EE140 Introduction to Communication Systems

Lecture 2

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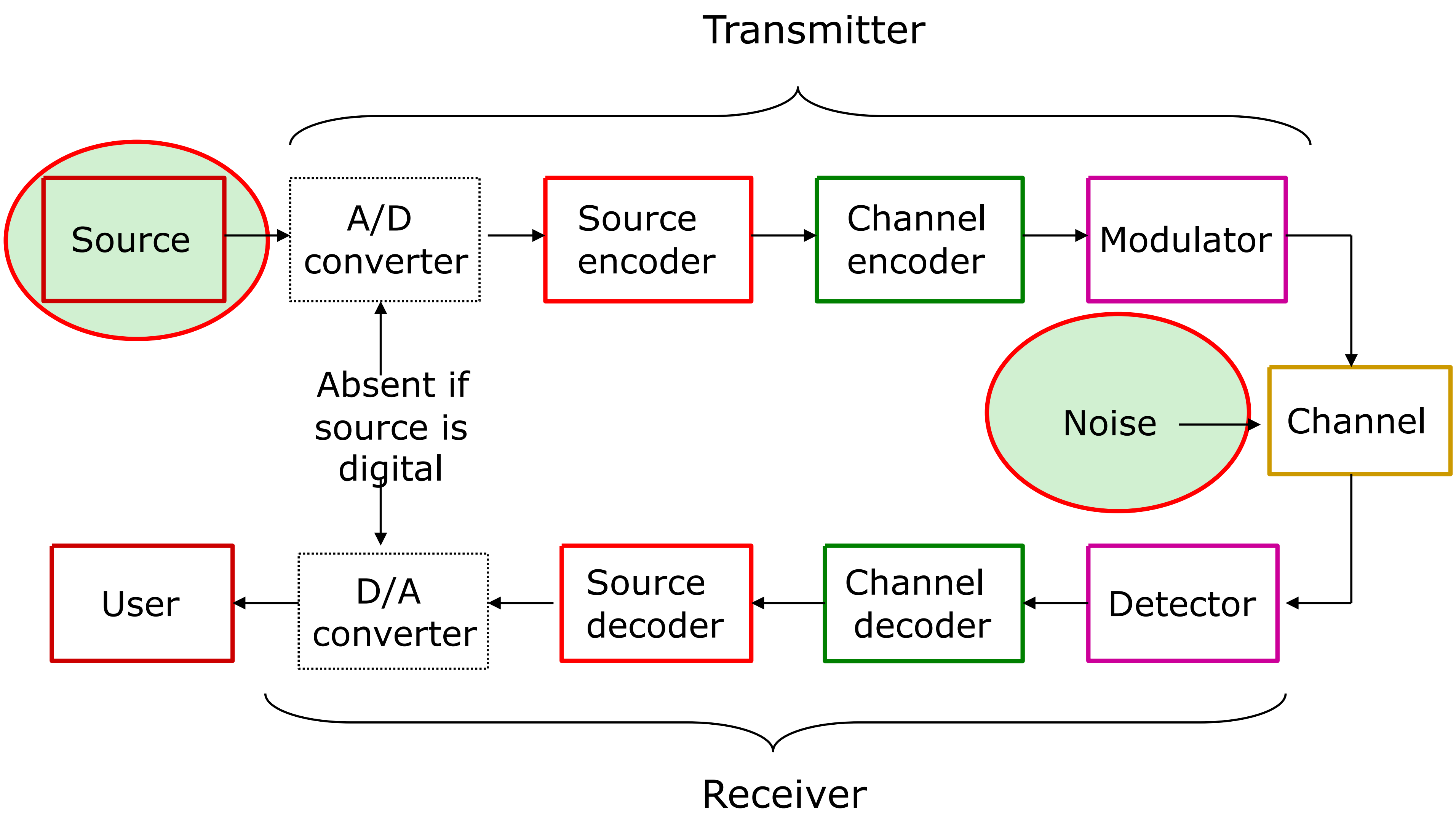
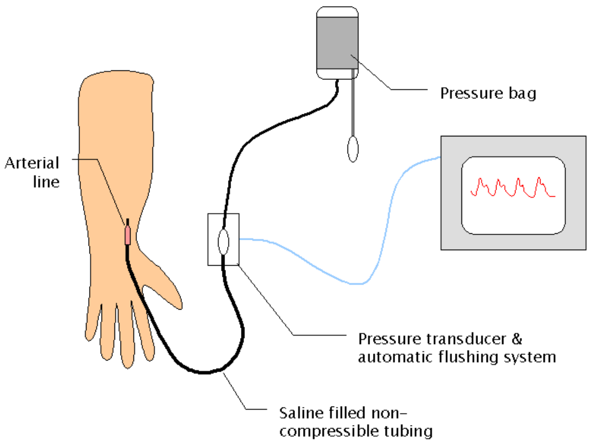
ShanghaiTech University, Fall 2025

Architecture of a Digital Communication System



Source Information

- *Message:* generated by source
- *Information:* the unpredictable part in a message
- *Signal:* a function that conveys information about the behavior or attributes of some phenomenon



Transducer: convert sensing signal to electric signal

Analog signal vs. digital signal

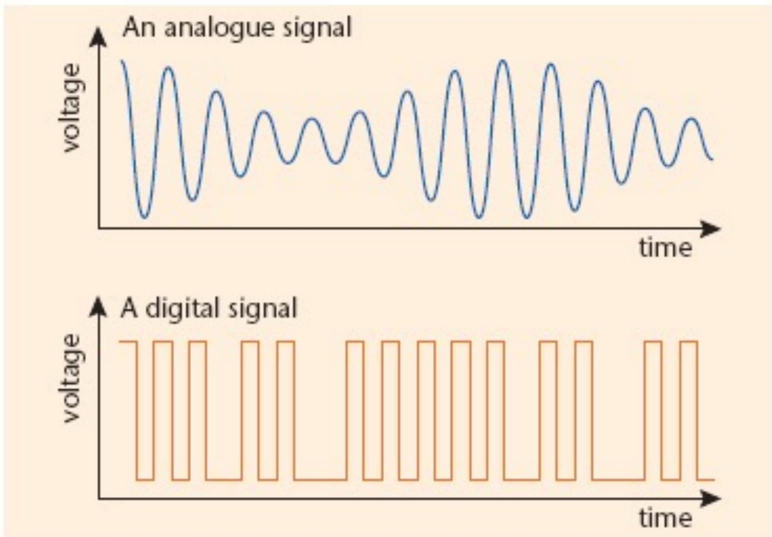


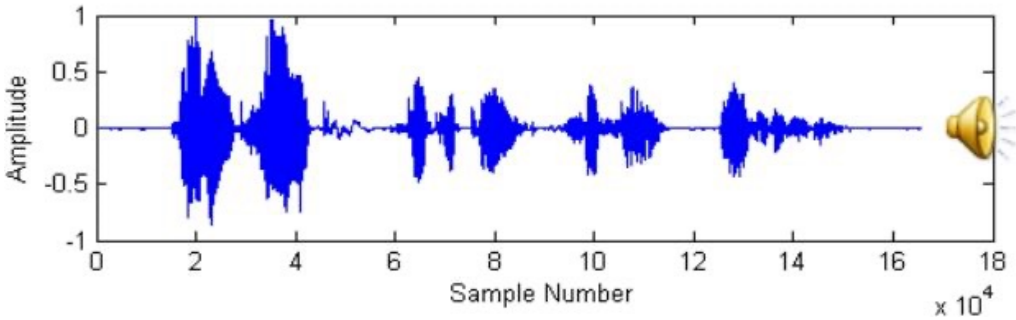
Fig. 12.4 How analogue and digital signals change with time.

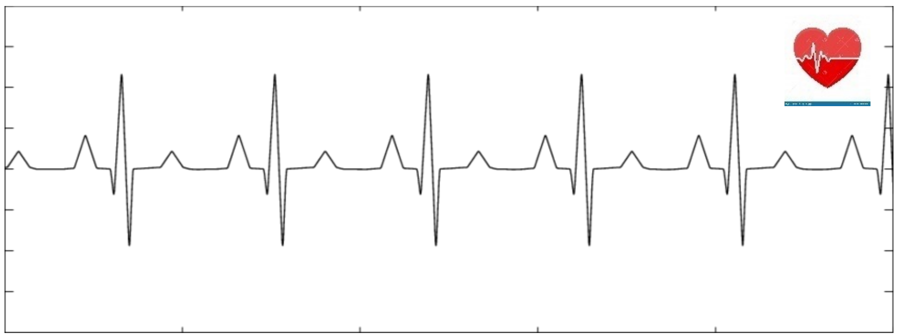
Contents

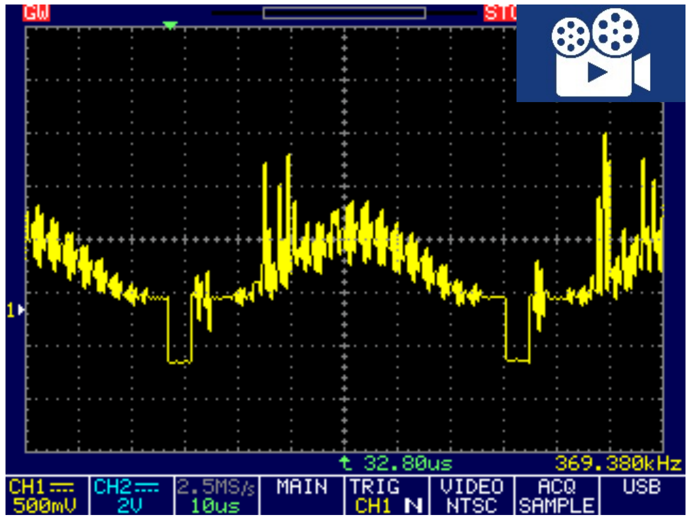
- Deterministic signals
  - Classification of signals
  - Review of Fourier Transform
  - Properties of the Fourier Transform
  - Fourier Transform of Periodic Signals
  - Sampling Theory
  - The Hilbert Transform

What is Signal?

- In communication systems, a signal is any function that carries information. Also called information bearing signal.

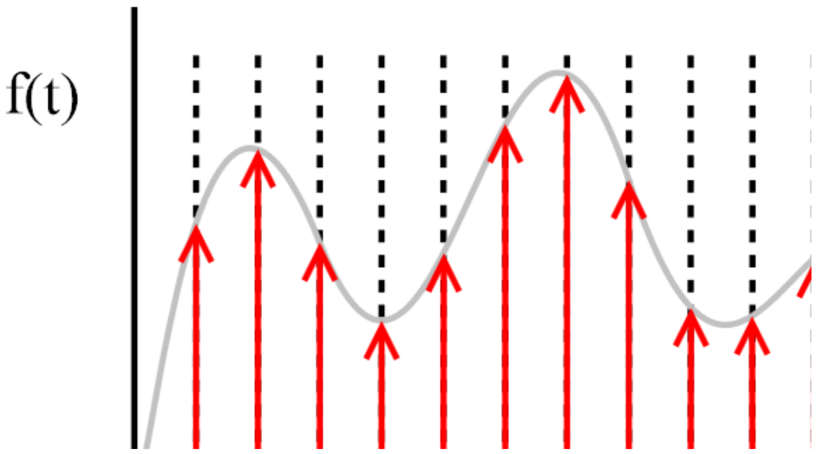




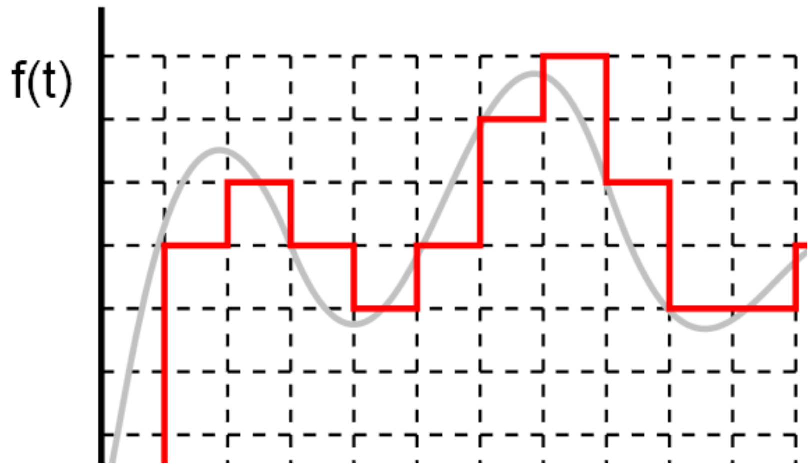


Classification of Signals

- Analog, discrete-time and digital signals
  - Analog signal: both time and value are continuous
  - Discrete-time signal: discrete time and continuous value
  - Digital signal: both time and value are discrete

