OIL FUTURES AND STORAGE OPTIONS

Project Report

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1 Documentation

1.1 Introduction

The oil market is critical to the global economy, affecting transportation costs and consumer goods prices. This report provides an introductory briefing on the market for oil futures, focusing on contracts traded on the Chicago Mercantile Exchange (CME). It covers the description of oil futures contracts, cash flows from long positions, the purpose of oil futures, evolution of trading volume and open interest, the concept of convenience yield, and market states like backwardation and contango.

1.2 Description of Oil Futures Contracts

Oil futures are standardized agreements to buy or sell crude oil at a set price on a future date, traded on exchanges like CME [2]. Each contract specifies the quality (e.g., West Texas Intermediate), quantity (typically 1,000 barrels), delivery location, and date. These contracts enable hedging against price volatility, speculating on price movements, and providing a mechanism for price discovery. Prices are determined through an auction process, reflecting market expectations based on supply and demand, geopolitical events, and economic indicators.

1.3 Cash Flows of a Long Position

When an investor takes a long position in an oil futures contract, they agree to buy oil at a set price on a future date. They must deposit an initial margin, a fraction of the contract's value, and the position is marked to market daily. This means the account is adjusted for daily gains or losses based on the contract's closing price. If the balance falls below the maintenance margin, additional funds are required. To maintain exposure, investors can roll over positions by closing the near-month contract and opening a new one for the next month, incurring rollover costs.

1.4 Purpose of Oil Futures

Oil futures exist to manage price risk. Producers and consumers hedge against adverse price movements, stabilizing revenues and costs. Speculators profit from price changes, providing market liquidity. Arbitrageurs exploit price discrepancies, ensuring market efficiency. Futures markets aid in price discovery, reflecting collective expectations of future supply and demand conditions, and serve as benchmarks for physical oil transactions. Including oil futures in investment portfolios can enhance returns and manage risk, hedging against inflation and economic uncertainty.

1.5 Evolution of Trading Volume and Open Interest

Trading volume and open interest in oil futures indicate market activity and liquidity. Higher volumes imply greater participation, facilitating transactions and accurate price discovery. Near-term contracts have the highest volumes due to active trading. Seasonal variations affect these metrics, driven by demand patterns like increased gasoline consumption in summer. Understanding these patterns helps in making informed trading and risk management decisions.

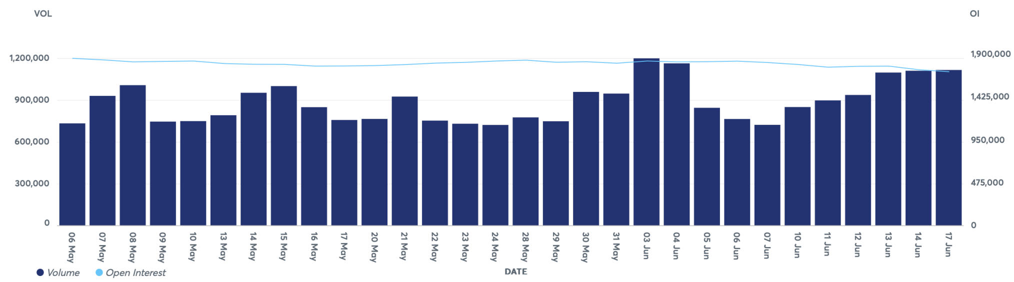

Figure 1: Evolution of Trading Volume and Open Interest

From 1, we notice that, in the period from early May to mid-June, the trading volume in oil futures exhibited notable fluctuations, with peaks around May 8, May 15, May 21, June 3, and June 13. These spikes likely reflect significant trading activity triggered by market events or anticipated changes in supply and demand. Open interest, which tracks the number of outstanding contracts, remained relatively stable, suggesting consistent engagement from market participants without significant liquidation or new positions being overwhelmingly added.

1.6 Convenience Yield and Market States

In futures markets, **backwardation** and **contango** describe the relationship between spot prices and futures prices of a commodity.

Backwardation occurs when the current spot price of a commodity is higher than its futures prices for later maturities.

Contango, on the other hand, occurs when the spot price is lower than the futures prices.Contango reflects expectations that prices will rise over time, often due to storage costs or anticipated increases in demand.

The convenience yield represents the non-monetary benefits of holding physical oil, influencing the relationship between spot and futures prices. In backwardation, high convenience yield leads to futures prices lower than spot prices, indicating tight supply. In contango, low convenience yield results in futures prices higher than spot prices, indicating ample supply. These states provide signals for trading and hedging strategies.

Based on the current data, the market is in **backwardation**. The spot price of crude oil is \$81.62 per barrel, which is higher than the futures prices for all listed contracts, indicating higher immediate demand or a shortage of supply. The following table summarizes the spot and futures prices:

This backwardation suggests a higher current demand for crude oil relative to future expectations, leading to higher spot prices compared to futures prices.

Contract	Settlement Date	Price ($)
Spot	Immediate	81.62
Aug 2024	Crude Oil Aug 24	80.75
Dec 2024	Crude Oil Dec 24	77.90
Jan 2025	Crude Oil Jan 25	77.33
Jun 2025	Crude Oil Jun 25	74.82

Table 1: Spot and Futures Prices for Crude Oil [?]

1.7 Other Oil Derivatives on CME

Besides oil futures, CME offers other derivatives such as options on oil futures, allowing for more complex trading strategies. These include call and put options, which give the right but not the obligation to buy or sell oil futures at a specified price before the option expires. These instruments provide additional tools for managing risk and speculating on price movements in the oil market.

2 Analysis

2.1 Spot price modelling

1. The dynamics of the processes are given by: The dynamics of the processes are given by:

$$dr_t = \lambda_r(\bar{r} - r_t)dt + \sigma_r^\top dB_t^Q \tag{1}$$

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = (r_t - \delta_t)dt + \sigma_\mathcal{O}^\top dB_t^Q \tag{2}$$

$$d\delta_t = \lambda_\delta(\bar{\delta} - \delta_t)dt + \sigma_\delta^\top dB_t^Q \tag{3}$$

where $\sigma_r, \sigma_\mathcal{O}, \sigma_\delta \in \mathbb{R}^3$ and B_t^Q is a 3-dimensional Brownian motion under the equivalent martingale measure Q.
The exposure to the Brownian motions can be rewritten as follows:

$$\sigma_\mathcal{O}^\top dB_t^Q = \left(\sum_{j=1}^{3} \sigma_{Oj}^2\right)^{1/2} dW_t^\mathcal{O} = \|\sigma_\mathcal{O}\| dW_t^\mathcal{O} \tag{4}$$

$$\sigma_r^\top dB_t^Q = \left(\sum_{j=1}^{3} \sigma_{rj}^2\right)^{1/2} dW_t^r = \|\sigma_r\| dW_t^r \tag{5}$$

$$\sigma_\delta^\top dB_t^Q = \left(\sum_{j=1}^{3} \sigma_{\delta j}^2\right)^{1/2} dW_t^\delta = \|\sigma_\delta\| dW_t^\delta \tag{6}$$

where the processes $W_t^{\mathcal{O}}, W_t^r, W_t^\delta$ are defined by:

$$W_t^{\mathcal{O}} = \int_0^t \frac{\sigma_{\mathcal{O}s}^\top dB_s^Q}{\|\sigma_{\mathcal{O}s}\|} \tag{7}$$

$$W_t^r = \int_0^t \frac{\sigma_{rs}^\top dB_s^Q}{\|\sigma_{rs}\|} \tag{8}$$

$$W_t^\delta = \int_0^t \frac{\sigma_{\delta s}^\top dB_s^Q}{\|\sigma_{\delta s}\|} \tag{9}$$

