EE140 Introduction to Communication Systems

Lecture 14

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ShanghaiTech University, Fall 2025

Signal Space and Vector Space

- Given an orthonormal basis $\{\phi_1(t), \phi_2(t), \dots\}$ of $\mathcal{L}_2$, any x(t) can be represented as:

$$x(t) = \sum_j x_j \phi_j(t)$$

Waveform $x(t) \leftrightarrow \{x_j\}$ Sequence of numbers

- Digital Modulation (Channel Encoding)
  - $\{0,1,0,1,\dots\} \rightarrow \{x_j\} \rightarrow x(t)$
  - Sequence of binary digits-> Sequence of real or complex numbers-> waveform

0101

$x(t) \longrightarrow \{x_j\} \longrightarrow \{\text{bits}\}$.

- Digital Demodulation (Channel Decoding)
  - Waveform (noisy)->sequence of received signal (numbers)-> Bit sequence
  - Minimize error rate, subject to power and bandwidth constraints.

$u(t) = \sum_k u_k p(t-kT_s)$

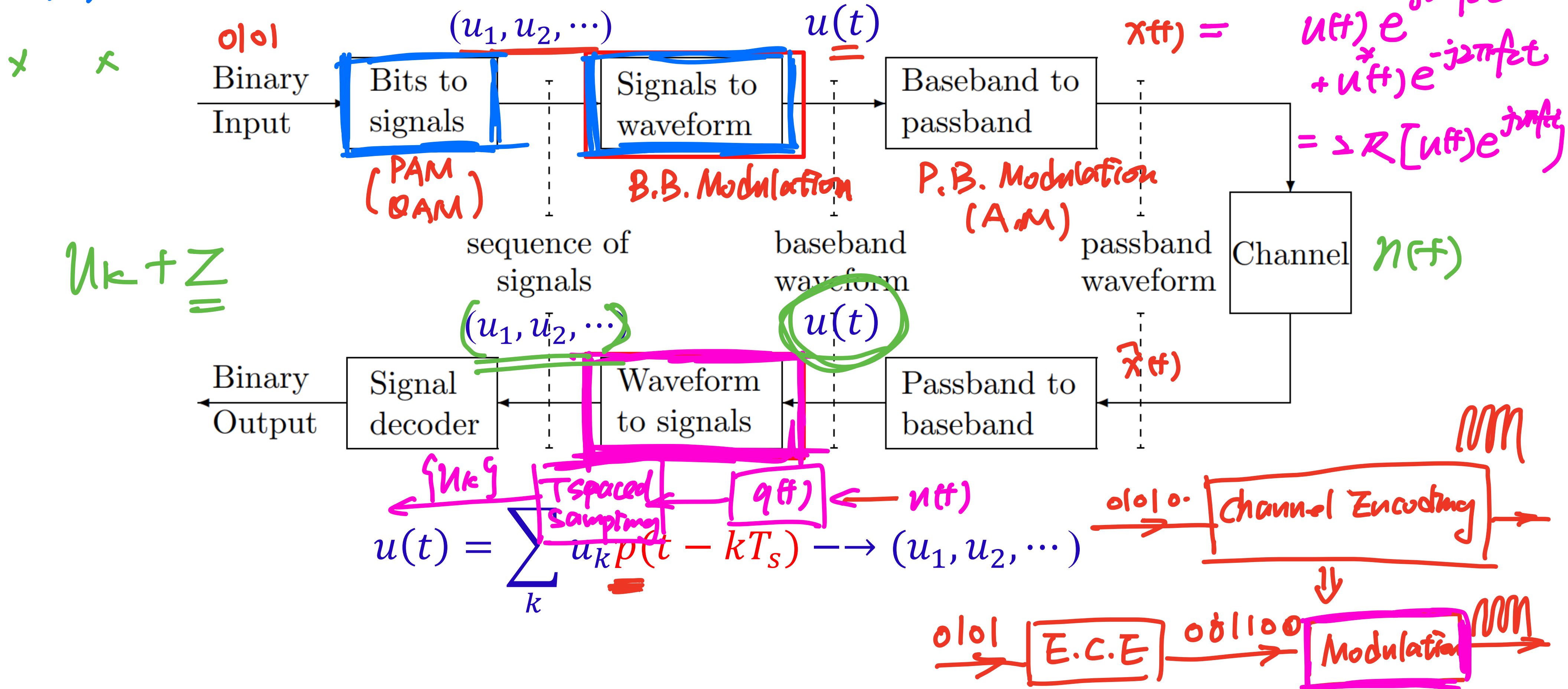
$\sum_k u_k \delta(t-kT_s) \rightarrow \boxed{p(t)} \rightarrow$

Detection

symbol/signal/number

Modulation and Demodulation

DSB-SC



$$u(t) = \sum_k u_k p(t - kT_s) \dashrightarrow (u_1, u_2, \cdots)$$

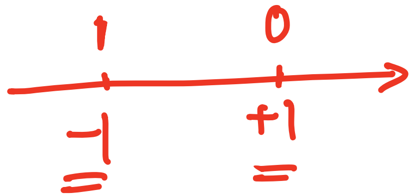
- Step 1: Binary digits $\rightarrow$ A sequence of signal[1] (numbers)
- Step 2: A sequence of numbers $\rightarrow$ Waveforms
- Step 3: baseband $\rightarrow$ pass band

Modulation and Demodulation

- Example 1:

$R_b = \frac{1}{T_b} \Leftarrow$ bit rate

  - Sequence of binary symbols enters the encoder at T_b-spaced instants of time. These symbols are mapped to real numbers using the mapping 0->+1, 1->-1. The resulting sequences u1,u2,… of real numbers is then mapped into a transmitted waveform $u(t) = \sum_k u_k p(t - kT_s)$.
  - $p(t)$: basic pulse waveform (sinc, rect)
  - $R_b = \frac{1}{T_b} \quad bits/s$.
  - $b = 1, T_s = T_b, R_s = R_b$ signals/s (symbols/s).



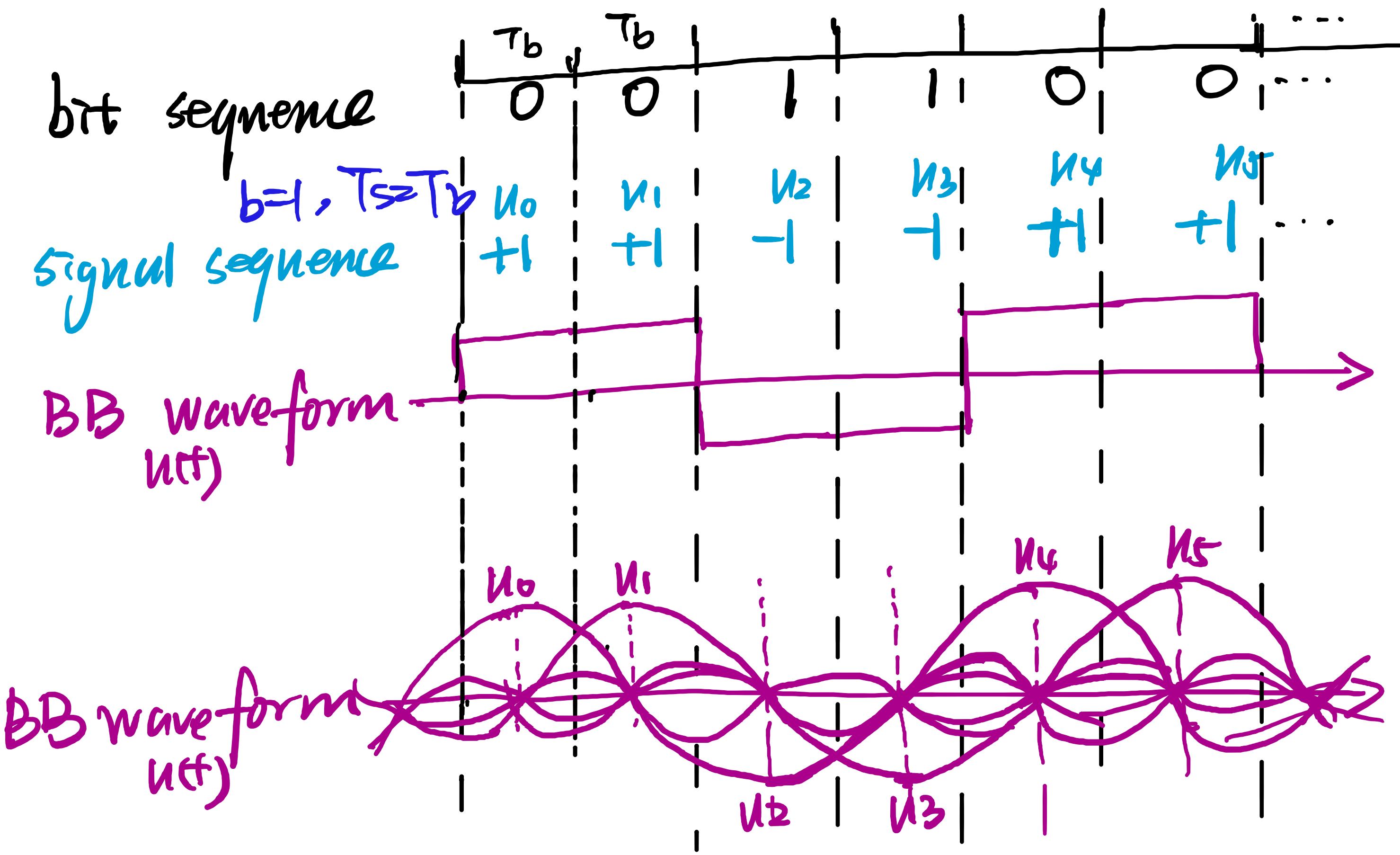
$= \frac{1}{Ts}$

Modulation and Demodulation

- Example 1:

$$u(t) = \sum_k u_k p(t - kT_s)$$

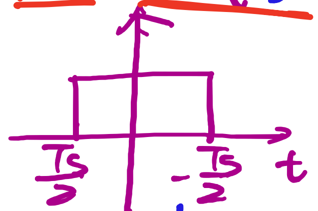
2-PAM



$B = \frac{1}{T_s}$ Bandwidth

$P(f) = T_s \mathrm{sinc}(T_s f)$

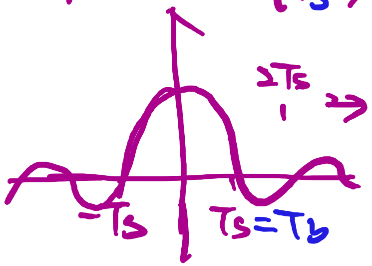
$p(t) = \mathrm{rect}\left(\frac{t}{T_s}\right)$



$B = \frac{1}{2T_s}$

$P(f) = T_s \mathrm{rect}(T_s f)$

$p(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$



Modulation and Demodulation

- Example 2:
  - Sequence of binary symbols enters the encoder at T_b-spaced instants of time.
  - Binary symbols are first segmented into b-bit blocks.
  - $\boxed{M = 2^b} \rightarrow$ signal constellation $\mathcal{A} = \{a_1, a_2, \dots, a_M\}$
    - b=2, 00->-3,01->-1,10->+1,11->+3.
    - $T_s = bT_b = 2T_b, R_s = \frac{1}{T_s} = \frac{R_b}{b} = \frac{R_b}{2}$ signals/s.
  - The resulting sequences u1,u2,... of real numbers is then mapped into a waveform $u(t) = \sum_k u_k p(t - kT_s)$.

0 1 0 1 0 1 ..

b bits

00 01 10 11

$A = \{-3, -1, +1, +3\}$

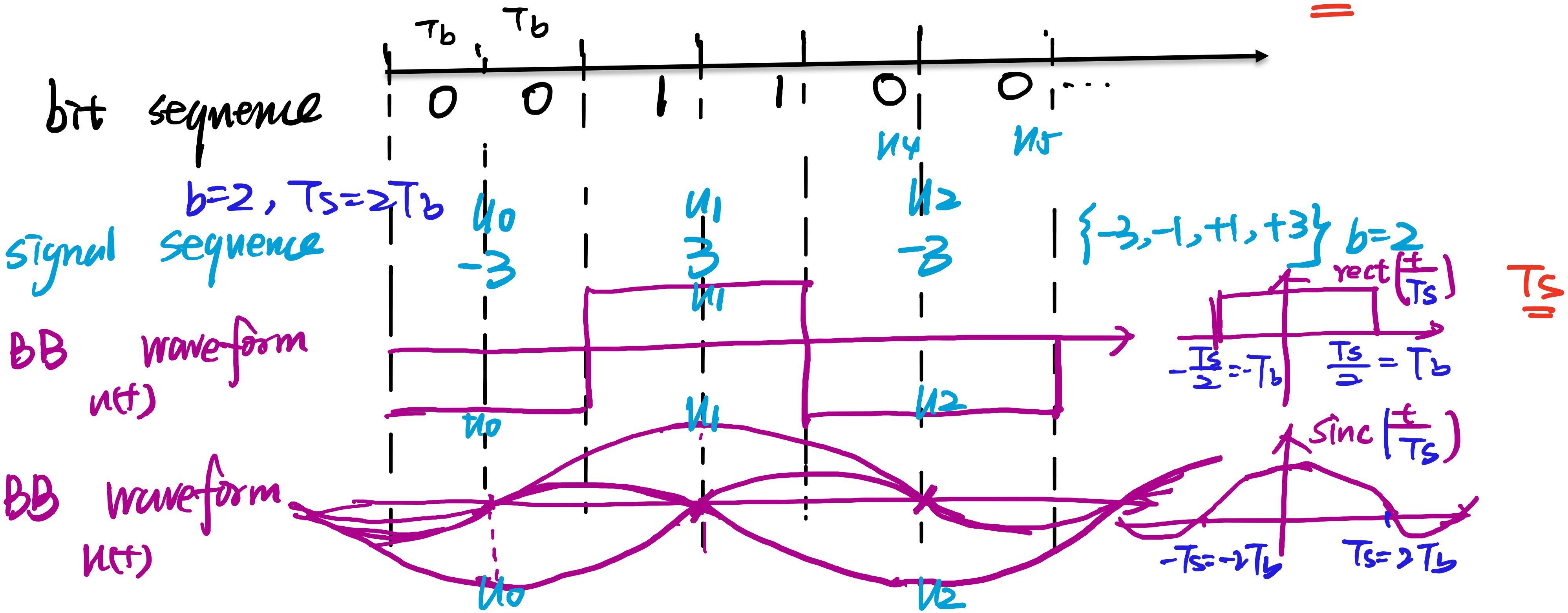
Modulation and Demodulation

- Example 2:

$$u(t) = \sum_k u_k p(t - kT_s)$$

M-PAM

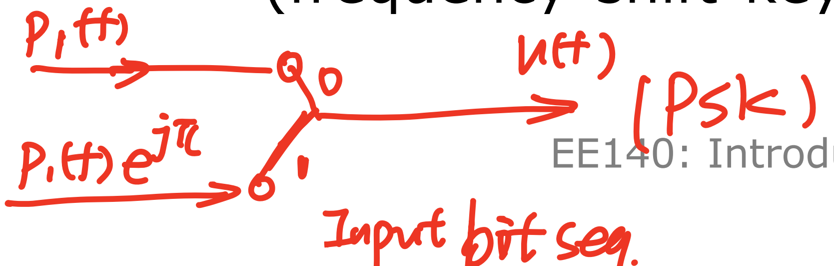
$M = 2^b$



Example: Modulation and Demodulation

1. Map information bits into $(u_1, u_2, \cdots)$.
   - (0010..) → (+1,+1,-1,+1,…)
2. Map sequence of numbers (or signals) into a waveform
   - $(u_1, u_2, \cdots) \rightarrow u_1 p(t), u_2 p(t - T_s), u_3 p(t - 3T_s), \dots$
   - $u(t) = \sum_k u_k p(t - kT_s)$, $u_k = \{+1, -1\}$, T_s is the signal interval
   - $\{p(t - kT_s)\}$: baseband basic pulse waveform (explain later)
3. Map a baseband waveform $u(t)$ into a passband waveform $x(t) = Re\left(u(t) e^{j2\pi f_c t}\right) = u(t)\cos 2\pi f_c t$ (DSB-AM).

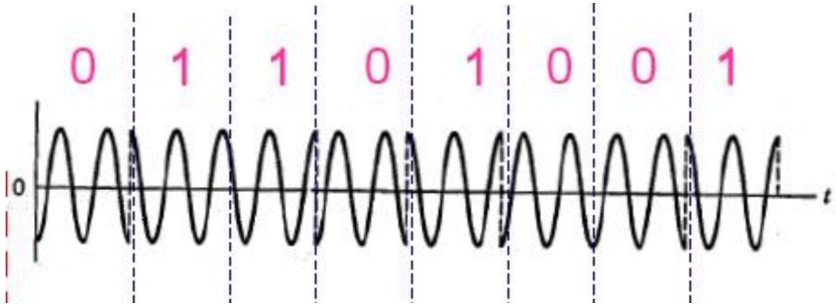
- Various Modulations: we can change
  - Amplitude: mapping {00,01,10,11} → -3,-1,+1,+3 (4PAM)
  - Phase: 0 → $p_1(t)$, 1 → $p_2(t) = p_1(t)e^{j\pi}$ (phase-shift-keying)
  - Frequency: 0 → $p_1(t) = a\cos(2\pi\Delta f t)$, 1 → $p_2(t) = a\cos(4\pi\Delta f t)$ (frequency-shift-keying)



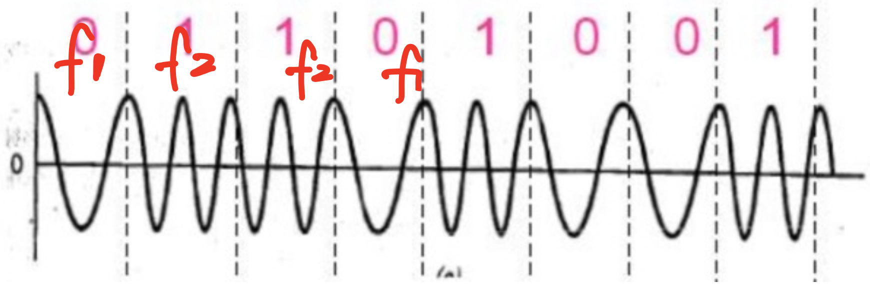
Example: Modulation and Demodulation

1. Map information bits into $(u_1, u_2, \cdots)$.
   - (0010..) → (+1,+1,-1,+1,…)
2. Map sequence of numbers (or signals) into a waveform
   - $(u_1, u_2, \cdots) \rightarrow u_1 p(t), u_2 p(t - T_s), u_3 p(t - 3T_s), \ldots$
   - $u(t) = \sum_k u_k p(t - kT_s)$, $u_k = \{+1, -1\}$, T_s is the signal interval
   - $\{p(t - kT_s)\}$: baseband pulse waveform (explain later)
3. Map a baseband waveform $u(t)$ into a passband waveform $x(t) = Re(u(t)e^{j2\pi f_c t})$.

- Various Modulations: we can change



PSK

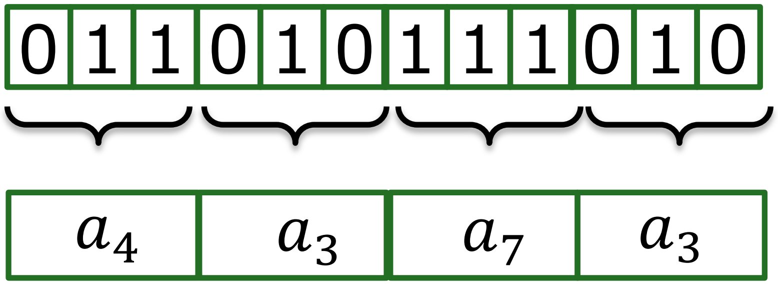


FSK

Pulse Amplitude Modulation (PAM)

Bit rate: R_b bps

Bit interval: $T_b = 1/R_b$ s.



b bits blocks (b=3)

$M = 2^b$

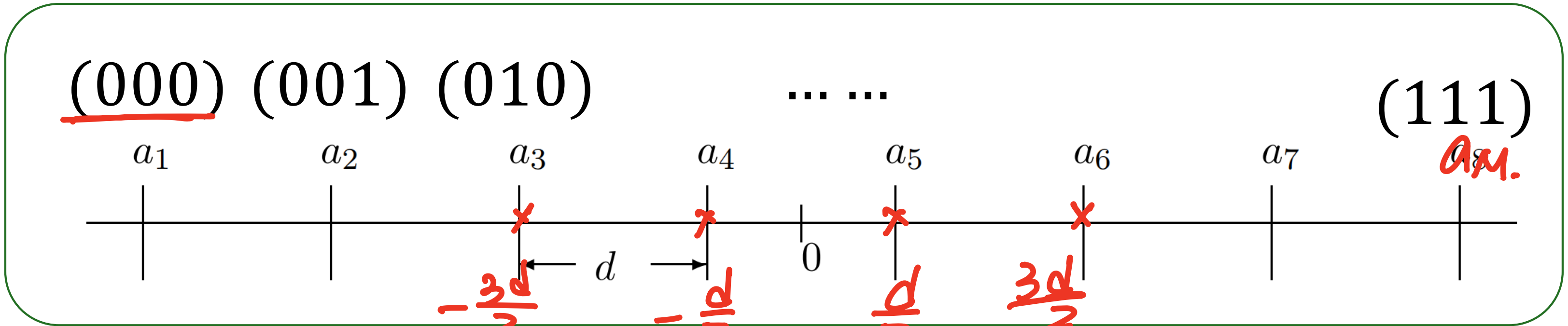
↓

signal constellation $\mathcal{A} = \{a_1, a_2, \dots, a_M\}$ of real numbers

Signal (Symbol) Rate: $R_s = R/b$

signals/s Baud/s

Signal (Symbol) interval: $T_s = \frac{1}{R_s} = bT_b$ s



Pulse Amplitude Modulation (PAM)

0	1	1	0	1	0	1	1	1	0	1	0

a_4	a_3	a_7	a_3

$\downarrow$

$$u(t) = \sum_k u_k p(t - kT_s)$$

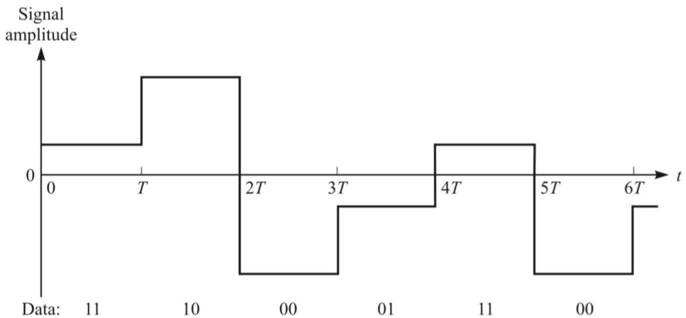
b bits blocks (b=3)

$M = 2^b$

$\downarrow$

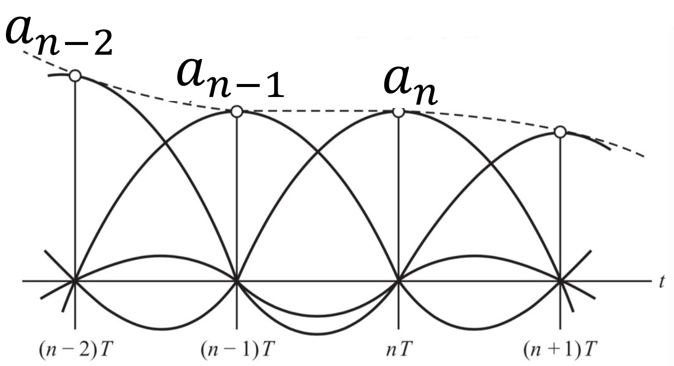
signal constellation $\mathcal{A} = \{a_1, a_2, \dots, a_M\},\ a_j \in R$

$T_b = \frac{1}{R}, T_s = \frac{1}{R_s} = \frac{b}{R} = bT_b$ (interval between signals)



(a) Baseband PAM signal

{00,01,10,11} → -3,-1,+1,+3



MPAM

(E.E., B.E.)

- How to design the M-PAM Modulation Scheme?

$M = 2^b$

$\mathcal{A}$: position, d

Mapping $\{\underbrace{010}_{b}\} \xrightarrow{?} \{a_m\}$

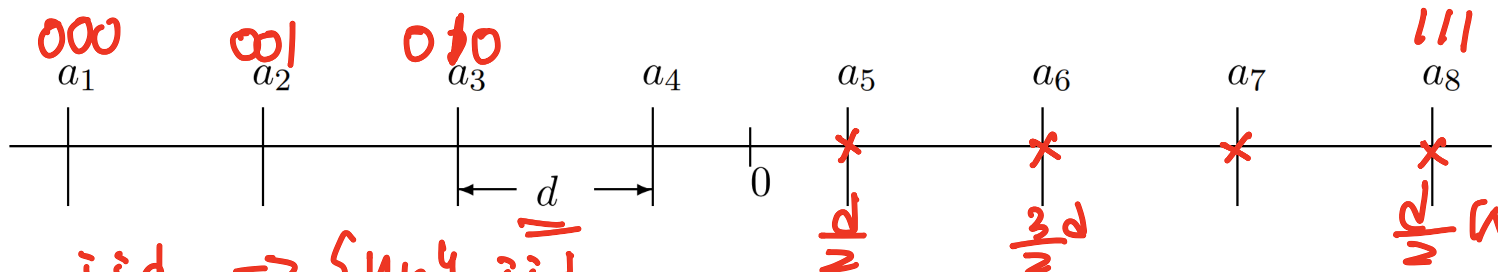
Efficiency ⟸ ① average Energy: E_b, E_s

② Bandwidth: $B \propto \frac{1}{T_s} = \frac{R_b}{b}$

③ Transmission rate: $R_s = \frac{R_b}{b}$

Reliability ④ Detection Error Prob: P_e

- ~~Average Energy~~
  - Standard M-PAM constellation $\mathcal{A}$



bit seq : iid $\Rightarrow \{u_k\}$ iid

  - Statistics of $\{u_k\}$

$P_m = P(u_k = a_m) = P(\qquad)$

$P(u_k = a_1) = P(\text{bit seq} = 000)$

$P(u_k = a_2) = P(\text{bit seq} = 001)$

$\boxed{\text{bit seq: iid } \Rightarrow P(0)=P(1) \Rightarrow P(u_k = a_m) = P_m = \frac{1}{M}}$

Pulse Amplitude Modulation (PAM)

