Cross-correlation methods and overlap reduction functions for stochastic GW background searches

Joe Romano, Texas Tech University

Friday, 22 July 2022

(HUST GW Summer School 2022, Lecture 3)

Outline

1. Cross-correlation methods
2. Optimal filtering examples
3. Response of a detector to a GWB
4. Overlap reduction functions

I. Cross-correlation methods

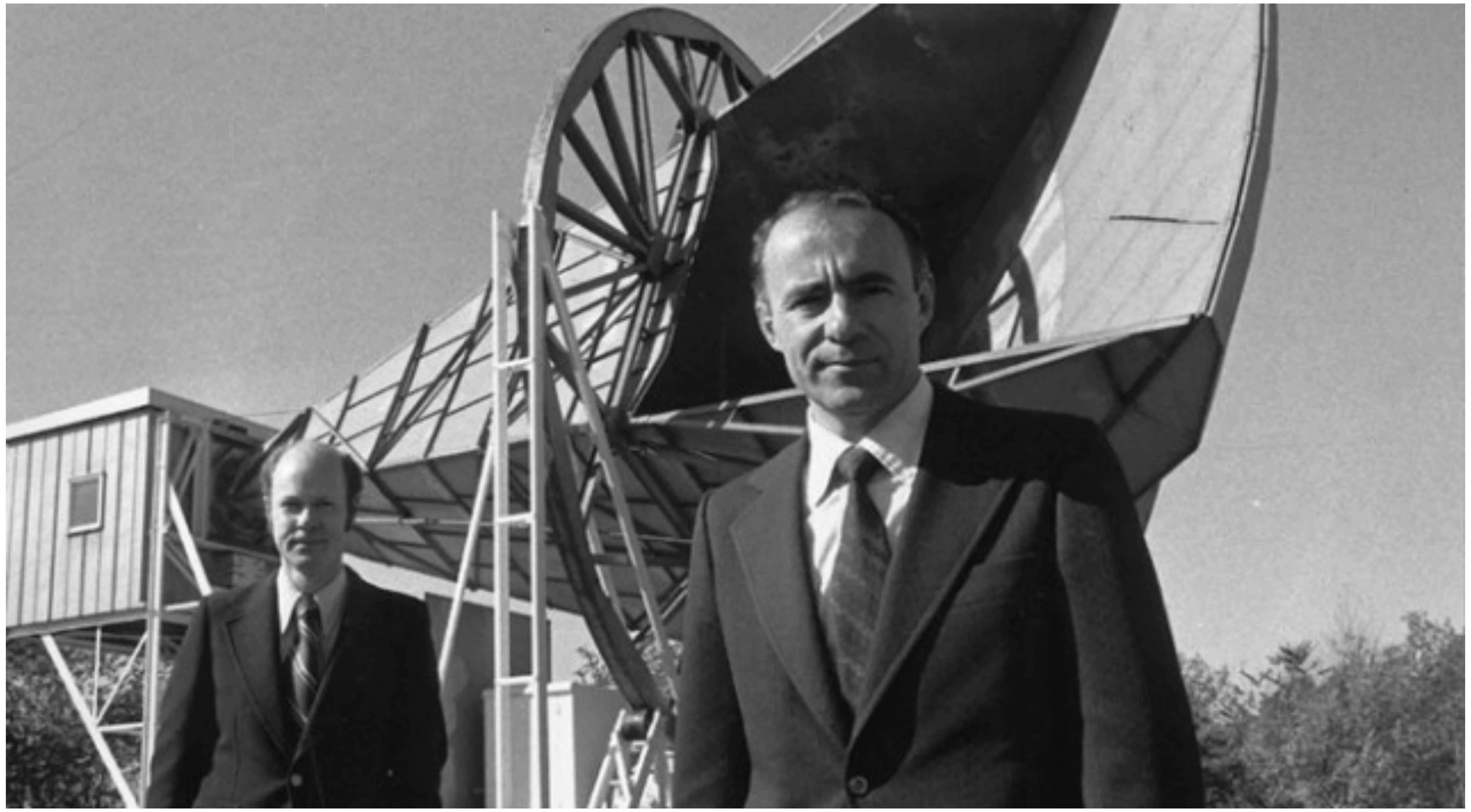
Single versus multiple detectors

- Initial detection of the CMB was in a **single detector** (excess noise that could not be attributed to any known noise source)
- Ground-based detectors currently **aren't sensitive enough** for expected GWBs to stand out above instrumental noise
- Instead, look for evidence of a common disturbance in multiple detectors -> **cross-correlation**
- signal might be weak, but you can **build up signal-to-noise ratio** by correlating for long periods of time

A MEASUREMENT OF EXCESS ANTENNA TEMPERATURE AT 4080 Mc/s

Measurements of the effective zenith noise temperature of the 20-foot horn-reflector antenna (Crawford, Hogg, and Hunt 1961) at the Crawford Hill Laboratory, Holmdel, New Jersey, at 4080 Mc/s have yielded a value about 3.5° K higher than expected. This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965). A possible explanation for the observed excess noise temperature is the one given by Dicke, Peebles, Roll, and Wilkinson (1965) in a companion letter in this issue.

(Penzias & Wilson, 1965)



Cross correlation — basic idea

- Data from two detectors:

$$d_1 = h + n_1$$
$$d_2 = h + n_2$$

common GW signal component

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- Expected value of cross-correlation:

$$\langle \hat{C}_{12} \rangle = \langle d_1 d_2 \rangle = \langle h^2 \rangle + \langle n_1 n_2 \rangle + \underbrace{\cancel{\langle h n_2 \rangle}}_{0} + \underbrace{\cancel{\langle n_1 h \rangle}}_{0} = \langle h^2 \rangle + \langle n_1 n_2 \rangle$$

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- Assuming detector noise is uncorrelated:

$$\boxed{\langle \hat{C}_{12} \rangle = \langle h^2 \rangle \equiv S_h}$$

Worked example: N samples, white GWB in white noise

$$\hat{S}_h \equiv \hat{C}_{12} = \frac{1}{N}\sum_{i=1}^{N} d_{1i} d_{2i} \qquad d_{1i} = h_i + n_{1i}\,, \quad d_{2i} = h_i + n_{2i}$$

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SNR:

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$$\rho \equiv \frac{\mu}{\sigma} \simeq \frac{S_h}{\sqrt{S_1S_2/N}} \quad \Longrightarrow \quad \boxed{\rho \simeq \sqrt{N}\frac{S_h}{S_n}} \qquad \text{where} \quad \sqrt{S_1S_2} \simeq \sqrt{S_{n_1}S_{n_2}} \equiv S_n$$

Cross-correlation estimators, optimal filtering

- More generally:

$$\hat{S}_h = \int_{-T/2}^{T/2} \mathrm{d}t \int_{-T/2}^{T/2} \mathrm{d}t' \, d_1(t) d_2(t') Q(t, t')$$

if stationary

$Q(t, t') = Q(t - t')$

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- Choose Q to maximize SNR for fixed spectral shape:

$$\tilde{Q}(f) \propto \frac{\Gamma_{12}(f) H(f)}{P_1(f) P_2(f)}$$

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when noise is large or
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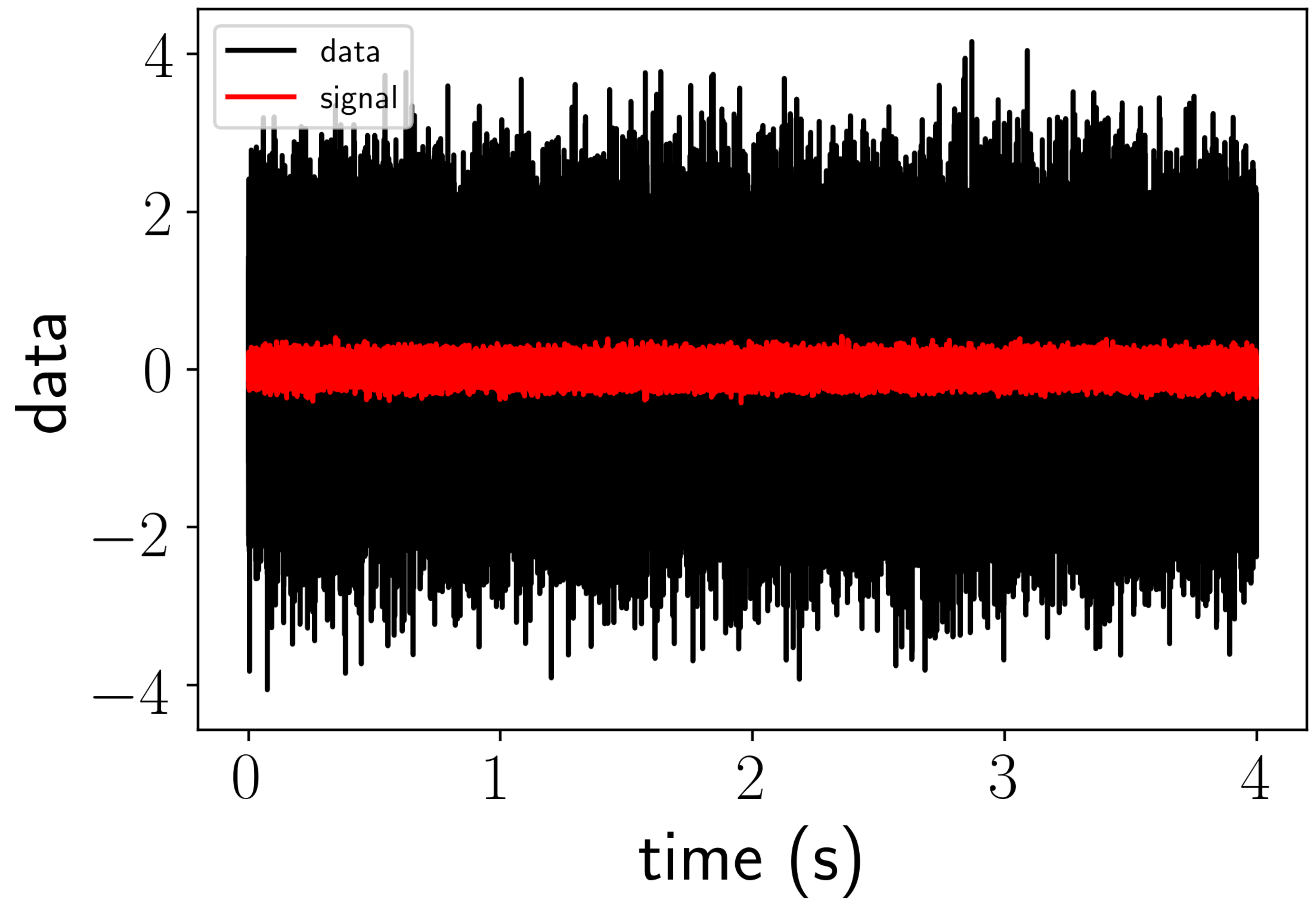
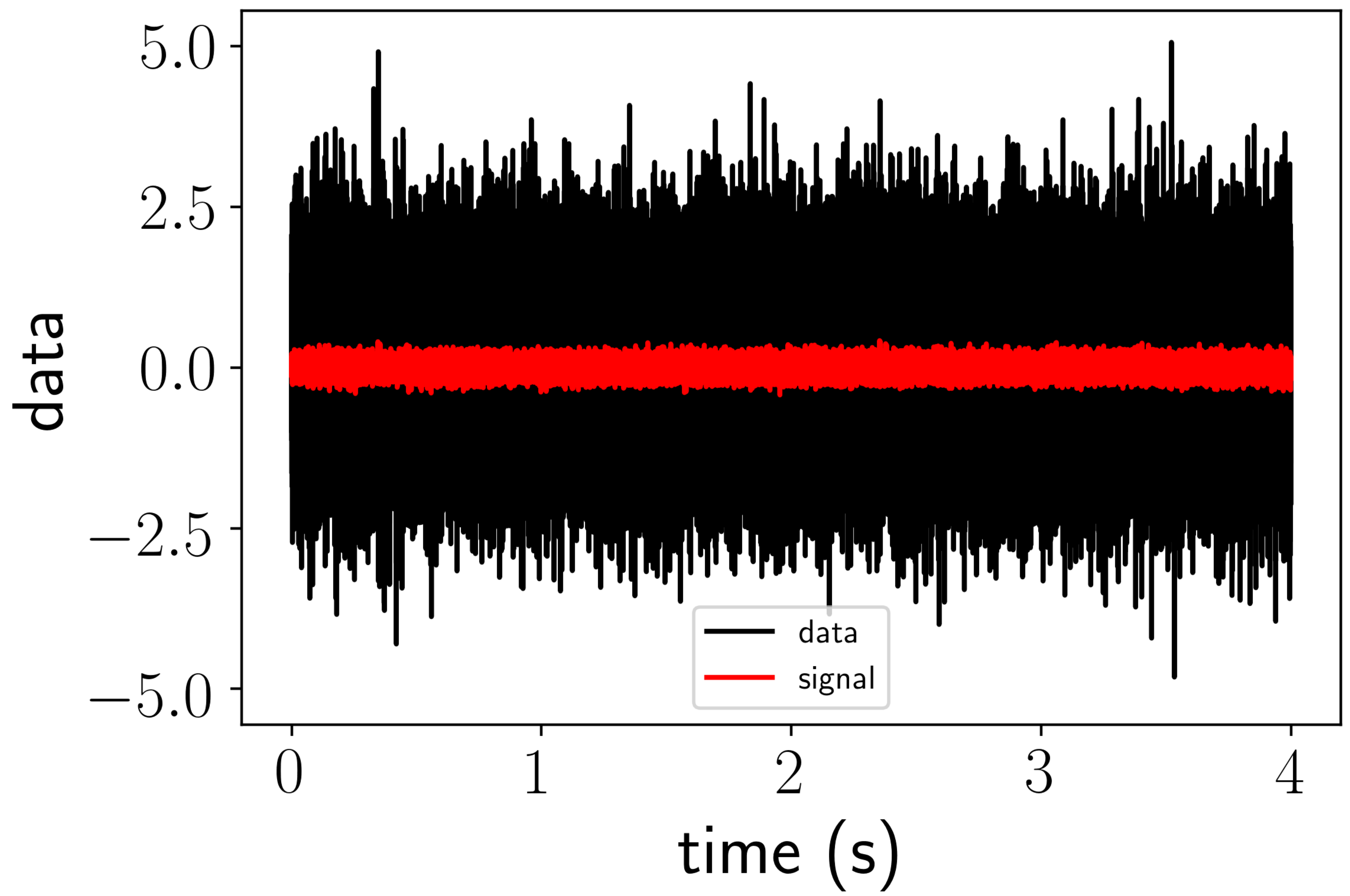
-> Will discuss detector response and overlap reductions in detail, later on

II. Optimal filtering examples
(code available at romano_code3.ipynb)

(i) white GWB in white noise



$$H(f) = 1$$



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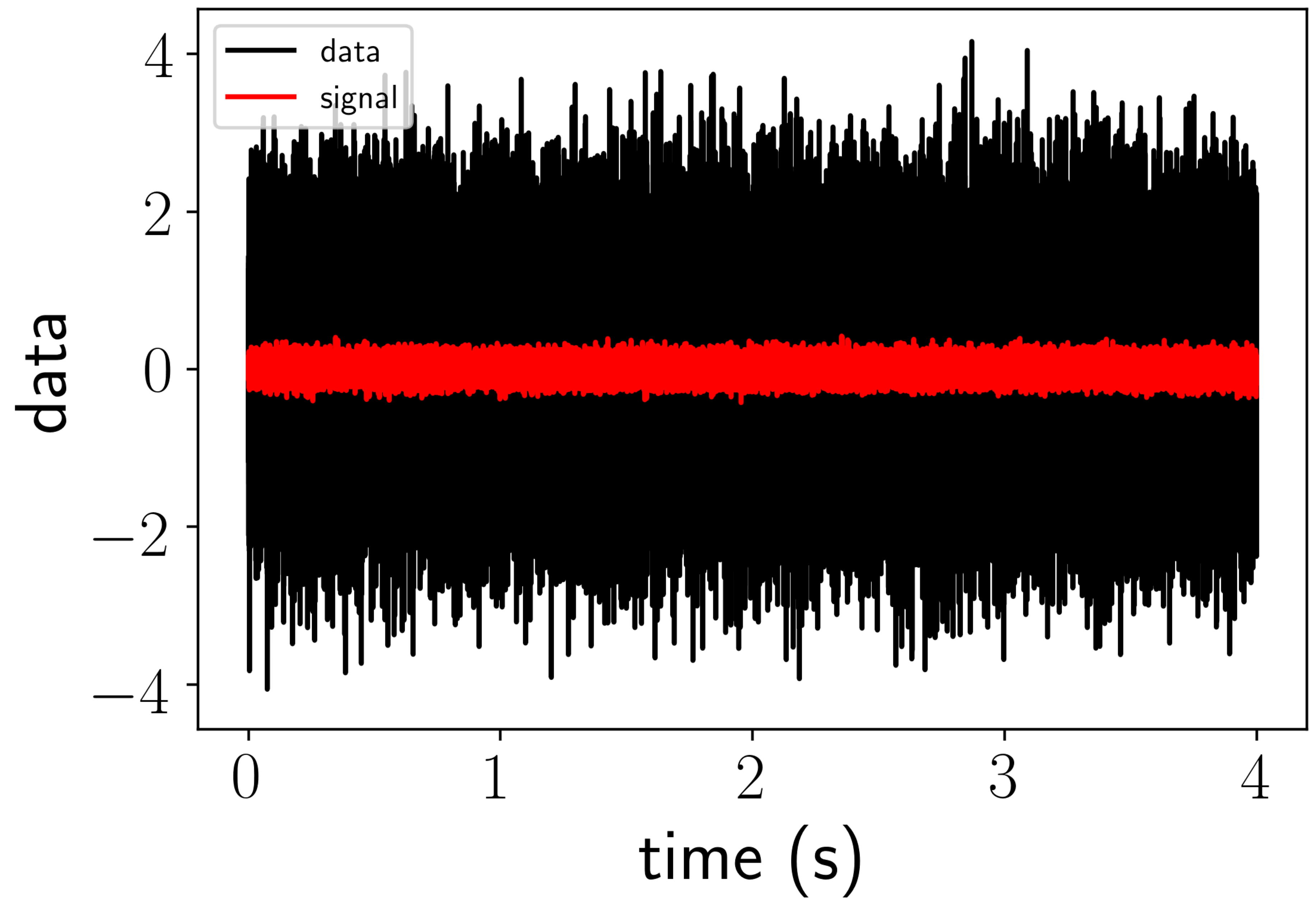

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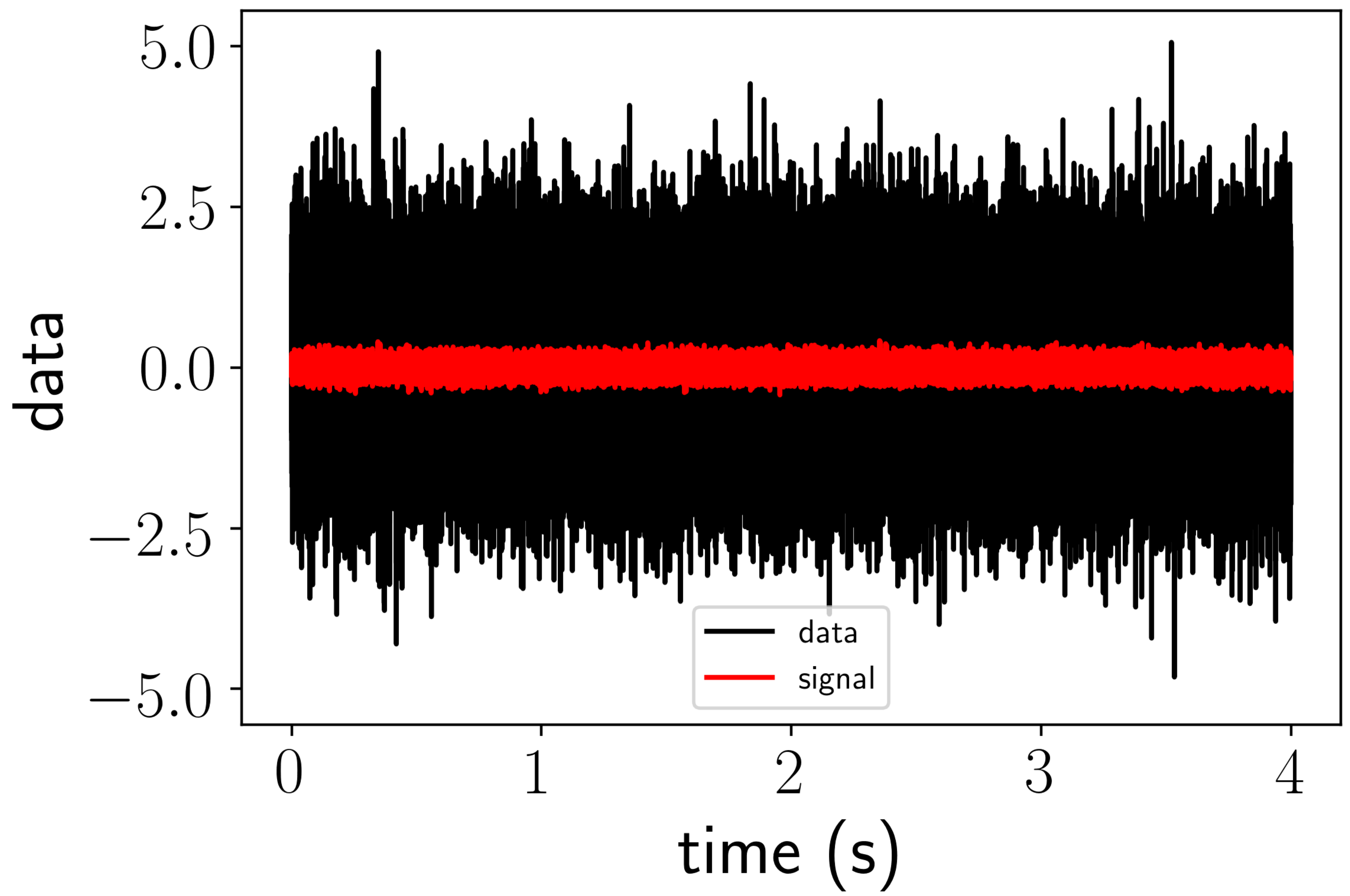
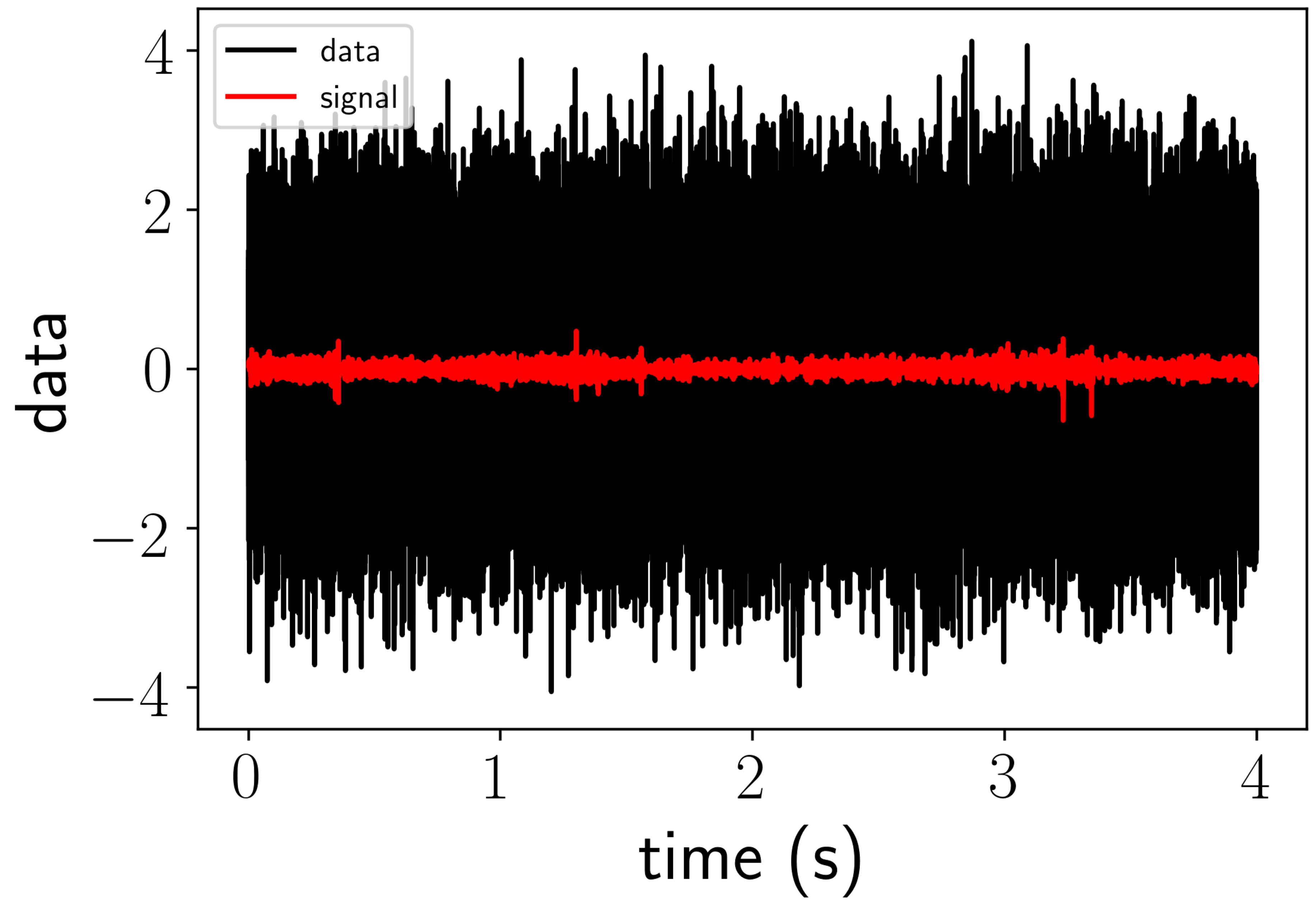