As is easily seen, these processes are continuous martingales with initial value zero and quadratic variation t, so they are Brownian motions by Lévy's characterization theorem. The correlation between the specific Brownian motions is:

$$\rho_{r\mathcal{O}} = \frac{d\langle W_t^r, W_t^{\mathcal{O}}\rangle_t}{\sqrt{d\langle W_t^r\rangle_t d\langle W_t^{\mathcal{O}}\rangle_t}} = \frac{\sigma_{rt}\sigma_{\mathcal{O}t}^\top}{\|\sigma_{rt}\|\|\sigma_{\mathcal{O}t}\|} \tag{10}$$

$$\rho_{r\delta} = \frac{d\langle W_t^r, W_t^{\delta}\rangle_t}{\sqrt{d\langle W_t^r\rangle_t d\langle W_t^{\delta}\rangle_t}} = \frac{\sigma_{rt}\sigma_{\delta t}^\top}{\|\sigma_{rt}\|\|\sigma_{\delta t}\|} \tag{11}$$

$$\rho_{\delta\mathcal{O}} = \frac{d\langle W_t^\delta, W_t^{\mathcal{O}}\rangle_t}{\sqrt{d\langle W_t^\delta\rangle_t d\langle W_t^{\mathcal{O}}\rangle_t}} = \frac{\sigma_{\delta t}\sigma_{\mathcal{O}t}^\top}{\|\sigma_{\delta t}\|\|\sigma_{\mathcal{O}t}\|} \tag{12}$$

These correlations take values in $[-1, 1]$ as a result of the Cauchy-Schwarz inequality, which states that for any $x, y \in \mathbb{R}^d$,

$$|x^\top y| \leq \sqrt{x^\top x}\sqrt{y^\top y} \tag{13}$$

2. By definition of the risk-neutral probability measure Q, we have that the process:

$$\hat{Y}_t = \hat{\mathcal{O}}_t + \int_0^t e^{-\int_0^u r_s ds}\delta_u \mathcal{O}_u du \tag{14}$$

is a Q-martingale. In particular, we have that:

$$\hat{Y}_t = \mathbb{E}_t^{\mathbb{Q}}[\hat{Y}_T] = \mathbb{E}_t^Q\left[\hat{\mathcal{O}}_T + \int_0^t e^{-\int_0^u r_s ds}\delta_u \mathcal{O}_u du\right]. \tag{15}$$

It follows that:

$$\hat{\mathcal{O}}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\hat{\mathcal{O}}_T + \int_0^T e^{-\int_0^u r_s ds}\delta_u \mathcal{O}_u du\right] - \int_0^t e^{-\int_0^u r_s ds}\delta_u \mathcal{O}_u du. \tag{16}$$

Which gives us

$$\hat{\mathcal{O}}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\hat{\mathcal{O}}_T + \int_t^T e^{-\int_0^u r_s ds}\delta_u \mathcal{O}_u du\right]. \tag{17}$$

or equivalently :

$$\mathcal{O}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u du}\delta_s \mathcal{O}_s ds + e^{-\int_t^T r_u du}\mathcal{O}_T\right], \quad 0 \leq t \leq T < \infty. \tag{18}$$

This completes the proof.
The relation $\mathcal{O}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u du}\delta_s \mathcal{O}_s ds + e^{-\int_t^T r_u du}\mathcal{O}_T\right]$ can be interpreted as follows: - The first term $\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u du}\delta_s \mathcal{O}_s ds\right]$ represents the present value of the expected future convenience yields of holding the oil, discounted at the risk-free rate. - The second term $\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\mathcal{O}_T\right]$ represents the present value of the expected spot price of oil at the terminal time T, discounted at the risk-free rate.
The Gibson and Schwartz model is referred to as a stochastic convenience yield model because it incorporates a convenience yield δ_t which varies stochastically over time.

3.To show the equivalence, we rewrite the Gibson and Schwartz model under the condition $\lambda_r = \|\sigma_r\| = 0$:

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = (r - \delta_t)dt + \sigma_{\mathcal{O}}^\top dB_t^Q,$$

$$d\delta_t = \lambda_\delta(\delta - \delta_t)dt + \sigma_\delta^\top dB_t^Q.$$

For equivalence with the Gabillon model, we need:

$$\phi \log\left(\frac{\ell_t}{\mathcal{O}_t}\right) = r - \delta_t.$$

This implies:

$$\delta_t = r - \phi \log\left(\frac{\ell_t}{\mathcal{O}_t}\right).$$

So, setting δ_t in the original model as:

$$\delta_t = r - \phi \log\left(\frac{\ell_t}{\mathcal{O}_t}\right),$$

we then have:

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = \phi \log\left(\frac{\ell_t}{\mathcal{O}_t}\right)dt + \sigma_{\mathcal{O}}^\top dB_t^Q,$$

which matches the Gabillon model under these transformations.

In the Schwartz and Smith model, the log spot price $\log \mathcal{O}_t$ is decomposed into:

$$\log \mathcal{O}_t = x_t + \ell_t,$$

where x_t and ℓ_t denote short-term and long-term components of the price, evolving as:

$$dx_t = \mu_x dt + \sigma_x dB_t^Q,$$

$$d\ell_t = \lambda_\ell(\bar{\ell} - \ell_t)dt + \sigma_\ell dB_t^Q.$$

We aim to show that the dynamics of $\log \mathcal{O}_t$ in the given model align with those in the

Schwartz and Smith model. Applying Ito's lemma to $\log \mathcal{O}_t$:

$$d(\log \mathcal{O}_t) = \frac{1}{\mathcal{O}_t} d\mathcal{O}_t - \frac{1}{2}\frac{1}{\mathcal{O}_t^2}(d\mathcal{O}_t)^2,$$

where:

$$d\mathcal{O}_t = \mathcal{O}_t(r_t - \delta_t)dt + \mathcal{O}_t \sigma_{\mathcal{O}} dB_t^Q,$$

$$(d\mathcal{O}_t)^2 = \mathcal{O}_t^2 \|\sigma_{\mathcal{O}}\| dt.$$

Thus:

$$d(\log \mathcal{O}_t) = (r_t - \delta_t)dt + \sigma_{\mathcal{O}} dB_t^Q - \frac{1}{2}\|\sigma_{\mathcal{O}}\| dt.$$

Comparing with Schwartz and Smith:

$$d(\log \mathcal{O}_t) = dx_t + d\ell_t = (\mu_x dt + \sigma_x dB_t^Q) + (\lambda_\ell(\bar{\ell} - \ell_t)dt + \sigma_\ell dB_t^Q),$$

$$d(\log \mathcal{O}_t) = (\mu_x + \lambda_\ell(\bar{\ell} - \ell_t))dt + \sigma_x dB_t^Q + \sigma_\ell dB_t^Q.$$