$U_k(t) = U_k p(t - kT_s)$

0 1 1 0 1 0 1 1 1 0 1 0

$U_1, U_2, \ldots$

a_4	a_3	a_7	a_3

↓

$$u(t) = \sum_k u_k p(t - kT_s)$$

if $U_k = a_m$

$U_k(t) = a_m p(t - kT_s)$

$E_m \stackrel{\text{if } U_k = a_m}{=} \int |U_k(t)|^2 dt$

$= a_m^2 \int |p(t - kT_s)|^2 dt$

$= a_m^2 E_p$

$E_{ave} = E\left[\int |U_k(t)|^2 dt\right]$

$= E[U_k^2] E_p$

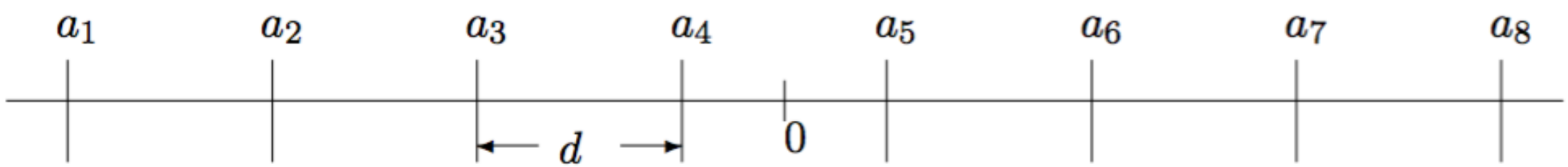
$= \sum_{m=1}^{M} P(U_k = a_m) E_m$

$= \boxed{\sum_{m=1}^{M} p_m a_m^2 E_p}$

$p_m = \frac{1}{M}$

- Assume: $U_1, U_2, \ldots,$ iid $\sim P(U_k = a_m) = p_m$ (k is time index, $m \in \{1, \ldots, M\}$ is index in $\mathcal{A}$)
- Energy $E_m = \int |a_m p(t)|^2 dt$
- Average signal energy $E_{\text{avg}} = \sum_{m=1}^{M} p_m E_m$
- Average energy per bit $E_{\text{bavg}} = \frac{E_{\text{avg}}}{b}$, power $P_{\text{avg}} = E_{\text{bavg}}/T_b$
- Binary PAM: $b = 1$; M-PAM : $M = 2^b$

PAM Signal Constellation



- Assume incoming bits are equiprobable RVs.
- Each signal $U_k = a_m$ is equiprobable in $\mathcal{A}$

$$\mathcal{A} = \{\frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2}\}$$

- The choose of values in $\mathcal{A}$ is similar to finding representation points in quantization problem
- Average power per signal: E_{avg}?

PAM Signal Constellation

$$\mathcal{A} = \{\frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2}\}$$

For symbol $U_k = a_m \in \mathcal{A} \Rightarrow$ send $u_k(t) = a_m p(t - kT)$,

$$E_m = \int_{-\infty}^{\infty} a_m^2 p^2(t-kT)dt = a_m^2 E_p$$

Thus,

$P_m = \frac{1}{M}$

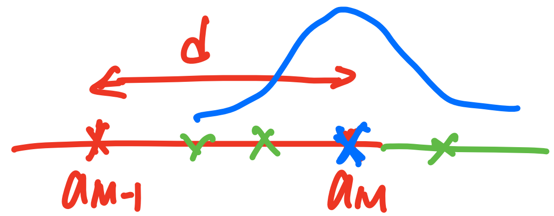
$$\begin{aligned} E_{\text{avg}} &= \sum_{m=1}^{M} \left(\frac{1}{M} \cdot a_m^2 E_p\right) \\ &= 2\frac{E_p}{M}\left(\frac{d}{2}\right)^2 (1^2 + 3^2 + \ldots + (M-1)^2) \\ &= \frac{d^2}{2}\frac{E_p}{M} \times \frac{M(M^2-1)}{6} \\ &= \frac{d^2(M^2-1)E_p}{12} \end{aligned}$$

PAM Signal Constellation

The signal energy, i.e., the mean square signal value assuming equiprobable signals is:

$$E_{\text{avg}} = \frac{d^2(M^2-1)E_p}{12}$$

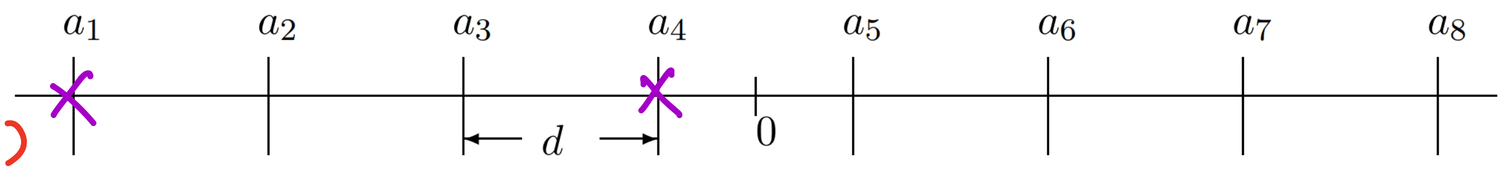
$d \downarrow$ $M \downarrow$ $E_{ave} \downarrow$



- E_{avg} increases as d^2 and M^2
- d is determined by the noise
- Errors is reception are primarily dues to noise exceeding $d/2$
- For many channels, the noise is independent of the signal, which explains the standard equal spacing between signal constellation values.

$P(000) > P(111)$

$P(0) > P(1)$



$P(0) = P(1)$

$P_m = \frac{1}{M}$

MPAM

$\begin{cases} M \\ d \quad \text{position} \\ \text{Mapping} \end{cases}$

$$E_s = \sum P_m |a_m|^2 E_p$$

Energy: $E_s = \frac{d^2(M^2-1)}{12} E_p \Longrightarrow \begin{cases} d \longrightarrow P_e : d\uparrow \; P_e \downarrow \\ M \xrightarrow[=2^b]{} \begin{cases} R_s = \frac{R_b}{\log_2 M} \\ \text{B.B. Bandwidth} \end{cases} \end{cases}$

$B_b \propto \frac{1}{T_s} = \frac{R_b}{b}$

E.E. $\frac{R_b}{E_b} = \frac{R_b \log_2 M}{E_s} = \frac{12 R_b \log_2 M}{d^2(M^2-1) E_p}$

$d\downarrow \; M\downarrow \; E.E.\uparrow$ $\qquad$ $M\uparrow \; \cancel{B}.E.\uparrow$

given M. $\boxed{\begin{array}{l} \min E_s \\ d.\text{mapping} \\ \text{s.t. } P_e \le \varepsilon \end{array}}$ $\Leftarrow$ mapping. $d = d_\varepsilon$ $\quad \Leftarrow d \ge d_\varepsilon$

$P_e \le \varepsilon_1 \Leftarrow d \ge d_{\varepsilon_1}$

$E_s \le \varepsilon_2 \Leftarrow d = d_{\varepsilon_1}$. $M \le M_{\varepsilon_2}$

$M = M_{\varepsilon_2}$

Quadrature Amplitude Modulation QAM

$$u(t) = \sum_k u_k p(t - kT) = \sum_k u_k(t)$$

- Segment the incoming binary symbols into b-bits blocks.
- Map the b-bits into a signal constellation $\mathcal{A} = \{a_1, \ldots, a_M\}$, $a_j \in \mathbb{C}$
- $u(t)$ is a complex baseband waveform
- Convert $u(t)e^{j2\pi f_c t}$ to a real waveform

$$u(t) = \sum_k u_k p(t - kT_s)$$

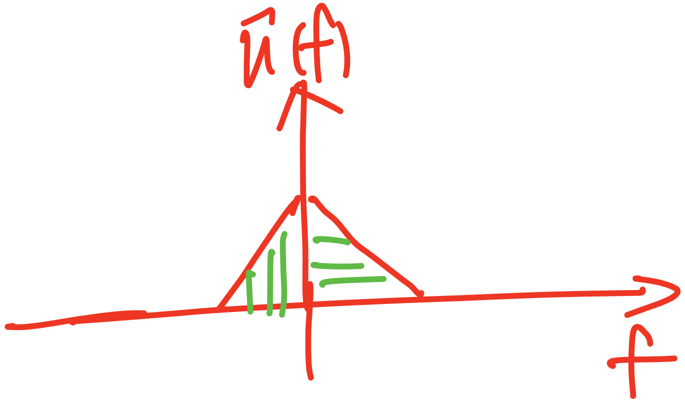
$$\begin{aligned} x(t) &= 2\text{Re}[u(t)e^{j2\pi f_c t}] \\ &= 2\text{Re}[u(t)]\cos(2\pi f_c t) - 2\text{Im}[u(t)]\sin(2\pi f_c t) \\ &= u(t)e^{j2\pi f_c t} + u^*(t)e^{-j2\pi f_c t} \end{aligned}$$

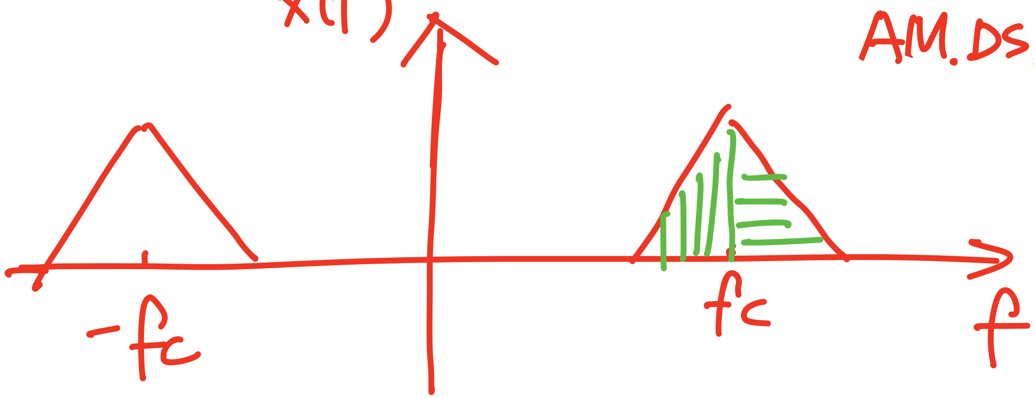
- QAM solves the frequency waste problem of DSB
- When $u(t)$ is real, QAM reduces to DSB PAM
- The factor of 2 is an arbitrary scale factor, and can be left out.

PAM $\quad u(t) = \sum u_k\, p(t-kT_s)$ real waveform

$$x(t) = u(t)\, e^{j2\pi f_c t} + u(t)\, e^{-j2\pi f_c t}$$
$$= 2u(t)\cos 2\pi f_c t$$

AM.DSB





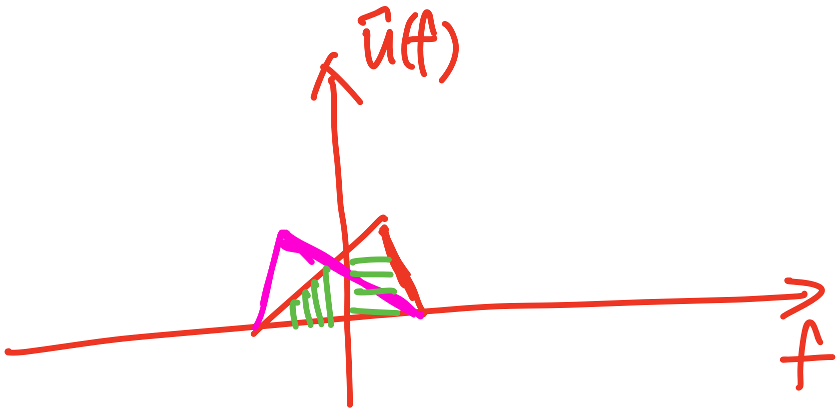
QAM $\quad u(t) = \sum_k u_k\, p(t-kT_s)$ complex waveform

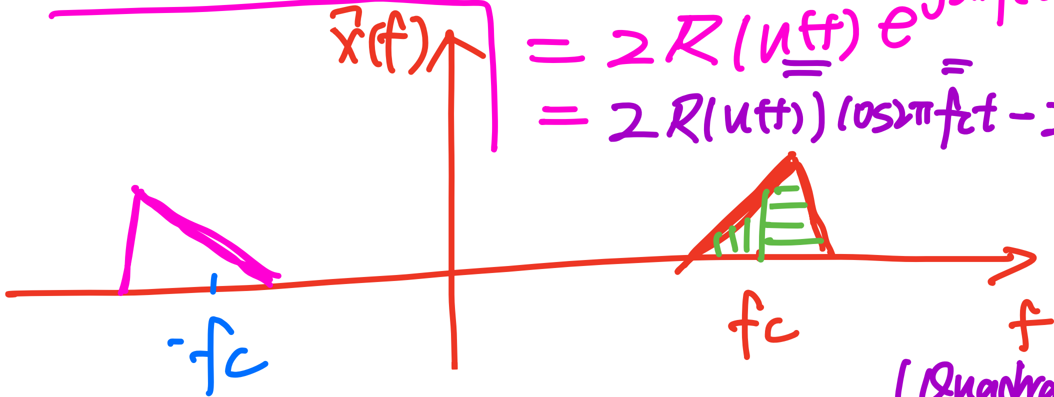
$$x(t) = u(t)\, e^{j2\pi f_c t} + u^*(t)\, e^{-j2\pi f_c t}$$
$$= 2\mathcal{R}\left(u(t)\, e^{j2\pi f_c t}\right)$$
$$= 2\mathcal{R}(u(t))\cos 2\pi f_c t - 2\mathcal{I}[u(t)]\sin 2\pi f_c t$$

$\Downarrow$

(Quadrature Carrier)