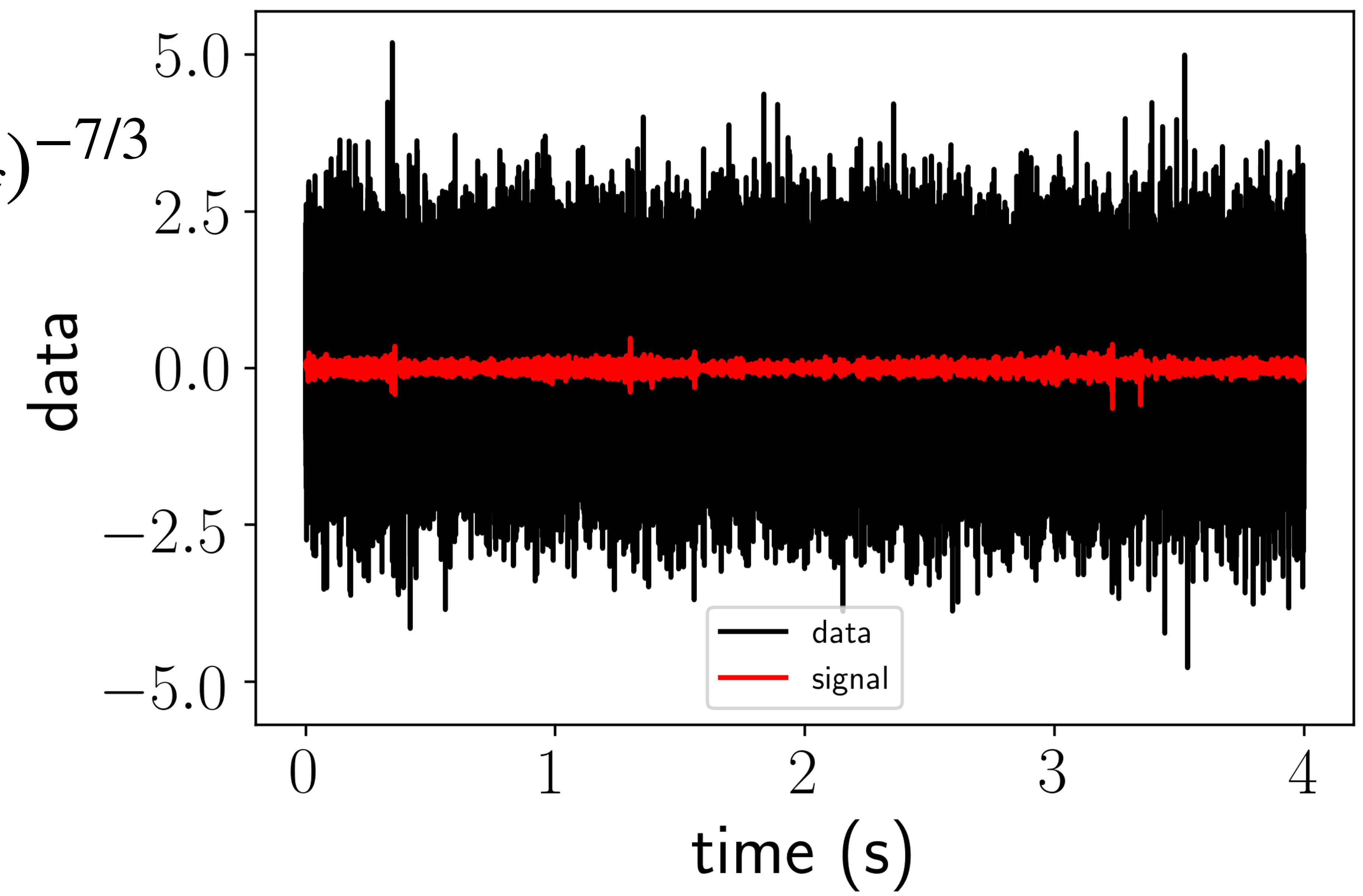
expected and estimated values of power agree to 3.5%, within 1 sigma

optimally filtered CC SNR = 2.9

(ii) BNS-confusion-limited GWB in white noise



$$H(f) = (f/f_{\mathrm{ref}})^{-7/3}$$



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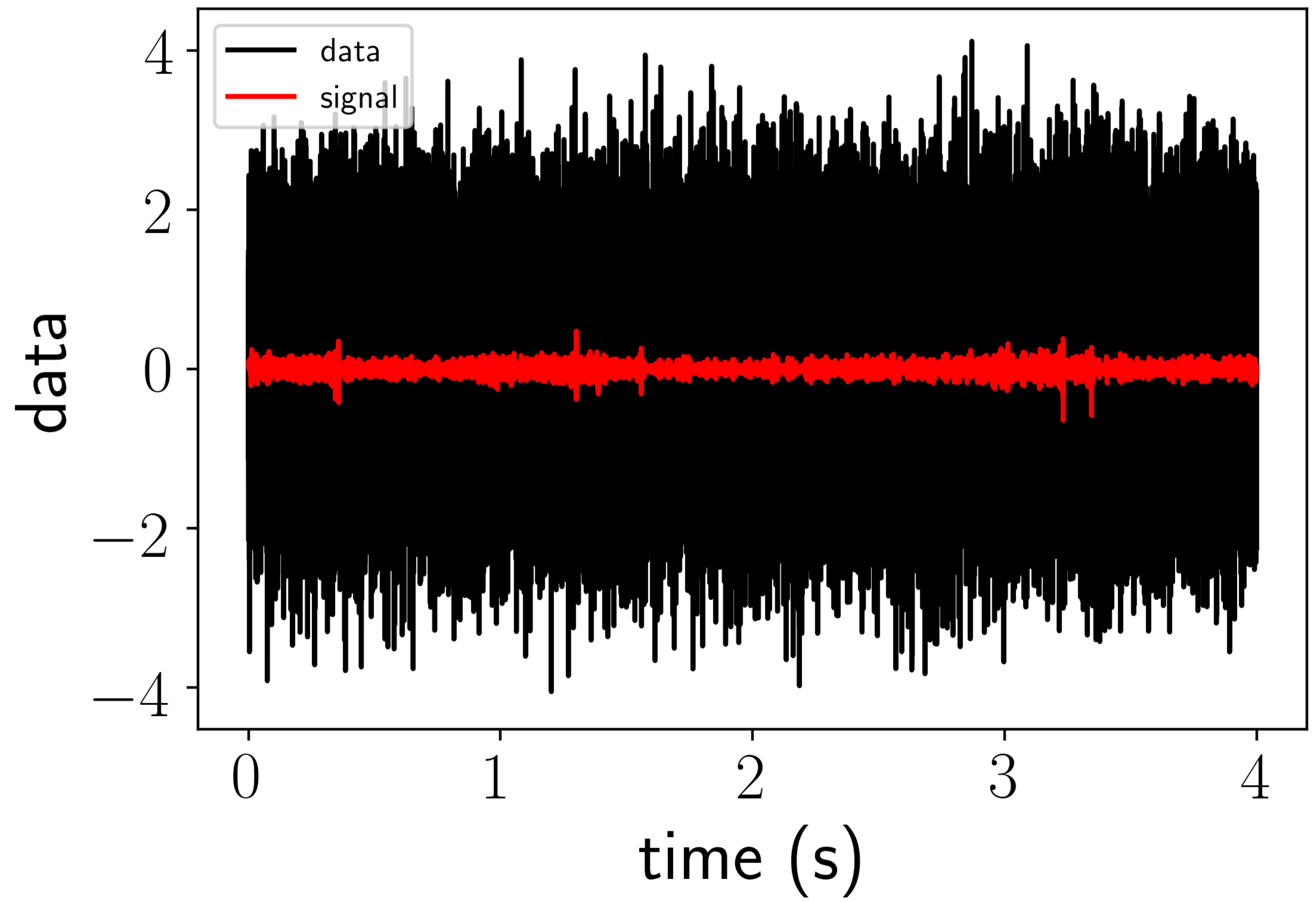

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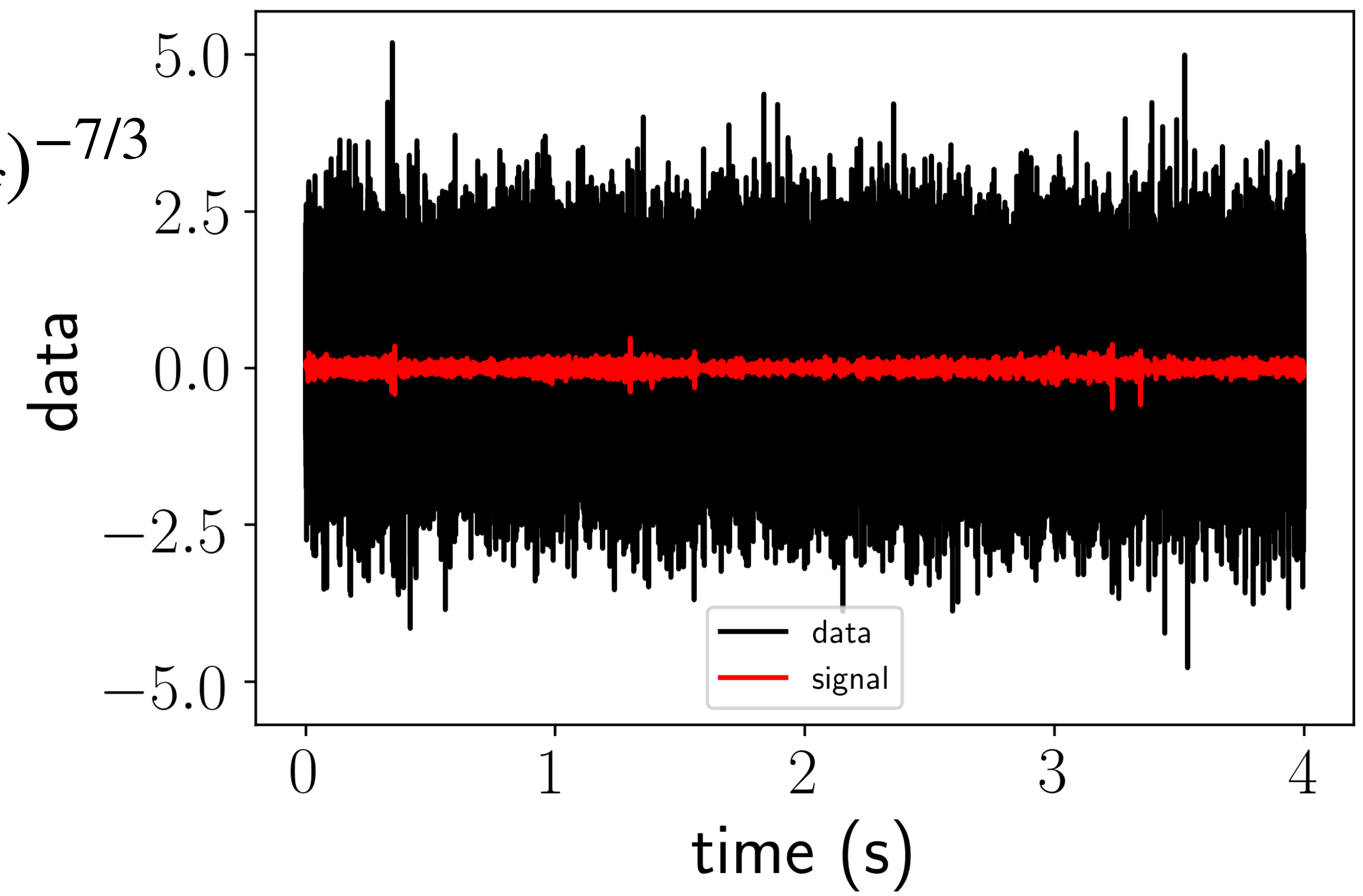
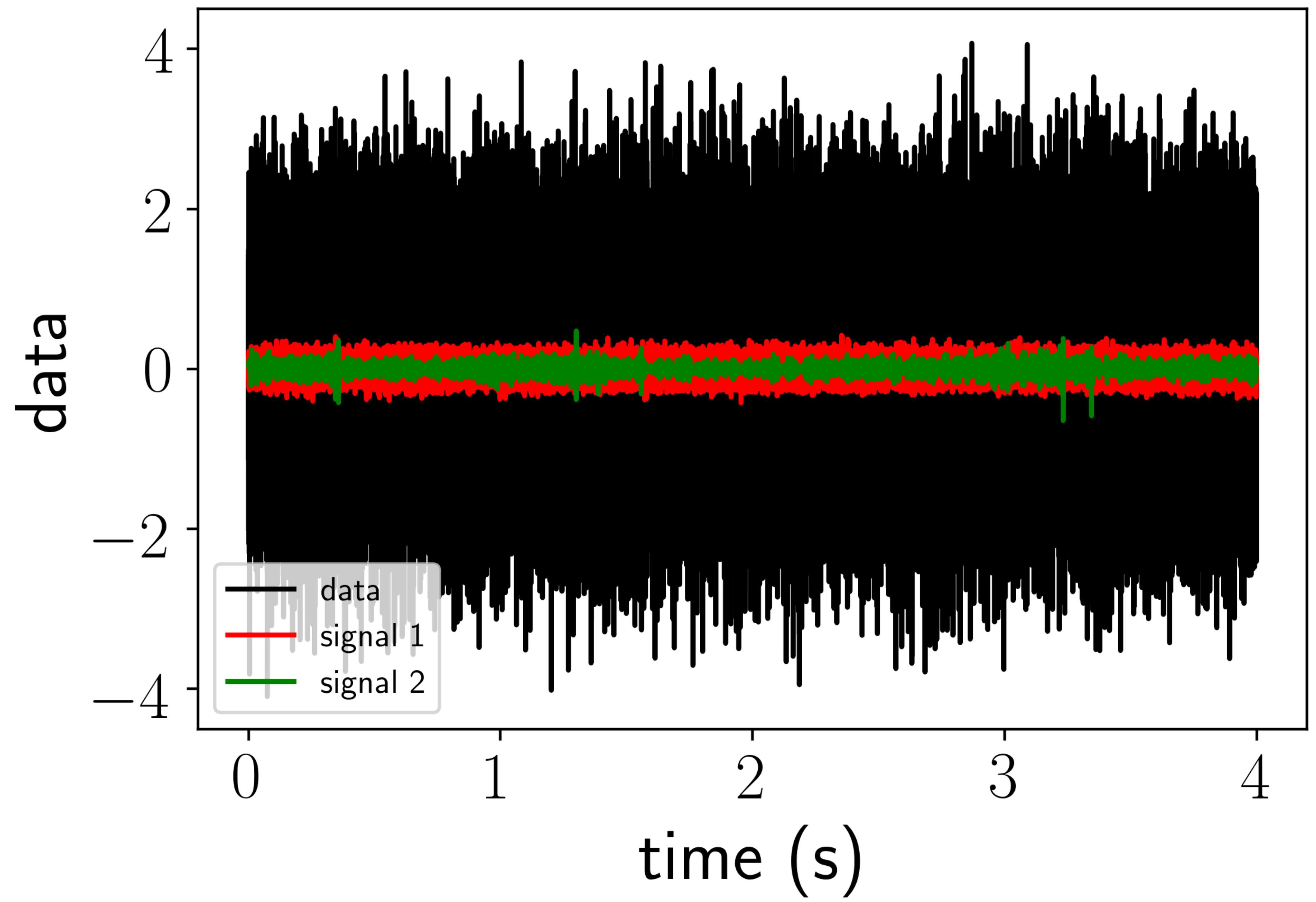


expected and estimated values of power agree to 2.7%, within 1 sigma

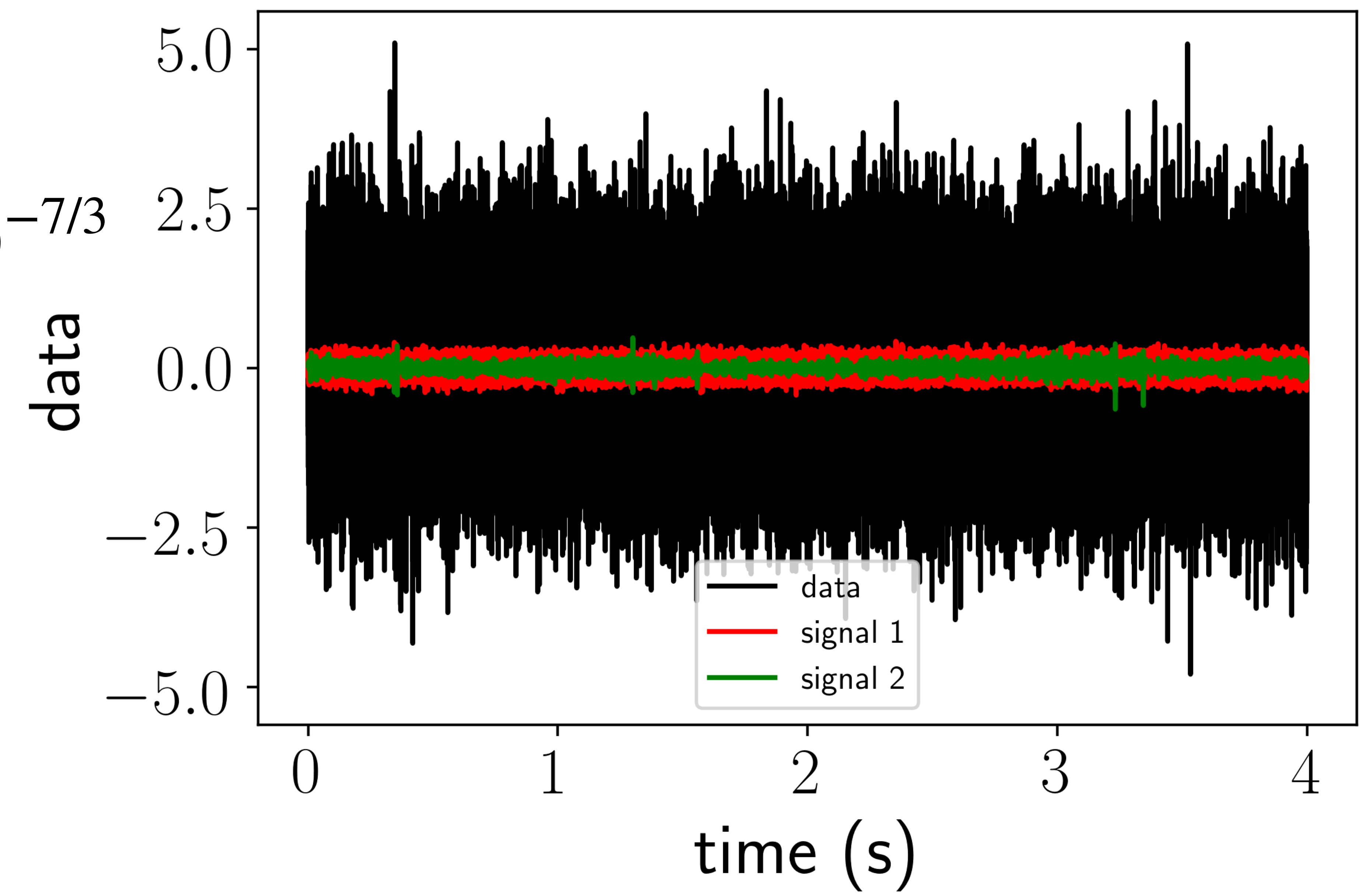
optimally filtered CC SNR = 12

(iii) Two-component GWB in white noise



$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\mathrm{ref}})^{-7/3}$$



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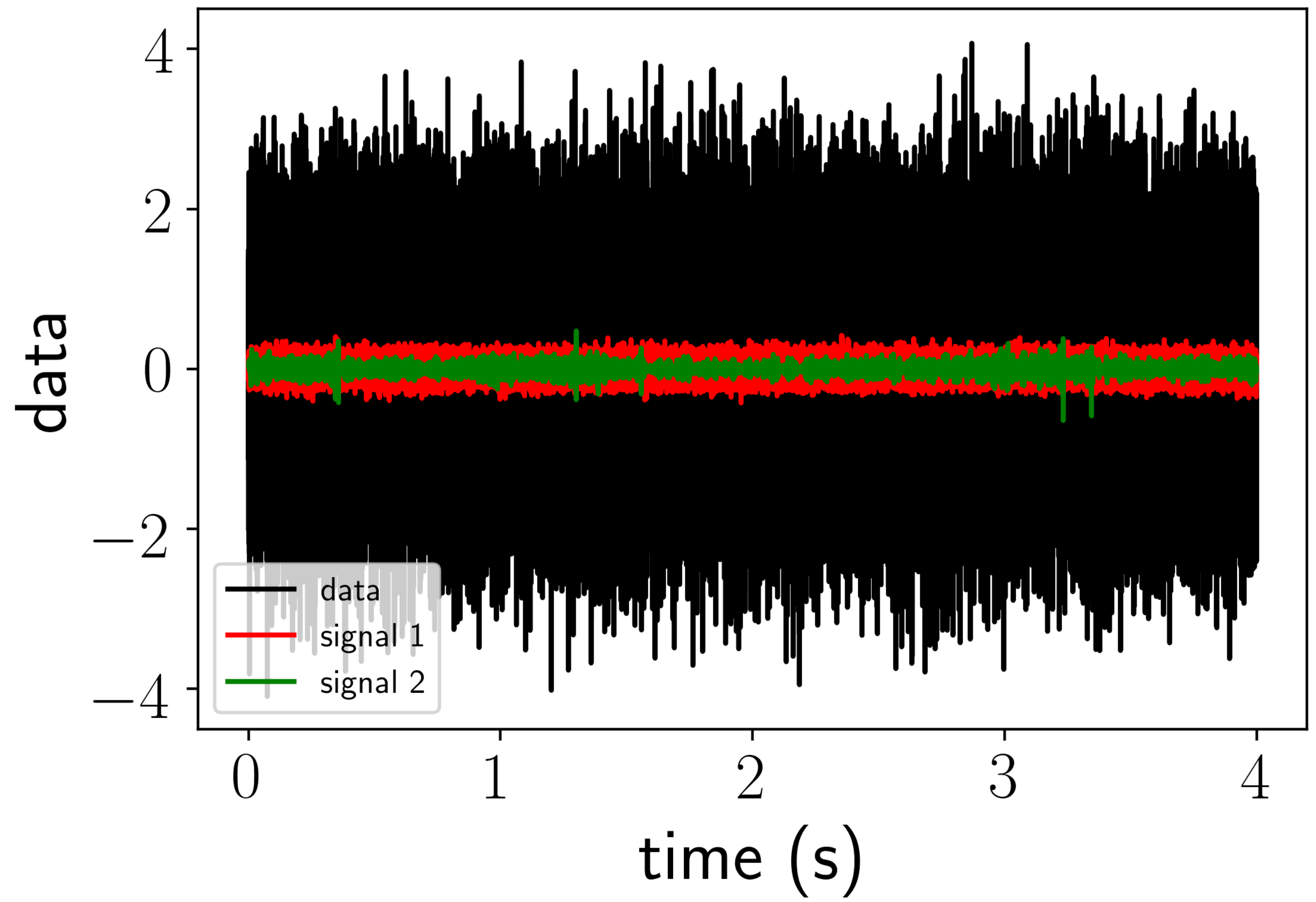

