

# Introduction to Statistical Learning and Machine Learning

## Chap 4 – Linear Classification

Yanwei Fu

School of Data Science, Fudan University



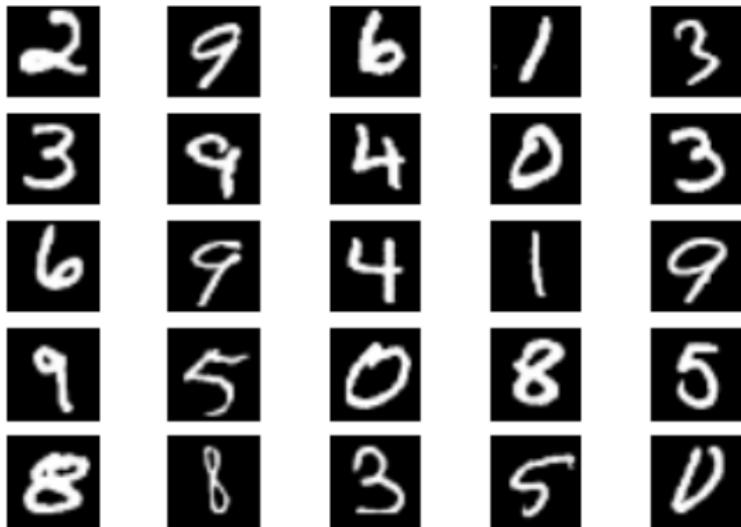
① Recap&Quick Overview of Linear Classification

② Logistic Regression

③ Linear Discriminant Analysis



# Examples of Classification



What digit is this?

How can I predict this? What are my input features?

# (Binary) Classification as Regression

- Can we do this task using what we have learned in previous lectures?
- Simple hack: Ignore that the input is categorical!
- Suppose we have a binary problem,  $t \in \{-1, 1\}$
- Assuming the standard model used for regression

$$y = f(\mathbf{x}, \mathbf{w}) = \mathbf{w}^T \mathbf{x}$$

- How can we obtain  $\mathbf{w}$ ?
- Use least squares,  $\mathbf{w} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{t}$ . How is  $\mathbf{X}$  computed? and  $\mathbf{t}$ ?
- Which loss are we minimizing? Does it make sense?

$$\ell_{square}(\mathbf{w}, t) = \frac{1}{N} \sum_{n=1}^N (t^{(n)} - \mathbf{w}^T \mathbf{x}^{(n)})^2$$

- How do I compute a label for a new example? Let's see an example

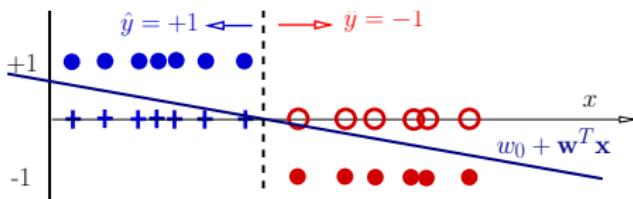


## Recap&Quick Overview of Linear Classification



# Classification as Regression





- Our classifier has the form

$$f(\mathbf{x}, \mathbf{w}) = w_o + \mathbf{w}^T \mathbf{x}$$

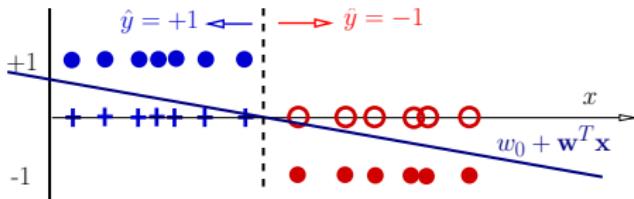
- A reasonable decision rule is

$$y = \begin{cases} 1 & \text{if } f(\mathbf{x}, \mathbf{w}) \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

- How can I mathematically write this rule?

$$y = \text{sign}(w_0 + \mathbf{w}^T \mathbf{x})$$

- How does this function look like?



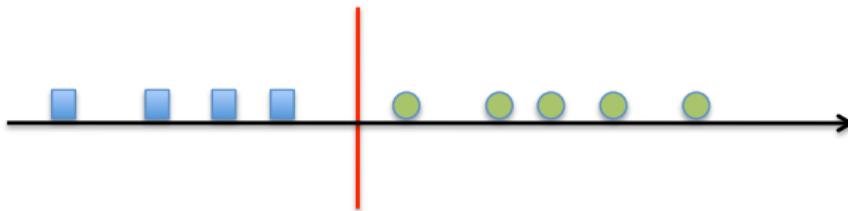
- How can I mathematically write this rule?

$$y = \text{sign}(w_0 + \mathbf{w}^T \mathbf{x})$$

- This specifies a **linear classifier**: it has a **linear boundary (hyperplane)**

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

which separates the space into two "half-spaces"

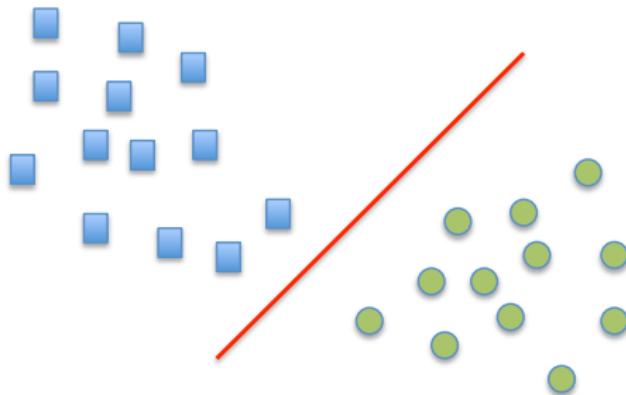


- The **linear classifier** has a **linear boundary** (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

which separates the space into two "half-spaces"

- In 1D this is simply a threshold

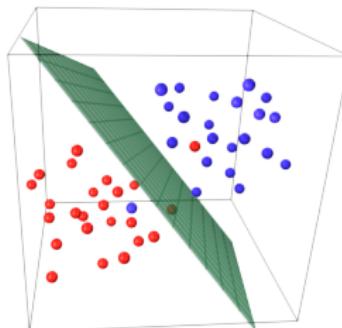


- The **linear classifier** has a **linear boundary** (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

which separates the space into two "half-spaces"

- In 2D this is a line



- The **linear classifier** has a **linear boundary (hyperplane)**

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

which separates the space into two "half-spaces"

- In 3D this is a plane
- What about higher-dimensional spaces?

