Fund Allocation Problems in Quantitative Trading

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Abstract

In this project, we investigate the challenge of fund allocation in various markets by introducing the information set $\mathcal{I}$, which forecasts the return rate, called alpha. Employing the loss function of the expected return rate, along with penalties for variance and the transfer of funds between markets, we formulate robust optimization problems under specific market constraints and apply numerical methods to derive solutions via computational programs. Ultimately, we achieve superior performance compared to conventional allocation methods and approximate the loss function of both intuitive and nearly optimal solutions (excluding the cost of allocation). This project is available at https://github.com/yanwugu2001/HKUST_ELEC5470_Project.

Keywords Portfolio Optimization · Asset Allocation · Quantitative Finance

1 Introduction

In the field of quantitative trading, a pre-trained automated program will use a number of different factors designed by researchers based on previous information on a single type of asset, as well as the whole market, to predict a **alpha** of the target asset, which is usually a prediction of return rate in a given horizon. Despite potential discrepancies between predicted and actual returns, alpha can also serve as the golden criterion to determine whether one should buy in the asset at this moment, which may last only microseconds, as long as it has a win rate greater than 50%. (This varies if the alpha has some potential style exposure.) This methodology follows a classical Capital Asset Pricing Model [1]. The connection between portfolio optimization and alpha investigation has been widely studied [2, 3] as well.

There are many implements for a good alpha, including estimating return rate, variance, and other quantities required in financial studies. One of the most famous and essential optimization models in finance is the portfolio selection model of Markowitz [4], that is, the Markowitz Mean-Variance model. However, the solution of Markowitz model, which is discussed in Section 2, requires an estimation of incoming data, which can be estimated by the alphas.

After doing portofolio optimization in an individual markets, a intuitive question emerges that how to do portfolio optimization in quantitative finance setting among different markets, buy in more assets in a market with higher return rate and stable properties, which can be derived by the historical data, and avoid the others, under the specific rules of different markets. How to allocate the money before the market "takes off" and grabs the opportunities? How to minimize the cost of allocation to gain the highest return?

This project focuses on the fund allocation problems in quantitative trading under the Markowiz model paradigm. Section 2 will give background information, notation and our optimization problems. The problem will be analyzed and solved theoretically in Section 3 and evaluated with experiments in Section 4. The conclusion will be made in 5.

2 Notations and Problem Definition

Suppose that we are combining n risky assets whose random returns are $r_1, r_2, \ldots, r_n$. Let $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}^T$. And denote $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ as the expected returns and the return covariance matrix, respectively. More specifically,

$$\boldsymbol{\mu}=\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{bmatrix}=\begin{bmatrix}\mathbb{E}r_1\\ \mathbb{E}r_2\\ \vdots\\ \mathbb{E}r_n\end{bmatrix},\quad \boldsymbol{V}=\begin{bmatrix}\sigma_{11}&\sigma_{12}&\cdots&\sigma_{1n}\\ \sigma_{21}&\sigma_{22}&\cdots&\sigma_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \sigma_{n1}&\sigma_{n2}&\cdots&\sigma_{nn}\end{bmatrix}=\begin{bmatrix}\text{Var}(r_1)&\text{Cov}(r_1,r_2)&\cdots&\text{Cov}(r_1,r_n)\\ \text{Cov}(r_2,r_1)&\text{Var}(r_2)&\cdots&\text{Cov}(r_2,r_n)\\ \vdots&\vdots&\ddots&\vdots\\ \text{Cov}(r_n,r_1)&\text{Cov}(r_n,r_2)&\cdots&\text{Var}(r_n)\end{bmatrix}$$

Given portfolio $\boldsymbol{x}=\begin{bmatrix}x_1 & x_2 & \cdots & x_n\end{bmatrix}^T$. Then, the expected value and variance are respectively

$$\boldsymbol{\mu}^T\boldsymbol{x}=\sum_{j=1}^{n}\mu_j x_j,\quad \boldsymbol{x^T V x}=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j.$$

