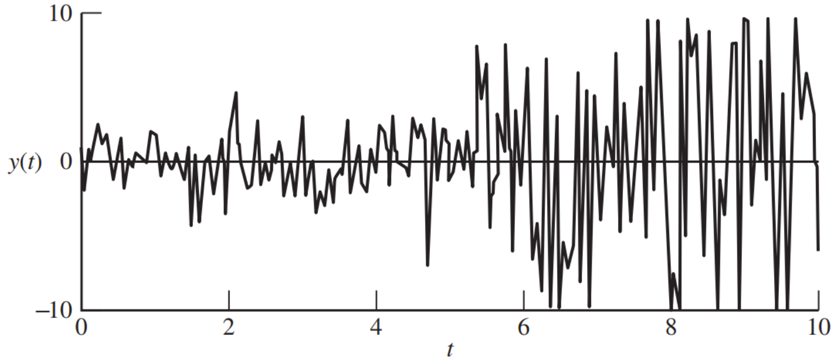
$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)] = E[B^2]\cos\omega_c t\cos\omega_c(t+\tau)$$
$$= b^2\cos\omega_c t\cos\omega_c(t+\tau)$$

Examples



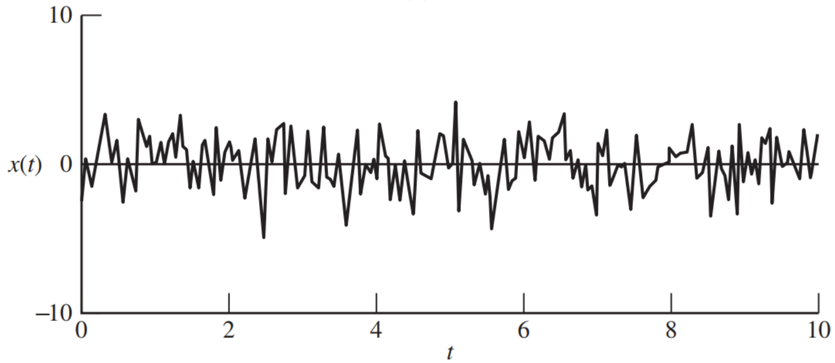
(a)

Time-varying mean



(b)

Time-varying variance



Stationary

$X(t,\xi)$

Averages and Ergodic

- Ensemble (or statistical) averaging

$$\overline{X(t)} \triangleq E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx, \quad (t) \quad m_x \text{ constant}$$

$$(t,\tau)$$

$$R_X(t,t+\tau) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2f_{X_1,X_2}(t,t+\tau)dx_1dx_2 \quad (\tau)$$

Function of time

- Time averaging

$$< X(t) > \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt \quad (\xi) \quad m_x \text{ constant}$$

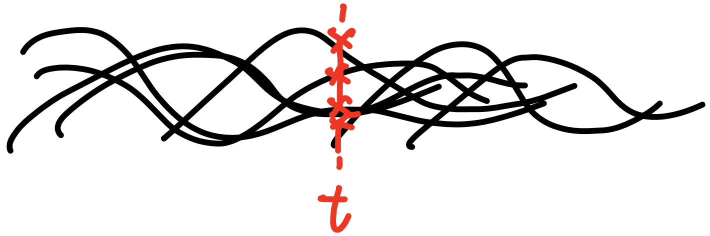
$$(\tau,\xi)$$

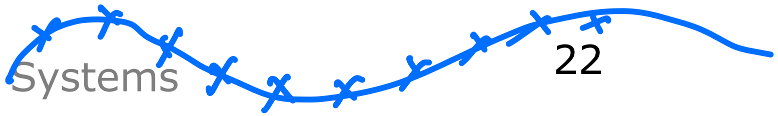
$$< X(t)X(t+\tau) >= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt \quad (\tau)$$

