One-Shot Coding over General Noisy Networks

ISIT 2024

Yanxiao Liu and Cheuk Ting Li

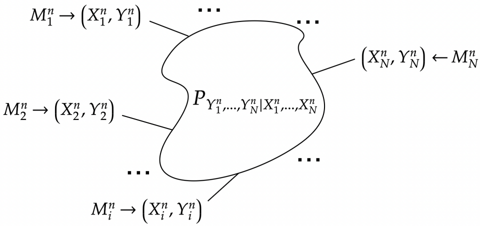
The Chinese University of Hong Kong

Overview: Our Contributions

Our Contributions

1. We consider the general **one-shot** coding problem.
2. We consider communication and compression of messages among multiple nodes across **general acyclic noisy networks**.
3. We design proof techniques based on Poisson functional representations.
4. Our coding framework is applicable to **any** combination of source coding, channel coding and coding for computing problems (with special cases presented).

Background: Noisy Network Coding

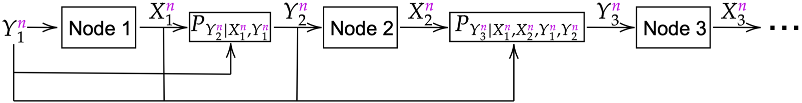


Noisy Network Coding

- **Noisy network coding**[a]: communicating messages between **multiple** sources and destinations over a general **noisy** network.
- Generalizing:
  1. Noiseless network coding (Ahlswede et al. [2000])
  2. Compress-forward coding for relay channels (Cover and El Gamal, [1979]).
  3. Coding for wireless relay networks and deterministic networks (Avestimehr et al. [2007]), coding for erasure networks (Dana et al. [2006]), etc.

[a] Lim, Sung Hoon, Young-Han Kim, Abbas El Gamal, and Sae-Young Chung. "Noisy network coding."IEEE Transactions on Information Theory 57, no. 5 (2011): 3132-3152.

Background: A Unified Random Coding Bound



A Unified Asymptotic Random Coding Bound

1. Unified random coding bound[a]: work for **any** combination of channel coding and source coding problems.
2. Unifying and generalizing known relaying strategies; can yield bounds without complicated error analysis.
3. Useful for designing automated theorem proving tools[b].

[a] Lee, Si-Hyeon, and Sae-Young Chung. "A unified random coding bound."IEEE Transactions on Information Theory 64, no. 10 (2018): 6779-6802.

[b] Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results."IEEE Transactions on Information Theory (2023).

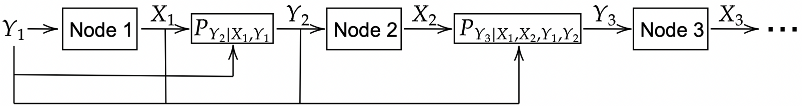
Background: One-Shot Information Theory

One-Shot Information Theory

What if each source and channel is only used once, i.e., $n = 1$ (Feinstein, [1954]; Shannon, [1957]; Verdú, [2012]; Yassaee el al. [2013]; Li and Anantharam [2021])?

1. Sources and channels can be **arbitrary**: no need to be memoryless or ergodic.
2. Goal: obtain one-shot results that can recover existing (first-order and second-order) **asymptotic** results when applied to memoryless sources and channels and also **finite blocklength** results (Polyanskiy elta. [2010]; Kostina and Verdú [2012]).

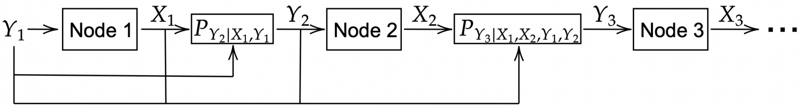
Overview



Our Contributions: One-Shot Coding Framework over Noisy Networks

1. A unified **one-shot** coding scheme
2. over general **noisy** acyclic discrete networks (ADN)
3. that is applicable to **any** combination of source coding, channel coding and coding for computing problems,
4. proved by our **exponential process refinement lemma**.

