

# **Cross-correlation methods and overlap reduction functions for stochastic GW background searches**

**Joe Romano, Texas Tech University**

**Friday, 22 July 2022**

**(HUST GW Summer School 2022, Lecture 3)**

# Outline

1. Cross-correlation methods
2. Optimal filtering examples
3. Response of a detector to a GWB
4. Overlap reduction functions

# I. Cross-correlation methods

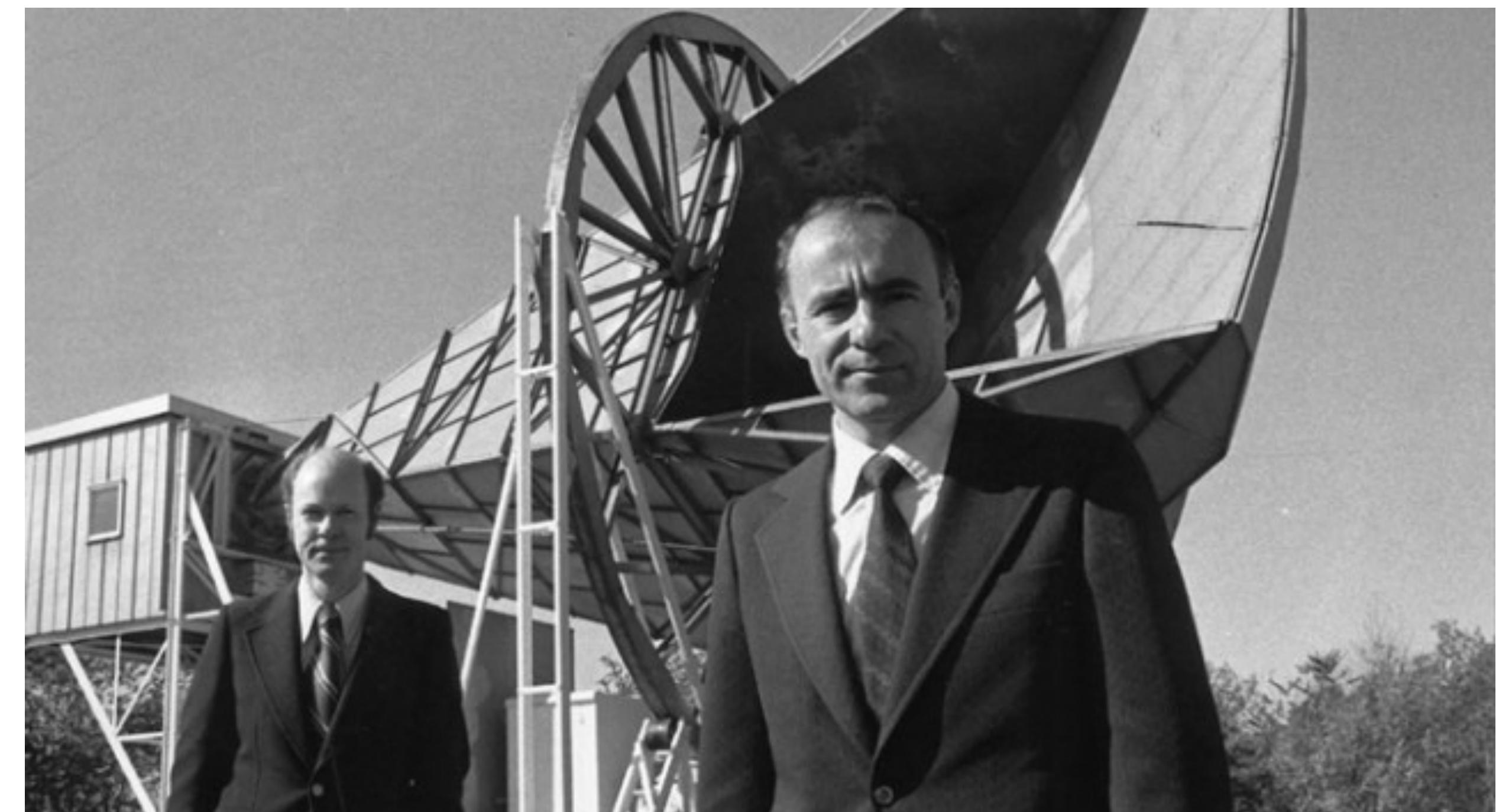
# Single versus multiple detectors

- Initial detection of the CMB was in a **single detector** (excess noise that could not be attributed to any known noise source)
- Ground-based detectors currently **aren't sensitive enough** for expected GWBs to stand out above instrumental noise
- Instead, look for evidence of a common disturbance in multiple detectors -> **cross-correlation**
- signal might be weak, but you can **build up signal-to-noise ratio** by correlating for long periods of time

## A MEASUREMENT OF EXCESS ANTENNA TEMPERATURE AT 4080 Mc/s

Measurements of the effective zenith noise temperature of the 20-foot horn-reflector antenna (Crawford, Hogg, and Hunt 1961) at the Crawford Hill Laboratory, Holmdel, New Jersey, at 4080 Mc/s have yielded a value about 3.5° K higher than expected. This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965). A possible explanation for the observed excess noise temperature is the one given by Dicke, Peebles, Roll, and Wilkinson (1965) in a companion letter in this issue.

(Penzias & Wilson, 1965)

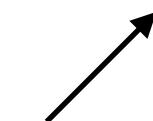


# Cross correlation – basic idea

- Data from two detectors:

$$d_1 = h + n_1$$

$$d_2 = h + n_2$$



common GW signal component

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$$\hat{C}_{12} \equiv d_1 d_2$$

- Expected value of cross-correlation:

$$\langle \hat{C}_{12} \rangle = \langle d_1 d_2 \rangle = \langle h^2 \rangle + \langle n_1 n_2 \rangle + \cancel{\langle hn_2 \rangle}^0 + \cancel{\langle n_1 h \rangle}^0 = \langle h^2 \rangle + \langle n_1 n_2 \rangle$$

# Cross correlation – basic idea

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- Assuming detector noise is uncorrelated:

$$\boxed{\langle \hat{C}_{12} \rangle = \langle h^2 \rangle \equiv S_h}$$

# Worked example: N samples, white GWB in white noise

$$\hat{S}_h \equiv \hat{C}_{12} = \frac{1}{N} \sum_{i=1}^N d_{1i} d_{2i} \quad d_{1i} = h_i + n_{1i}, \quad d_{2i} = h_i + n_{2i}$$

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Variance:  $\sigma^2 \equiv \langle \hat{C}_{12}^2 \rangle - \langle \hat{C}_{12} \rangle^2 = \left( \frac{1}{N} \right)^2 \sum_{i=1}^N \sum_{j=1}^N \left( \langle d_{1i} d_{2i} d_{1j} d_{2j} \rangle - \langle d_{1i} d_{2i} \rangle \langle d_{1j} d_{2j} \rangle \right)$

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$\downarrow \quad \langle abcd \rangle = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle$

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where  $S_1 = S_{n_1} + S_h$   
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SNR:

$$\rho \equiv \frac{\mu}{\sigma} \simeq \frac{S_h}{\sqrt{S_1 S_2 / N}}$$

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$$\rho \equiv \frac{\mu}{\sigma} \simeq \frac{S_h}{\sqrt{S_1 S_2 / N}} \quad \Rightarrow \quad \boxed{\rho \simeq \sqrt{N} \frac{S_h}{S_n}} \quad \text{where} \quad \sqrt{S_1 S_2} \simeq \sqrt{S_{n_1} S_{n_2}} \equiv S_n$$

# Cross-correlation estimators, optimal filtering

- More generally:

$$\hat{S}_h = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' d_1(t)d_2(t')Q(t, t')$$

if stationary  
 $Q(t, t') = Q(t - t')$

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correlation coeff (overlap)  
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$$\langle \tilde{r}_1(f)\tilde{r}_2^*(f') \rangle = \frac{1}{2}\delta(f-f')\Gamma_{12}(f)S_h(f)$$

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de-weight correlation  
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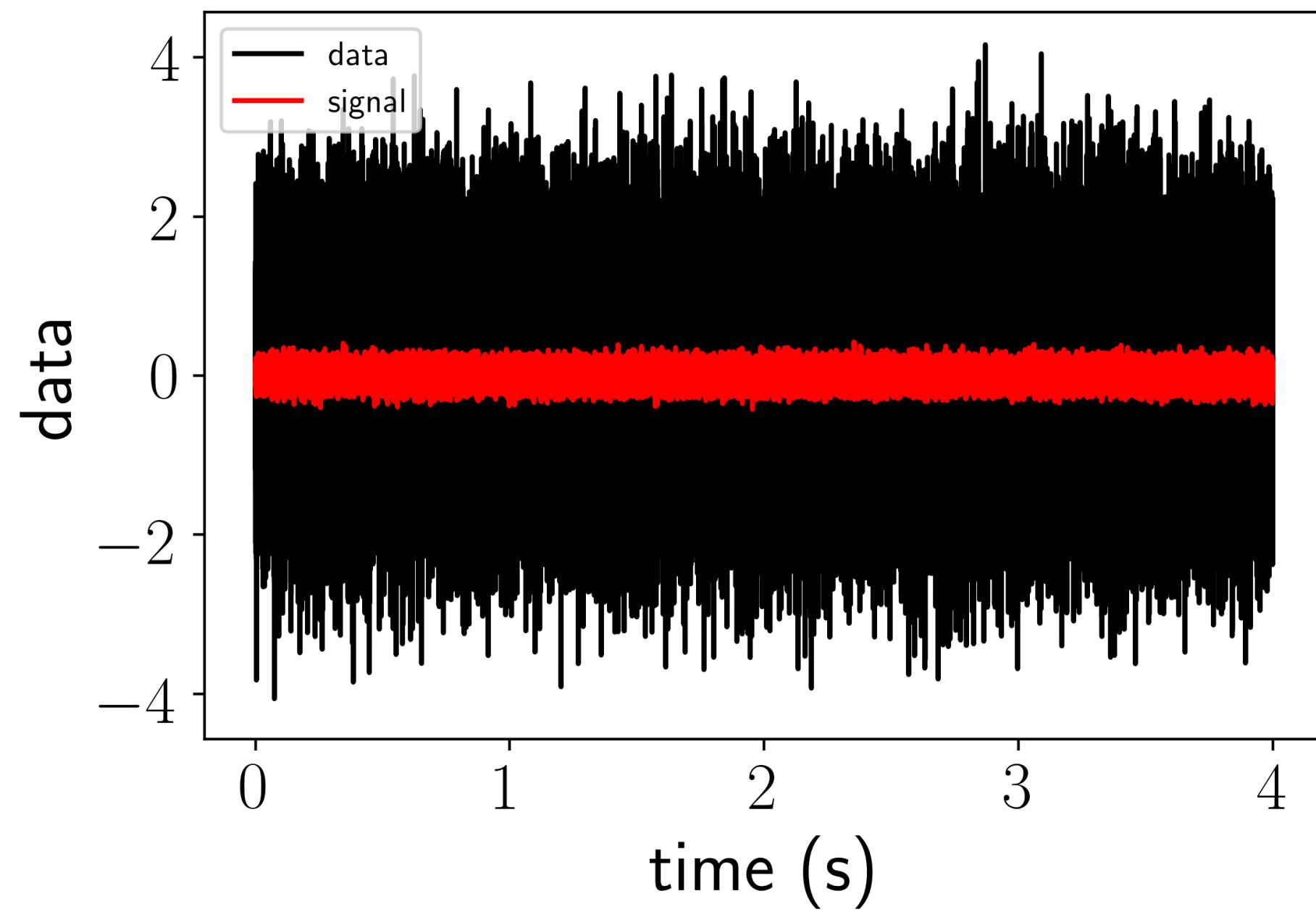
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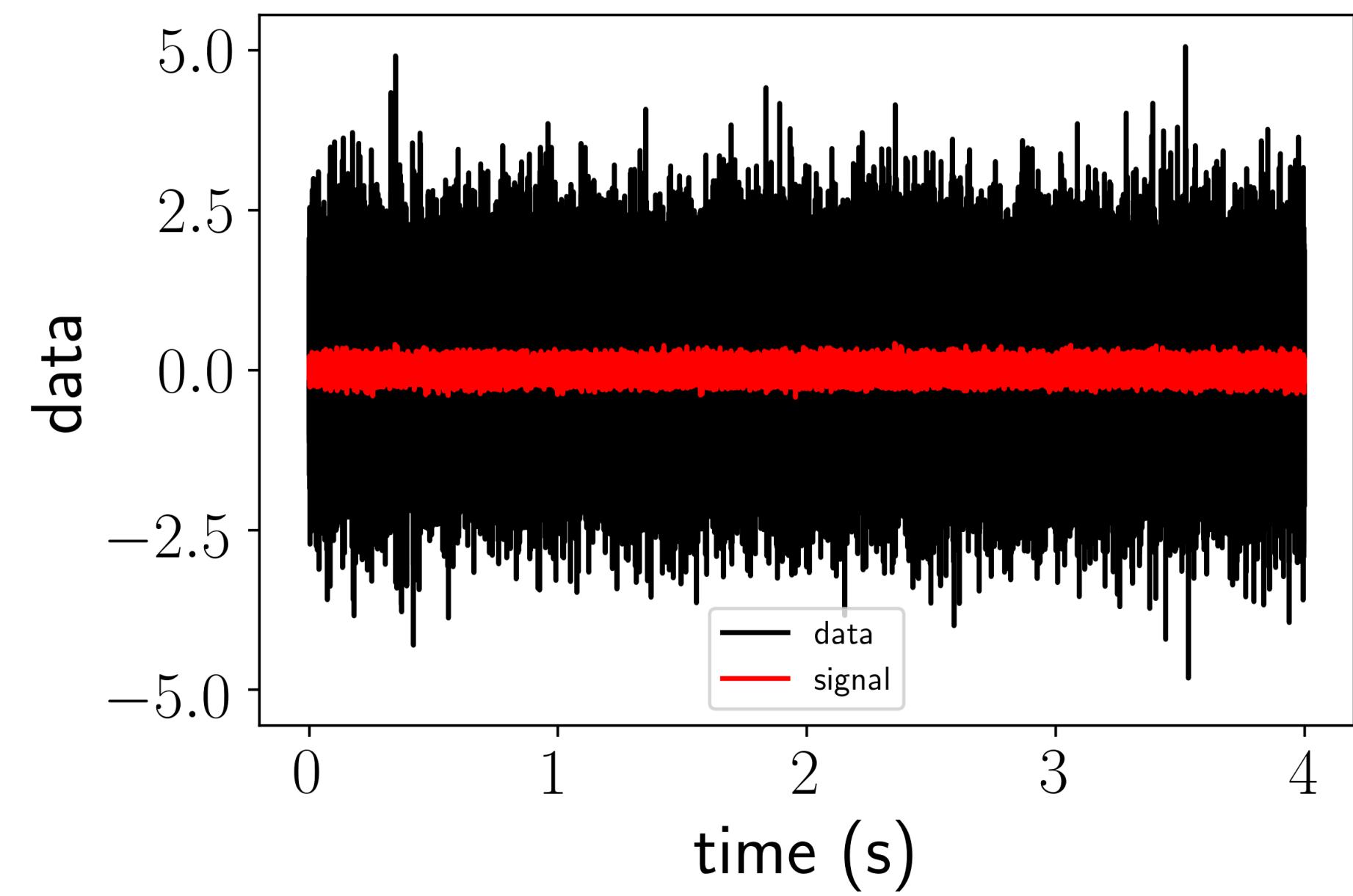
-> Will discuss detector response and overlap reductions in detail, later on