For equivalence:

$$\mu_x + \lambda_\ell(\bar{\ell} - \ell_t) = r_t - \delta_t - \frac{1}{2}\|\sigma_{\mathcal{O}}\|,$$

$$\sigma_x + \sigma_\ell = \sigma_{\mathcal{O}}.$$

which implies that for equivalence, we have to set

$$\mu_x = r_t - \delta_t - \frac{1}{2}\|\sigma_{\mathcal{O}}\| - \lambda_\ell(\bar{\ell} - \ell_t),$$

$$\sigma_x + \sigma_\ell = \sigma_{\mathcal{O}}.$$

2.2 Bond pricing

4. Consider a zero-coupon bond with maturity date T. The price of the bond at time t is denoted by $B_t(T)$ and can be written as a function of t and r_t, i.e., $B_t(T) = B(t, r_t)$.
To derive the PDE, we start with the discounted bond price process $\widehat{B}_t(T) = e^{-\int_0^t r_s ds} B(t, r_t)$.
Using Itô's Lemma, we have:

$$\begin{aligned}
d\widehat{B}_t(T) &= d\left(e^{-\int_0^t r_s ds} B(t, r_t)\right) \\
&= d\left(e^{-\int_0^t r_s ds} B(t, r_t)\right) + e^{-\int_0^t r_s ds} dB(t, r_t) \\
&= -r_t e^{-\int_0^t r_s ds} B(t, r_t)dt + e^{-\int_0^t r_s ds} dB(t, r_t) \\
&= -r_t e^{-\int_0^t r_s ds} B(t, r_t)dt \\
&\quad + e^{-\int_0^t r_s ds}\left(\frac{\partial B(t, r_t)}{\partial t}dt + \frac{\partial B(t, r_t)}{\partial r_t}dr_t + \frac{1}{2}\frac{\partial^2 B(t, r_t)}{\partial r_t^2}\sigma(t, r_t)^2(dr_t)^2\right) \\
&= -r_t e^{-\int_0^t r_s ds} B(t, r_t)dt \\
&\quad + e^{-\int_0^t r_s ds}\left(\frac{\partial B(t, r_t)}{\partial t}dt + \frac{\partial B(t, r_t)}{\partial r_t}(\mu(t, r_t)dt + \sigma(t, r_t)dB_t) + \frac{1}{2}\frac{\partial^2 B(t, r_t)}{\partial r_t^2}\sigma(t, r_t)^2 dt\right) \\
&= e^{-\int_0^t r_s ds}\left((-r_t B(t, r_t) + \frac{\partial B(t, r_t)}{\partial t} + \mu(t, r_t)\frac{\partial B(t, r_t)}{\partial r_t} + \frac{1}{2}\frac{\partial^2 B(t, r_t)}{\partial r_t^2}\sigma(t, r_t)^2)dt + \sigma(t, r_t)\frac{\partial B(t, r_t)}{\partial r_t}dB_t\right).
\end{aligned}$$

Since $\widehat{B}_t(T)$ is a martingale under the risk-neutral measure Q, the drift term must be zero. Thus, we obtain the PDE:

$$\frac{\partial B(t,r_t)}{\partial t} + \mu(t,r_t)\frac{\partial B(t,r_t)}{\partial r_t} + \frac{1}{2}\sigma(t,r_t)^2\frac{\partial^2 B(t,r_t)}{\partial r_t^2} - r_t B(t,r_t) = 0.$$

In our model, the dynamics of r_t are given by:

$$dr_t = \lambda_r(\bar{r} - r_t)dt + \sigma_r dB_t^Q,$$

which means:

$$\mu(t,r_t) = \lambda_r(\bar{r} - r_t) \quad \text{and} \quad \sigma(t,r_t) = \sigma_r.$$

Substituting these into the PDE, we get:

$$\frac{\partial B(t,r_t)}{\partial t} + \lambda_r(\bar{r} - r_t)\frac{\partial B(t,r_t)}{\partial r_t} + \frac{1}{2}\|\sigma_r\|^2\frac{\partial^2 B(t,r_t)}{\partial r_t^2} - r_t B(t,r_t) = 0,$$

with the terminal condition:

$$B(T,r_T) = 1.$$

5. To show that the zero-coupon bond price is explicitly given by

$$B(t,r_t) = e^{b_0(T-t)+b_r(T-t)r_t},$$

we assume this form and substitute it into the PDE derived in question 4:

$$\frac{\partial B(t,r_t)}{\partial t} + \lambda_r(\bar{r} - r_t)\frac{\partial B(t,r_t)}{\partial r_t} + \frac{1}{2}\sigma_r^2\frac{\partial^2 B(t,r_t)}{\partial r_t^2} - r_t B(t,r_t) = 0.$$

First, compute the partial derivatives of $B(t,r_t)$:

$$B(t,r_t) = e^{b_0(T-t)+b_r(T-t)r_t}.$$

$$\frac{\partial B(t,r_t)}{\partial t} = \left(-b_0'(T-t) - b_r'(T-t)r_t\right) e^{b_0(T-t)+b_r(T-t)r_t},$$

$$\frac{\partial B(t,r_t)}{\partial r_t} = b_r(T-t)e^{b_0(T-t)+b_r(T-t)r_t},$$

$$\frac{\partial^2 B(t,r_t)}{\partial r_t^2} = b_r^2(T-t)e^{b_0(T-t)+b_r(T-t)r_t}.$$

Substitute these into the PDE:

$$\begin{aligned}
&\left(-b_0'(T-t) - b_r'(T-t)r_t\right) e^{b_0(T-t)+b_r(T-t)r_t} \\
&\quad + \lambda_r(\bar{r} - r_t)b_r(T-t)e^{b_0(T-t)+b_r(T-t)r_t} \\
&\quad + \frac{1}{2}\sigma_r^2 b_r^2(T-t)e^{b_0(T-t)+b_r(T-t)r_t} - r_t e^{b_0(T-t)+b_r(T-t)r_t} = 0.
\end{aligned}$$

Factor out $e^{b_0(T-t)+b_r(T-t)r_t}$:

$$-b_0'(T-t) - b_r'(T-t)r_t + \lambda_r\bar{r}b_r(T-t) - \lambda_r r_t b_r(T-t) + \frac{1}{2}\|\sigma_r\|^2 b_r^2(T-t) - r_t = 0.$$

Rearrange the terms:

$$-b_0'(T-t) + \lambda_r\bar{r}b_r(T-t) + \frac{1}{2}\|\sigma_r\|^2 b_r^2(T-t) - r_t(b_r'(T-t) + \lambda_r b_r(T-t) + 1) = 0.$$