Overview



Our Contributions: Specific Network Information Theory Settings

- **Novel** one-shot achievability results for:
  1. One-shot relay channels
  2. One-shot primitive relay channels
     - Compress-and-forward bound
     - Partial-decode-and-forward bound
- **Recovered** one-shot & asymptotic results for:
  1. Source and channel coding
  2. Gelfand-Pinsker, Wyner-Ziv and coding for computing
  3. Multiple access channels
  4. Broadcast channels

Preliminaries: Poisson Functional Representation

Poisson Functional Representation

- For a finite set $\mathcal{U}$, let $\mathbf{U} := (Z_u)_{u \in \mathcal{U}}$ be i.i.d. $\mathrm{Exp}(1)$ random variables[a].
- Given a distribution P over $\mathcal{U}$, **Poisson functional representation**[b]:

$$\mathbf{U}_P := \mathrm{argmin}_u \frac{Z_u}{P(u)} \tag{1}$$

- $\mathbf{U}_P \sim P$
- Various applications: minimax learning, neural network compression, etc.

[a] When the space $\mathcal{U}$ is continuous, a Poisson process is used instead.

[b] Li, Cheuk Ting, and Abbas El Gamal. "Strong functional representation lemma and applications to coding theorems."IEEE Transactions on Information Theory 64, no. 11 (2018): 6967-6978.

Preliminaries: Poisson Matching Lemma

Poisson Functional Representation

- Given a distribution P over $\mathcal{U}$, **Poisson functional representation**:

$$\mathbf{U}_P := \mathrm{argmin}_u \frac{Z_u}{P(u)}$$

Generalized Poisson Matching Lemma

- Let $\mathbf{U}_P(1), \ldots, \mathbf{U}_P(|\mathcal{U}|) \in \mathcal{U}$ be the elements of $\mathcal{U}$ sorted in ascending order of $Z_u/P(u)$, let $\mathbf{U}_P^{-1} : \mathcal{U} \to [|\mathcal{U}|]$ for the inverse function of $i \mapsto \mathbf{U}_P(i)$.
- **Generalized Poisson matching lemma**[a]: For distributions P, Q over $\mathcal{U}$, we have the following almost surely:

$$\mathbf{E}\left[\mathbf{U}_Q^{-1}(\mathbf{U}_P) \,\middle|\, \mathbf{U}_P\right] \le \frac{P(\mathbf{U}_P)}{Q(\mathbf{U}_P)} + 1.$$

[a] Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma."IEEE Transactions on Information Theory 67, no. 5 (2021): 2624-2651.

New Techniques

Refining a distribution by an exponential process

- For a joint distribution $Q_{V,U}$ over $\mathcal{V} \times \mathcal{U}$, the **refinement** of $Q_{V,U}$ by $\mathbf{U}$:

$$Q^{\mathbf{U}}_{V,U}(v,u) := \frac{Q_V(v)}{\mathbf{U}^{-1}_{Q_{U|V}(\cdot|v)}(u) \sum_{i=1}^{|\mathcal{U}|} i^{-1}} \tag{2}$$

  for all (v,u) in the support of $Q_{V,U}$.
- The refinement is for the **soft decoding**.
- If the distribution $Q_{V,U}$ represents our "**prior distribution**" of (V,U), then the refinement $Q^{\mathbf{U}}_{V,U}$ is our updated "**posterior distribution**" after taking the exponential process $\mathbf{U}$ into account.

New Techniques

Exponential Process Refinement Lemma

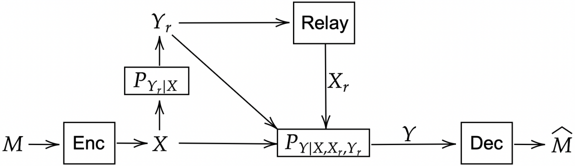
- For a distribution P over $\mathcal{U}$ and a joint distribution $Q_{V,U}$ over a finite $\mathcal{V} \times \mathcal{U}$, for every $v \in \mathcal{V}$, we have, almost surely,

$$\mathbf{E}\left[\frac{1}{Q^{\mathbf{U}}_{V,U}(v, \mathbf{U}_P)} \middle| \mathbf{U}_P\right] \le \frac{\ln|\mathcal{U}| + 1}{Q_V(v)} \left(\frac{P(\mathbf{U}_P)}{Q_{U|V}(\mathbf{U}_P|v)} + 1\right). \tag{3}$$

Purpose

It keeps track of the evolution of the "posterior probability" of the correct values of a large number of random variables through the refinement process.