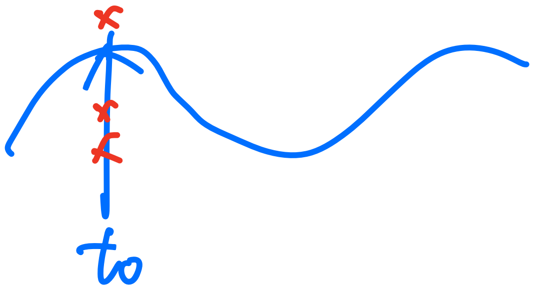


Discrete-time signal



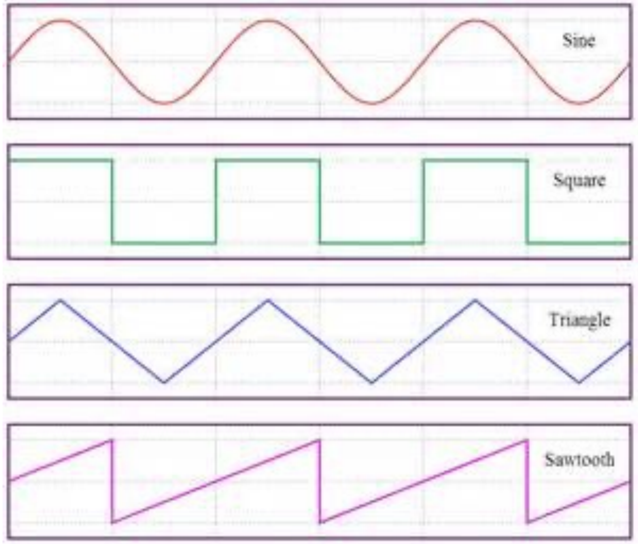
Digital signal

Classification of Signals

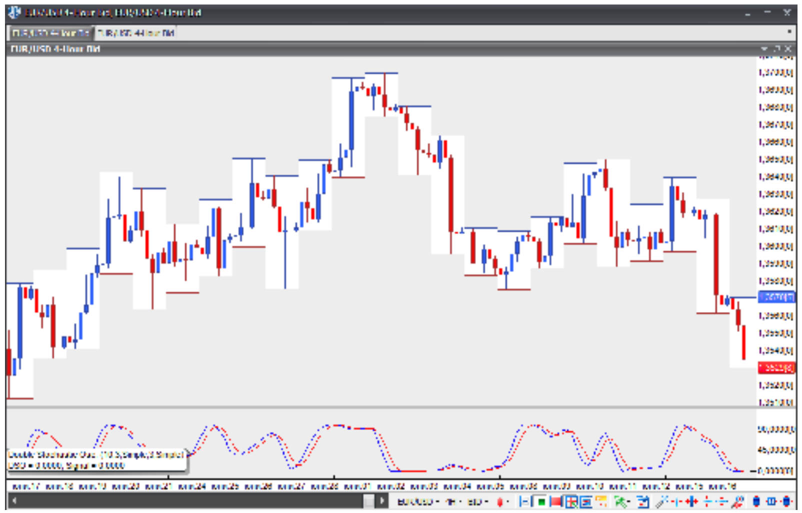


- Periodic and non-periodic signals

$$x(t + T_0) = x(t), \quad -\infty < t < \infty$$



- Random and deterministic
  - Deterministic signal: no uncertainty in value. It can be modeled or expressed by an explicit mathematical function of time.
  - Random (stochastic) signal: its value is uncertain or unpredictable. Probability distribution MUST be used to model it.

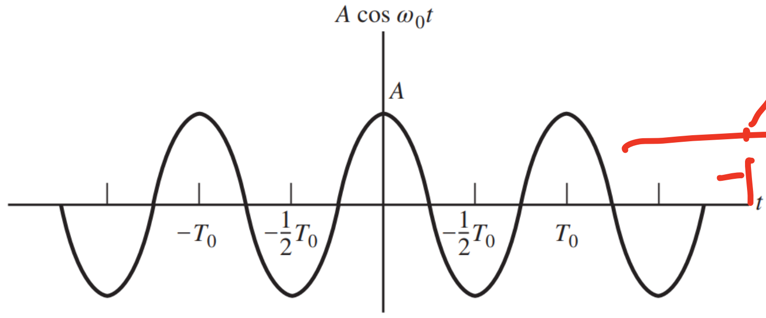


Classification of Signals

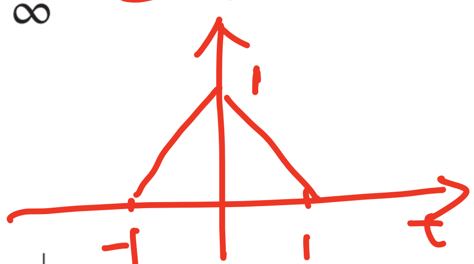
$A\Pi\left(\frac{t-t_0}{\tau}\right)$

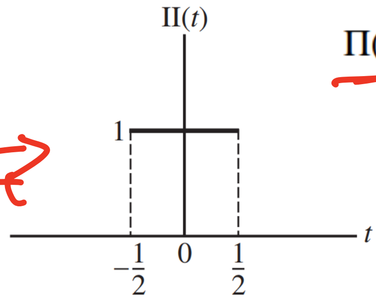
$\Lambda(t)$ $\quad A\Lambda\left(\frac{t-t_0}{\tau}\right)$

$$x(t) = A\cos\left(\omega_0 t\right), \quad -\infty < t < \infty$$



(a)



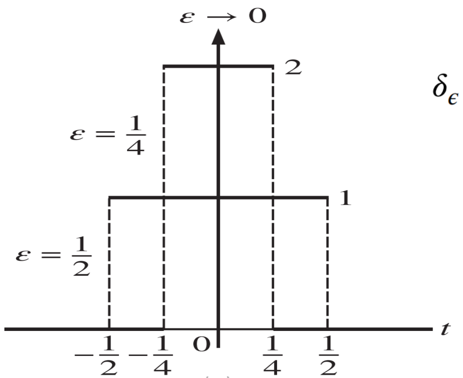


(b)

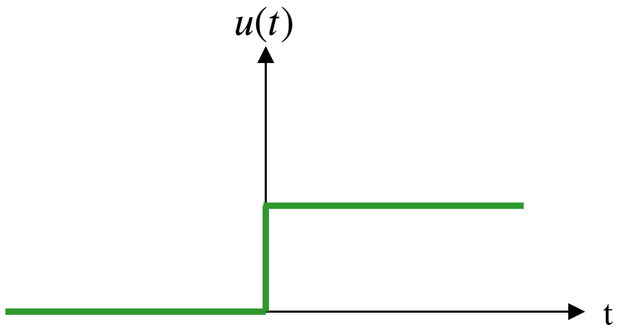
$$\Pi(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Deterministic (sinusoidal) signal

Unit rectangular pulse signal



$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & \text{otherwise} \end{cases}$$

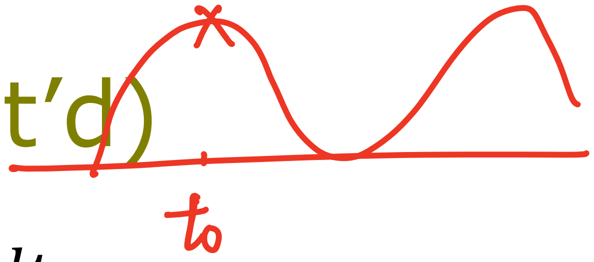


Unit impulse function (delta function)

Unit step function

$x(t_0)$

Dirac Delta Function (cont'd)



- Sifting property: $x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt$

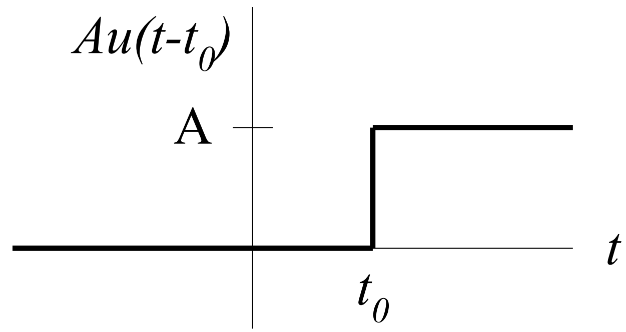
  because

  $$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

  – The impulse function selects a particular value of the function $x(t)$ in the integration process

- Unit step function

$$u(t-t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$



- Relationship between $\delta(t)$ and $u(t)$

$$\delta(t-t_0) = \frac{d}{dt}u(t-t_0) \quad \Longleftrightarrow \quad u(t-t_0) = \int_{-\infty}^{t} \delta(\tau-t_0)\, d\tau$$

Classification of Signals

- Energy and power signals
  - Total Energy of a signal: $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$ joules
  - Signal $x(t)$ is an energy signal if $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$
  - Average Power of a signal: $P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$ watts
  - Signal $x(t)$ is a power signal $0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt < \infty$
  - Remark:
    - ➢ Energy signal: P=0; Power signal: E=∞.
    - ➢ Discrete Signal: $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$; $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

Exercise: Question

Classify the following signals as energy signals or power signals. Find the normalized energy or normalized power of each.

(1)

$$x(t) = \begin{cases} A\cos(2\pi f_0 t) & \mathrm{for}\ \ -\mathrm{T_0/2 \le t \le T_0/2, where\ T_0 = 1/f_0} \\ 0 & \mathrm{elsewhere} \end{cases}$$

(2)

$$x(t) = \begin{cases} A\exp(-at) & \mathrm{for\ t > 0, a > 0} \\ 0 & \mathrm{elsewhere} \end{cases}$$

Exercise: Solution

Classify the following signals as energy signals or power signals. Find the normalized energy or normalized power of each.

(1)

$$x(t) = \begin{cases} A\cos(2\pi f_0 t) & \text{for } -\mathrm{T_0/2 \le t \le T_0/2, where\ T_0 = 1/f_0} \\ 0 & \text{elsewhere} \end{cases}$$

(2)

$$x(t) = \begin{cases} A\exp(-at) & \text{for } \mathrm{t > 0, a > 0} \\ 0 & \text{elsewhere} \end{cases}$$

Solution 1

(1) Energy signal. $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-T_0/2}^{T_0/2} A^2\cos^2(2\pi f_0 t)\,dt = \frac{A^2 T_0}{2}$.

(2) Energy signal. $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{0}^{\infty} A^2\exp(-2at)\,dt = \frac{A^2\exp(-2at)}{-2a}\Big|_0^{\infty} = \frac{A^2}{2a}$.

Contents

- Deterministic signals
  - Classification of signals
  - Review of Fourier Transform
    - Fourier Transform
    - Discrete-time Fourier Transform (skip)
    - Discrete Fourier Series (skip)
    - Discrete Fourier Transform (skip)
    - Fast Fourier Transform (skip)
  - Properties of the Fourier Transform
  - Fourier Transform of Periodic Signals
  - Sampling Theory
  - The Hilbert Transform

Fourier Series

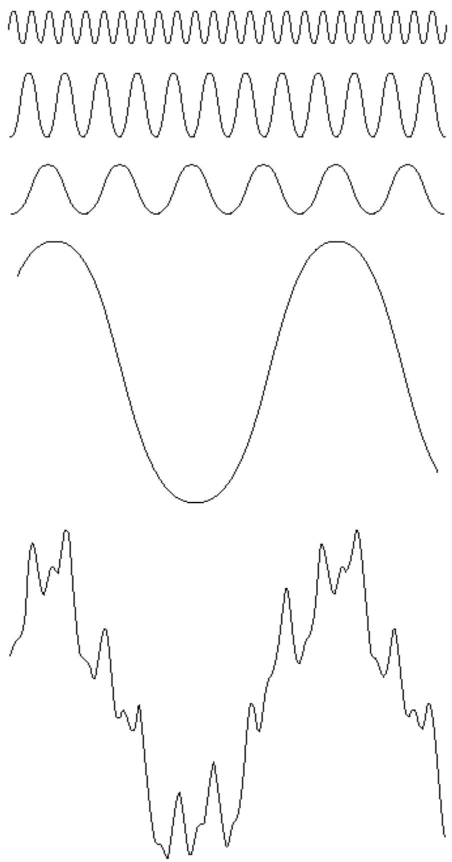


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

A periodic function = superposition or linear combination of simple sine and cosine functions

- Jean-Baptiste Joseph Fourier: 1768-1830
- Student of Laplace and Lagrange
- 1807: introduced the Fourier series expansion

Fourier Transform

w

f : Hz

- Fourier Transform of a continuous-time signal

$$X(f) = \Im[x(t)] \Longleftrightarrow X(f) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f \lambda}\, d\lambda$$

if x(t) is absolutely integrable $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