For the equality to hold for all r_t, both parts must equal zero:

$$-b_0'(T-t) + \lambda_r\bar{r}b_r(T-t) + \frac{1}{2}\|\sigma_r\|^2 b_r^2(T-t) = 0,$$

$$b_r'(T-t) + \lambda_r b_r(T-t) + 1 = 0.$$

Thus, we have the system of ODEs:

$$\begin{cases} b_0'(s) = \lambda_r\bar{r}b_r(s) + \frac{1}{2}\|\sigma_r\|^2 b_r^2(s), \\ b_r'(s) = -1 - \lambda_r b_r(s), \end{cases}$$

where $s = T - t$ with the boundary condition:

$$b_0(0) = 0 \quad \text{and} \quad b_r(0) = 0,$$

6. The solution to the above system is given by:

$$\begin{cases} b_r(T-t) = \frac{e^{-\lambda_r(T-t)}-1}{\lambda_r}, \\ b_0(T-t) = \frac{e^{-2\lambda_r(T-t)}\left(e^{\lambda_r(T-t)}-1\right)\left(\|\sigma_r\|^2+e^{\lambda_r(T-t)}\left(4\lambda_r^2\bar{r}-3\|\sigma_r\|^2\right)\right)+2\lambda_r(T-t)\left(\|\sigma_r\|^2-2\lambda_r^2\bar{r}\right)}{4\lambda_r^3}, \end{cases}$$

2.3 Futures pricing

7. Because $(r_t, \mathcal{O}_t, \delta_t)$ is the solution to a three-dimensional SDE under $\mathbb{Q}$, we know that it is a Markov process and it follows that

$$f_t = \mathbb{E}_t^{\mathbb{Q}}[\mathcal{O}_T] = f(t, r_t, \mathcal{O}_t, \delta_t)$$

for some function $f(t, r, \mathcal{O}, \delta)$ such that $f(T, r, \mathcal{O}, \delta) = \mathcal{O}$. Since the process f_t is by definition a martingale under the risk-neutral measure, we know that its drift should be equal to zero and it follows from an application of Itô's lemma that, when regrouping only terms with dt, the function should satisfy the partial differential equation

$$0 = \frac{\partial f}{\partial t} + (r-\delta)\mathcal{O}\frac{\partial f}{\partial \mathcal{O}} + \lambda_r(\bar{r}-r)\frac{\partial f}{\partial r} + \lambda_\delta(\bar{\delta}-\delta)\frac{\partial f}{\partial \delta} + \frac{1}{2}\|\sigma_\mathcal{O}\|^2\mathcal{O}\frac{\partial^2 f}{\partial \mathcal{O}^2} + \frac{1}{2}\|\sigma_r\|^2\frac{\partial^2 f}{\partial r^2} + \frac{1}{2}\|\sigma_\delta\|^2\frac{\partial^2 f}{\partial \delta^2} \tag{19}$$

$$+(\sigma_r^\top\sigma_\mathcal{O})\mathcal{O}\frac{\partial^2 f}{\partial \mathcal{O}\partial r} + (\sigma_\delta^\top\sigma_\mathcal{O})\mathcal{O}\frac{\partial^2 f}{\partial \mathcal{O}\partial \delta} + (\sigma_r^\top\sigma_\delta)\frac{\partial^2 f}{\partial r\partial \delta} \tag{20}$$

subject to the terminal boundary condition

$$f(T, r, \mathcal{O}_T) = \mathcal{O}_T.$$

8. We conjecture that the solution is of the form:

$$f_t = f(t, r_t, \delta_t, \mathcal{O}_t) = e^{\phi_0(T-t)+\phi_r(T-t)r_t+\phi_\delta(T-t)\delta_t}\mathcal{O}_t$$

for some functions ϕ_0, ϕ_r, and ϕ_δ that solve a system of ODEs that we are going to determine. We have to plug this conjecture into the above equation of question 7 (eq(20)), this will lead after some basic derivative calculation, to the equation:

$$(-\phi_0' - r\phi_r' - \delta\phi_\delta')f + \lambda_r(\bar{r} - r)\phi_r f + \lambda_\delta(\bar{\delta} - \delta)\phi_\delta f + (r - \delta)f + \sigma_r^\top\sigma_\mathcal{O}\phi_r f + \phi_\delta\sigma_\delta^\top\sigma_\mathcal{O} f$$
$$+ \frac{1}{2}\|\sigma_r\|^2\phi_r^2 f + \frac{1}{2}\|\sigma_\delta\|^2\phi_\delta^2 f = 0 \tag{21}$$

Now we regroup terms with only f, only rf and only δf, leading to the three-dimensional system of ODEs:

$$\begin{cases} \phi_r(\tau)' = 1 - \lambda_r\phi_r(\tau), \\ \phi_\delta(\tau)' = -1 - \lambda_\delta\phi_\delta(\tau), \\ \phi_\mathcal{O}(\tau)' = \lambda_r\bar{r}\phi_r + \lambda_\delta\bar{\delta}\phi_\delta + \frac{\sigma_r^2}{2}\phi_r^2 + \frac{\sigma_\delta^2}{2}\phi_\delta^2 + \sigma_\mathcal{O}\sigma_r\phi_r + \sigma_r\sigma_\delta\phi_r\phi_\delta + \sigma_\delta\sigma_\mathcal{O}\phi_\delta \end{cases}$$

with terminal boundary conditions $\phi_\mathcal{O}(0) = \phi_r(0) = \phi_\delta(0) = 0$

9. The solution to the above system is given by:

$$\begin{cases} \phi_r(\tau) = \frac{1-e^{-\lambda_r\tau}}{\lambda_r}, \\ \phi_\delta(\tau) = \frac{e^{-\lambda_\delta\tau}-1}{\lambda_\delta}, \\ \phi_0(\tau) = \int_0^\tau \frac{1}{2}\left(\phi_\delta(s)\left(2\lambda_\delta\bar{\delta} + \|\sigma_\mathcal{O}\|^2\phi_\delta(s) + 2\sigma_\delta^\top\sigma_\mathcal{O}\right) + 2\phi_r(s)\left(\lambda_r\bar{r} + \phi_\delta(s)\sigma_r^\top\sigma_\mathcal{O} + \sigma_r^\top\sigma_\delta\right) + \|\sigma_r\|^2\phi_r(s)^2\right) ds \end{cases}$$

To describe how the futures price evolves under the measure Q, one might apply Ito's lemma to $f_t = e^{\phi_0(\tau)+\phi_r(\tau)r_t+\phi_\delta(\tau)\delta_t}\mathcal{O}_t$. Applying Ito's lemma reveals that the risk-neutral dynamics of the martingale f_t is the following:

$$df_t = e^{\phi_0(\tau)+\phi_r(\tau)r_t+\phi_\delta(\tau)\delta_t}\left[\sigma_\mathcal{O}\mathcal{O}_t dB_t^{Q,\mathcal{O}} + \phi_r(\tau)\mathcal{O}_t\sigma_r dB_t^{Q,r} + \phi_\delta(\tau)\mathcal{O}_t\sigma_\delta dB_t^{Q,\delta}\right]$$

Which simplifies to:

$$df_t = f_t\left[\sigma_\mathcal{O} + \phi_r(\tau)\sigma_r + \phi_\delta(\tau)\sigma_\delta\right]dB_t^Q$$

10. Having either a changing convenience yield or a changing interest rate (or both) is necessary to explain why we see periods of backwardation and contango for different reasons. Firstly, the convenience yield represents the benefit of having the physical commodity immediately. When this benefit changes, it affects the value of holding the commodity now versus in the future. High convenience yields usually lead to backwardation, while low or negative yields lead to contango.

Secondly, interest rates affect the cost of financing and storing the commodity. When interest rates change, they alter these costs, impacting futures prices relative to spot prices. Higher interest rates can cause contango, while lower interest rates can cause backwardation. Therefore, it seems necessary to account for changing interest rates.