A fully invested portfolio is efficient if it has minimum risk for a given level of return, or equivalently, if it has maximum expected return for a given level of risk. A fully invested efficient portfolio can be characterized as the solution to the following quadratic program:

$$\begin{aligned}\max_{\boldsymbol{x}}\quad & \boldsymbol{\mu}^T\boldsymbol{x}-\frac{1}{2}\gamma\boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}\\ \text{s.t.}\quad & \boldsymbol{1}^T\boldsymbol{x}=1\end{aligned}\tag{1}$$

for domr risk0aversion coefficient $\gamma>0$. The bounds of return and variance can controlled respectively by

$$\begin{aligned}\max_{\boldsymbol{x}}\quad & \boldsymbol{\mu}^T\boldsymbol{x}\\ \text{s.t.}\quad & \boldsymbol{1}^T\boldsymbol{x}=1\\ & \boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}\le\sigma_0^2\end{aligned}$$

and

$$\begin{aligned}\min_{\boldsymbol{x}}\quad & \boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}\\ \text{s.t.}\quad & \boldsymbol{1}^T\boldsymbol{x}=1\\ & \boldsymbol{\mu}^T\boldsymbol{x}\ge\mu_0\end{aligned}$$

It can be solved that the Equation (1) has KKT condition

$$\begin{cases}\nabla L=\boldsymbol{\mu}-\gamma\boldsymbol{V}\boldsymbol{x}+\nu\boldsymbol{1}\\ \boldsymbol{1}^T\boldsymbol{x}=1\end{cases}$$

and the solution is

$$\boldsymbol{x}^*=\gamma^{-1}\boldsymbol{V}^{-1}(\boldsymbol{\mu}+\frac{\gamma-\boldsymbol{1}^T\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\boldsymbol{1}^T\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{1}).\tag{2}$$

This is apparently a simple and beautiful result, but may face problems when implementing to the real market, especially the quantitative trade fields.

Firstly, in the real market, the return $\boldsymbol{\mu}$ and the covariate $\boldsymbol{V}$ are unknown, making the expression of $\boldsymbol{x}^*$ unachievable.

Secondly, the assumption of identically and independently distribution of the same asset at different time does not hold, since the condition of each single asset will change at any time and can be influenced by other assets as well.

Thirdly, within distinct asset classes, such as the stock market in the United States and commodity futures in China, the risk associated with each individual asset may adhere to their own market trends. Consequently, concentrating investments in a singular market presents substantial risk, which requires a diversified investment strategy across various asset types to mitigate these risks. However, the divergent characteristics and regulations of these markets have yet to be adequately addressed.

Lastly, but most importantly, the expected return rate is not equivalent to the real return rate we get under our rules of trade in quantitative finance setting.

For simplicity, we assume that there are p possible asset markets that contain $n_1, n_2, \ldots, n_p$ individual assets, respectively. The superscripts j are used to denote these two different markets. $r_i^j(t), \mu_i^j(t), \sigma_{i_1,i_2}^j(t)$ are corresponding return rate, expected value, and covariate of the superscripts and subscripts of the variables, at time t. Meanwhile, we introduce the return rates, expected values, and variances of p markets using capital letters $R^j(t), M^j(t)$ and $\Sigma^j(t)$.

Then, we introduce some constraints studied by preliminaries [5, 6] and select necessary one to implement into our problems:

Neutral Constraint At time t, the whole market j follows the market neutral constraint, that is,

$$(\boldsymbol{v}^j(t))^T(\boldsymbol{\mu}^j(t) - M^j(t)\mathbf{1}) = 0, \quad j = 1, \ldots, p \tag{3}$$

where $\boldsymbol{v}^j$ is the corresponding market values of assets. This implies that inside a specific market, if we get rid of the external factors, which influence the whole market, like the flow of funds through the market, this whole market is neutral. Meanwhile, for all the markets, we have that

$$\boldsymbol{v}(t)\boldsymbol{M}(t) = 0, \tag{4}$$

where $\boldsymbol{v}$ is the corresponding market values. This implies that the entire financial system is neutral.

This constraint is hard to implement into the solution procudure of optimization, since the expected return rate $\boldsymbol{\mu}$ and $\boldsymbol{M}$ is hard to estimate by the received return rate $\boldsymbol{r}$ and $\boldsymbol{R}$. Implementing these variables deliberately will be violated to this constraint. Instead, we use this constraint to generate simulation data.

