# Universal Exact Compression of Differentially Private Mechanisms

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## Introduction

**Local differential privacy (DP) [1].**

Local randomizer $\mathcal{A}: \mathcal{X} \to \mathcal{Z}$ with distribution $P_{Z|X}$ satisfies $(\varepsilon, \delta)$-local DP if for any $x, x' \in \mathcal{X}$ and measurable set $\mathcal{S} \subseteq \mathcal{Z}$,

$$\Pr(Z \in \mathcal{S} | X = x) \le e^{\varepsilon} \cdot \Pr(Z \in \mathcal{S} | X = x') + \delta.$$

**Compression of DP mechanisms.**

**Objective**: Compress DP mechanisms exactly (i.e., $Z \sim P_{Z|X}$) to near-optimal sizes, while ensuring privacy guarantees.

**Prior works**:

· [2-5]: Compress $\varepsilon$-local DP mechanism **approximately**.

· [6,7]: Dithered quantization tools ensure a correct simulated distribution, but only for additive noise mechanisms.

**Poisson Functional Representation (PFR) [8]**

Let $(T_i)_i$ be a Poisson process with rate 1, independent of $Z_i \overset{\text{i.i.d.}}{\sim} Q$. Then $(Z_i, T_i)_i$ is a Poisson process with intensity measure $Q \times \lambda_{[0,\infty)}$. Fix distribution $P$ absolutely continuous w.r.t $Q$. Let

$$\tilde{T}_i \triangleq T_i \cdot \left(\frac{\mathrm{d}P}{\mathrm{d}Q}(Z_i)\right)^{-1}.$$

**Theorem**: $K \triangleq \arg\min_i \tilde{T}_i$ and $Z = Z_K$, then $Z \sim P$.

**Our contributions**: Poisson private representation, which is:

(a) **Exact**: simulates $P_{Z|X}$ exactly;

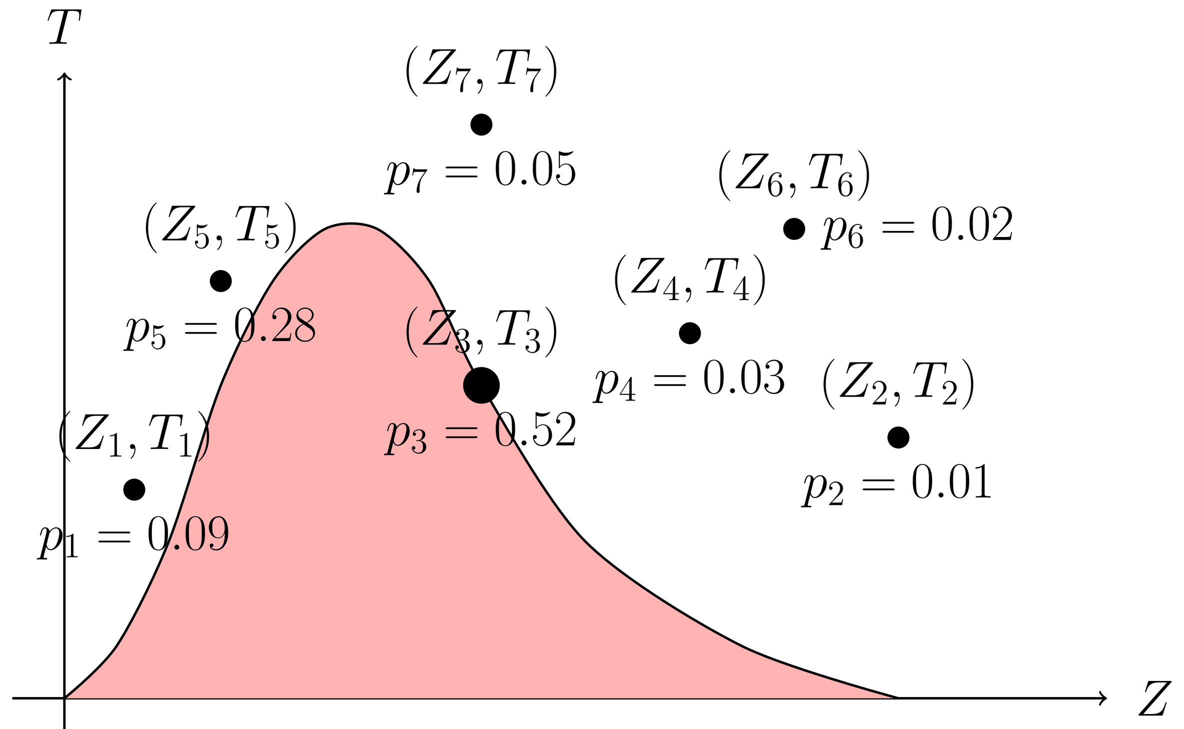
(b) **Universal**: simulates *any* DP mechanism;

(c) **Communication-efficient**: compresses $P_{Z|X}$ to

$$I(X;Z) + \log\left(I(X;Z)+1\right) + O(1) \text{ bits.}$$

(d) **Private**: ensures both local and central DP.