Lastly, when both the convenience yield and interest rates change, their combined effect creates a situation where the market can switch between backwardation and contango based on these changes. This illustrates the real-world complexity and variability in commodity markets.

11. To find the relation $Y_t(T)$, which is the implied yield and the instantaneous yield δ_t, one need to find an expression for $Y_t(T)$ in terms of δ_t. Given $f_t(T) = e^{(r-Y_t(T))\tau}\mathcal{O}_t$ and $f_t(T) = \mathbb{E}_t^Q[\mathcal{O}_T]$ from above questions, the futures price can be written as a function of δ_t like so: $f_t(T) = \mathcal{O}_t e^{(r-\delta_t)\tau}$.

When putting together both equations for $f_t(T)$, one obtain:

$$\mathcal{O}_t e^{(r-\delta_t)\tau} = e^{(r-Y_t(T))\tau}\mathcal{O}_t$$

Hence, the relation between the implied yield $Y_t(T)$ and the instantaneous yield δ_t is:

$$Y_t(T) = \delta_t$$

which holds if and only if δ_t is deterministic and constant over time. If the instantaneous convenience yield δ_t is non-stochastic, the process under the risk-neutral measure Q becomes deterministic. This implies that there is no stochastic variation for δ_t and that $\delta_t = \delta$.

2.4 Storage options

12.

We start from the given expression for $\mathcal{P}_{T_0}$:

$$\mathcal{P}_{T_0} \equiv E_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u\,du}\mathcal{O}_{T_1}\right] - \mathcal{O}_{T_0} - E_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\xi_s\,ds\right]$$

We have that $\mathcal{P}_{T_0}^{+} = \max\{0, \mathcal{P}_{T_0}\}$.

Rewriting $\mathcal{P}_{T_0}$:

$$\mathcal{P}_{T_0} = E_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u\,du}\mathcal{O}_{T_1}\right] - \mathcal{O}_{T_0} - E_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\xi_s\,ds\right]$$

We then combine the two expectation terms:

$$\mathcal{P}_{T_0} = -\left[\mathcal{O}_{T_0} + E_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\xi_s\,ds\right] - E_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u\,du}\mathcal{O}_{T_1}\right]\right]$$

Now we introduce the convenience yield $\delta_s\mathcal{O}_s$:

$$\mathcal{P}_{T_0} = -E_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}(\xi_s + \delta_s\mathcal{O}_s)\,ds\right]$$

Given $\mathcal{P}_{T_0} = -E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\, du}(\xi_s + \delta_s \mathcal{O}_s)\, ds\right]$, we now have:

$$\mathcal{P}^+_{T_0} = \max\{0, \mathcal{P}_{T_0}\}$$

Substituting $\mathcal{P}_{T_0}$ into the max function, we have shown that:

$$\mathcal{P}^+_{T_0} = \max\left\{0, -E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\, du}\left(\xi_s + \delta_s \mathcal{O}_s\right)\, ds\right]\right\}$$

δt is the convenience yield. It can be seen as a cost or a negative yield on holding the commodity because it represents the benefits forgone by not having immediate access to the physical commodity. It increases the cost of holding the option. Therefore the buyer will exercise the storage option if and only if this value is positive, i.e., the storage costs are sufficiently low compared to the benefits of holding the oil.

13.

We have that

$$\mathcal{O}_{T_0} = e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}$$

and we want that

$$e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1} = e^{-\int_{T_0}^{T_1} \delta_u du}\mathcal{O}_{T_0}\frac{Z_{T_1}}{Z_{T_0}}$$

Thus

$$Z_t = \frac{\mathcal{O}_t}{\mathcal{O}_0} \cdot e^{\int_{T_0}^{t}(\delta_u - r_u)du}$$

which is a $\mathbb{Q}$-Martingale because

$$E^{\mathbb{Q}}_{T_0}\left[\frac{Z_{T_1}}{Z_{T_0}}\right] = 1$$

Therefore we can construct our new Martingale measure $\bar{\mathbb{Q}}$:

$$E^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] = \mathcal{O}_{T_0} E^{\bar{\mathbb{Q}}}_{T_0}\left[e^{-\int_{T_0}^{T_1} \delta_u du}\right]$$

$$dZ_t = Z_t \sigma^T_{\mathbb{O}} dB^{\mathbb{Q}}_t$$

$$\mathcal{P}_{T_0} = E^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] - \mathcal{O}_{T_0} - E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$
$$\mathcal{P}_{T_0} = \mathcal{O}_{T_0}\left[E^{\bar{\mathbb{Q}}}_{T_0}\left[e^{-\int_{T_0}^{T_1} \delta_u du}\right] - 1\right] - E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$

δ_t behaves like:

$$d\delta_t = (\lambda_\delta(\bar{\delta} - \delta) + \sigma^T_\delta \cdot \sigma_{\mathbb{O}})dt + \sigma^T_\delta dB^{\bar{\mathbb{Q}}}_t \tag{22}$$

14.
We set up the differential equations for $\psi_0(s)$ and $\psi_\delta(s)$:

$$\begin{cases} \psi_0'(s) = \lambda_\delta \psi_\delta(s) + \frac{1}{2}\|\sigma_\delta\|^2 \psi_\delta^2(s) \\ \psi_\delta'(s) = -1 - \lambda_\delta \psi_\delta(s) \end{cases}$$

with boundary conditions $\psi_0(0) = 0$ and $\psi_\delta(0) = 0$.
Solving these ODEs similarly to the bond price case, we get:

$$\begin{cases} \psi_\delta(s) = \frac{e^{-\lambda_\delta s} - 1}{\lambda_\delta} \\ \psi_0(s) = \frac{\left(e^{-2\lambda_\delta s}(1 - e^{\lambda_\delta s})\left((3e^{\lambda_\delta s} - 1)\|\sigma_\delta\|^2 - 4e^{\lambda_\delta s}\lambda_\delta \bar{\delta}\lambda_\delta^2 + 4e^{\lambda_\delta s}\lambda_\delta \sigma_\delta^T \sigma_O\right) + 2\lambda_\delta s\left(\|\sigma_\delta\|^2 - 2\bar{\delta}\lambda_\delta^2 + 2\lambda_\delta \sigma_\delta^T \sigma_O\right)\right)}{4\lambda_\delta^3} \end{cases}$$

15.
Since $\mathcal{O}_s = \mathcal{O}_{T_0} e^{\int_{T_0}^s (r_u - \delta_u) du + \sigma_O^T \left(B_s^{\mathbb{Q}} - B_{T_0}^{\mathbb{Q}}\right)}$, we can substitute this into the integral. However, for simplicity, we use the expectation in exponential form derived earlier:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^s r_u du} \mathcal{O}_s\right] = \mathcal{O}_{T_0} e^{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}}$$

So,

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u du} \xi_s ds\right] = \alpha \mathcal{O}_{T_0} \int_{T_0}^{T_1} \mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^s r_u du}\right] e^{\psi_0(s - T_0) + \psi_\delta(s - T_0)\delta_{T_0}} ds$$

By changing variables, $\tau = s - T_0$:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u du} \alpha \mathcal{O}_s ds\right] = \alpha \mathcal{O}_{T_0} \int_0^\Delta e^{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}} d\tau$$