DSB-QC





$$F[u^*(t)] = \hat{u}^*(-f)$$

QAM Constellation

- Bit rate: R bps
- Segment binary b bits, map into complex numbers $u_k \in \mathcal{A}$, $|\mathcal{A}| = M = 2^b$.
- Signal (symbol) rate: $R_s = R/b$.
- Standard QAM Constellation
  - Cartesian product: $\mathcal{A} \times \mathcal{B}$, e.g.,

$$\mathcal{A} = \{1,2\}, \mathcal{B} = \{3,4\}$$
$$\mathcal{A} \times \mathcal{B} = \{(1,3),(1,4),(2,3),(2,4)\}$$
$$\mathcal{B} \times \mathcal{A} = \{(3,1),(3,2),(4,1),(4,2)\}$$

$$\underbrace{0\,1\,1}_{a_4}\;\underbrace{0\,1\,0}_{a_3}\;\underbrace{1\,1\,1}_{a_7}\;\underbrace{0\,1\,0}_{a_3}$$

complex ↓

$$u(t) = \sum_k u_k p(t - kT_s)$$

QAM Constellation

- Standard QAM Constellation
  - A standard $(\sqrt{M}, \sqrt{M})$-QAM signal set is the Cartesian product of two $\sqrt{M}$-PAM set, i.e.,

$$\mathcal{A} = \{(a' + ja'') | a' \in \mathcal{A}', a'' \in \mathcal{A}'\}$$

where $\mathcal{A}' = \{\frac{-d(\sqrt{M}-1)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(\sqrt{M}-1)}{2}\}$

  - It is a square array signal points located as below for $M = 16$

$$E_s = E[|u_k|^2] E_p$$
$$= E[|u_k' + u_k'' j|^2] E_p$$
$$= \left(E[|u_k'|^2] + E[|u_k''|^2]\right) E_p$$
$$= \left(\frac{((\sqrt{M})^2 - 1) d^2}{12}\right) \times 2 E_p$$
$$= \frac{(M-1) d^2}{6} E_p$$

$\left(-\frac{3}{2}d, \frac{3}{2}d\right)$ $\left(\frac{3}{2}d, \frac{3}{2}d\right)$

(4, 4)-QAM

4-PAM

$\left\{-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}\right\}$

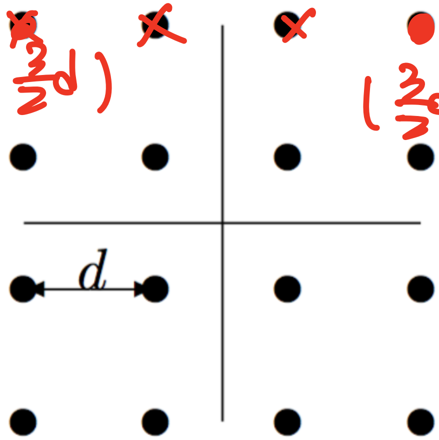
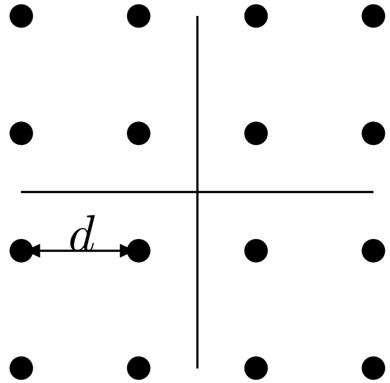


Figure: Standard 16QAM constellation [Gallagar'Book]

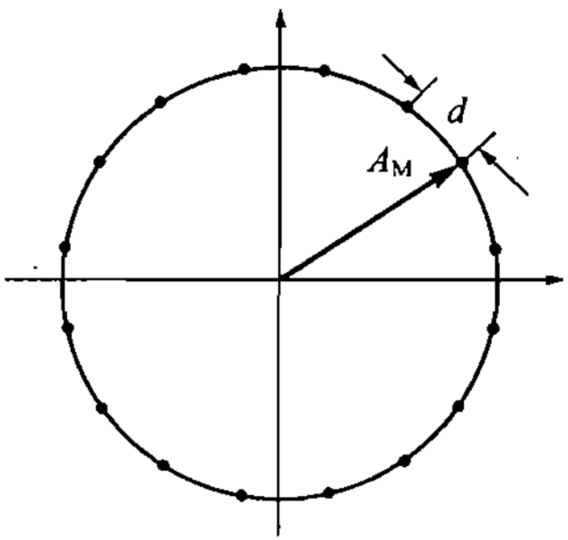
QAM Constellation

- Standard QAM Constellation
  - Average energy per 2D signal:

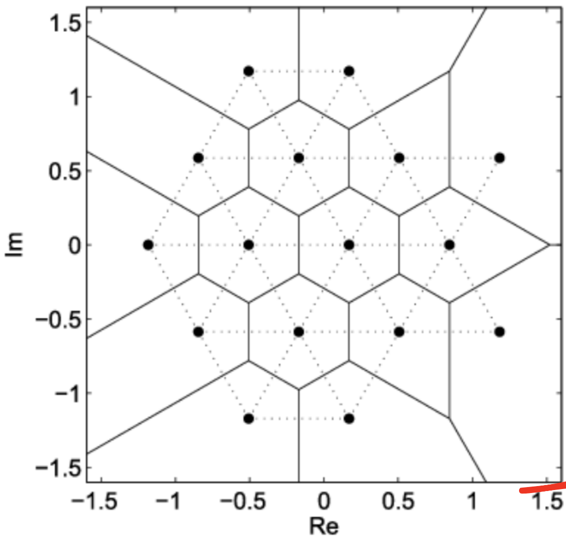
$$E_s = \frac{d^2(M'^2-1)E_p}{6} = \frac{d^2(M-1)E_p}{6}$$



- Q: How to arrange the signal points
  - Choose constellations that minimize Es given d and M.
  - MPSK



(b) 16PSK



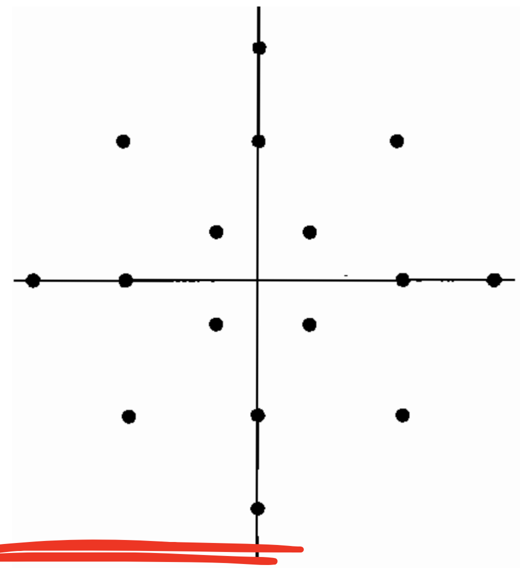


图 8－5　改进的 16QAM 方案

same Pe

Es(M, d)

same Es

Pe(M)

$E_s(QAM) = \frac{(M-1)d}{6}$, $E_s(PAM) = \frac{(M^2-1)d}{12}$

d_{PAM}

$d_{QAM} >$

(AM) PAM

QAM Constellation

$E_{s,QAM} < E_{s,PAM}$ $\quad P_{e,QAM} < P_{e,PAM}$

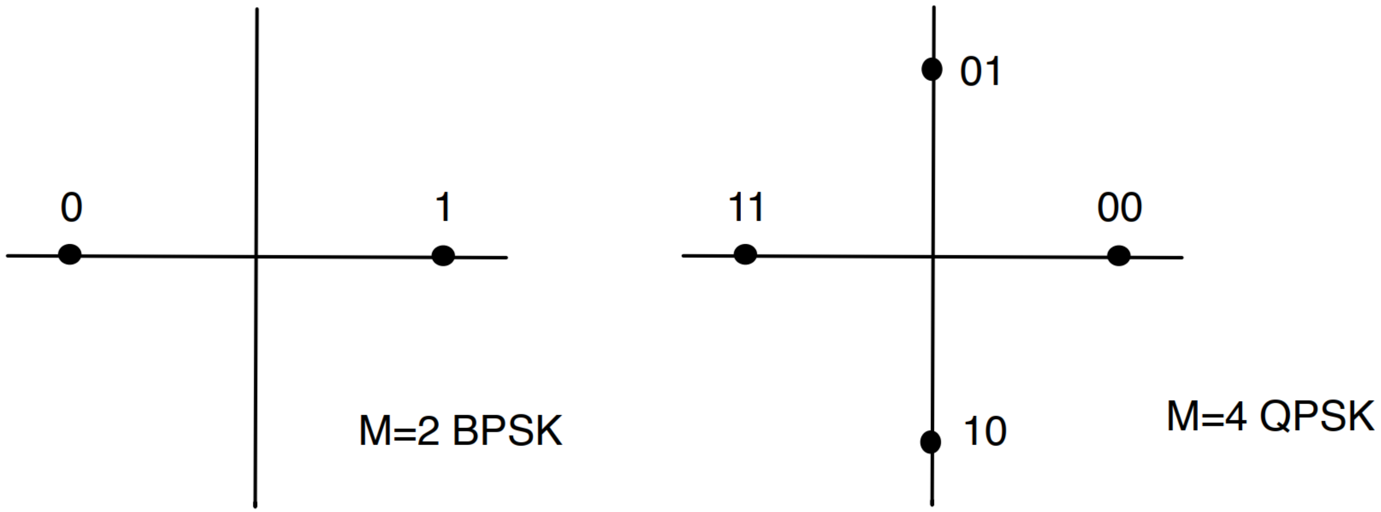
M is large (AM, PM) QAM

- PSK: Special Case of QAM
  - $u_k \in \mathcal{A}$

(PM) PSK $\quad E_{s,QAM} < E_{s,PSK}$

$P_{e,QAM} < P_{e,PSK}$

$$\mathcal{A} = \{e^{j\frac{2\pi(m-1)}{M}}, \text{ for } m = 1, \ldots, M\}$$



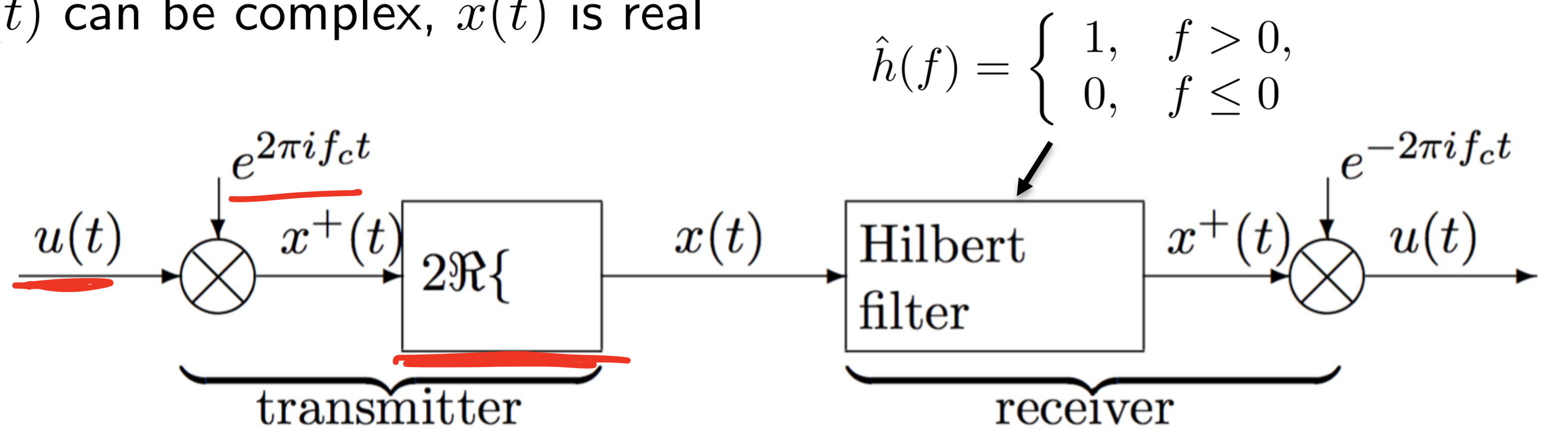
- Signal points have same amplitude $\quad E_s = E[|u_k|^2]E_p = r^2 E_p$
- PSK is rarely used for large M (signal points are very closed)
- Combing PSK and PAM

QAM: baseband to passband

Shift complex $u(t)$ to f_c, then add complex conjugate to form real $x(t)$

$$\begin{aligned} x(t) &= u(t)e^{j2\pi f_c t} + u^*(t)e^{-j2\pi f_c t} \\ &= x^+(t) + (x^+(t))^* \\ &= 2\text{Re}[u(t)e^{j2\pi f_c t}] \end{aligned}$$

- Baseband Bandwidth: $B_b < f_c$
  - $u(t)$ can be complex, $x(t)$ is real



  - This is nice for analysis, but not usually so for implenmentation

QAM: baseband to passband

- Easier way

$$u(t) = \sum_k u_k p(t - kT)$$

$$\begin{aligned} x(t) &= 2\text{Re}[u(t)e^{j2\pi f_c t}] \\ &= 2\text{Re}[u(t)]\cos(2\pi f_c t) - 2\text{Im}[u(t)]\sin(2\pi f_c t) \end{aligned}$$