## **II. Optimal filtering examples (code available at [romano\\_code3.ipynb](#))**

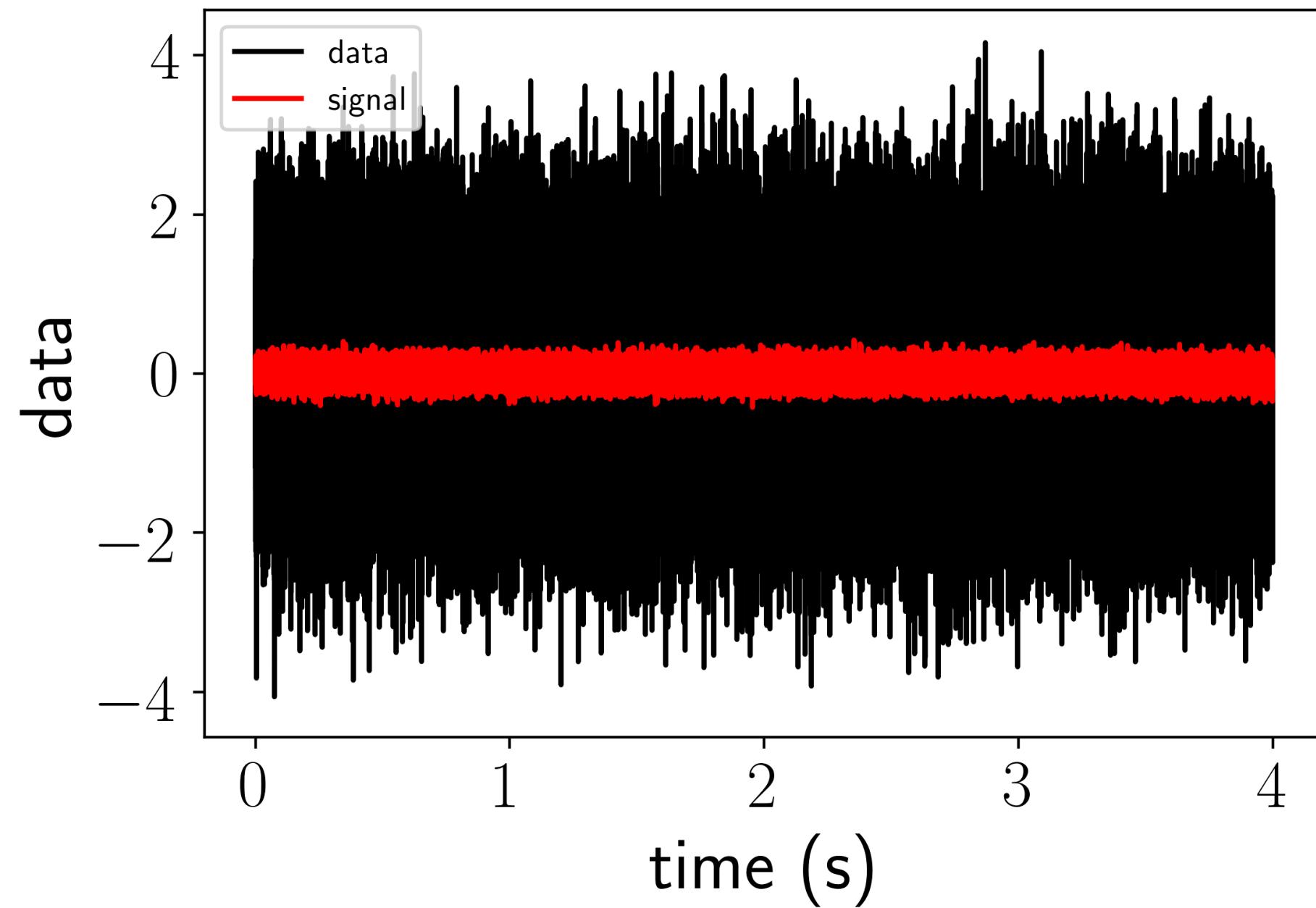
# (i) white GWB in white noise



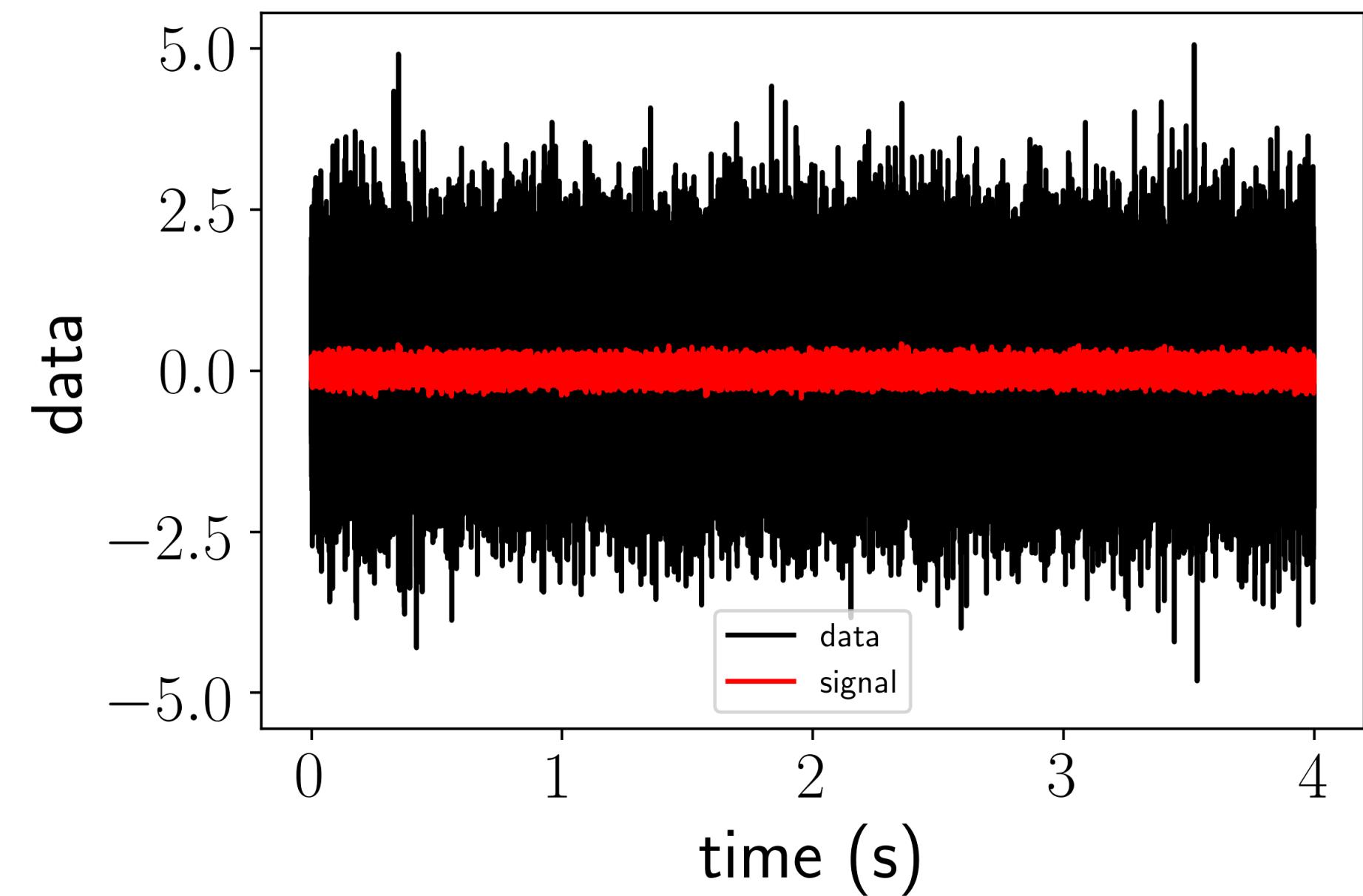
$$H(f) = 1$$



# (i) white GWB in white noise



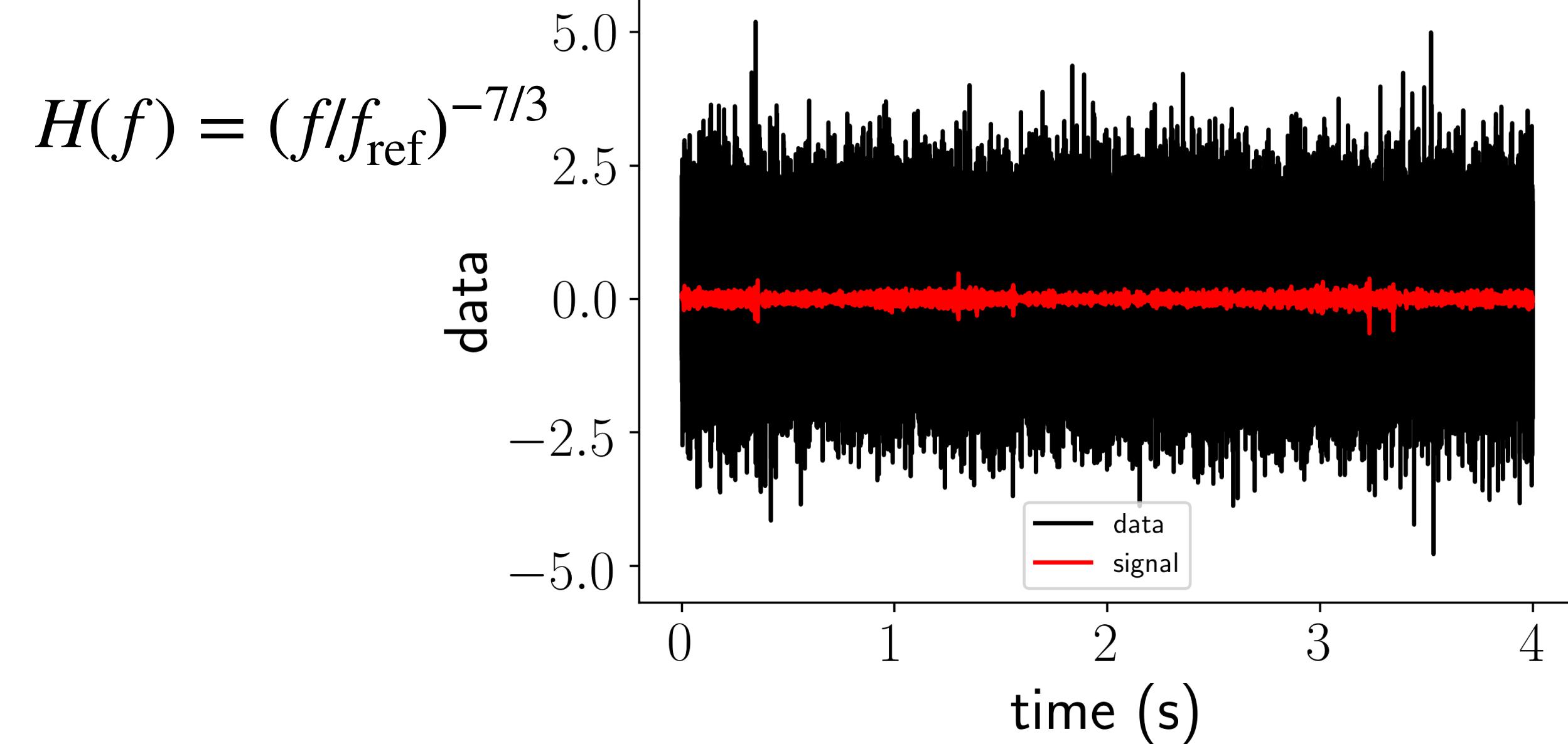
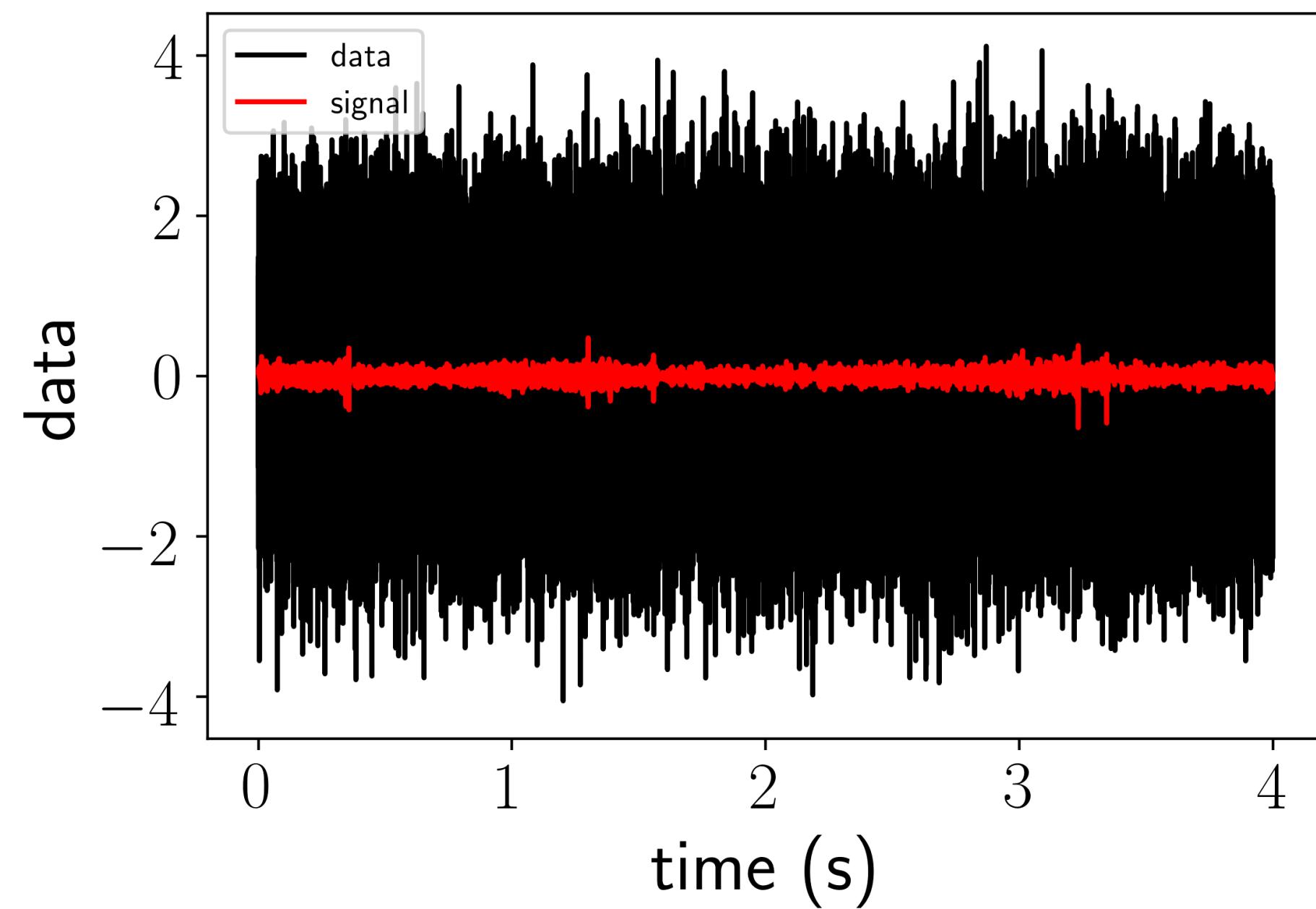
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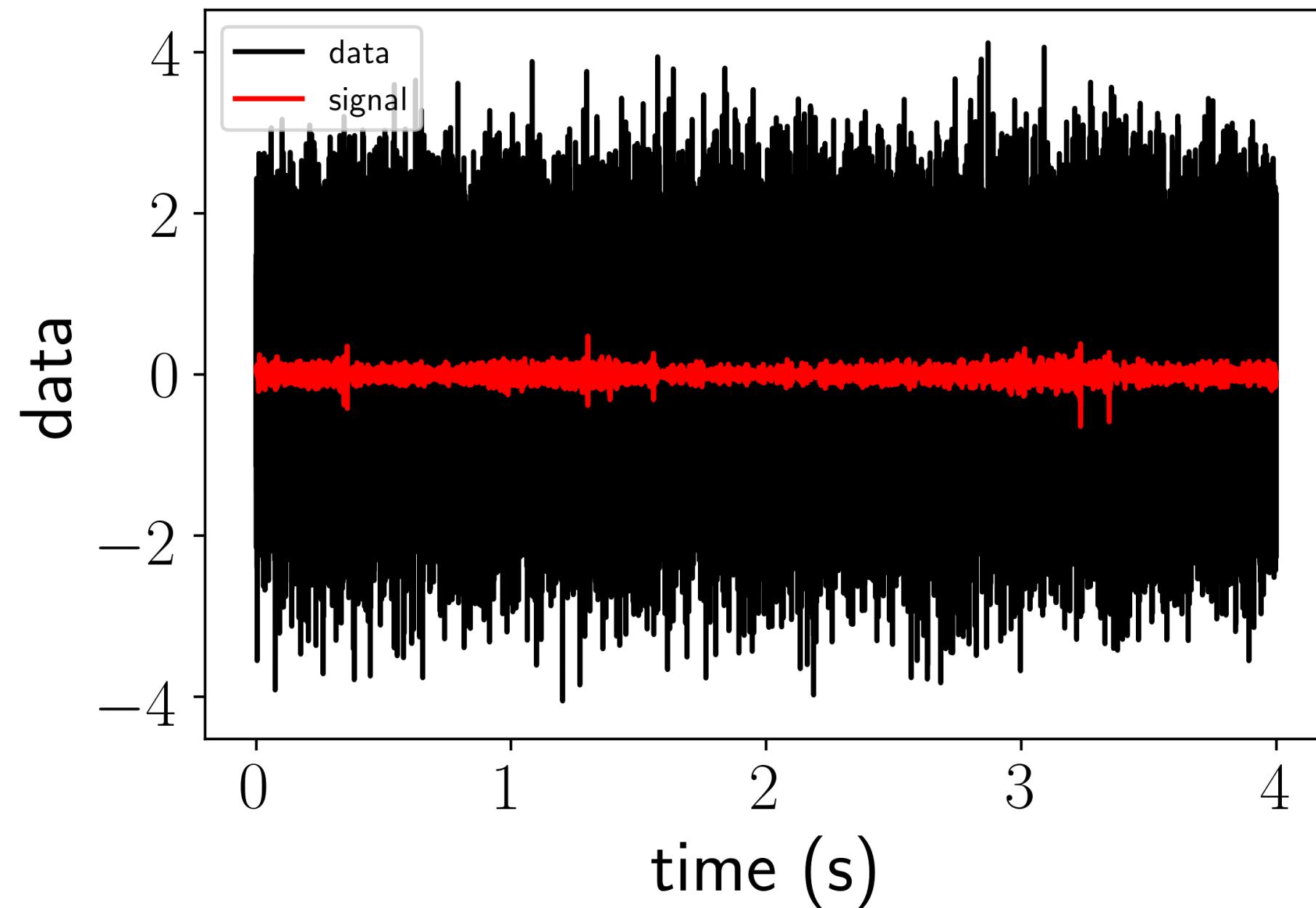
expected and estimated values of power agree to 3.5%, within 1 sigma

optimally filtered CC SNR = 2.9

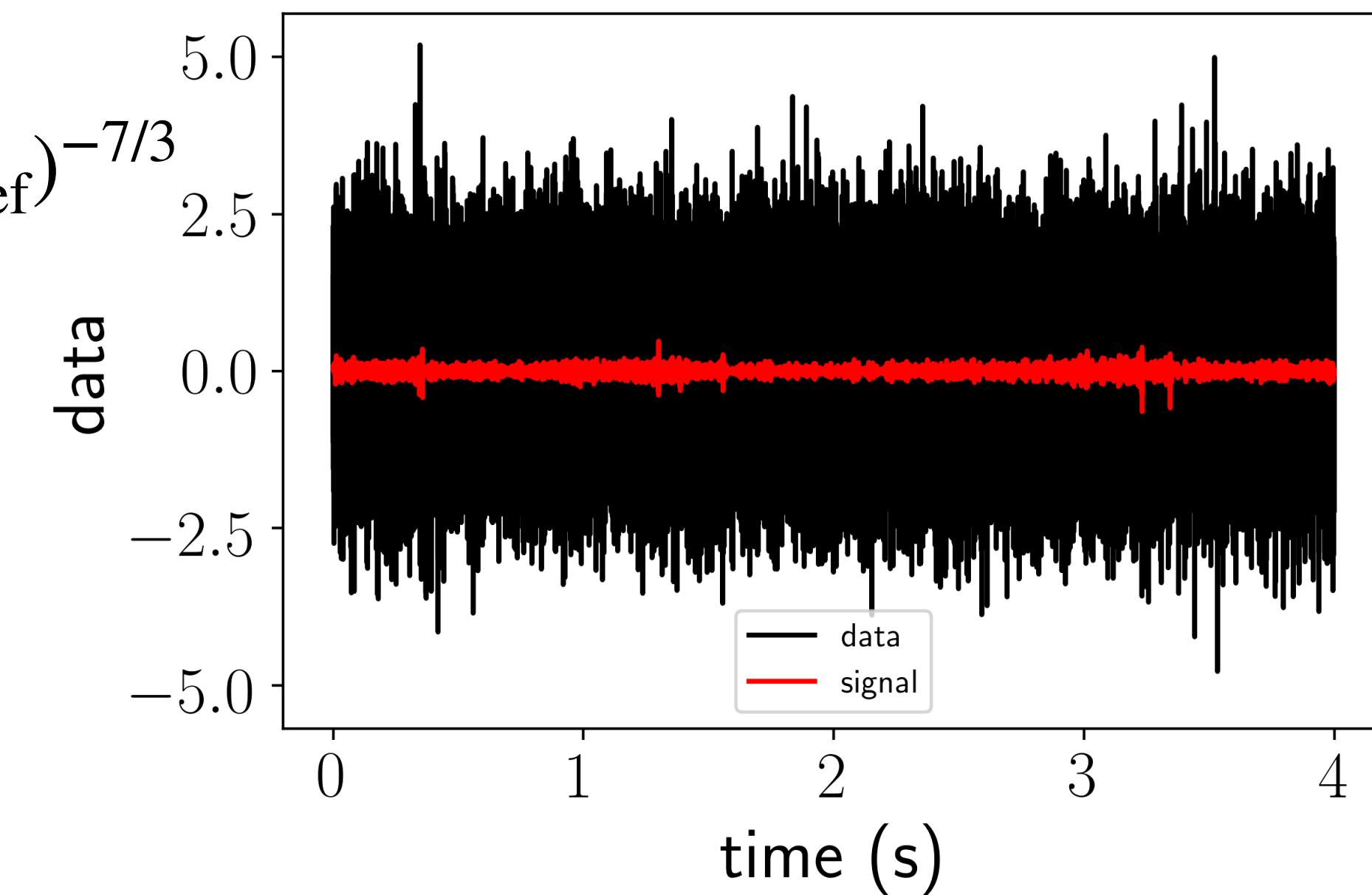
## (ii) BNS-confusion-limited GWB in white noise



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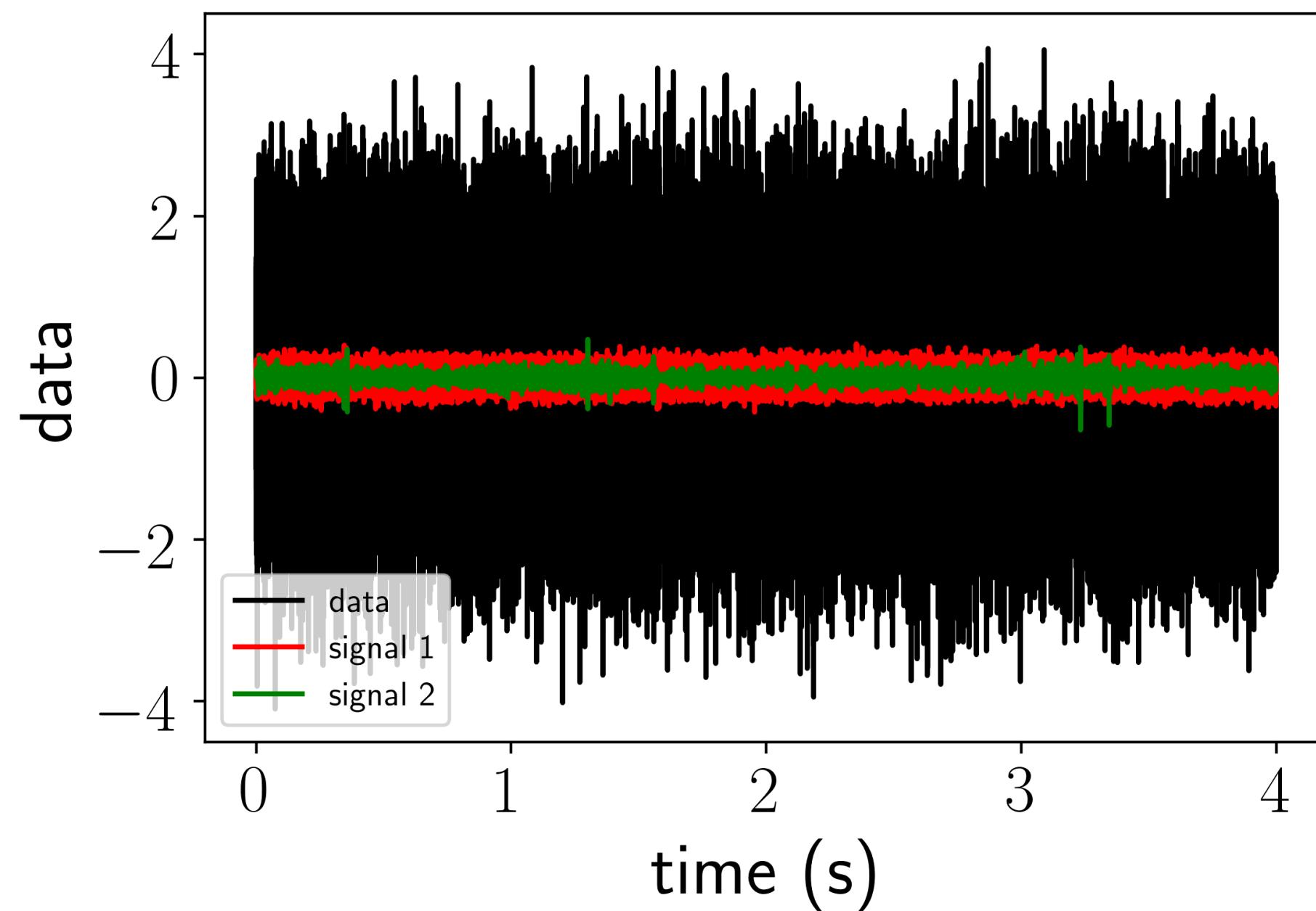
$$H(f) = (f/f_{\text{ref}})^{-7/3}$$