Thus,

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u du} \xi_s ds\right] = \alpha \mathcal{O}_{T_0} \int_0^\Delta e^{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}} d\tau$$

We combine this result with the storage option valuation formula:

$$\mathbb{E}_T^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du} P_{T_0}^+\right] = O_t \mathbb{E}_T^{\mathbb{Q}}\left[e^{-\int_t^{T_0} \delta_u du} H\left(\delta_{T_0}\right)\right]$$

To show the value of the storage option at $t \leq T_0$:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du} O_{T_1}\right] - O_{T_0} - \mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u du} \xi_s ds\right]$$

By using the derived expectation:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du} O_{T_1}\right] - O_{T_0} - \alpha O_{T_0} \int_0^\Delta e^{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}} d\tau$$

Since:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du} O_{T_1}\right] = O_{T_0} e^{\psi_0(\Delta) + \psi_\delta(\Delta)\delta_{T_0}}$$

We can substitute this into the storage option valuation formula:

$$O_{T_0} e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - O_{T_0} - \alpha O_{T_0} \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T_0}} d\tau$$

We factor out O_{T_0} and we identify:

$$H(\delta_{T_0}) = e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T_0}} d\tau$$

The price of the storage option is independent of the risk-free rate process r_t because the valuation uses the stochastic convenience yield δ_t instead. When changing the measure to $\bar{\mathbb{Q}}$, the calculations focus on δ_t, which discounts storage costs and future oil prices. This approach isolates the option's value from the risk-free rate, concentrating on the convenience yield that directly affects the storage option's price. Therefore, the risk-free rate r_t does not influence the final valuation.

16.

$$H'(\delta) = \psi_\delta(\Delta) e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta} - \alpha \int_0^\Delta \psi_\delta(\tau) e^{\psi_0(\tau)+\psi_\delta(\tau)\delta} d\tau$$

and

$$\psi_\delta(\Delta) H(\delta) = \psi_\delta(\Delta) \left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta} - 1 - \alpha \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta} d\tau \right)$$

We know that $\psi_\delta(s) \leq 0$ and is an decreasing function of s. Thus $\psi_\delta(\Delta) < \psi_\delta(s)$ for 0 < s < Δ. Implying

$$-\psi_\delta(\Delta) - \psi_\delta(\Delta)\alpha \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta} d\tau > -\psi_\delta(\Delta)\alpha \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta} d\tau > -\alpha \int_0^\Delta \psi_\delta(\tau) e^{\psi_0(\tau)+\psi_\delta(\tau)\delta} d\tau$$

which leads to

$$H'(\delta) < \psi_\delta(\Delta) H(\delta)$$

Now, we use this inequality to prove that $\{\delta \in \mathbb{R} : H(\delta) > 0\} = (-\infty, \delta^*)$ for some threshold δ^*.

The inequality gives

$$\frac{H'(\delta)}{H(\delta)} < \psi_\delta(\Delta)$$

We integrate:

$$\int \frac{H'(\delta)}{H(\delta)} d\delta < \int \psi_\delta(\Delta) \, d\delta$$

Then we have:

$$\ln |H(\delta)| < \psi_\delta(\Delta)\delta + C$$

Leading to:

$$|H(\delta)| < e^{\psi_\delta(\Delta)\delta + C}$$

This implies that $H(\delta)$ decays faster than at an exponential rate $e^{\psi_\delta(\Delta)\delta}$.

For large negative δ, $H(\delta)$ becomes negative and for large positive δ, $H(\delta)$ becomes positive. Thus, there exists a threshold δ^* such that $H(\delta) > 0$ for $\delta \in (-\infty, \delta^*)$ and $H(\delta) \leq 0$ for $\delta \geq \delta^*$.
Using Lemma 1, we replace x_t with δ_{T_0}, so we can write:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}\mathcal{P}_{T_0}^{+}\right] = \mathcal{O}_t \int_{-\infty}^{\delta^*}\left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha\int_0^{\Delta} e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T_0}}d\tau\right) f_{\delta_{T_0}|\delta_t}(\delta_{T_0})d\delta_{T_0}$$

where the conditional density function $f_{\delta_{T_0}|\delta_t}(\delta_{T_0})$ is then the density of a normal distribution:

$$f_{\delta_{T_0}|\delta_t}(\delta_{T_0}) = \frac{1}{(2\pi)^{3/2}|V|^{1/2}}\exp\left(-\frac{1}{2}(\delta_{T_0} - m_t)^T V^{-1}(\delta_{T_0} - m_t)\right)$$

with mean vector:

$$m_t = \begin{pmatrix} e^{-\lambda_\delta\Delta}\delta_t + (1-e^{-\lambda_\delta\Delta})\bar{\delta} \\ \frac{\bar{\delta}(1-e^{-\lambda_\delta\Delta})+(1-e^{-\lambda_\delta\Delta})(\delta_t-\bar{\delta})}{\lambda_\delta} \end{pmatrix}$$

and Variance-Covariance matrix:

$$V = \begin{pmatrix} \frac{\sigma_\delta^2(1-e^{-2\lambda_\delta\Delta})}{2\lambda_\delta} & \frac{\sigma_\delta^2(1-e^{-\lambda_\delta\Delta})^2}{2\lambda_\delta^2} \\ \frac{\sigma_\delta^2(1-e^{-\lambda_\delta\Delta})^2}{2\lambda_\delta^2} & \frac{\sigma_\delta^2 e^{-2\lambda_\delta\Delta}(e^{2\lambda_\delta\Delta}(2\lambda_\delta\Delta-3)+4e^{\lambda_\delta\Delta}-1)}{2\lambda_\delta^3} \end{pmatrix}$$

17.
As in question 15. we have:

$$O_{T_0}\bar{H} = \mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du}O_{T_1}\right] - O_{T_0} - \mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$

$$= O_{T_0}\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du}\right] - O_{T_0} - \alpha O_{T_0}\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}ds\right]$$

r is constant so $\int_{T_0}^{s} r_u du = \mathrm{r(s-}T_0)$
Thus, we eliminate O_{T_0} and

$$\bar{H} = e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 + \frac{\alpha}{r}(e^{-r(T_1-T_0)} - 1)$$