- Inverse Fourier Transform

$$x(t) = \Im^{-1}[X(f)] \Longleftrightarrow x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

- Spectrum
  - The Fourier transform of a continuous-time signal is a complex signal

$$X(f) = |X(f)| e^{j\angle X(f)}$$

x(t) real signal

real + even

real + odd

Parseval's Theorem

- Parseval's Theorem

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df$$

  – When $x_1(t) = x_2(t)$, we have

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy

Energy Spectral Density

- Energy Spectral Density

$$G(f) = |X(f)|^2 \text{ Joules/Hz}$$

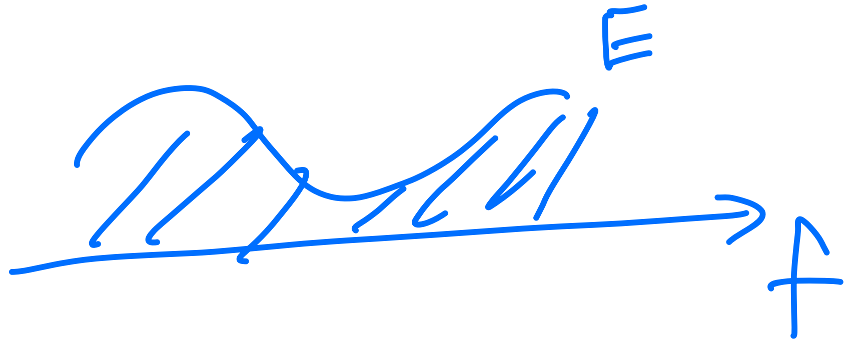
$$\underline{\int x_1(t)\, x_2^*(t)\, dt} = \int x_1(t) \int X_2^*(f)\, e^{-j2\pi f t}\, df\, dt$$

$$= \int X_2^*(f) \boxed{\int x_1(t)\, e^{-j2\pi f t}\, dt}\, df$$

$$= \int X_2^*(f)\, X_1(f)\, df$$

if $x_2(t) = x_1(t)$

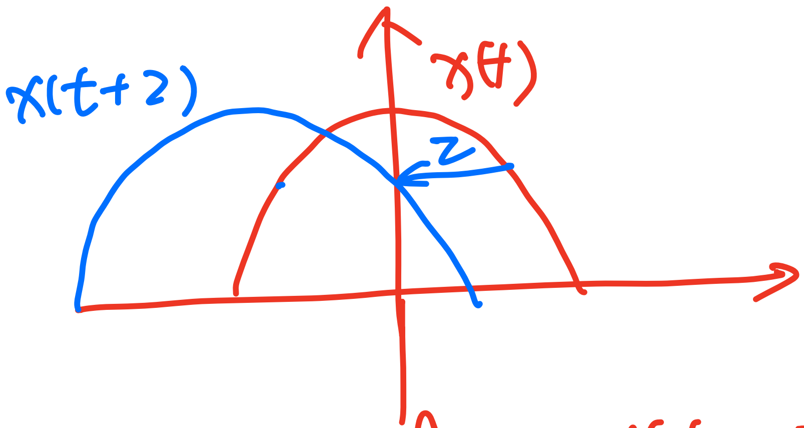
$$E = \boxed{\int |x_1(t)|^2 dt} = \int \underbrace{\boxed{|X_1(f)|^2}}_{G(f)} df$$



Power Spectral Density and Correlation

	Energy Signal	Power Signal
Spectral density	Energy Spectral Density $E = \int_{-\infty}^{\infty} \lvert x(t) \rvert^2 \, dt = \int_{-\infty}^{\infty} \lvert X(f) \rvert^2 \, df$ $G(f) = \lvert X(f) \rvert^2$	$P = \int_{-\infty}^{\infty} S(f) \, df = \langle x^2(t) \rangle$
Time-average autocorrelation function	$\phi(\tau) = x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(\lambda) x(\lambda + \tau) \, d\lambda$ $= \lim_{T \to \infty} \int_{-T}^{T} x(\lambda) x(\lambda + \tau) \, d\lambda$ (energy sign	$R(\tau) = \langle x(t) x(t+\tau) \rangle$ $\triangleq \boxed{\lim_{T \to \infty} \frac{1}{2T}} \int_{-T}^{T} x(t) x(t+\tau) \, \boxed{dt}$
Relationship	$\phi(t) = \mathfrak{F}^{-1}(G(f))$ $G(f) = \mathfrak{F}(\phi(t))$	$R(\tau) = \mathfrak{F}^{-1}[S(f)] = \int_{-\infty}^{\infty} S(f) e^{j2\pi f \tau} \, df$ $S(f) = \mathfrak{F}[R(\tau)] = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f \tau} \, d\tau$

$$\phi(\tau) = x(\tau) * x(-\tau) = \int_{-\infty}^{+\infty} \underset{\sim}{x(t)}\, x(t+\tau)\, dt$$



$\tau = 0 \quad \phi(0) = E$

$\tau = \infty \quad \phi(\infty) = 0$

$\phi(\tau)$ even

$$\phi(-\tau) = \int x(t)\, x(t-\tau)\, dt$$

$t' = t - \tau$

$$= \int x(t'+\tau)\, x(t')\, dt' = \phi(\tau)$$

$$\boxed{\mathcal{F}(x(-t)) = X^*(f) \quad x(t) \text{ is real signal}}$$

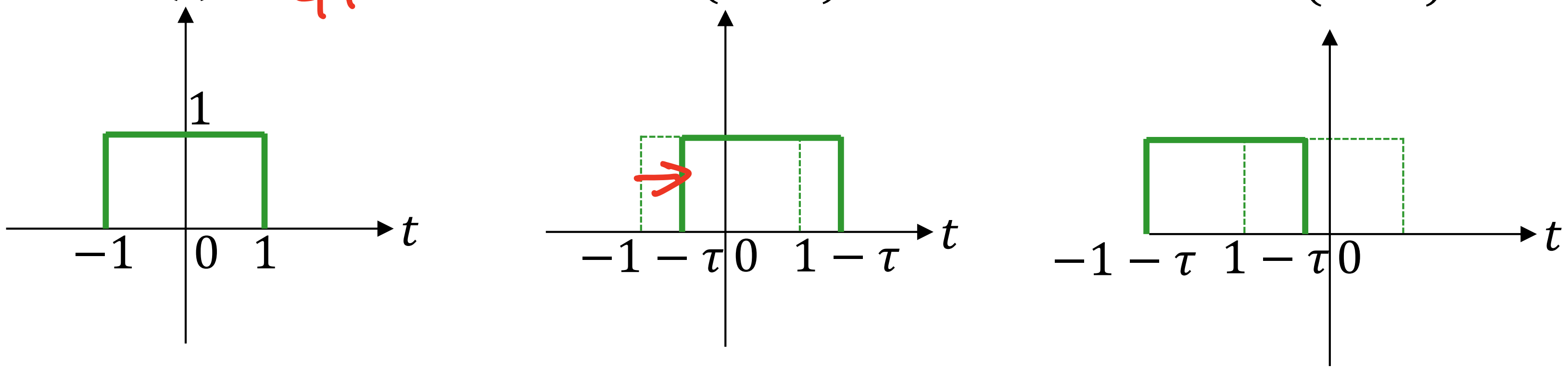
$$\mathcal{F}[\phi(\tau)] = \mathcal{F}\left(x(\tau) * x(-\tau)\right) = \underline{X(f)}X^*(f)$$

$$= G(f)$$

Example: Autocorrelation Function

- Energy Signal

$\phi(\tau) = x(\tau) * x(-\tau)$

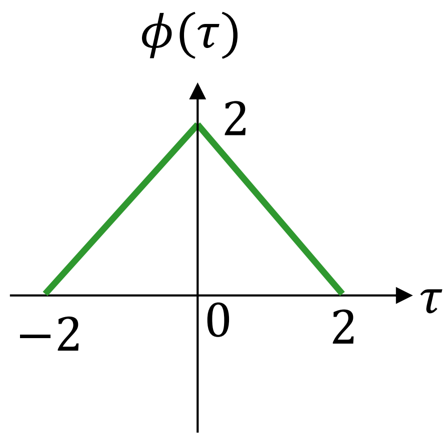


$$\tau = 0, \phi(0) = \int_{-1}^{1} x(t)x(t)dt = 2$$

$$-2 < \tau < 0, \phi(\tau) = \int_{-1-\tau}^{1} x(t)x(t+\tau)dt = 2+\tau$$

$$0 < \tau < 2, \phi(\tau) = \int_{-1}^{1-\tau} x(t)x(t+\tau)dt = 2-\tau$$

$$|\tau| \geq 2, \phi(\tau) = 0$$



1. $\phi(\tau)$ is even.
2. $\phi(0) \geq |\phi(\tau)|$.

Example: Rectangular Pulse

$$x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right) \quad \Longrightarrow \quad X(f) = \int_{-\infty}^{\infty} A\Pi\left(\frac{t-t_0}{\tau}\right) e^{j2\pi f t}\, dt$$

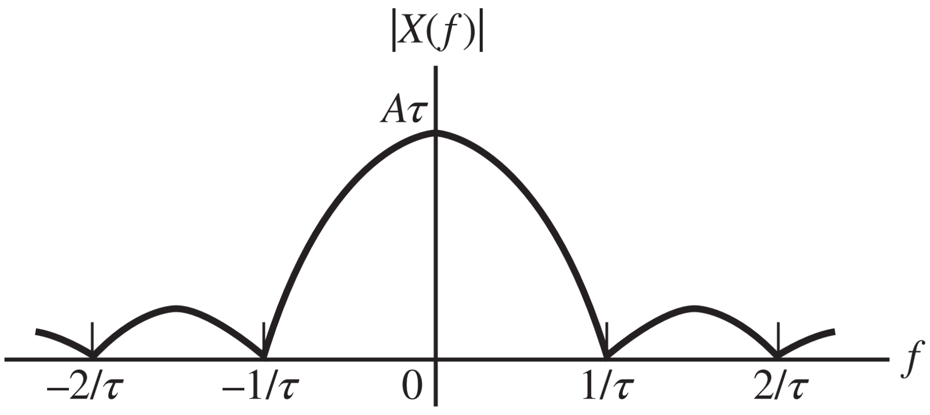
$$= A\int_{t_0-\tau/2}^{t_0+\tau/2} e^{-j2\pi f t}\, dt = A\tau \,\mathrm{sinc}\,(f\tau)\, e^{-j2\pi f t_0}$$

Amplitude spectrum

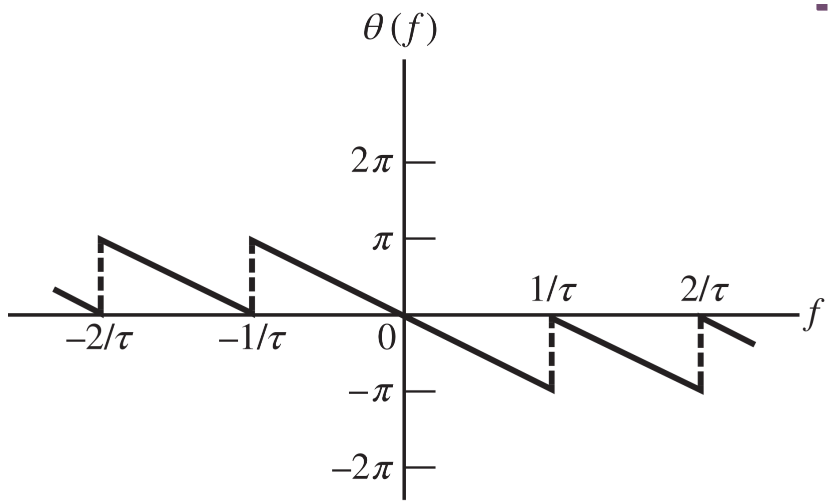
$$|X(f)| = A\tau|\mathrm{sinc}\,(f\,\tau)\,|$$

Phase spectrum

$$\theta(f) = \begin{cases} -2\pi t_0 f & \text{if } \mathrm{sinc}\,(f\tau) > 0 \\ -2\pi t_0 f \pm \pi & \text{if } \mathrm{sinc}\,(f\tau) < 0 \end{cases}$$



(a)



