Weighted Parity-Check Codes for Channels with State and Asymmetric Channels

Chih Wei Ling, Yanxiao Liu and Cheuk Ting Li
chihweiLing@link.cuhk.edu.hk
yanxiaoliu@link.cuhk.edu.hk
ctli@ie.cuhk.edu.hk

Department of Information Engineering, The Chinese University of Hong Kong

1st July 2022

Table of Contents

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse
  - Martinian and Wainwright [2006] used sparse graphical code to generate nested linear code which is practical

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse
  - Martinian and Wainwright [2006] used sparse graphical code to generate nested linear code which is practical
  - Li and Anantharam [2021] proposed Poisson functional representation construction, with the best known second-order error bound [Scarlett, 2015] compared to other finite-blocklength schemes, but is impractical

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse
  - Martinian and Wainwright [2006] used sparse graphical code to generate nested linear code which is practical
  - Li and Anantharam [2021] proposed Poisson functional representation construction, with the best known second-order error bound [Scarlett, 2015] compared to other finite-blocklength schemes, but is impractical
- Our goal is to construct a code that is:
  - Practical

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse
  - Martinian and Wainwright [2006] used sparse graphical code to generate nested linear code which is practical
  - Li and Anantharam [2021] proposed Poisson functional representation construction, with the best known second-order error bound [Scarlett, 2015] compared to other finite-blocklength schemes, but is impractical
- Our goal is to construct a code that is:
  - Practical
  - Applicable to asymmetric channels (unlike Barron et al. [2003])

Introduction and Motivation

- Background: Find practical construction for the Gelfand-Pinsker setting with cost constraint and for the asymmetric channels
  - Barron et al. [2003] proposed nested linear code, but not sparse
  - Martinian and Wainwright [2006] used sparse graphical code to generate nested linear code which is practical
  - Li and Anantharam [2021] proposed Poisson functional representation construction, with the best known second-order error bound [Scarlett, 2015] compared to other finite-blocklength schemes, but is impractical
- Our goal is to construct a code that is:
  - Practical
  - Applicable to asymmetric channels (unlike Barron et al. [2003])
  - Having error performance as good as (and sometimes better than) the construction in Barron et al. [2003]

Query Functions

- Let $\mathbf{H} \in \mathbb{F}_2^{n \times n}$ be a full-rank matrix, called the *full parity check matrix*
  - $\mathbf{H}$ uniformly chosen random full-rank matrix
  - Also works for sparse $\mathbf{H}$, but the analysis is left for future study

Query Functions

- Let $\mathbf{H} \in \mathbb{F}_2^{n \times n}$ be a full-rank matrix, called the *full parity check matrix*
  - $\mathbf{H}$ uniformly chosen random full-rank matrix
  - Also works for sparse $\mathbf{H}$, but the analysis is left for future study
- For a *bias vector* $\mathbf{q} = [q_1, \ldots, q_n] \in [0, 1]^n$, define the $\mathbf{q}$-*weight* of a vector $\mathbf{u} \in \mathbb{F}_2^n$ as

$$w_{\mathbf{q}}(\mathbf{u}) := \prod_{i=1}^{n} q_i^{u_i}(1 - q_i)^{1-u_i} = P_{x_i \sim \mathrm{Bern}(q_i)}(\mathbf{x} = \mathbf{u})$$

Query Functions

- Let $\mathbf{H} \in \mathbb{F}_2^{n\times n}$ be a full-rank matrix, called the *full parity check matrix*
    - $\mathbf{H}$ uniformly chosen random full-rank matrix
    - Also works for sparse $\mathbf{H}$, but the analysis is left for future study
- For a *bias vector* $\mathbf{q} = [q_1, \ldots, q_n] \in [0,1]^n$, define the $\mathbf{q}$-*weight* of a vector $\mathbf{u} \in \mathbb{F}_2^n$ as

$$w_{\mathbf{q}}(\mathbf{u}) := \prod_{i=1}^{n} q_i^{u_i}(1-q_i)^{1-u_i} = P_{x_i\sim\mathrm{Bern}(q_i)}(\mathbf{x}=\mathbf{u})$$