expected and estimated values of  
power agree to 2.7%, within 1 sigma

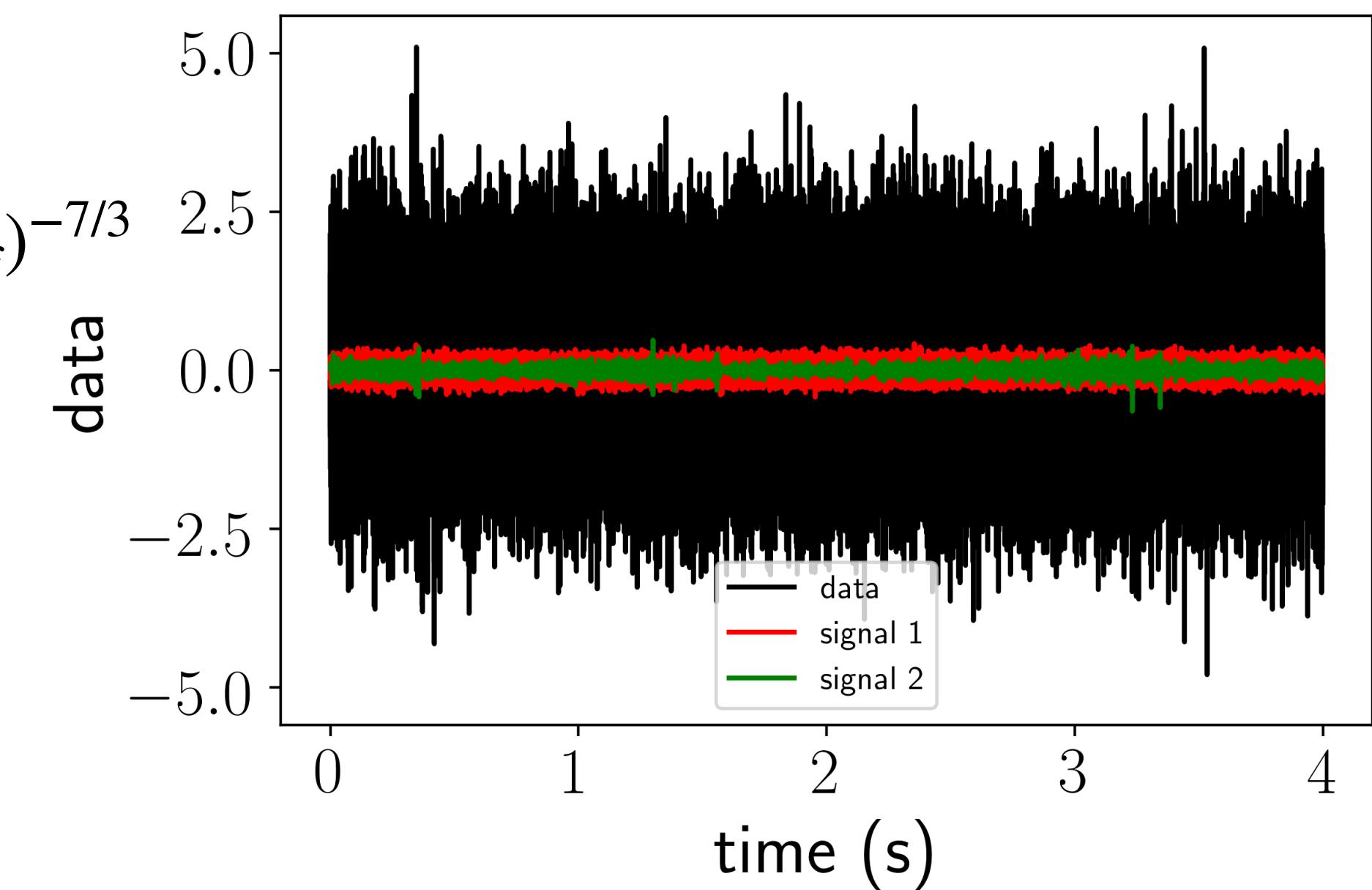
optimally filtered CC SNR = 12

### (iii) Two-component GWB in white noise

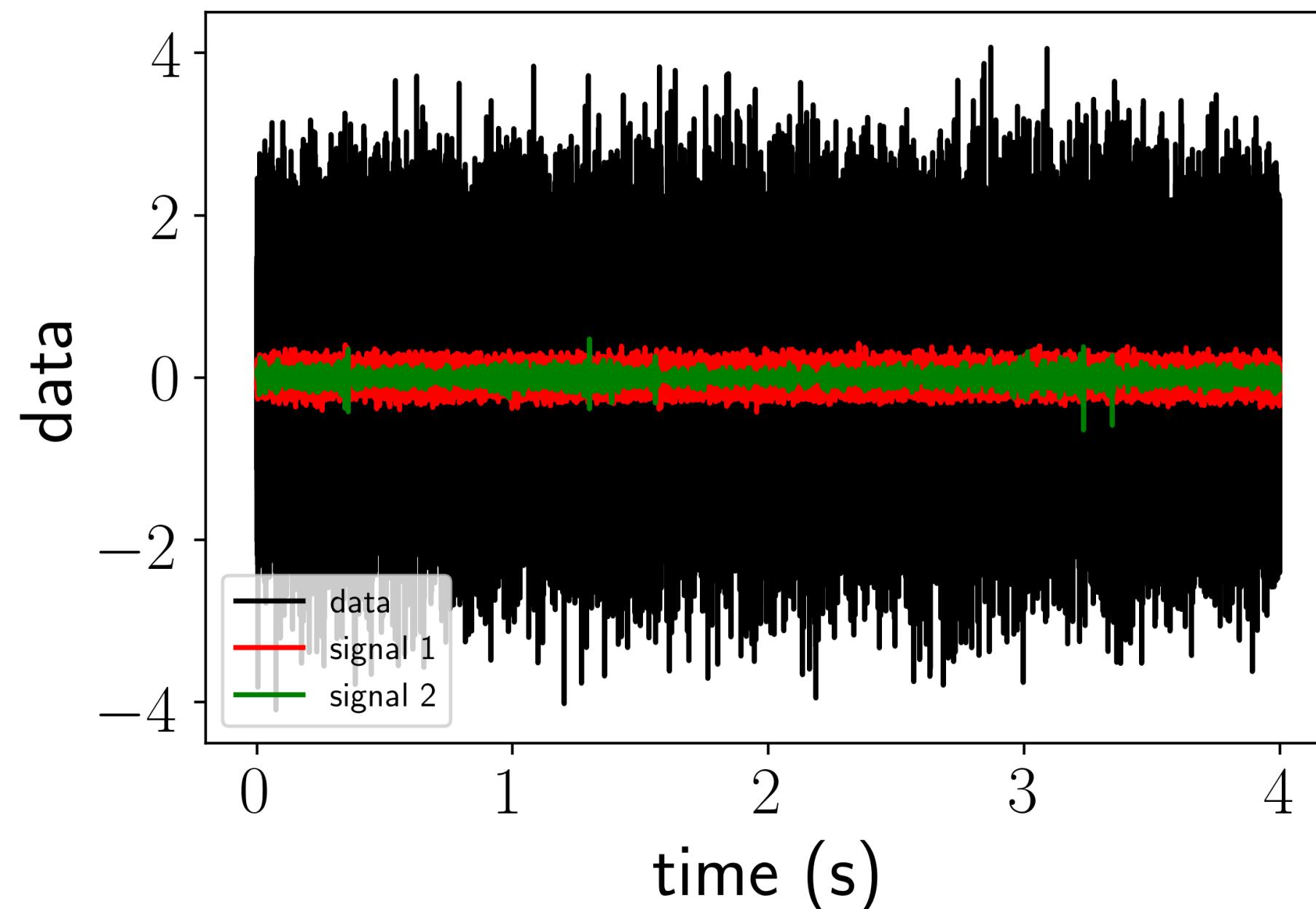


$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\text{ref}})^{-7/3}$$

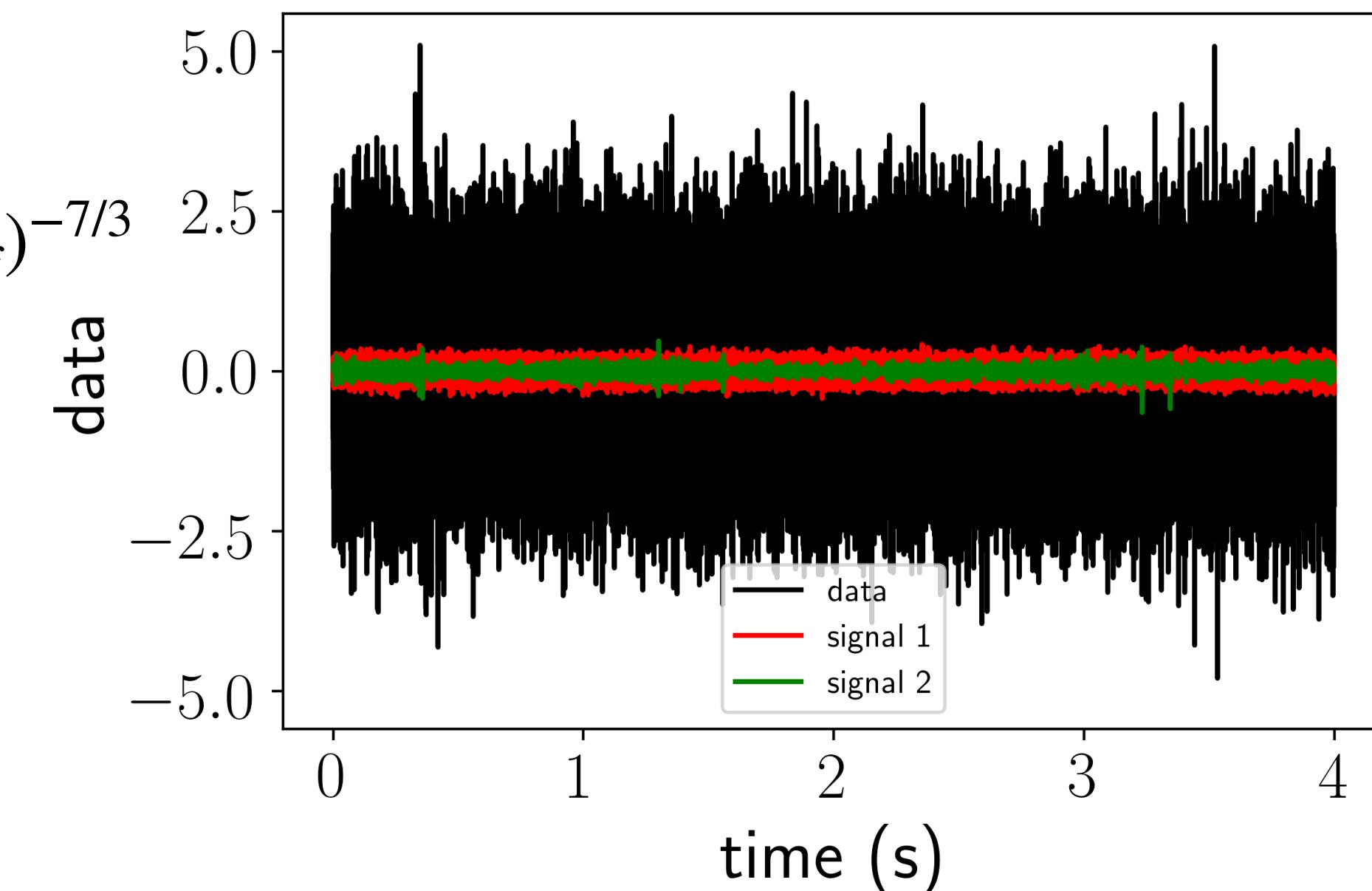


### (iii) Two-component GWB in white noise



$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\text{ref}})^{-7/3}$$



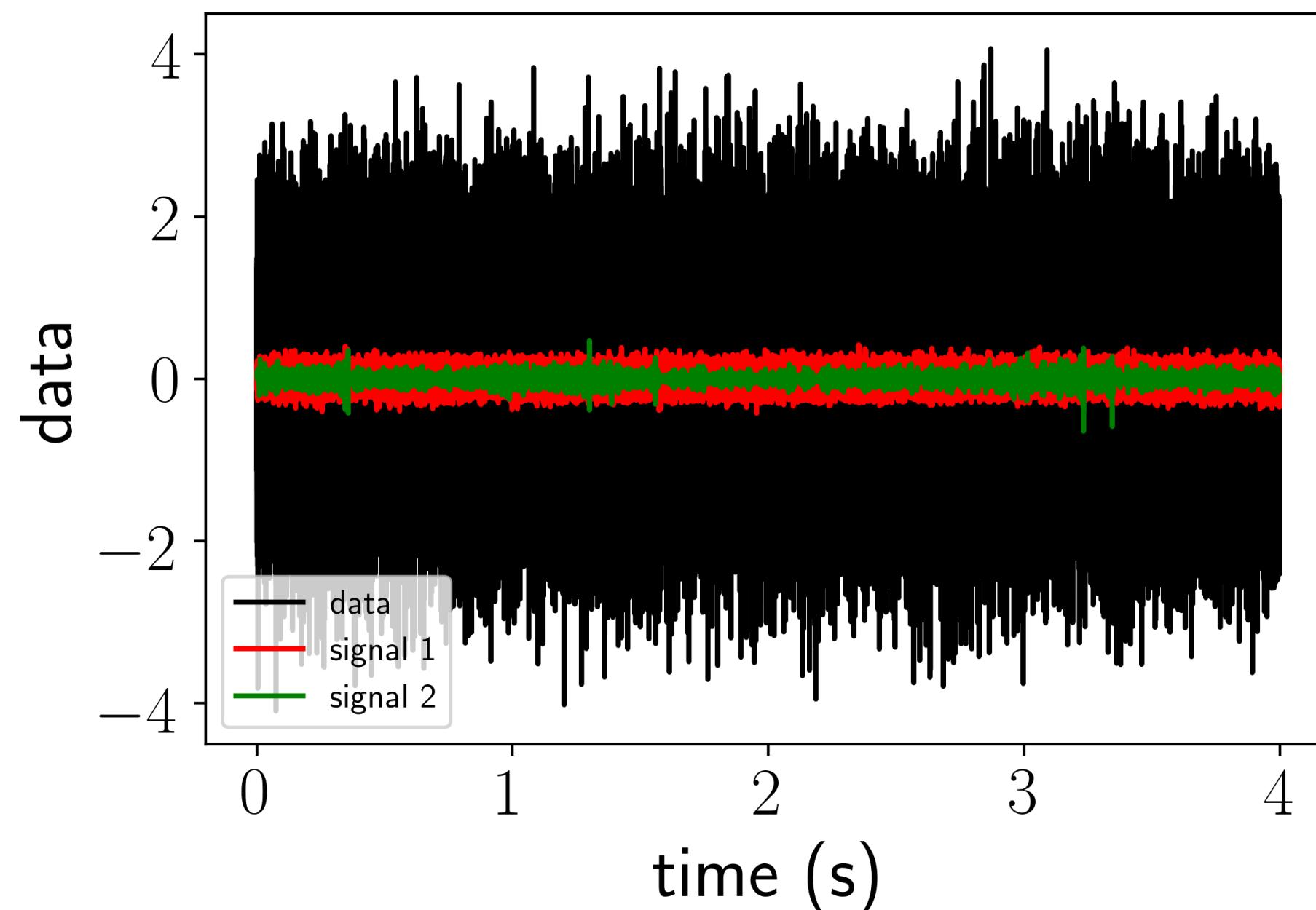
optimal-filtering for each component separately:

white GWB: 48% overestimate, > 1 sigma

BNS GWB: 6.9% overestimate, within 1 sigma

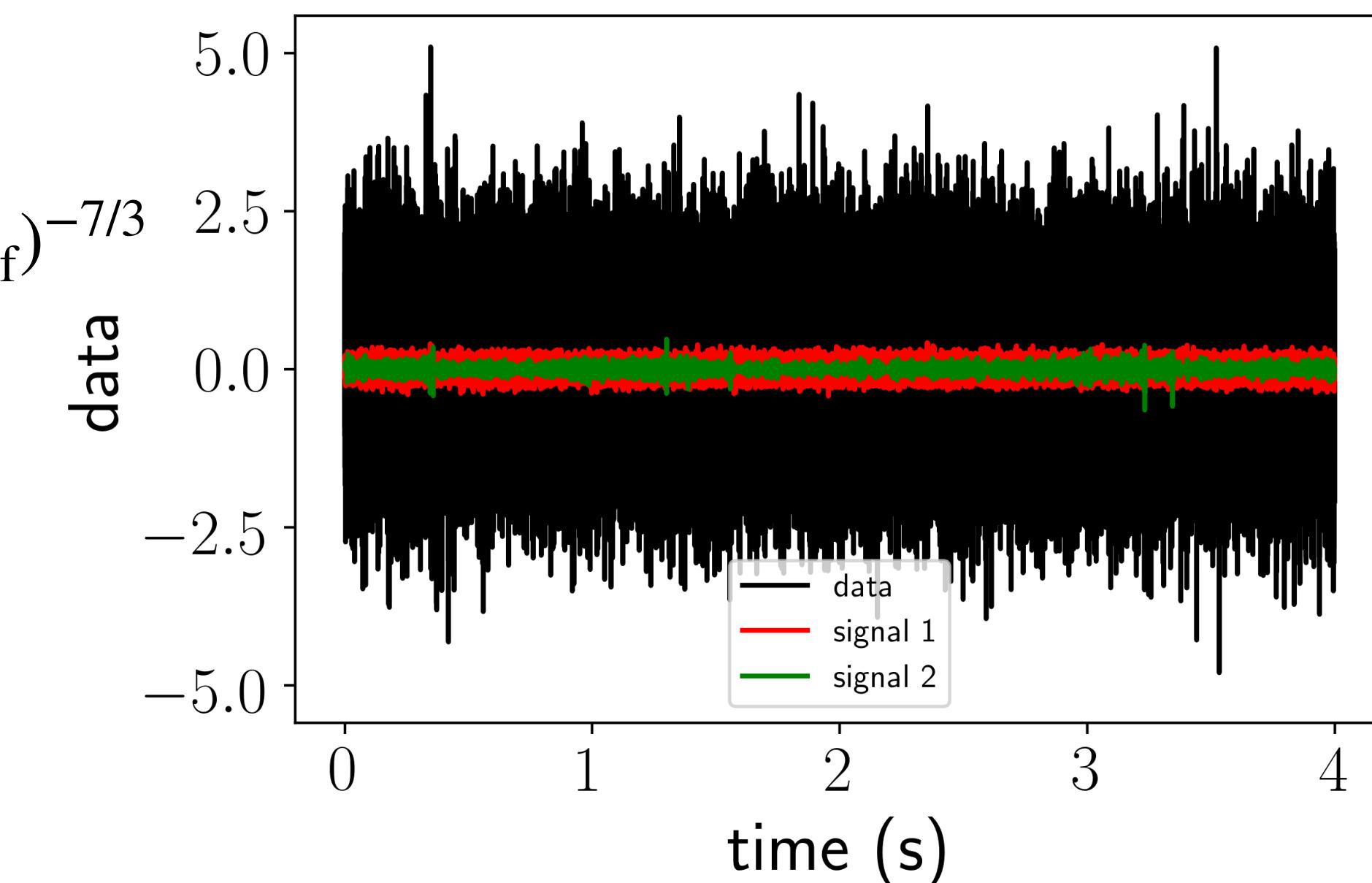
separate analyses **overestimate strength** of each GWB component and **underestimate error bars**

### (iii) Two-component GWB in white noise



$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\text{ref}})^{-7/3}$$



optimal-filtering for each component separately:

white GWB: 48% overestimate, > 1 sigma

BNS GWB: 6.9% overestimate, within 1 sigma

joint multi-component analysis:

white GWB: agreement to 7.3%, SNR=1.4

BNS GWB: agreement to 3.8%, SNR=6.0

separate analyses **overestimate strength** of each GWB component and **underestimate error bars**

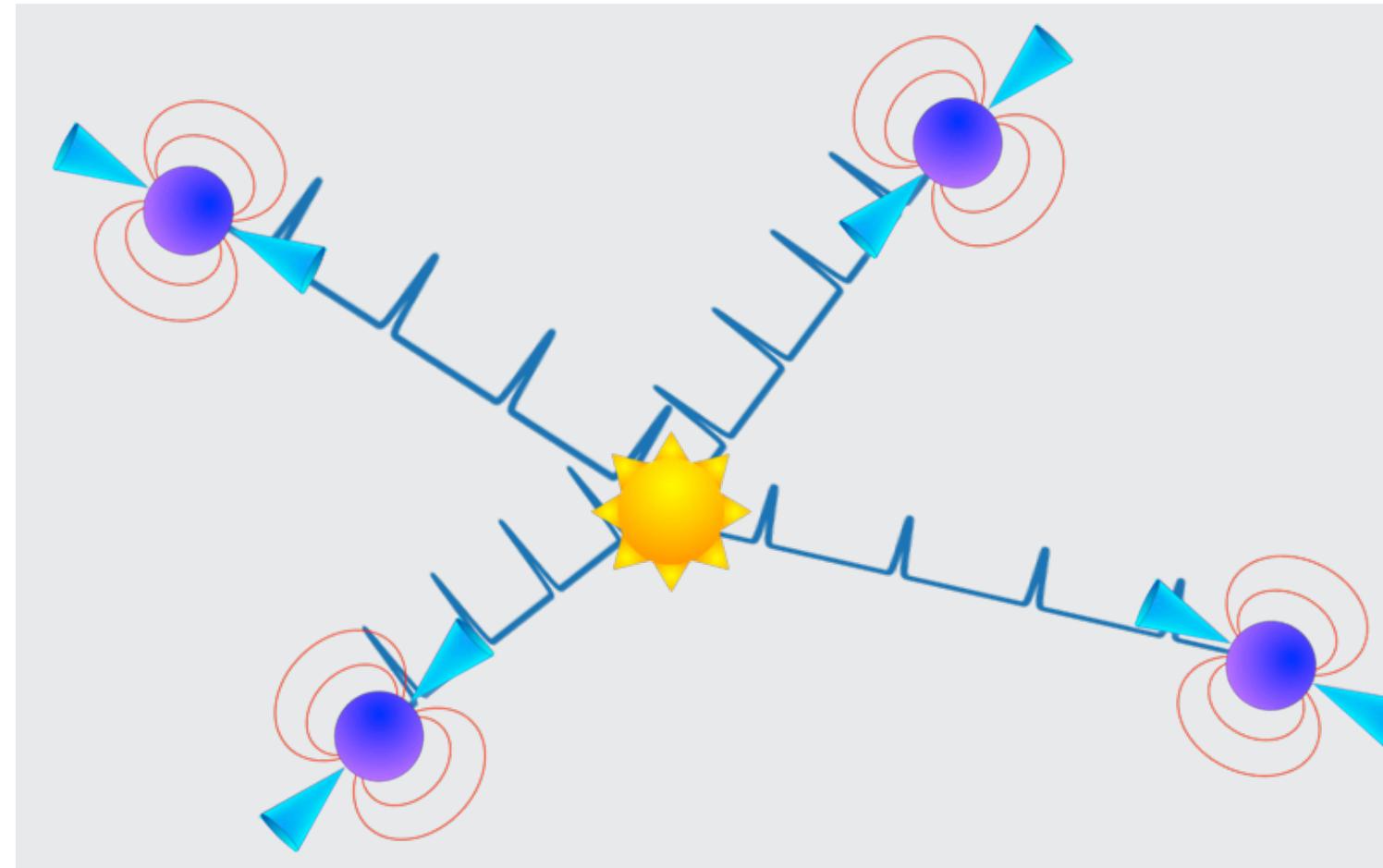
joint analysis properly takes into account the **covariance** between the component spectral shapes