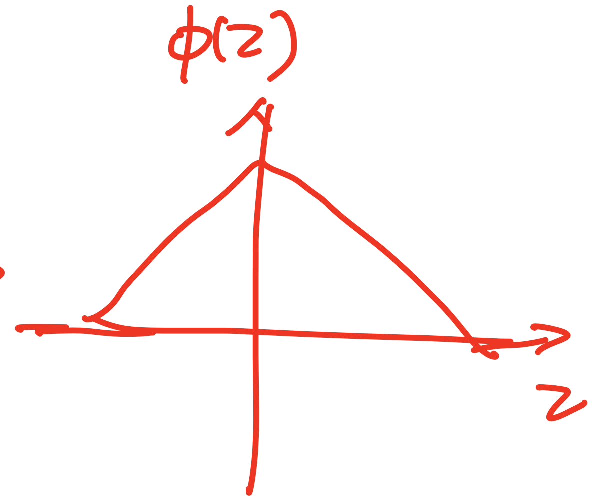
(b)

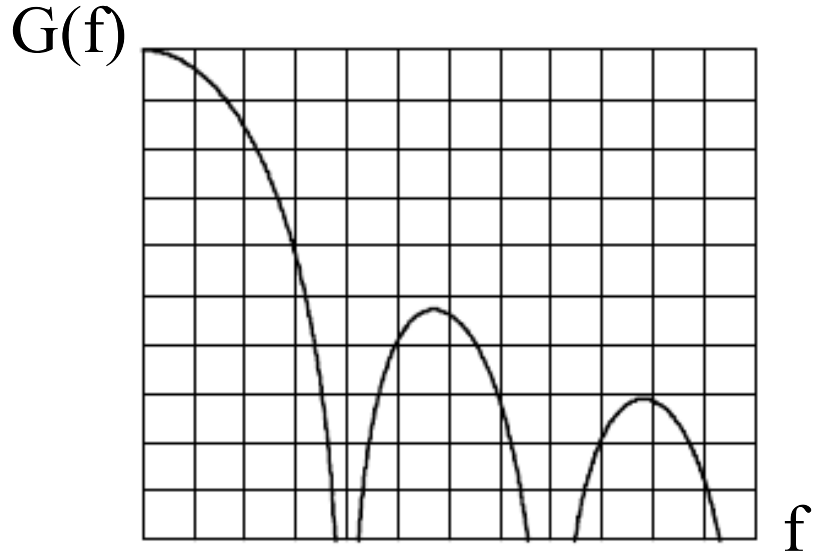
Example: Rectangular Pulse

- Energy spectral density

$$G(f) = |X(f)|^2 = A^2\tau^2\mathrm{sinc}^2(f\tau)$$

IFT

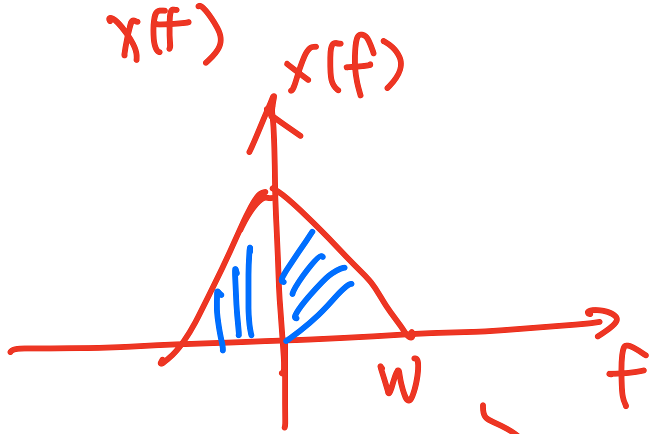


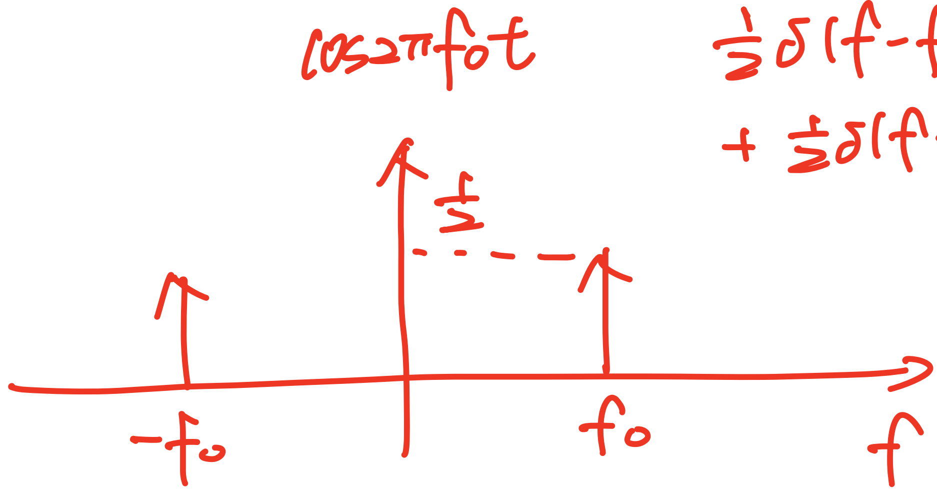


Contents

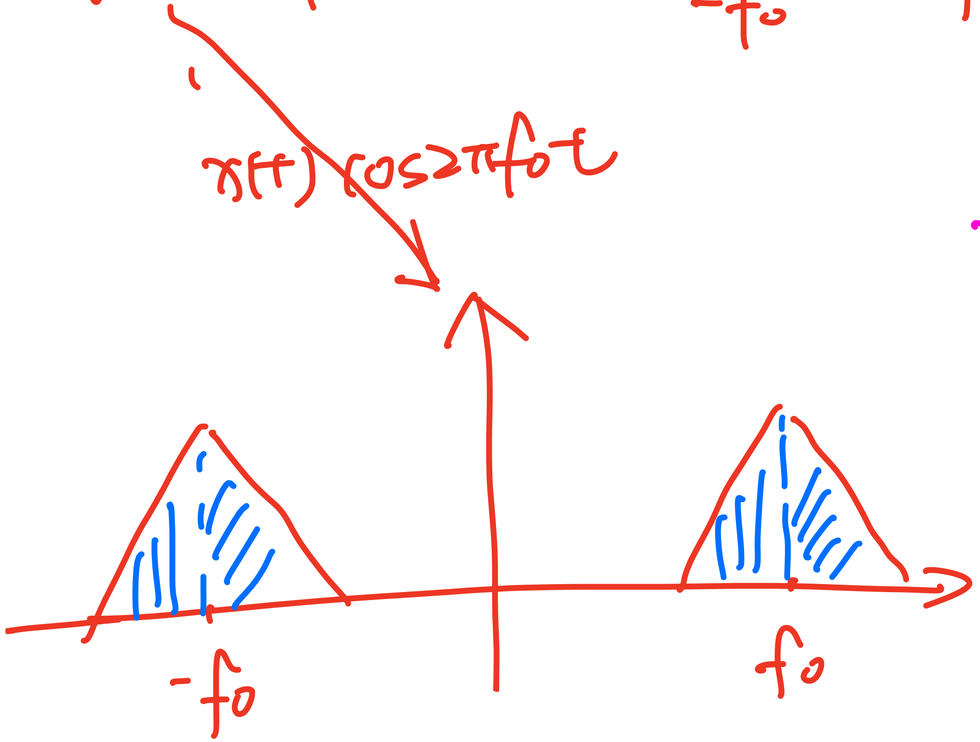
Properties of Fourier Transform

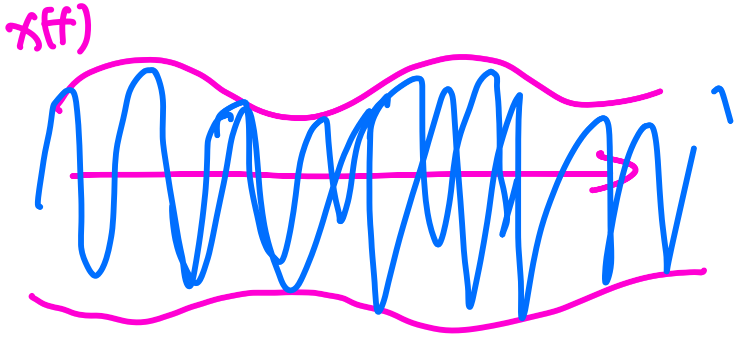
Operation	$x(t)$	$X(f)$
1. Superposition	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(f)+a_2X_2(f)$
2. Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$
3. Time shifting	$x(t-t_0)$	$X(f)\exp(-j2\pi f t_0)$
4. Frequency shifting	$x(t)\exp(j2\pi f_0 t)$	$X(f-f_0)$
5. Duality theorem	$X(t)$	$x(-f)$
6. Modulation Theorem	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0)$
7. Time differentiation	$\frac{d^n x}{dt^n}$	$(j2\pi f)^n X(f)$
8. Frequency differentiation	$(-jt)^n x(t)$	$\frac{d^n X}{df^n}$
9. Time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
10. Time convolution	$x_1(t)*x_2(t)$	$X_1(f)X_2(f)$
11. Multiplication	$x_1(t)x_2(t)$	$X_1(f)*X_2\,(f)$





$$\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$





Example: Dirac Delta Function

- Dirac delta function

$$\delta(t) = \begin{cases} \infty & t = 0, \\ 0 & t \neq 0, \end{cases}$$

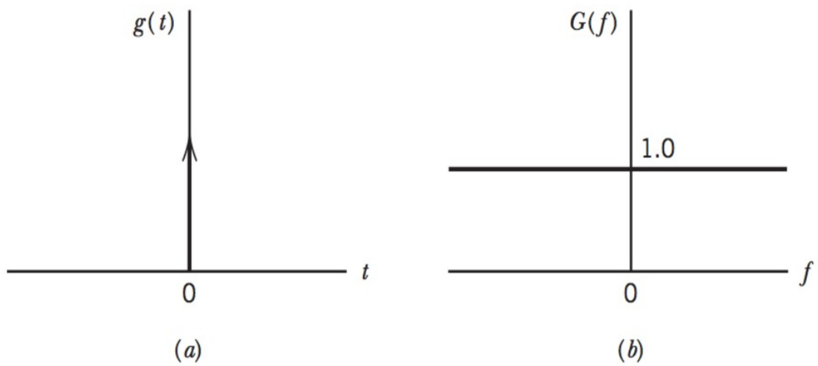
and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- Fourier transform

$$G(f) = \mathrm{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f 0} dt = e^{0} = 1$$



Example: Dirac Delta Function

- Application of the delta function

1. $A\delta(t) \longleftrightarrow A$
2. $A\delta(t - t_0) \longleftrightarrow Ae^{-j2\pi f t_0}$
3. $A \longleftrightarrow A\delta(f)$
4. $Ae^{j2\pi f_0 t} \longleftrightarrow A\delta(f - f_0)$

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$$

Definition of delta function

$$\int_{-\infty}^{\infty} \cos(2\pi f t)\, dt = \delta(f)$$

$$\int_{-\infty}^{\infty} \exp(-j2\pi f t)\, dt = \delta(f)$$

$$\int \cos 2\pi f t - j \sin 2\pi f t \, dt$$

$$= \delta(f)$$

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Line Spectra for Periodic Signal

- Periodic signal $x(t)$ with period T_0 $f_0 = \frac{1}{T_0}$

$$
\begin{aligned}
X(f) = \mathrm{F}[x(t)] &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \right) e^{-j2\pi f t}\, dt \\
&= \sum_{n=-\infty}^{\infty} X_n \int_{-\infty}^{\infty} e^{-j2\pi (f - n f_0) t}\, dt \\
&= \sum_{n=-\infty}^{\infty} X_n \delta(f - n f_0)
\end{aligned}
$$