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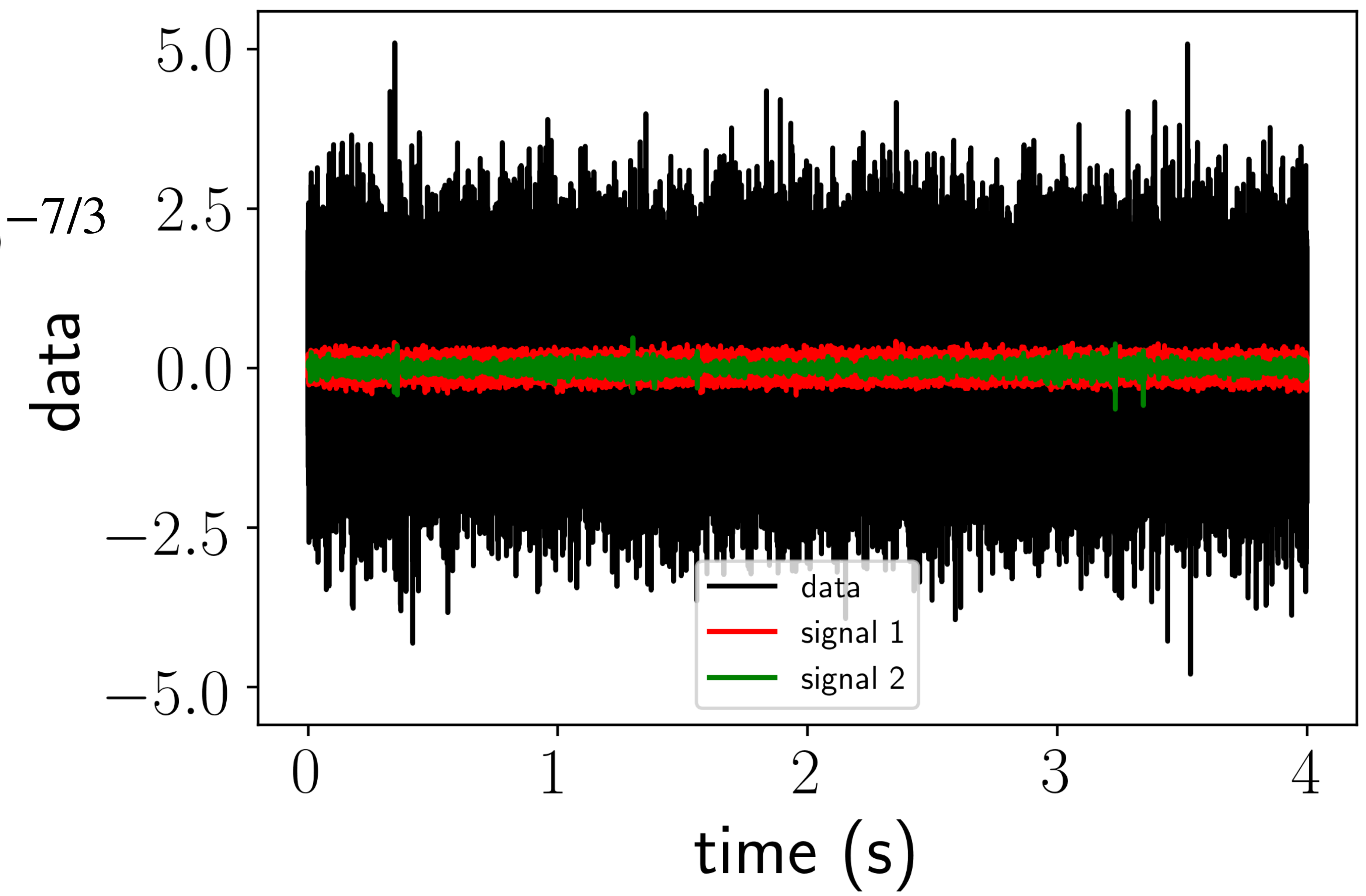
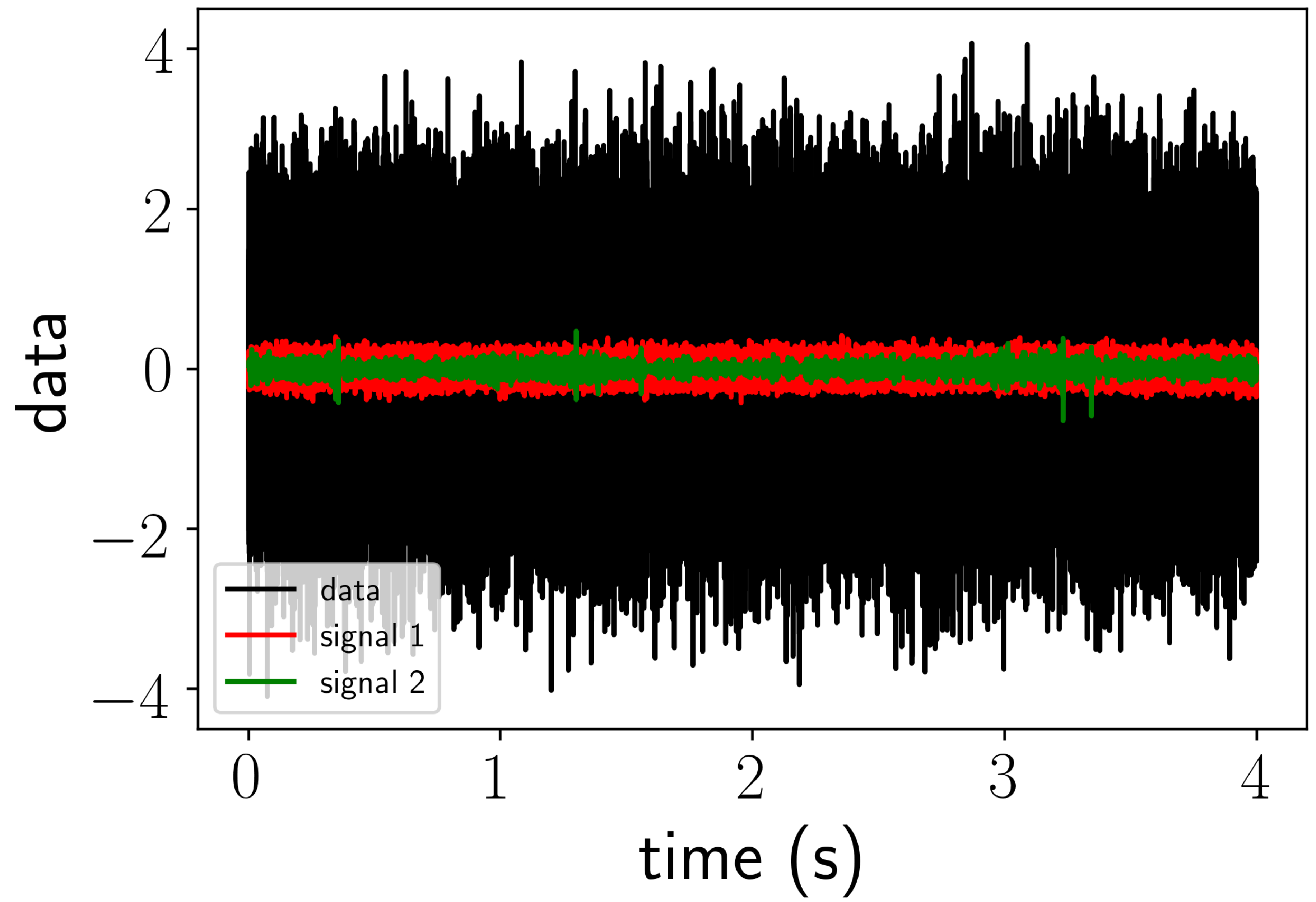


<u>optimal-filtering for each component separately</u>:
white GWB: 48% overestimate, > 1 sigma
BNS GWB: 6.9% overestimate, within 1 sigma

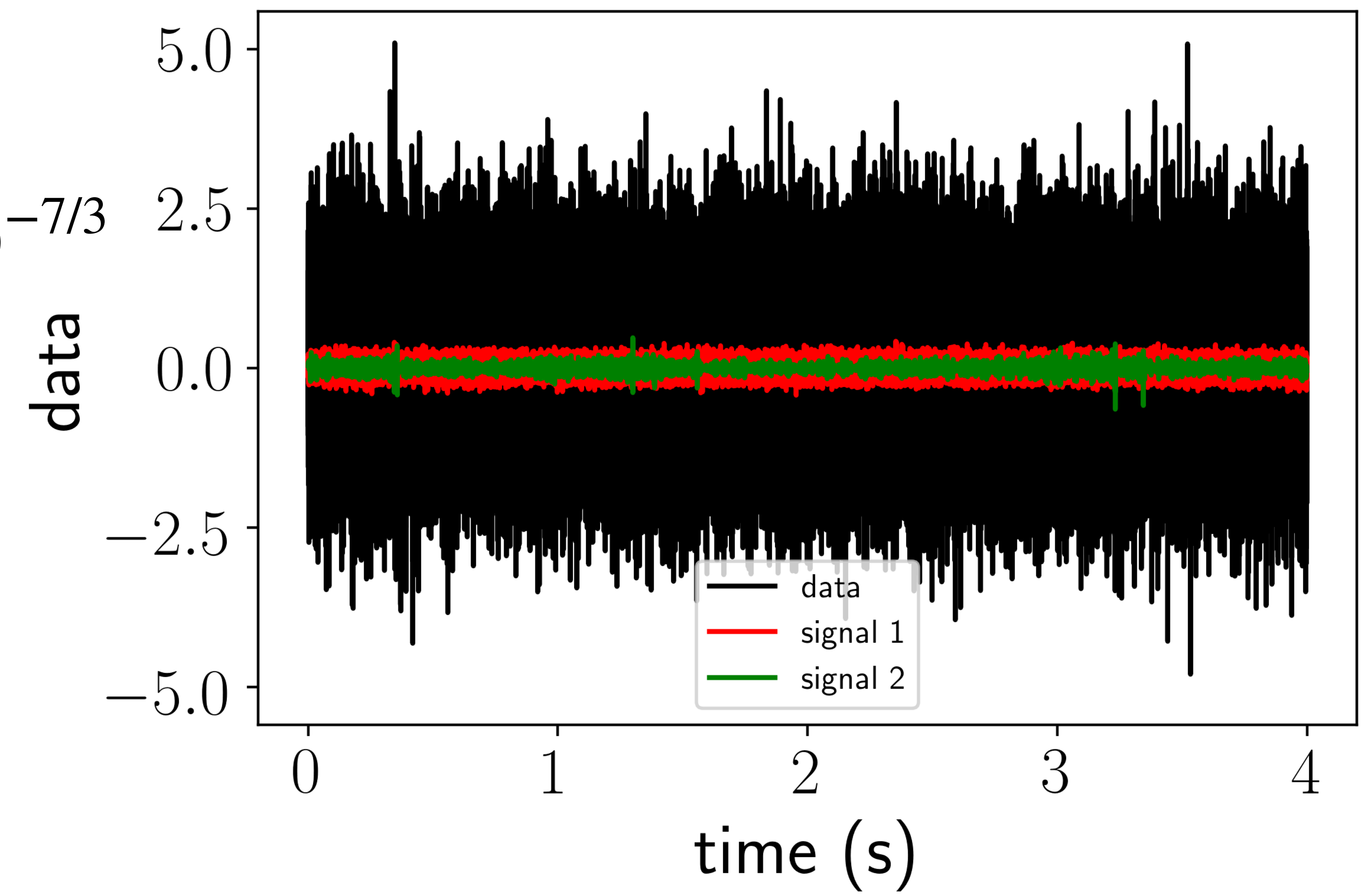
separate analyses overestimate strength of each GWB component and underestimate error bars

(iii) Two-component GWB in white noise



$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\text{ref}})^{-7/3}$$



<u>optimal-filtering for each component separately</u>:
white GWB: 48% overestimate, > 1 sigma
BNS GWB: 6.9% overestimate, within 1 sigma

<u>joint multi-component analysis:</u>
white GWB: agreement to 7.3%, SNR=1.4
BNS GWB: agreement to 3.8%, SNR=6.0

separate analyses overestimate strength of each GWB component and underestimate error bars

joint analysis properly takes into account the covariance between the component spectral shapes

Parida et al, JCAP 024, 2106

Joint multi-component analysis

Joint multi-component analysis

Data are cross-correlation estimates: $\hat{C}_{12}(f) \equiv \frac{2}{T}\tilde{d}_1(f)\tilde{d}_2^*(f)$

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Expectation value: $\langle \hat{C}_{12}(f) \rangle = \sum_\alpha A_\alpha \Gamma_{12}(f) H_\alpha(f) \equiv \sum_\alpha M_\alpha(f) A_\alpha$

spectral shapes

amplitudes

Joint multi-component analysis

Data are cross-correlation estimates:

$$\hat{C}_{12}(f) \equiv \frac{2}{T}\tilde{d}_1(f)\tilde{d}_2^*(f)$$

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amplitudes

Noise covariance matrix:

$$N_{12}(f, f') \equiv \langle \hat{C}_{12}(f)\hat{C}_{12}^*(f') \rangle - \langle \hat{C}_{12}(f) \rangle \langle \hat{C}_{12}^*(f') \rangle$$
$$\simeq \delta_{ff'} P_1(f) P_2(f)$$

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Likelihood function: $$p(\hat{C}\,|\,A, N) \propto \exp\left[-\frac{1}{2}(\hat{C} - MA)^\dagger N^{-1}(\hat{C} - MA)\right]$$

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$$p(\hat{C}|A,N) \propto \exp\left[-\frac{1}{2}(\hat{C}-MA)^\dagger N^{-1}(\hat{C}-MA)\right]$$

Maximum-likelihood estimators:

$$\hat{A} = F^{-1}X$$
$$F \equiv M^\dagger N^{-1} M\,, \qquad X \equiv M^\dagger N^{-1}\hat{C}$$

$$F_{\alpha\beta} = \int_{-\infty}^{\infty} \mathrm{d}f\, \frac{H_\alpha(f)\Gamma_{12}^2(f)H_\beta(f)}{P_1(f)P_2(f)}$$

(noise-weighted inner product of spectral shapes; inverse covariance matrix for A)

Fisher matrix

Joint multi-component analysis

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$$\hat{A} = F^{-1}X$$
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<u>Exercise</u>: Verify the expression for the ML estimators.

$$F_{\alpha\beta} = \int_{-\infty}^{\infty} df\, \frac{H_\alpha(f)\Gamma_{12}^2(f)H_\beta(f)}{P_1(f)P_2(f)}$$

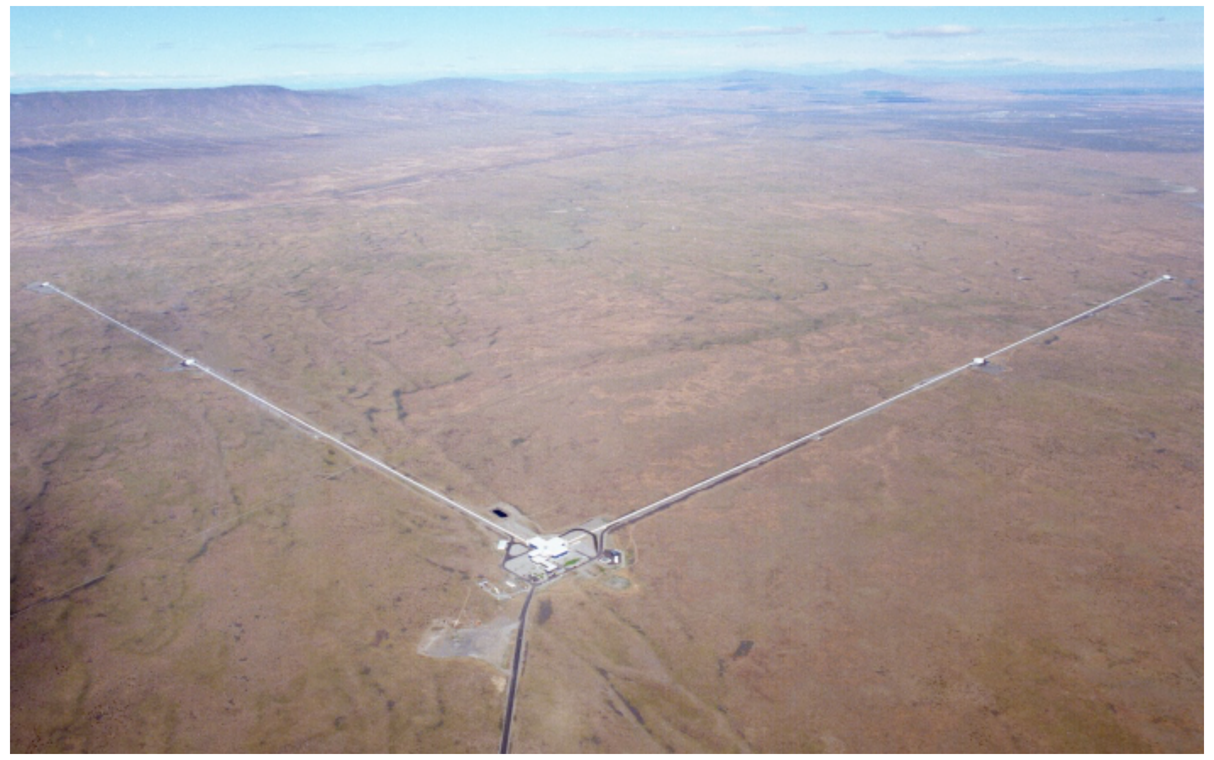
Fisher matrix

(noise-weighted inner product of spectral shapes; inverse covariance matrix for A)

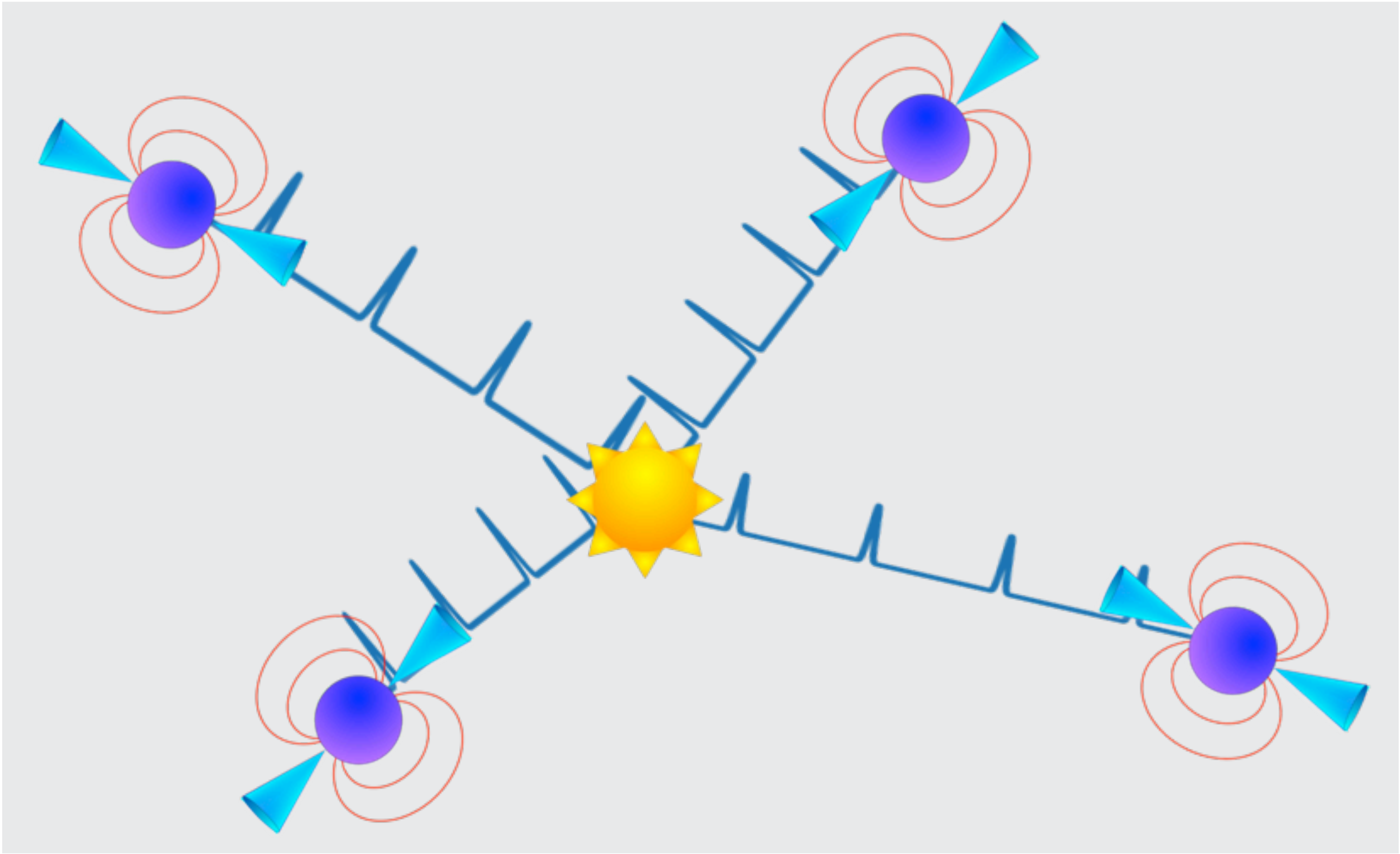
III. Response of a detector to a GW background