Definition

Given the bias vectors $\mathbf{p}, \mathbf{q} \in [0,1]^n$ (we call $\mathbf{p}$ the *codeword bias*, and $\mathbf{q}$ the *parity bias*), the *query function* with respect to $\mathbf{H}$ is given by

$$f_{\mathbf{H}}(\mathbf{p},\mathbf{q}) := \mathrm{argmax}_{\mathbf{x}\in\mathbb{F}_2^n}\, w_{\mathbf{p}}(\mathbf{x})w_{\mathbf{q}}(\mathbf{x}\mathbf{H}^T) \tag{1}$$

Weighted Parity-Check Codes (WPC)

Definition: Encoder

Given the *encoder codeword bias function* $\mathbf{p}_e : \mathbb{F}_2^k \to [0,1]^n$, which maps a message $\mathbf{m} \in \mathbb{F}_2^k$ (and other information available at the encoder) to a bias vector $\mathbf{p}_e(\mathbf{m})$, and the *encoder parity bias function* $\mathbf{q}_e : \mathbb{F}_2^k \to [0,1]^n$. The encoding function is

$$\mathbf{m} \mapsto \mathbf{x} = f_{\mathbf{H}}\left(\mathbf{p}_e(\mathbf{m}), \mathbf{q}_e(\mathbf{m})\right) \quad (2)$$

Weighted Parity-Check Codes (WPC)

Definition: Encoder

Given the *encoder codeword bias function* $\mathbf{p}_e : \mathbb{F}_2^k \to [0,1]^n$, which maps a message $\mathbf{m} \in \mathbb{F}_2^k$ (and other information available at the encoder) to a bias vector $\mathbf{p}_e(\mathbf{m})$, and the *encoder parity bias function* $\mathbf{q}_e : \mathbb{F}_2^k \to [0,1]^n$. The encoding function is

$$\mathbf{m} \mapsto \mathbf{x} = f_{\mathbf{H}}\left(\mathbf{p}_e(\mathbf{m}),\, \mathbf{q}_e(\mathbf{m})\right) \tag{2}$$

Definition: Decoder

Similarly, given the *decoder codeword and parity bias functions* $\mathbf{p}_d, \mathbf{q}_d : \mathbb{F}_2^n \to [0,1]^n$. For a corrupted version $\mathbf{y}$ of $\mathbf{x}$, the decoding function is

$$\mathbf{y} \mapsto \hat{\mathbf{m}} = \left[(\hat{\mathbf{x}}\mathbf{H}^T)_1, \ldots, (\hat{\mathbf{x}}\mathbf{H}^T)_k\right], \tag{3}$$

where

$$\hat{\mathbf{x}} := f_{\mathbf{H}}\left(\mathbf{p}_d(\mathbf{y}),\, \mathbf{q}_d(\mathbf{y})\right) \tag{4}$$

Recovering Conventional Linear Codes by WPC

- Binary symmetric channel with parameter β, i.e., $P(y_i|x_i)$ is $\mathrm{BSC}(\beta)$

Recovering Conventional Linear Codes by WPC

- Binary symmetric channel with parameter β, i.e., $P(y_i|x_i)$ is $\mathrm{BSC}(\beta)$
- To recover the conventional linear code, we take

$$\mathbf{p}_e(\mathbf{m}) = \frac{1}{2}\mathbf{1}^n, \qquad \mathbf{q}_e(\mathbf{m}) = [\mathbf{m},\, \mathbf{0}^{n-k}],$$

$$\mathbf{p}_d(\mathbf{y}) = \beta\mathbf{1}^n + (1-2\beta)\mathbf{y}, \qquad \mathbf{q}_d(\mathbf{y}) = \frac{1}{2}\mathbf{1}^n,$$

and substitute into Equations (2) and (4)