Assume $p(t)$ is real

$$\text{Re}[u(t)] = \sum_k \text{Re}[u_k]p(t - kT) \qquad \text{B.B.}$$

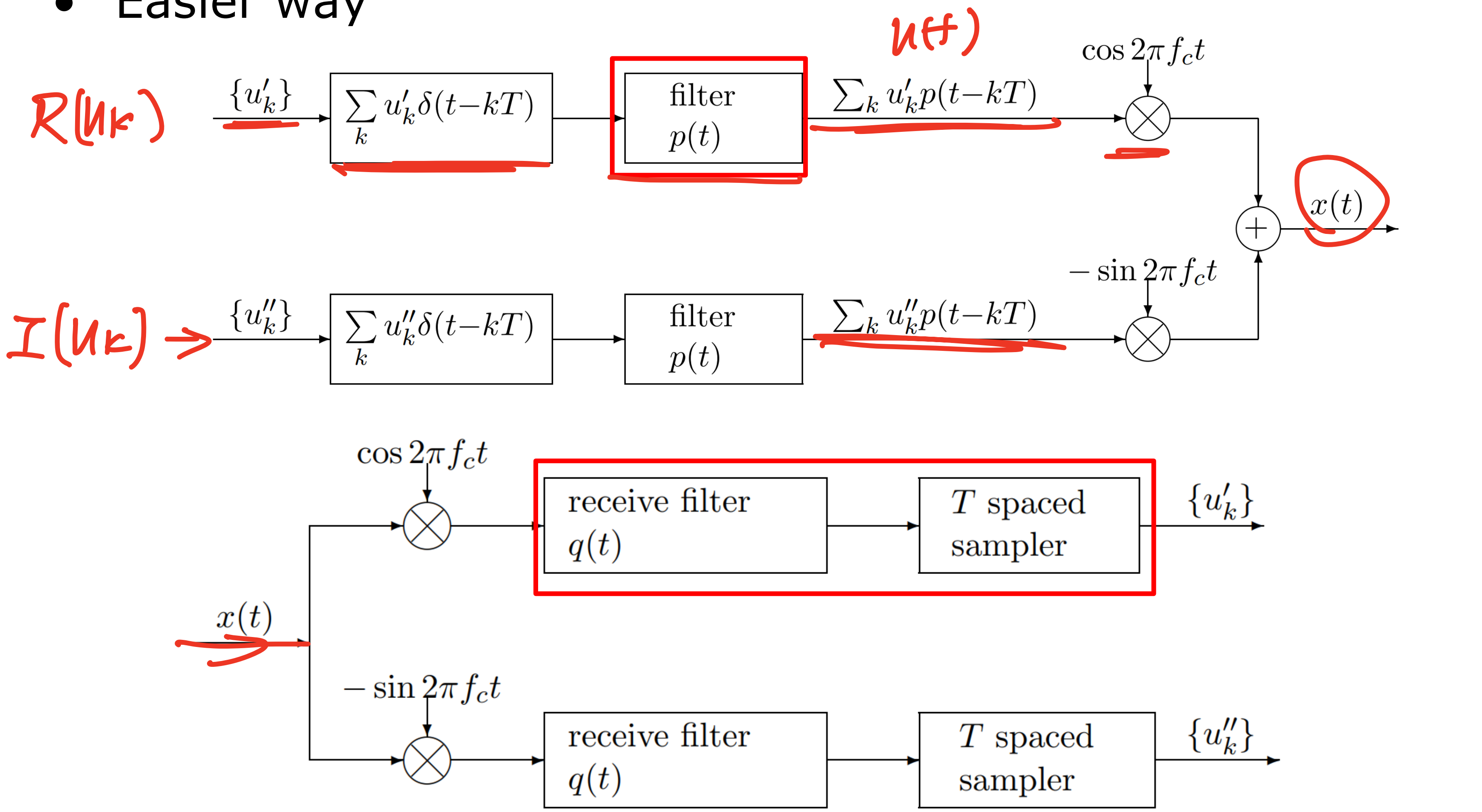
$$\text{Im}[u(t)] = \sum_k \text{Im}[u_k]p(t - kT) \qquad \text{B.B.}$$

with $u'_k = \text{Re}[u_k]$ and $u''_k = \text{Im}[u_k]$

$$x(t) = 2\left(\sum_k u'_k p(t - kT)\right)\cos 2\pi f_c t - 2\left(\sum_k u''_k p(t - kT)\right)\sin 2\pi f_c t$$

QAM: baseband to passband

- Easier way



q(t) should be chosen to such that $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$ satisfy the Nyquist criterion. (Explained Later)

QAM: Implementation

PAM is a special case of QAM

Transmitter:

- Binary digits $\xRightarrow{b \text{ bits}}$ symbols $\xRightarrow{\text{Constellation}}$ $\boxed{\text{complex signals} \xRightarrow{\text{multiply } p(t)} u(t)}$ $\xRightarrow{\text{shift to } f_c}$ $u(t)e^{j2\pi f_c t}$ $\xRightarrow{\text{add } u^*(t)e^{-j2\pi f_c t}}$ $x(t)$

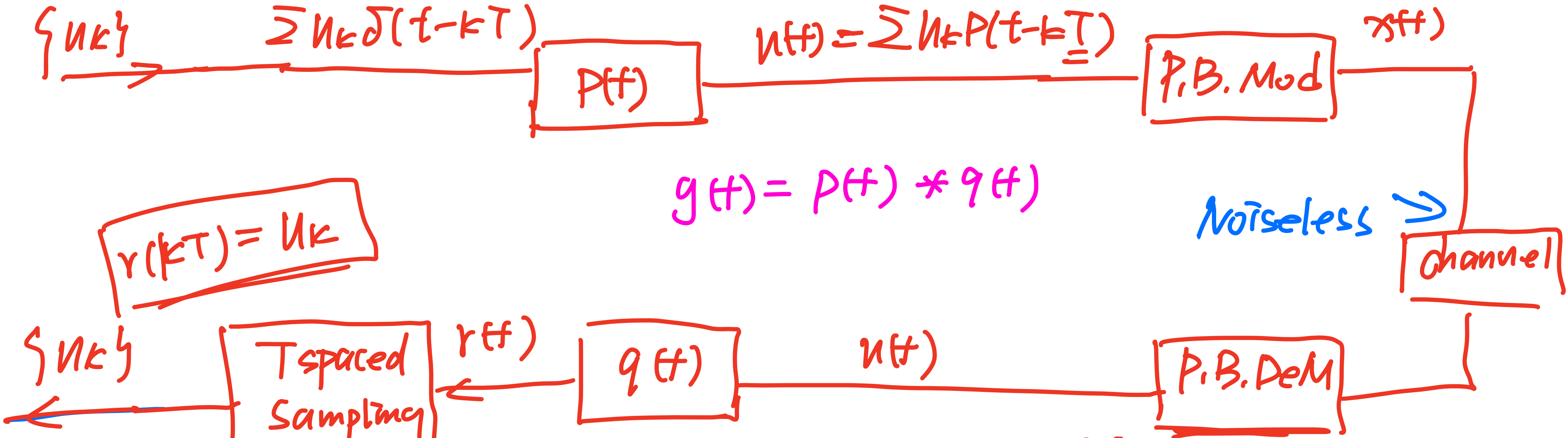
Receiver:

- $x(t) \xRightarrow{\text{positive part}} u(t)e^{j2\pi f_c t} \xRightarrow{\text{remove} f_c}$ $\boxed{u(t) \xRightarrow{\text{filter and sampling}} \text{complex signals}}$ $\xRightarrow{\text{Constellation}}$ symbols $\xRightarrow{b \text{ bits}}$ Binary digits

Q: How to choose pulse $p(t)$ and receive filter $q(t)$, so that there is no inter-symbol interference (ISI)??

PAM QAM $\quad u(t) = \sum u_k \, p(t - kT_s)$

$$\begin{cases} T_s: \text{ symbol duration} \qquad R_s = \dfrac{1}{T_s} \qquad T_s = bT_b \\ u_k: \\ p(t): \end{cases}$$

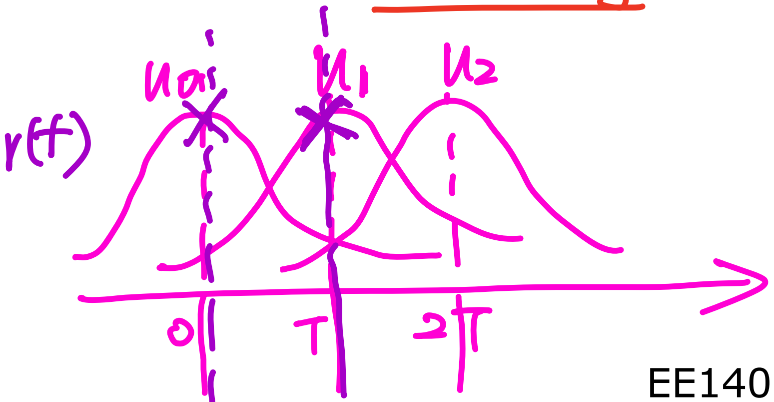


$$r(t) = u(t) * q(t)$$
$$= \sum u_k \, p(t-kT) * q(t)$$
$$= \sum u_k \, g(t-kT)$$

$p(t) * q(t)$

objective: $p(t), q(t) \Leftrightarrow g(t) =$

s.t. $r(kT) = u_k, \ \forall k$



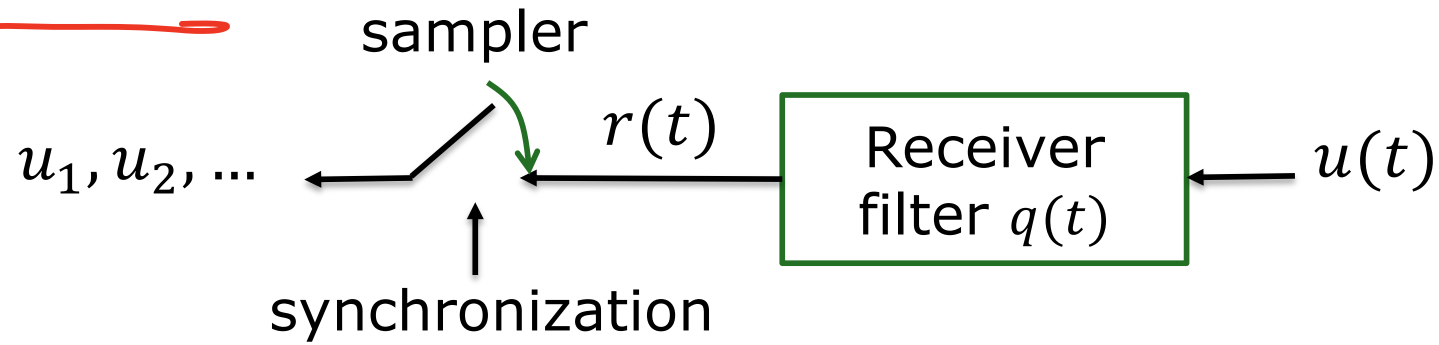
$r(0) = u_0 \quad r(T) = u_1 \quad r(2T) = u_2$ (Zero ISI)

(zero ISI)

The Nyquist Criterion

$$u(t) = \sum_k u_k p(t - kT)$$

- Assume no noise in channel
- Received baseband waveform u(t), retrieve the signals $\{u_1, u_2, \ldots\}$



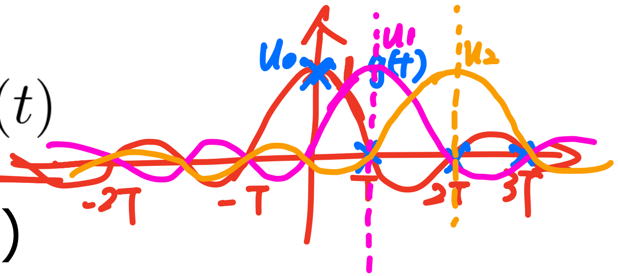
- $r(t) = \int_{-\infty}^{\infty} u(\tau) q(t - \tau) d\tau$ is sampled as $r(0), r(T), \ldots$
- Objective: choose $p(t)$ and $q(t)$ so that $r(kT) = u_k$ (No ISI)

channel: $h(t)$ $g(t) = p(t) * h(t) * q(t)$

The Nyquist Criterion

$$r(t) = \int u(\tau)q(\tau - t)d\tau = \int \sum_k u_k p(\tau - kT)q(t-\tau)d\tau \quad \text{sinc}\left(\frac{t}{T}\right)$$

$$= \sum_k u_k g(t - kT) \quad \text{where } g(t) = p(t) * q(t)$$



- While ignoring noise, r(t) is determined by g(t)
- Ideal Nyquist: a wave form g(t) is ideal Nyquist with period T if $g(0) = 1$ *and* $g(kT) = 0$ for $k \neq 0$ (same property as sinc function)

$t = jT$ $j = k$

$$r(jT) = \sum_k u_k g(jT - kT) \overset{g(t) \text{ is ideal Nyquist}}{=} u_j$$

- If g(t) is ideal Nyquist, then $r(kT) = u_k$ for all k. If g(t) is not ideal Nyquist, then $r(kT) \neq u_k$ for some u_k (inter symbol interference)

$$g(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Recall: Aliasing Problem

Given a signal $u(t)$, its sampling approximation

$$s(t) = \sum_k u(kT)\,\mathrm{sinc}(\frac{t}{T} - k),$$

and the Fourier transform of $s(t)$ satisfies:

$$\hat{s}(f) = \mathcal{F}(s(t)) = \sum_k \hat{u}(f + \frac{k}{T})\mathrm{rect}(fT)$$