Parida et al, JCAP 024, 2106

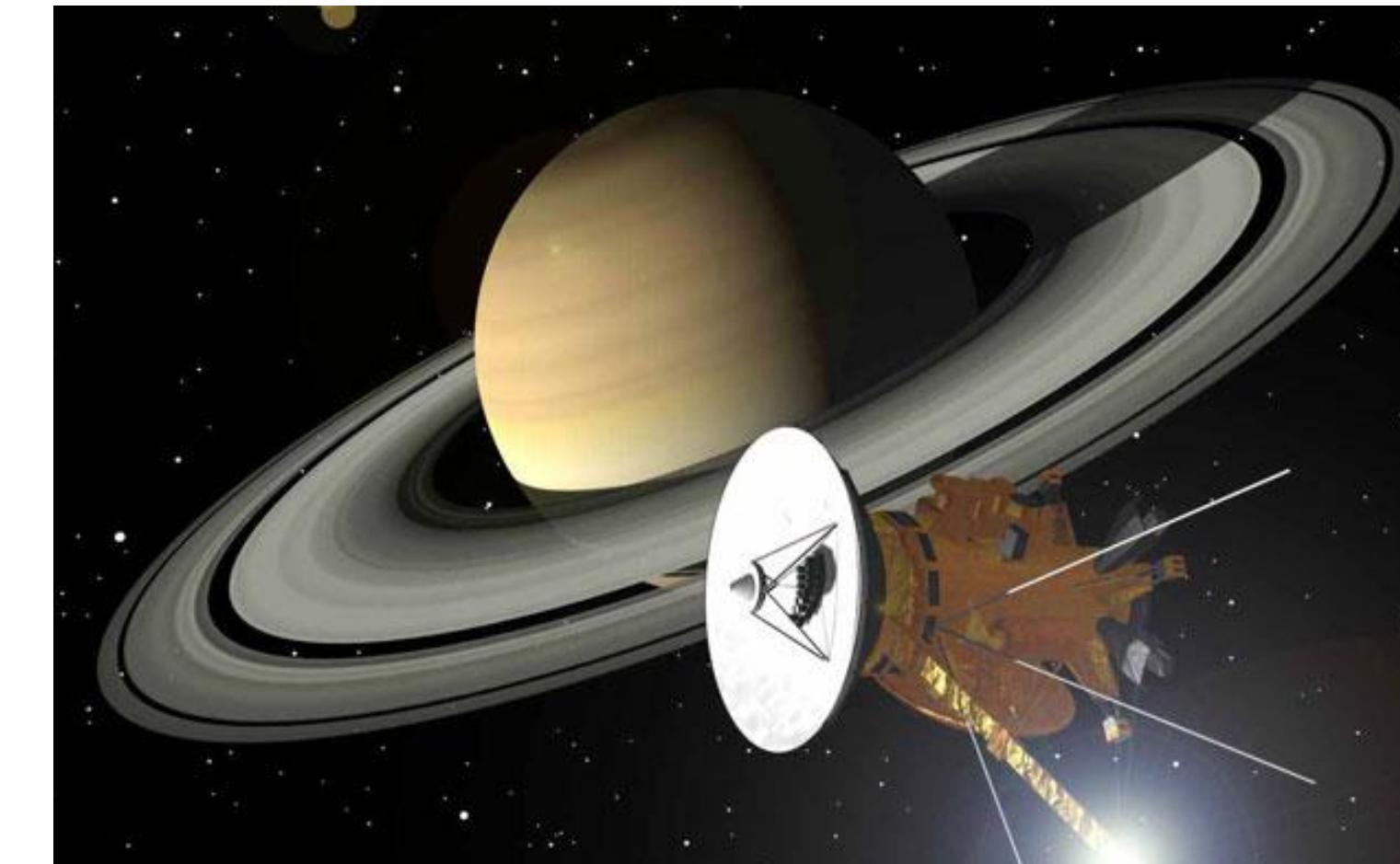
### **III. Response of a detector to a GW background**

# Beam detectors (use EM radiation to monitor the separation between two test masses)

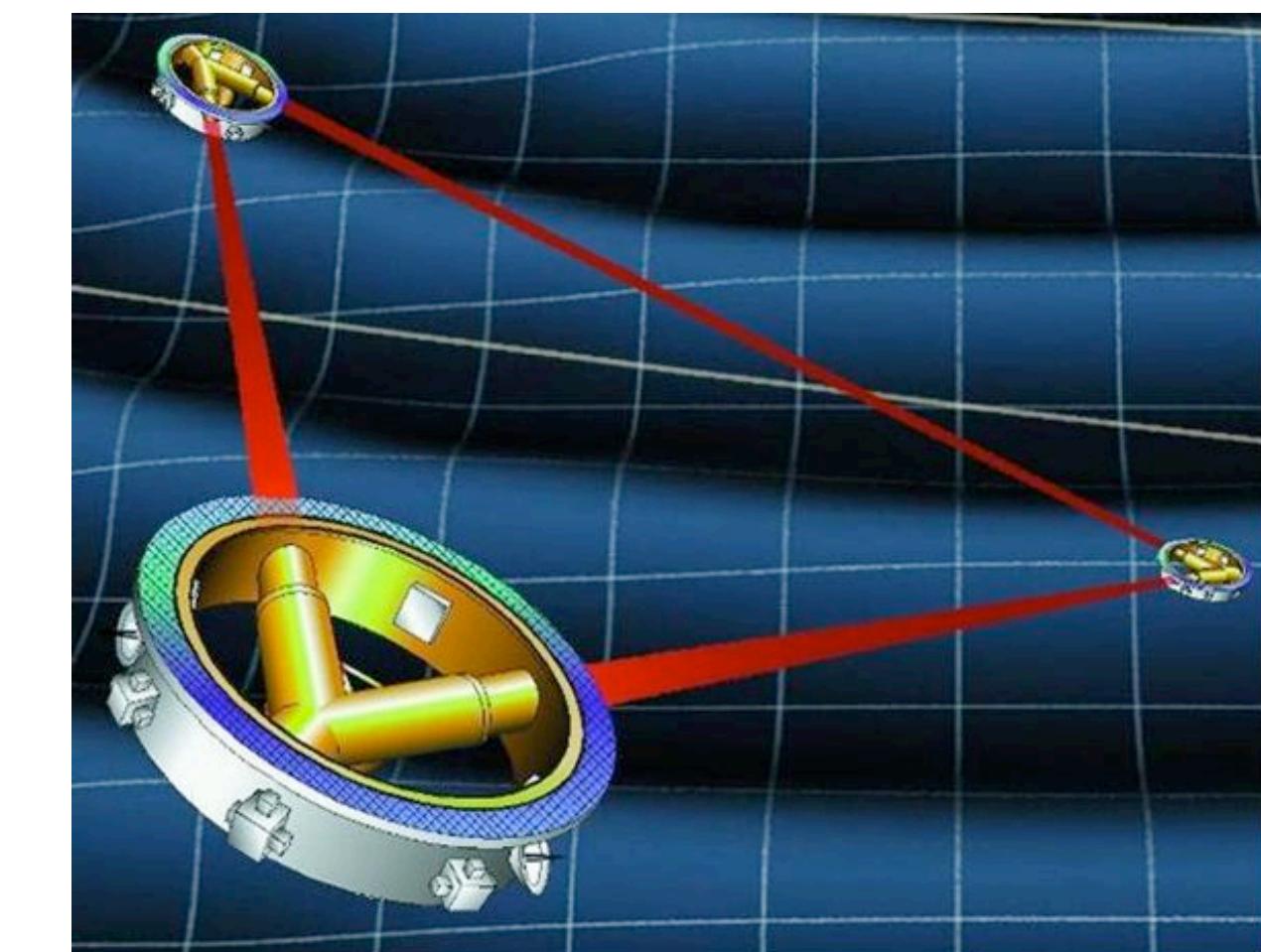
pulsar timing



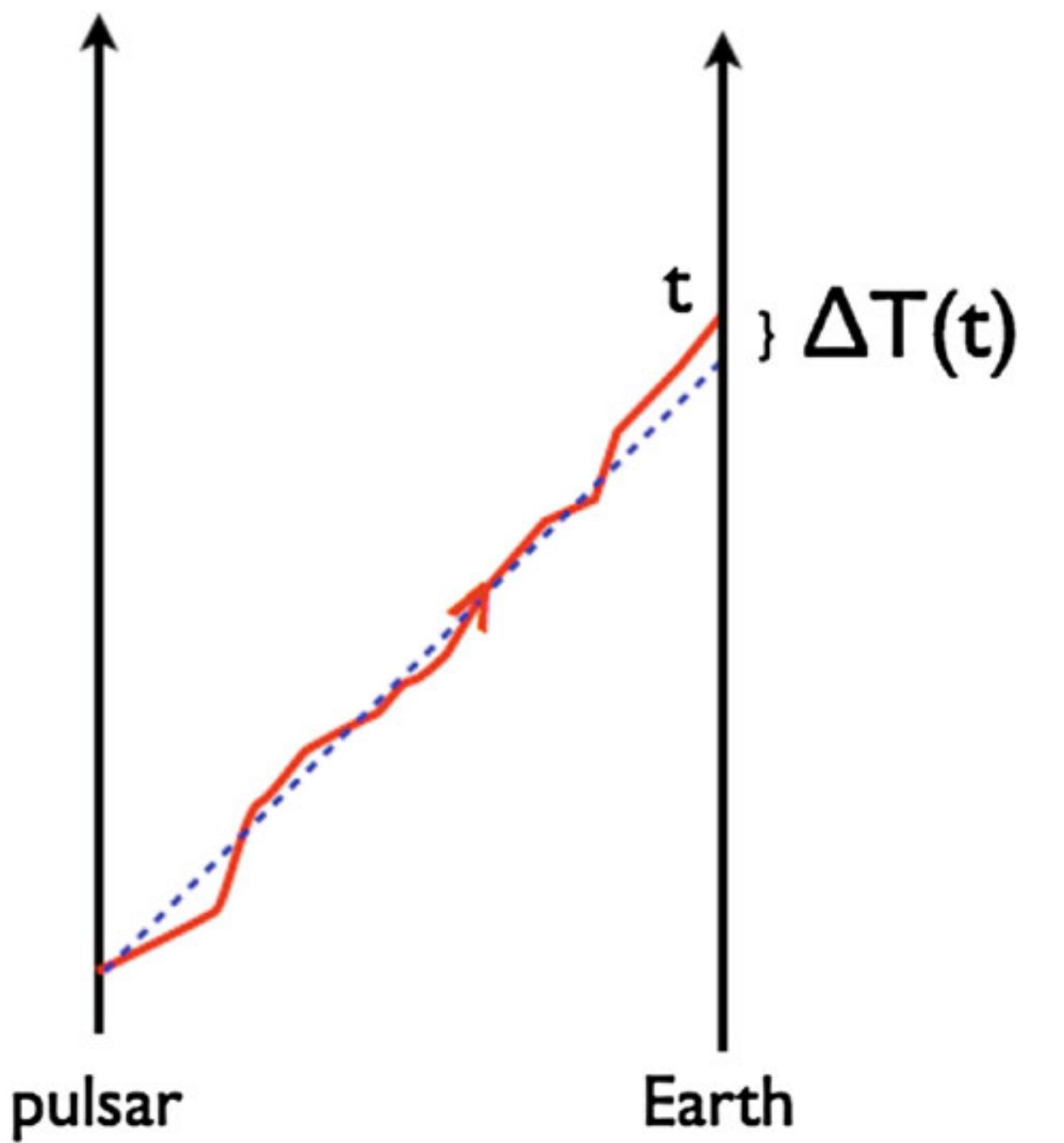
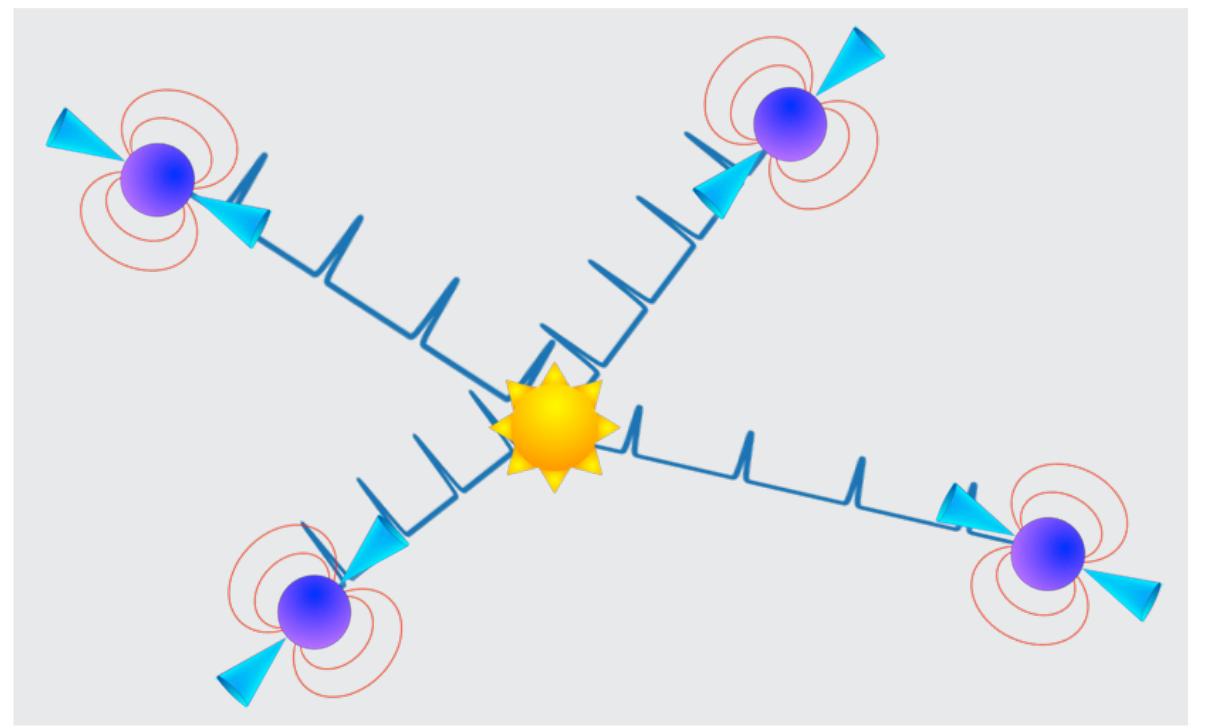
spacecraft Doppler tracking



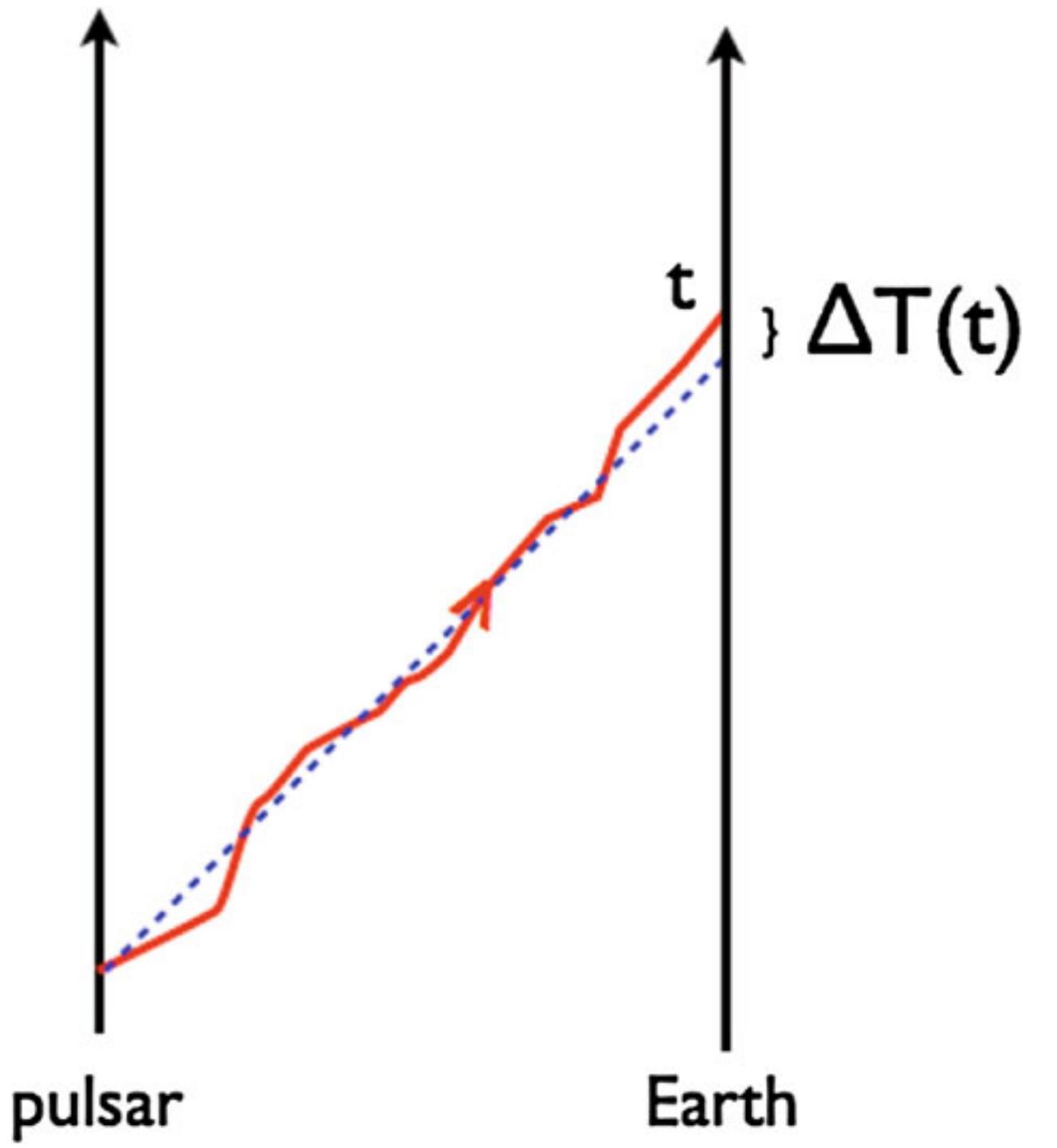
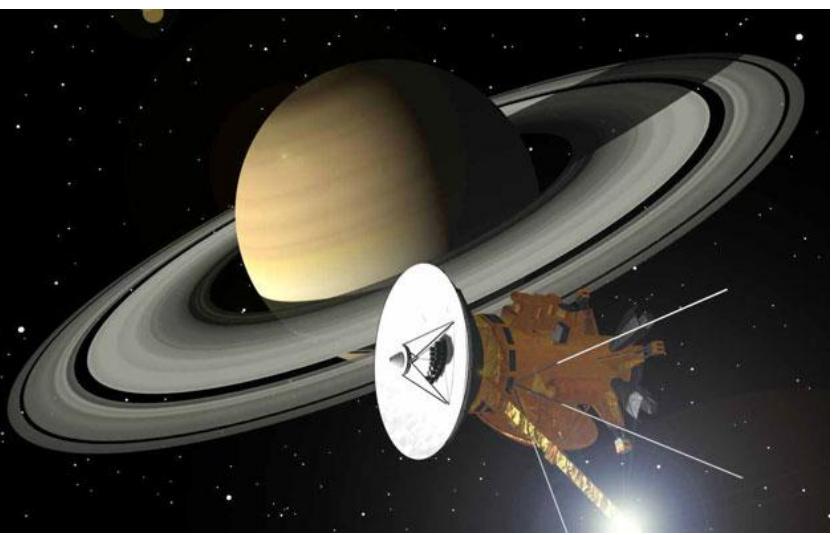
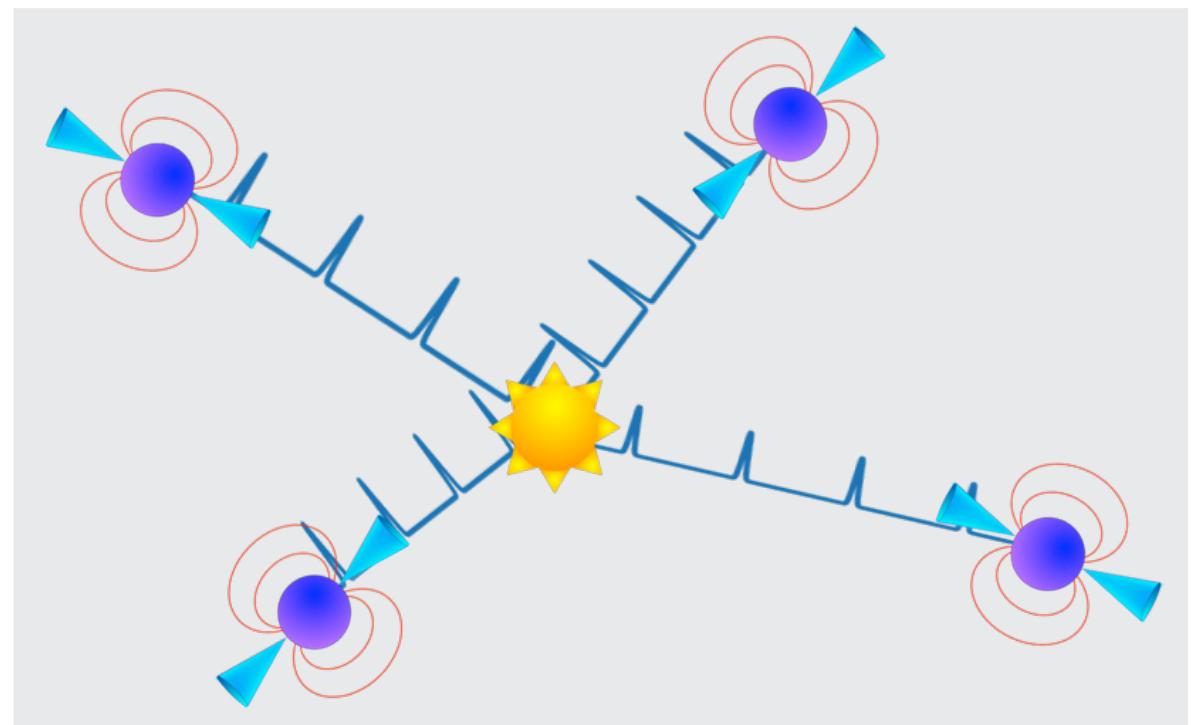
laser interferometers