$\mathbf{w}^T \mathbf{x} = 0$  a line passing through the origin and orthogonal to  $\mathbf{w}$   
 $\mathbf{w}^T \mathbf{x} + w_0 = 0$  shifts it by  $w_0$

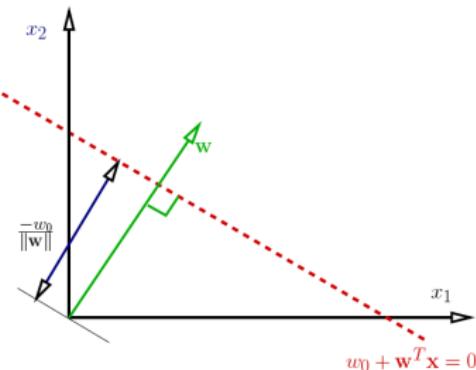
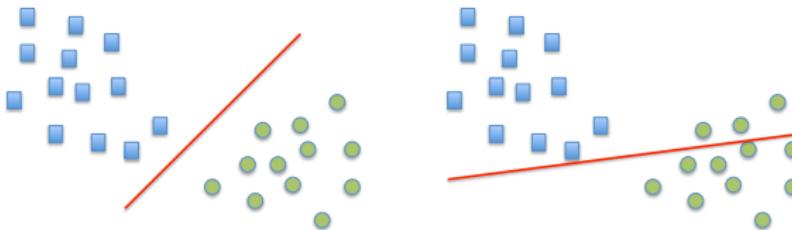


Figure from G. Shakhnarovich

- Learning consists in estimating a "good" decision boundary
- We need to find  $w$  (direction) and  $w_0$  (location) of the boundary
- What does "good" mean?
- Is this boundary good?



- We need a criteria that tell us how to select the parameters
- Do you know any?

# Other Loss functions

- Classifying using a linear decision boundary reduces the data dimension to 1

$$y(\mathbf{x}) = \text{sign}(w_0 + \mathbf{w}^T \mathbf{x})$$

- What is the cost of being wrong?
- Loss function:**  $L(y, t)$  is the loss incurred for predicting  $y$  when correct answer is  $t$
- For medical diagnosis: For a diabetes screening test is it better to have false positives or false negatives?
- For movie ratings: The "truth" is that Alice thinks E.T. is worthy of a 4. How bad is it to predict a 5? How about a 2?



# More complex Loss Functions

- A possible loss to minimize is the **zero/one loss**

$$L(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Is this minimization easy to do? why?



# Can we always separate the classes?

- Zero/one loss for a classifier

$$L_{0-1}(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Asymmetric Binary Loss

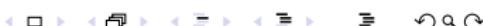
$$L_{ABL}(y(\mathbf{x}), t) = \begin{cases} \alpha & \text{if } y(\mathbf{x}) = 1 \wedge t = 0 \\ \beta & \text{if } y(\mathbf{x}) = 0 \wedge t = 1 \\ 0 & \text{if } y(\mathbf{x}) = t \end{cases}$$

- Squared (quadratic) loss

$$L_{squared}(y(\mathbf{x}), t) = (t - y(\mathbf{x}))^2$$

- Absolute Error

$$L_{quadratic}(y(\mathbf{x}), t) = |t - y(\mathbf{x})|$$

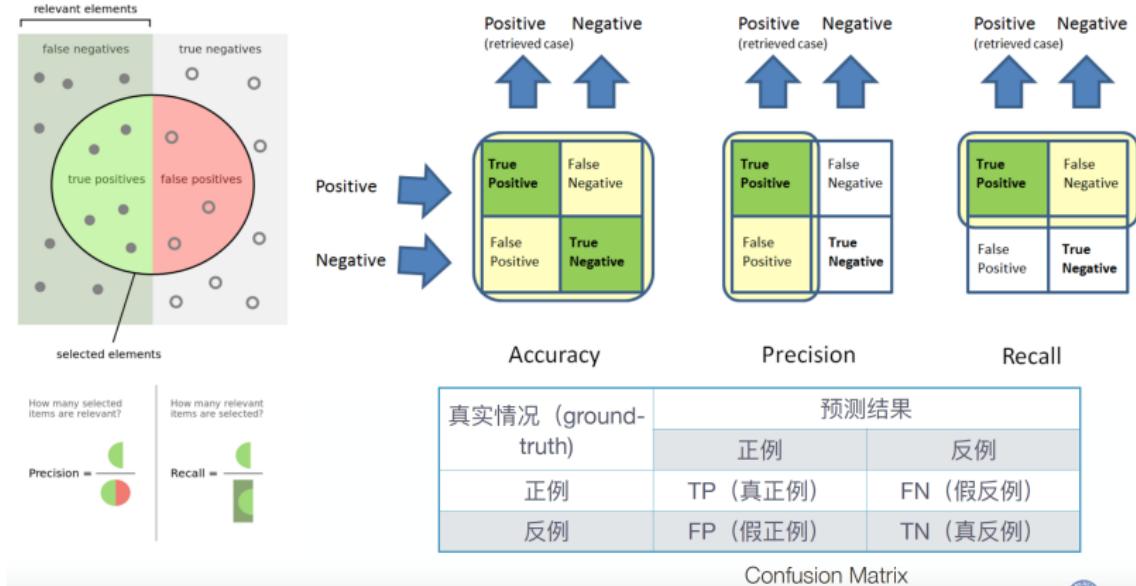


# Can we always separate the classes?

- What if the movie predictions are used for rankings? Now the predicted ratings don't matter, just the order that they imply.
- In what order does Alice prefer E.T., Amelie and Titanic?
- Possibilities:
  - ▶ 0-1 loss on the winner
  - ▶ Permutation distance
  - ▶ Accuracy of top K movies.