Random variable

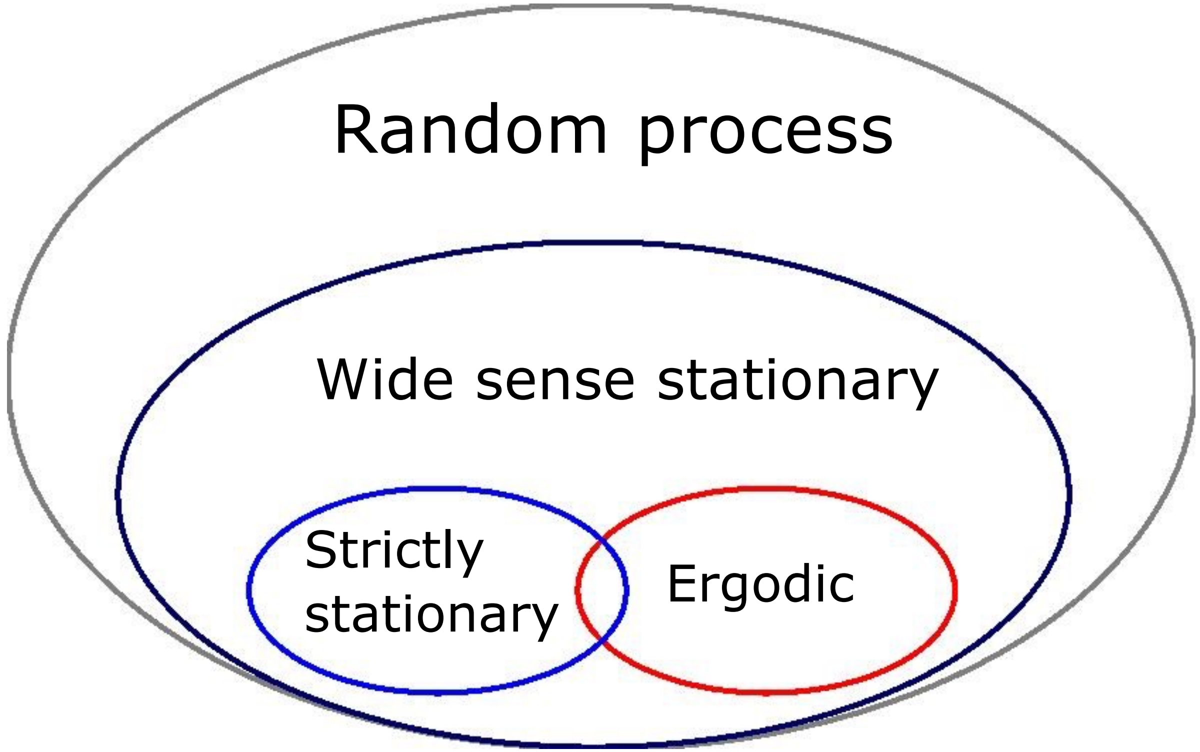
- If ensemble average = time average, $X(t)$ is said to be Ergodic (各态历经)

ergodic RP $\Longrightarrow$ WSS RP





Applications of Random Process



- Applications
  - Signal: WSS
  - Noise: (strictly) stationary
  - Time-varying channel: ergodic

Examples

- Example 1: Determine if $X(t)$ is Ergodic

$$X(t) = A\cos(2\pi f t + \theta), \text{ where } \theta \sim U(-\pi, \pi)$$

Statistical average

$$E[X(\mathrm{t})] = 0$$

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

Time average

$$< X(\mathrm{t}) >= \frac{1}{T}\int_0^T A\cos(2\pi f t + \theta)\,dt = 0$$

$$< X(t)X(t+\tau) >$$

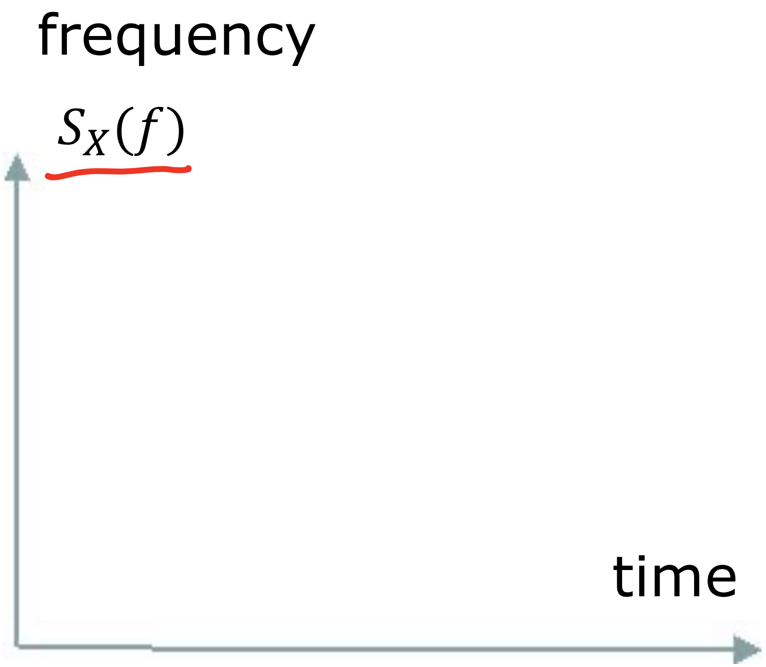
$$= \frac{1}{T}\int_0^T A^2\cos(2\pi f t + \theta)\cos(2\pi f(t+\tau) + \theta)\,dt$$

$$= \frac{A^2}{2}\cos(2\pi f \tau)$$

$X(t)$ is Ergodic

Frequency Domain Characteristics of Random Process

- Power Spectral Density



if $x(t)$ energy signal (T)