**Poisson Private Representation** ($p_k \triangleq \Pr(K = k)$):




https://github.com/cheuktingli/PoissonPrivateRepr

## Poisson Private Representation (PPR)

**Algorithm 1 (PPR).**

**Input:** private $x \in \mathcal{X}$, $(\varepsilon, \delta)$-local DP mechanism $P_{Z|X}$, reference distribution $Q$, parameter $\alpha > 1$.

(a) Generate shared randomness between user and server

$$(Z_i)_{i=1,2,\ldots} \overset{\text{i.i.d.}}{\sim} Q.$$

(b) The user knows $(Z_i)_i$, $x$, $P_{Z|X}$ and performs:

(1) Generate the Poisson process $(T_i)_i$ with rate 1.

(2) Compute $\tilde{T}_i \triangleq T_i \cdot \left(\frac{dP_{Z|X}(\cdot|x)}{dQ}(Z_i)\right)^{-1}$.

(3) Generate $K \in \mathbb{Z}_+$ with

$$\Pr(K = k) = \tilde{T}_k^{-\alpha} \Big/ \left(\sum_{i=1}^{\infty} \tilde{T}_i^{-\alpha}\right).$$

(4) Compress and send $K$ (e.g., by Elias delta code).

(c) The server, which knows $(Z_i)_i, K$, outputs $Z = Z_K$.

### Privacy guarantees

❶ **Thm 4.5**: If the mechanism $P_{Z|X}$ is $\varepsilon$-DP, then PPR $P_{(Z_i)_i, K|X}$ with $\alpha > 1$ is $2\alpha\varepsilon$-DP.

❷ **Thm 4.8**: If $P_{Z|X}$ is $(\varepsilon, \delta)$-DP, then PPR $P_{(Z_i)_i, K|X}$ is $(\alpha\varepsilon + \tilde{\varepsilon}, 2(\delta + \tilde{\delta}))$-DP, for $\alpha > 1$, $\tilde{\varepsilon} \in (0, 1]$ and $\tilde{\delta} \in (0, 1/3]$ s.t.

$$\alpha \le e^{-4.2}\tilde{\delta}\tilde{\varepsilon}^2/(-\ln\tilde{\delta}) + 1.$$

### Exactness

The output $Z$ of PPR follows $P_{Z|X}$ exactly.

### Communication Efficiency

**Thm 4.3**: For PPR with $\alpha > 1$, message $K$ satisfies

$$\mathbb{E}[\log_2 K] \le D_{\mathsf{KL}}(P(\cdot|x) \| Q(\cdot)) + \log_2(3.56)/\min((\alpha-1)/2, 1).$$

$K$ can be encoded by a prefix-free code with expected length $\approx D_{\mathsf{KL}}(P(\cdot|x)\|Q(\cdot))$ bits within a log gap. If $X \sim P_X$ is random, take $Q = P_Z$ and the expected length $\approx I(X;Z)$ (near-optimal).

**Corollary 4.4**: For $P_{Z|X}$ with $\varepsilon$-local DP, the compression size

$$\le \ell + \log_2(\ell+1) + 2 \text{ (bits)},$$

where $\ell \triangleq \varepsilon \log_2 e + \log_2(3.56)/\min((\alpha-1)/2, 1)$.

### Remarks

- The exactness of PPR follows from the PFR [8].
- While the algorithm requires infinite samples, it can be reparametrized to terminate in finite steps.
- When $\alpha = \infty$, PPR reduces to PFR.

## Application: Metric Privacy and Laplace Mechanism

For a mechanism $\mathcal{A}$ with $P_{Z|X}$ and a metric $d_{\mathcal{X}}$ over $\mathcal{X}$, it satisfies $\varepsilon \cdot d_{\mathcal{X}}$-privacy [9] if $\forall x, x' \in \mathcal{X}$, $\mathcal{S} \subseteq \mathcal{Z}$, we have