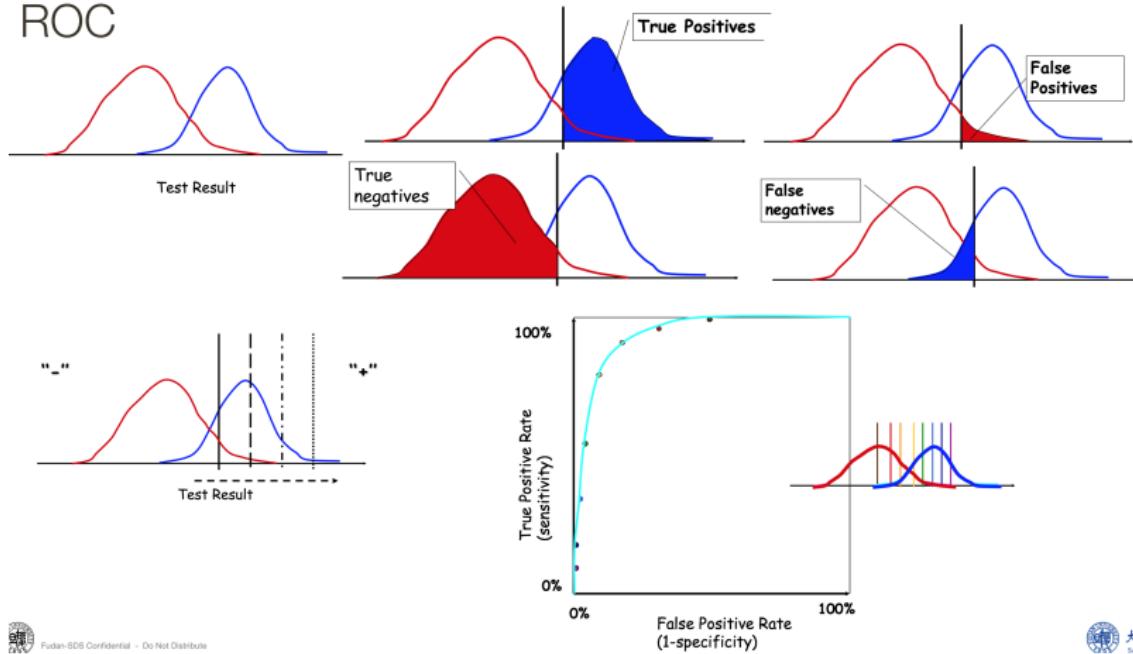


## Metrics-Revisit



# ROC Curve

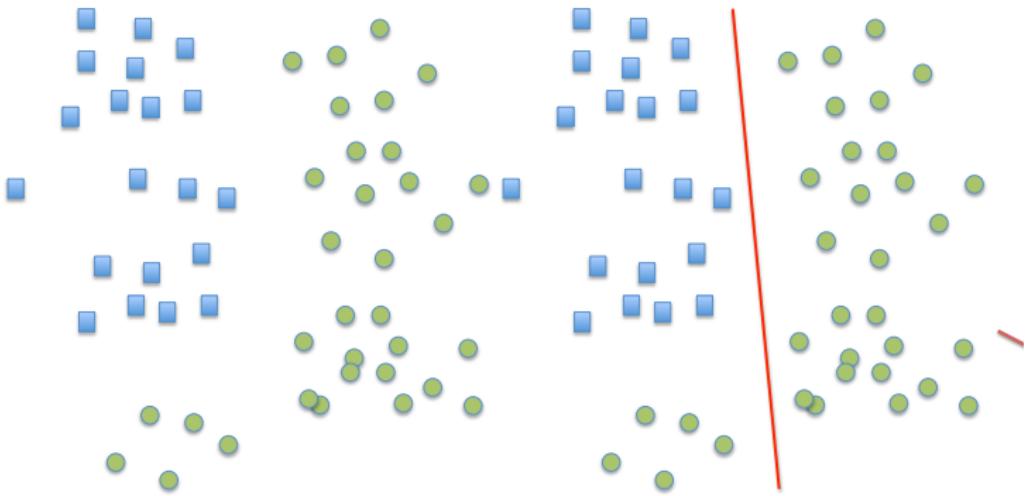
ROC



Fudan-SDS Confidential - Do Not Distribute



- If we can separate the classes, the problem is **linearly separable**



Causes of non perfect separation:

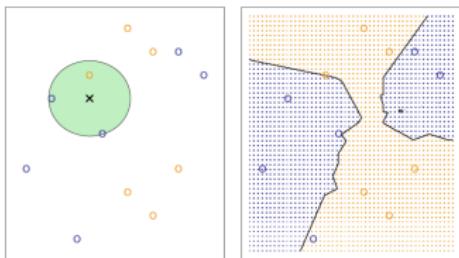
- Model is too simple
- Noise in the inputs (i.e., data attributes)
- Simple features that do not account for all variations
- Errors in data targets (miss labelings)

Should we make the model complex enough to have perfect separation in the training data?



## K-Nearest Neighbors (Non-parametric Method)

$$\Pr(Y = j | X = x_0) = \frac{1}{K} \sum_{i \in \mathcal{N}_0} I(y_i = j).$$



### *k*-Nearest Neighbor – Training

**input** dataset  $\mathcal{D} = \{(x^1, y^1), \dots, (x^n, y^n)\} \subset \mathbb{R}^d \times \mathcal{Y}$   
store all examples  $(x^1, y^1), \dots, (x^n, y^n)$ .

### *k*-Nearest Neighbor – Classification

**input** new example  $x$   
for each training example  $(x^i, y^i)$  compute  $d_i(x) = \|x - x^i\|$   
(Euclidean distance)  
sort  $d_i$  in increasing order  
**output** majority vote among  $y^i$ 's within the  $k$  smallest  $d^i$

"nearest neighbor" is 1-nearest neighbour.

# Log-Linear Regression

For dataset  $D = \{(x, y)\}$ , we have linear regression  $y = w^T x + b$ .

How to make the model directly model the nonlinear data?

log-linear regression:

$$\ln(y) = w^T x + b$$

$$y = e^{w^T x + b}$$

More generally, we can assume monotonically, differentiable function  $g(\cdot)$ , i.e. Generalised Linear Model (GLM),

$$y = g^{-1}(w^T x + b)$$



# Logistic Regression

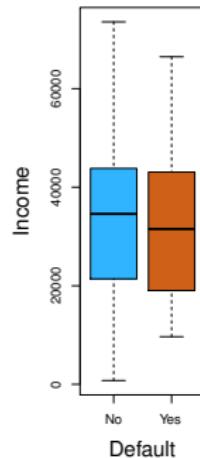
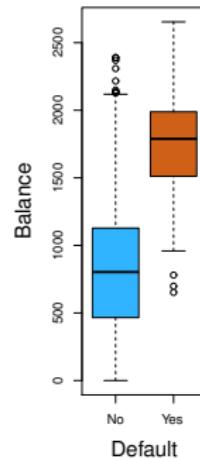
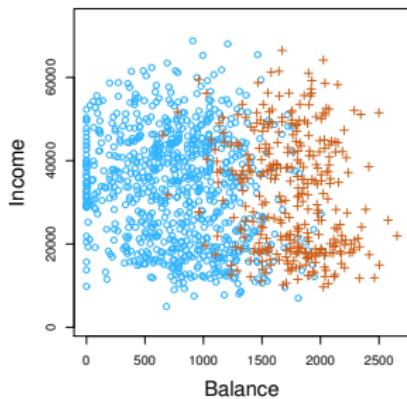


- Qualitative variables take values in an unordered set  $\mathcal{C}$ , such as:  
`eye color ∈ {brown, blue, green}`  
`email ∈ {spam, ham}.`
- Given a feature vector  $X$  and a qualitative response  $Y$  taking values in the set  $\mathcal{C}$ , the classification task is to build a function  $C(X)$  that takes as input the feature vector  $X$  and predicts its value for  $Y$ ; i.e.  $C(X) \in \mathcal{C}$ .
- Often we are more interested in estimating the *probabilities* that  $X$  belongs to each category in  $\mathcal{C}$ .

For example, it is more valuable to have an estimate of the probability that an insurance claim is fraudulent, than a classification fraudulent or not.



# Example: Credit Card Default



# Can we use Linear Regression?

Suppose for the **Default** classification task that we code

$$Y = \begin{cases} 0 & \text{if } \text{No} \\ 1 & \text{if } \text{Yes.} \end{cases}$$

Can we simply perform a linear regression of  $Y$  on  $X$  and classify as **Yes** if  $\hat{Y} > 0.5$ ?