$x(t) \xrightarrow{FT} X(f) \rightarrow G(f) = |X(f)|^2$

$S_X(f) = \frac{G(f)}{T} = \frac{|X(f)|^2}{T}$

$m(t) = E[X(\mathrm{t})]$

$R_X(t, \tau) = E[X(t)X(t+\tau)]$

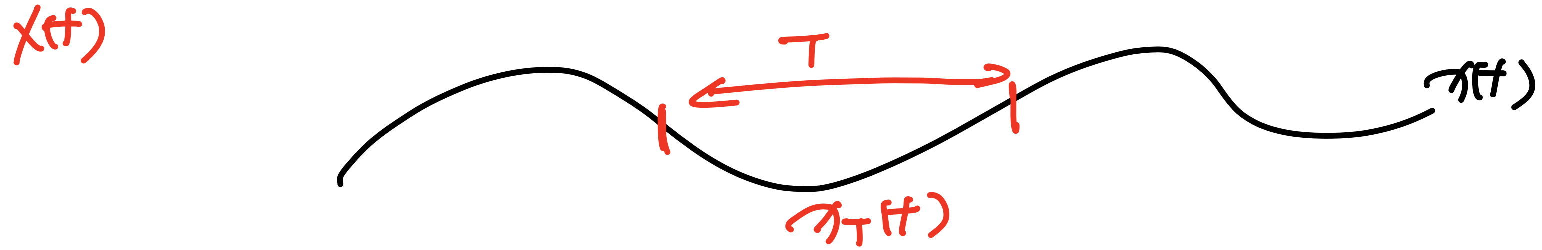
PSD of Random Process

- PSD of deterministic signal

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

- Consider x(t) as a sample function of a random process X(t). The PSD of X(t) is given by

$$S_X(f) = \lim_{T\to\infty} \frac{E\{|X_T(f)|^2\}}{T}$$



$$X(t) \xrightarrow{\text{sample function}} x(t) \xrightarrow{\text{Truncate}} x_T(t) \xrightarrow{FT} X_T(f) \longrightarrow |X_T(f)|^2$$

$$S_X(f) = \lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T}$$

PSD of WSS Process

- Wiener-Khinchine theorem (Page 318)
- For WSS process

$\tau = 0$ $P = R_x(0) = \int S_x(f)\, df$

$$S_X(f) \leftrightarrow R_X(\tau) \begin{cases} R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau) df \\ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \end{cases}$$

- Property:

Autocorrelation function $R_X(\tau)$	PSD $S_X(f)$
Total power $R_X(0) = E[X(t)^2] = \int_{-\infty}^{\infty} S_X(f) df$	$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$
$\lvert R(\tau)\rvert \leq R(0)$	$S_X(f) \geq 0, \forall f$
$R(-\tau) = R(\tau)$	$S_X(f) = S_X(-f)$
$\lim_{\lvert\tau\rvert\to\infty} R(\tau) = \overline{X(t)}^2$ if X(t) does not contain a periodic component	

$|R(\tau)| \le R(0)$

proof: $E\left[|x(t) \pm x(t+\tau)|^2\right] \ge 0$

$\Rightarrow \quad R_x(0) + R_x(0) \pm 2R_x(\tau) \ge 0$

$\Rightarrow \quad -R_x(0) \le R_x(\tau) \le R_x(0) \quad \Rightarrow \quad |R_x(\tau)| \le R_x(0)$

$R(-\tau) = R(\tau)$

proof: $R(\tau) = E[x(t)\,x(\underbrace{t+\tau}_{t'})] = E[x(t'-\tau)\,x(t')] = R_x(-\tau)$