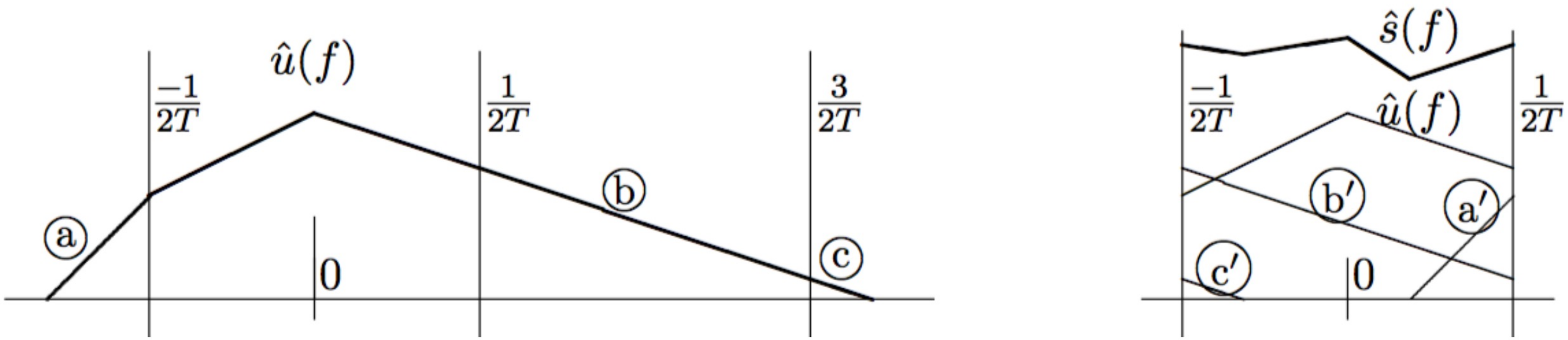


Figure: Aliasing when $1/(2T) < B_b$ [Gallagar'Book]

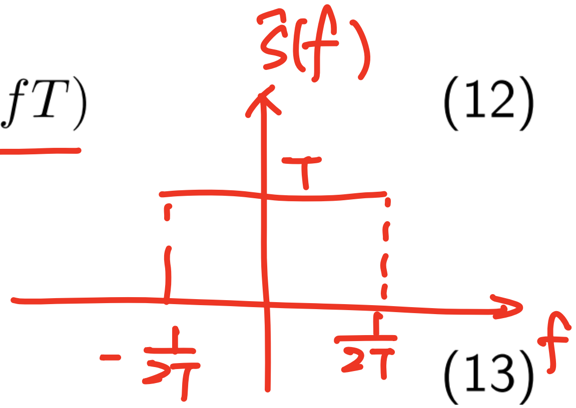
The Nyquist Criterion

Let $s(t)$ be signal reconstructed by samples of $g(t)$:

$$s(t) = \sum_k g(kT)\,\mathrm{sinc}(\frac{t}{T} - k) \tag{11}$$

g(t) is idea Nyquist: $g(kT) = \delta(k)$, substitute it into (11), we have

$$s(t) = \mathrm{sinc}(\frac{t}{T}) \longrightarrow \hat{s}(f) = T\mathrm{rect}(fT) \tag{12}$$



From (11) and the aliasing theorem

$$\hat{s}(f) = \sum_k \hat{g}(f + \frac{k}{T})\mathrm{rect}(fT) \tag{13}$$

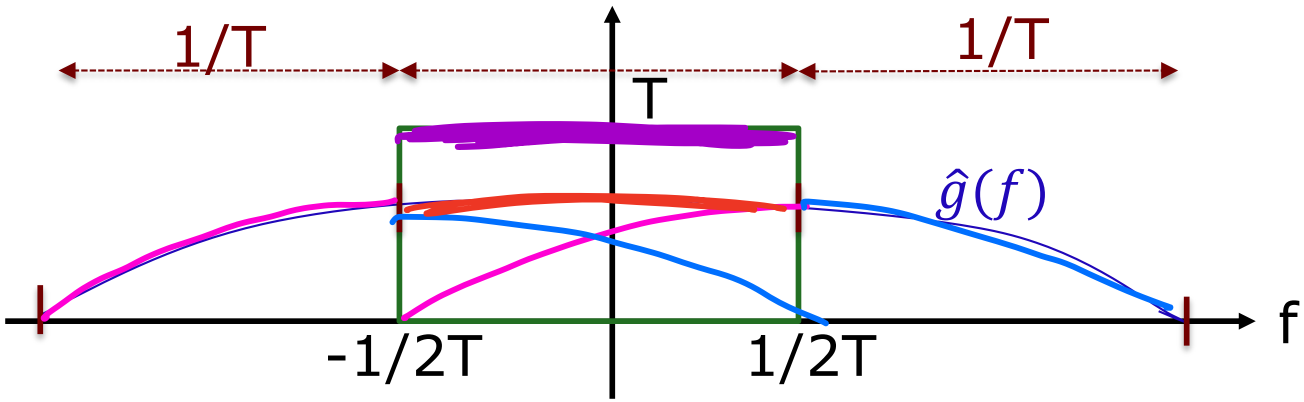
Thus from (12) and (13), we have $g(t)$ is ideal Nyquist iff

$$\sum_k \hat{g}(f + \frac{k}{T})\mathrm{rect}(fT) = T\mathrm{rect}(fT)$$

The Nyquist Criterion

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$



The Nyquist Criterion

$u(t) = \sum_k u_k p(t-kT)$

B_b

$g(t) = p(t) * q(t)$

$\hat{g}(f) = \hat{p}(f)\hat{q}(f)$

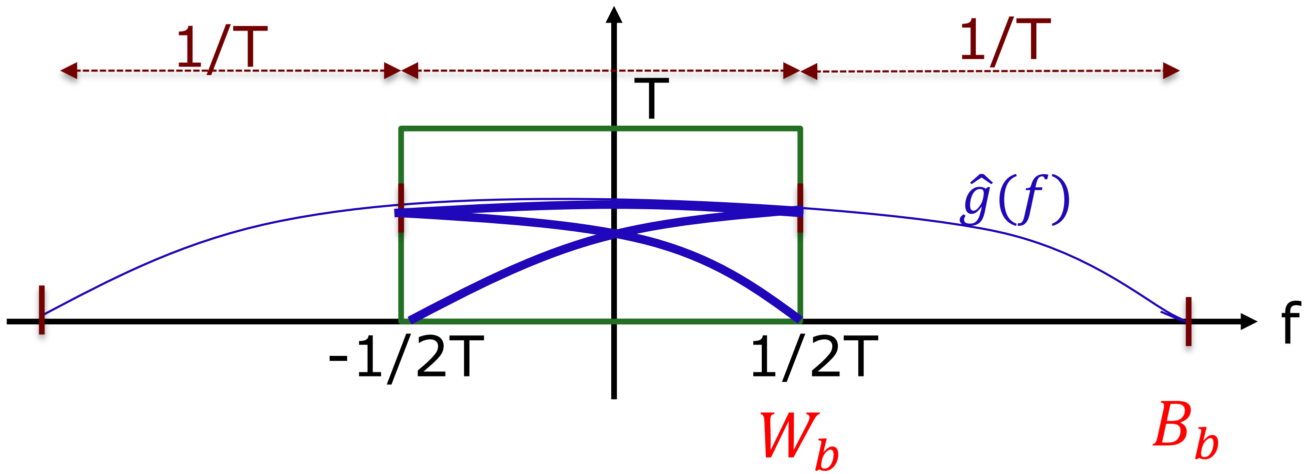
$B_b \quad B_b \quad B_b$

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

$R_s = \frac{1}{T} \qquad W_b = \frac{1}{2T} = \frac{R_s}{2}$

  - Signal Interval of g(t): T
  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)



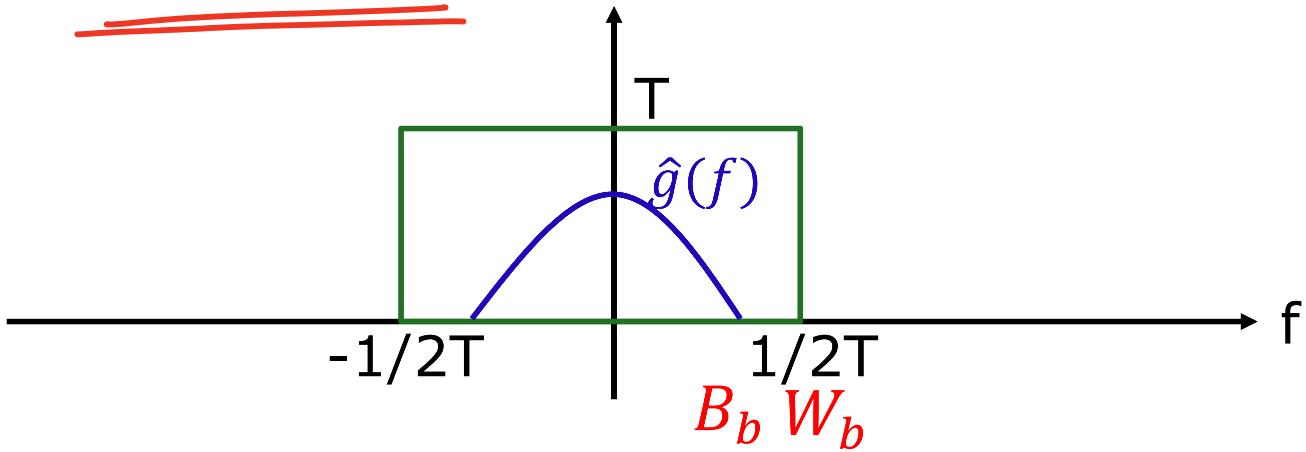
The Nyquist Criterion

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case1: $B_b < W_b$, ISI is not avoidable

$B_b \geq W_b = \frac{1}{2T} = \frac{R_s}{2}$



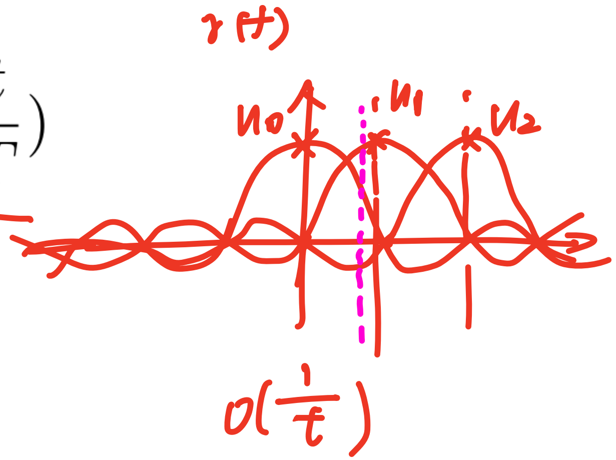
The Nyquist Criterion

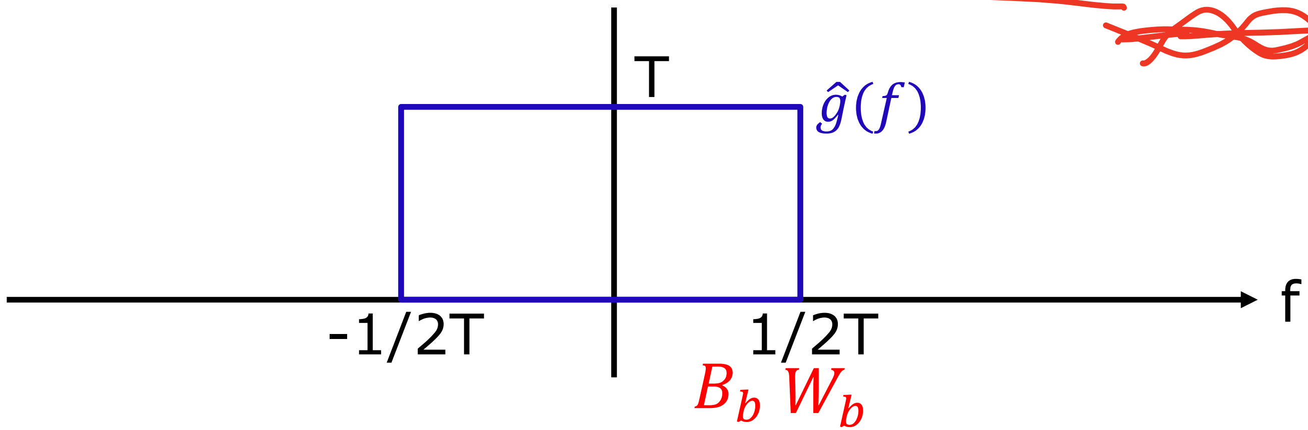
- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\mathrm{rect}(fT) = T\mathrm{rect}(fT)}$$

  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case2: $B_b = W_b$

$$\hat{g}(f) = T\mathrm{rect}(fT) \leftrightarrow g(t) = \mathrm{sinc}(\frac{t}{T})$$





The Nyquist Criterion

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f+\frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

$\text{B.E.} = \frac{R_s}{B_b}$

  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case3: $B_b > W_b$ $\quad B_b \gg W_b = \frac{R_s}{2}$ $\quad$ given B_b $\quad T\downarrow$ $R_s\uparrow$ $W_b\uparrow \to B_b$
    - If B_b is much larger than W_b, than W_b can be increased (T can be decreased), thus increasing the rate Rs at which signal can be transmitted.
    - g(t) should be chosen B_b exceed W_b by a relatively small amount.
    - $W_b \leq B_b < 2W_b$: Keep $\hat{g}(f)$ almost baseband limited to 1/(2T)

$\alpha \to 0$ $\quad B_b \to W_b$, $W_b \to B_b$ $\quad$ adv: B.E.$\uparrow$ $\quad$ dis: slow time decay.