Beam detectors (use EM radiation to monitor the separation between two test masses)

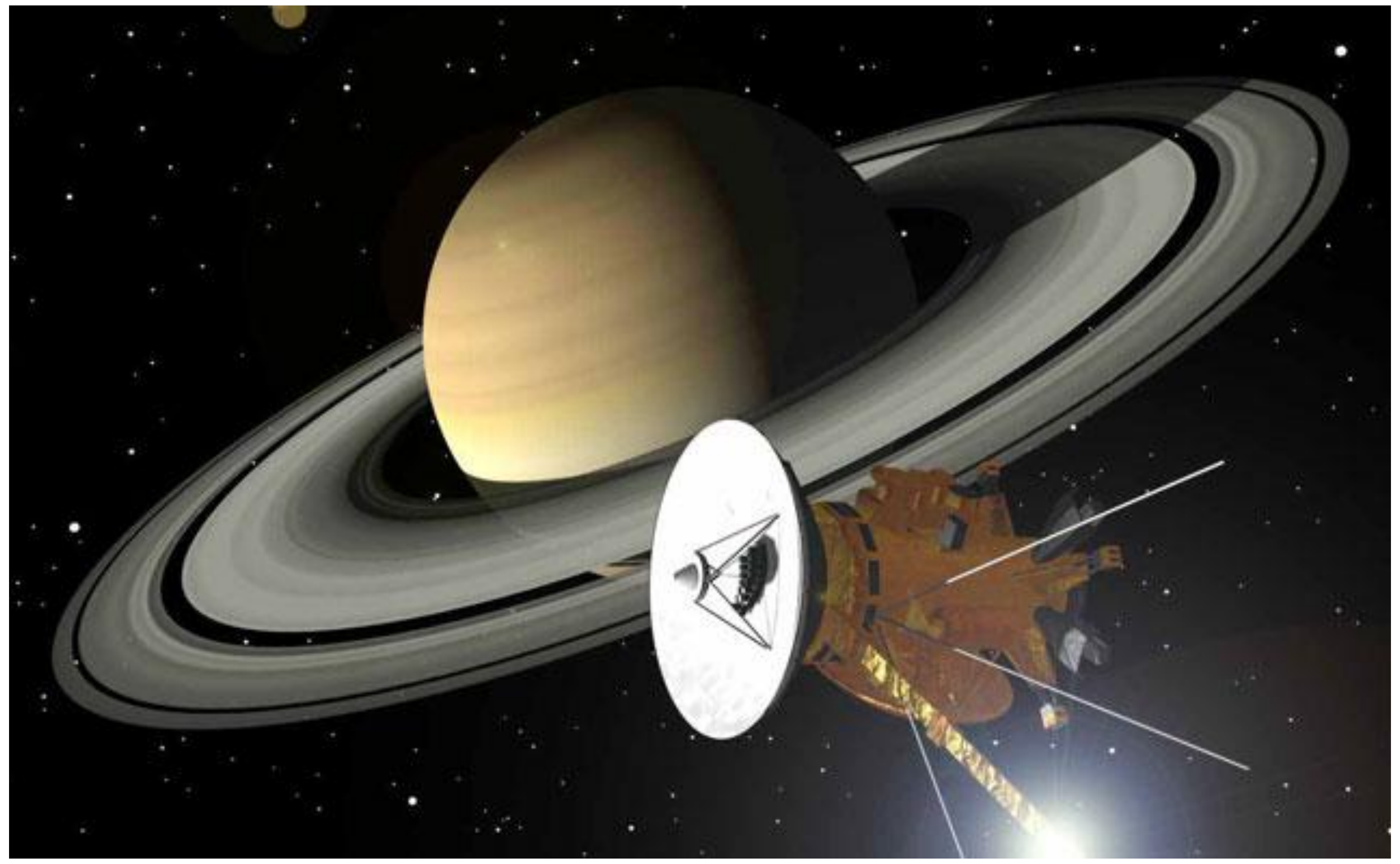
laser interferometers

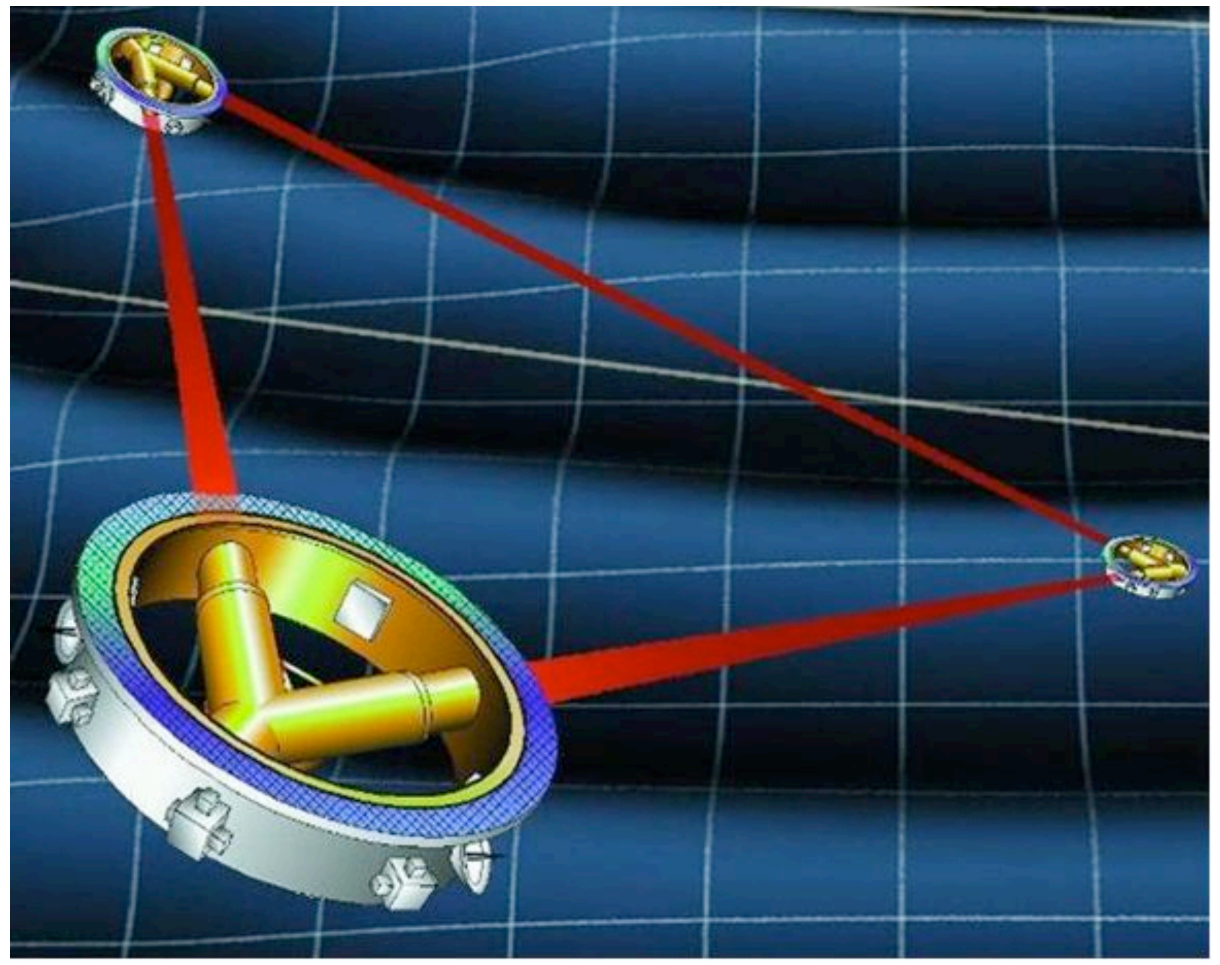


pulsar timing

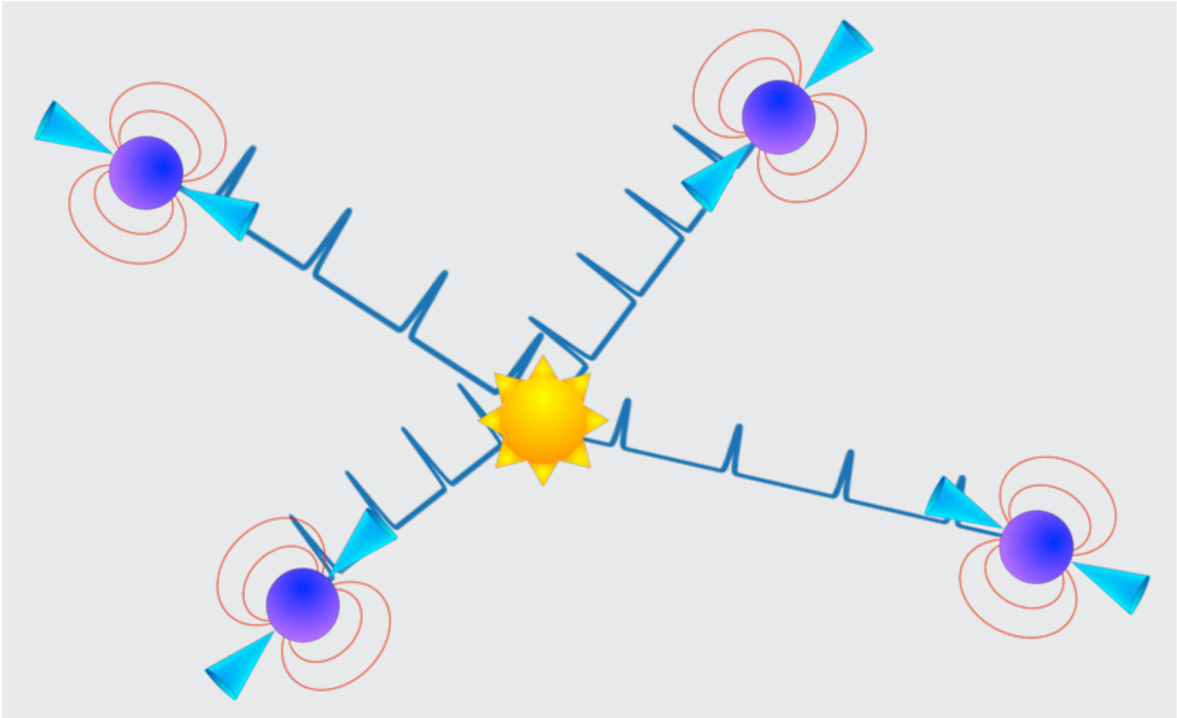


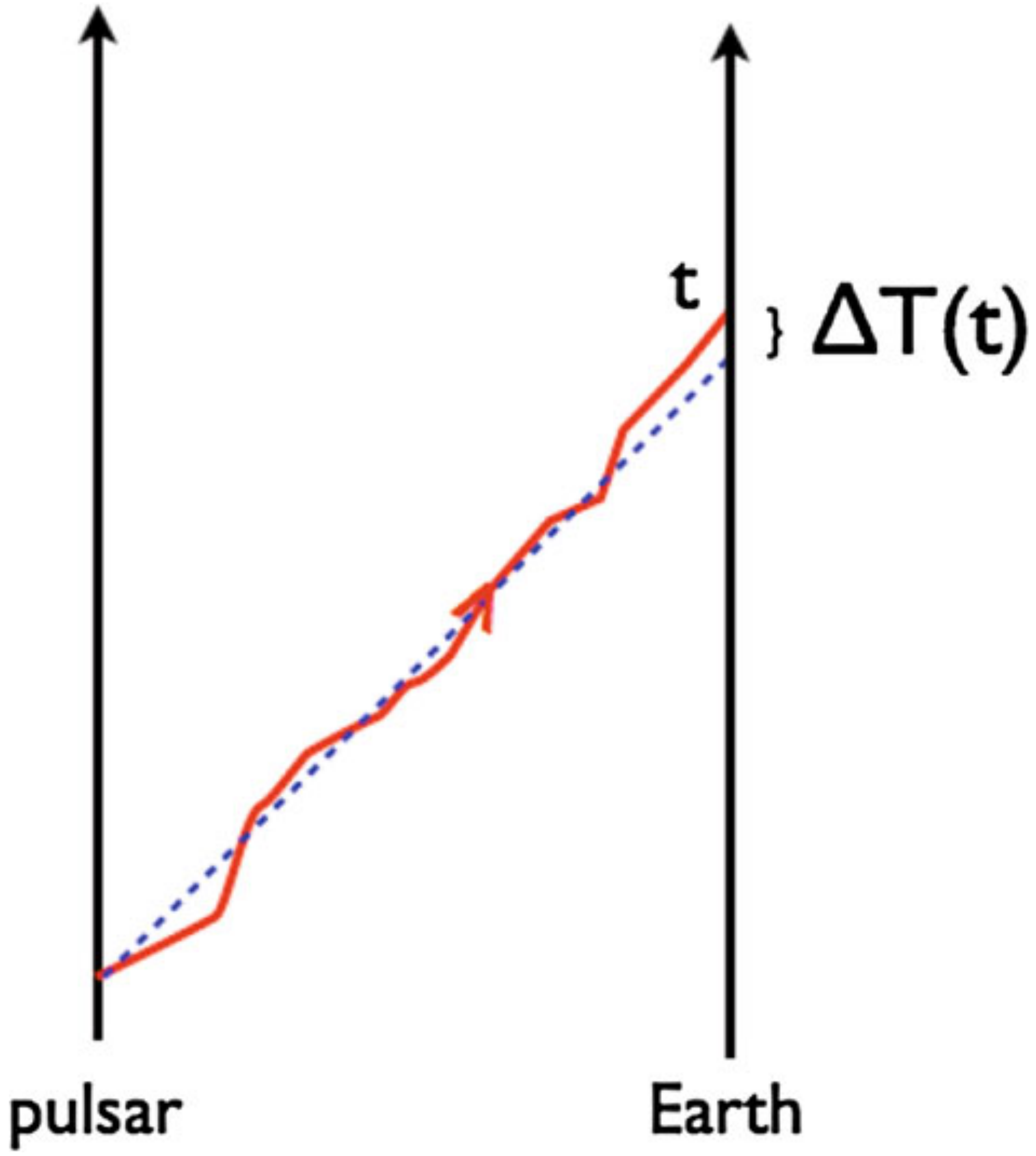
spacecraft Doppler tracking



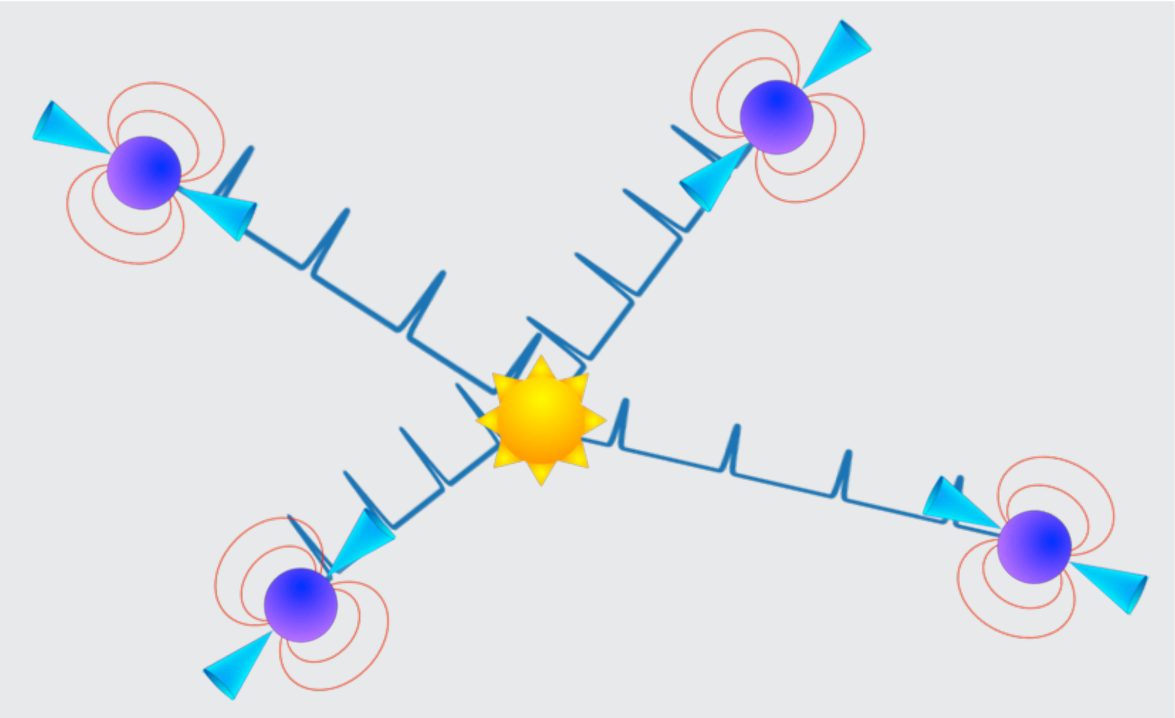


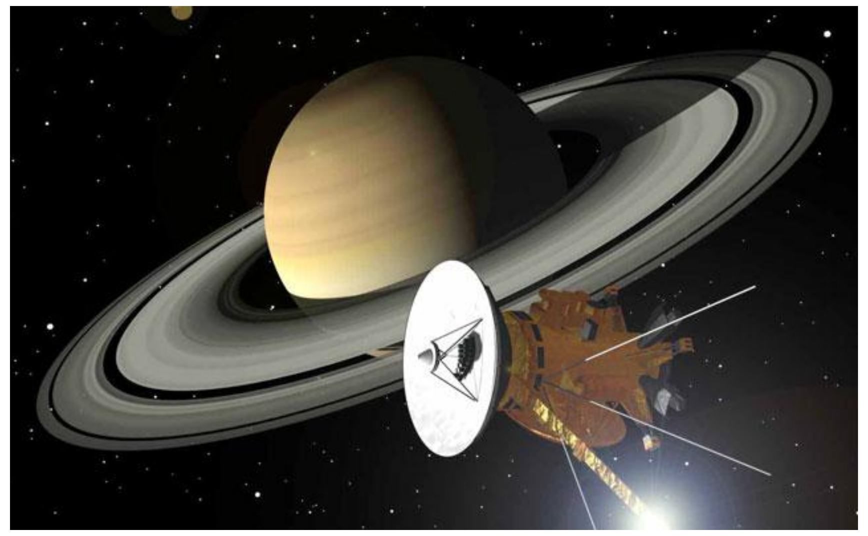
GW perturbs the photon propagation time between the test masses

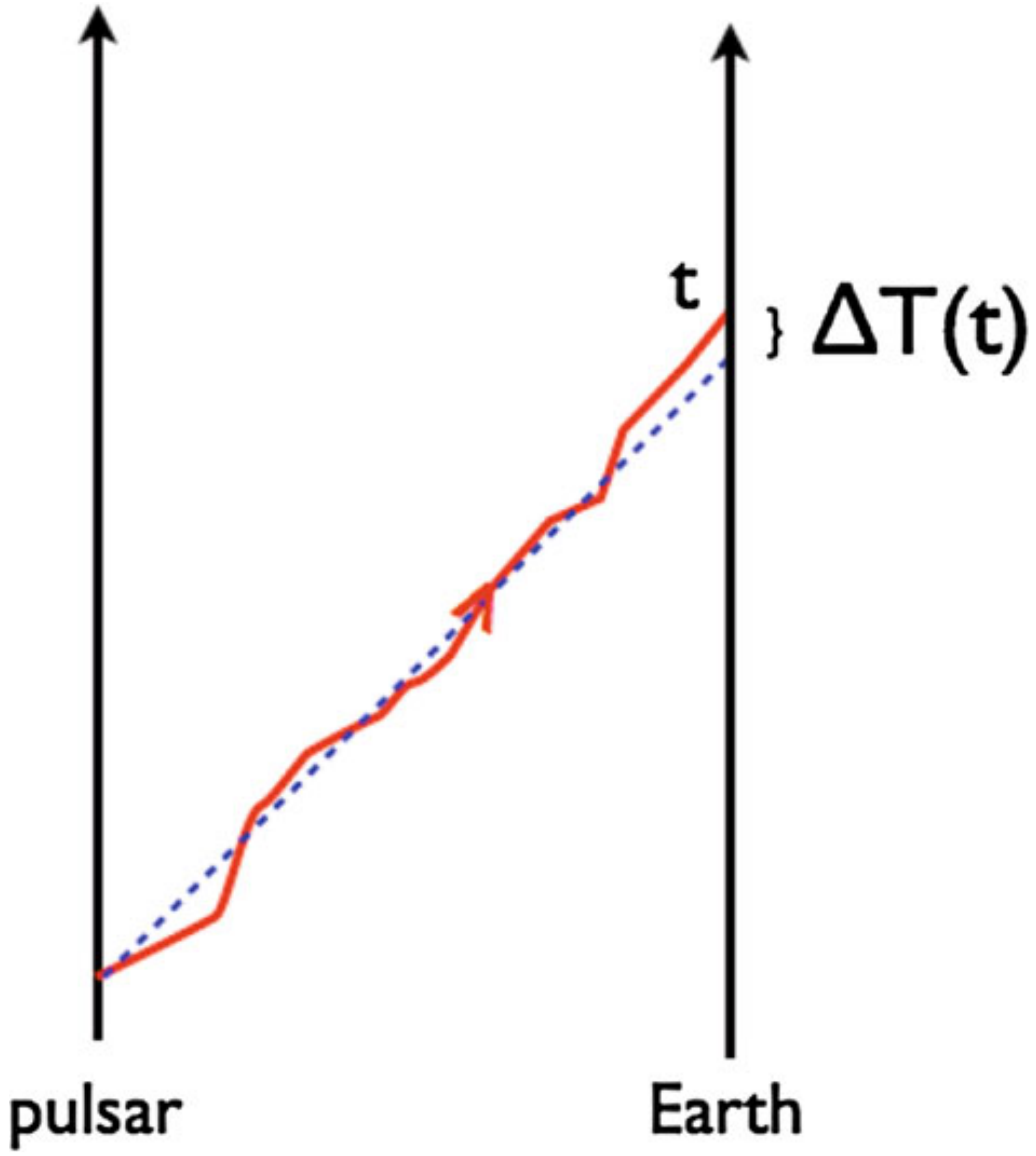




(1-arm, 1-way)