One-Shot Relay Channel

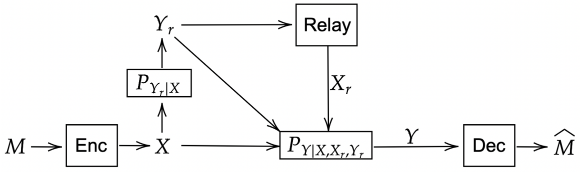


One-Shot Relay Channel

- One-shot version of **relay-with-unlimited-look-ahead**[a]
- Limitation of one-shot settings: unable to model "networks with causality", e.g., conventional relay channel (Van Der Meulen, [1971]; Cover and El Gamal, [1979]; Kim, [2007])
- "**Best one-shot approximation**" of the conventional relay channel

[a] El Gamal, Abbas, Navid Hassanpour, and James Mammen. "Relay networks with delays."IEEE Transactions on Information Theory 53, no. 10 (2007): 3413-3431.

One-Shot Relay Channel

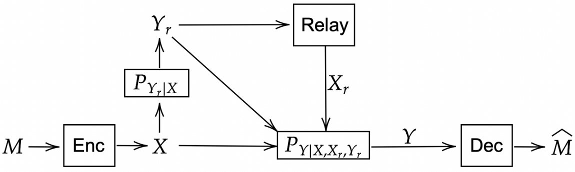


One-Shot Relay Channel

1. Encoder observes $M \sim \mathrm{Unif}[\mathsf{L}]$ and outputs X, which is passed through the channel $P_{Y_\mathrm{r}|X}$.
2. Relay observes Y_r and outputs X_r.
3. $(X, X_\mathrm{r}, Y_\mathrm{r})$ is passed through the channel $P_{Y|X,X_\mathrm{r},Y_\mathrm{r}}$.
   - Y depends on all of $X, X_\mathrm{r}, Y_\mathrm{r}$. X_r may interfere with (X, Y_r).
4. Decoder observes Y and recovers $\hat{M}$.

Practical in scenarios where the relay outputs X_r instantaneously or the channel has a long memory, or it is a storage device.

One-Shot Relay Channel



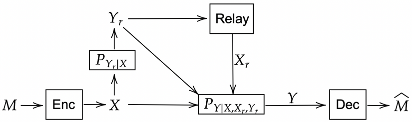
Theorem (One-Shot Achievable Bound)

For any P_X, $P_{U|Y_\mathrm{r}}$, function $x_\mathrm{r}(y_\mathrm{r}, u)$, there is a coding scheme for the one-shot relay channel such that the error probability satisfies

$$P_e \le \mathbf{E}\left[\min\left\{\gamma \mathsf{L} 2^{-\iota(X;U,Y)}\left(2^{-\iota(U;Y)+\iota(U;Y_\mathrm{r})}+1\right), 1\right\}\right],$$

where $(X, Y_\mathrm{r}, U, X_\mathrm{r}, Y) \sim P_X P_{Y_\mathrm{r}|X} P_{U|Y_\mathrm{r}} \delta_{x_\mathrm{r}(Y_\mathrm{r},U)} P_{Y|X,Y_\mathrm{r},X_\mathrm{r}}$, *and* $\gamma := \ln|\mathcal{U}| + 1$.

One-Shot Relay Channel



Proof

1. "Random codebooks" $\mathbf{U}_1$, $\mathbf{U}_2$: independent exponential processes.
2. Encoder: $U_1 = (\mathbf{U}_1)_{P_{U_1} \times \delta_M}$.
3. Relay: $U_2 = (\mathbf{U}_2)_{P_{U_2|Y_\mathrm{r}}(\cdot|Y_\mathrm{r})}$, then outputs $X_\mathrm{r} = x_\mathrm{r}(Y_\mathrm{r}, U_2)$.
4. Decoder observes Y, and:
   - Refine $P_{U_2|Y}(\cdot|Y)$ to $Q_{U_2} := P_{U_2|Y}^{\mathbf{U}_2}$. By Exponential Process Refinement Lemma:

$$\mathbf{E}\left[\frac{1}{Q_{U_2}(U_2)}\,\middle|\, U_2, Y, Y_\mathrm{r}\right] \le (\ln|\mathcal{U}_2| + 1)\left(\frac{P_{U_2|Y_\mathrm{r}}(U_2)}{P_{U_2|Y}(U_2)} + 1\right).$$

   - Compute $Q_{U_2} P_{U_1|U_2,Y}$ over $\mathcal{U}_1 \times \mathcal{U}_2$, and let its U_1-marginal be $\tilde{Q}_{U_1}$.
   - Let $\tilde{U}_1 = (\mathbf{U}_1)_{\tilde{Q}_{U_1} \times P_M}$, and output its M-component.