$\lim_{|\tau|\to\infty} R(\tau) = \lim_{|\tau|\to\infty} E[x(t)\,x(t+\tau)]$

$= E[x(t)]\,E[x(t+\tau)]$

$= m_x^2$

PSD of Ergodic Random Process

- By definition:
  - the time average of the auto-correlation function of the sample function equals the auto-correlation function of the random process, or

$$< X(t)X(t+\tau) > = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$\Updownarrow$$

$$R_X(\tau) = \overline{X(t)X(t+\tau)}$$

- We have

$$< X(t)X(t+\tau) > \Leftrightarrow S_X(f)$$

Example 1

- For the random process

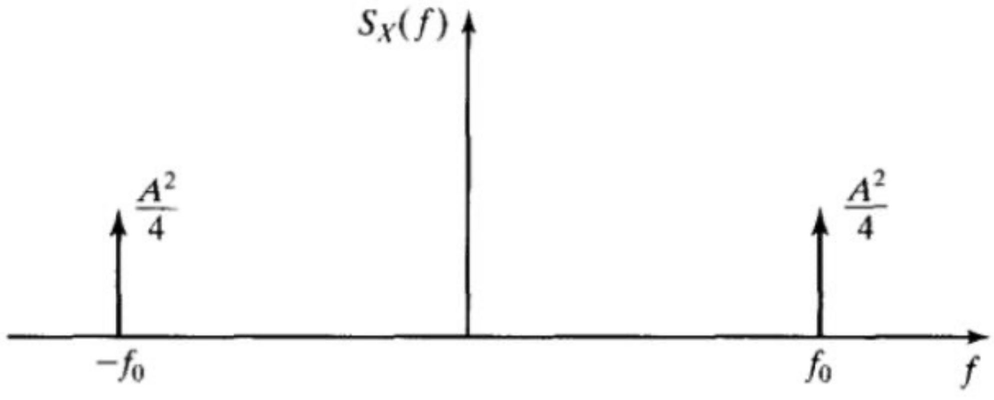
$$X(t) = A\cos(2\pi f_0 t + \theta)$$

- Autocorrelation

$$\Rightarrow R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

- PSD

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

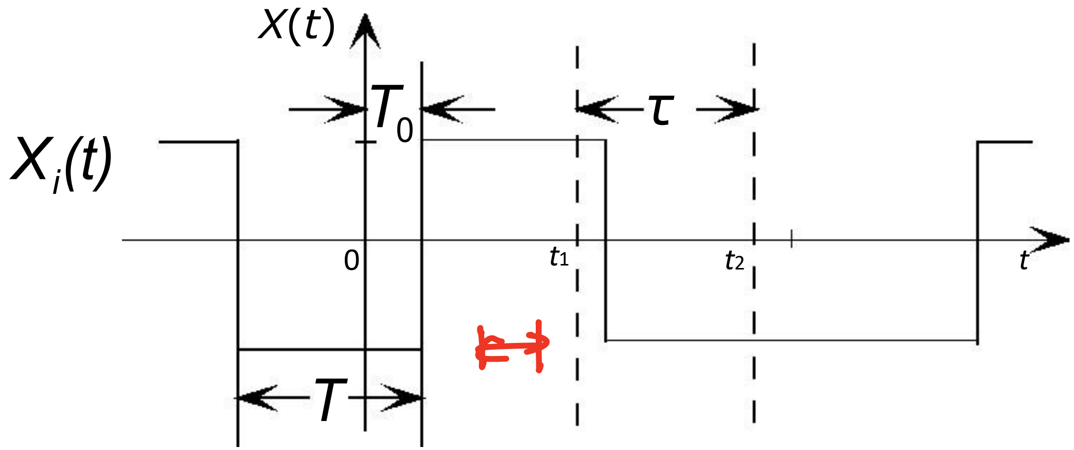


Example 2

- Given a binary random signal

$$X(t) = \sum_n a_n p(t - nT - T_0)$$

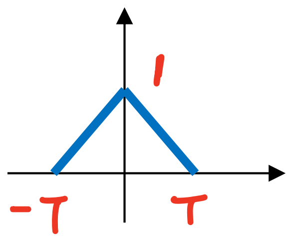
  - $p(t)$ is a rectangular pulse shaping function with width T
  - a_n is a random variable that takes +1 or -1 with equal probability, and it is independent for different n
  - T_0 is a random time delay uniformly distributed within $[0, T]$
  - A typical sample function of X(t) is



Example (cont'd)