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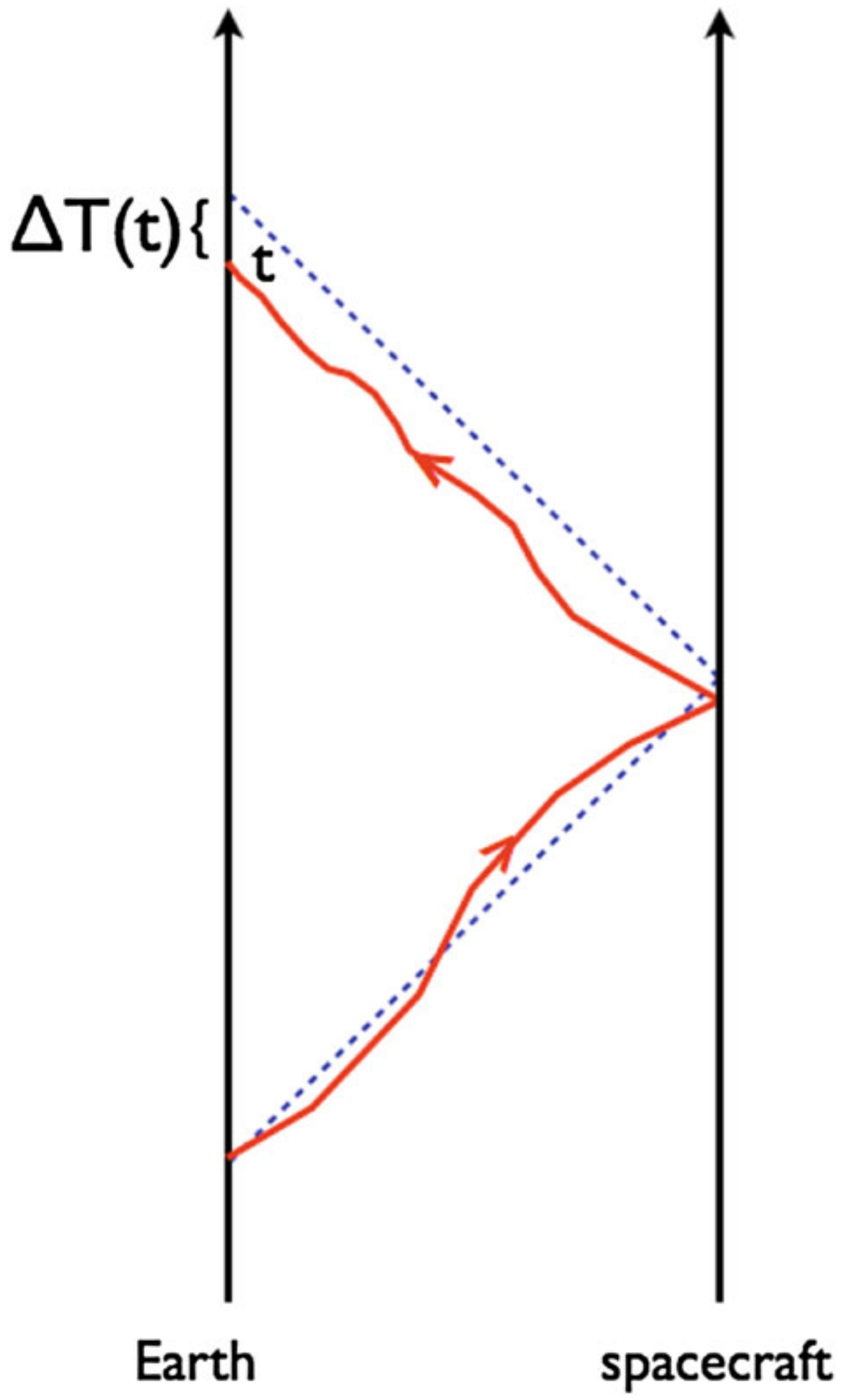
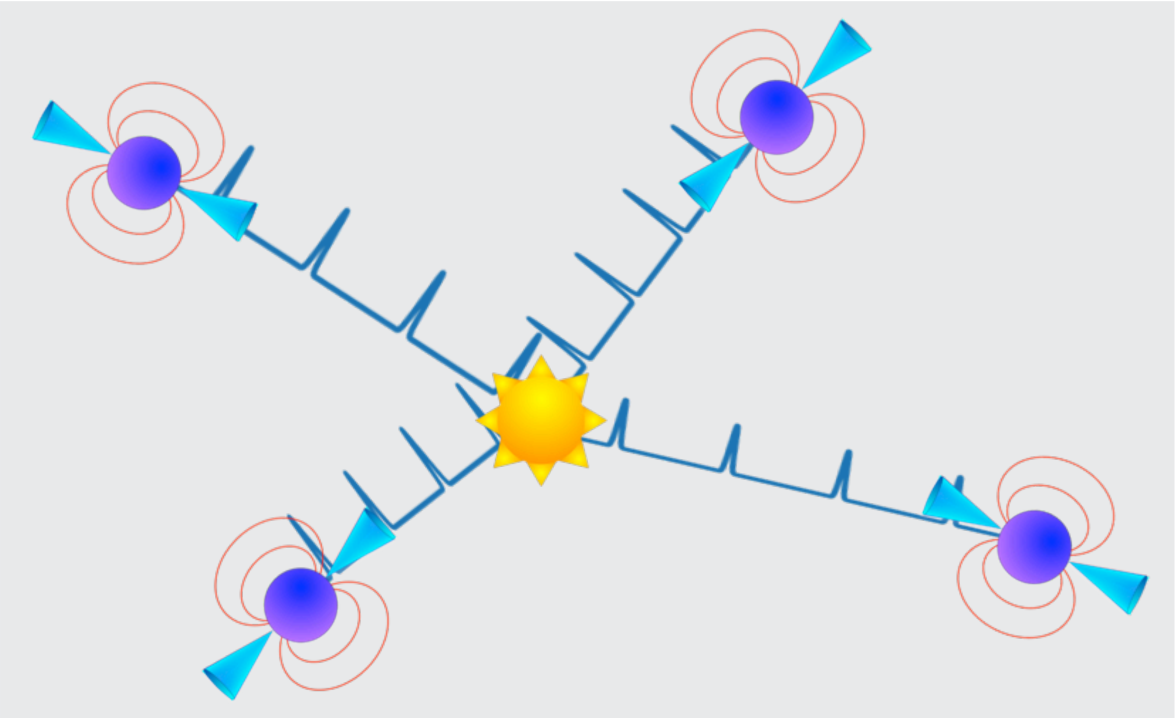
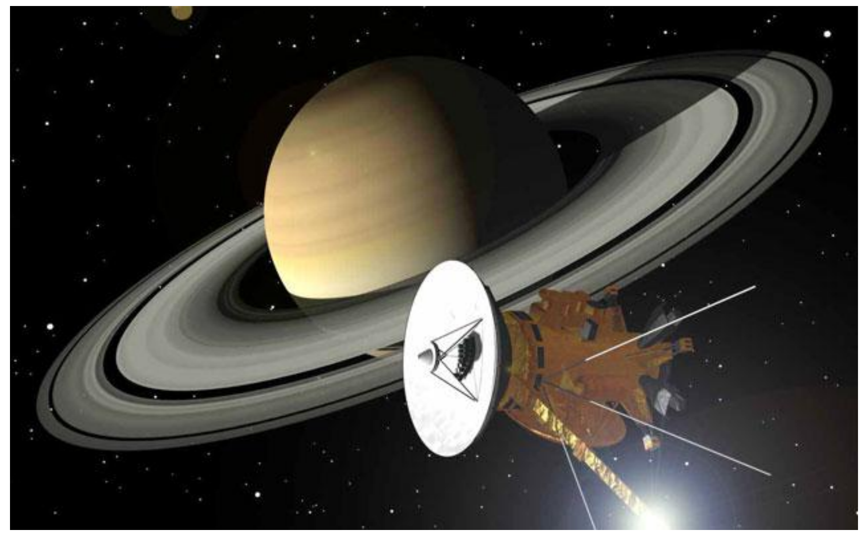
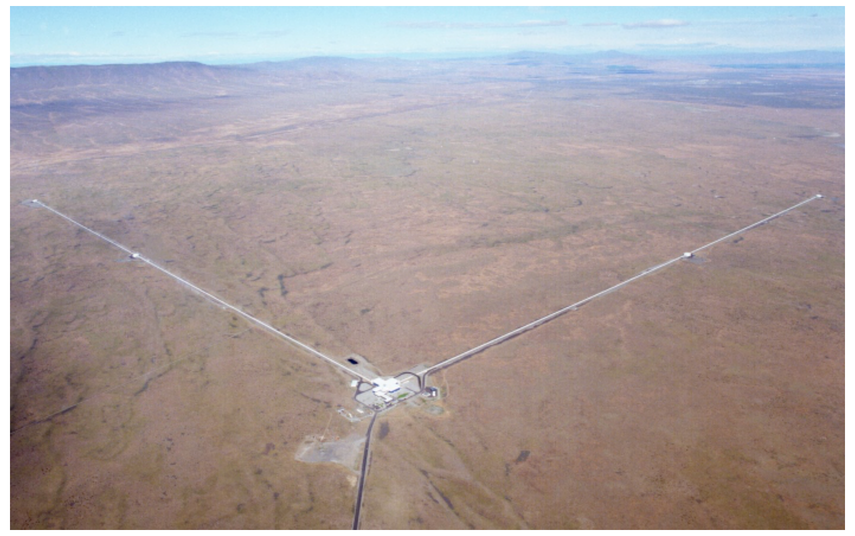
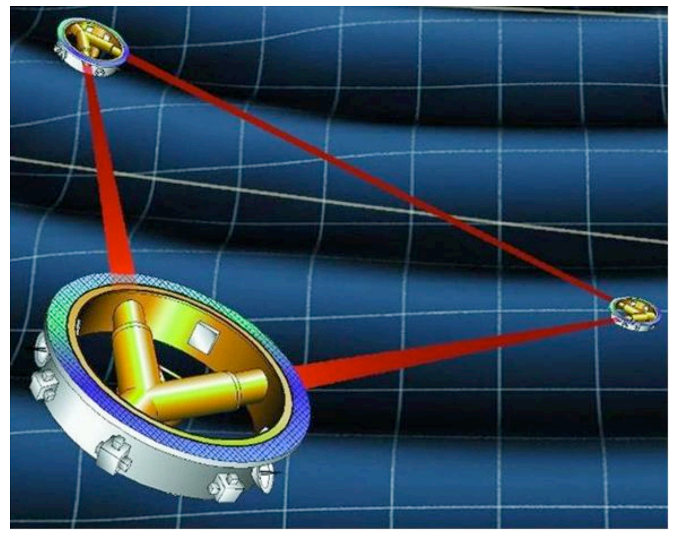


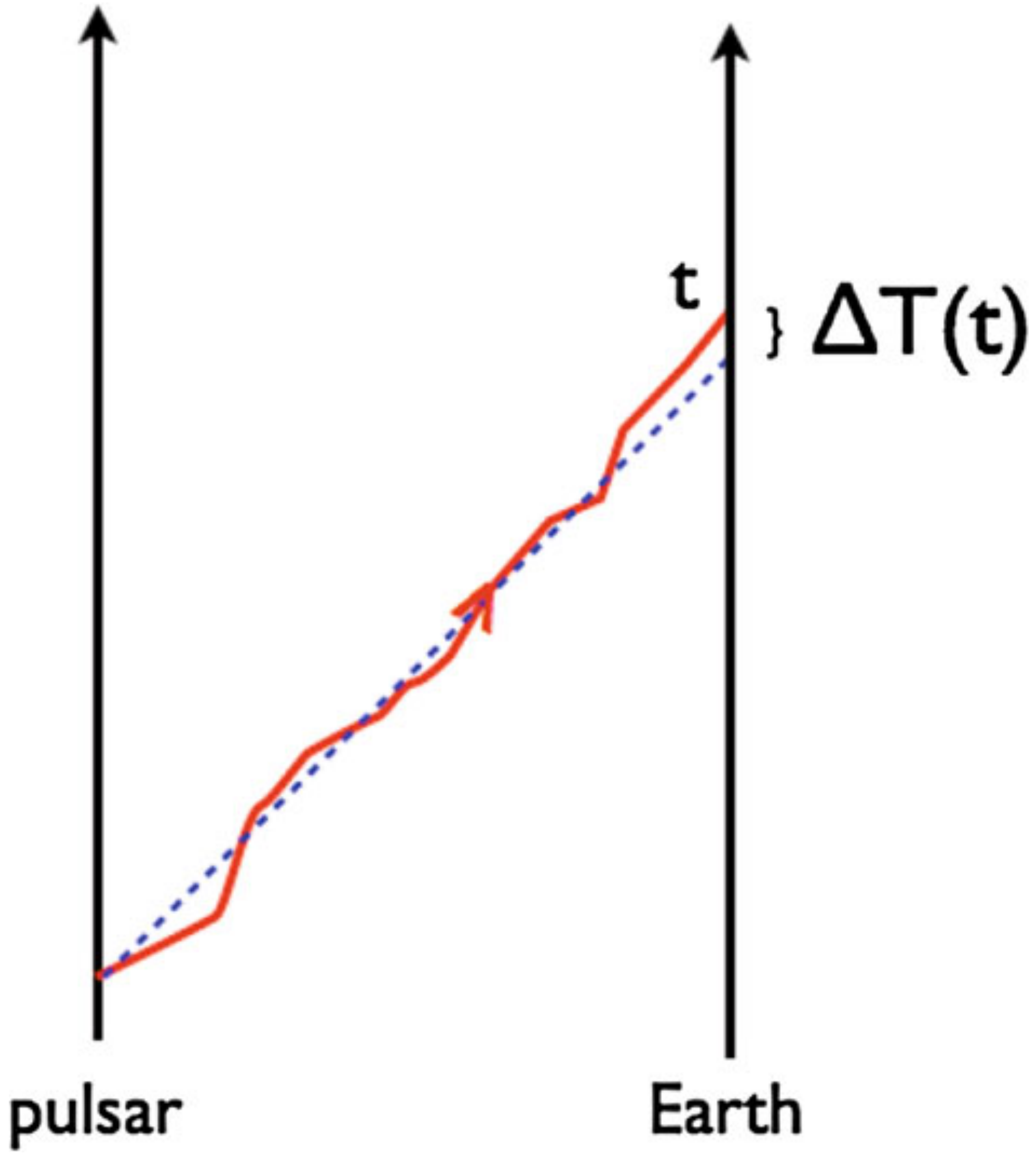
(1-arm, 2-way)



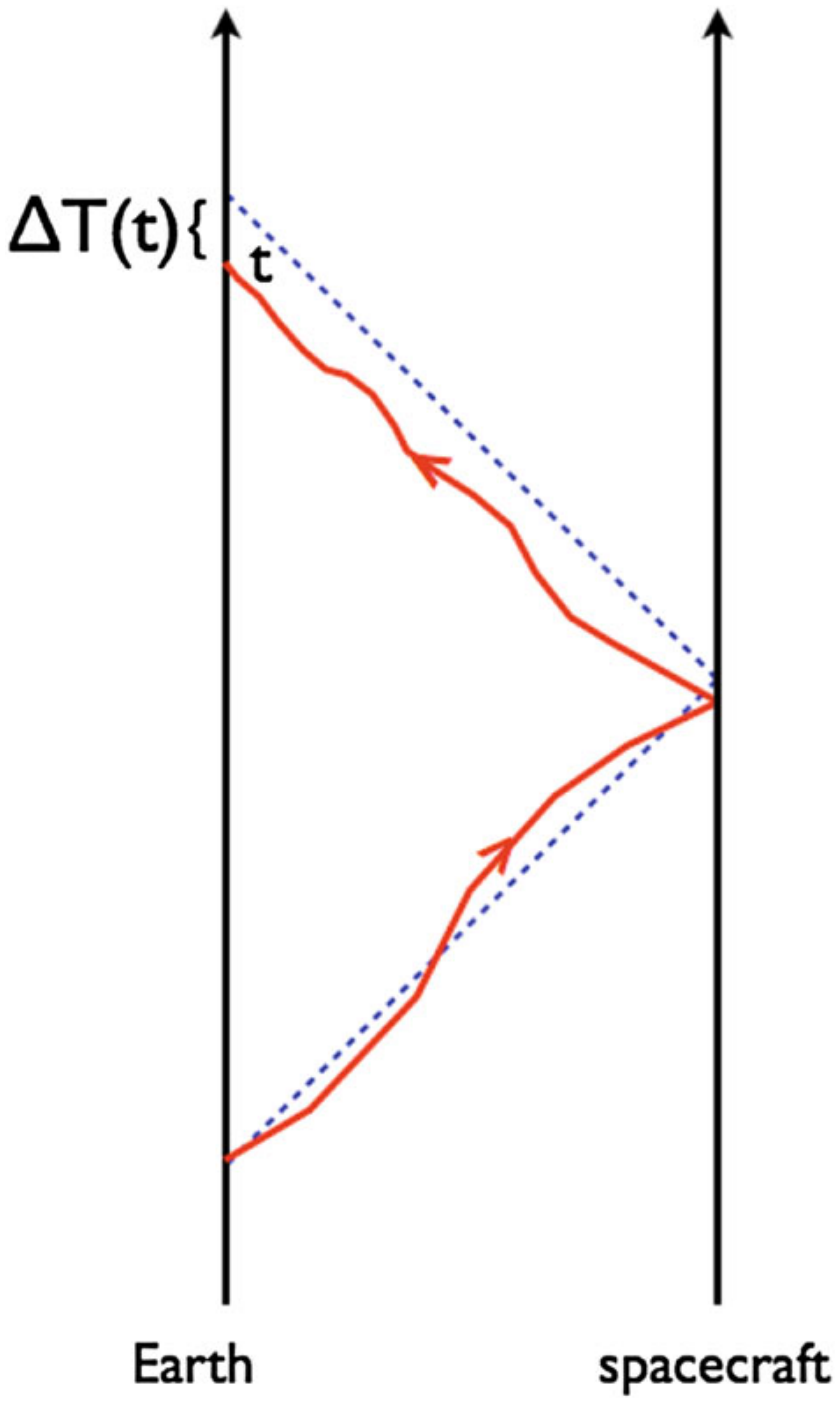




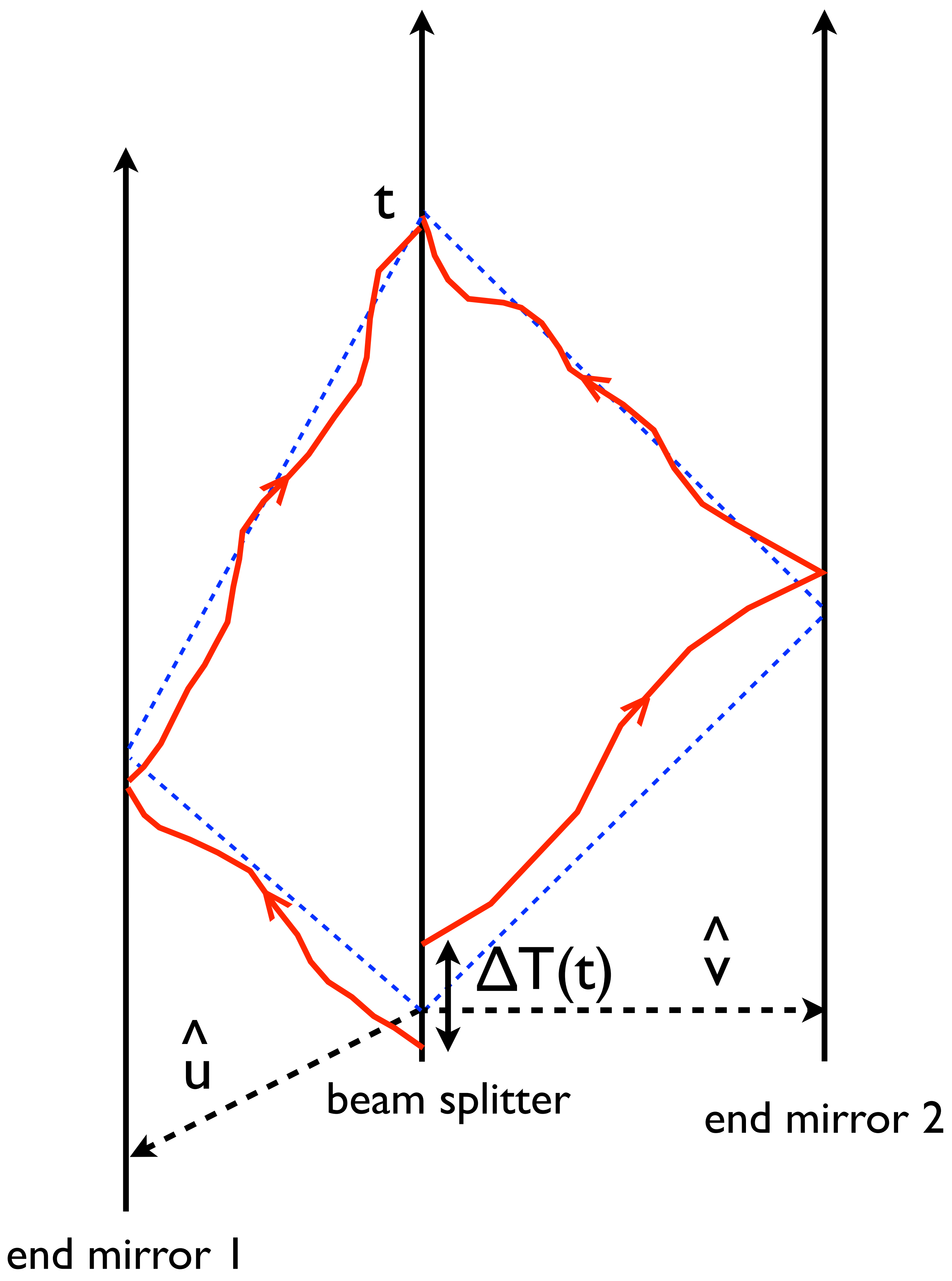




(1-arm, 1-way)



(1-arm, 2-way)



(2-arm, 2-way)

Detector response function

- GW detector is a linear system that converts metric perturbations to the output of a detector

detector

$$h_{ab}(t, \vec{x}) \longrightarrow \boxed{R^{ab}(\tau, \vec{y})} \longrightarrow h(t)$$