One-Shot Relay Channel

Proof

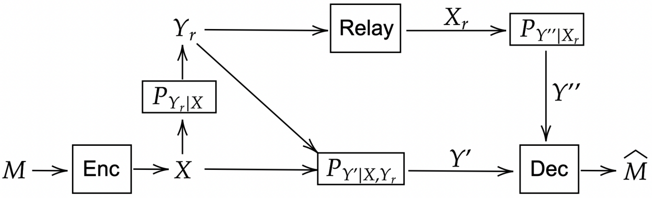
$$
\begin{aligned}
&\mathbf{P}(\tilde{U}_1 \neq U_1 \,|\, X, Y_{\mathrm{r}}, U_2, X_{\mathrm{r}}, Y, M) \\
&\overset{(a)}{\leq} \mathbf{E}\left[\min\left\{\frac{P_{U_1}(U_1)\delta_M(M)}{P_{U_1|U_2,Y}(U_1|U_2,Y)Q_{U_2}(U_2)P_M(M)}, 1\right\} \,\middle|\, X, Y_{\mathrm{r}}, U_2, X_{\mathrm{r}}, Y, M\right] \\
&\overset{(b)}{\leq} \min\left\{\mathrm{L}\frac{P_{U_1}(U_1)}{P_{U_1|U_2,Y}(U_1|U_2,Y)}(\ln|\mathcal{U}_2|+1)\left(\frac{P_{U_2|Y_{\mathrm{r}}}(U_2)}{P_{U_2|Y}(U_2)}+1\right), 1\right\} \\
&= \min\left\{(\ln|\mathcal{U}_2|+1)\mathrm{L}2^{-\iota(X;U_2,Y)}\left(2^{-\iota(U_2;Y)+\iota(U_2;Y_{\mathrm{r}})}+1\right), 1\right\}.
\end{aligned}
$$

(a) is by the Poisson matching lemma;
(b) is by the refinement step and Jensen's inequality.

For some $P_{U|Y_{\mathrm{r}}}$ and function $x_{\mathrm{r}}(y_{\mathrm{r}}, u_2)$, it yields the asymptotic achievable rate:

$$R \leq I(X; U, Y) - \max\left\{I(U; Y_{\mathrm{r}}) - I(U; Y), 0\right\}.$$

One-Shot Primitive Relay Channel



One-shot version of primitive relay channels (Kim, [2007]; Mondelli et al. [2019]; El Gamal et al. [2021]; El Gamal et al. [2022]):
$Y = (Y', Y'')$ and the channel $P_{Y|X,X_\mathrm{r},Y_\mathrm{r}} = P_{Y'|X,Y_\mathrm{r}} P_{Y''|X_\mathrm{r}}$ can be decomposed into two orthogonal components.

Theorem

For any P_X, P_{X_r}, $P_{U'|Y_\mathrm{r}}$, there is a coding scheme for the one-shot primitive relay channel with $M \sim \mathrm{Unif}[\mathsf{L}]$ such that

$$P_e \le \mathbf{E}\left[\min\left\{ \left(\ln(|\mathcal{U}'||\mathcal{X}_\mathrm{r}|)+1\right) \mathsf{L} 2^{-\iota(X;U',Y')} \left(2^{-\iota(X_\mathrm{r};Y'')+\iota(U';Y_\mathrm{r}|Y')}+1\right), 1\right\}\right],$$

$(X, Y_\mathrm{r}, U', Y') \sim P_X P_{Y_\mathrm{r}|X} P_{U'|Y_\mathrm{r}} P_{Y'|X,Y_\mathrm{r}}$ *independent of* $(X_\mathrm{r}, Y'') \sim P_{X_\mathrm{r}} P_{Y''|X_\mathrm{r}}$.

One-Shot Primitive Relay Channel

Theorem

For any P_X, P_{X_r}, $P_{U'|Y_\mathrm{r}}$, there is a coding scheme for the one-shot primitive relay channel with $M \sim \mathrm{Unif}[\mathsf{L}]$ such that

$$P_e \le \mathbf{E}\left[\min\left\{ \left(\ln(|\mathcal{U}'||\mathcal{X}_\mathrm{r}|)+1\right) \mathsf{L} 2^{-\iota(X;U',Y')}\left(2^{-\iota(X_\mathrm{r};Y'')+\iota(U';Y_\mathrm{r}|Y')}+1\right), 1\right\}\right],$$

$(X, Y_\mathrm{r}, U', Y') \sim P_X P_{Y_\mathrm{r}|X} P_{U'|Y_\mathrm{r}} P_{Y'|X,Y_\mathrm{r}}$ *independent of* $(X_\mathrm{r}, Y'') \sim P_{X_\mathrm{r}} P_{Y''|X_\mathrm{r}}$.