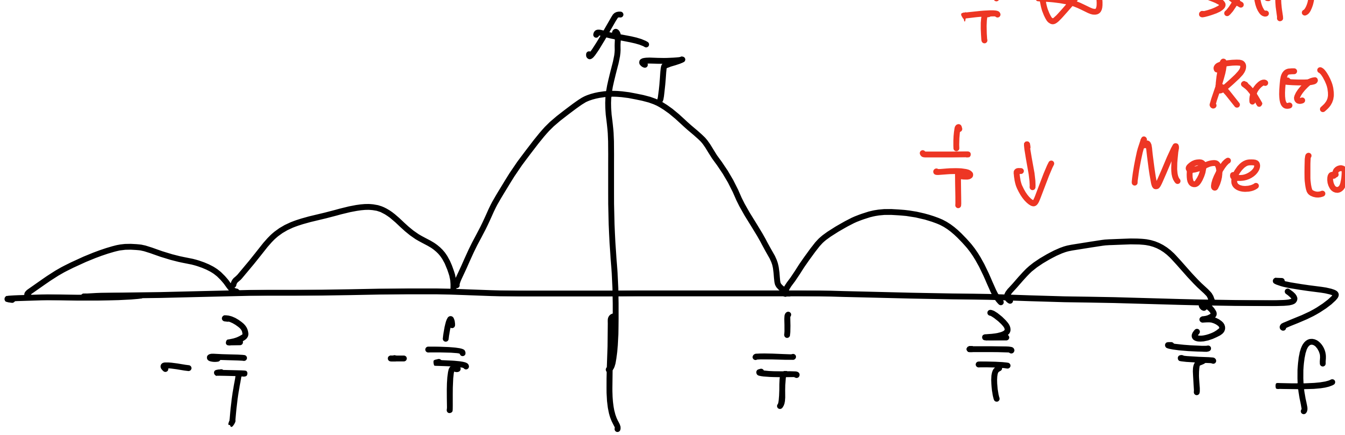
- Autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, -T < \tau < T \\ 0, \text{ otherwise} \end{cases}$$



- PSD

$$S_X(f) = T sinc^2(fT)$$



correlation ↓

$\frac{1}{T}$ ↑ More High freq.

$\frac{1}{T}$ ∞ $S_X(f)$ = constant

$R_X(\tau) = \delta(\tau)$ correlation = 0

$\frac{1}{T}$ ↓ More Low freq.

correlation ↑

$\frac{1}{T} = 0$ $S_X(f) = \delta(f)$

$R_X(\tau) = C$

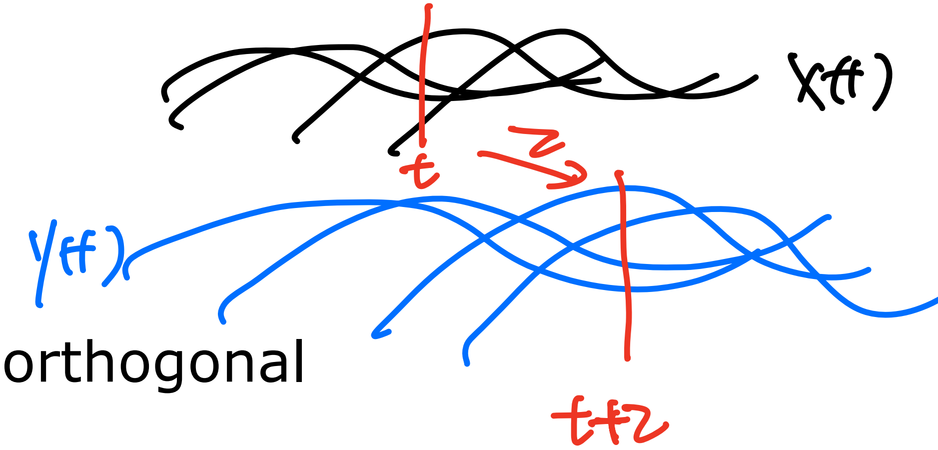
Cross Correlation

- X(t), Y(t): each WSS, jointly WSS.
- $n(t) = X(t) + Y(t)$, calculate the power of n(t)

$$E[n^2(t)] = E\{[X(t) + Y(t)]^2\} = P_X + 2E[X(t)Y(t)] + P_Y$$

- Cross-correlation function

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$



  – $R_{XY}(\tau) = 0, \forall\tau \Longrightarrow$ X(t) and Y(t) are orthogonal
  – Property: $R_{XY}(\tau) = R_{YX}(-\tau)$