Again, we using Lemma 1, so we can write:

$$\mathbb{E}_t^{\overline{\mathbb{Q}}}\left[e^{-\int_t^{T_0}\delta_u du}\bar{H}(\delta_{T_0})^{+}\right] = \int_{-\infty}^{\delta^*} H(\delta_{T_0})f_{\delta_{T_0}|\delta_t}(\delta_{T_0})d\delta_{T_0}$$

where $f_{\delta_{T_0}|\delta_t}(\delta_{T_0})$ is the conditional density of δ_{T_0} given δ_t. which gives:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}\mathcal{P}_{T_0}^{+}\right] = \mathcal{O}_t \int_{-\infty}^{\delta^*}\left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 + \frac{\alpha}{r}(e^{-r(T_1-T_0)} - 1)\right) f_{\delta_{T_0}|\delta_t}(\delta_{T_0})d\delta_{T_0}$$

where the conditional density function $f_{\delta_{T_0}|\delta_t}(\delta_{T_0})$ is then the density of a normal distribution:

$$f_{\delta_{T_0}|\delta_t}\left(\delta_{T_0}\right) = \frac{1}{(2\pi)^{3/2}|V|^{1/2}} \exp\left(-\frac{1}{2}\left(\delta_{T_0} - m_t\right)^T V^{-1}\left(\delta_{T_0} - m_t\right)\right)$$

with mean vector:

$$m_t = \begin{pmatrix} e^{-\lambda_\delta \Delta}\delta_t + (1 - e^{-\lambda_\delta \Delta})\bar{\delta} \\ \frac{\bar{\delta}(1-e^{-\lambda_\delta \Delta})+(1-e^{-\lambda_\delta \Delta})(\delta_t - \bar{\delta})}{\lambda_\delta} \end{pmatrix}$$

and Variance-Covariance matrix:

$$V = \begin{pmatrix} \frac{\sigma_\delta^2(1-e^{-2\lambda_\delta \Delta})}{2\lambda_\delta} & \frac{\sigma_\delta^2(1-e^{-\lambda_\delta \Delta})^2}{2\lambda_\delta^2} \\ \frac{\sigma_\delta^2(1-e^{-\lambda_\delta \Delta})^2}{2\lambda_\delta^2} & \frac{\sigma_\delta^2 e^{-2\lambda_\delta \Delta}(e^{2\lambda_\delta \Delta}(2\lambda_\delta \Delta - 3)+4e^{\lambda_\delta \Delta}-1)}{2\lambda_\delta^3} \end{pmatrix}$$

Changing r_t means that the value of the storage option depends on the entire path of r_t over time, not just its value at specific points.
The changing of r_t represents extra randomness to our calculations. Incorporating this randomness makes the calculations more complex. Indeed, with a variable r_t, we can't just factor it out easily. Instead, we have to integrate over its changing values, which is much more complicated.
18. The calibration method utilized here involves a Nelder-Mead optimization approach to fit a term structure model to market data by minimizing the squared error between model-predicted prices and observed market prices through adjusting key parameters such as the mean reversion intensity ($\lambda_r \approx 29.91\%$), long-term mean ($\bar{r} \approx 1.00\%$), and initial interest rate ($r_0 \approx 1.46\%$).

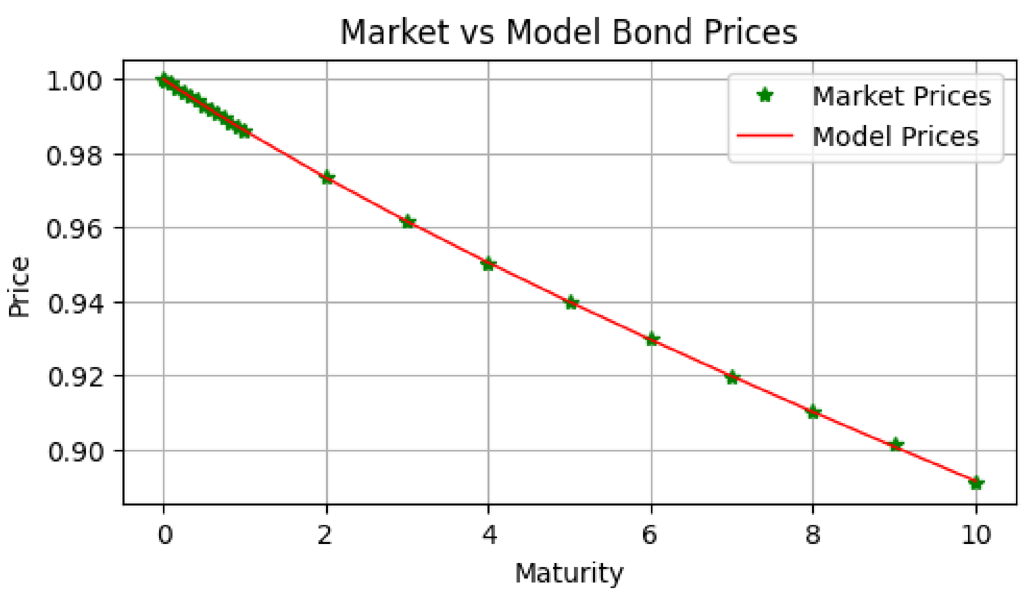

Figure 2: Actusal Vs Predicted Bond prices

The resulting RMSE of 0.000204 indicates a highly accurate calibration. This suggests that the model effectively captures the underlying dynamics of the term structure, making it reliable for forecasting. See 2.
19. The calibration method employed involves minimizing the Mean Squared Error (MSE)

to fit a futures pricing model to observed market data, focusing on parameters such as the mean reversion intensity of the spread ($\lambda_\delta \approx 1.3225$), the long-term spread mean ($\bar{\delta} \approx 0.0260$), and the initial spread value ($\delta_0 \approx 0.004$).

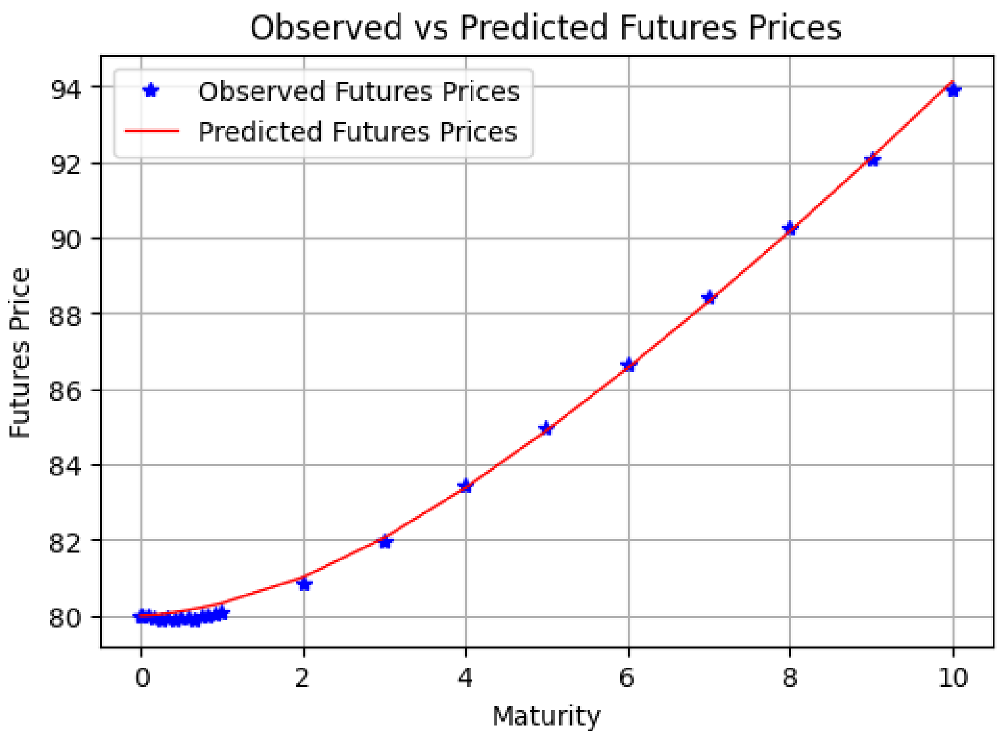

Figure 3: Predicted VS Actual Futures prices

Extensive tweaking of initial values was performed to achieve the lowest possible Root Mean Square Error (RMSE), which resulted in an RMSE of 0.1609. While the model provides a reasonably good fit, some variance between predicted and observed prices suggests room for further refinement in model assumptions or additional calibration. See 3.