Asymptotic rate

$$R \le I(X;U',Y') - \max\{I(U';Y_\mathrm{r}|Y') - C_\mathrm{r}, 0\}$$

where $C_\mathrm{r} = \max_{P_{X_\mathrm{r}}} I(X_\mathrm{r};Y'')$.

- It recovers the **compress-and-forward bound**[a].

[a] Kim, Young-Han. "Coding techniques for primitive relay channels."In Proc. Forty-Fifth Annual Allerton Conf. Commun., Contr. Comput, p. 2007. 2007.

One-Shot Relay Channel

Corollary (Partial-Decode-and-Forward Bound)

Fix any $P_{X,V}$, $P_{U|Y_\mathrm{r},V}$, *function* $x_\mathrm{r}(y_\mathrm{r},u,v)$, *and* J *which is a factor of* L. *There exists a deterministic coding scheme for the one-shot relay channel with*

$$P_e \le \mathbf{E}\Big[\min\Big\{\mathsf{J}2^{-\iota(V;Y_\mathrm{r})} + (\ln(\mathsf{J}|\mathcal{U}|)+1)(\ln(\mathsf{J}|\mathcal{V}|)+1)\mathsf{L}\mathsf{J}^{-1}2^{-\iota(X;U,Y|V)} \cdot \left(2^{-\iota(U;V,Y)+\iota(U;V,Y_\mathrm{r})}+1\right)\left(\mathsf{J}2^{-\iota(V;Y)}+1\right), 1\Big\}\Big],$$

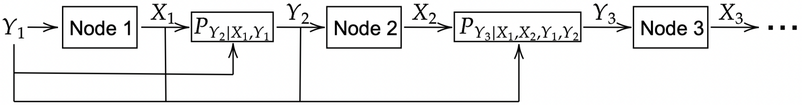
where $(X,V,Y_\mathrm{r},U,X_\mathrm{r},Y) \sim P_{X,V}P_{Y_\mathrm{r}|X,V}P_{U|Y_\mathrm{r},V}\delta_{x_\mathrm{r}(Y_\mathrm{r},U,V)}P_{Y|X,Y_\mathrm{r},X_\mathrm{r}}$.

- It recovers existing asymptotic partial-decode-and-forward bounds on primitive relay channel[a] and on relay-with-unlimited-look-ahead[b].

[a] Cover, Thomas, and Abbas El Gamal. "Capacity theorems for the relay channel."IEEE Transactions on information theory 25, no. 5 (1979): 572-584.

[b] El Gamal, Abbas, Navid Hassanpour, and James Mammen. "Relay networks with delays."IEEE Transactions on Information Theory 53, no. 10 (2007): 3413-3431.

General Acyclic Discrete Network



Acyclic discrete network (ADN)

- Nodes are labelled by $1, \ldots, N$; node i sees $Y_i \in \mathcal{Y}_i$ and produces $X_i \in \mathcal{X}_i$.
- Y_i depends on all previous inputs and outputs X^{i-1}, Y^{i-1}.
- **ADN**: a collection of channels $(P_{Y_i|X^{i-1},Y^{i-1}})_{i\in[N]}$, where $P_{Y_i|X^{i-1},Y^{i-1}}$ is a conditional distribution from $\prod_{j=1}^{i-1} \mathcal{X}_j \times \prod_{j=1}^{i-1} \mathcal{Y}_j$ to $\mathcal{Y}_i$.