Information set $\mathcal{I}$ The information set $\mathcal{I}$ contains two series of data, the return rate $r_i^j(t)$, and alpha $\alpha_i^j(t)$, that is, the prediction of the return rate of the i-th asset of the j-th market, at time t. Despite the assumption that the return rate has fixed expectations and variances, we just assume that the alpha has a relatively stable correlation coefficient and Sharpe ratio during a fixed period of time in the past, which is defined as

$$IC_i^j = \frac{\sum_{\tau=1}^T \alpha_i^j(t-\tau) r_i^j(t-\tau)}{\sqrt{\sum_{\tau=1}^T [\alpha_i^j(t-\tau)]^2}\sqrt{\sum_{\tau=1}^T [r_i^j(t-\tau)]^2}}, \tag{5}$$

and

$$SR_i^j = \frac{\sum_{\tau=1}^T \alpha_i^j(t-\tau) r_i^j(t-\tau)}{\sqrt{\sum_{\tau=1}^T [\alpha_i^j(t-\tau)]^2}}. \tag{6}$$

Unlike the traditional computation method of correlation and covariance, in order to avoid the leak of future information and intuitively express the power of alpha, we do not use the sample mean of the past time window. In other words, we simply assume that the average alpha and the return rate are approximately zero.

Intuitively, if α is exactly the same as the return rate, the Sharpe ratio is the standard variance of the return rate. When they are independent, the Sharpe ratio is zero, since we assume that $\mathbb{E}r \approx 0$. The Sharpe ratio can be considered as the intuitive return taken by alpha, if the trading process is ideal, then the expected return is $SR_i^j(t)$ for time t.

To estimate the return rate under the information set $\mathcal{I}$, we first introduce the trade criterion briefly. In a real trade scenario, we receive all market data at time $t-1$ and use these data to predict the alpha α_i^j at time t before we receive the true return rate at time t. If α_i^j is large enough to cover the trade cost, that is, the expected return is larger than the transaction fee, then we choose to buy in now and sell out after the fixed horizon. Although the real world application involves the time and criterion to sell out, we just focus on the fixed horizon. So, rather than treating r_i^j at time t as our true return rate, we receive $r_i^j(t)\mathbb{I}(\alpha_i^j(t) > 0)$ instead. For the estimation of variance, we just use the definition of covariance matrix.

To compute the expected return under the information set $\mathcal{I}$ and above trade criterion, we suppose the return rate follows the following model proposed by Fernholz [7]:

$$r(t) = b(t) + \sigma W(t) \tag{7}$$

where $W(t)$ is a Brownian motion. We simply assume that $b(t) = a\alpha(t)$ and we gain that

$$\hat{a} = \frac{\text{corr}(\alpha, r)\text{s.e.}(r)}{\text{s.e.}(\alpha)}, \quad \widehat{\sigma^2} = \text{Var}(r) - \hat{a}^2\text{Var}(\alpha).$$

Then we have

$$\begin{aligned}\mathbb{E}\text{Ret}_i^j(t) &= \mathbb{E}r_i^j\mathbb{I}(\alpha_i^j(t) > 0) \\ &= \mathbb{E}(a\alpha + \sigma W)I(\alpha > 0) \\ &= \hat{a}\mathbb{E}(\alpha|\alpha > 0)\end{aligned} \tag{8}$$

Here, $\hat{a}$ is a variant of the Sharpe ratio and the latter expectation is based on realistic worlds.

Capacity Constraint For each asset, there is a capacity limit which means that investors cannot buy in more than this amount. Or, they are required to declare this trade or meet other compliance conditions, which takes more cost and may encounter with higher risk than the nature risk of this asset. In mathematical language, we write it as

$$0 \leq x_i^j \leq u_i^j, \quad 1 \leq i \leq n_j, 1 \leq j \leq p, \tag{9}$$

where u_i^j is the corresponding upper bound. In real setting, the amount of buying in is apparently not continuous, and has a minimum limit of absolute value. However, we omit these two conditions for simplicity and choose the closest feasible solution in real trading.

Meanwhile, there is no need to invest all the funds in the market. If the excess return of a certain asset or market with respect to the risk-free rate cannot beat the risk, then the investor would keep the money in risk-free products rather than invest in this asset or market. Since there exists a cost in transferring money between different markets, the usual operation is keeping the money of a certain market account rather than buying in risky assets. In mathematical language:

$$\begin{cases} \sum_{i=1}^{n_j} x_i^j \leq x^j, & 1 \leq j \leq p \\ \sum_{j=1}^{p} x^j \leq 1 \end{cases} \tag{10}$$

Leverage For a multitude of derivatives, including commodity futures and bound futures contraries, given their inherently minimal volatility and the substantial pricing of each contrary, the exchange has implemented a margin trading system. More specifically, the investor is required to commit only a predetermined proportion of the total value of the asset to initiate a purchase of this financial instrument. In this way, the return and the risk are proportionally enlarged, and this proportion is certain inside a market.