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- In this case of a binary outcome, linear regression does a good job as a classifier, and is equivalent to *linear discriminant analysis* which we discuss later.
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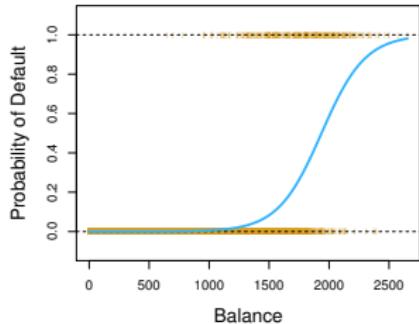
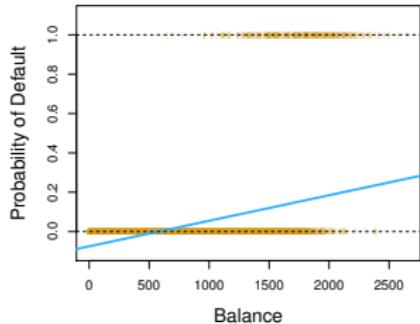
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- In this case of a binary outcome, linear regression does a good job as a classifier, and is equivalent to *linear discriminant analysis* which we discuss later.
- Since in the population  $E(Y|X = x) = \Pr(Y = 1|X = x)$ , we might think that regression is perfect for this task.
- However, *linear* regression might produce probabilities less than zero or bigger than one. *Logistic regression* is more appropriate.



# Linear versus Logistic Regression



The orange marks indicate the response  $Y$ , either 0 or 1. Linear regression does not estimate  $\Pr(Y = 1|X)$  well. Logistic regression seems well suited to the task.

# Logistic Regression

Let's write  $p(X) = \Pr(Y = 1|X)$  for short and consider using **balance** to predict **default**. Logistic regression uses the form

$$p(X) = \frac{e^{\beta_0 + \beta_1 X}}{1 + e^{\beta_0 + \beta_1 X}}.$$

( $e \approx 2.71828$  is a mathematical constant [Euler's number.])

It is easy to see that no matter what values  $\beta_0$ ,  $\beta_1$  or  $X$  take,  $p(X)$  will have values between 0 and 1.



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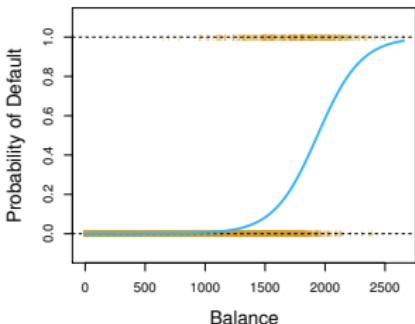
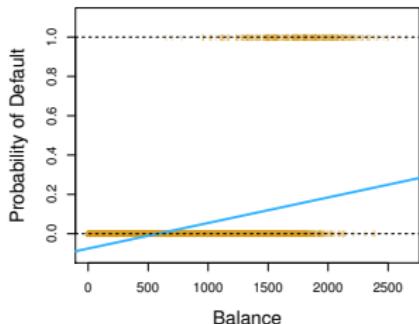
A bit of rearrangement gives

$$\log\left(\frac{p(X)}{1 - p(X)}\right) = \beta_0 + \beta_1 X.$$

This monotone transformation is called the *log odds* or *logit* transformation of  $p(X)$ .



# Linear versus Logistic Regression



Logistic regression ensures that our estimate for  $p(X)$  lies between 0 and 1.

# Maximum Likelihood

We use maximum likelihood to estimate the parameters.

$$\ell(\beta_0, \beta) = \prod_{i:y_i=1} p(x_i) \prod_{i:y_i=0} (1 - p(x_i)).$$

This *likelihood* gives the probability of the observed zeros and ones in the data. We pick  $\beta_0$  and  $\beta_1$  to maximize the likelihood of the observed data.



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Most statistical packages can fit linear logistic regression models by maximum likelihood. In R we use the `glm` function.

	Coefficient	Std. Error	Z-statistic	P-value
Intercept	-10.6513	0.3612	-29.5	< 0.0001
balance	0.0055	0.0002	24.9	< 0.0001



What is our estimated probability of **default** for someone with a balance of \$1000?

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 X}} = \frac{e^{-10.6513 + 0.0055 \times 1000}}{1 + e^{-10.6513 + 0.0055 \times 1000}} = 0.006$$

## The Generalisation of Logistic Regression(LR)

- 1, generalize the LR with multi-dimension cases.
- 2, generalize the LR to multi-class cases.



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With a balance of \$2000?

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 X}} = \frac{e^{-10.6513 + 0.0055 \times 2000}}{1 + e^{-10.6513 + 0.0055 \times 2000}} = 0.586$$

## The Generalisation of Logistic Regression(LR)

- 1, generalize the LR with multi-dimension cases.
- 2, generalize the LR to multi-class cases.



# Logistic regression with more than two classes

So far we have discussed logistic regression with two classes.  
It is easily generalized to more than two classes. One version  
(used in the R package **glmnet**) has the symmetric form

$$\Pr(Y = k|X) = \frac{e^{\beta_{0k} + \beta_{1k}X_1 + \dots + \beta_{pk}X_p}}{\sum_{\ell=1}^K e^{\beta_{0\ell} + \beta_{1\ell}X_1 + \dots + \beta_{p\ell}X_p}}$$

Here there is a linear function for *each* class.



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(The *mathier* students will recognize that some cancellation is possible, and only  $K - 1$  linear functions are needed as in 2-class logistic regression.)



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Multiclass logistic regression is also referred to as *multinomial regression*.



# Linear Discriminant Analysis



Here the approach is to model the distribution of  $X$  in each of the classes separately, and then use *Bayes theorem* to flip things around and obtain  $\Pr(Y|X)$ .

When we use normal (Gaussian) distributions for each class, this leads to linear or quadratic discriminant analysis.

However, this approach is quite general, and other distributions can be used as well. We will focus on normal distributions.



# Bayes theorem for classification

Thomas Bayes was a famous mathematician whose name represents a big subfield of statistical and probabilistic modeling. Here we focus on a simple result, known as Bayes theorem:

$$\Pr(Y = k|X = x) = \frac{\Pr(X = x|Y = k) \cdot \Pr(Y = k)}{\Pr(X = x)}$$



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One writes this slightly differently for discriminant analysis:

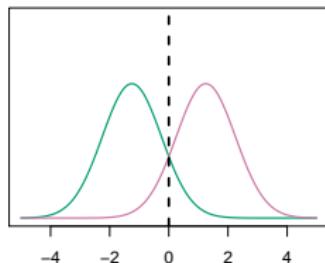
$$\Pr(Y = k|X = x) = \frac{\pi_k f_k(x)}{\sum_{l=1}^K \pi_l f_l(x)}, \quad \text{where}$$

- $f_k(x) = \Pr(X = x|Y = k)$  is the *density* for  $X$  in class  $k$ . Here we will use normal densities for these, separately in each class.
- $\pi_k = \Pr(Y = k)$  is the marginal or *prior* probability for class  $k$ .