Recovering Conventional Linear Codes by WPC

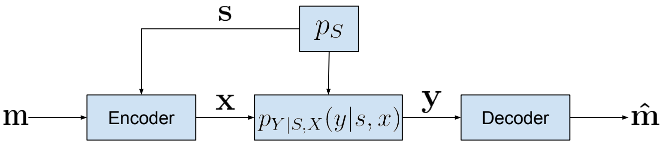
- Binary symmetric channel with parameter β, i.e., $P(y_i|x_i)$ is $\mathrm{BSC}(\beta)$
- To recover the conventional linear code, we take

$$\mathbf{p}_e(\mathbf{m}) = \frac{1}{2}\mathbf{1}^n, \qquad \mathbf{q}_e(\mathbf{m}) = [\mathbf{m}, \mathbf{0}^{n-k}],$$

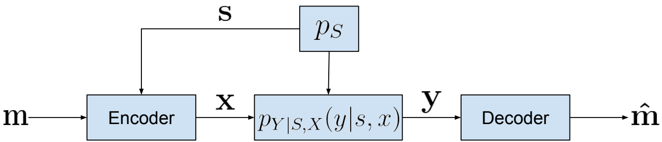
$$\mathbf{p}_d(\mathbf{y}) = \beta\mathbf{1}^n + (1-2\beta)\mathbf{y}, \qquad \mathbf{q}_d(\mathbf{y}) = \frac{1}{2}\mathbf{1}^n,$$

  and substitute into Equations (2) and (4)
- Note that $w_{\mathbf{p}_d(\mathbf{y})}(\mathbf{x}) = P(\mathbf{x}|\mathbf{y})$ is the posterior distribution of $\mathbf{x}$

WPC for Gelfand-Pinsker Setting



WPC for Gelfand-Pinsker Setting

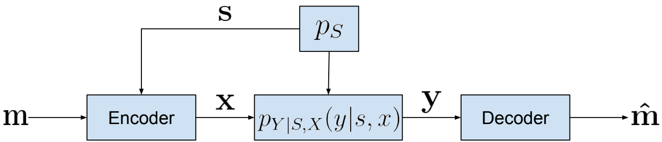


- We construct the *weighted parity-check codes with state* as follows
- Encoder: after observing **m** and **s**, takes

$$\mathbf{p}_e(\mathbf{m},\mathbf{s}) = [p_e(s_1),\ldots,p_e(s_n)], \quad \mathbf{q}_e(\mathbf{m},\mathbf{s}) = [\mathbf{m},\ \mathbf{q}], \tag{5}$$

where we choose $p_e(s) = P_{X|S}(1|s)$ so **x** approximately follows $P_{X|S}$

WPC for Gelfand-Pinsker Setting



- We construct the *weighted parity-check codes with state* as follows
- Encoder: after observing $\mathbf{m}$ and $\mathbf{s}$, takes

$$\mathbf{p}_e(\mathbf{m},\mathbf{s}) = [p_e(s_1),\ldots,p_e(s_n)], \quad \mathbf{q}_e(\mathbf{m},\mathbf{s}) = [\mathbf{m},\, \mathbf{q}], \tag{5}$$

where we choose $p_e(s) = P_{X|S}(1|s)$ so $\mathbf{x}$ approximately follows $P_{X|S}$

- Decoder: after observing $\mathbf{y}$, takes

$$\mathbf{p}_d(\mathbf{y}) = [p_d(y_1),\ldots,p_d(y_n)], \quad \mathbf{q}_d(\mathbf{y}) = [\frac{1}{2}\mathbf{1}^k,\, \mathbf{q}], \tag{6}$$

and outputs $\hat{\mathbf{m}} = [(\hat{\mathbf{x}}\mathbf{H}^T)_1,\ldots,(\hat{\mathbf{x}}\mathbf{H}^T)_k]$, where $p_d(y) = P_{X|Y}(1|y)$

WPC is Capacity-Achieving

- We first state our main result as follows:

Theorem 1

- Assume $\mathbf{q} \sim P_Q$ i.i.d., where P_Q is a discrete distribution over $[0, 1]$ with finite support satisfying

$$\mathbf{E}[H_b(Q)] = \frac{1 - H(X|S)}{1 - R}, \tag{7}$$

where $H_b : [0, 1] \to [0, 1]$ is the binary entropy function

WPC is Capacity-Achieving

- We first state our main result as follows:

Theorem 1

- Assume $\mathbf{q} \sim P_Q$ i.i.d., where P_Q is a discrete distribution over $[0,1]$ with finite support satisfying

$$\mathbf{E}[H_b(Q)] = \frac{1 - H(X|S)}{1 - R}, \tag{7}$$

where $H_b : [0,1] \to [0,1]$ is the binary entropy function
- Then, for any $R < I(X;Y) - I(X;S)$, the probability of error of the code tends to 0, and the empirical joint distribution of $\{(s_i, x_i)\}_{i=1,\ldots,n}$ tends to $P_S P_{X|S}$ in probability as $n \to \infty$

WPC is Capacity-Achieving

- We first state our main result as follows:

Theorem 1

- Assume $\mathbf{q} \sim P_Q$ i.i.d., where P_Q is a discrete distribution over $[0, 1]$ with finite support satisfying

$$\mathbf{E}[H_b(Q)] = \frac{1 - H(X|S)}{1 - R}, \tag{7}$$

where $H_b : [0, 1] \to [0, 1]$ is the binary entropy function
- Then, for any $R < I(X; Y) - I(X; S)$, the probability of error of the code tends to 0, and the empirical joint distribution of $\{(s_i, x_i)\}_{i=1,\ldots,n}$ tends to $P_S P_{X|S}$ in probability as $n \to \infty$

- Unlike nested linear codes, WPC also works for asymmetric $P_{X|S}$

WPC is Capacity-Achieving

- We first state our main result as follows:

Theorem 1

- Assume $\mathbf{q} \sim P_Q$ i.i.d., where P_Q is a discrete distribution over $[0, 1]$ with finite support satisfying

$$\mathbf{E}[H_b(Q)] = \frac{1 - H(X|S)}{1 - R}, \tag{7}$$

where $H_b : [0, 1] \to [0, 1]$ is the binary entropy function

- Then, for any $R < I(X; Y) - I(X; S)$, the probability of error of the code tends to 0, and the empirical joint distribution of $\{(s_i, x_i)\}_{i=1,\dots,n}$ tends to $P_S P_{X|S}$ in probability as $n \to \infty$

- Unlike nested linear codes, WPC also works for asymmetric $P_{X|S}$
- Proof uses Sanov's theorem and robust typicality

How to Choose P_Q satisfying Equation (7)

- To achieve capacity, we need $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$

How to Choose P_Q satisfying Equation (7)

- To achieve capacity, we need $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$
- (Threshold) Take $P_Q(0) = P_Q(1) = (1-\gamma)/2$, $P_Q(1/2) = \gamma$, where $\gamma = (1 - H(X|S))/(1-R)$
  - Essentially equivalent to the nested linear code

How to Choose P_Q satisfying Equation (7)

- To achieve capacity, we need $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$
- (Threshold) Take $P_Q(0) = P_Q(1) = (1-\gamma)/2$, $P_Q(1/2) = \gamma$, where $\gamma = (1 - H(X|S))/(1-R)$
  - Essentially equivalent to the nested linear code
- (Linear) Take P_Q to be the uniform distribution $\mathrm{Unif}[0, 1]$
  - $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$ may not hold, not capacity achieving

How to Choose P_Q satisfying Equation (7)

- To achieve capacity, we need $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$
- (Threshold) Take $P_Q(0) = P_Q(1) = (1-\gamma)/2$, $P_Q(1/2) = \gamma$, where $\gamma = (1-H(X|S))/(1-R)$
  - Essentially equivalent to the nested linear code
- (Linear) Take P_Q to be the uniform distribution $\mathrm{Unif}[0,1]$
  - $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$ may not hold, not capacity achieving
- (Threshold linear) Construct P_Q using the cdf

$$F_Q(t) := \begin{cases} 0 & \text{if } t < 0 \\ \max\{\theta/2, 0\} & \text{if } 0 \le t < |\theta|/2 \\ t & \text{if } |\theta|/2 \le t < 1-|\theta|/2 \\ 1-\max\{\theta/2, 0\} & \text{if } 1-|\theta|/2 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases} \tag{8}$$

where $\theta \in [-1,1]$ is chosen s.t. $\mathbf{E}[H_b(Q)] = \frac{1-H(X|S)}{1-R}$