General Acyclic Discrete Network

1. $\tilde{X}_i, \tilde{Y}_i$: **actual** random variables from the coding scheme.
2. X_i, Y_i: random variables following an **ideal** distribution.
   - Example 1 (channel coding): the ideal distribution is $Y_1 = X_2 \sim \mathrm{Unif}[L]$ (decoding without error), independent of $(X_1, Y_2) \sim P_{X_1} P_{Y_2|X_1}$. If we ensure $\tilde{X}^2, \tilde{Y}^2$ is "close to" the ideal X^2, Y^2, it implies $\tilde{Y}_1 = \tilde{X}_2$ with high probability, i.e., a small error probability.
3. Take an "error set" $\mathcal{E}$ that we do not want $(\tilde{X}^N, \tilde{Y}^N)$ to fall into.
   - Example 2 (channel coding): $\mathcal{E}$ is the set where $\tilde{Y}_1 \neq \tilde{X}_2$, i.e., an error occurs.
   - Example 3 (lossy source coding): $\mathcal{E}$ is the set where $d(\tilde{Y}_1, \tilde{X}_2) > \mathsf{D}$, i.e., the distortion exceeds the limit.
4. **Goal**: make $P_{\tilde{X}^N, \tilde{Y}^N}$ "approximately as good as" the P_{X^N, Y^N}, i.e.,

$$\mathbf{P}\Big((\tilde{X}^N, \tilde{Y}^N) \in \mathcal{E}\Big) \lesssim \mathbf{P}\Big((X^N, Y^N) \in \mathcal{E}\Big), \tag{4}$$

which can be guaranteed by ensuring the closeness in TV distance:

$$\delta_{\mathrm{TV}}\Big(P_{X^N, Y^N},\, P_{\tilde{X}^N, \tilde{Y}^N}\Big) \approx 0. \tag{5}$$

Coding Scheme

Deterministic coding scheme $(f_i)_{i\in[N]}$

A sequence of encoding functions $(f_i)_{i\in[N]}$, where $f_i : \mathcal{Y}_i \to \mathcal{X}_i$. For $i = 1, \ldots, N$:

- $\tilde{X}_i = f_i(\tilde{Y}_i)$.
- $\tilde{Y}_i$ follows $P_{Y_i|X^{i-1},Y^{i-1}}$ conditional on $\tilde{X}^{i-1}, \tilde{Y}^{i-1}$.

Goal: $\mathbf{P}\big((\tilde{X}^N, \tilde{Y}^N) \in \mathcal{E}\big) \lesssim \mathbf{P}\big((X^N, Y^N) \in \mathcal{E}\big)$

To construct a deterministic coding scheme, we first construct a randomized coding scheme:

Public-randomness coding scheme $(P_W, (f_i)_{i\in[N]})$

1. Generate **public randomness** $W \in \mathcal{W}$ available to all nodes;
2. **Encoding function** of node i: $f_i : \mathcal{Y}_i \times \mathcal{W} \to \mathcal{X}_i$, $\tilde{X}_i = f_i(\tilde{Y}_i, W)$.

Goal: $\delta_{\mathrm{TV}}\big(P_{X^N,Y^N},\ P_{\tilde{X}^N,\tilde{Y}^N}\big) \approx 0$

If there is a good public-randomness coding scheme, then there is a good deterministic coding scheme by fixing the value of W.

Main Theorem

Theorem

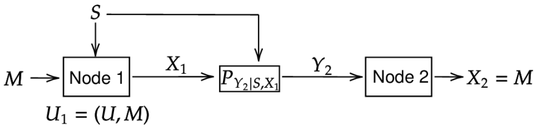
Fix an ADN $(P_{Y_i|X^{i-1},Y^{i-1}})_{i\in[N]}$. For any collection of indices $(a_{i,j})_{i\in[N],j\in[d_i]}$ where $(a_{i,j})_{j\in[d_i]}$ is a sequence of distinct indices in $[i-1]$ for each i, any sequence $(d_i')_{i\in[N]}$ with $0\le d_i'\le d_i$ and any collection of conditional distributions $(P_{U_i|Y_i,\overline{U}_i'}, P_{X_i|Y_i,U_i,\overline{U}_i'})_{i\in[N]}$ (where $\overline{U}_{i,\mathcal{S}} := (U_{a_{i,j}})_{j\in\mathcal{S}}$ for $\mathcal{S}\subseteq[d_i]$ and $\overline{U}_i' := \overline{U}_{i,[d_i']}$), which induces the joint distribution of X^N, Y^N, U^N (the "ideal distribution"), there exists a public-randomness coding scheme s.t.