To describe this condition, we apply the leverage rate L^j to the market j. It is worth mentioning that the rules behind the leverage is too complicated to be involved in the optimization problem, so we choose to integrate the leverage into the information set $\mathcal{I}$.

Problem Description We have plenty preliminaries [8, 3] discuss about the portfolio optimization inside a market whose fund can circulate inside freely. But there is little discussion about how to allocate funds among different markets, based on the previous information of returns and alphas. In other words, if we have ten billion dollars, how much should be invested to stocks, futures, options, and other derivatives of different countries and regions, in order to gain the highest return and bear the smallest risk under the rules?

$$\begin{aligned} \max \quad & \sum_{j=1}^{p} [\boldsymbol{r}^j(t,\mathcal{I})^T \boldsymbol{y}^{j*}(x^j) - \gamma_1 \boldsymbol{y}^{j*}(x^j)^T \boldsymbol{V}^j(t,\mathcal{I}) \boldsymbol{y}^{j*}(x^j)] - \gamma_2 d(\boldsymbol{x}, \boldsymbol{x}_0) \\ \text{s.t.} \quad & \boldsymbol{y}^{j*}(x^j) = \arg \max_{\boldsymbol{1}^T \boldsymbol{y} \leq x_j, \boldsymbol{0} \preceq \boldsymbol{y}} \frac{\boldsymbol{y}^T \boldsymbol{r}^j(t,\mathcal{I})}{\sqrt{\boldsymbol{y}^T \boldsymbol{V}^j(t,\mathcal{I}) \boldsymbol{y}}} \\ & \sum_{j=1}^{p} x^j = 1 \end{aligned} \tag{11}$$

where $\boldsymbol{r}^j(t,\mathcal{I})$ and $\boldsymbol{V}^j(t,\mathcal{I})$ are the estimation of the return and variance of assets in the j-th market, based on the information $\mathcal{I}$. $d(\boldsymbol{x}, \boldsymbol{x}_0)$ refers to the cost of transferring funds between different markets. For large hedge funds that participate in a large number of markets around the world with different currencies, d would be a complex matching optimization problem. We just assume it as the norm of $\boldsymbol{x} - \boldsymbol{x}_0$, that is, $d(\boldsymbol{x}, \boldsymbol{x_0}) = \gamma \|\boldsymbol{x} - \boldsymbol{x_0}\|_2$, with a coefficient. Intuitively, the cost of transferring funds should be substituted for the final return rate.

3 Methodology and Solution

To solve Problem (11), we first estimate the return rate $\boldsymbol{r}^j$ and variance $\boldsymbol{V}^j$ under the information $\mathcal{I}$, and then solve the subproblem:

$$\begin{aligned} \max \quad & \frac{\boldsymbol{y}^T \boldsymbol{r}^j(t,\mathcal{I})}{\sqrt{\boldsymbol{y}^T \boldsymbol{V}^j(t,\mathcal{I}) \boldsymbol{y}}} \\ \text{s.t.} \quad & \boldsymbol{1}^T \boldsymbol{y} \leq x_j \\ & \boldsymbol{0} \preceq \boldsymbol{y} \preceq \boldsymbol{u}^j \end{aligned} \tag{12}$$

To estimate the return rate $\boldsymbol{r}^j$ and covariance matrix $\boldsymbol{V}^j$, we use the information set $\mathcal{I}$ which contains the previous alpha $\alpha_i^j(t)$ and return rate $r_i^j(t)$.

$$\begin{aligned}\{\boldsymbol{r}^j(t,\mathcal{I})\}_i &= T^{-1}\sum_{\tau=t-1}^{t-T} r_i^j(\tau)\mathbb{I}(\alpha_i^j(\tau)>0)\\ \{\boldsymbol{V}^j(t,\mathcal{I})\}_{m,n} &= T^{-1}\sum_{\tau=t-1}^{t-T}[r_m^j(\tau)r_n^j(\tau)\mathbb{I}(\alpha_m^j(\tau)>0)\mathbb{I}(\alpha_n^j(\tau)>0)-\{\boldsymbol{r}^j(\tau,\mathcal{I})\}_m\{\boldsymbol{r}^j(\tau,\mathcal{I})\}_n]\end{aligned} \tag{13}$$

For this optimal problem (12), if the second constraint does not exist, then the optimal solution is

$$\boldsymbol{y}_0^* = \frac{x_j}{\mathbf{1}^T\boldsymbol{V}^{-1}\boldsymbol{r}}\boldsymbol{V}^{-1}\boldsymbol{r}.$$

If $\boldsymbol{y}_0^*$ satisfies the second condition, then the optimal solution of (12) is just $\boldsymbol{y}_0^*$. This condition holds if the amount of fund does not exceed the capacity of the market and the market allows for selling short.