  - Combines the linear method for t close to $1/2$, and the threshold method for smaller and larger t's

Simulation Result

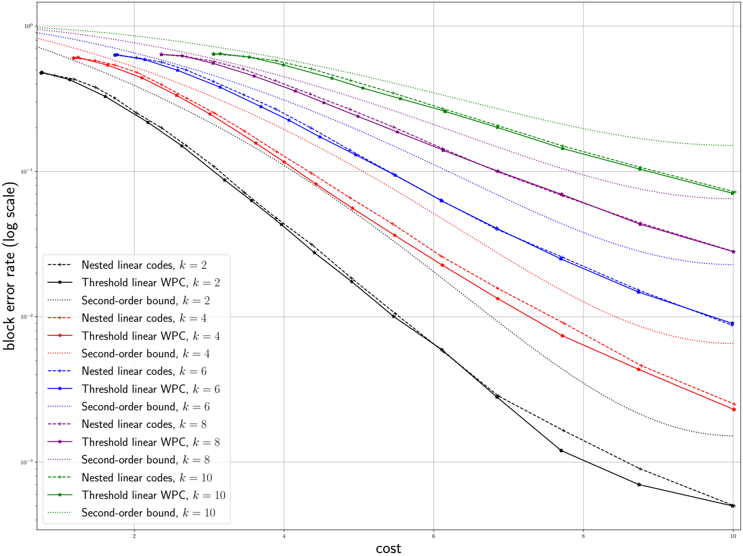


Figure: Performance evaluation with $n = 20$, $\beta = 0.05$

Conclusion and Discussions

- We propose the weighted parity check code, applicable to channels with state and asymmetric channels

Conclusion and Discussions

- We propose the weighted parity check code, applicable to channels with state and asymmetric channels
- Simulation results show that WPC achieves a smaller error probability compared to nested linear codes

Conclusion and Discussions

- We propose the weighted parity check code, applicable to channels with state and asymmetric channels
- Simulation results show that WPC achieves a smaller error probability compared to nested linear codes
- In the full paper [Ling et al., 2022], we show that our weighted construction also applies to the Wyner-Ziv setting [Wyner and Ziv, 1976]

Conclusion and Discussions

- We propose the weighted parity check code, applicable to channels with state and asymmetric channels
- Simulation results show that WPC achieves a smaller error probability compared to nested linear codes
- In the full paper [Ling et al., 2022], we show that our weighted construction also applies to the Wyner-Ziv setting [Wyner and Ziv, 1976]
- The code can be made more practical by considering a sparse parity-check matrix, though this is left for future work

Reference

Richard J Barron, Brian Chen, and Gregory W Wornell. The duality between information embedding and source coding with side information and some applications. *IEEE Transactions on Information Theory*, 49 (5):1159–1180, 2003.

Cheuk Ting Li and Venkat Anantharam. A unified framework for one-shot achievability via the Poisson matching lemma. *IEEE Transactions on Information Theory*, 67(5):2624–2651, 2021.

Chih Wei Ling, Yanxiao Liu, and Cheuk Ting Li. Weighted parity-check codes for channels with state and asymmetric channels. *arXiv preprint arXiv:2201.10171*, 2022.

Emin Martinian and Martin J Wainwright. Low-density constructions can achieve the Wyner-Ziv and Gelfand-Pinsker bounds. pages 484–488, 2006.

Jonathan Scarlett. On the dispersions of the Gel'fand–Pinsker channel and dirty paper coding. *IEEE Trans. Inf. Theory*, 61(9):4569–4586, 2015.

A. D. Wyner and J. Ziv. The rate-distortion function for source coding with side information at the decoder. *IEEE Trans. Inf. Theory*, 22(1):