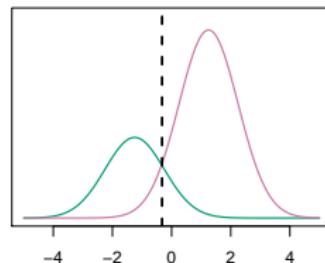


# Classify to the highest density

$$\pi_1=.5, \quad \pi_2=.5$$



$$\pi_1=.3, \quad \pi_2=.7$$

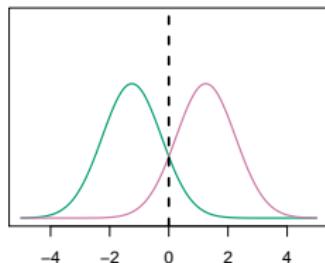


We classify a new point according to which density is highest.

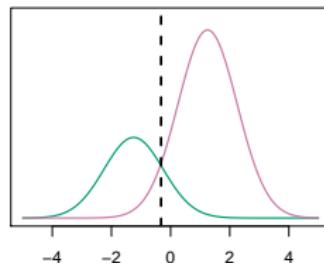


# Classify to the highest density

$$\pi_1=.5, \quad \pi_2=.5$$



$$\pi_1=.3, \quad \pi_2=.7$$



We classify a new point according to which density is highest.

When the priors are different, we take them into account as well, and compare  $\pi_k f_k(x)$ . On the right, we favor the pink class — the decision boundary has shifted to the left.

# Why discriminant analysis?

- When the classes are well-separated, the parameter estimates for the logistic regression model are surprisingly unstable. Linear discriminant analysis does not suffer from this problem.
- If  $n$  is small and the distribution of the predictors  $X$  is approximately normal in each of the classes, the linear discriminant model is again more stable than the logistic regression model.
- Linear discriminant analysis is popular when we have more than two response classes, because it also provides low-dimensional views of the data.



# Linear Discriminant Analysis when $p = 1$

The Gaussian density has the form

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{1}{2}\left(\frac{x-\mu_k}{\sigma_k}\right)^2}$$

Here  $\mu_k$  is the mean, and  $\sigma_k^2$  the variance (in class  $k$ ). We will assume that all the  $\sigma_k = \sigma$  are the same.



# Linear Discriminant Analysis when $p = 1$

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Here  $\mu_k$  is the mean, and  $\sigma_k^2$  the variance (in class  $k$ ). We will assume that all the  $\sigma_k = \sigma$  are the same.

Plugging this into Bayes formula, we get a rather complex expression for  $p_k(x) = \Pr(Y = k|X = x)$ :

$$p_k(x) = \frac{\pi_k \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu_k}{\sigma}\right)^2}}{\sum_{l=1}^K \pi_l \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu_l}{\sigma}\right)^2}}$$

Happily, there are simplifications and cancellations.



# Discriminant functions

To classify at the value  $X = x$ , we need to see which of the  $p_k(x)$  is largest. Taking logs, and discarding terms that do not depend on  $k$ , we see that this is equivalent to assigning  $x$  to the class with the largest *discriminant score*:

$$\delta_k(x) = x \cdot \frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$

Note that  $\delta_k(x)$  is a *linear* function of  $x$ .



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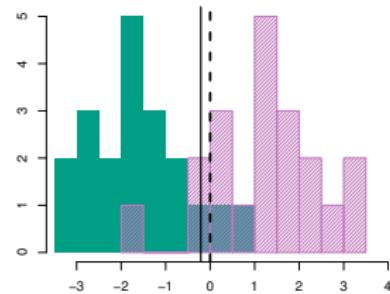
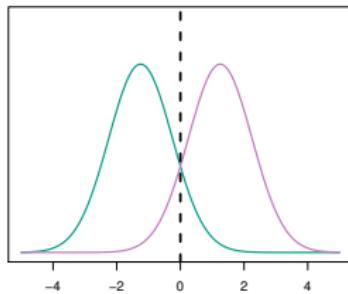
Note that  $\delta_k(x)$  is a *linear* function of  $x$ .

If there are  $K = 2$  classes and  $\pi_1 = \pi_2 = 0.5$ , then one can see that the *decision boundary* is at

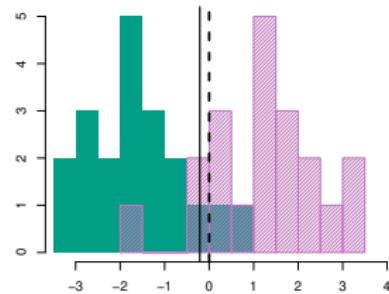
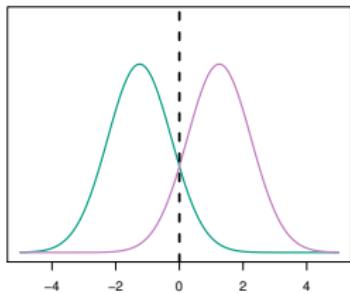
$$x = \frac{\mu_1 + \mu_2}{2}.$$

(See if you can show this)





Example with  $\mu_1 = -1.5$ ,  $\mu_2 = 1.5$ ,  $\pi_1 = \pi_2 = 0.5$ , and  $\sigma^2 = 1$ .



Example with  $\mu_1 = -1.5$ ,  $\mu_2 = 1.5$ ,  $\pi_1 = \pi_2 = 0.5$ , and  $\sigma^2 = 1$ .

Typically we don't know these parameters; we just have the training data. In that case we simply estimate the parameters and plug them into the rule.

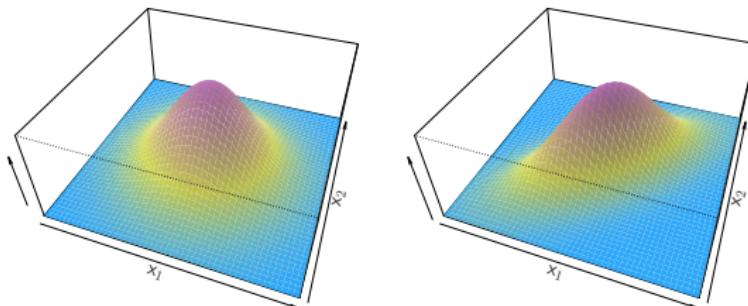
# Estimating the parameters

$$\begin{aligned}\hat{\pi}_k &= \frac{n_k}{n} \\ \hat{\mu}_k &= \frac{1}{n_k} \sum_{i: y_i=k} x_i \\ \hat{\sigma}^2 &= \frac{1}{n-K} \sum_{k=1}^K \sum_{i: y_i=k} (x_i - \hat{\mu}_k)^2 \\ &= \sum_{k=1}^K \frac{n_k - 1}{n - K} \cdot \hat{\sigma}_k^2\end{aligned}$$

where  $\hat{\sigma}_k^2 = \frac{1}{n_k - 1} \sum_{i: y_i=k} (x_i - \hat{\mu}_k)^2$  is the usual formula for the estimated variance in the  $k$ th class.