$$\delta_{\mathrm{TV}}\left(P_{X^N,Y^N},\, P_{\tilde{X}^N,\tilde{Y}^N}\right) \le \mathbf{E}\left[\min\left\{\sum_{i=1}^{N}\sum_{j=1}^{d_i'} B_{i,j},\, 1\right\}\right],$$

where $\gamma_{i,j} := \prod_{k=j+1}^{d_i}\left(\ln|\mathcal{U}_{a_{i,k}}|+1\right)$ and

$$B_{i,j} := \gamma_{i,j}\prod_{k=j}^{d_i}\left(2^{-\iota(\overline{U}_{i,k};\overline{U}_{i,[d_i]\setminus[j..k]},Y_i)+\iota(\overline{U}_{i,k};\overline{U}'_{a_{i,k}},Y_{a_{i,k}})} + \mathbf{1}\{k>j\}\right).$$

ADN: Gelfand-Pinsker Problem (Gelfand and Pinsker [1980], Heegard and El Gamal [1983])



Gelfand-Pinsker Problem

- **ADN**: $Y_1 := (M, S)$, $Y_2 := Y$, $P_{Y_2|Y_1,X_1}$ be $P_{Y|S,X}$, and $X_2 := M$.
- **Auxiliary** on node 1: $U_1 = (U, M)$ for some U following $P_{U|S}$ given S.
- **Decoding order**: on node 2 "U_1" (i.e., it only wants U_1).

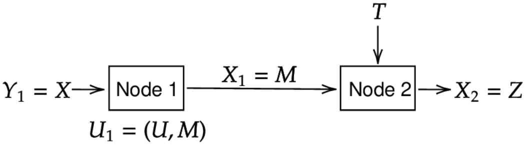
Corollary (Gelfand-Pinsker)

Fix $P_{U|S}$ and function $x : \mathcal{U} \times \mathcal{S} \to \mathcal{X}$. There exists a coding scheme for the channel $P_{Y|X,S}$ with $S \sim P_S$, $M \sim \mathrm{Unif}[\mathsf{L}]$ such that

$$P_e \leq \mathbf{E}\left[\min\left\{\mathsf{L}2^{-\iota(U;Y)+\iota(U;S)}, 1\right\}\right],$$

where $S, U, X, Y \sim P_S P_{U|S} \delta_{x(U,S)} P_{Y|X,S}$.

ADN: Wyner-Ziv Problem (Wyner and Ziv [1976])



Corollary (Wyner-Ziv)

Fix $P_{U|X}$ and function $z : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$. There exists a coding scheme s.t.

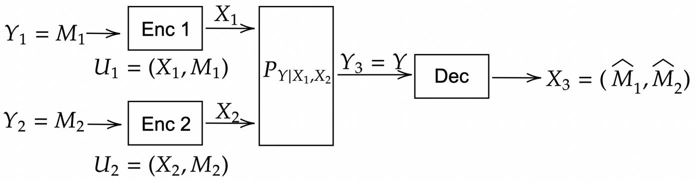
$$P_e \leq \mathbf{E}\left[\min\left\{\mathbf{1}\{d(X,Z) > \mathsf{D}\} + \mathsf{L}^{-1}2^{-\iota(U;T)+\iota(U;X)}, 1\right\}\right],$$

where $X, Y, U, Z \sim P_X P_{Y|X} P_{U|X} \delta_{z(U,Y)}$.

Coding for Computing (Yamamoto, [1982])

- Coding for computing: node 2 recovers a function $f(X, T)$, $P_e \leq \mathbf{E}[\min\{\mathbf{1}\{d(f(X,T), Z) > \mathsf{D}\} + \mathsf{L}^{-1}2^{-\iota(U;T)+\iota(U;X)}, 1\}]$.

ADN: Multiple Access Channel (Liao, [1972]; Ahlswede, [1974])



Corollary (Multiple Access Channel)

Fix P_{X_1}, P_{X_2}. There exists a coding scheme for the multiple access channel $P_{Y|X_1,X_2}$ with

$$P_e \leq \mathbf{E}\left[\min\left\{\gamma \mathsf{L}_1\mathsf{L}_2 2^{-\iota(X_1,X_2;Y)} + \gamma \mathsf{L}_2 2^{-\iota(X_2;Y|X_1)} + \mathsf{L}_1 2^{-\iota(X_1;Y|X_2)}, 1\right\}\right],$$

where $\gamma := \ln(\mathsf{L}_1|\mathcal{X}_1|) + 1$, $(X_1, X_2, Y) \sim P_{X_1}P_{X_2}P_{Y|X_1,X_2}$.