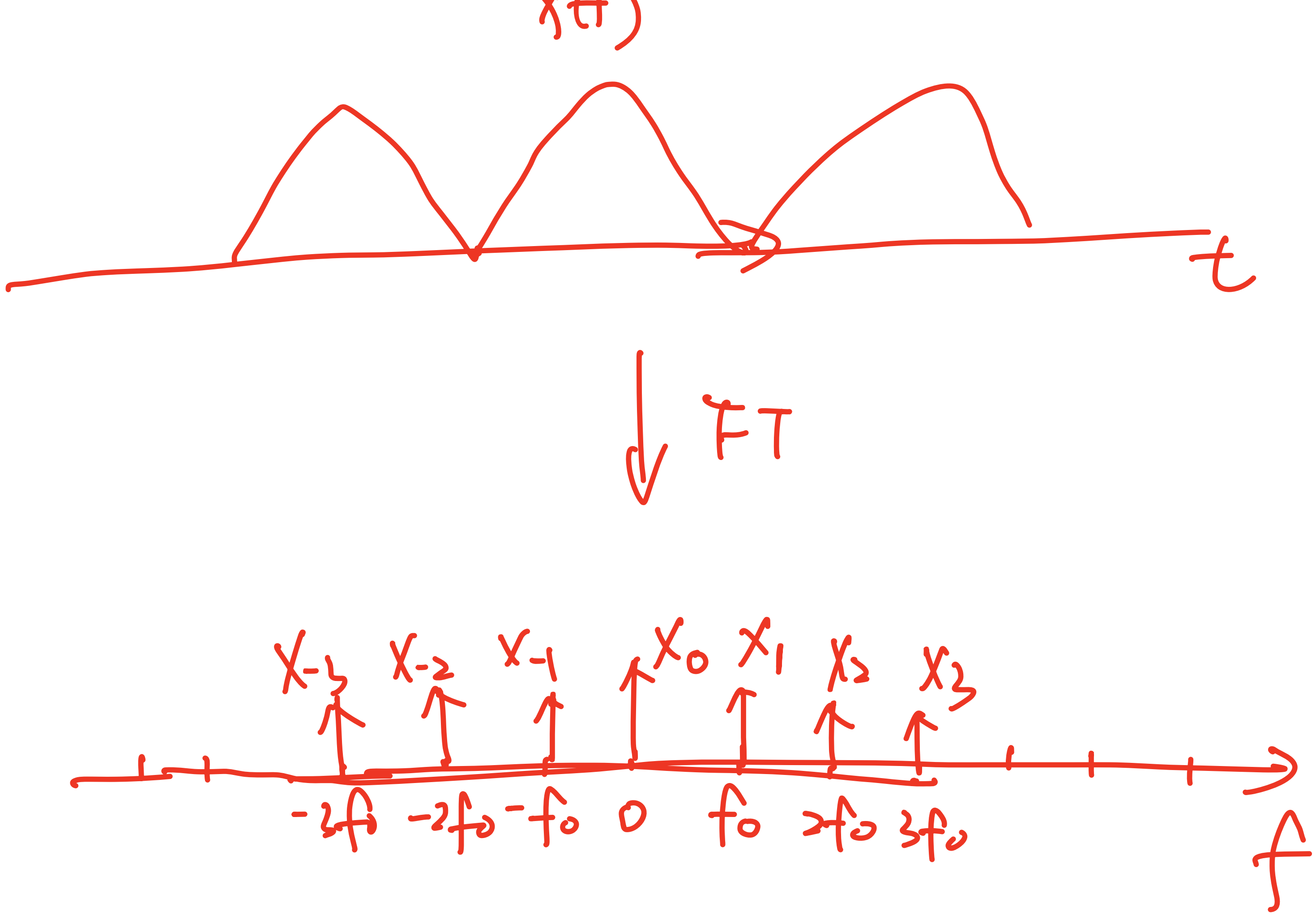
Continuous Periodic

↕ FT

Discrete

Fourier series

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$


x(t)
t
FT
X_{-3} X_{-2} X_{-1} X_0 X_1 X_2 X_3
$-3f_0$ $-2f_0$ $-f_0$ 0 f_0 $2f_0$ $3f_0$
f

Example: The "Comb" Function

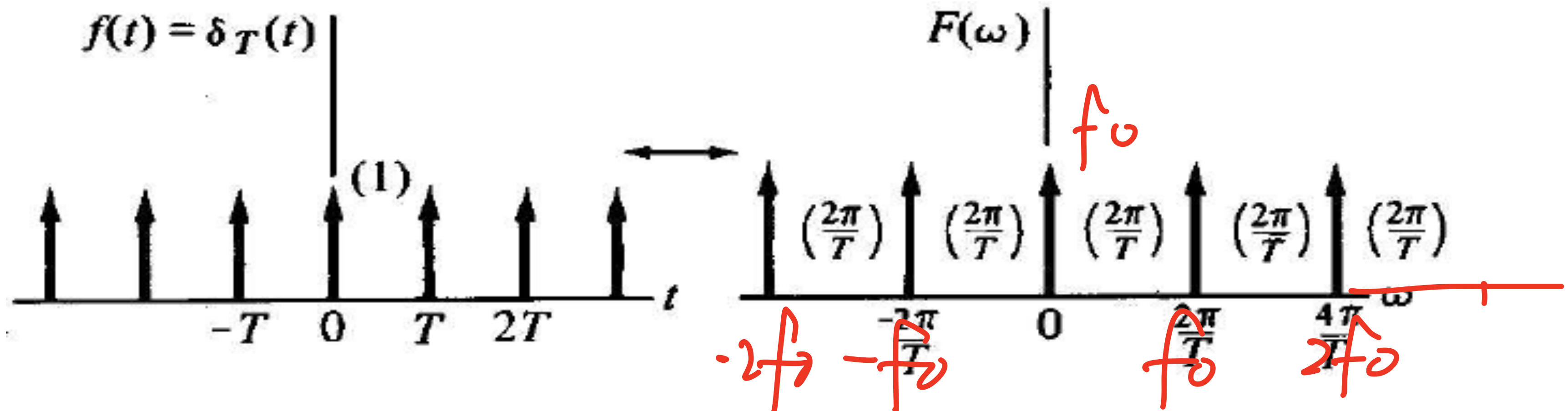
- The "comb" function:

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

- FT 1: $\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn2\pi f_0 t}, f_0 = \dfrac{1}{T}$

$$X_n = \frac{1}{T}\int_T \delta(t)\, e^{-jn2\pi f_0 t} dt = f_0$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} f_0\, e^{jn2\pi f_0 t} \Longleftrightarrow \mathfrak{F}[\delta_T(t)] = \sum_{n=-\infty}^{\infty} f_0\, \delta(f - nf_0)$$



Example: The "Comb" Function

- The "comb" function:

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

- FT 2:

$$\Im\left[\sum_{m=-\infty}^{\infty} \delta(t - mT\,)\right] = \int_{-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] e^{-j2\pi ft}\,dt$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - mT_s) e^{-j2\pi ft}\,dt$$

$$= \sum_{m=-\infty}^{\infty} e^{-j2\pi mT_s f}$$

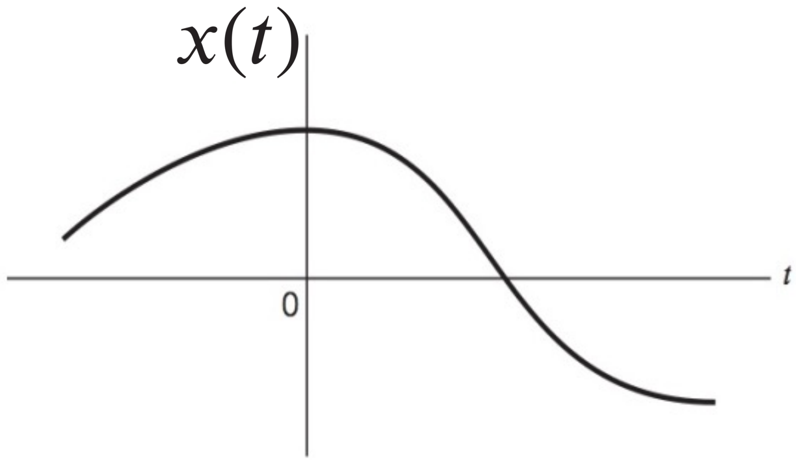
$$\boxed{\sum_{m=-\infty}^{\infty} e^{j2\pi mTf} = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$$

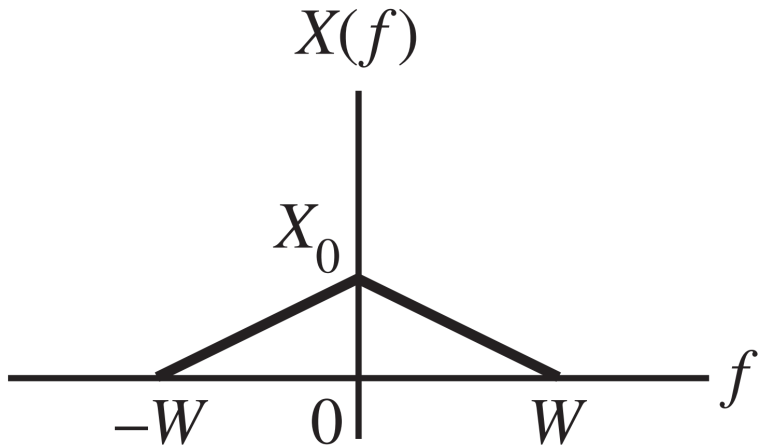
Fourier Series

Contents

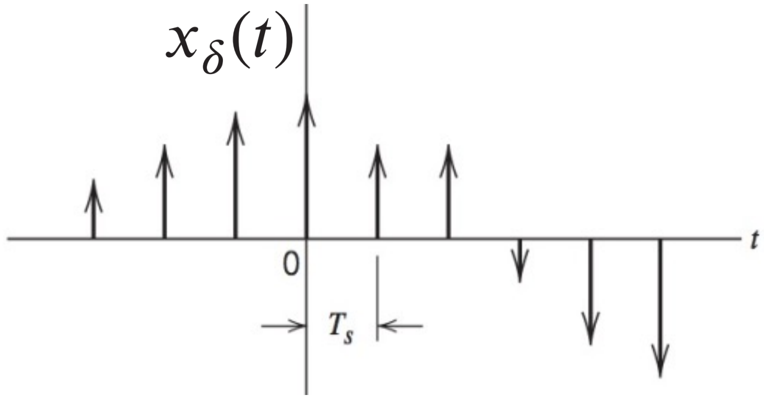
- Deterministic signals
  - Classification of signals
  - Review of Fourier Transform
  - Properties of the Fourier Transform
  - Fourier Transform of Periodic Signals
  - Sampling Theory → Digital Com
  - The Hilbert Transform → Analog

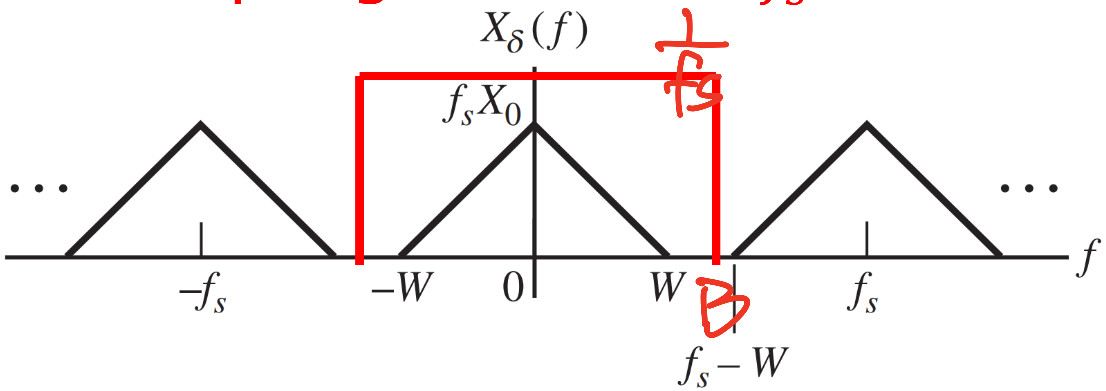
Sampling Theory





Sampling theorem: $f_s \geq 2W$





$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

$$x_\delta(t) = x(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \xrightarrow{F}$$