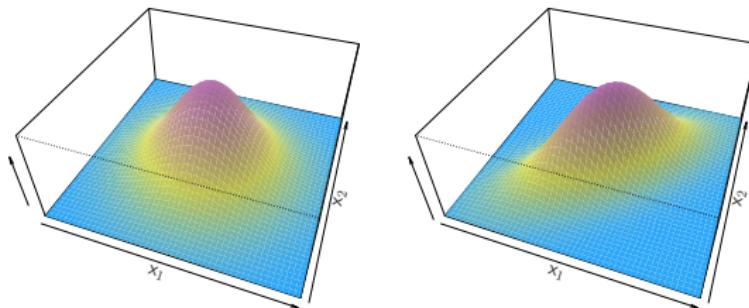


# Linear Discriminant Analysis when $p > 1$



$$\text{Density: } f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

# Linear Discriminant Analysis when $p > 1$

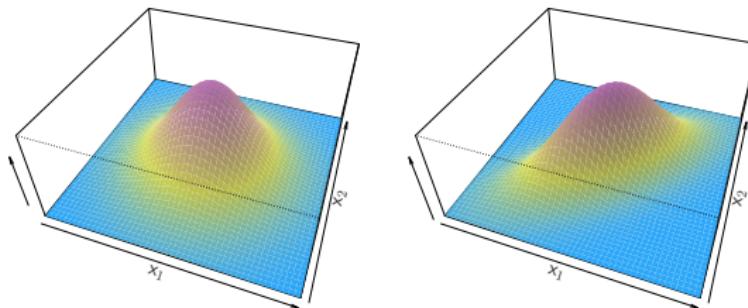


Density:  $f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$

Discriminant function:  $\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + \log \pi_k$



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$$\text{Discriminant function: } \delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + \log \pi_k$$

Despite its complex form,

$\delta_k(x) = c_{k0} + c_{k1}x_1 + c_{k2}x_2 + \dots + c_{kp}x_p$  — a linear function.

# Other forms of Discriminant Analysis

$$\Pr(Y = k|X = x) = \frac{\pi_k f_k(x)}{\sum_{l=1}^K \pi_l f_l(x)}$$

When  $f_k(x)$  are Gaussian densities, with the same covariance matrix  $\Sigma$  in each class, this leads to linear discriminant analysis. By altering the forms for  $f_k(x)$ , we get different classifiers.

- With Gaussians but different  $\Sigma_k$  in each class, we get *quadratic discriminant analysis*.



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- Many other forms, by proposing specific density models for  $f_k(x)$ , including nonparametric approaches.



Assumes features are independent in each class.

Useful when  $p$  is large, and so multivariate methods like QDA and even LDA break down.

- Gaussian naive Bayes assumes each  $\Sigma_k$  is diagonal:

$$\delta_k(x) \propto \log \left[ \pi_k \prod_{j=1}^p f_{kj}(x_j) \right] = -\frac{1}{2} \sum_{j=1}^p \frac{(x_j - \mu_{kj})^2}{\sigma_{kj}^2} + \log \pi_k$$

- can use for *mixed* feature vectors (qualitative and quantitative). If  $X_j$  is qualitative, replace  $f_{kj}(x_j)$  with probability mass function (histogram) over discrete categories.

Despite strong assumptions, naive Bayes often produces good classification results.



# Logistic Regression versus LDA

For a two-class problem, one can show that for LDA

$$\log \left( \frac{p_1(x)}{1 - p_1(x)} \right) = \log \left( \frac{p_1(x)}{p_2(x)} \right) = c_0 + c_1 x_1 + \dots + c_p x_p$$

So it has the same form as logistic regression.

The difference is in how the parameters are estimated.



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- Despite these differences, in practice the results are often very similar.

Footnote: logistic regression can also fit quadratic boundaries like QDA, by explicitly including quadratic terms in the model.



- Logistic regression is very popular for classification, especially when  $K = 2$ .
- LDA is useful when  $n$  is small, or the classes are well separated, and Gaussian assumptions are reasonable. Also when  $K > 2$ .
- Naive Bayes is useful when  $p$  is very large.
- See Section 4.5 for some comparisons of logistic regression, LDA and KNN.



## Acknowledgement

Some slides are in courtesy of

- (1) Chap 4 of James *et. al.* "An Introduction to Statistical Learning with applications in R", 2011;
- (2) Lecture 03, CSC 411 by Raquel Urtasun & Rich Zemel, University of Toronto



Something more from the slides of textbook.



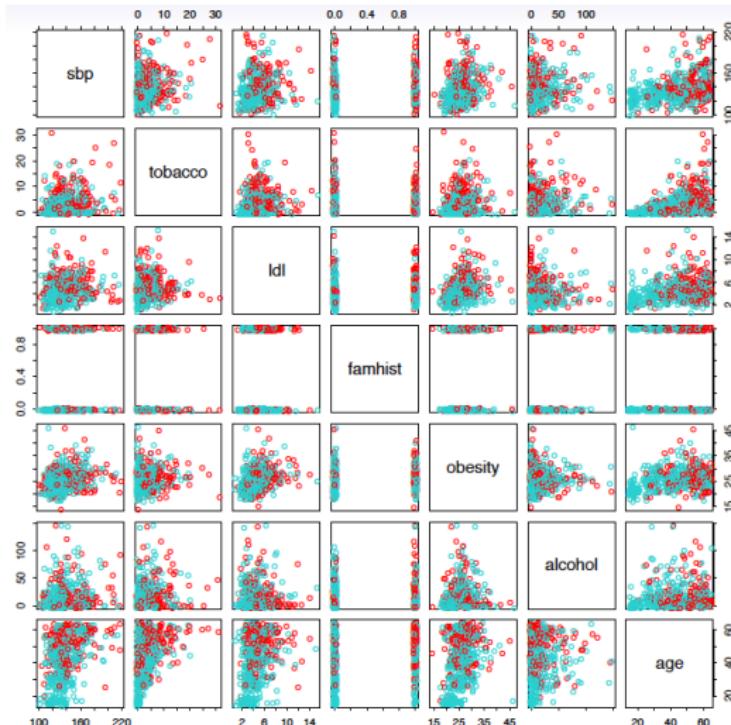
# Example: South African Heart Disease

Note that: this example comes from the book “The Elements of Statistical learning ...”(print 4. Sec. 4.4.2, Page 121).

- 160 cases of MI (myocardial infarction) and 302 controls (all male in age range 15-64), from Western Cape, South Africa in early 80s.
- Overall prevalence very high in this region: 5.1%.
- Measurements on seven predictors (risk factors), shown in scatterplot matrix.
- Goal is to identify relative strengths and directions of risk factors.
- This was part of an intervention study aimed at educating the public on healthier diets.