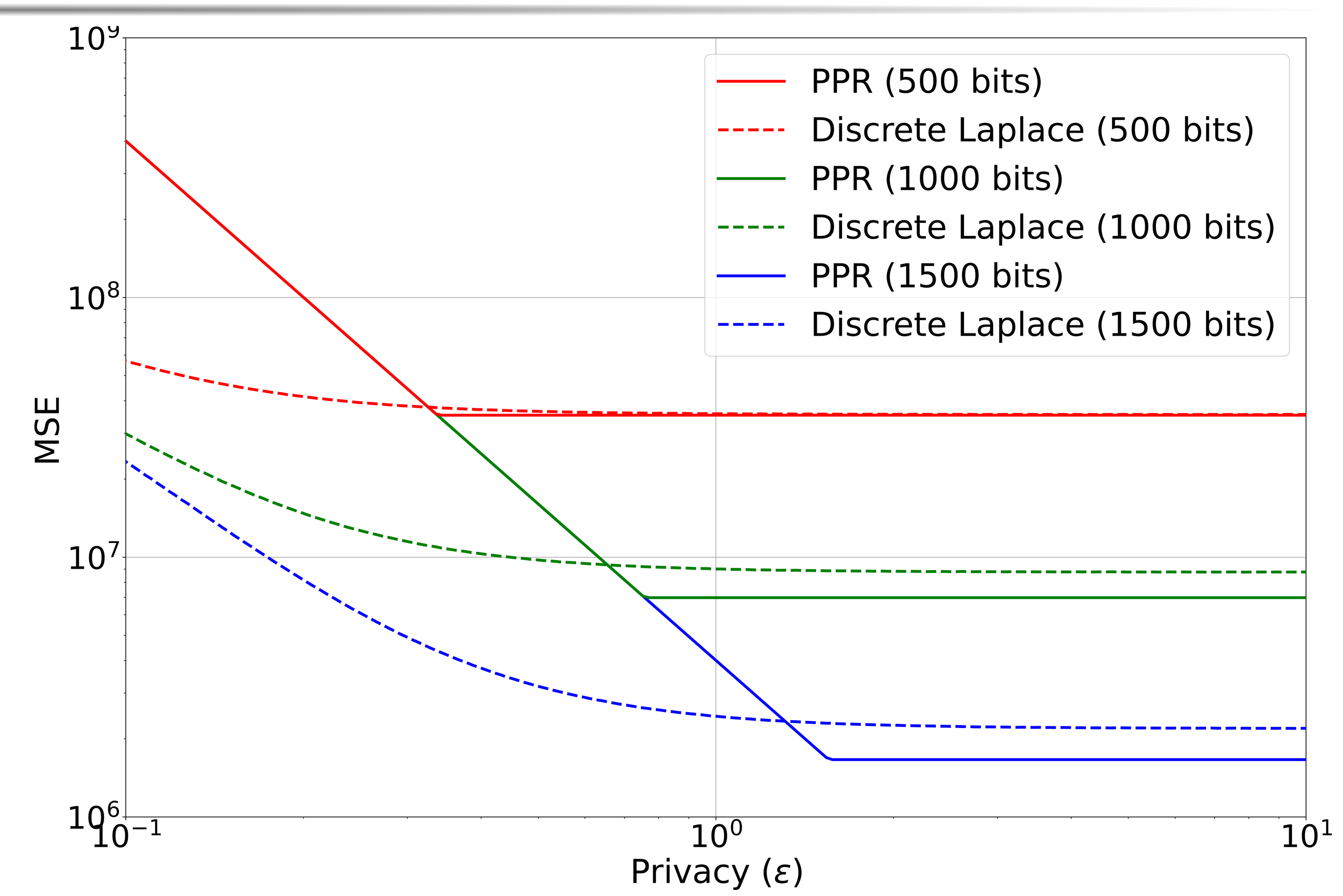
$$\Pr(Z \in \mathcal{S} \mid X = x) \le e^{\varepsilon \cdot d_{\mathcal{X}}(x,x')} \Pr(Z \in \mathcal{S} \mid X = x').$$

**PPR-compressed Laplace mechanism**:

For Laplace mechanism $P_{Z|X}$ with $X \in \{x \in \mathbb{R}^d \,|\, \|x\|_2 \le C\}$ and proposal distribution $Q = \mathcal{N}(0, (\frac{C^2}{d} + \frac{d+1}{\varepsilon^2})\mathbb{I}_d)$, the output of PPR has MSE $\frac{d(d+1)}{\varepsilon^2}$, $2\alpha\epsilon \cdot d_{\mathcal{X}}$-privacy and compression size $\le \ell + \log_2(\ell+1) + 2$ bits, where $\ell \triangleq$

$$\frac{d}{2}\log_2\left(\frac{2}{e}\left(\frac{C^2\varepsilon^2}{d} + d + 1\right)\right) - \log_2\frac{\Gamma(d+1)}{\Gamma(\frac{d}{2}+1)} + \frac{\log_2(3.56)}{\min\{\frac{\alpha-1}{2}, 1\}}.$$

We compare with the discrete Laplace mechanism [9], $d = 500$.