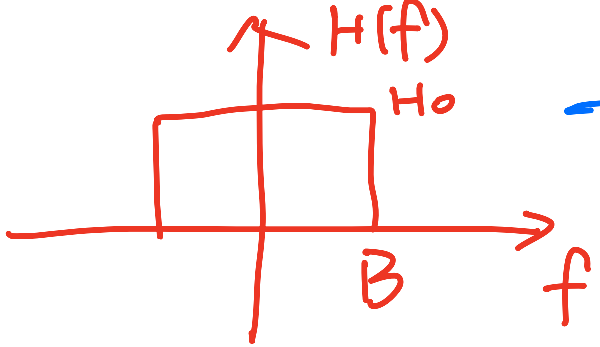
$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{+\infty} \delta(f - \underline{nf_s})$$

$$= \sum_{n=-\infty}^{+\infty} f_s \, X(f - nf_s)$$

$$x_\delta(t) \longrightarrow \boxed{\text{LPF}} \longrightarrow \tilde{x}(t) \quad (x(t))$$

$$H(f) = H_0\, \Pi\left(\frac{f}{2B}\right)$$

$$h(t) = 2H_0B\,\mathrm{sinc}(2Bt)$$



$$H_0 = \frac{1}{f_s}$$

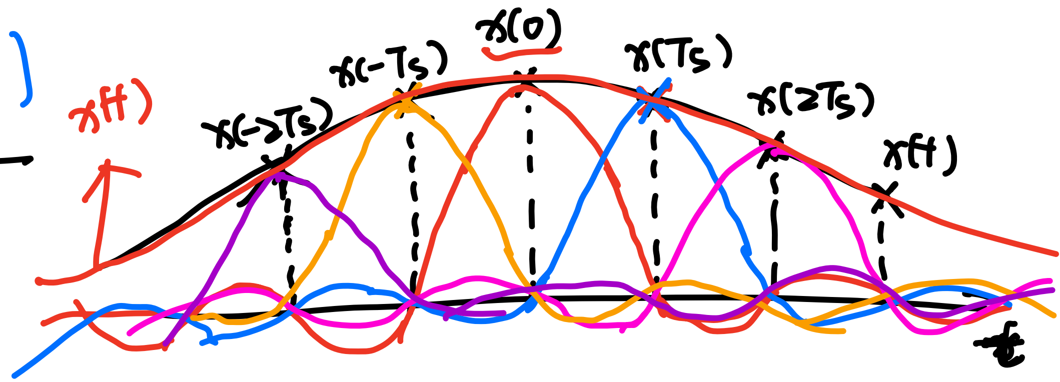
$$B = \frac{f_s}{2}$$

$$\tilde{x}(t) = x_\delta(t) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) * 2H_0B\,\mathrm{sinc}(2Bt)$$

$$= \sum_{n=-\infty}^{+\infty} 2H_0B\, x(nT_s)\,\mathrm{sinc}\left(2B(t-nT_s)\right)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT_s)\,\mathrm{sinc}\left(f_s(t-nT_s)\right)$$



$$x(t) \Longleftrightarrow \{x_1, x_2 \cdots x_n\}$$

Contents

- Deterministic signals
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  - Review of Fourier Transform
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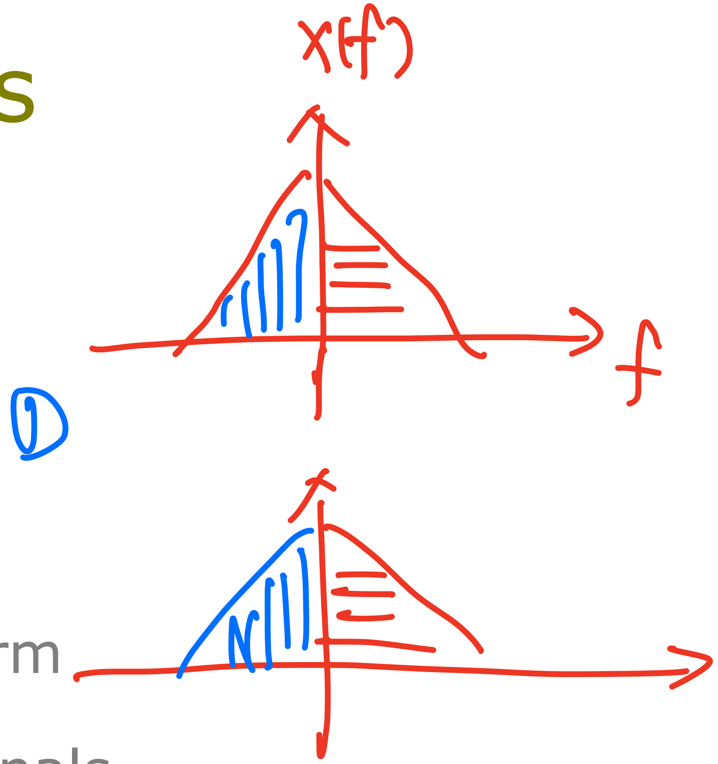
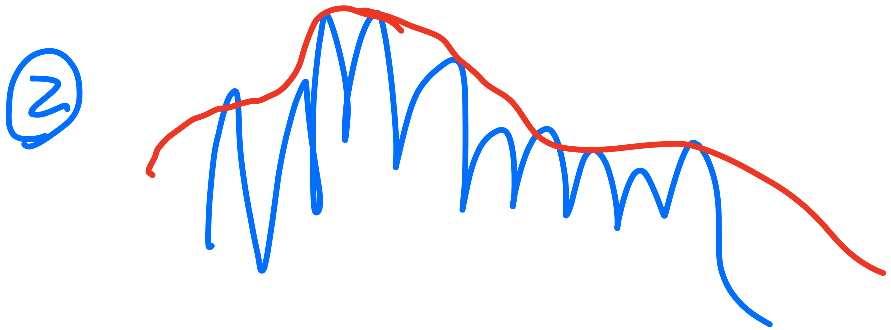
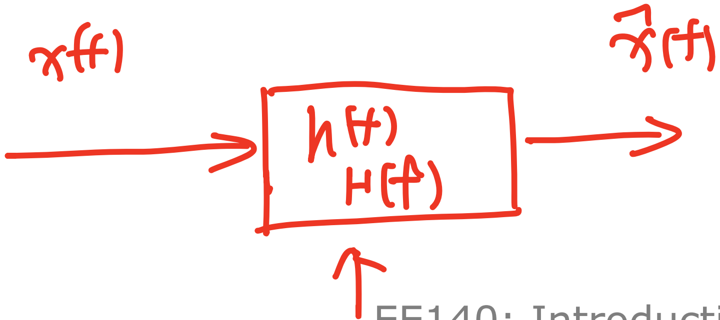
  - Fourier Transform of Periodic Signals
  - Sampling Theory
  - The Hilbert Transform





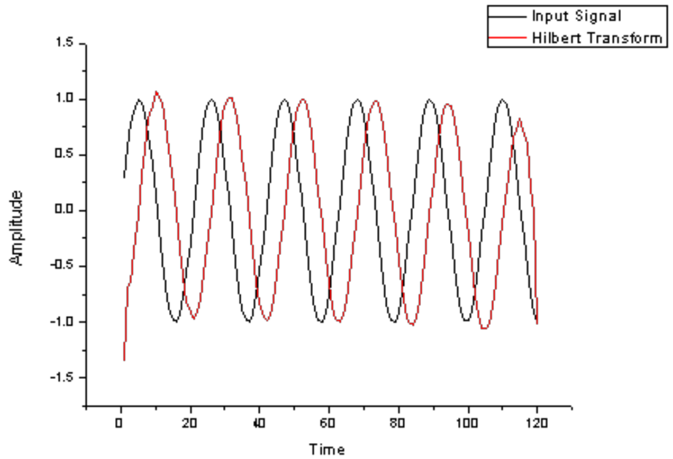


Hilbert Transform

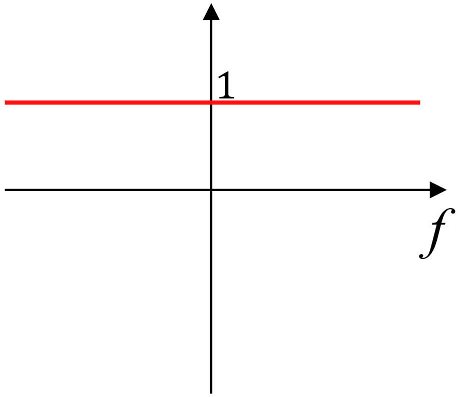
- Frequency response function

$$H(f) = \boxed{-j \operatorname{sgn} f} \qquad \operatorname{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

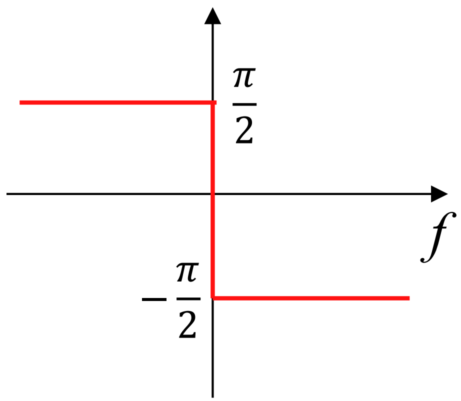
$$= \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$



$|H(f)|$



$\underline{/H(f)}$

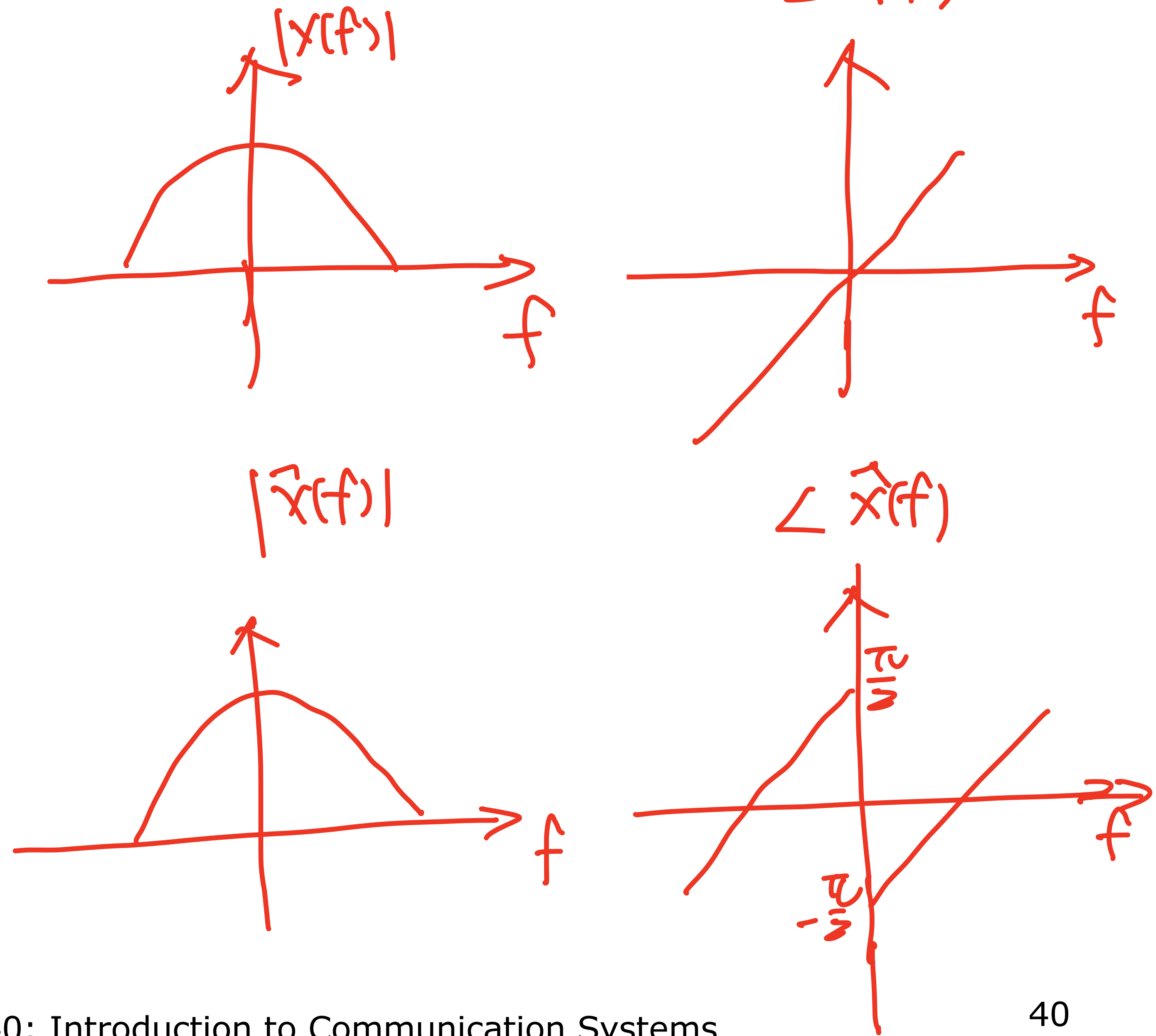


$$\boxed{\frac{j}{\pi t} \longleftrightarrow \operatorname{sgn}(f)}$$

- Hilbert Transform of x(t): $\hat{x}(t) = \mathfrak{F}^{-1}[-j \operatorname{sgn}(f) X(f)]$

$$= h(t) * x(t) = x(t) * \left(\frac{1}{\pi t}\right)$$

$$\underline{\hat{X}(f)} = X(f)\underline{H(f)} = -j\,\mathrm{sgn}(f)\,\underline{X(f)}$$



Hilbert Transform

- Hilbert Transform of x(t):

$$\hat{x}(t) = \mathfrak{I}^{-1}[-j\ \mathrm{sgn}\,(f)X(f)]$$

$$= h(t) * x(t)$$

$$\hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\lambda)}{\pi(t-\lambda)} d\lambda = \int_{-\infty}^{\infty} \frac{x(t-\eta)}{\pi\eta} d\eta$$