20. To determine the critical value of the convenience yield (δ_{T_0}),we used the `brentq` method to solve the equation defined by $H = 0$. (See code)

21. From graph 4, we can say that the initial option price is increasing in storage period, storage cost and correlation coefficient. However, it is decreasing in initial convenience yield, long term mean of the convenience yield and volatility of the convenience yield.

From graph 5, we can say that the exercise threshold is increasing in storage cost parameter and and the correlation. However, it is decreasing in initial convenience yield, long term mean of the convenience yield, volatility of the convenience yield and storage period.

References

[1] CME Group, *What is Contango and Backwardation*,`https://www.cmegroup.com/education/courses/introduction-to-ferrous-metals/what-is-contango-and-backwardation.html`. Accessed 8 June 2024

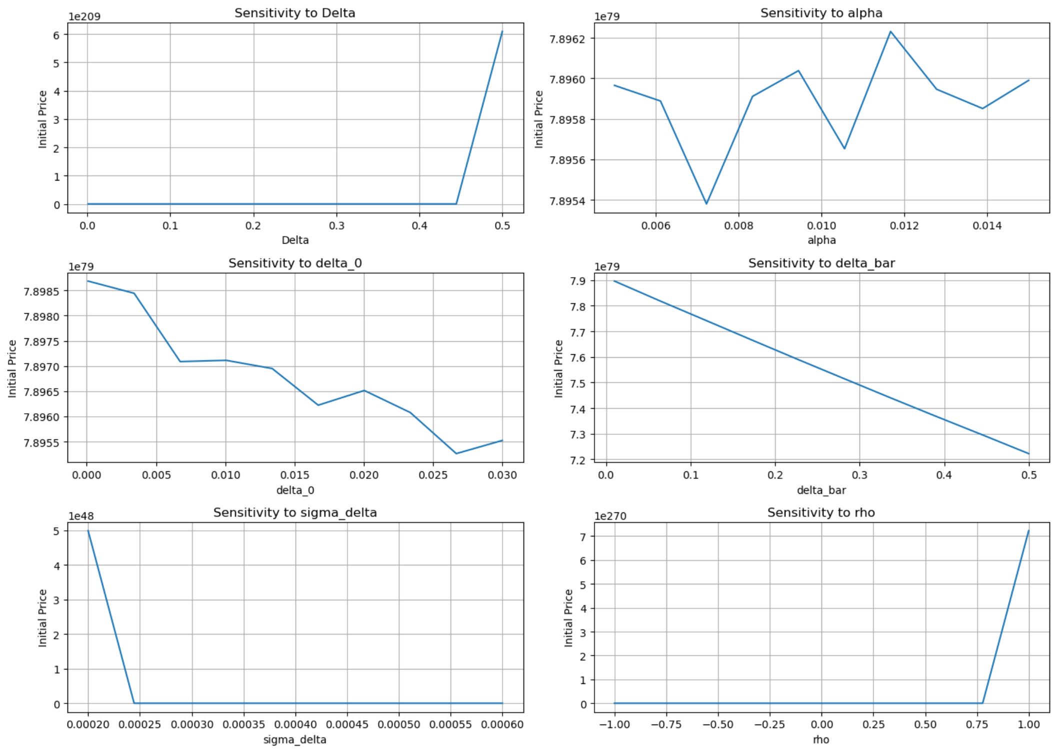

Figure 4: Sensitivity Analysis for the Initial Option Price ($T_0 = 0.5, \alpha = 0.01, \Delta = 1/4$)

[2] `https://www.cmegroup.com/markets/energy/crude-oil/light-sweet-crude.contractSpecs.html`. Accessed 19 June 2024

[3] CME Group, *A Trader's Guide to Futures*, `http://cmegroup.com/education`. Accessed 15 May 2024.

[4] CME Group, *CME Commodity Trading Manual*, Chicago Mercantile Exchange, `https://www.cmegroup.com/education`. Accessed 15 May 2024.

[5] Victor Bernal A. *Calibration of the Vasicek Model: An Step by Step Guide*, `https://victor-bernal.weebly.com/uploads/5/3/6/9/53696137/projectcalibration.pdf`. Accessed 19 May 2024.

[6] `https://finance.yahoo.com/quote/CL%3DF/futures`. Accessed 19 June 2024.

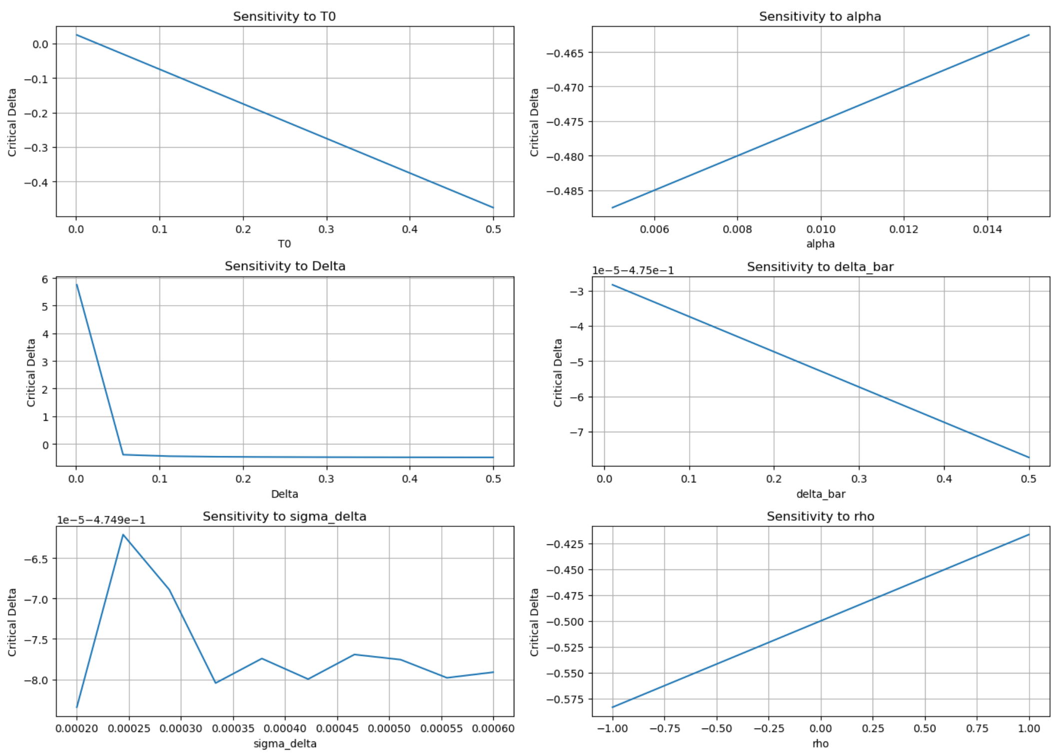

Figure 5: Sensitivity Analysis for the Exercise Threshold on the Storage Period ($T_0 = 0.5, \alpha = 0.01, \Delta = 1/4$)