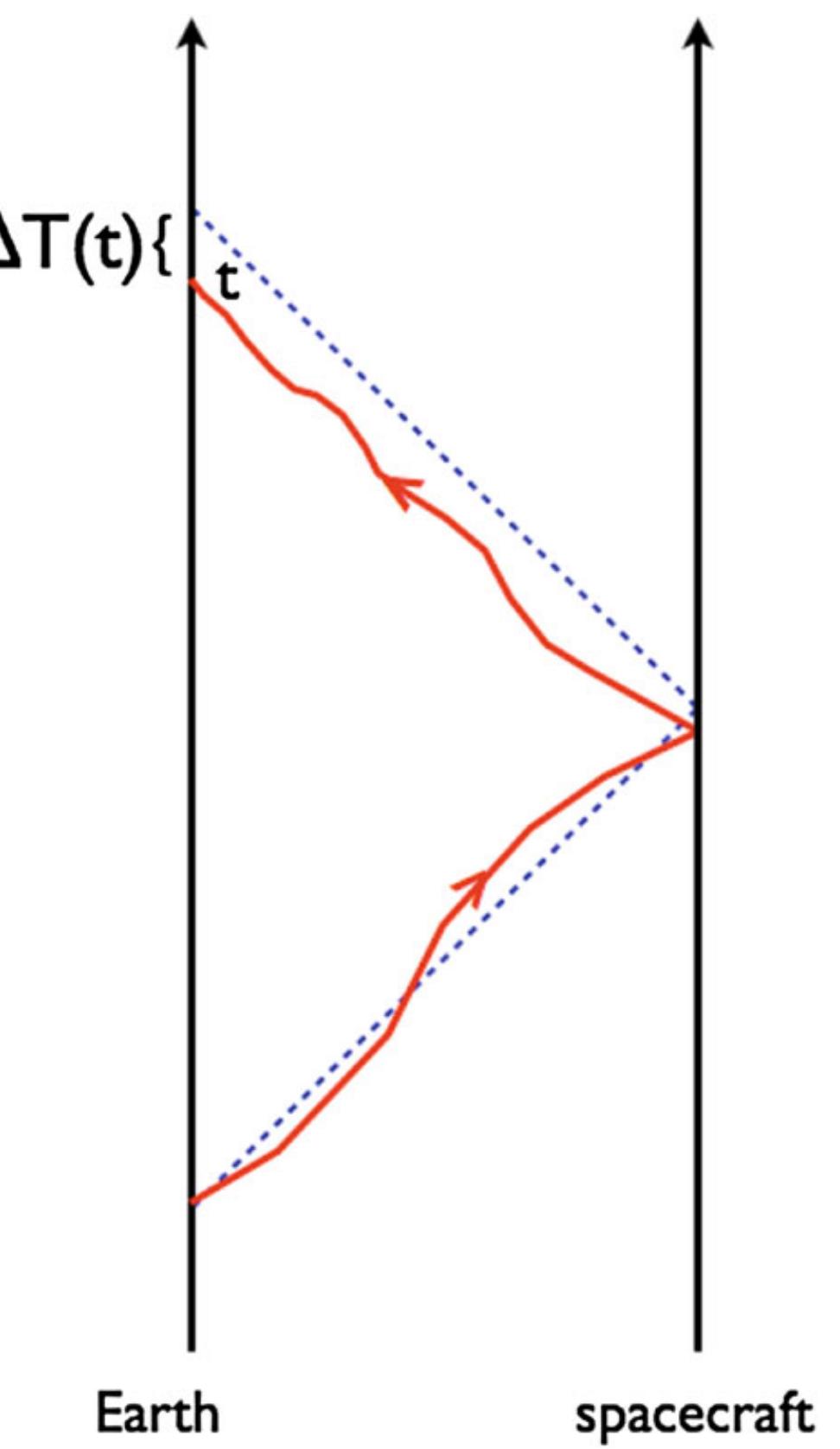
GW perturbs the photon propagation time between the test masses



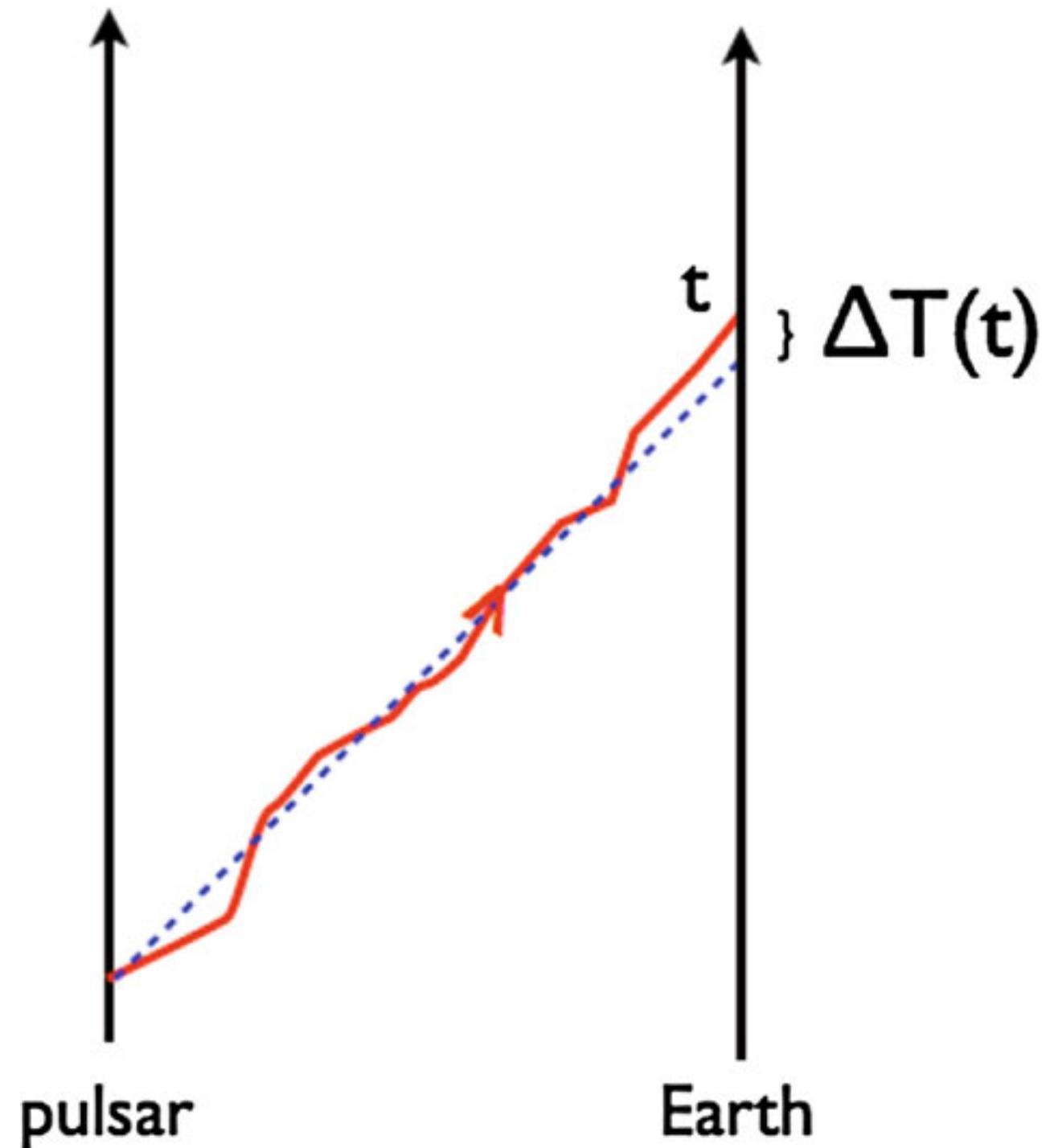
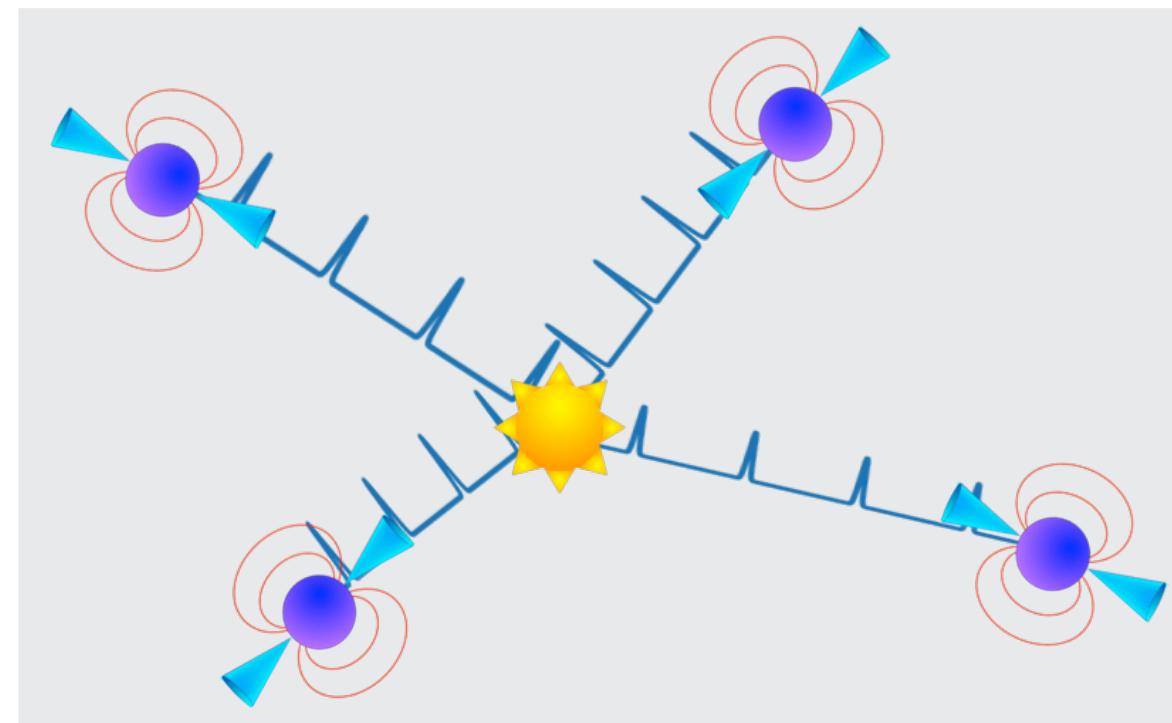
(1-arm, 1-way)



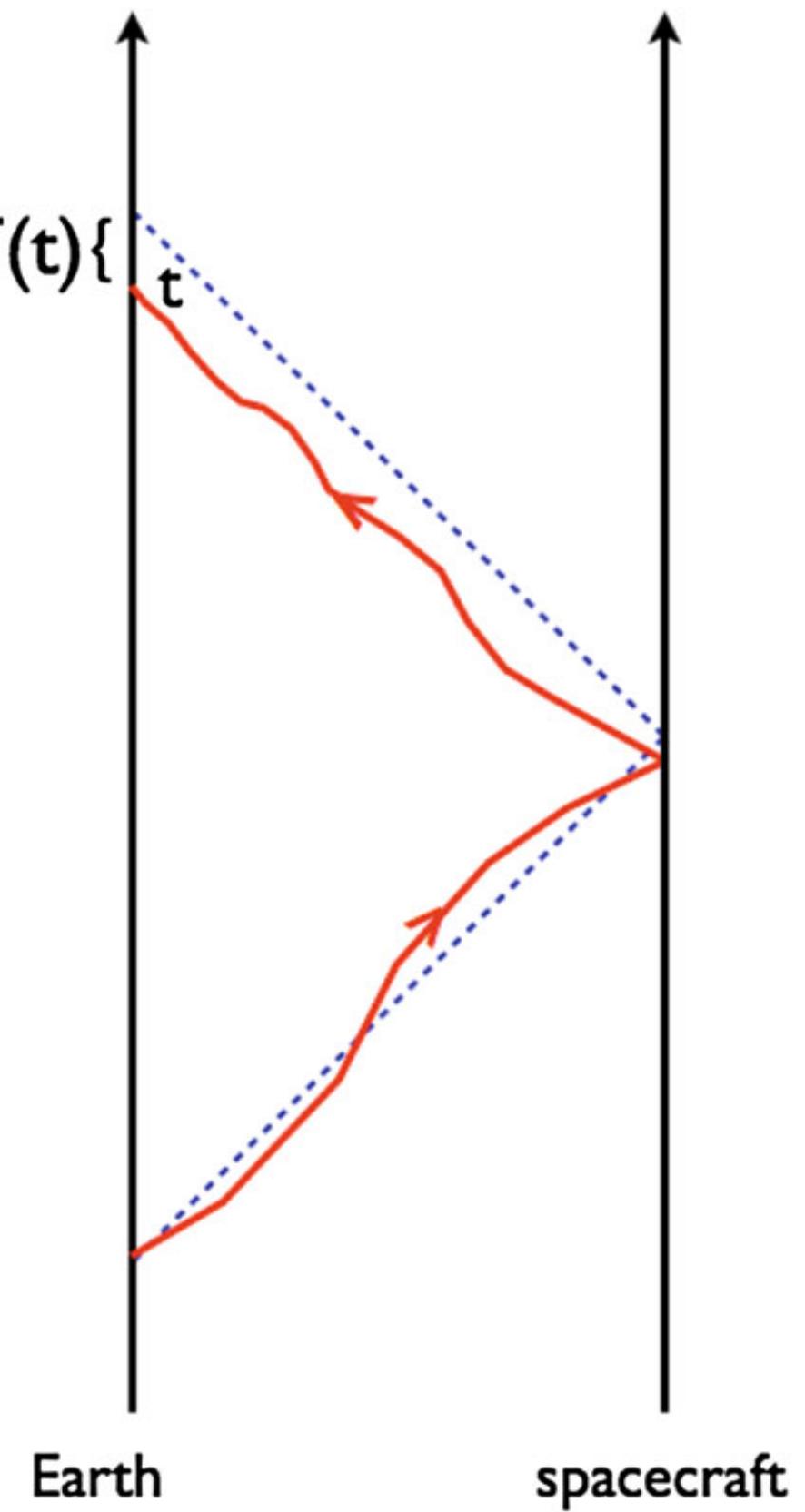
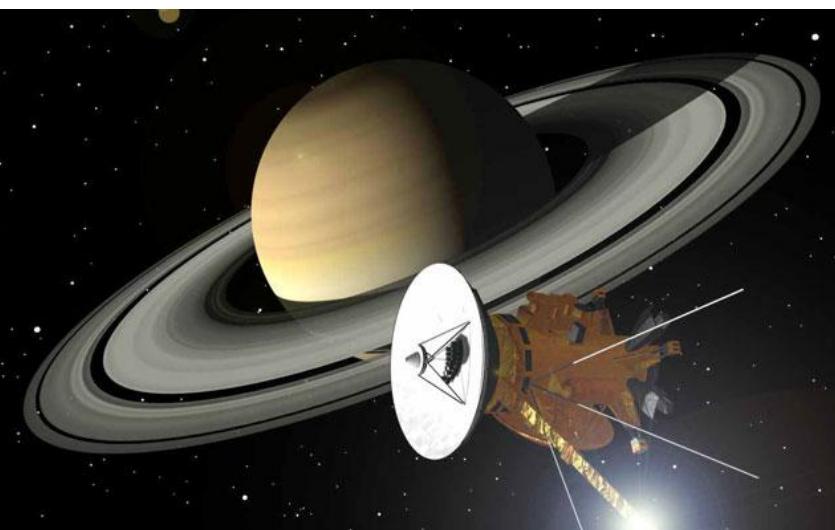
(1-arm, 1-way)



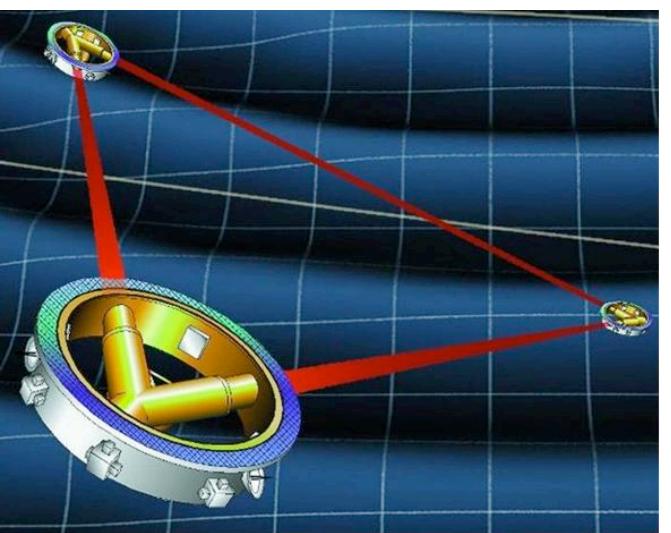
(1-arm, 2-way)



(1-arm, 1-way)



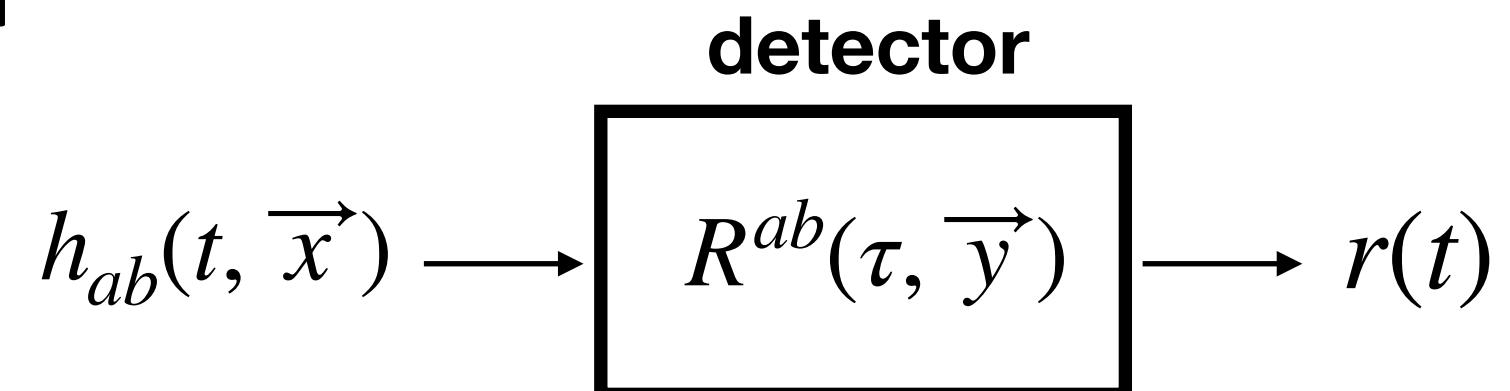
(1-arm, 2-way)



(2-arm, 2-way)

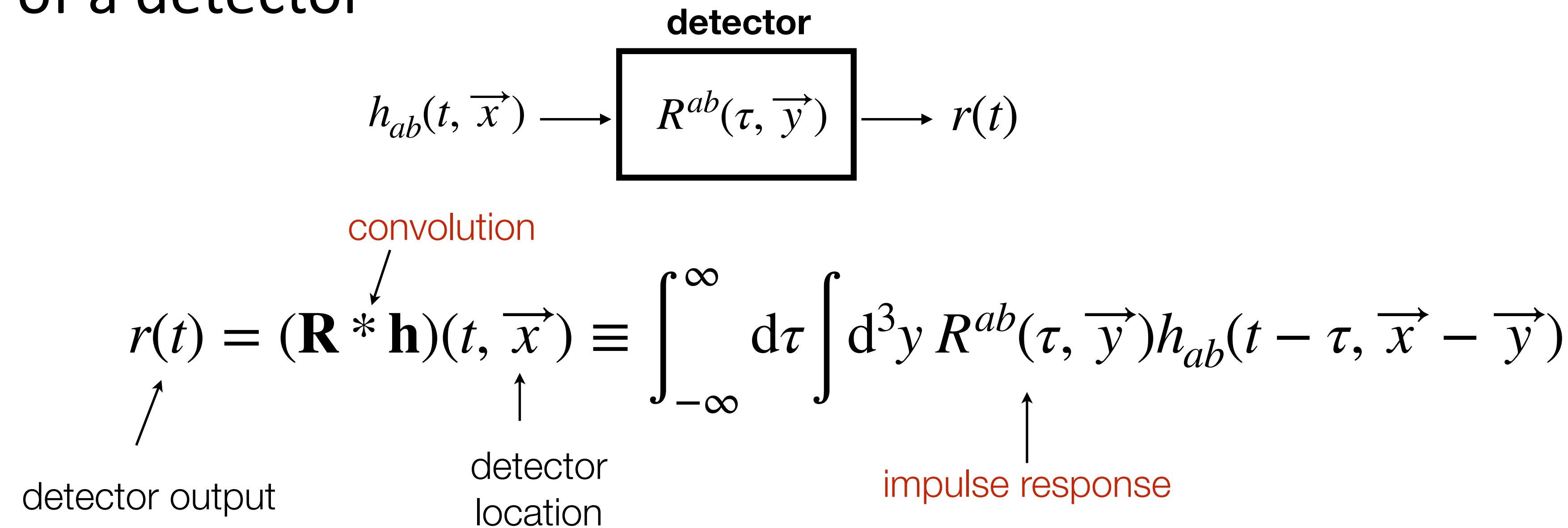
# Detector response function

- GW detector is a linear system that converts metric perturbations to the output of a detector