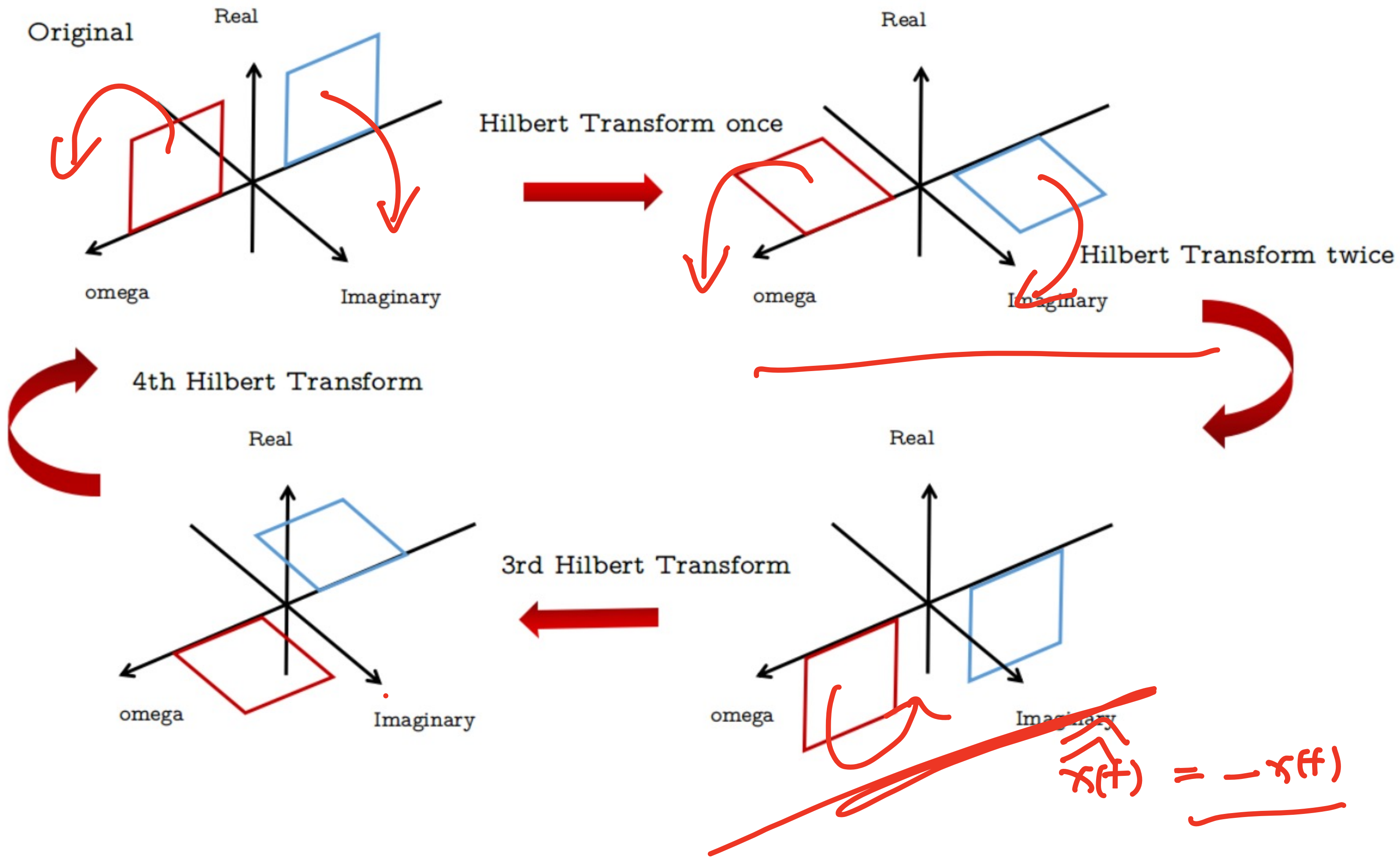
- Hilbert transform of Hilbert transform

$$(-j\ \mathrm{sgn}\ f)^2 = -1$$

$$\hat{\hat{x}}(t) = -x(t).$$

Hilbert Transform

- Hilbert Transform of x(t):



Example

- Calculate the Hilbert transform filter of

$$x(t) = \cos\left(2\pi f_0 t\right)$$

- Solution:

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$\hat{X}(f) = \frac{1}{2}\delta(f - f_0)e^{-j\pi/2} + \frac{1}{2}\delta(f + f_0)e^{j\pi/2}$$

$$\hat{x}(t) = \frac{1}{2}e^{j2\pi f_0 t}e^{-j\pi/2} + \frac{1}{2}e^{-j2\pi f_0 t}e^{j\pi/2}$$

$$\widehat{\cos(2\pi f_0 t)} = \sin\left(2\pi f_0 t\right)$$

Properties (P84)

1. Energy in a signal and its Hilbert transform are equal.

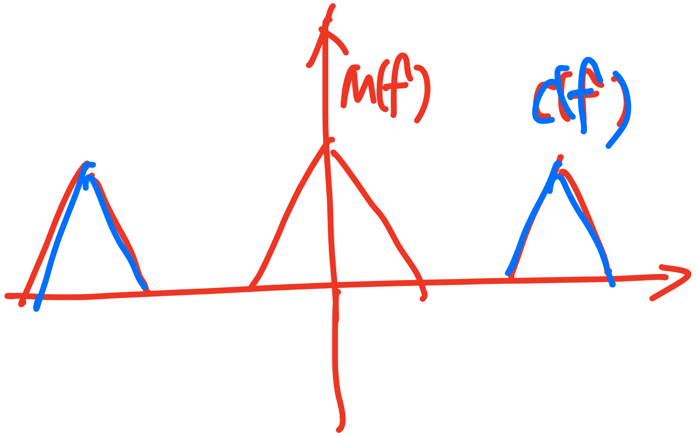
$$\left|\hat{X}(f)\right|^2 \triangleq \left|\mathfrak{F}\left[\hat{x}(t)\right]\right|^{2} = \underbrace{|-j \operatorname{sgn}(f)|^2}_{=1} |X(f)|^2 = |X(f)|^2$$

2. A signal and its Hilbert transform are orthogonal

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)\, dt = 0 \text{ (energy signals)}$$

3. If c(t) and m(t) are signals with nonoverlapping spectra, where m(t) is lowpass and c(t) is highpass, then

$$\widehat{m(t)c(t)} = m(t)\hat{c}(t)$$



$m(t) = \cos 2\pi f_1 t \quad f_1 = 100\,Hz$

$c(t) = \cos 2\pi f_2 t \quad f_2 = 500\,Hz$

$\widehat{m(t)c(t)} = m(t)\,\widehat{c(t)}$

$= \cos 2\pi f_1 t \sin 2\pi f_2 t$

Analytic Signals

- The analytic signal of the real signal x(t) is

$$x_p(t) = x(t) + j\hat{x}(t)$$

$$X_p(f) = X(f) + j(-j\,\mathrm{sgn}(f))X(f)$$

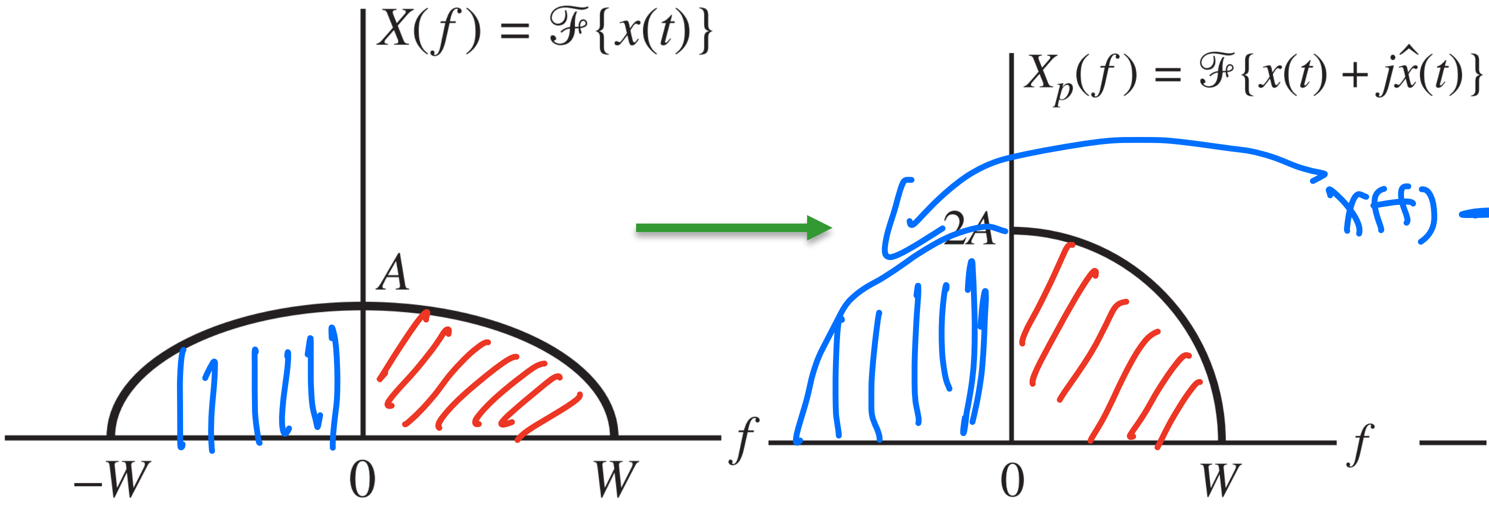
$$= (1 + \mathrm{sgn}(f))X(f)$$

if $f > 0$ $X_p(f) = 2X(f)$

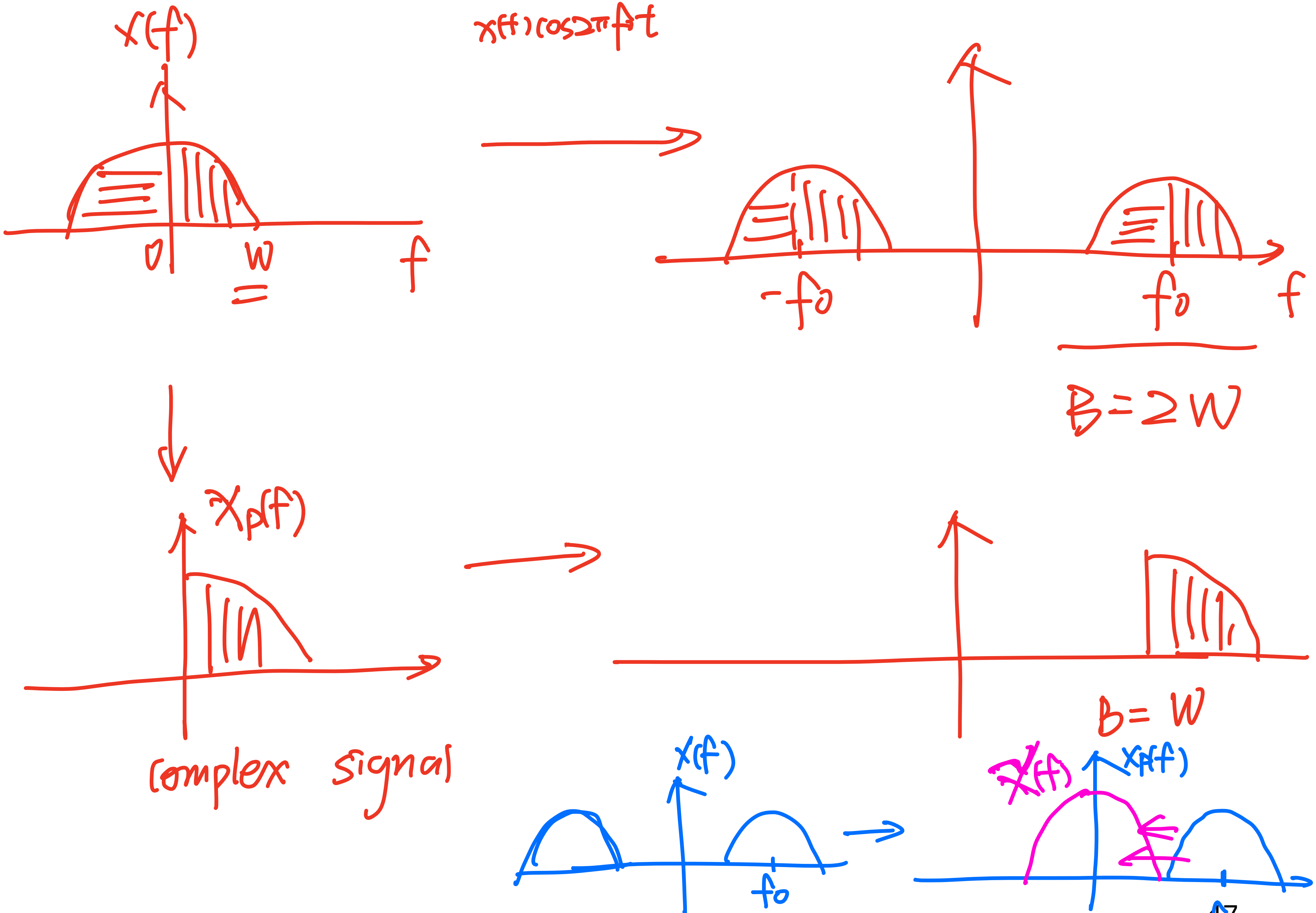
if $f < 0$ $X_p(f) = 0$

- Envelope: $|x_p(t)| = \sqrt{x(t)^2 + \hat{x}(t)^2}$
- Spectrum:

$$X_p(f) = X(f)\left[1 + \mathrm{sgn}\, f\right]$$



(a) (b)


x(f)
x(f) cos2πf₀t
0
W
f
-f₀
f₀
f
B=2W
xₚ(f)
B= W
complex signal
x(f)
f₀
x(f)
xₚ(f)
f₀