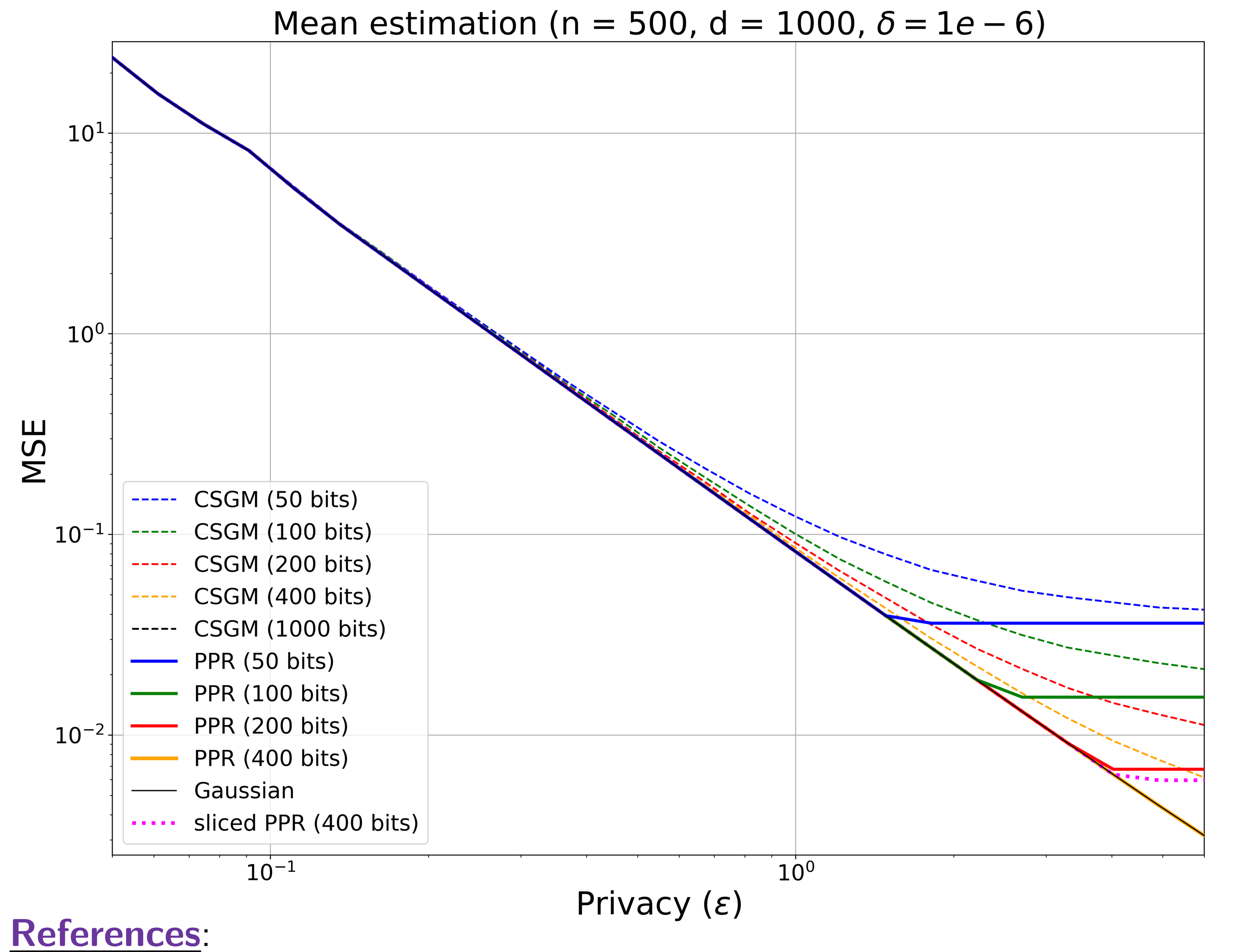

## Distributed Mean Estimation

Consider $n$ users, each with data $X_i \in \mathbb{R}^d$. They use **Gaussian mechanism** and send $Z_i \sim \mathcal{N}(X_i, \frac{\sigma^2}{n}\mathbb{I}_d)$ to server, where $\sigma \ge C\sqrt{2\ln(1.25/\delta)}/\varepsilon$. Server estimates mean as $\hat{\mu}(Z^n) = \frac{1}{n}\sum_i Z_i$. Using PPR to compress the Gaussian mechanism:

- $\hat{\mu}(Z^n) = \frac{1}{n}\sum_i Z_i$ is unbiased, has $(\varepsilon, \delta)$-central DP.
- PPR satisfies $(2\alpha\sqrt{n}\varepsilon, 2\delta)$-local DP for $\epsilon < 1/\sqrt{n}$.
- The average per-client communication cost is at most

$$\frac{d}{2}\log\left(\frac{n\varepsilon^2}{2d\log(1.25/\delta)} + 1\right) + \frac{\log_2(3.56)}{\min\{(\alpha-1)/2, 1\}} \text{ bits.}$$

Compare to CSGM [10] on distributed mean estimation:

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