$\alpha \to 1$ $\quad B_b \leftarrow W_b$ $\quad$ adv: fast time decay

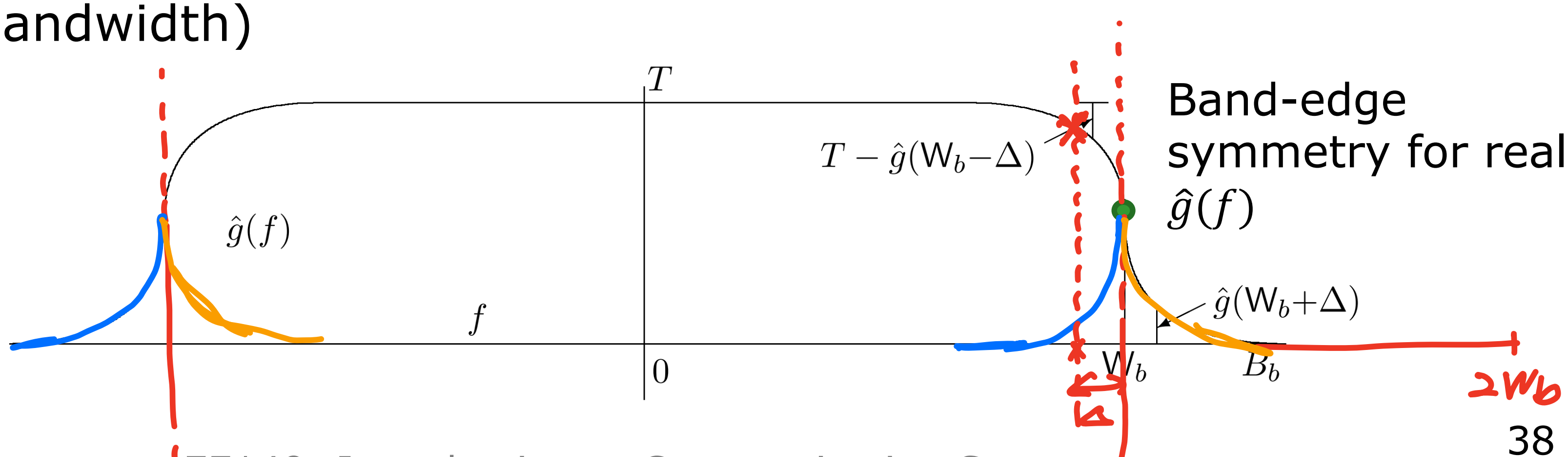
$W_b \leftarrow B_b$ | dis : B.Z. ↓

The Nyquist Criterion

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}\left(f + \frac{k}{T}\right)\text{rect}(fT) = T\,\text{rect}(fT)}$$

  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case3: $W_b \leq B_b < 2W_b$
    - Rolloff factor: $\alpha = \frac{B_b}{W_b} - 1$ $\quad 0 \leq \alpha < 1$
    - Tradeoff between rolloff and smoothness (slow time decay and bandwidth)



$\hat{g}(W_b - \Delta) + \hat{g}(W_b + \Delta) = T$

The Nyquist Criterion

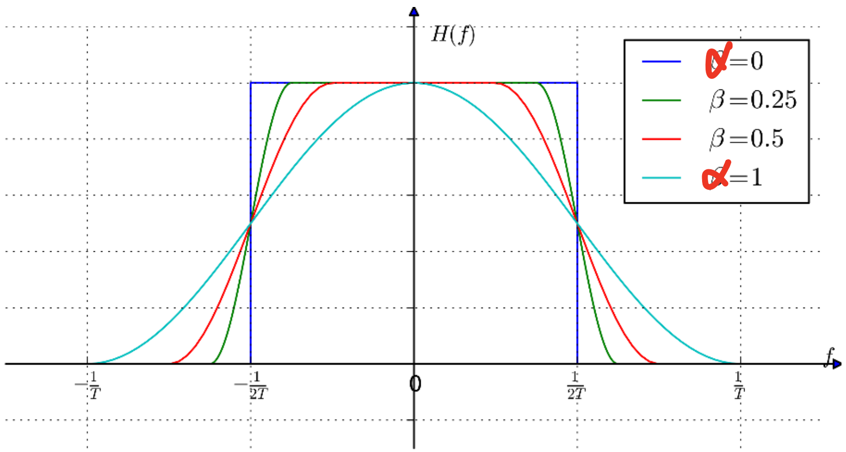
- Nyquist Criterion
  - PAM filters in practice often have raised cosine transform

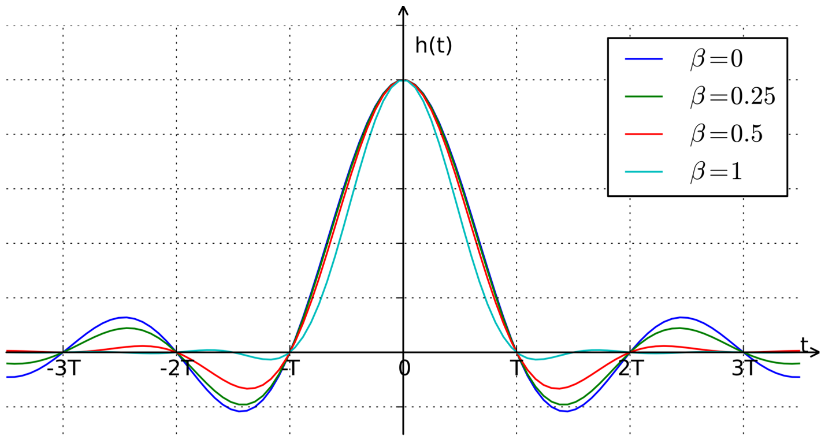
$$\hat{g}_\alpha(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T}; \\ T \ \cos^2\left[\frac{\pi T}{2\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T}; \\ 0, & |f| \ge \frac{1+\alpha}{2T}. \end{cases}$$

  - The inverse transform of $\hat{g}_\alpha(f)$

$$\hat{g}_\alpha(t) = \mathrm{sinc}\left(\frac{t}{T}\frac{\cos(\pi\alpha\frac{t}{T})}{1 - 4\alpha^2 t^2/T^2}\right)$$

$o(\frac{1}{t^3})$





$$\{u_k\} \longleftarrow \boxed{\text{Sampling}} \xleftarrow{\ r(t)\ } \boxed{q(t)} \longleftarrow u(t) = \sum u_k p(t-kT)$$

T , $R_s = \frac{1}{T}$ symbol rate (Baud/s)

B_b : Baseband Bandwidth.

$$r(t) = \sum_k u_k g(t-kT)$$

$$g(t) = p(t) * q(t)$$

$$\underbrace{\hat{g}(f)}_{B_b} = \underbrace{\hat{P}(f)}_{B_b}\, \underbrace{\hat{q}(f)}_{B_b}$$

$$\boxed{\begin{array}{c} \text{No ISI } (u_k = r(kT)) \\ \Updownarrow \\ g(t) \text{ ideal Nyquist } (g(0)=1,\ g(kT)=0,\ k \neq 0) \\ \Updownarrow \\ \hat{g}(f) \text{ Nyquist Criterion} \end{array}}$$

$$W_b = \frac{1}{2T} = \frac{R_s}{2}$$

$$B_b \geq W_b \Longrightarrow \underbrace{\boxed{B_b \geq \frac{R_s}{2}}}_{\text{min B.B.}} \Longleftrightarrow \underbrace{\boxed{R_s \leq 2B_b}}_{\text{max. symbol rate}}$$

Baseband: $B.E. = \dfrac{R_s}{B_b} \leq 2$ (Baud/s/Hz) (No ISI)

Passband: $B.E. = \dfrac{R_s}{2B_b} \leq 1$ (No ISI)

Example.

Digital B.B.

$R_s = 100$ Baud/s

$B_b \geq \frac{R_s}{2} = 50\,Hz$

Pass Band

$B \geq R_s = 100\,Hz$.

or $B_b = 50\,Hz$

$R_s \leq 2B_b = 100$ Baud/s

$B = 100 \text{ Hz}$

$R_s \leq B = 100 \text{ Baud/s}$

The Nyquist Criterion

- Q: Restrict $\hat{g}(f)$ real and nonnegative, $\hat{g}(f) > 0$, how to choose $p(t)$ and $q(t)$, s.t. $\hat{p}(f)\hat{q}(f) = \hat{g}(f)$
- Choose $\boxed{\hat{q}(f) = \hat{p}^*(f)}$ → $|\hat{p}(f)| = |\hat{q}(f)| = \sqrt{\hat{g}(f)}$ and $\boxed{q(t) = p^*(-t).}$
  - $p(t)$: square root of Nyquist;
  - $q(t)$: matched filter to $p(t)$
  - Same bandwidth for $\hat{p}(f)$, $\hat{q}(f)$ and $\hat{g}(f)$ (slightly larger than 1/2T), truncated in time to allow finite delay.
- Orthonormal Shifts:
  - Let $p(t) \in \mathcal{L}_2$ and $\hat{g}(f) = |\hat{p}(f)|^2$ satisfies the Nyquist criterion for T. Then $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions. Conversely, if $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions, then $|\hat{p}(f)|^2$ satisfies the Nyquist criterion.
  - Proof:

$\hat{g}(f) \geq 0$. $\quad \hat{g}(f) = \hat{p}(f)\hat{q}(f)$

$\hat{q}(f)$: Matched filter to $\hat{p}(f)$

$$\boxed{\begin{array}{l} \hat{q}(f) = \hat{p}^*(f) \;/\; \hat{p}(f) = \hat{q}^*(f) \\ q(t) = p^*(-t) \;/\; p(t) = q^*(-t) \end{array}}$$

given $\hat{g}(f)$

① $\hat{p}(f)$: $|\hat{p}(f)| = \sqrt{\hat{g}(f)}$, $\forall$ phase.

② $\begin{cases} \hat{q}(f) : |\hat{q}(f)| = |\hat{p}(f)|, \quad \angle \hat{q}(f) = -\angle \hat{p}(f) \\ q(t) = p^*(-t) \end{cases}$

---

① If $\hat{q}(f) = \hat{p}^*(f)$ & $\quad g(f) = |P(f)|^2$ satisfy Nyquist criterion

$\Rightarrow \quad \{p(t-kT)\}_k$ orthonormal functions.

② if $\{p(t-kT)\}_k$ orthonormal functions. & $\quad \hat{q}(f) = \hat{p}^*(f) \;/\; q(t) = p^*(-t)$

$\Rightarrow \quad \hat{g}(f)$ satisfies Nyquist criterion (zero ISI)

$q(t) = p^*(-t)$

The Nyquist Criterion

$$g(t) = \int p(\tau)q(t-\tau)d\tau = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(t) = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(kT) = \int p(\tau)p^*(\tau - kT)d\tau$$

- Shift τ by jT for any integer j $(\tau = \tau - jT)$

$$g(kT) = \int p(\tau - jT)p^*(\tau - (k+j)T)d\tau$$
$$= \int p(t - jT)p^*(t - (k+j)T)dt$$

- If $g(t)$ is idea Nyquist, then $g(kT) = 1$ for $k = 0$ and 0 otherwise. Thus,

$$g(kT) = \int p(t - jT)p^*(t - (k+j)T)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$\Longrightarrow \{p(t - kT)\}$ is a set of orthonormal functions

The Nyquist Criterion

- if $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions, then let $\hat{q}(f) = \hat{p}^*(f)$, that is $q(t) = p^*(-t)$. We have

$$g(t) = \int p(\tau)q(t-\tau)d\tau = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(kT) = \int p(\tau)p^*(\tau - kT)d\tau$$

- Shift τ by jT for any integer j $(\tau = \tau - jT)$

$$\begin{aligned} g(kT) &= \int p(\tau - jT)p^*(\tau - (k+j)T)d\tau \\ &= \int p(t - jT)p^*(t - (k+j)T)dt \end{aligned}$$

$$g(kT) = \int p(t - jT)p^*(t - (k+j)T)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$\Longrightarrow$ $g(t)$ is idea Nyquist

The Nyquist Criterion

- Orthonormal Shifts → recover u_k from vector space perspective

$\{u_k\} \longrightarrow u(t)$

$\{u_k\} \longleftarrow u(t)$

$$\hat{s}(t) = \sum_{k=1}^{N} s_k \phi_k(t)$$

$$s_k = \int_{-\infty}^{\infty} s(t)\phi_k^*(t)dt = \langle s(t), \phi_k(t) \rangle$$

- Since $\{p(t-kT)\}$ is an orthonormal basis and $u(t) = \sum u_k p(t-kT)$ is the orthonormal expansion, thus

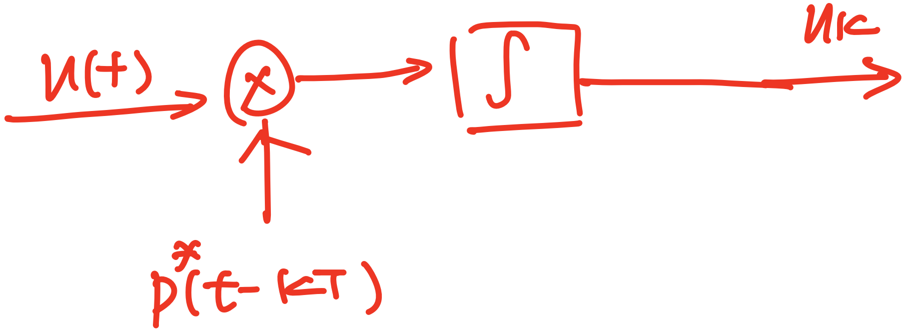
$$u_k = \langle u(t), p(t-kT) \rangle$$

- Retrieving u_k corresponding to projecting $u(t)$ onto $p(t-kT)$

- Note this projection is done by $u(t) * q(t)$ and then sampling at time kT

$$u(t) = \sum u_k p(t-kT)$$

$\{p(t-kT)\}$ is orthonormal Basis



$$u_k = \int u(t)\, p^*(t-kT)\, dt$$

$$q(t) = p^*(-t)$$

$$r(t) = u(t) * q(t) = \int u(\tau)\, q(t-\tau)\, d\tau$$

$$r(kT) = \int u(\tau)\, q(kT-\tau)\, d\tau = \int u(\tau)\, p^*(\tau-kT)\, d\tau = \int u(t)\, p^*(t-kT)\, dt = u_k$$