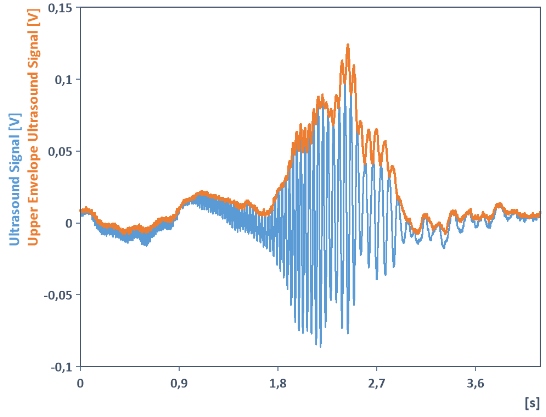
Complex Envelope

$\tilde{x}(t) = x_p(t)\, e^{-j2\pi f_0 t}$

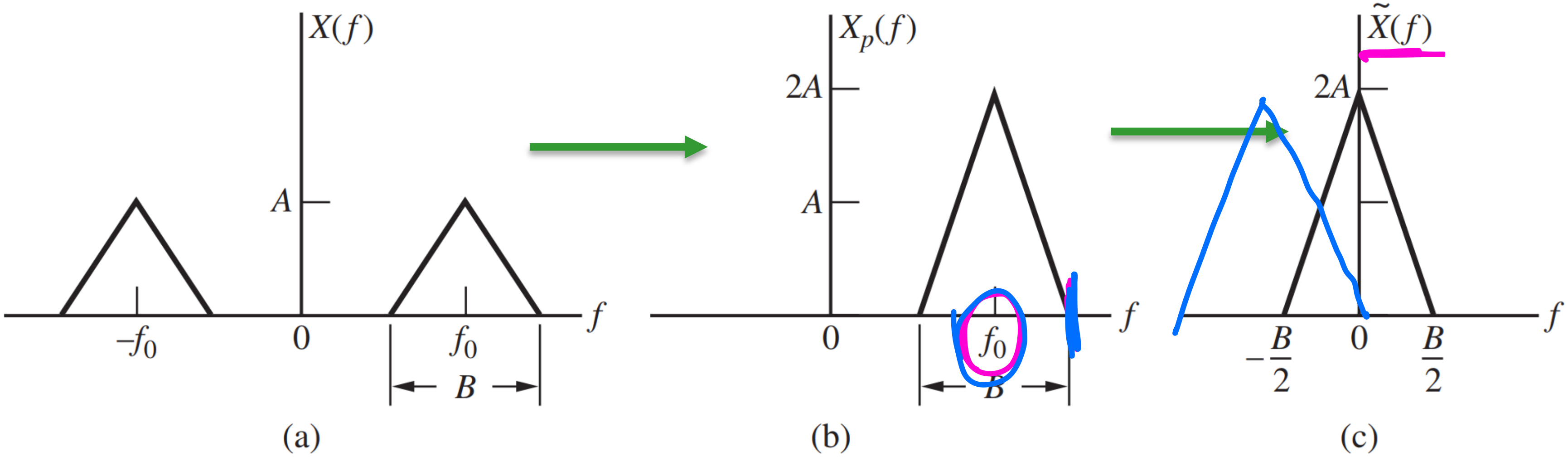
$\tilde{X}(f) = X_p(f + f_0)$

- The analytic signal can be written as

$$x_p(t) = \tilde{x}(t) e^{j2\pi f_0 t}$$



- $\tilde{x}(t)$: complex envelope of x(t)
- f_0: reference frequency
- Spectrum:



Bandpass signal → lowpass signal

Inphase and Quadrature Components

- Since

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} \triangleq x(t) + j\hat{x}(t)$$

- Thus

Re $(x_p(t))$

$$x(t) = \text{Re}\left[\tilde{x}(t)e^{j2\pi f_0 t}\right]$$
$$= \text{Re}\left[\tilde{x}(t)\right]\cos\left(2\pi f_0 t\right) - \text{Im}\left[\tilde{x}(t)\right]\sin\left(2\pi f_0 t\right)$$
$$= x_R(t)\cos(2\pi f_0 t) - x_I(t)\sin(2\pi f_0 t)$$

Inphase component of x(t)

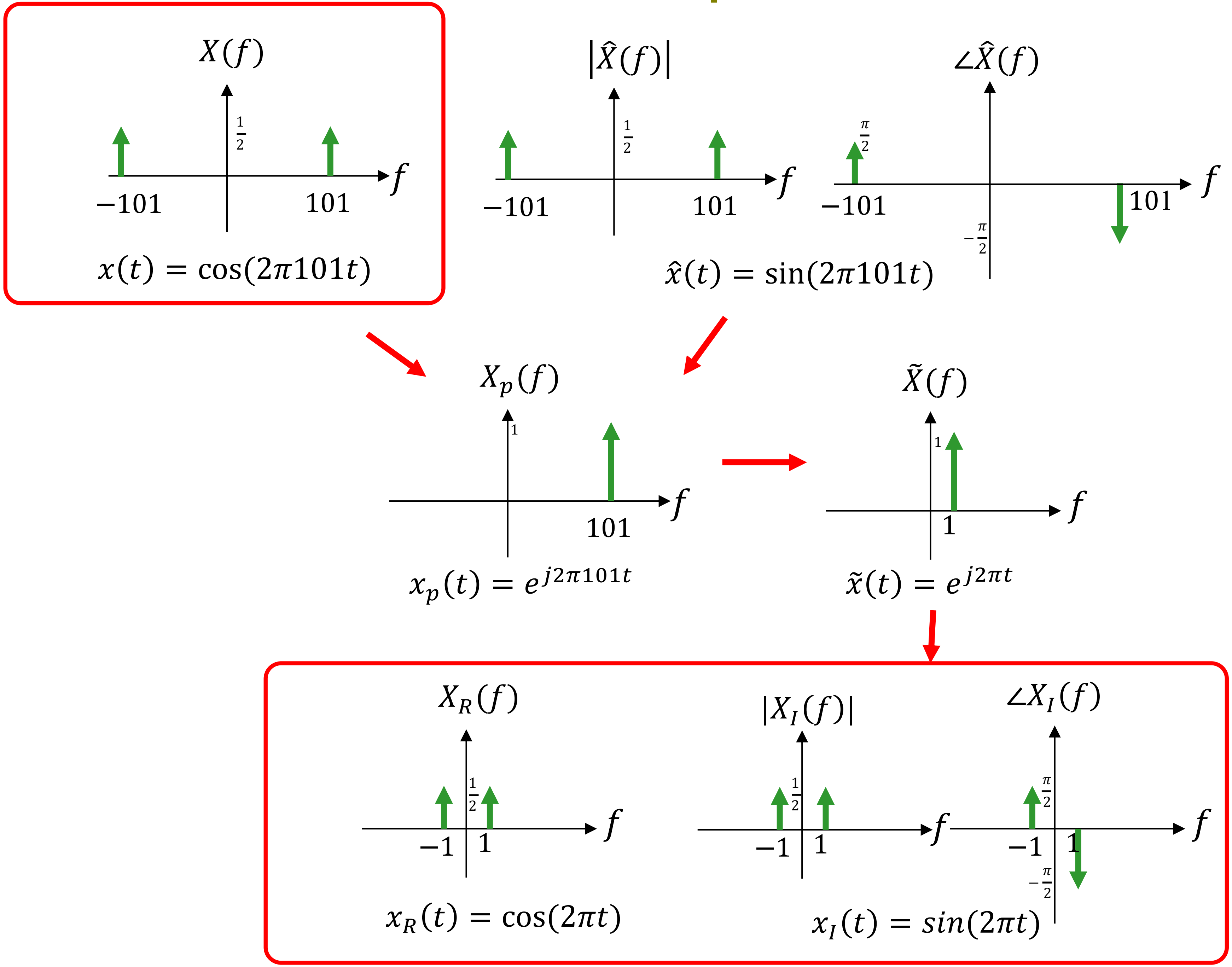
Lowpass

Quadrature component of x(t)

Lowpass

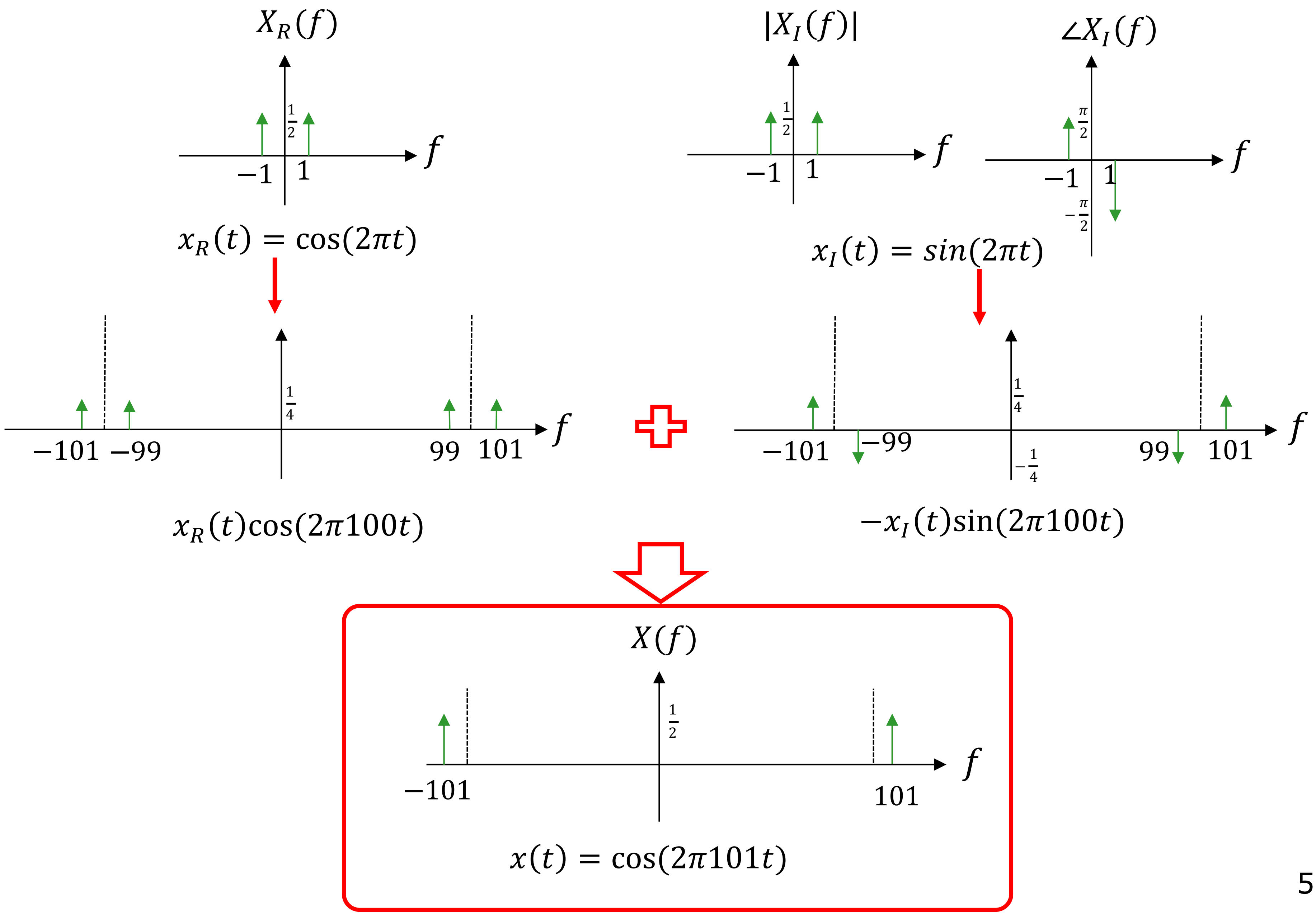
Example

$f_0 = 100Hz$



Example

$f_0 = 100Hz$



Thanks for your kind attention!

Questions?