Detector response function

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$$h_{ab}(t, \vec{x}) \longrightarrow \boxed{R^{ab}(\tau, \vec{y})} \longrightarrow h(t)$$

convolution

$$r(t) = (\mathbf{R} * \mathbf{h})(t, \vec{x}) \equiv \int_{-\infty}^{\infty} \mathrm{d}\tau \int \mathrm{d}^3 y \, R^{ab}(\tau, \vec{y}) h_{ab}(t - \tau, \vec{x} - \vec{y})$$

detector output

detector location

impulse response

Detector response function

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detector output

detector location

impulse response

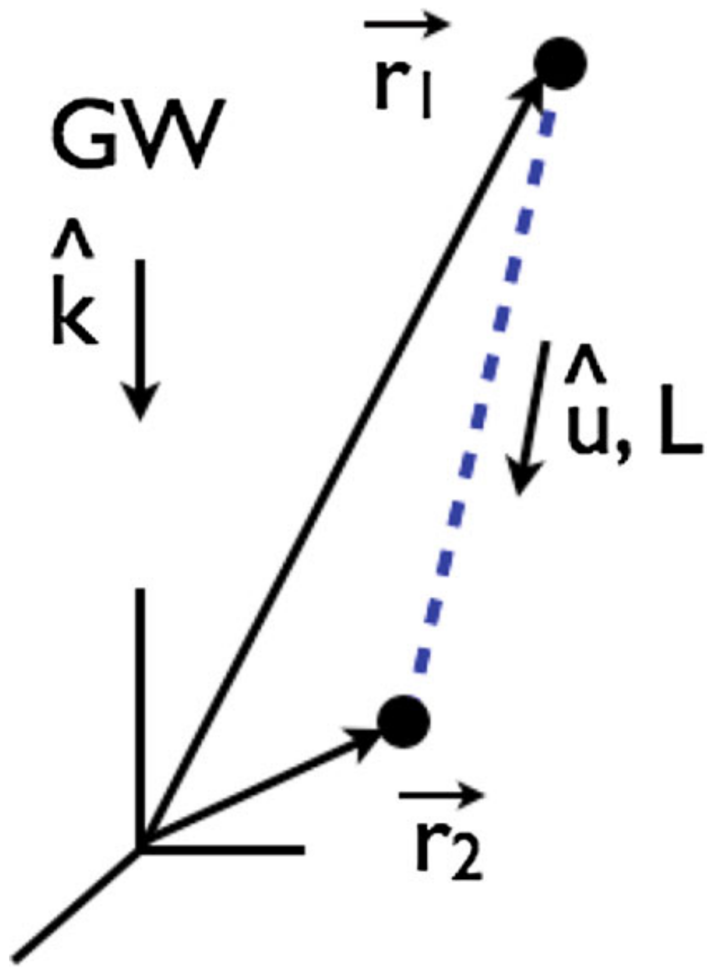
$$\Longrightarrow \quad \tilde{r}(f) = \int \mathrm{d}^2\Omega_{\hat{k}} \sum_A \boxed{R^A(f, \hat{k})} h_A(f, \hat{k})$$

detector response for a plane-wave
with frequency f, direction n, polarization A

$$R^A(f, \hat{k}) \equiv R^{ab}(f, \hat{k}) e^A_{ab}(\hat{k})$$

$$R^{ab}(f, \hat{k}) \equiv e^{-i2\pi f \hat{k}\cdot\vec{x}/c} \int_{-\infty}^{\infty} \mathrm{d}\tau \int \mathrm{d}^3 y \, R^{ab}(\tau, \vec{y}) \, e^{-i2\pi f(\tau - \hat{k}\cdot\vec{y}/c)}$$

Example: One-arm, one-way detector (e.g., pulsar timing)



$$r(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L \mathrm{d}s \, h_{ab}(t(s), \overrightarrow{x}(s))$$

$$t(s) = (t - L/c) + s/c \,, \qquad \overrightarrow{x}(s) = \vec{r}_1 + s\hat{u}$$

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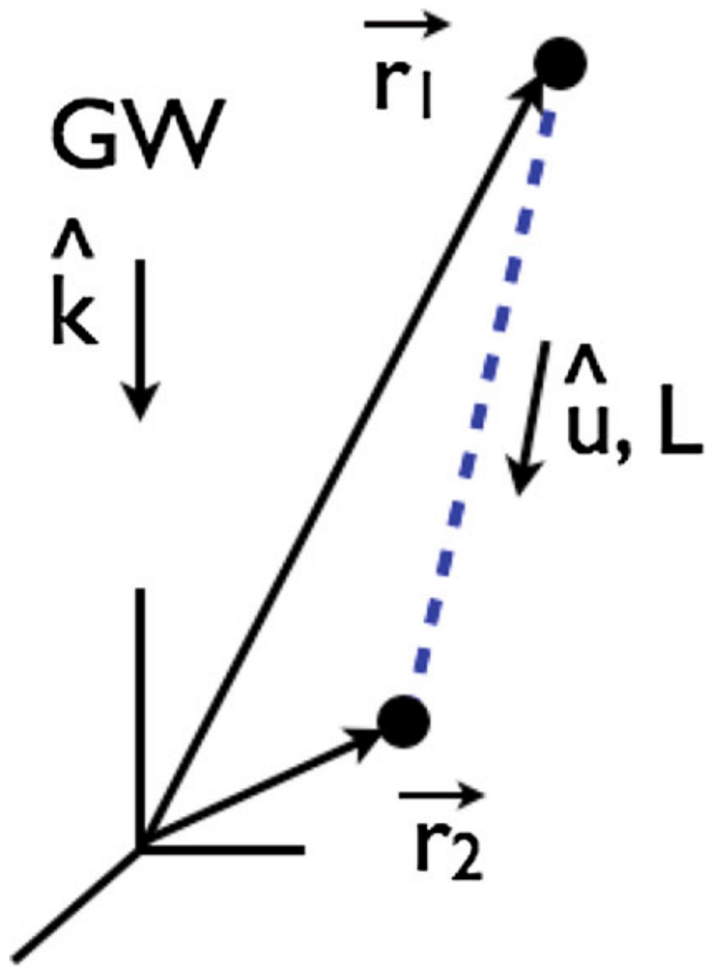

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$$R^A(f, \hat{k}) = \frac{1}{i2\pi f} \frac{1}{2} u^a u^b e^A_{ab}(\hat{k}) \frac{1}{1 - \hat{k}\cdot\hat{u}} \left[1 - e^{-\frac{i2\pi f L}{c}(1-\hat{k}\cdot\hat{u})}\right] e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}$$

<u>Exercise</u>: Verify the above expression for the response function

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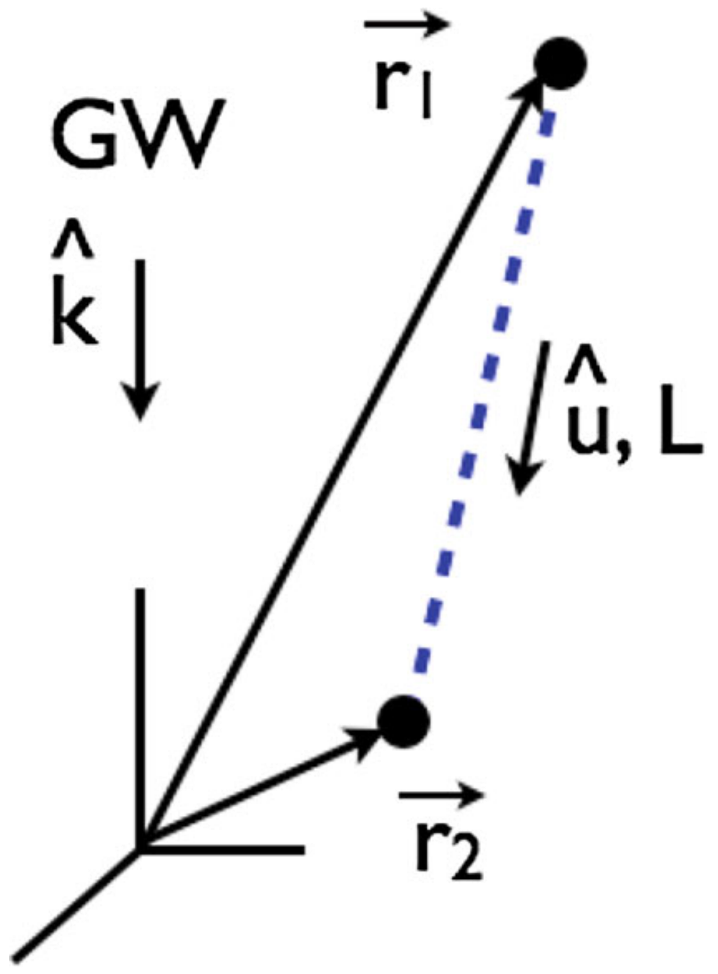

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earth term

pulsar term

(typically ignored for pulsar timing)

$$R^A(f, \hat{k}) = \frac{1}{i2\pi f} \underbrace{\boxed{\frac{1}{2} u^a u^b e^A_{ab}(\hat{k}) \frac{1}{1 - \hat{k}\cdot\hat{u}}}}_{F^A(\hat{k})} \left[1 - e^{-\frac{i2\pi f L}{c}(1 - \hat{k}\cdot\hat{u})}\right] e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}$$

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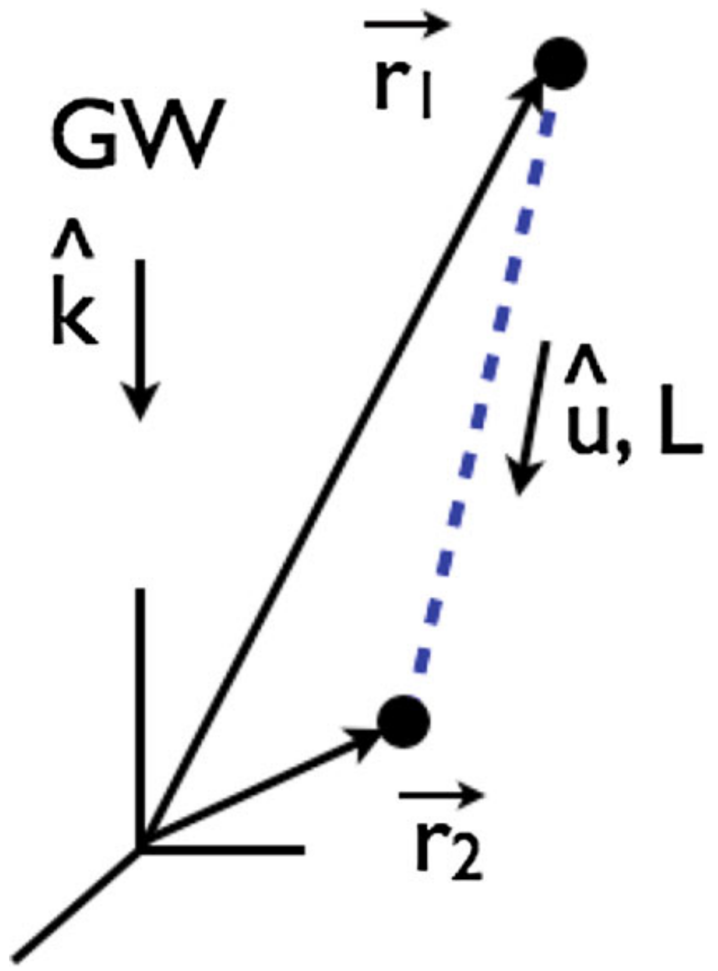

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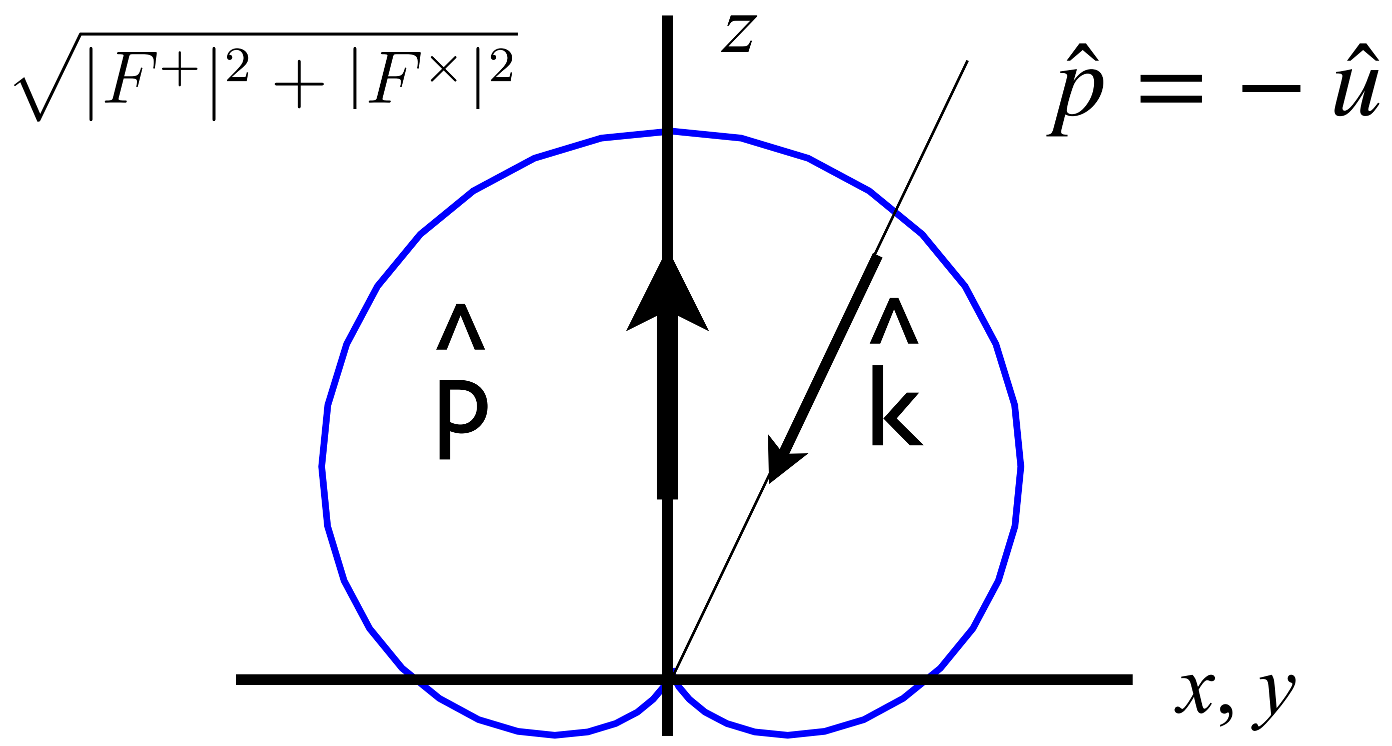
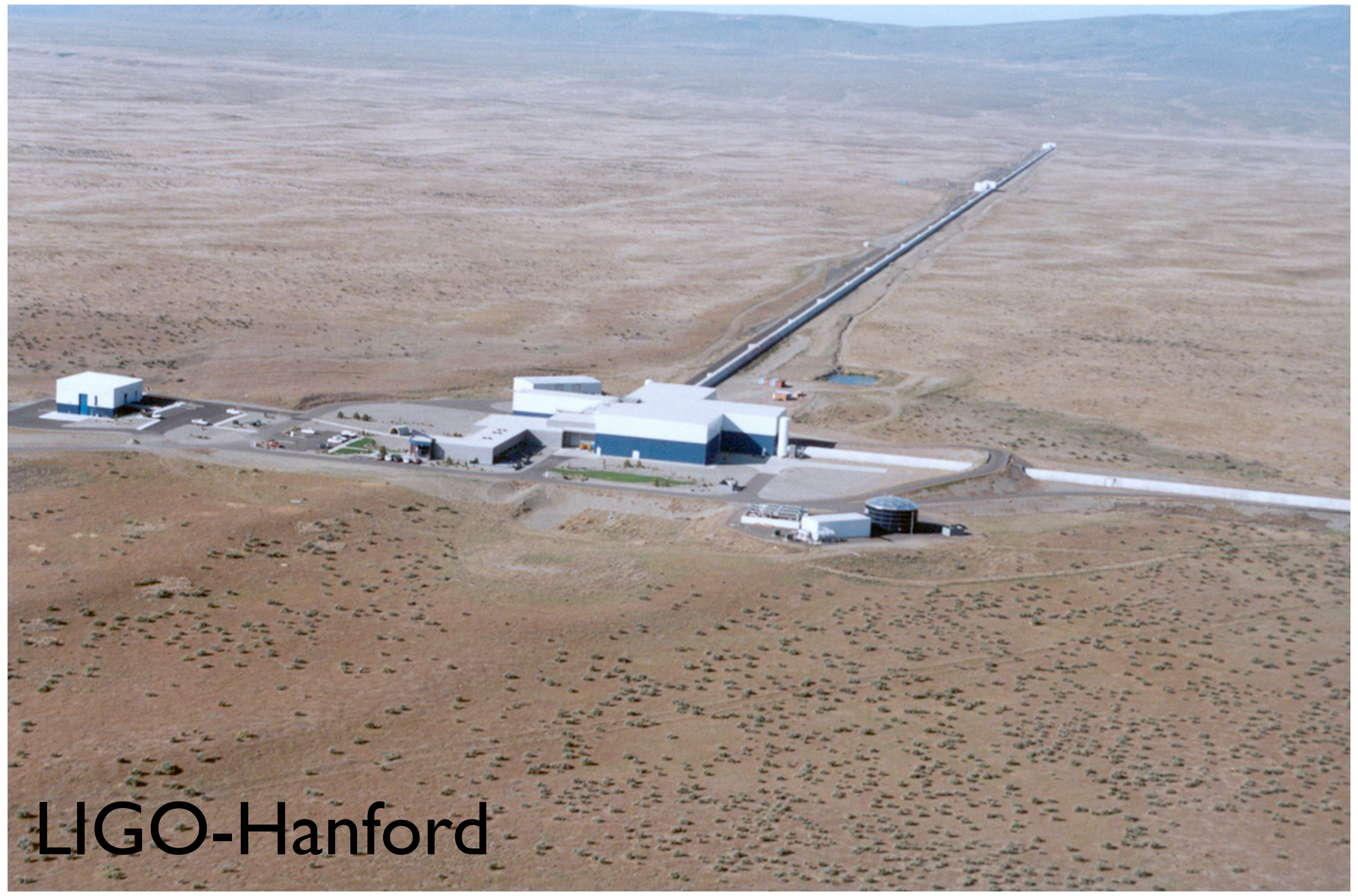


Exercise: Verify the above expression for the response function

Example: Ground-based interferometers (short-antenna, equal arm limit)



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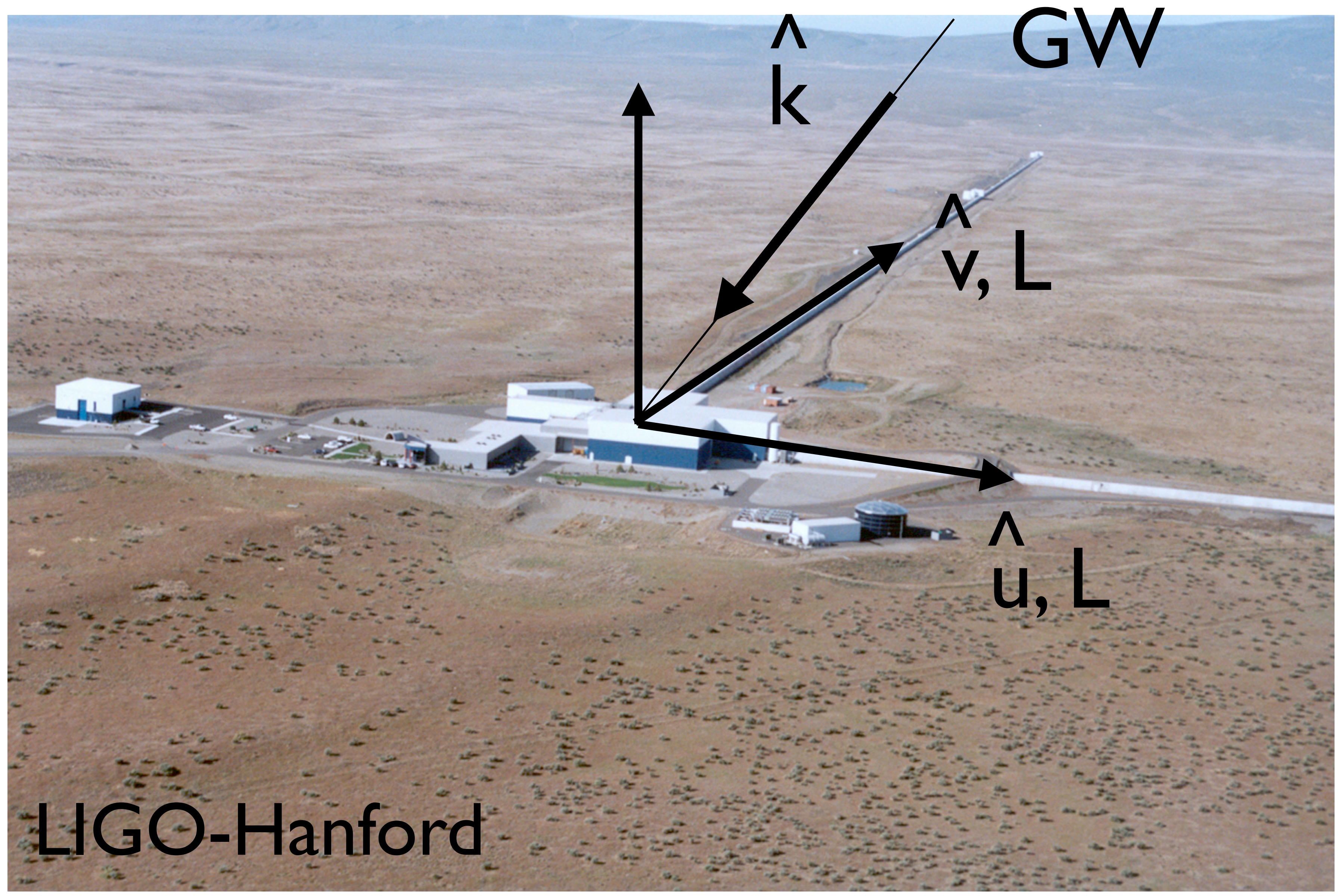

$$r(t) = \frac{1}{2}\left(\frac{\Delta T_{\vec{u},\,\mathrm{roundtrip}}(t)}{T} - \frac{\Delta T_{\vec{v},\,\mathrm{roundtrip}}(t)}{T}\right)$$

$$R^A(f,\hat{k}) \simeq \underbrace{\boxed{\frac{1}{2}\left(u^a u^b - v^a v^b\right)}}_{\text{detector tensor}}\, e^A_{ab}(\hat{k})\mathrm{e}^{-i2\pi f\hat{k}\cdot\vec{x}/c}$$