Asymptotic region: $R_1 < I(X_1; Y|X_2)$, $R_2 < I(X_2; Y|X_1)$, $R_1 + R_2 < I(X_1, X_2; Y)$.

Summary

Summary

- We provide a **unified one-shot coding framework** for communication and compression over general noisy networks.
- We design a proof technique "**exponential process refinement lemma**" that can keep track of a large number of auxiliary random variables.
- We provide **novel one-shot results** for various multi-hop settings.
- We recover existing one-shot and asymptotic results on various settings.

Future Directions

- A unified coding scheme is useful to design automated theorem proving tools, e.g., PSITIP[a]. Extensions to one-shot results is left for future study.

[a] Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results."IEEE Transactions on Information Theory (2023).

Acknowledgement

This work was partially supported by two grants from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No.s: CUHK 24205621 (ECS), CUHK 14209823 (GRF)].

References

- Feinstein, Amiel. "A new basic theorem of information theory."(1954).
- Shannon, Claude E. "Certain results in coding theory for noisy channels."Information and control 1.1 (1957): 6-25.
- Verdú, Sergio. "Non-asymptotic achievability bounds in multiuser information theory."2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton). IEEE, 2012.
- Yassaee, Mohammad Hossein, Mohammad Reza Aref, and Amin Gohari. "A technique for deriving one-shot achievability results in network information theory."2013 IEEE International Symposium on Information Theory. IEEE, 2013.
- Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma."IEEE Transactions on Information Theory 67.5 (2021): 2624-2651.
- Polyanskiy, Yury, H. Vincent Poor, and Sergio Verdú. "Channel coding rate in the finite blocklength regime."IEEE Transactions on Information Theory 56.5 (2010): 2307-2359.
- Kostina, Victoria, and Sergio Verdú. "Fixed-length lossy compression in the finite blocklength regime."IEEE Transactions on Information Theory 58.6 (2012): 3309-3338.

References

- Ahlswede, Rudolf, et al. "Network information flow."IEEE Transactions on information theory 46.4 (2000): 1204-1216.
- Avestimehr, Amir Salman, Suhas N. Diggavi, and D. N. C. Tse. "Wireless network information flow."arXiv preprint arXiv:0710.3781 (2007).
- Dana, Amir F., et al. "Capacity of wireless erasure networks."IEEE Transactions on Information Theory 52.3 (2006): 789-804.
- Van Der Meulen, Edward C. "Three-terminal communication channels."Advances in applied Probability 3.1 (1971): 120-154.
- Cover, Thomas, and Abbas El Gamal. "Capacity theorems for the relay channel."IEEE Transactions on information theory 25.5 (1979): 572-584.
- Kim, Young-Han. "Coding techniques for primitive relay channels."Proc. Forty-Fifth Annual Allerton Conf. Commun., Contr. Comput. 2007.
- Mondelli, Marco, S. Hamed Hassani, and Rüdiger Urbanke. "A new coding paradigm for the primitive relay channel."Algorithms 12.10 (2019): 218.
- El Gamal, Abbas, Amin Gohari, and Chandra Nair. "Achievable rates for the relay channel with orthogonal receiver components."2021 IEEE Information Theory Workshop (ITW). IEEE, 2021.
- El Gamal, Abbas, Amin Gohari, and Chandra Nair. "A strengthened cutset upper bound on the capacity of the relay channel and applications."IEEE Transactions on Information Theory 68.8 (2022): 5013-5043.

References

- S. I. Gelfand and M. S. Pinsker. Coding for channel with random parameters. Problems of Control and Information Theory, 9(1):19–31, 1980
- Heegard, Chris, and A. El Gamal. "On the capacity of computer memory with defects."IEEE transactions on Information theory 29.5 (1983): 731-739.
- Wyner, Aaron, and Jacob Ziv. "The rate-distortion function for source coding with side information at the decoder."IEEE Transactions on information Theory 22.1 (1976): 1-10.
- Yamamoto, Hirosuke. "Wyner-Ziv theory for a general function of the correlated sources (corresp.)."IEEE Transactions on Information Theory 28.5 (1982): 803-807.
- Ahlswede, Rudolf. "The capacity region of a channel with two senders and two receivers."The annals of probability (1974): 805-814.
- Liao, H. "Multiple access channels (Ph. D. Thesis abstr.)."IEEE Transactions on Information Theory 19.2 (1973): 253-253.