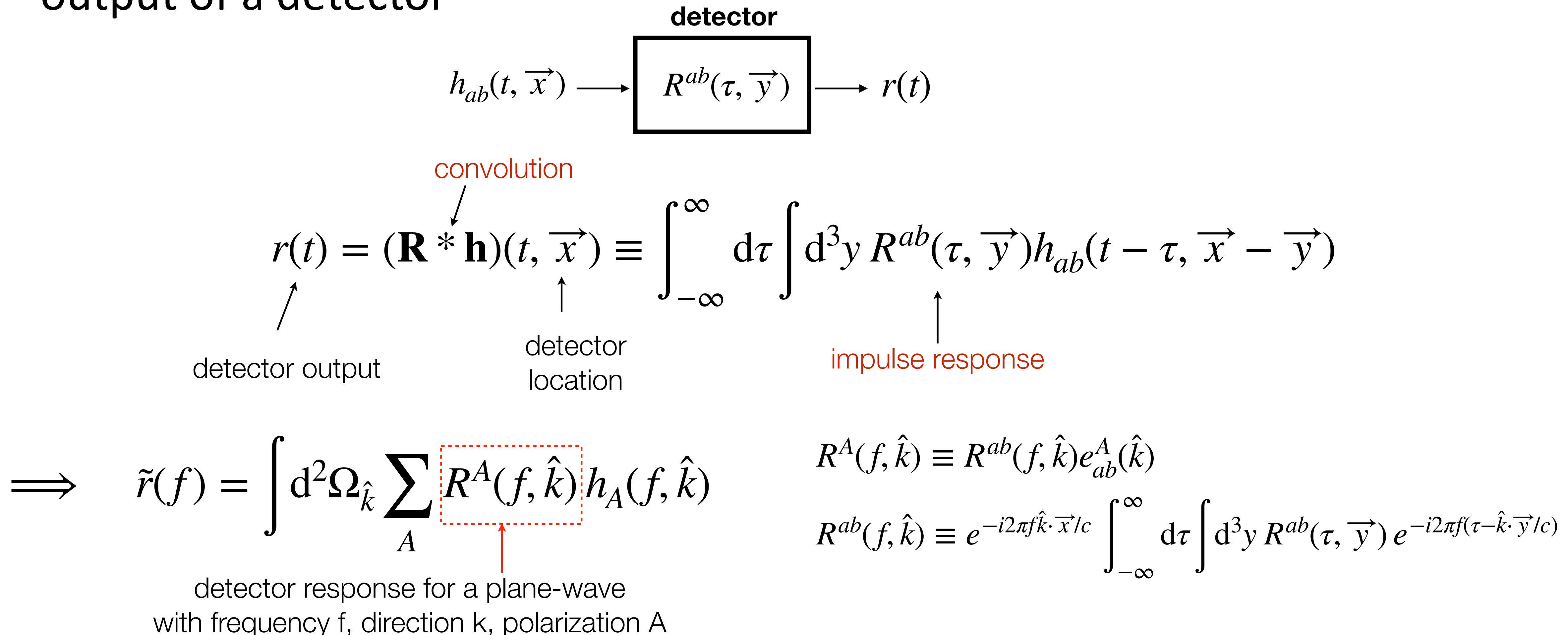
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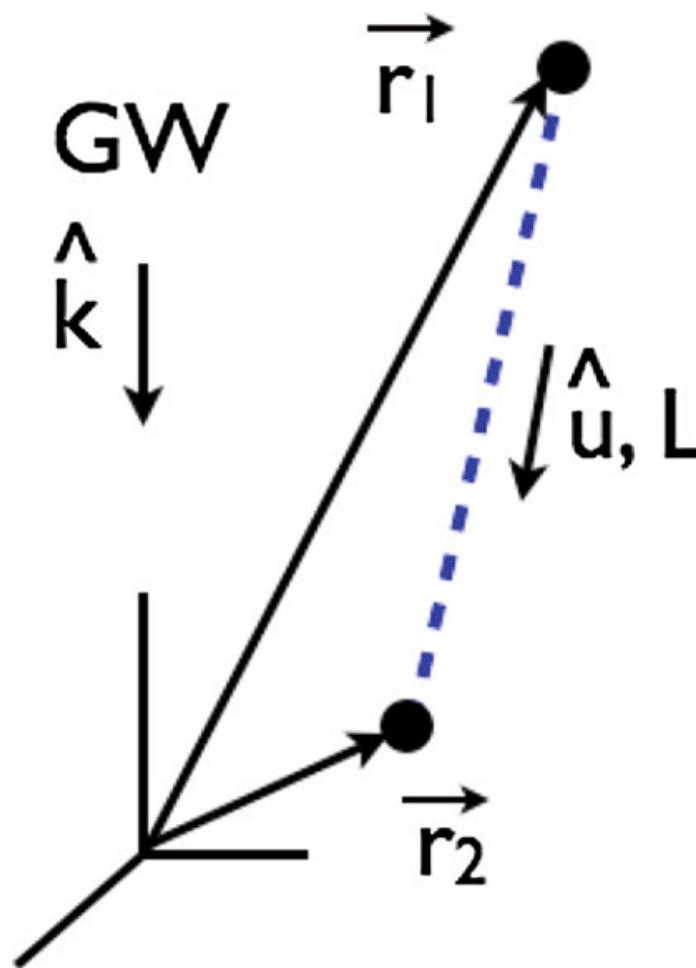


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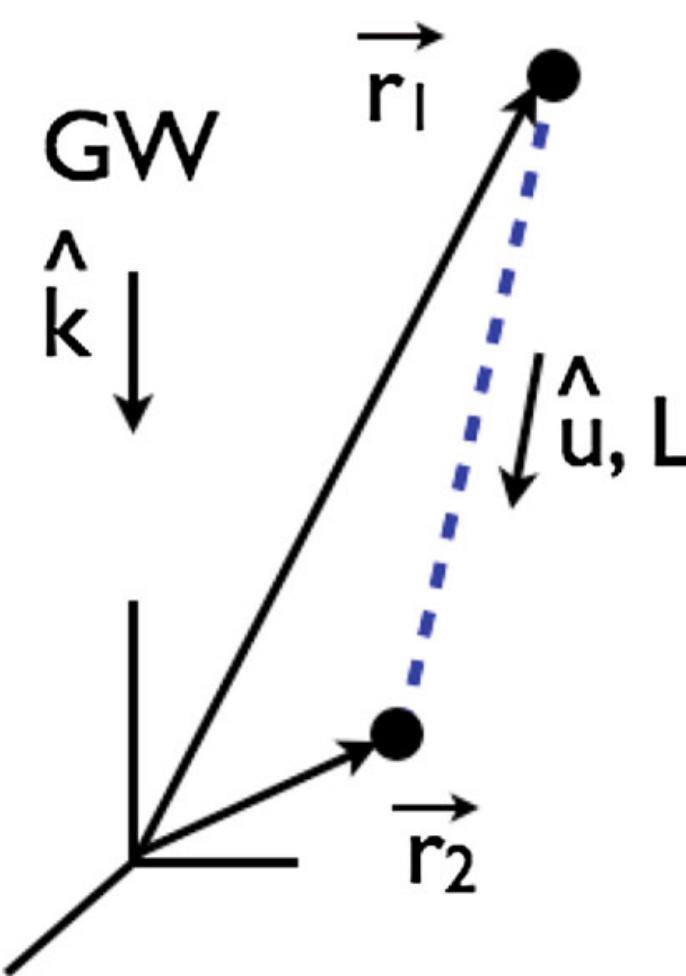


# Example: One-arm, one-way detector (e.g., pulsar timing)



$$r(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L ds h_{ab}(t(s), \vec{x}(s))$$
$$t(s) = (t - L/c) + s/c, \quad \vec{x}(s) = \vec{r}_1 + s\hat{u}$$

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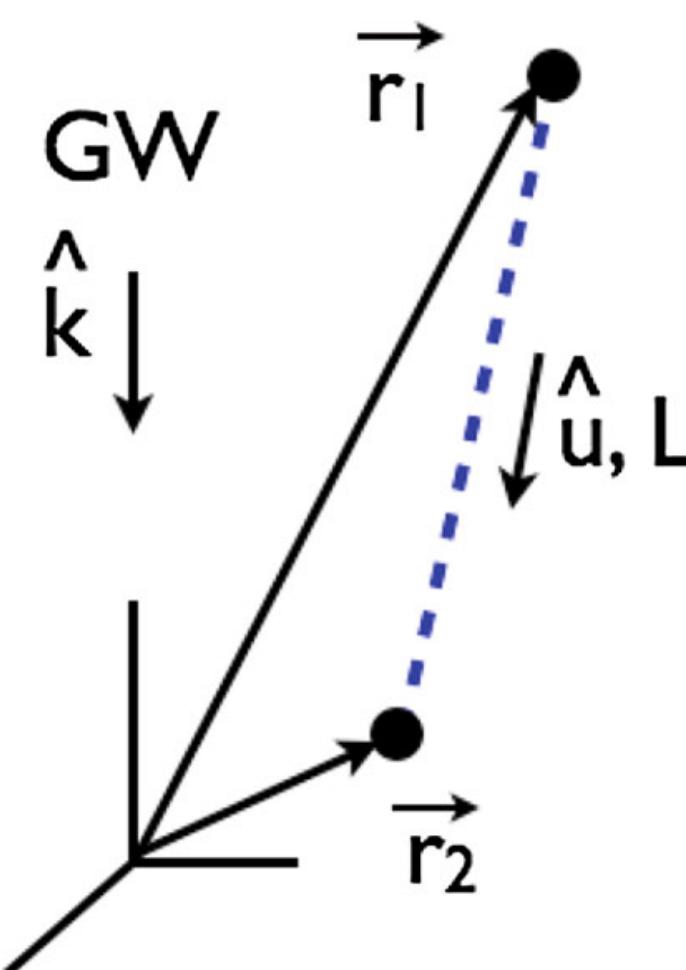


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Exercise: Verify the above expression for the response function

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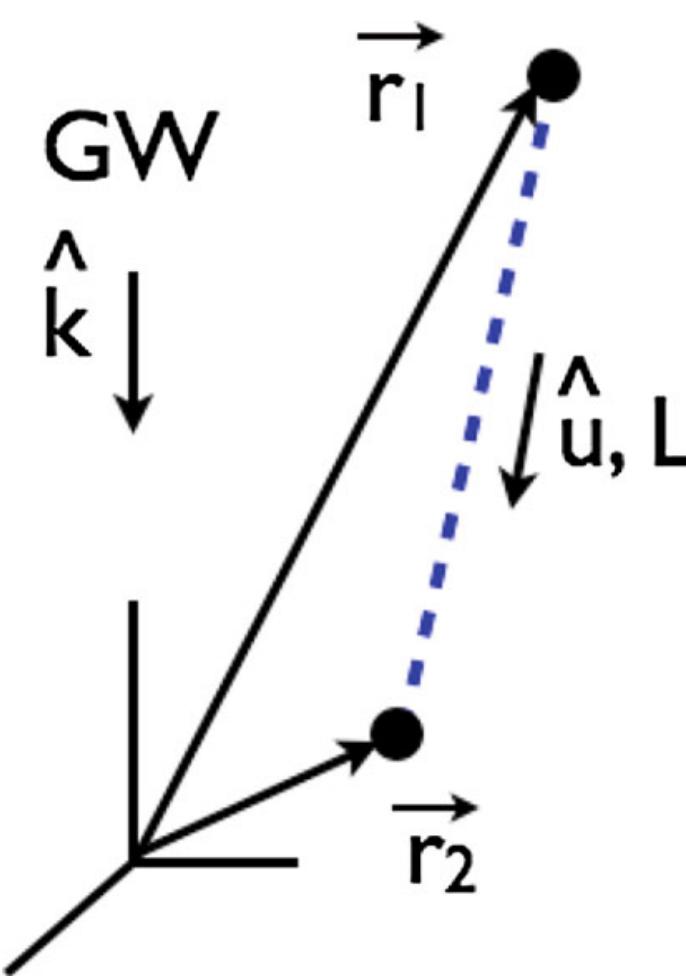
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earth term      pulsar term  
(typically ignored for pulsar timing)

$\uparrow$   
 $F^A(\hat{k})$

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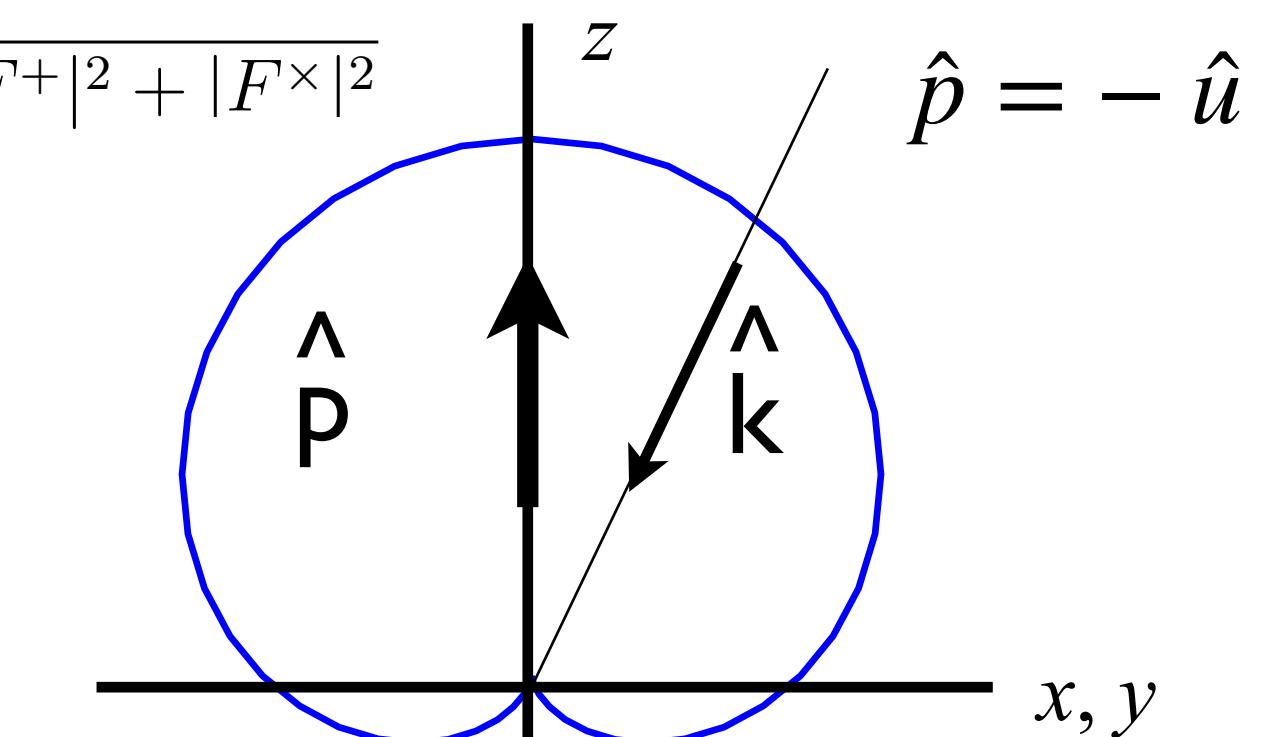
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Exercise: Verify the above expression for the response function



## IV. Overlap reduction functions

# General definition of an ORF (correlation coefficient)

- The ORF between two detectors I and J is defined as the transfer function relating the **expected cross-correlated power** in the two detectors to the strain power in the GWB:

$$C_{IJ}(f) = \Gamma_{IJ}(f)S_h(f) \quad \text{where } I, J \text{ label detectors}$$

- The ORF encodes the reduction in sensitivity due to the physical **separation and misalignment** of the two detectors
- Can be written as the **sky and polarization-averaged product of detector response functions**:

$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_I^A(f, \hat{k}) R_J^{A\star}(f, \hat{k}) \quad \text{where} \quad \tilde{r}_I(f) = \int d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f, \hat{k}) R_I^A(f, \hat{k})$$

# Simple example: Omni-directional detectors

$$R_I^A(f, \hat{k}) = G e^{-i2\pi f \hat{k} \cdot \vec{x}_I / c}$$

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$$= \frac{G^2}{4\pi} \int d^2\Omega_{\hat{k}} e^{-i\frac{2\pi f D_{IJ}}{c} \hat{k} \cdot \hat{\Delta x}_{IJ}}$$

$$= \frac{G^2}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) e^{i\frac{2\pi f D_{IJ}}{c} \cos \theta}$$

...

$$= G^2 \operatorname{sinc} \left( \frac{2\pi f D_{IJ}}{c} \right)$$

$$\vec{x}_I - \vec{x}_J \equiv D_{IJ} \hat{\Delta x}_{IJ}$$

$$\operatorname{sinc} x \equiv \frac{\sin x}{x}$$

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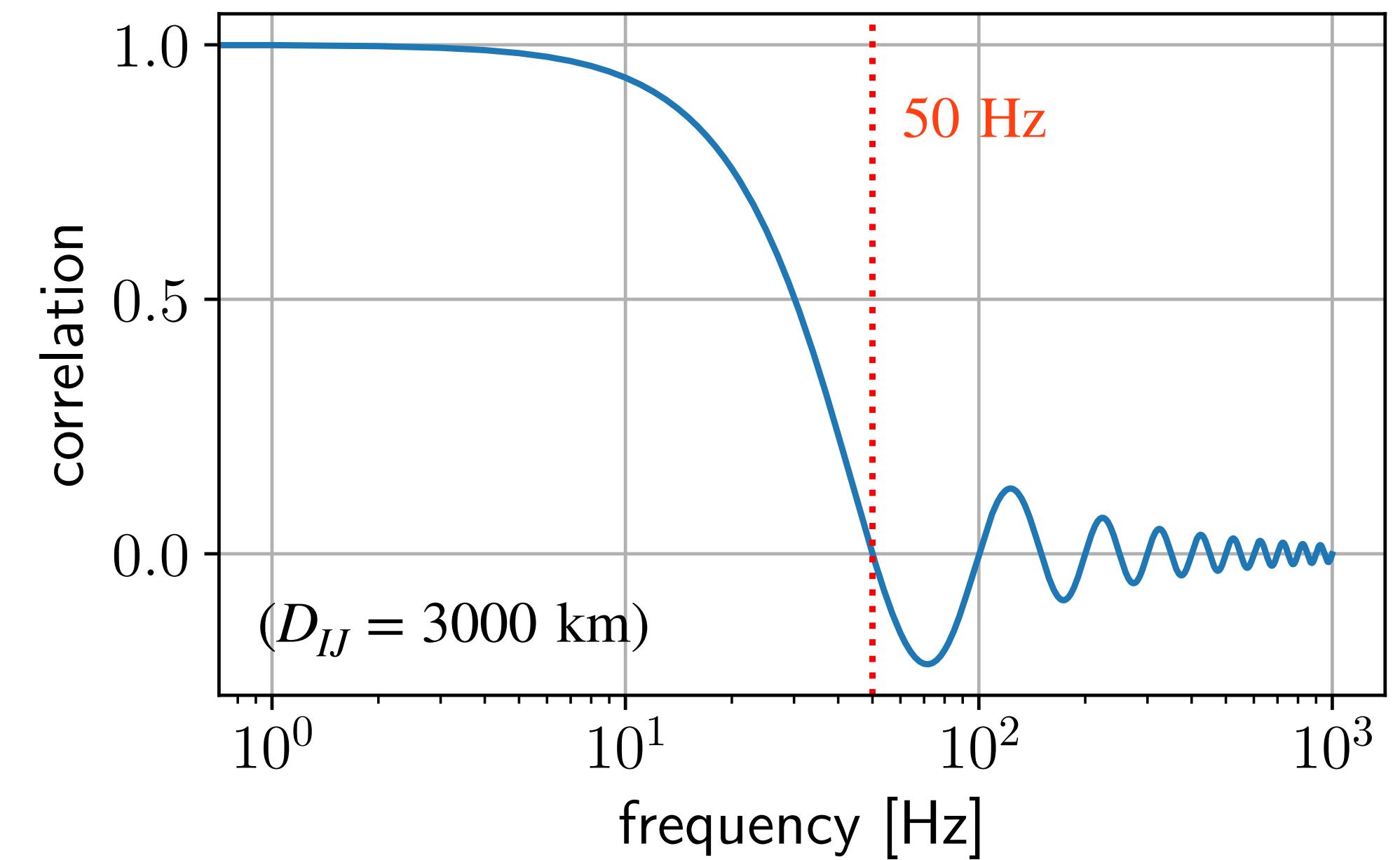
...