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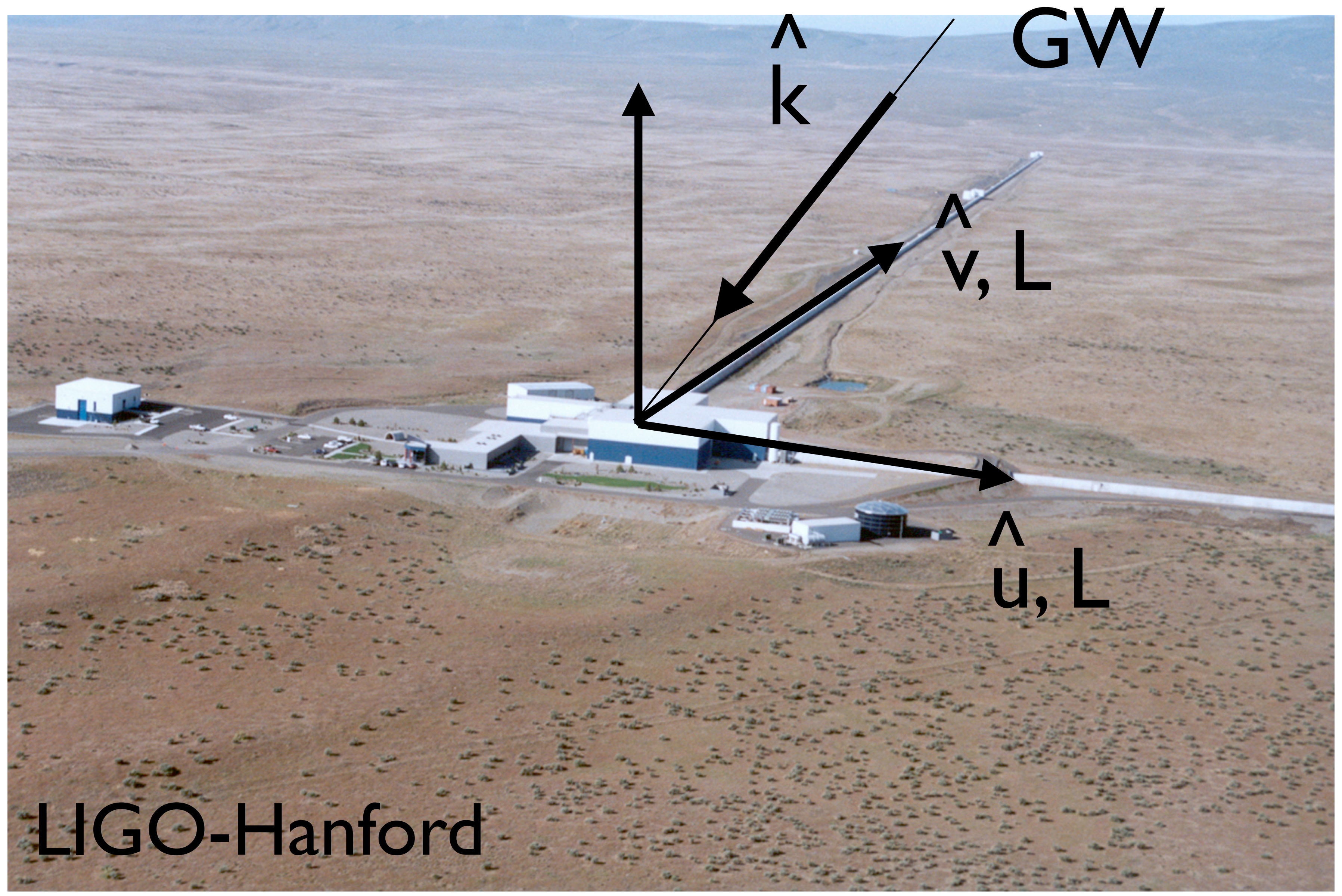

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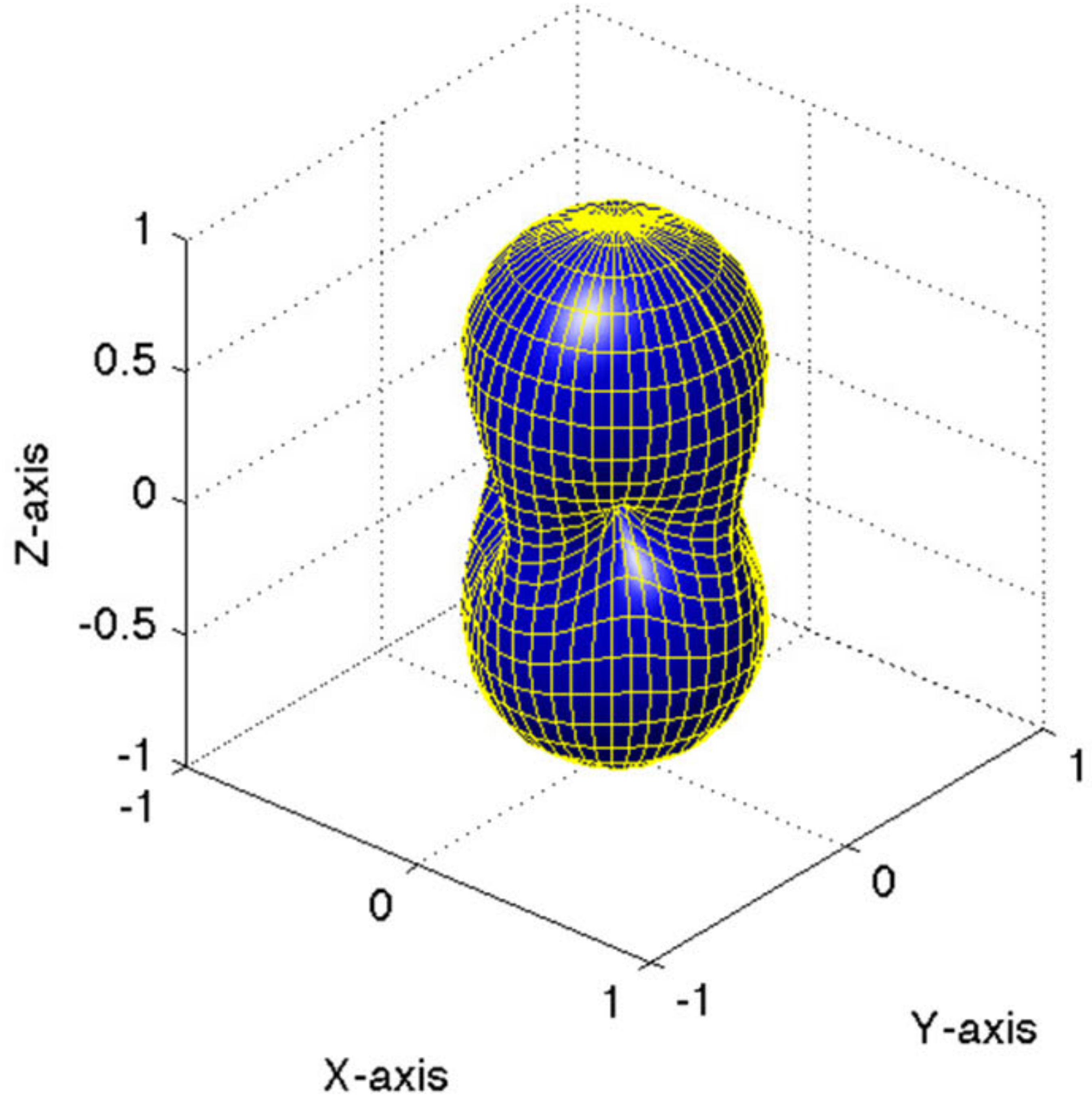
$$\sqrt{|R^+|^2 + |R^\times|^2}$$



IV. Overlap reduction functions

General definition of an ORF (correlation coefficient)

- The ORF between two detectors a and b is defined as the transfer function relating the **expected cross-correlated power** in the two detectors to the strain power in the GWB:

$$C_{IJ}(f) = \Gamma_{IJ}(f) S_h(f) \quad \text{where } I, J \text{ label detectors}$$

- The ORF encodes the reduction in sensitivity due to the physical **separation** and **misalignment** of the two detectors
- Can be written as the **sky and polarization-averaged product of detector response functions**:

$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_I^A(f, \hat{k}) R_J^{A\star}(f, \hat{k}) \qquad \text{where} \quad \tilde{r}_I(f) = \int d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f, \hat{k}) R_I^A(f, \hat{k})$$

Simple example: Omni-directional detectors

$$R_I^A(f,\hat{k}) = G\,\mathrm{e}^{-i2\pi f\hat{k}\cdot\vec{x}_I/c}$$

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$$= \frac{G^2}{4\pi}\int \mathrm{d}^2\Omega_{\hat{k}}\,\mathrm{e}^{-i\frac{2\pi f D_{IJ}}{c}\hat{k}\cdot\hat{\Delta}x_{IJ}}$$

$$= \frac{G^2}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}(\cos\theta)\,\mathrm{e}^{i\frac{2\pi f D_{IJ}}{c}\cos\theta}$$

$$\ldots$$

$$= G^2\,\mathrm{sinc}\left(\frac{2\pi f D_{IJ}}{c}\right)$$

where

$$\vec{x}_I - \vec{x}_J \equiv D_{IJ}\,\hat{\Delta}x_{IJ}$$

$$\mathrm{sinc}\,x \equiv \frac{\sin x}{x}$$

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$$\ldots$$

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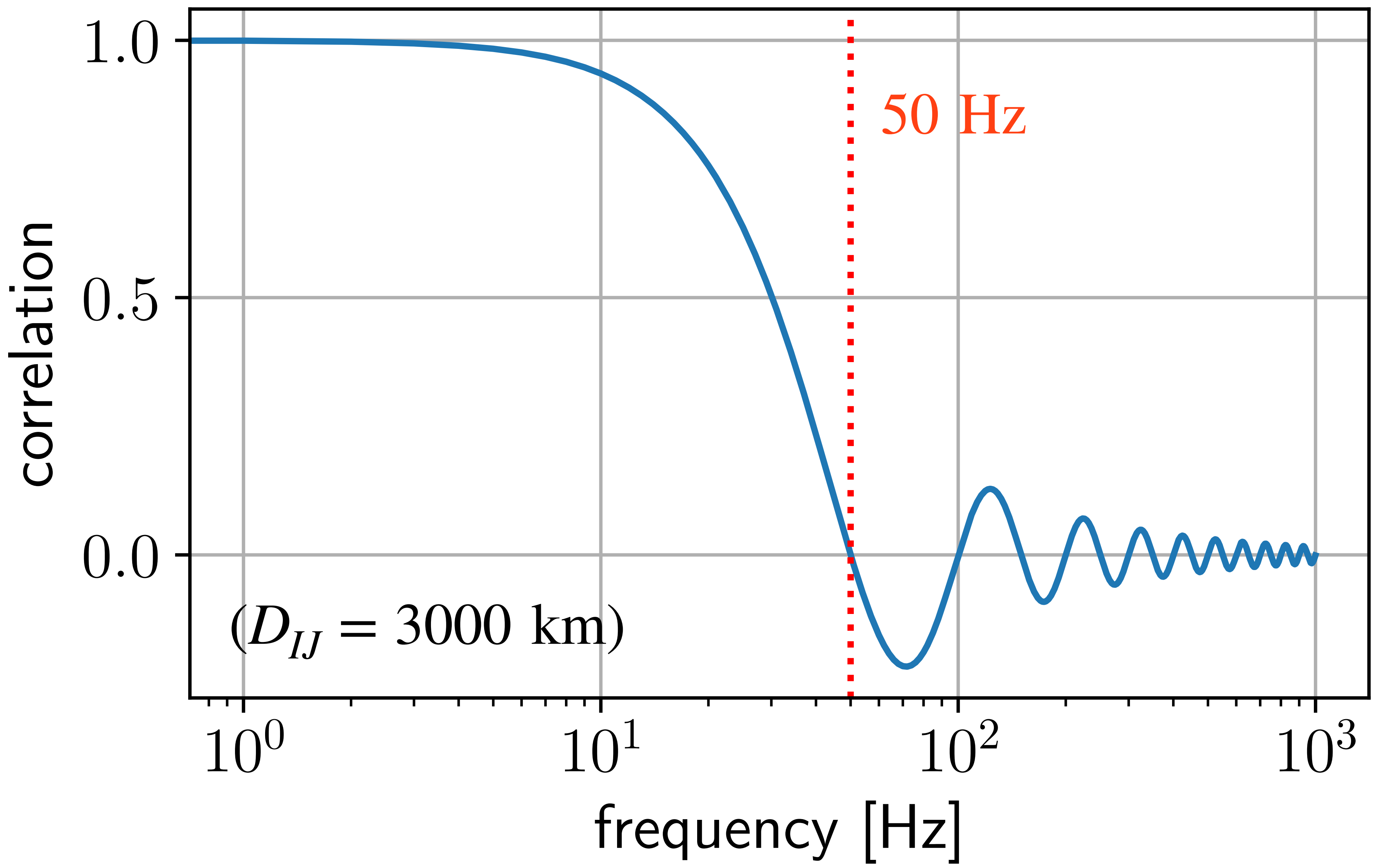
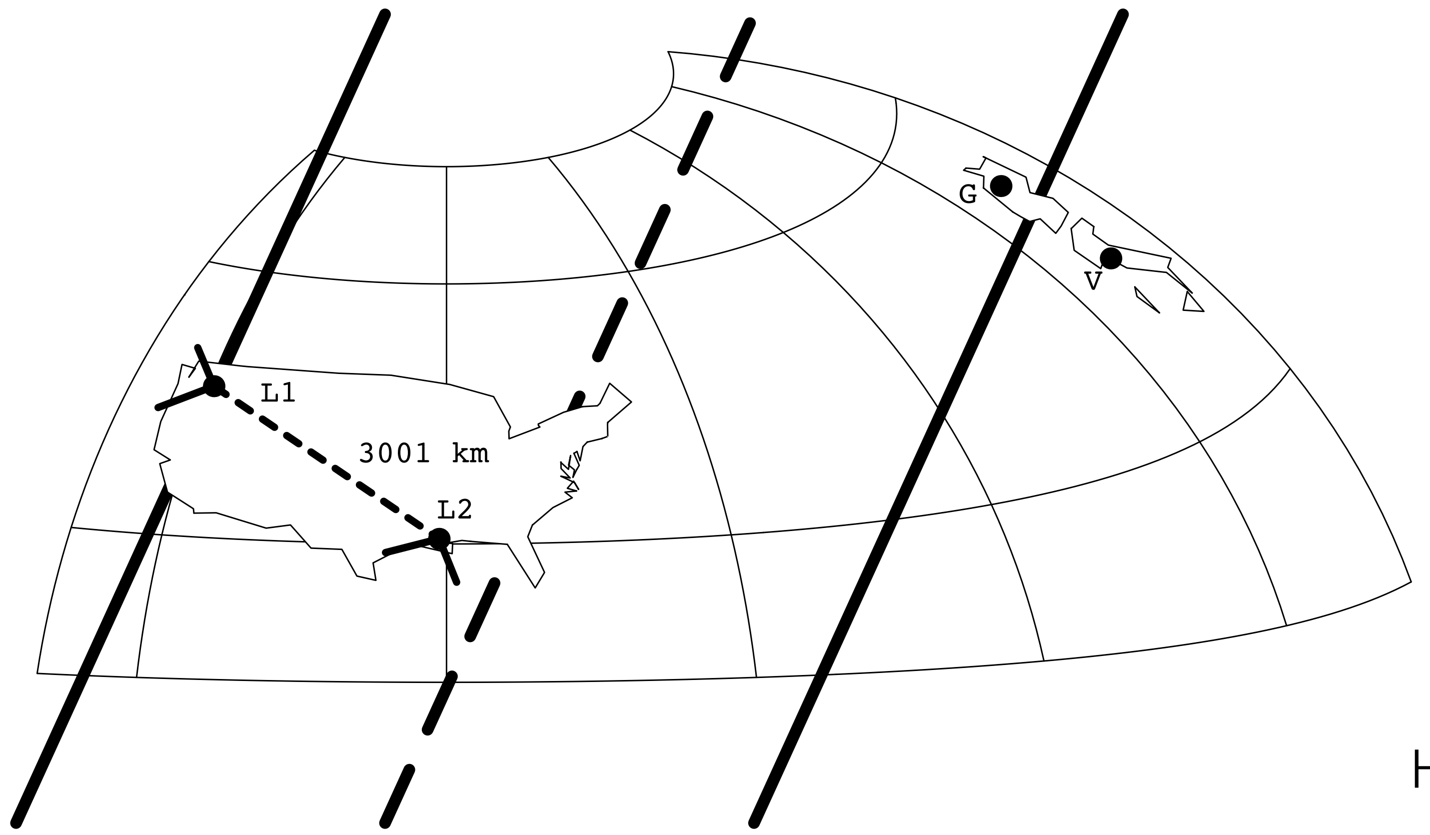


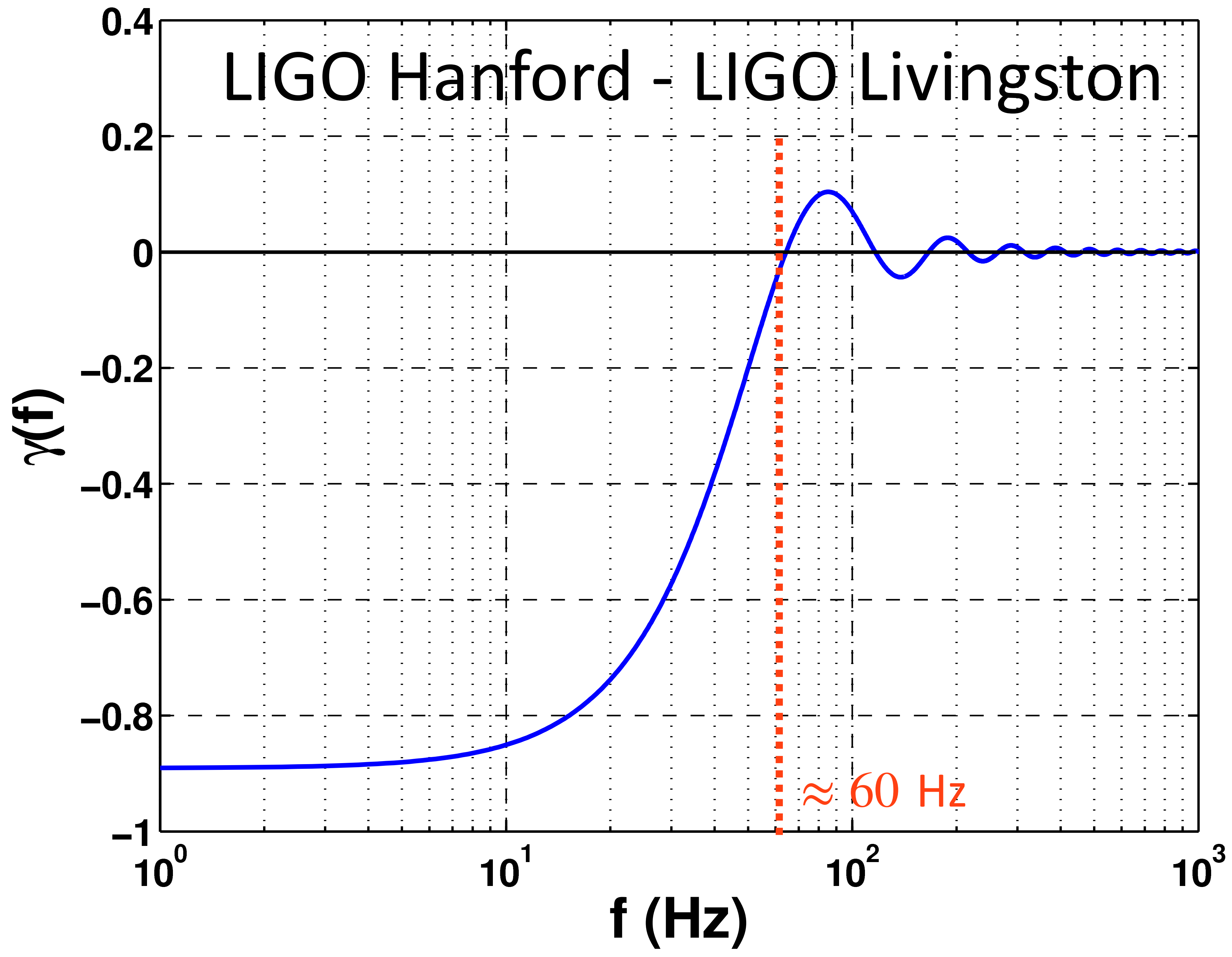
Example: Ground-based interferometers

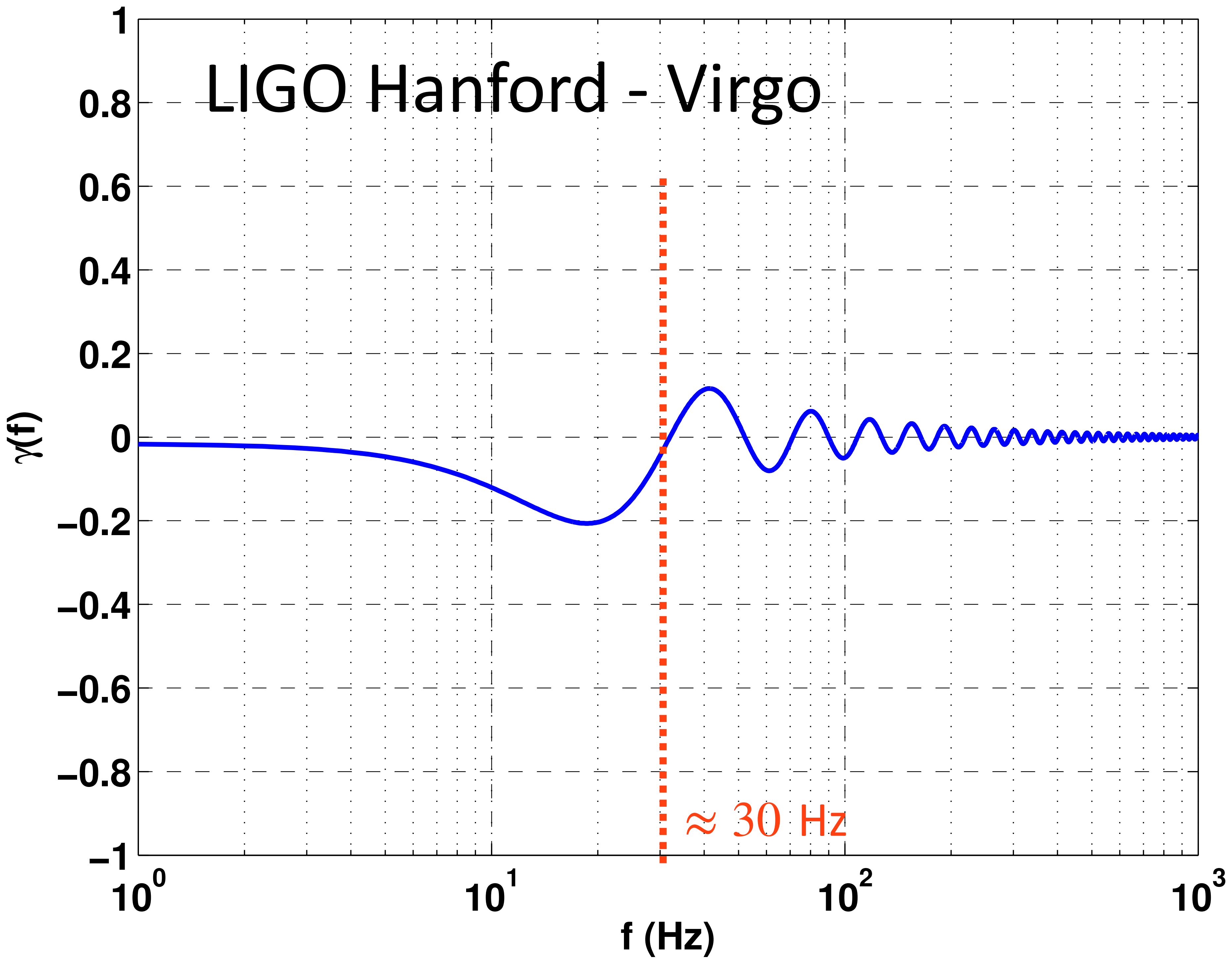
$$R_I^A(f,\hat{k}) \simeq \frac{1}{2}(u_I^a u_I^b - v_I^a v_I^b)e_{ab}^A(\hat{k})\,\mathrm{e}^{-i2\pi f\hat{k}\cdot\vec{x}_I/c}$$