$R_{XY}(\tau) = E[X(t)Y(\underbrace{t+\tau}_{t'})] = E[X(t'-\tau)Y(t')]$

$= E[Y(t')X(t'-\tau)]$

$= R_{YX}(-\tau)$

orthogonal + $E[X(t)]E[Y(t+\tau)] = 0$

$\Downarrow$

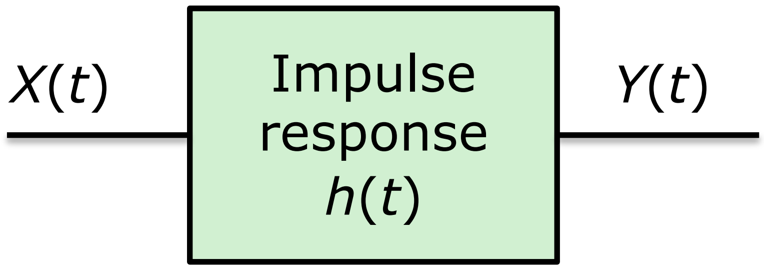
uncorrelated

Independent

- Cross PSD: $S_{XY}(f) = \mathfrak{F}[R_{XY}(\tau)]$

Random Process Transmission Through Linear Systems

- Consider a linear system (channel)



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau) d\tau$$

- Mean

$$\overline{Y}(t) = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)] d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \overline{X}(t-\tau) d\tau$$

If $X(t)$ is WSS

$$= \overline{X} \int_{-\infty}^{\infty} h(\tau)\, d\tau = \overline{X} \cdot H(0)$$

Random Process Transmission Through Linear Systems

- Autocorrelation of Y(t)

$$R_Y(t,u) = E[Y(t)Y(u)]$$
$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1) d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, X(u-\tau_2) d\tau_2\right]$$
$$= \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) E[X(t-\tau_1)X(u-\tau_2)] d\tau_2$$

If $X(t)$ is WSS $\Downarrow$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\, R_X(\tau-\tau_1+\tau_2) d\tau_1 d\tau_2$$

If input is a WSS process, the output is also a WSS process!

Relation Among the Input-Output PSD

- Autocorrelation of Y(t)

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$

$$= \int h(\tau_2)[h(\tau) * R_X(\tau + \tau_2)]d\tau_2$$

$$= h(-\tau) * h(\tau) * R_X(\tau)$$

$$x(-\tau) * h(\tau)$$
$$= \int x(t_1)h(\tau + t_1)dt_1$$

- PSD of Y(t): $H^*(f)\, H(f)\, S_X(f)$

$\Downarrow$

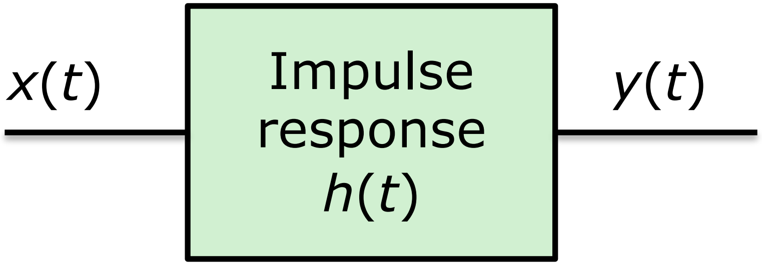
h(t) real signal

$\mathcal{F}(h(-t)) = H^*(f)$

$F(R_Y(\tau)) =$ $S_Y(f) = |H(f)|^2 S_X(f)$

Deterministic vs. Random

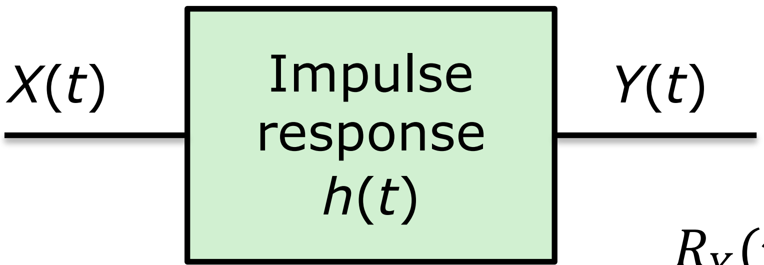
Deterministic signal



$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

Radom signal (WSS)



$Y(t)$ is WSS

$E[Y(t)] = \bar{X} \cdot H(0)$

$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$$

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

Example

- LTI = a differentiator $H(f) = j2\pi f$
- Input random signal $X(t) = A\cos(2\pi f_0 t + \theta)$
- Output PSD

P[Y(f)] =

$$S_Y(f)$$
$$= 4\pi^2 f^2 \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$
$$= A^2\pi^2 {f_0}^2[\delta(f - f_0) + \delta(f + f_0)]$$