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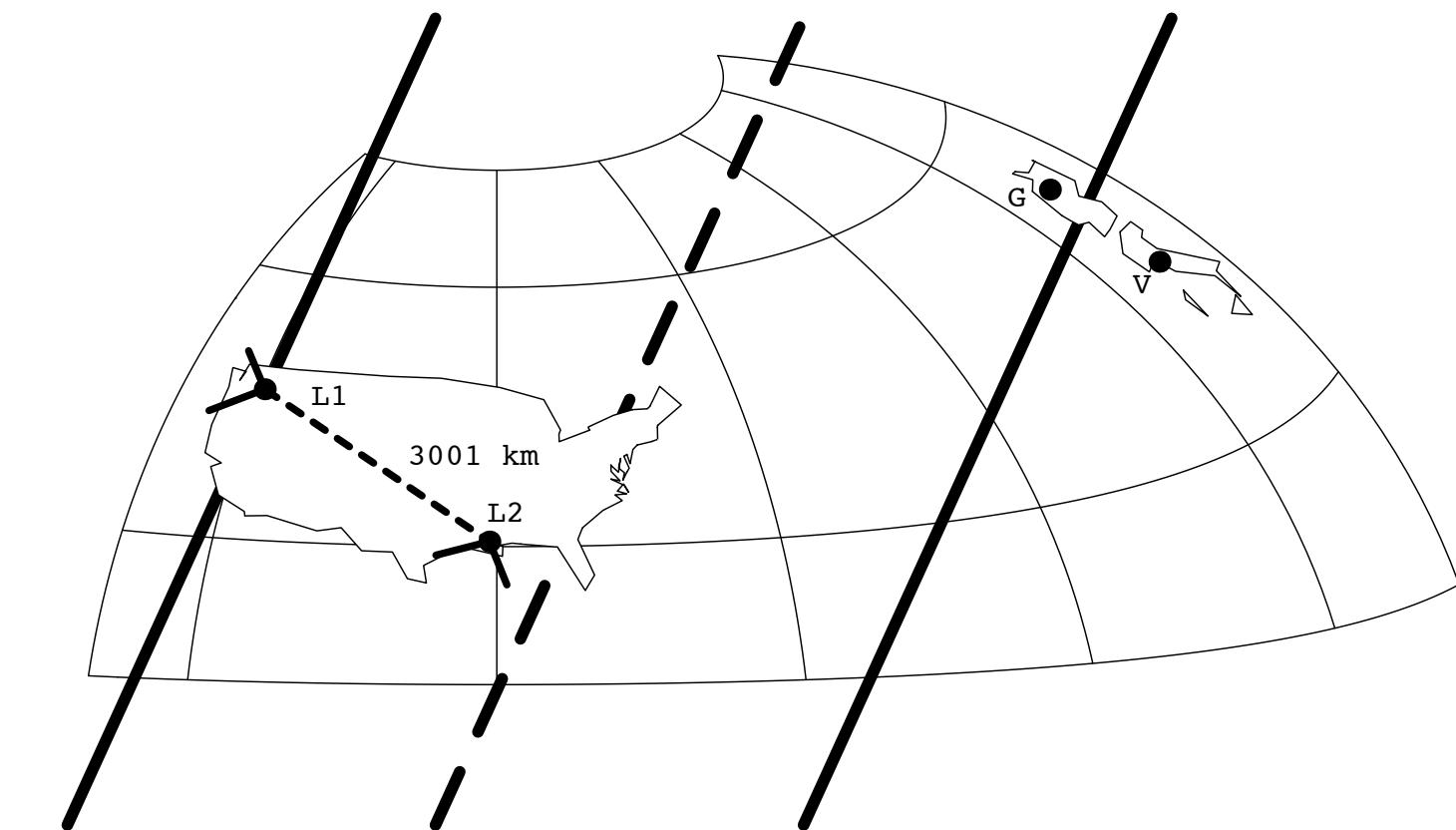
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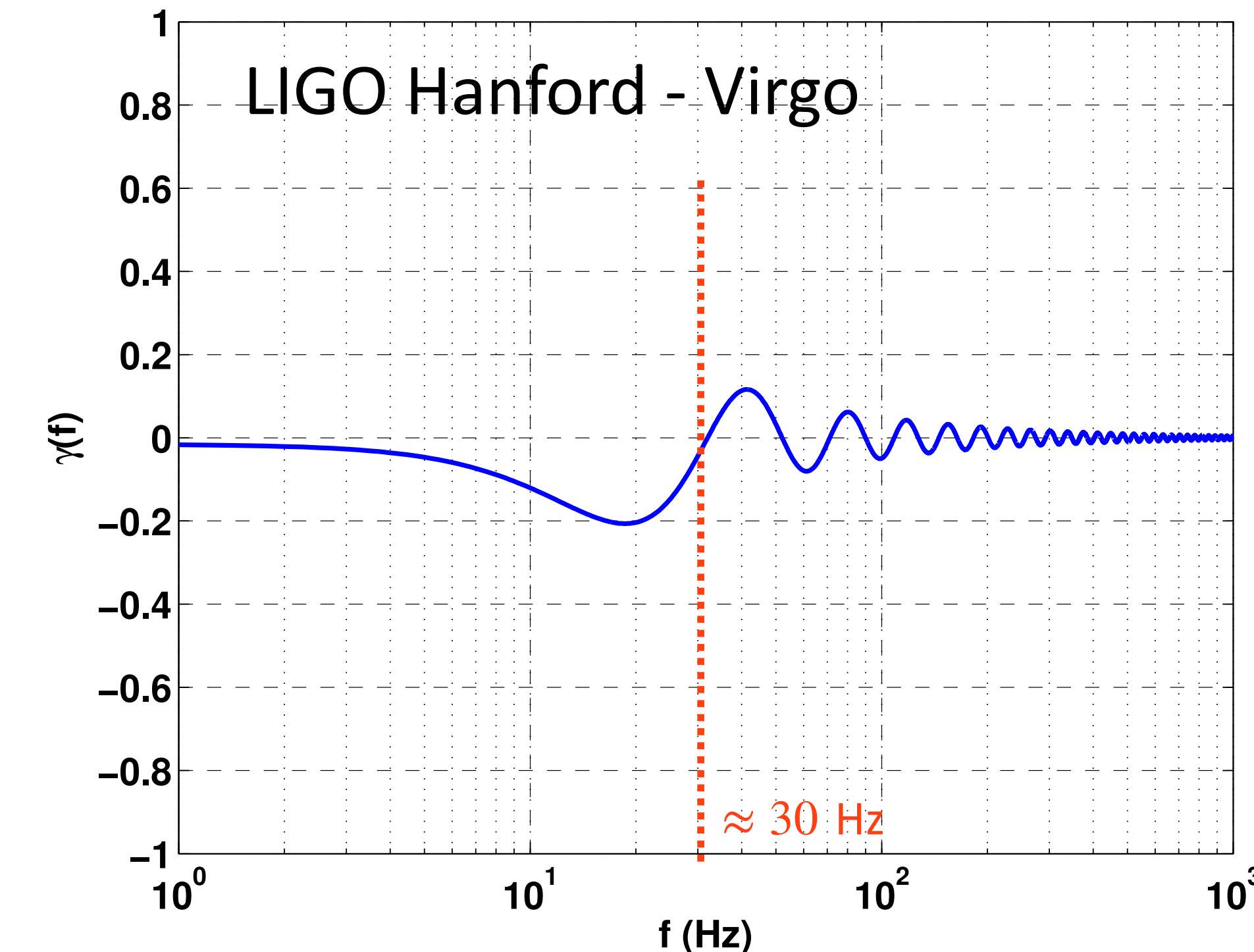
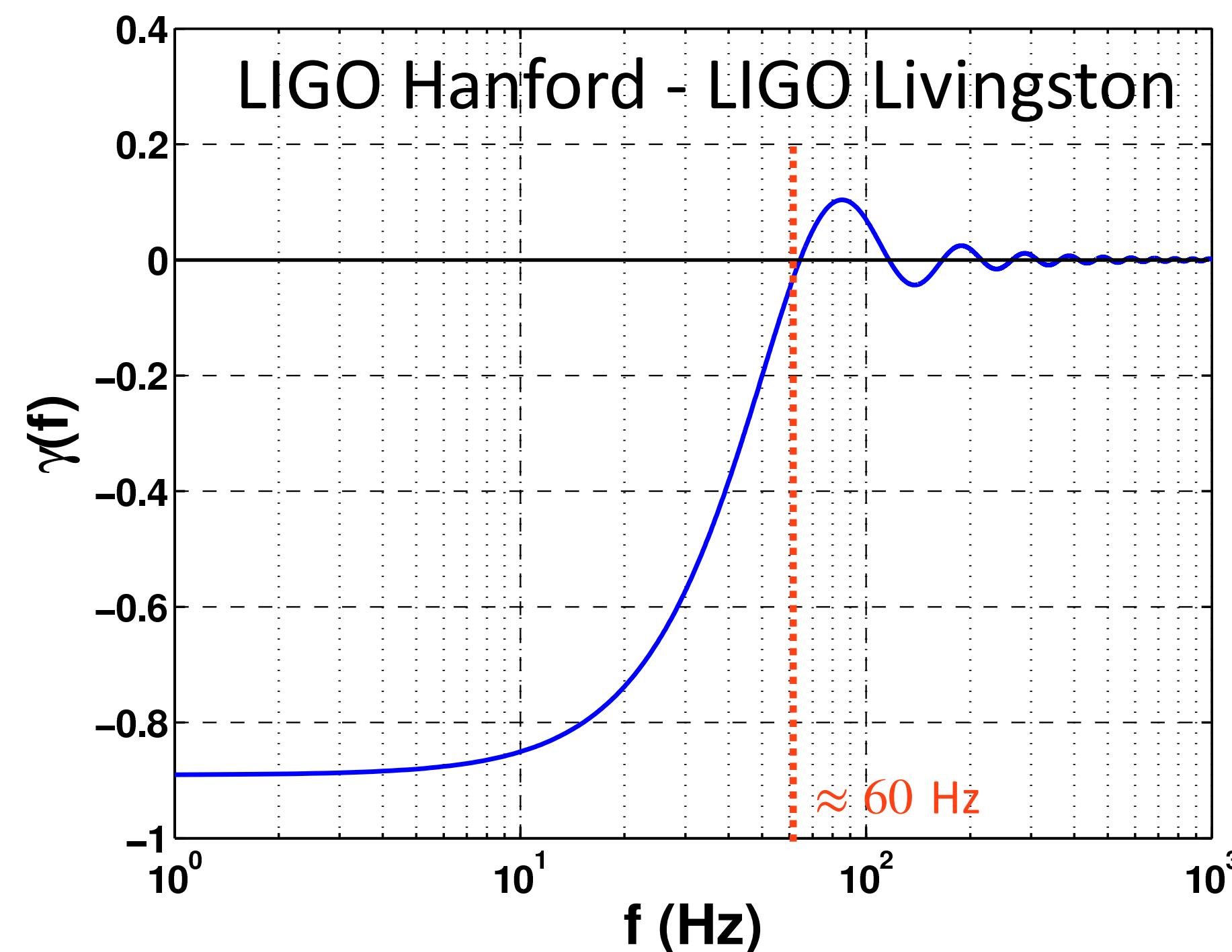
# Example: Ground-based interferometers

$$R_I^A(f, \hat{k}) \simeq \frac{1}{2}(u_I^a u_I^b - v_I^a v_I^b) e_{ab}^A(\hat{k}) e^{-i2\pi f \hat{k} \cdot \vec{x}_I/c}$$

(short-antenna limit)

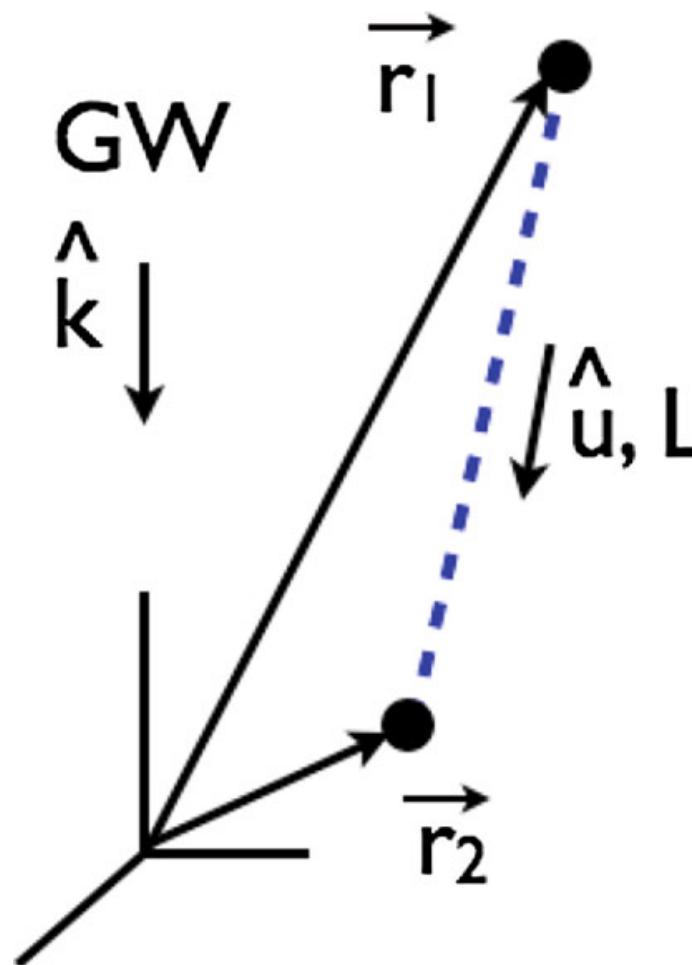


(B. Allen, Les Houches 1995)



# Example: Pulsar timing arrays

- Recall timing residual response:

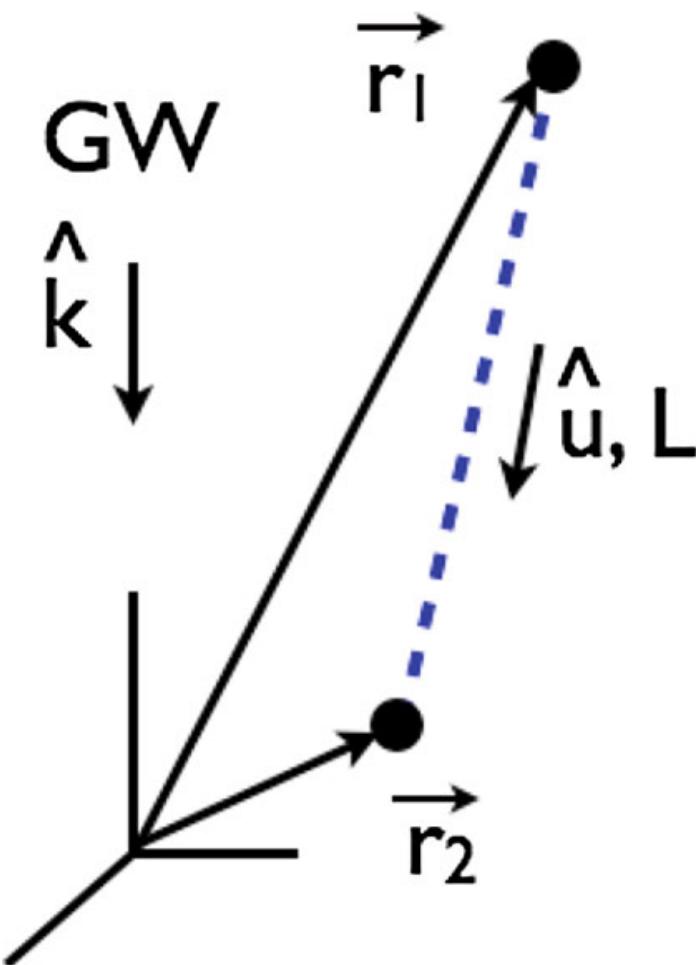


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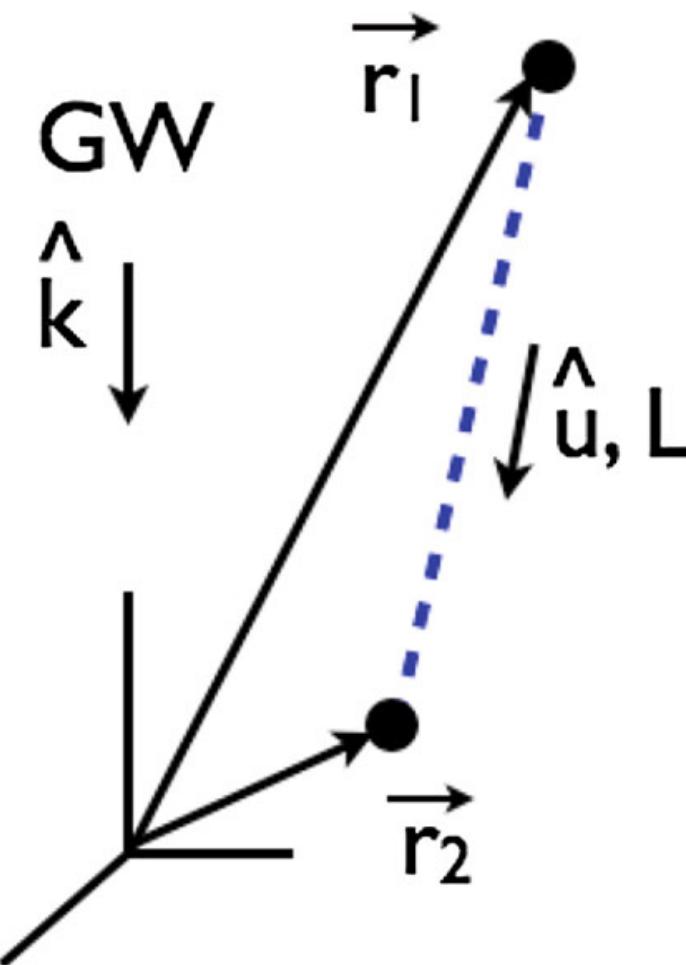
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- Simplification for redshift response  $\Delta\nu(t)/\nu \equiv d\Delta T/dt$ :

$$R_I^A(f, \hat{k}) = F_I^A(k) \left[ 1 - e^{-\frac{i2\pi f L_I}{c}(1 + \hat{k} \cdot \hat{p}_I)} \right] \quad F_I^A(\hat{k}) \equiv \frac{1}{2} \frac{p_I^a p_I^b}{1 + \hat{k} \cdot \hat{p}_I} e_{ab}^A(\hat{k})$$

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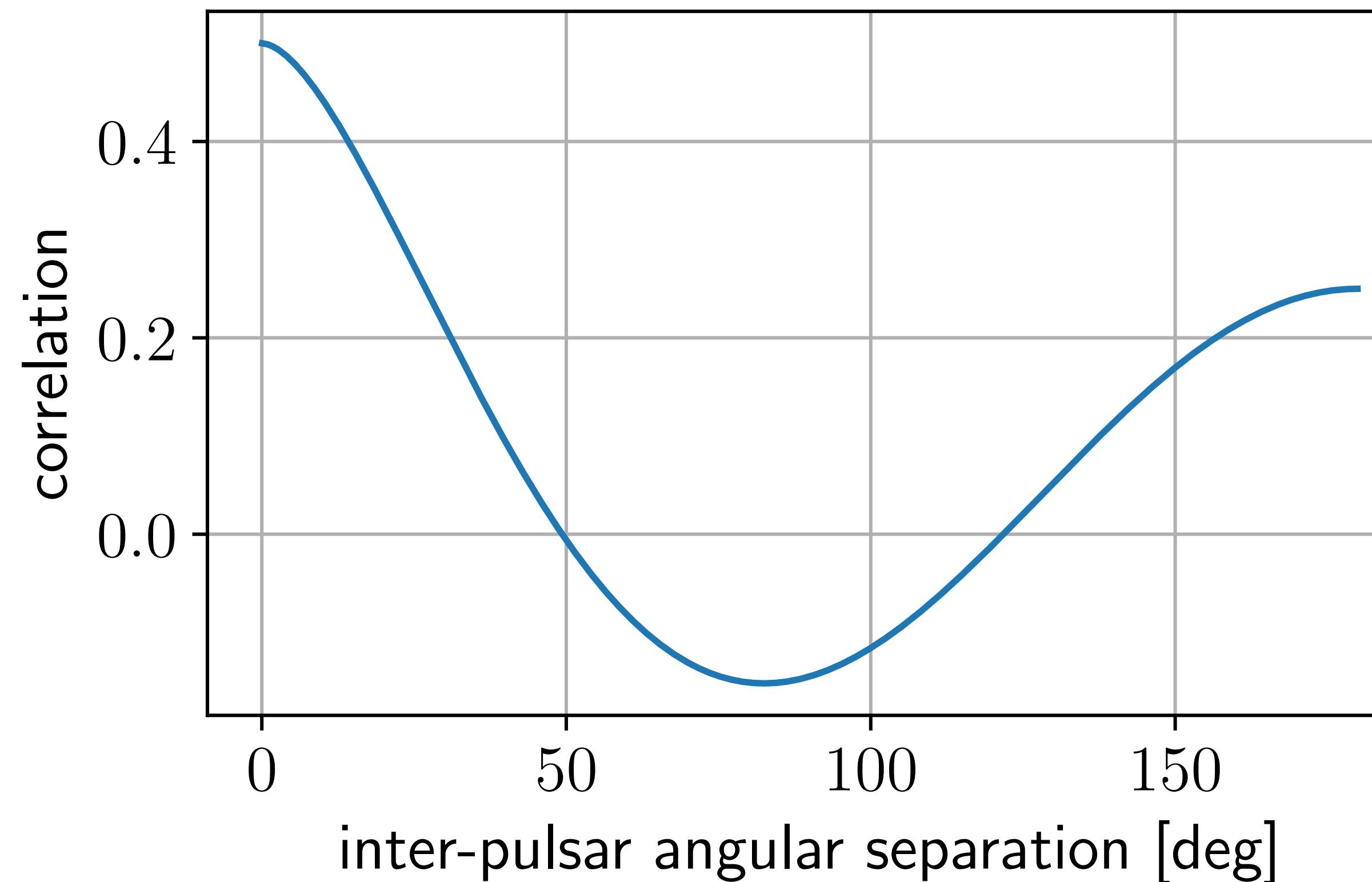
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$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A F_I^A(\hat{k}) F_J^A(\hat{k}) \left[ 1 - e^{-\frac{i2\pi f L_I}{c}(1 + \hat{k} \cdot \hat{p}_I)} \right] \left[ 1 - e^{+\frac{i2\pi f L_J}{c}(1 + \hat{k} \cdot \hat{p}_J)} \right] \simeq \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A F_I^A(\hat{k}) F_J^A(\hat{k}) [1 + \delta_{IJ}]$$

# Hellings and Downs correlation

- Normalizing to 0.5 for zero angular separation:

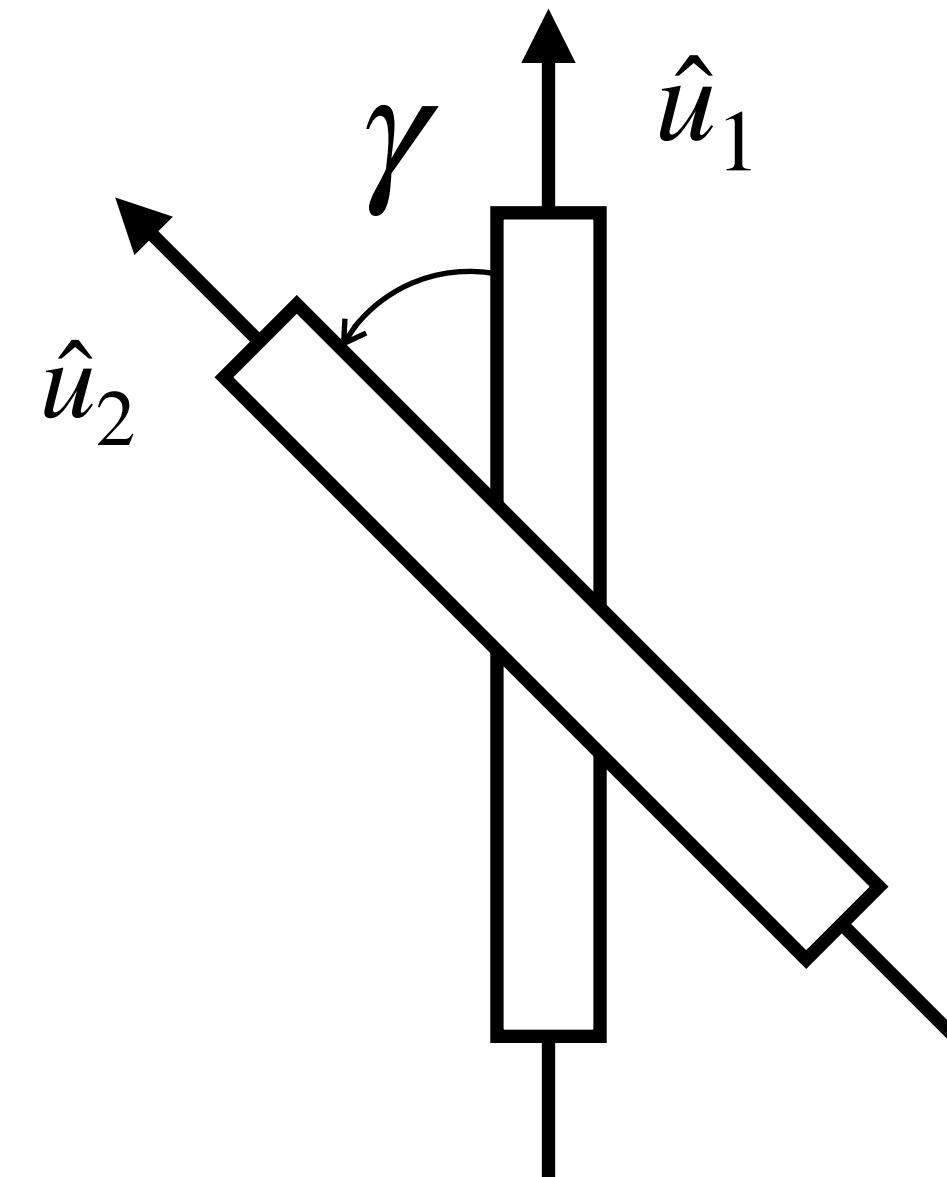
$$\chi_{IJ} = \frac{3}{2} \left( \frac{1 - \cos \gamma_{IJ}}{2} \right) \left[ \ln \left( \frac{1 - \cos \gamma_{IJ}}{2} \right) - \frac{1}{6} \right] + \frac{1}{2} + \frac{1}{2} \delta_{IJ}$$



**Exercises / worked examples  
(please see exercises.pdf)**

# Exercise

- Calculate the value of the ORF for a pair of short electric dipole antennae pointing in directions  $\hat{u}_1$  and  $\hat{u}_2$  for an unpolarized, isotropic electromagnetic field



$$r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x} = \vec{0}) \quad \text{for } I = 1, 2$$

$$\vec{E}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{k}} \sum_{\alpha=1,2} \tilde{E}_{\alpha}(f, \hat{k}) \hat{e}_{\alpha}(\hat{k}) e^{i2\pi f(t - \hat{k} \cdot \vec{x}/c)}$$

$$\hat{e}_1(\hat{k}) = -\hat{\phi}, \quad \hat{e}_2(\hat{k}) = -\hat{\theta}$$

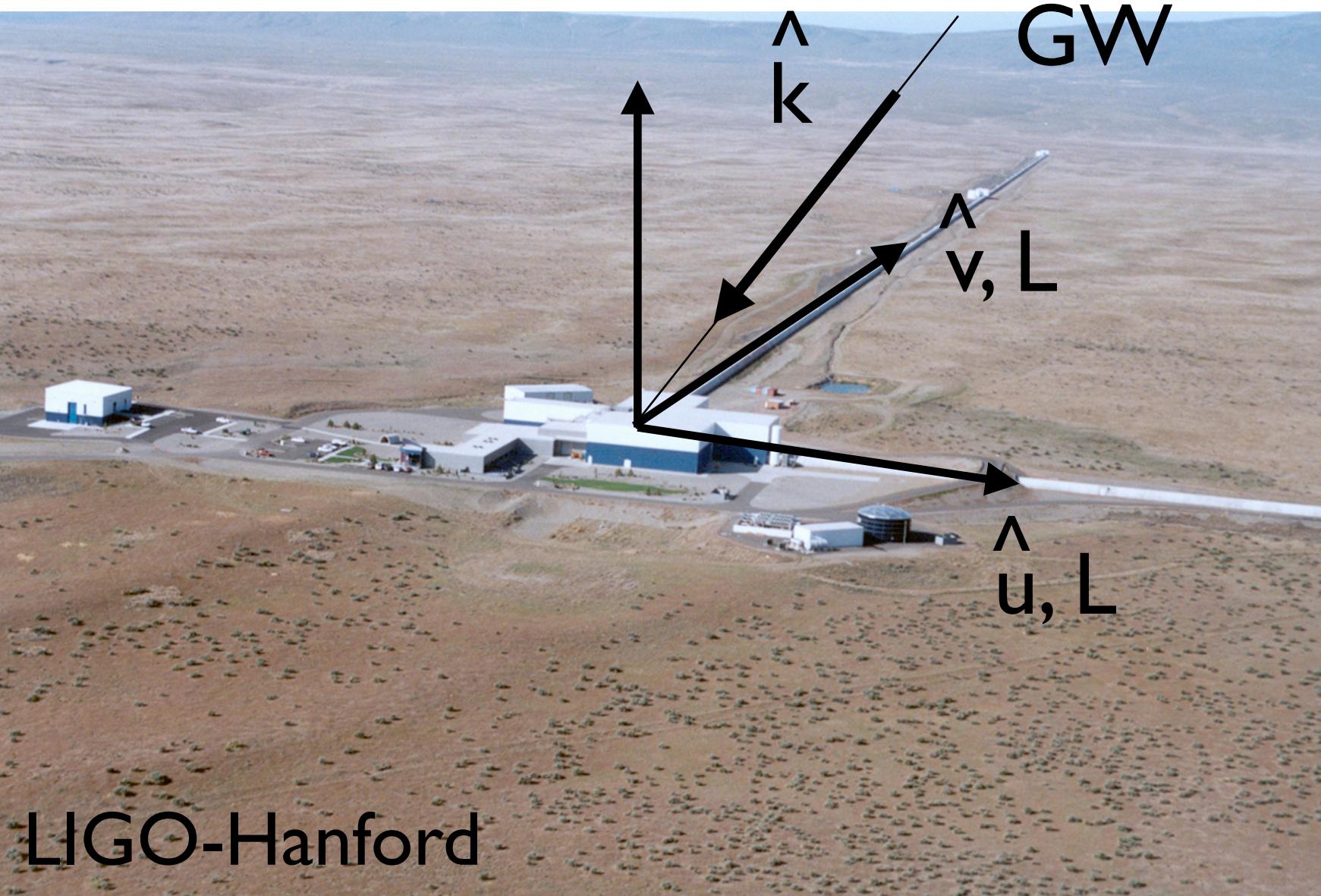
- The final result should depend very simply on  $\gamma$  !

# Extra slides

# Example: Ground-based interferometers (short-antenna, equal arm limit)



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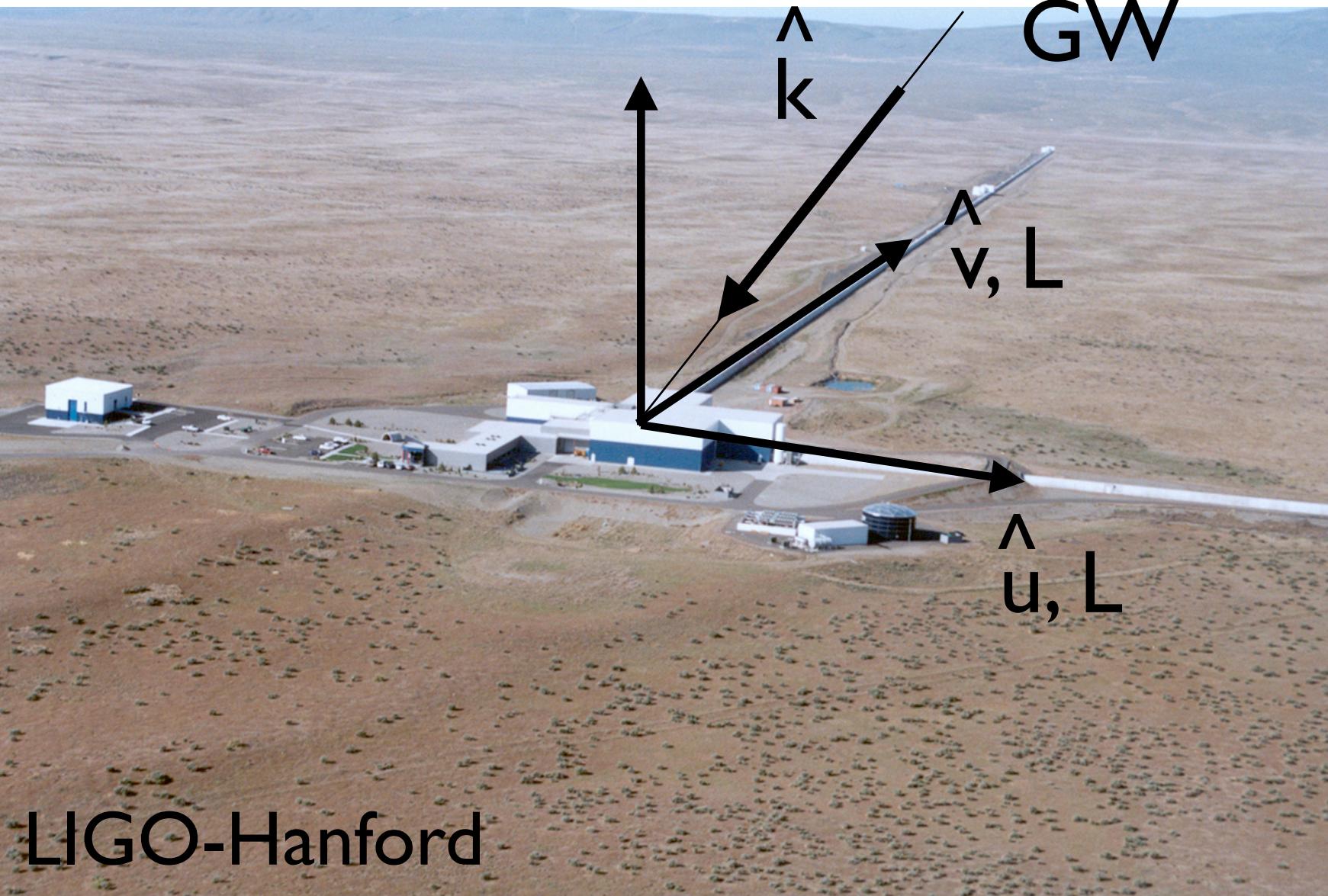


$$r(t) = \frac{1}{2} \left( \frac{\Delta T_{\vec{u}, \text{roundtrip}}(t)}{T} - \frac{\Delta T_{\vec{v}, \text{roundtrip}}(t)}{T} \right)$$

$$R^A(f, \hat{k}) \simeq \boxed{\frac{1}{2} (u^a u^b - v^a v^b)} e_{ab}^A(\hat{k}) e^{-i 2\pi f \hat{k} \cdot \vec{x}/c}$$

detector tensor

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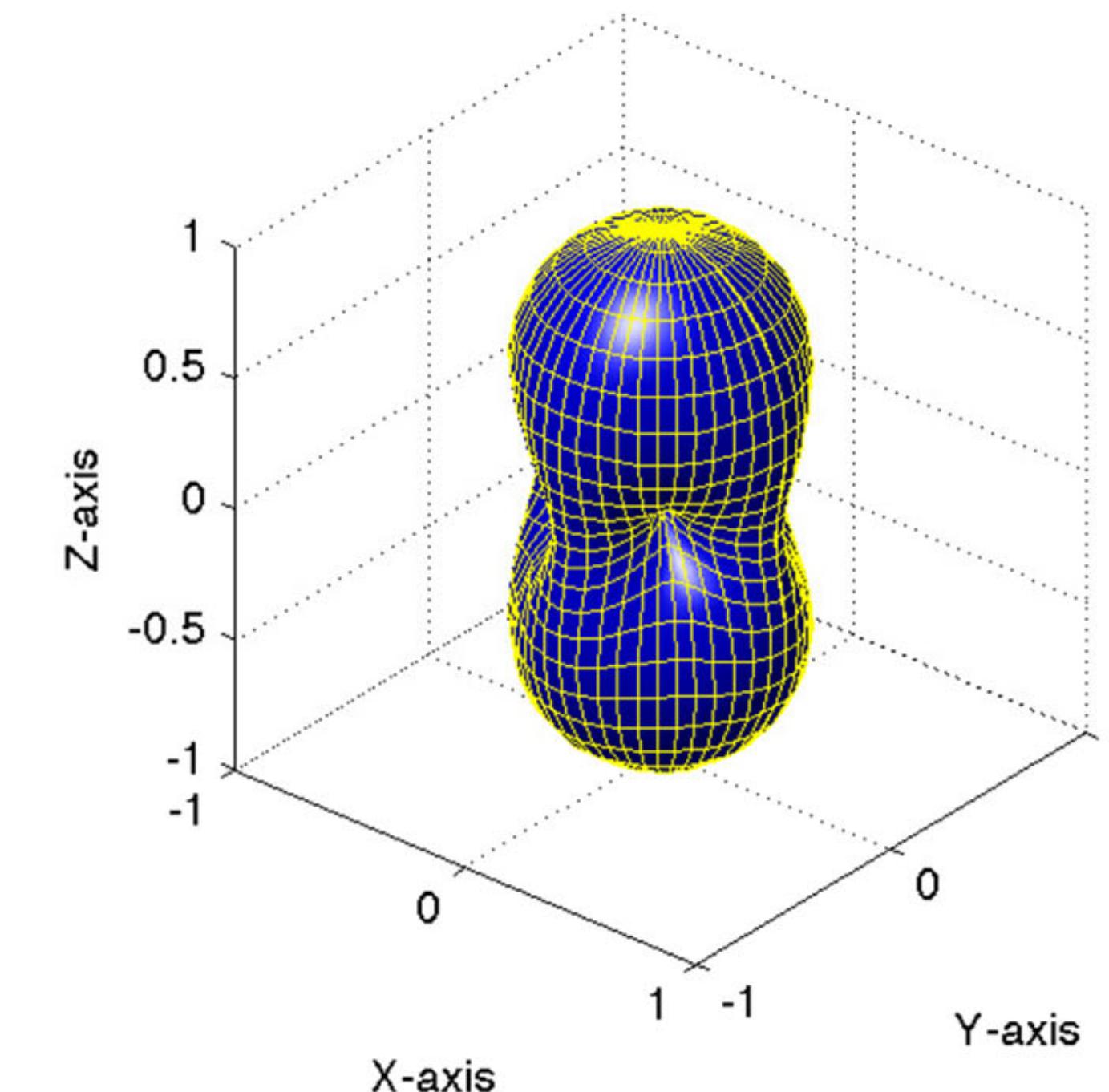


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detector tensor

$$\sqrt{|R^+|^2 + |R^\times|^2}$$



# Joint multi-component analysis

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Data are cross-correlation estimates:

$$\hat{C}_{12}(f) \equiv \frac{2}{T} \tilde{d}_1(f) \tilde{d}_2^*(f)$$

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spectral shapes

amplitudes

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↑  
amplitudes

Noise covariance matrix:

$$N_{12}(f, f') \equiv \langle \hat{C}_{12}(f) \hat{C}_{12}^*(f') \rangle - \langle \hat{C}_{12}(f) \rangle \langle \hat{C}_{12}^*(f') \rangle$$
$$\simeq \delta_{ff'} P_1(f) P_2(f)$$

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Maximum-likelihood estimators:

$$\hat{A} = F^{-1} X \\ F \equiv M^{\dagger} N^{-1} M, \quad X \equiv M^{\dagger} N^{-1} \hat{C}$$

$$F_{\alpha\beta} = \int_{-\infty}^{\infty} df \frac{H_{\alpha}(f) \Gamma_{12}^2(f) H_{\beta}(f)}{P_1(f) P_2(f)}$$

(noise-weighted inner product of spectral shapes;  
inverse covariance matrix for A )

Fisher matrix

# Joint multi-component analysis

Data are cross-correlation estimates:

$$\hat{C}_{12}(f) \equiv \frac{2}{T} \tilde{d}_1(f) \tilde{d}_2^*(f)$$

Expectation value:

$$\langle \hat{C}_{12}(f) \rangle = \sum_{\alpha} A_{\alpha} \Gamma_{12}(f) H_{\alpha}(f) \equiv \sum_{\alpha} M_{\alpha}(f) A_{\alpha}$$

↑  
amplitudes

Noise covariance matrix:

$$N_{12}(f, f') \equiv \langle \hat{C}_{12}(f) \hat{C}_{12}^*(f') \rangle - \langle \hat{C}_{12}(f) \rangle \langle \hat{C}_{12}^*(f') \rangle \\ \simeq \delta_{ff'} P_1(f) P_2(f)$$

Likelihood function:

$$p(\hat{C} | A, N) \propto \exp \left[ -\frac{1}{2} (\hat{C} - MA)^{\dagger} N^{-1} (\hat{C} - MA) \right]$$

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**Exercise:** Verify the expression  
for the ML estimators.

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## **IV. Cross correlation in frequentist (maximum-likelihood) and Bayesian analyses**

# Likelihood functions

- Recall that a likelihood function is a starting point for both frequentist & Bayesian analyses:

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- N samples of white noise, white GWB, in two colocated and coaligned detectors:

$$C_n = \begin{bmatrix} S_{n_1} \mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & S_{n_2} \mathbf{1}_{N \times N} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} (S_{n_1} + S_h) \mathbf{1}_{N \times N} & S_h \mathbf{1}_{N \times N} \\ S_h \mathbf{1}_{N \times N} & (S_{n_2} + S_h) \mathbf{1}_{N \times N} \end{bmatrix}$$

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$$\hat{S}_h \equiv \frac{1}{N} \sum_{i=1}^N d_{1i} d_{2i} \quad \longleftarrow \text{cross-correlation estimator}$$

$$\hat{S}_{n_1} \equiv \frac{1}{N} \sum_{i=1}^N d_{1i}^2 - \hat{S}_h$$

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Exercise: Verify the expression for the detection statistic  $2 \ln(\Lambda_{\text{ML}}(d))$

# Bayesian analyses

- Use Bayes' theorem to calculate posterior distributions for parameter estimation and odds ratios (Bayes factors) for model selection

$$p(\theta | d, \mathcal{M}) = \frac{p(d | \theta, \mathcal{M}) p(\theta | \mathcal{M})}{p(d | \mathcal{M})}$$

The diagram shows the Bayes' theorem formula  $p(\theta | d, \mathcal{M}) = \frac{p(d | \theta, \mathcal{M}) p(\theta | \mathcal{M})}{p(d | \mathcal{M})}$ . Four arrows point to different parts of the equation: 'posterior' points to the leftmost term  $p(\theta | d, \mathcal{M})$ ; 'likelihood' points to the first term in the numerator  $p(d | \theta, \mathcal{M})$ ; 'prior' points to the second term in the numerator  $p(\theta | \mathcal{M})$ ; and 'evidence' points to the denominator  $p(d | \mathcal{M})$ .

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Diagram illustrating the components of Bayes' Theorem:

- Posterior**:  $p(\theta | d, \mathcal{M})$
- likelihood**:  $p(d | \theta, \mathcal{M})$
- prior**:  $p(\theta | \mathcal{M})$
- evidence**:  $p(d | \mathcal{M})$

Arrows point from the labels to their corresponding terms in the equation.

- **Posteriors:**

$$p(S_{n_1}, S_{n_2}, S_h | d, \mathcal{M}_1) = \frac{p(d | S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1)p(S_{n_1}, S_{n_2}, S_h | \mathcal{M}_1)}{p(d | \mathcal{M}_1)}$$

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posterior →      likelihood      prior  
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- Model selection:

$$\frac{p(\mathcal{M}_1 | d)}{p(\mathcal{M}_0 | d)} = \frac{p(d | \mathcal{M}_1) p(\mathcal{M}_1)}{p(d | \mathcal{M}_0) p(\mathcal{M}_0)}$$

Bayes factor  $\mathcal{B}_{10}(d)$

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Bayes factor  $\mathcal{B}_{10}(d)$

- Relationship to frequentist approach:

$$\mathcal{B}_{10}(d) \equiv \frac{p(d | \mathcal{M}_1)}{p(d | \mathcal{M}_0)} = \frac{\int dS_{n_1} \int dS_{n_2} \int dS_h p(d | S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1) p(S_{n_1}, S_{n_2}, S_h | \mathcal{M}_1)}{\int dS_{n_1} \int dS_{n_2} p(d | S_{n_1}, S_{n_2}, \mathcal{M}_0) p(S_{n_1}, S_{n_2} | \mathcal{M}_0)} \simeq \Lambda_{\text{ML}}(d) \frac{\Delta V_1 / V_1}{\Delta V_0 / V_0}$$

