Weighted Parity-Check Codes for Channels with State and Asymmetric Channels

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Table of Contents

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- Our goal is to construct a code that is:
  - Practical
  - Applicable to asymmetric channels (unlike Barron et al. [2003])
  - Having error performance as good as (and sometimes better than) the construction in Barron et al. [2003]

Query Functions

- Let $\mathbf{H} \in \mathbb{F}_2^{n \times n}$ be a full-rank matrix, called the *full parity check matrix*
  - $\mathbf{H}$ uniformly chosen random full-rank matrix
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- For a *bias vector* $\mathbf{q} = [q_1, \ldots, q_n] \in [0,1]^n$, define the $\mathbf{q}$-*weight* of a vector $\mathbf{u} \in \mathbb{F}_2^n$ as

$$w_{\mathbf{q}}(\mathbf{u}) := \prod_{i=1}^{n} q_i^{u_i}(1-q_i)^{1-u_i} = P_{x_i\sim\mathrm{Bern}(q_i)}(\mathbf{x} = \mathbf{u})$$

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Definition

Given the bias vectors $\mathbf{p}, \mathbf{q} \in [0,1]^n$ (we call $\mathbf{p}$ the *codeword bias*, and $\mathbf{q}$ the *parity bias*), the *query function* with respect to $\mathbf{H}$ is given by

$$f_{\mathbf{H}}(\mathbf{p}, \mathbf{q}) := \mathrm{argmax}_{\mathbf{x} \in \mathbb{F}_2^n} w_{\mathbf{p}}(\mathbf{x}) w_{\mathbf{q}}(\mathbf{x}\mathbf{H}^T) \tag{1}$$

Weighted Parity-Check Codes (WPC)

Definition: Encoder

Given the *encoder codeword bias function* $\mathbf{p}_e : \mathbb{F}_2^k \to [0,1]^n$, which maps a message $\mathbf{m} \in \mathbb{F}_2^k$ (and other information available at the encoder) to a bias vector $\mathbf{p}_e(\mathbf{m})$, and the *encoder parity bias function* $\mathbf{q}_e : \mathbb{F}_2^k \to [0,1]^n$. The encoding function is

$$\mathbf{m} \mapsto \mathbf{x} = f_{\mathbf{H}}\left(\mathbf{p}_e(\mathbf{m}), \mathbf{q}_e(\mathbf{m})\right) \tag{2}$$

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Definition: Decoder

Similarly, given the *decoder codeword and parity bias functions* $\mathbf{p}_d, \mathbf{q}_d : \mathbb{F}_2^n \to [0,1]^n$. For a corrupted version $\mathbf{y}$ of $\mathbf{x}$, the decoding function is

$$\mathbf{y} \mapsto \hat{\mathbf{m}} = \left[(\hat{\mathbf{x}}\mathbf{H}^T)_1, \ldots, (\hat{\mathbf{x}}\mathbf{H}^T)_k\right], \tag{3}$$

where

$$\hat{\mathbf{x}} := f_{\mathbf{H}}\left(\mathbf{p}_d(\mathbf{y}),\, \mathbf{q}_d(\mathbf{y})\right) \tag{4}$$

Recovering Conventional Linear Codes by WPC

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$$\mathbf{p}_d(\mathbf{y}) = \beta\mathbf{1}^n + (1-2\beta)\mathbf{y}, \qquad \mathbf{q}_d(\mathbf{y}) = \frac{1}{2}\mathbf{1}^n,$$

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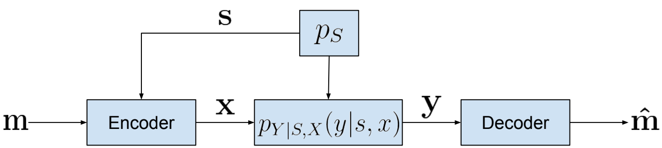
and substitute into Equations (2) and (4)
- Note that $w_{\mathbf{p}_d(\mathbf{y})}(\mathbf{x}) = P(\mathbf{x}|\mathbf{y})$ is the posterior distribution of $\mathbf{x}$

WPC for Gelfand-Pinsker Setting



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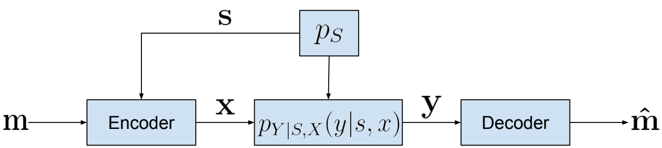