# An Example



Scatterplot matrix of the South African Heart Disease data. The response is color coded | The cases (MI) are red, the controls turquoise. famhist is a binary variable, with 1 indicating family history of MI.



```
> heartfit<-glm(chd~.,data=heart,family=binomial)
> summary(heartfit)

Call:
glm(formula = chd ~ ., family = binomial, data = heart)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.1295997  0.9641558 -4.283 1.84e-05 ***
sbp          0.0057607  0.0056326  1.023  0.30643
tobacco      0.0795256  0.0262150  3.034  0.00242 **
ldl          0.1847793  0.0574115  3.219  0.00129 **
famhistPresent 0.9391855  0.2248691  4.177 2.96e-05 ***
obesity      -0.0345434  0.0291053 -1.187  0.23529
alcohol       0.0006065  0.0044550  0.136  0.89171
age           0.0425412  0.0101749  4.181 2.90e-05 ***

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 596.11    on 461    degrees of freedom
Residual deviance: 483.17    on 454    degrees of freedom
AIC: 499.17
```



# Case-control sampling and logistic regression

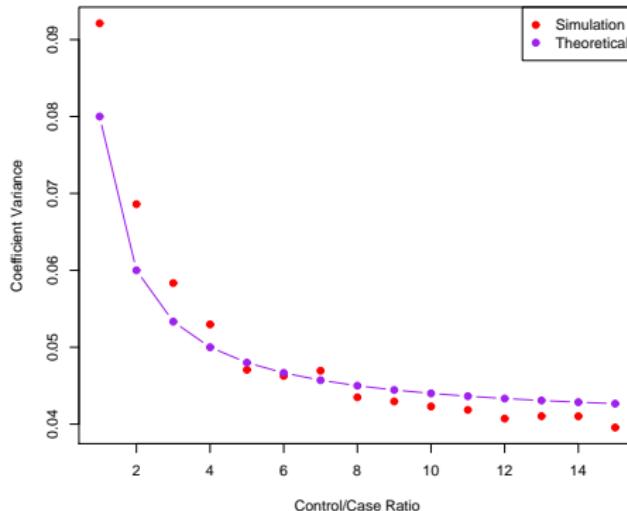
- In South African data, there are 160 cases, 302 controls —  $\tilde{\pi} = 0.35$  are cases. Yet the prevalence of MI in this region is  $\pi = 0.05$ .
- With case-control samples, we can estimate the regression parameters  $\beta_j$  accurately (if our model is correct); the constant term  $\beta_0$  is incorrect.
- We can correct the estimated intercept by a simple transformation

$$\hat{\beta}_0^* = \hat{\beta}_0 + \log \frac{\pi}{1 - \pi} - \log \frac{\tilde{\pi}}{1 - \tilde{\pi}}$$

- Often cases are rare and we take them all; up to five times that number of controls is sufficient. See next frame

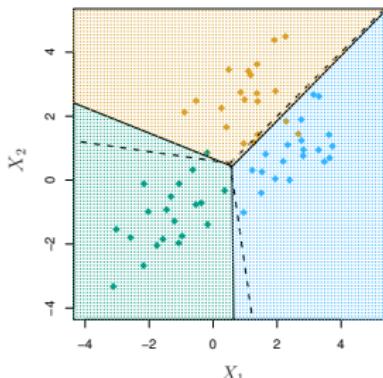
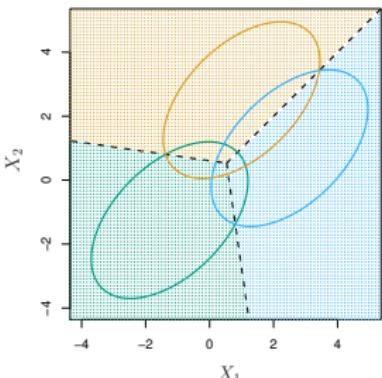


# Diminishing returns in unbalanced binary data



Sampling more controls than cases reduces the variance of the parameter estimates. But after a ratio of about 5 to 1 the variance reduction flattens out.

# Illustration: $p = 2$ and $K = 3$ classes



Here  $\pi_1 = \pi_2 = \pi_3 = 1/3$ .

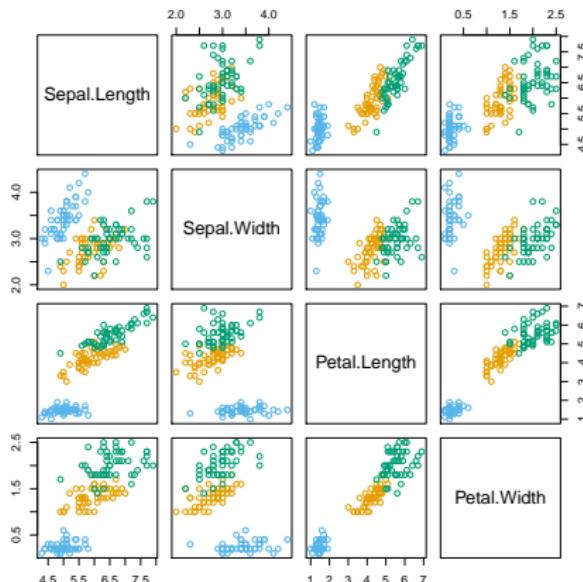
The dashed lines are known as the *Bayes decision boundaries*.

Were they known, they would yield the fewest misclassification errors, among all possible classifiers.

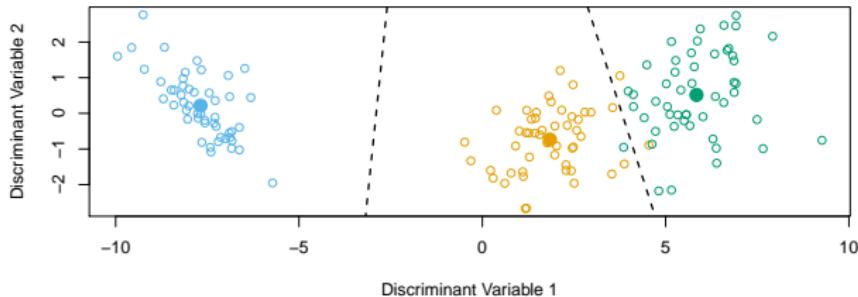
4 variables  
3 species  
50 samples/class

- Setosa
- Versicolor
- Virginica

LDA classifies all but 3 of the 150 training samples correctly.



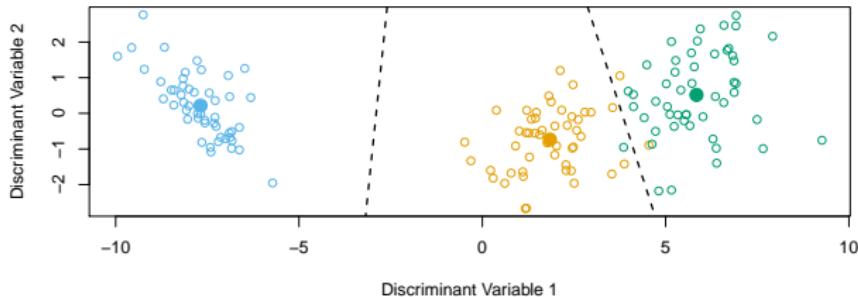
# Fisher's Discriminant Plot



When there are  $K$  classes, linear discriminant analysis can be viewed exactly in a  $K - 1$  dimensional plot.

Why?