(short-antenna limit)



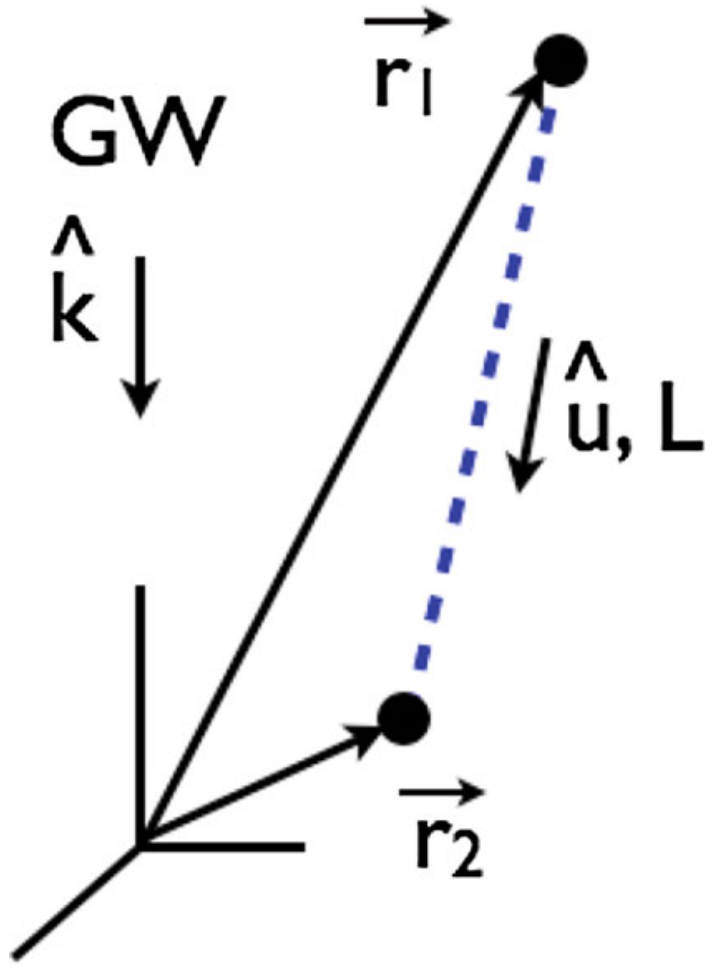
(B. Allen, Les Houches 1995)





Example: Pulsar timing arrays

- Recall timing residual response:



$$r(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L \mathrm{d}s \, h_{ab}(t(s), \vec{x}(s))$$

$$R^A(f, \hat{k}) = \frac{1}{i2\pi f} \frac{1}{2} \frac{u^a u^b}{1 - \hat{k} \cdot \hat{u}} e^A_{ab}(\hat{k}) \left[1 - \mathrm{e}^{-\frac{i2\pi f L}{c}(1 - \hat{k} \cdot \hat{u})} \right] \mathrm{e}^{-i2\pi f \hat{k} \cdot \vec{r}_2 / c}$$

Example: Pulsar timing arrays

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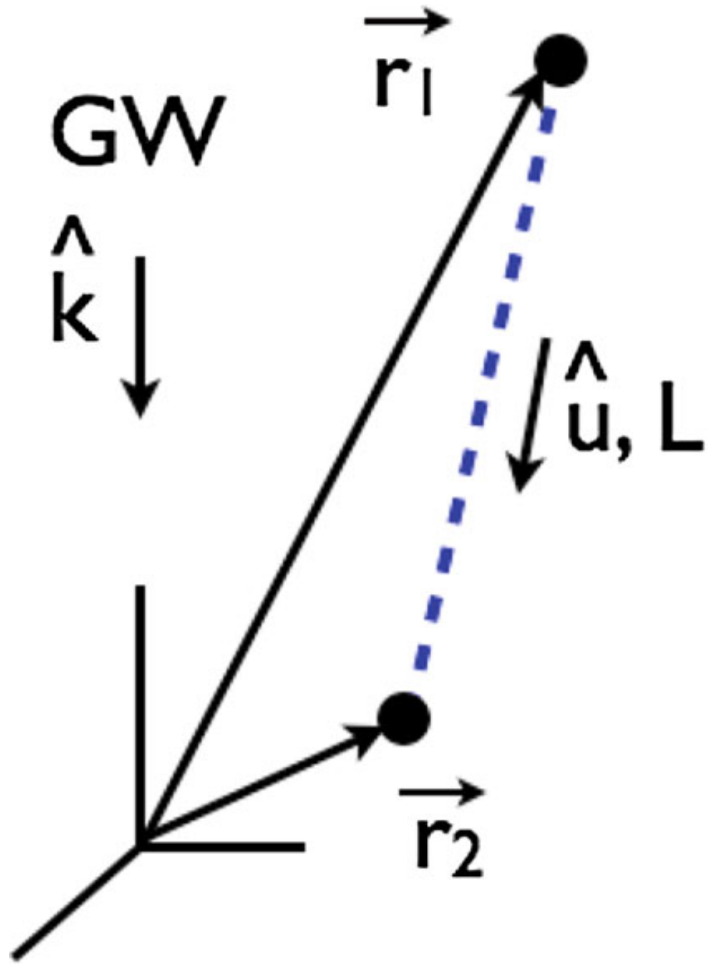

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- Simplification for redshift response $\Delta\nu(t)/\nu \equiv \mathrm{d}\Delta T/\mathrm{d}t$:

$$R^A_I(f, \hat{k}) = F^A_I(k)\left[1 - \mathrm{e}^{-\frac{i2\pi f L_I}{c}(1+\hat{k}\cdot\hat{p}_I)}\right] \qquad F^A_I(\hat{k}) \equiv \frac{1}{2} \frac{p^a_I p^b_I}{1 + \hat{k}\cdot\hat{p}_I} e^A_{ab}(\hat{k})$$

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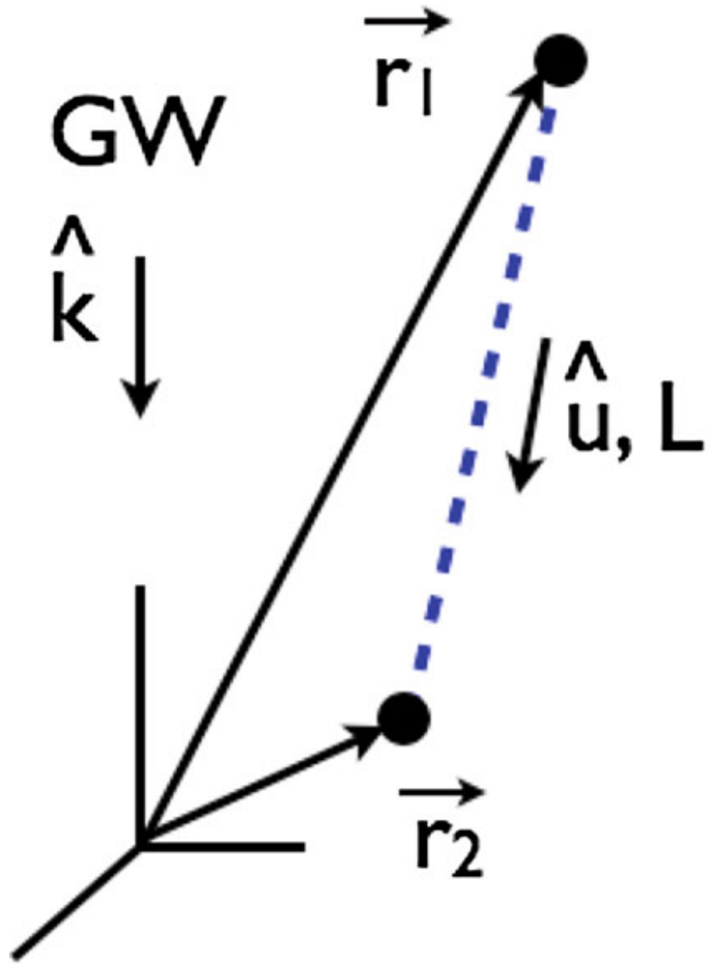

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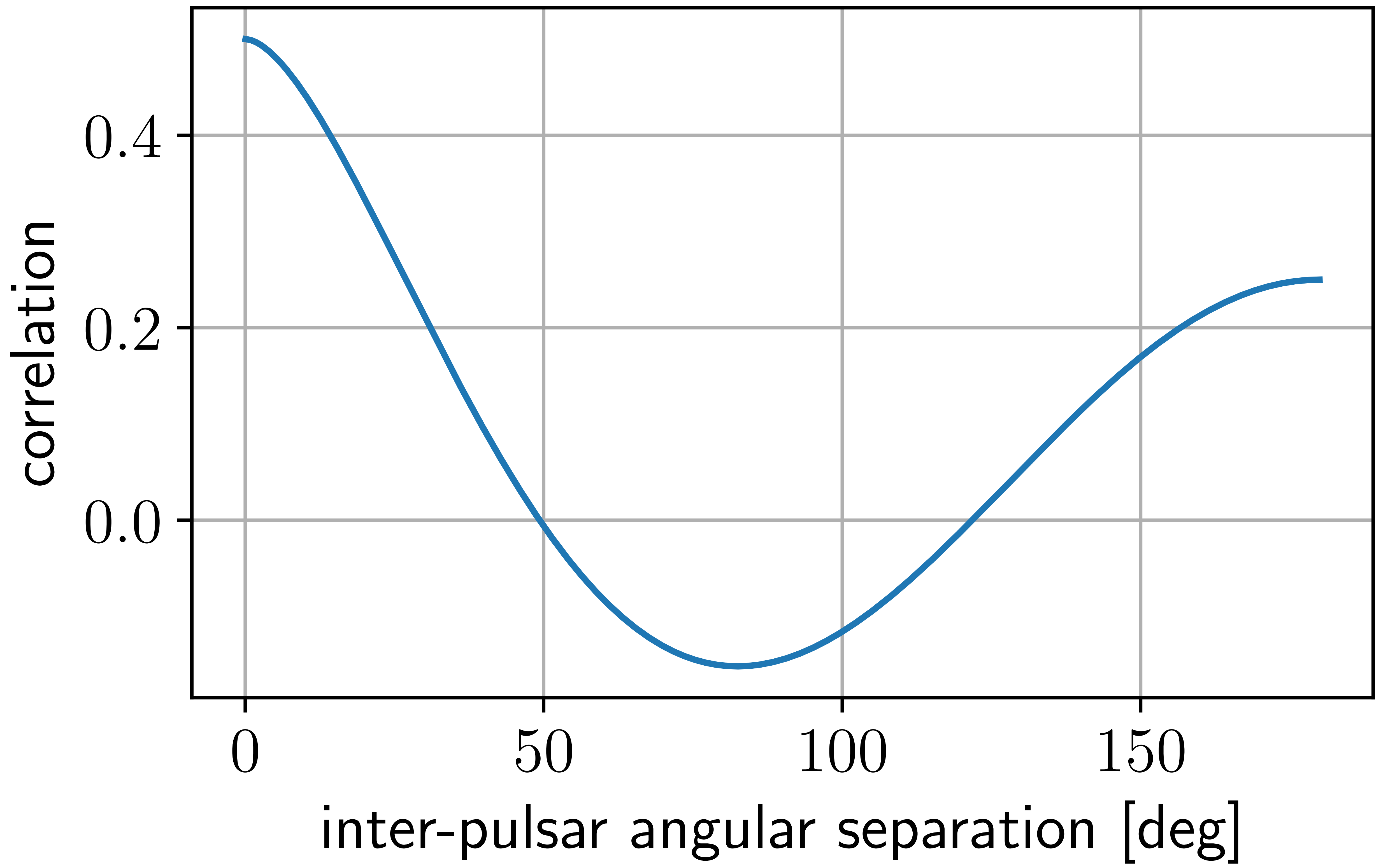
$$\Gamma_{IJ}(f) = \frac{1}{8\pi}\int \mathrm{d}^2\Omega_{\hat{k}} \sum_A F^A_I(\hat{k}) F^A_J(\hat{k})\left[1 - \cancel{\mathrm{e}^{-\frac{i2\pi f L_I}{c}(1 + \hat{k}\cdot\hat{p}_I)}}\right]\left[1 - \cancel{\mathrm{e}^{+\frac{i2\pi f L_J}{c}(1 + \hat{k}\cdot\hat{p}_J)}}\right] \simeq \frac{1}{8\pi}\int \mathrm{d}^2\Omega_{\hat{k}} \sum_A F^A_I(\hat{k}) F^A_J(\hat{k})[1 + \delta_{IJ}]$$

Hellings and Downs correlation

- Normalizing to 0.5 for zero angular separation:

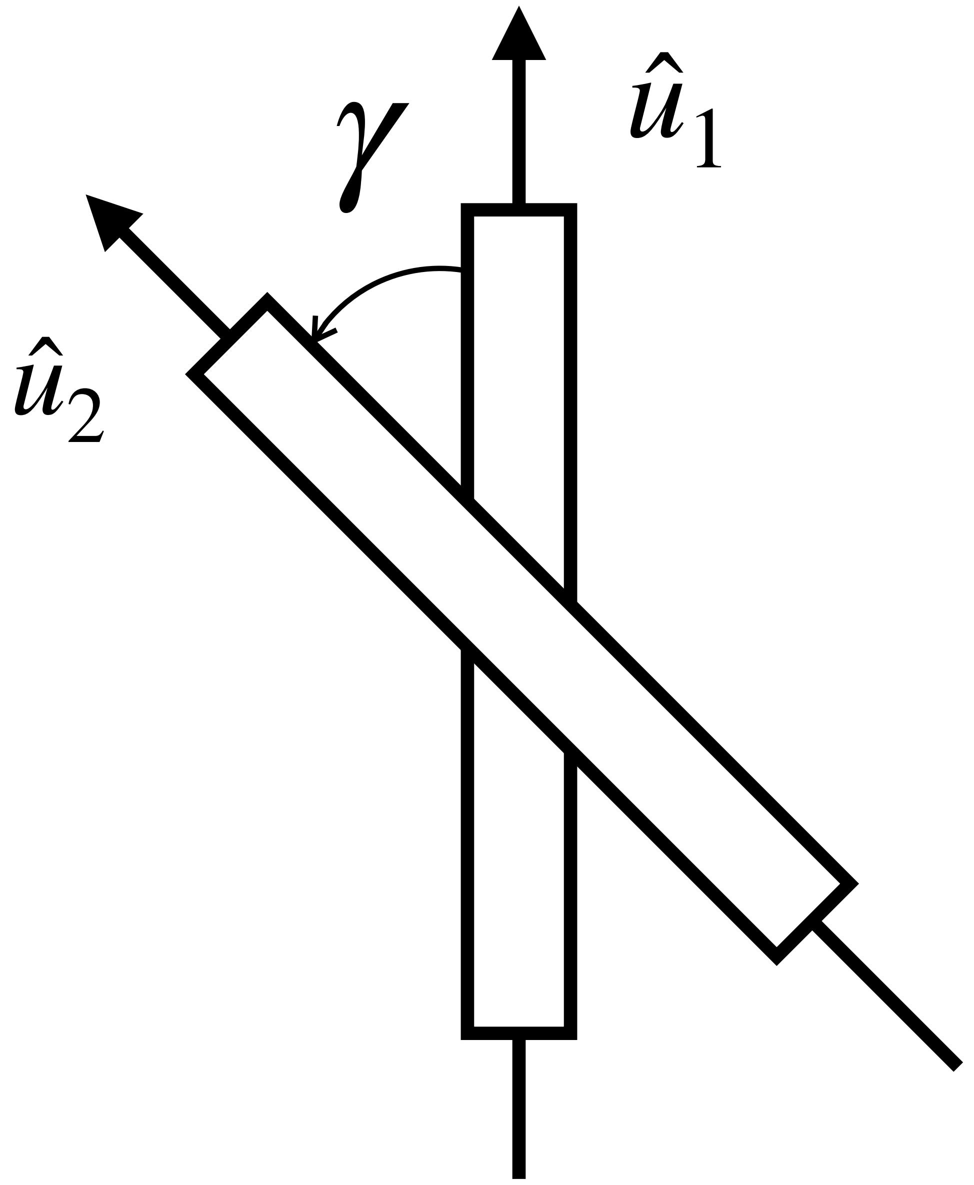
$$\chi_{IJ} = \frac{3}{2}\left(\frac{1-\cos\gamma_{IJ}}{2}\right)\left[\ln\left(\frac{1-\cos\gamma_{IJ}}{2}\right)-\frac{1}{6}\right]+\frac{1}{2}+\frac{1}{2}\delta_{IJ}$$



Exercises / worked examples
(please see exercises.pdf)

Exercise

- Calculate the value of the ORF for a pair of short electric dipole antennae pointing in directions $\hat{u}_1$ and $\hat{u}_2$ for an unpolarized, isotropic electromagnetic field



$$r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x} = \vec{0}) \quad \text{for} \quad I = 1,2$$

$$\vec{E}(t, \vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{\alpha=1,2} \tilde{E}_\alpha(f, \hat{k})\hat{\epsilon}_\alpha(\hat{k})\, \mathrm{e}^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi}\,, \quad \hat{\epsilon}_2(\hat{k}) = -\hat{\theta}$$

- The final result should depend very simply on γ !

Extra slides

IV. Cross correlation in frequentist (maximum-likelihood) and Bayesian analyses

Likelihood functions

- Recall that a likelihood function is a starting point for both frequentist & Bayesian analyses:

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$$p(d \,|\, S_{n_1}, S_{n_2}, \mathcal{M}_0) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp\left[-\frac{1}{2} d^T C_n^{-1} d\right]$$ **(noise-only model)**

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- N samples of white noise, white GWB, in two colocated and coaligned detectors:

$$C_n = \begin{bmatrix} S_{n_1} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & S_{n_2} \mathbb{1}_{N\times N} \end{bmatrix} \qquad \& \qquad C = \begin{bmatrix} (S_{n_1} + S_h)\, \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & (S_{n_2} + S_h)\, \mathbb{1}_{N\times N} \end{bmatrix}$$

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$$\hat{S}_h \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}d_{2i} \longleftarrow \text{cross-correlation estimator}$$

$$\hat{S}_{n_1} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}^2 - \hat{S}_h$$

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<u>Exercise</u>: Verify the expressions for the ML estimators.

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<u>Exercise</u>: Verify the expressions for the ML estimators.

<u>Exercise</u>: Verify the expression for the detection statistic $2\ln(\Lambda_{\rm ML}(d))$

Bayesian analyses

- Use Bayes' theorem to calculate posterior distributions for parameter estimation and odds ratios (Bayes factors) for model selection

likelihood

prior

$$p(\theta \mid d, \mathscr{M}) = \frac{p(d \mid \theta, \mathscr{M})\, p(\theta \mid \mathscr{M})}{p(d \mid \mathscr{M})}$$

posterior

evidence

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$$\underbrace{p(\theta|d,\mathcal{M})}_{\text{posterior}} = \frac{\overbrace{p(d|\theta,\mathcal{M})}^{\text{likelihood}}\,\overbrace{p(\theta|\mathcal{M})}^{\text{prior}}}{\underbrace{p(d|\mathcal{M})}_{\text{evidence}}}$$

- Posteriors:

$$p(S_{n_1}, S_{n_2}, S_h|d, \mathcal{M}_1) = \frac{p(d|S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1)p(S_{n_1}, S_{n_2}, S_h|\mathcal{M}_1)}{p(d|\mathcal{M}_1)}$$

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Bayes factor $\mathcal{B}_{10}(d)$

$$\frac{p(\mathcal{M}_1|d)}{p(\mathcal{M}_0|d)} = \boxed{\frac{p(d|\mathcal{M}_1)}{p(d|\mathcal{M}_0)}}\frac{p(\mathcal{M}_1)}{p(\mathcal{M}_0)}$$

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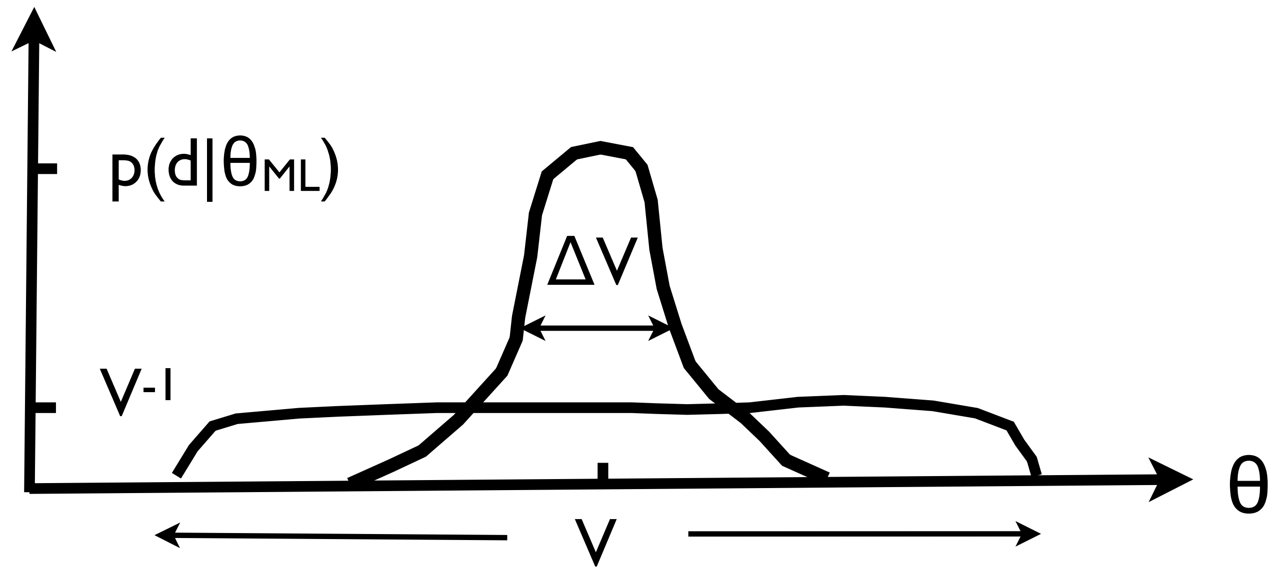
- Relationship to frequentist approach:



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