Example :

$$p(t) = \frac{1}{\sqrt{T}} \operatorname{rect}\left(\frac{t}{T}\right)$$

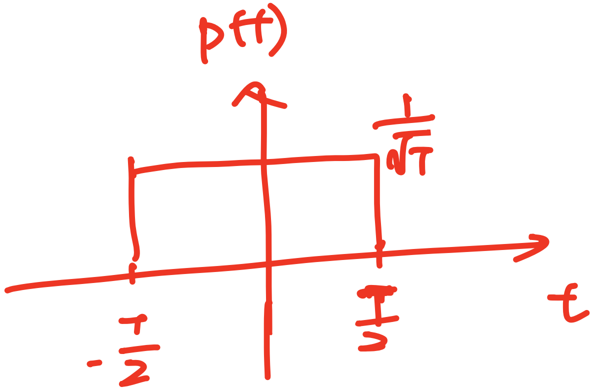
$$q(t) = p^*(-t) = \frac{1}{\sqrt{T}} \operatorname{rect}\left(\frac{t}{T}\right)$$

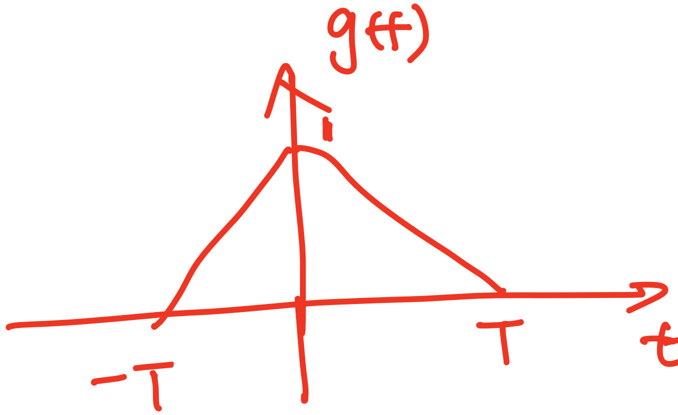
$$g(t) = p(t) * q(t) = \Lambda\left(\frac{t}{T}\right)$$

$$\hat{p}(f) = \frac{1}{\sqrt{T}} T \operatorname{sinc}(Tf)$$

$$\hat{q}(f) = \frac{1}{\sqrt{T}} T \operatorname{sinc}(Tf)$$

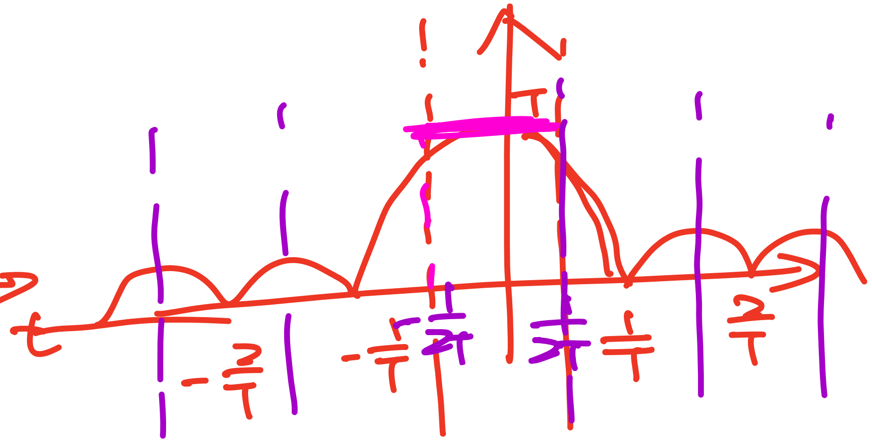
$$\hat{g}(f) = T \operatorname{sinc}^2(Tf)$$





ideal Nyquist

B.E. $\downarrow \Leftarrow B_b \rightarrow \infty$



Nyquist criterion

$p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t}{T}\right)$

$q(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t}{T}\right)$

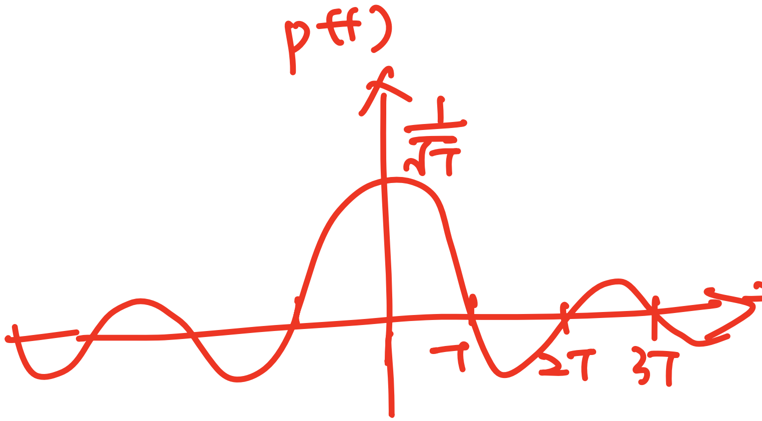
$g(t) = \frac{1}{T} \operatorname{sinc}\left(\frac{t}{T}\right) * \operatorname{sinc}\left(\frac{t}{T}\right)$

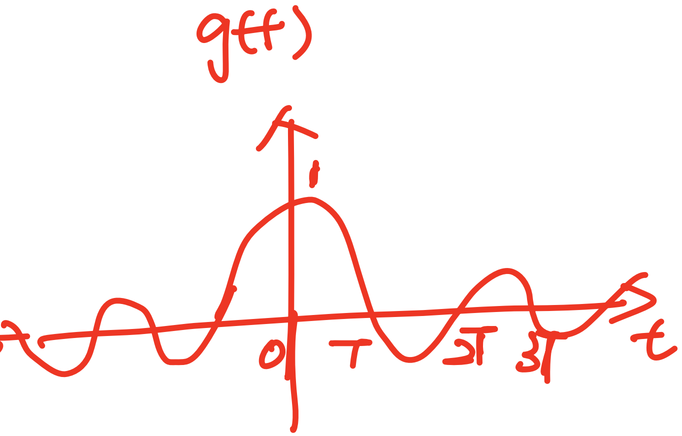
$= \operatorname{sinc}\left(\frac{t}{T}\right)$

$\tilde{P}(f) = \sqrt{T}\, \operatorname{rect}(Tf)$

$\tilde{q}(f) = \sqrt{T}\, \operatorname{rect}(Tf)$

$\tilde{g}(f) = T\, \operatorname{rect}(Tf)$



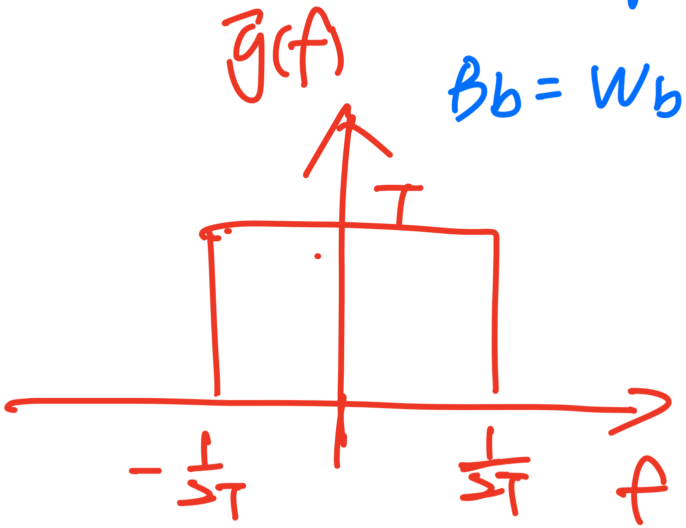


$O\left(\frac{1}{t}\right)$

ideal Nyquist

B.E. ↑

$B_b = W_b$



Nyquist criterion

Summarize

- Bit sequence to symbol sequence
  - PAM: b-tuple of bits → one of $M = 2^b$ signal points in $\mathbb{R}^1$
  - QAM: b-tuple of bits → one of $M = 2^b$ signal points in $\mathbb{C}^1$
  - Signal constellation: small average energy with a large distance between points
  - Standard Mapping
- Symbol sequences to baseband waveform
  - $u(t) = \sum_k u_k p(t - kT)$
  - How to choose p(t): $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$ must satisfy Nyquist criterion to avoid ISI

$$\boxed{\text{if } g(kT) = 1 \text{ for } k = 0, \text{ and } 0 \text{ for } k \in \mathbb{Z} \neq 0}$$

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

Summarize

- Symbol sequences to baseband waveform

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

  - $\hat{g}(f)$: almost baseband limited to 1/2T; smooth such that g(t) goes to zero quickly Bb
  - If $\hat{g}(f) > 0$: $|\hat{p}(f)| = |\hat{q}(f)| = \sqrt{\hat{g}(f)}$. In this case, $\{p(t - kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions
- Baseband to passband: $x(t) = 2R\left(u(t)e^{j2\pi f_c t}\right)$

$$x(t) = 2\left(\sum_k u_k' p(t - kT)\right)\cos 2\pi f_c t - 2\left(\sum_k u_k'' p(t - kT)\right)\sin 2\pi f_c t$$

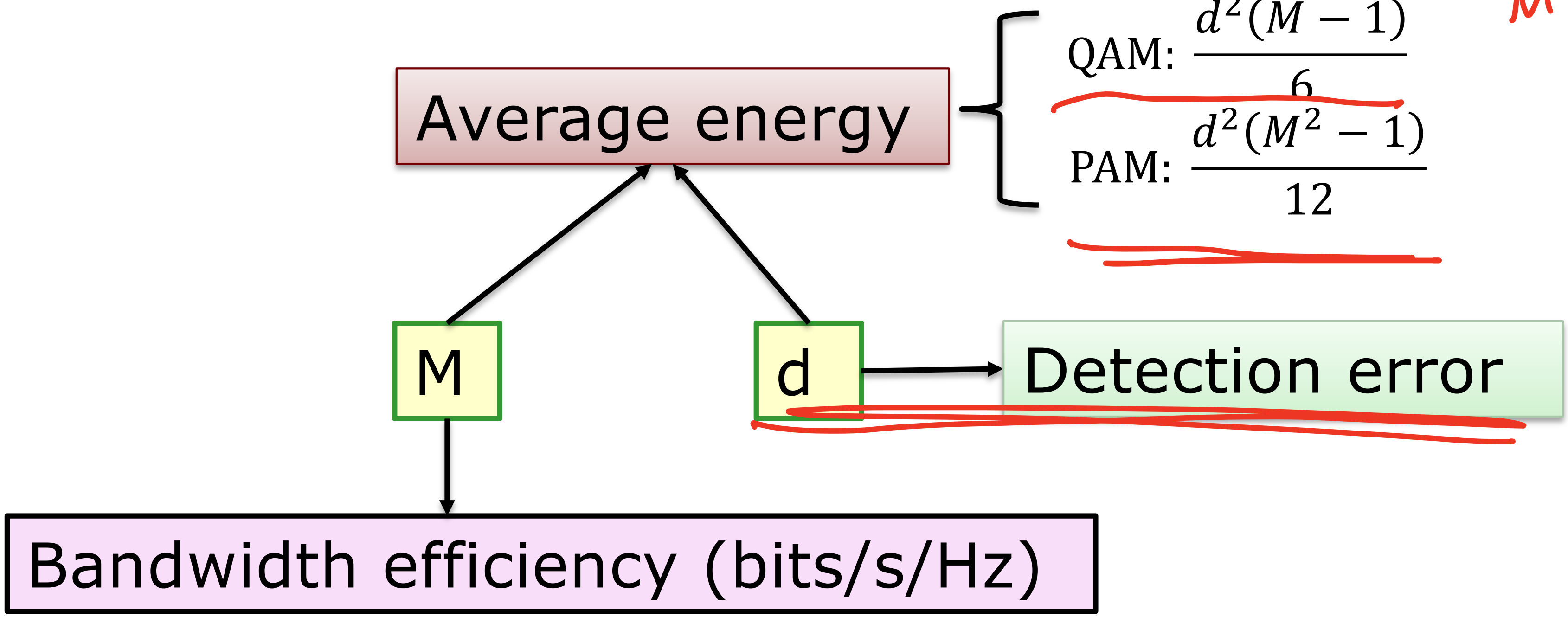
DSB-QC

- Tradeoff: Bandwidth Efficiency (M), Energy (M,d), Detection Error (d)

Summarize

- Tradeoff: Bandwidth Efficiency (M), Energy (M,d), Detection Error (d)

M↓ d↓



Bandwidth=1/(2T)=Rs/2=R/(2logM)

M ↗ Bandwidth ↘ (same R), Bandwidth efficiency ↗

Thanks for your kind attention!

Questions?