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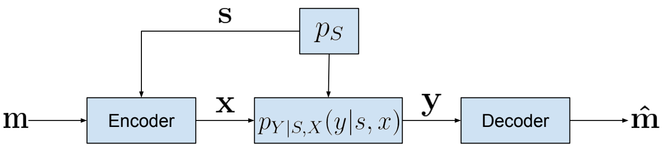

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- Encoder: after observing $\mathbf{m}$ and $\mathbf{s}$, takes

$$\mathbf{p}_e(\mathbf{m},\mathbf{s}) = [p_e(s_1),\ldots,p_e(s_n)], \quad \mathbf{q}_e(\mathbf{m},\mathbf{s}) = [\mathbf{m},\ \mathbf{q}], \tag{5}$$

where we choose $p_e(s) = P_{X|S}(1|s)$ so $\mathbf{x}$ approximately follows $P_{X|S}$

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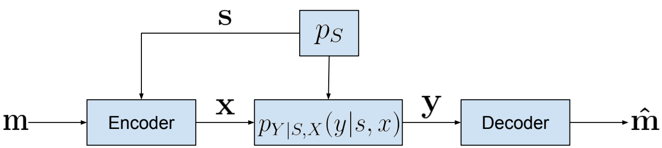

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and outputs $\hat{\mathbf{m}} = [(\hat{\mathbf{x}}\mathbf{H}^T)_1,\ldots,(\hat{\mathbf{x}}\mathbf{H}^T)_k]$, where $p_d(y) = P_{X|Y}(1|y)$

WPC is Capacity-Achieving

- We first state our main result as follows:

Theorem 1

- Assume $\mathbf{q} \sim P_Q$ i.i.d., where P_Q is a discrete distribution over $[0,1]$ with finite support satisfying

$$\mathbf{E}[H_b(Q)] = \frac{1 - H(X|S)}{1 - R}, \tag{7}$$

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- Then, for any $R < I(X; Y) - I(X; S)$, the probability of error of the code tends to 0, and the empirical joint distribution of $\{(s_i, x_i)\}_{i=1,\ldots,n}$ tends to $P_S P_{X|S}$ in probability as $n \to \infty$

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- Proof uses Sanov's theorem and robust typicality

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  - $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$ may not hold, not capacity achieving
- (Threshold linear) Construct P_Q using the cdf

$$F_Q(t) := \begin{cases} 0 & \text{if } t < 0 \\ \max\{\theta/2,\, 0\} & \text{if } 0 \le t < |\theta|/2 \\ t & \text{if } |\theta|/2 \le t < 1 - |\theta|/2 \\ 1 - \max\{\theta/2,\, 0\} & \text{if } 1 - |\theta|/2 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases} \tag{8}$$

where $\theta \in [-1, 1]$ is chosen s.t. $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$

- Combines the linear method for t close to $1/2$, and the threshold method for smaller and larger t's

Simulation Result

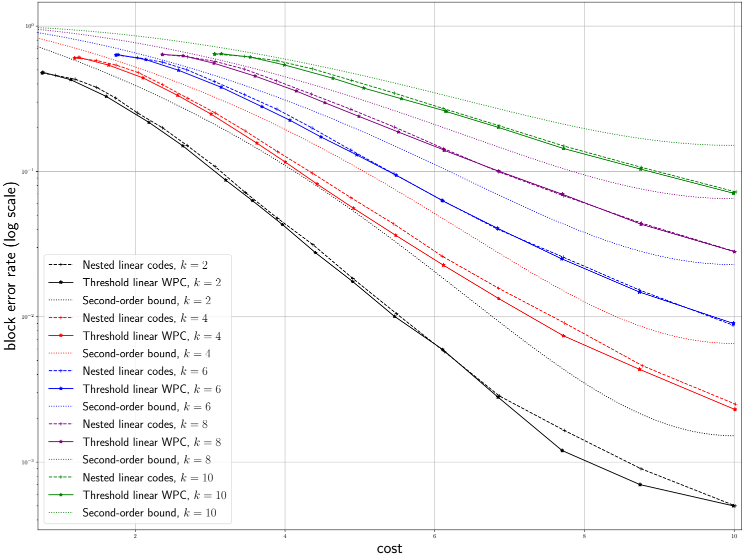

Figure: Performance evaluation with $n = 20$, $\beta = 0.05$

Conclusion and Discussions

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- Simulation results show that WPC achieves a smaller error probability compared to nested linear codes
- In the full paper [Ling et al., 2022], we show that our weighted construction also applies to the Wyner-Ziv setting [Wyner and Ziv, 1976]
- The code can be made more practical by considering a sparse parity-check matrix, though this is left for future work

Acknowledgments

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