Ignoring the capacity of the market and we can rewrite the problem as a QP problem

$$\begin{aligned}\min \quad & \tilde{\boldsymbol{y}}^T\boldsymbol{V}^j(t,\mathcal{I})\tilde{\boldsymbol{y}}\\ \text{s.t.} \quad & t = 1/\boldsymbol{y}^T\boldsymbol{r}^j(t,\mathcal{I})>0\\ & \tilde{\boldsymbol{y}} = t\boldsymbol{y}\\ & \mathbf{1}^T\tilde{\boldsymbol{y}} = t > 0\\ & \boldsymbol{y}\succeq 0\end{aligned} \tag{14}$$

with auxiliary variable t, by scaling y with x_j. This is a QP problem, which can be solved in polynomial time since V is positive definite. After solving it with a QP solver, we test whether $\boldsymbol{r}^T\boldsymbol{y}^*$ is positive, if so, we get corresponding optimal return, risk and portfolio of market j. Otherwise, we simply give the return rate and variance as zero. Then, we do the global optimization.

$$\begin{aligned}\max \quad & \boldsymbol{x}^T\boldsymbol{r}^* - \frac{1}{2}\gamma_1\sqrt{\boldsymbol{x}^T\boldsymbol{V}^*\boldsymbol{x}} - \gamma_2\|\boldsymbol{x}-\boldsymbol{x}_0\|_2\\ \text{s.t.} \quad & \mathbf{1}^T\boldsymbol{x} = 1\\ & \mathbf{0}\preceq\boldsymbol{x}\end{aligned} \tag{15}$$

The reason why we do not use Sharpe ratio to describe this optimization is for the simplicity to treat the cost of fund allocation, as well as the explainability since the cost and return rate are with the same order. Since the objective function is concave (i.e., convex for minimization), we can use barrier method to deal with the inequality restriction and finally solve this problem with numerical methods. The implementation of all these numerical algorithms has been accomplished in the course's homework, and we use more advanced and encapsulated libraries to reach our targets.

The pseudo code of our methodology is listed in Algorithm 1.

4 Experiments and Analysis

In this section, we utilize the A-share data from mainland China, characterized by an inter-day frequency. The dataset spans from March 17, 2017, to August 31, 2021, encompassing 3045 distinct stocks. Owing to the absence of diverse markets, we partitioned these stocks into six distinct clusters, each comprising roughly 500 assets. Concurrently, an alpha, predicted utilizing 32 distinct features via the XGBoost algorithm, was generated and subsequently validated through a backtesting procedure within the specified periods.

Upon obtaining the alphas, we employ the strategy delineated in Section 3 to allocate capital across various markets. Subsequently, we conduct a comparative analysis of the expected returns (inclusive of the actual return and the penalties associated with fund allocation) against the losses (comprising the expected return and the penalties arising from variance). The entities subject to comparison include:

- Mean allocation, namely, assigning equal weights to all markets;
- Outside allocation, namely, utilizing our method to determine the weights and allocate the funds;

Algorithm 1 Algorithm of fund allocation problems

Require: Alpha series $\alpha_i^j(t)$, return rate series $r_i^j(t)$
for $t = T$ **to** max time **do**
    Solve sub-optimal problem and gain return rate and variance of individual markets
    Estimate alpha-oriented trading excepted return and variance $\boldsymbol{r}(t), \boldsymbol{V}(t)$
    Solve main optimal problem and gain allocation solution
end for
for $t = T + 1$ **to** max time **do**
    if Yesterday receive any positive return **then**
        Allocate yesterday's allocation strategy and compute the return rate today
    else
        Remain the strategy, invest no money today and record the return as zero
    end if
    Compute the accumulated return, plot and compare
end for