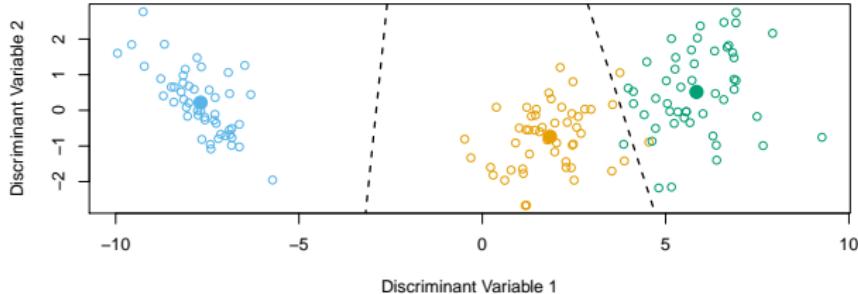
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Even when  $K > 3$ , we can find the “best” 2-dimensional plane for visualizing the discriminant rule.

# From $\delta_k(x)$ to probabilities

Once we have estimates  $\hat{\delta}_k(x)$ , we can turn these into estimates for class probabilities:

$$\widehat{\Pr}(Y = k|X = x) = \frac{e^{\hat{\delta}_k(x)}}{\sum_{l=1}^K e^{\hat{\delta}_l(x)}}.$$

So classifying to the largest  $\hat{\delta}_k(x)$  amounts to classifying to the class for which  $\widehat{\Pr}(Y = k|X = x)$  is largest.



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When  $K = 2$ , we classify to class 2 if  $\widehat{\Pr}(Y = 2|X = x) \geq 0.5$ , else to class 1.



# LDA on Credit Data

		True Default Status		
		No	Yes	Total
Predicted Default Status	No	9644	252	9896
	Yes	23	81	104
Total		9667	333	10000

$(23 + 252)/10000$  errors — a 2.75% misclassification rate!

Some caveats:

- This is *training* error, and we may be overfitting.



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- If we classified to the prior — always to class **No** in this case — we would make  $333/10000$  errors, or only 3.33%.
- Of the true **No**'s, we make  $23/9667 = 0.2\%$  errors; of the true **Yes**'s, we make  $252/333 = 75.7\%$  errors!



# Types of errors

**False positive rate:** The fraction of negative examples that are classified as positive — 0.2% in example.

**False negative rate:** The fraction of positive examples that are classified as negative — 75.7% in example.

We produced this table by classifying to class **Yes** if

$$\widehat{\Pr}(\text{Default} = \text{Yes} | \text{Balance}, \text{Student}) \geq 0.5$$

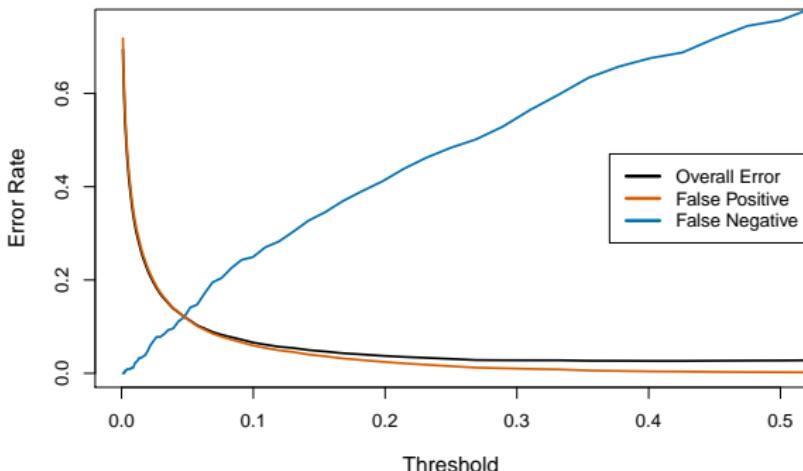
We can change the two error rates by changing the threshold from 0.5 to some other value in  $[0, 1]$ :

$$\widehat{\Pr}(\text{Default} = \text{Yes} | \text{Balance}, \text{Student}) \geq \text{threshold},$$

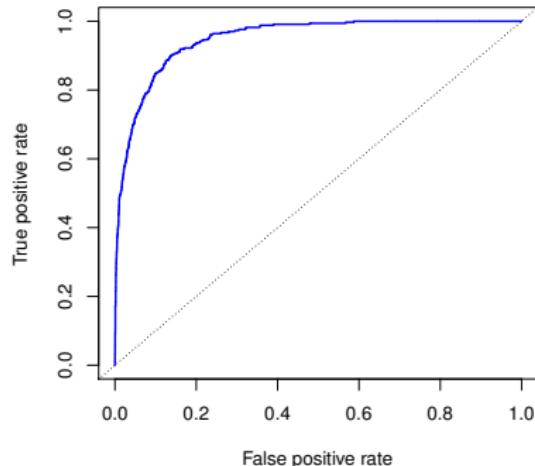
and vary *threshold*.



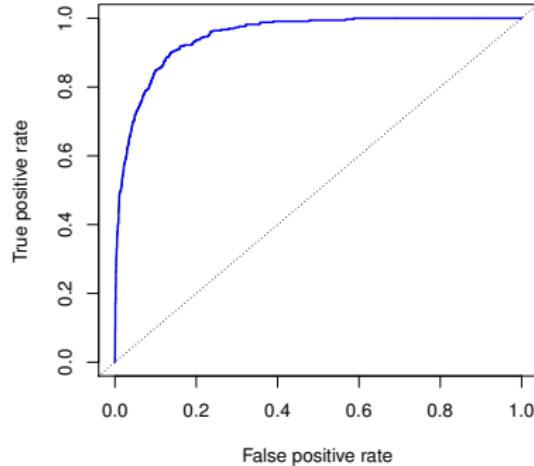
# Varying the threshold



In order to reduce the false negative rate, we may want to reduce the threshold to 0.1 or less.



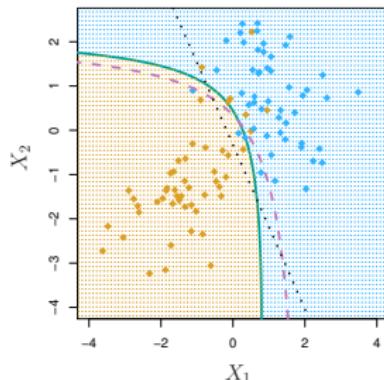
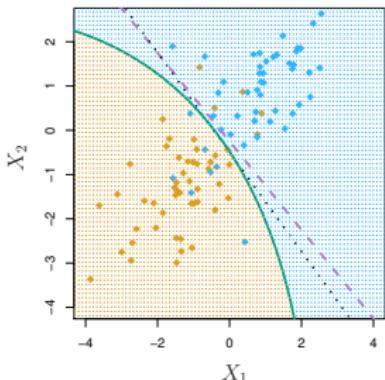
The *ROC plot* displays both simultaneously.



The *ROC plot* displays both simultaneously.

Sometimes we use the *AUC* or *area under the curve* to summarize the overall performance. Higher *AUC* is good.

# Quadratic Discriminant Analysis



$$\delta_k(x) = -\frac{1}{2}(x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k) + \log \pi_k$$

Because the  $\Sigma_k$  are different, the quadratic terms matter.