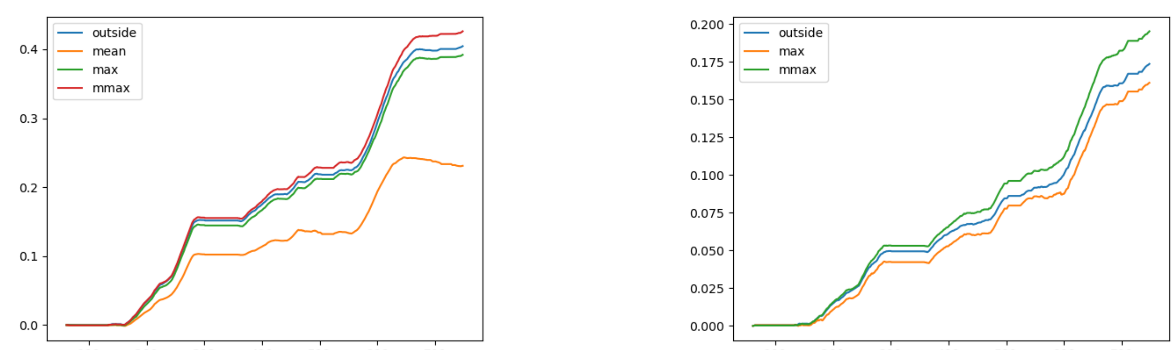
- Max allocation, namely, directing all funds to the market that yielded the highest returns the previous day;
- Mmax allocation, namely, presupposing foresight and allocating all funds to the market that yields the highest returns on the current day.

It is pertinent to note that the Mmax allocation, while theoretically intriguing, is impracticable in actual trading scenarios and serves solely as a benchmark for evaluating our method.

The allocation strategy within a distinct market adheres to the traditional portfolio optimization framework, which is not the primary focus of this discussion. To mitigate the adverse effects of market trends and alpha exposure, we implement a strategy whereby, should today's backtest of alphas yield no positive returns across any market, we maintain the existing capital allocation and abstain from investing on the subsequent day.

However, constrained by the mere daily frequency of data, the trading strategy suffers from inadequate adjustments, precipitating a marked instability in returns. In essence, possessing alphas at a substantially higher frequency would not only seize a greater array of superior trading opportunities but also significantly enhance both the expected return and the Sharpe ratio.

The comparative analysis of actual returns and the penalization of allocations is depicted in Figure 1, wherein Figure 1a delineates the cumulative sum of anticipated returns, and Figure 1b illustrates the disparities among three methodologies: outside, max, and mmax, relative to the baseline method's mean. Examination of these trajectories reveals that our approach not only markedly surpasses the mean method but also outperforms the max method, approaching the performance of mmax, an optimal yet unattainable benchmark under identical investment conditions within individual markets.

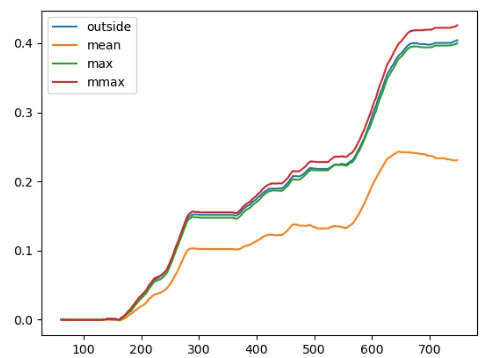


(a) Accumulated sum of expected return

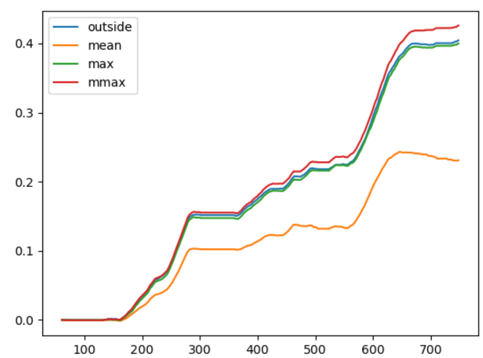
(b) Difference of accumulate sum w.r.t mean allocation

Figure 1: Comparison of true return and punishment of allocation

By incorporating punitive measures into the loss function, as delineated in Figure 2, and accounting for norm variance as depicted in Figure 2a, along with a scenario of 100-fold variance representing an extreme case in Figure 1b, our methodology consistently sustains its efficacy. This outcome demonstrates the substantial robustness of our approach against risks.



(a) Normal Variance



(b) 100x Variance

Figure 2: Comparison of true return, punishment of allocation and variance

5 Conclusions

In conclusion, this project proposes a novel approach to fund allocation in various markets by introducing a forecasting information set called alpha. We formulates robust optimization problems under specific market constraints and applies numerical methods to derive solutions via computational programs. The results show that the proposed approach achieves superior performance compared to conventional allocation methods and approximates the loss function of both intuitive and nearly optimal solutions. The project contributes to the field of fund allocation by providing a more effective and efficient approach that can be applied to various markets. Further studies can focus on the high frequency intraday situation and apply more real markets information, assiciated with more realistic constraints.

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