# One-Shot Coding over General Noisy Networks


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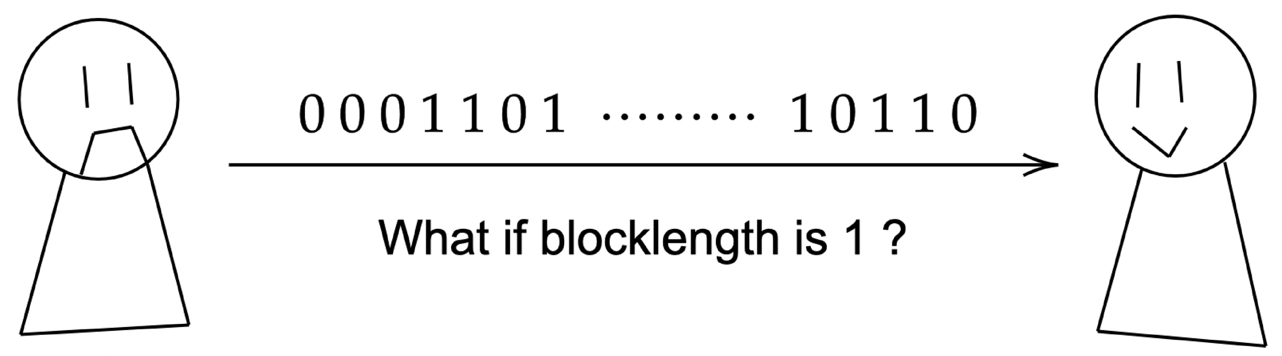
## Key Questions in Information Theory

Some key questions at the heart of information theory.

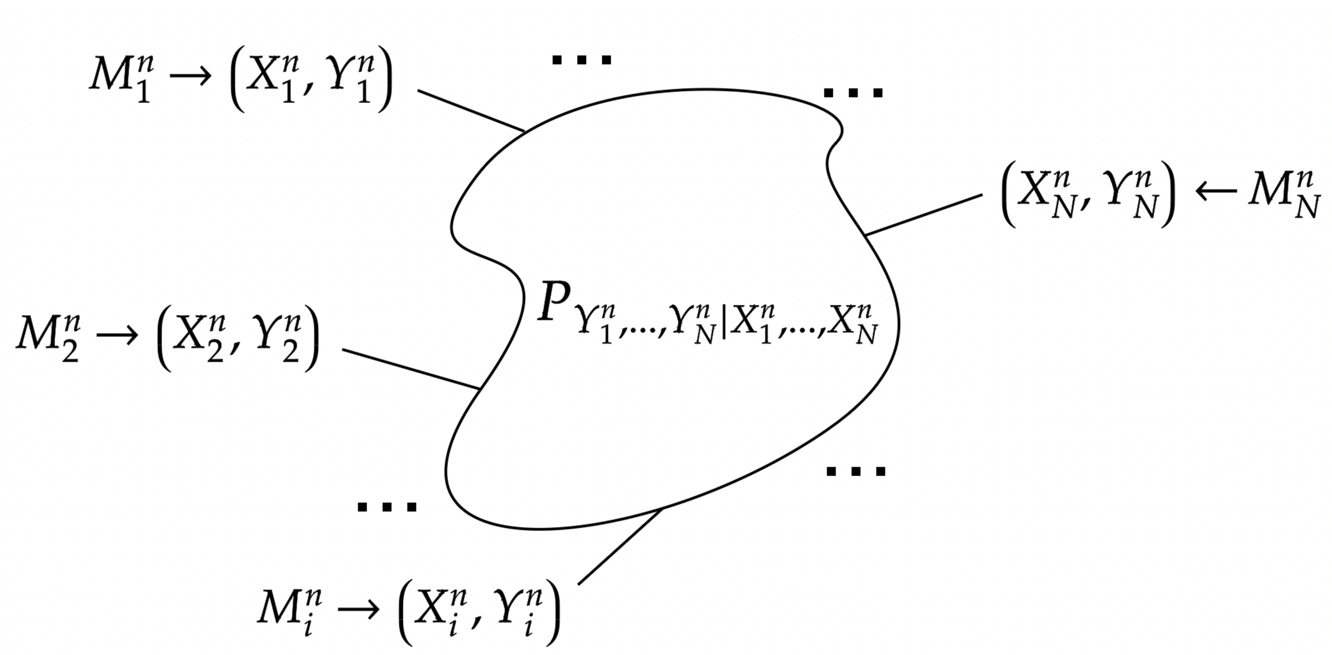
### Key Question 1: Blocklength in Information Theory




**One-shot information theory** [2, 3]: network is only used **once**!

- Error probability cannot be driven to zero!
- No law of large number $\to$ no typicality!
- No time-sharing!
- No memoryless/ergodic assumption!
- **Objective:** One-shot achievabilities that can imply existing (first/second order) asymptotic/finite-blocklength bounds?

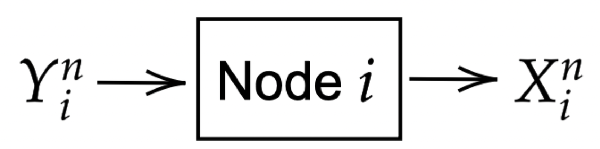
### Key Question 2: Noisy network coding




**Noisy network coding** [4]:

- What is the capacity of a **noisy network**?
- What coding scheme can achieve the capacity?

### Key Question 3: Unified Coding Scheme




node operation

**A unified coding schemes** [1]:

- A unified **node operation** in networks?
- Unify channel coding, source coding, and coding for computing?

## A Unified One-Shot Coding Scheme

To answer the key questions on the left, our scheme [5] combines:

1. One-shot/finite-blocklength network information theory
2. Noisy network coding
3. Unified scheme (source coding/channel coding/coding for computing)

### Main Theorem

For any acyclic discrete network $(P_{Y_i|X^{i-1},Y^{i-1}})_{i\in[N]}$, we provide a one-shot achievability result: For any collection of indices $(a_{i,j})_{i\in[N],j\in[d_i]}$ where $(a_{i,j})_{j\in[d_i]}$ is a sequence of distinct indices in $[i-1]$ for each $i$, any sequence $(d_i')_{i\in[N]}$ with $0 \le d_i' \le d_i$ and any collection of conditional distributions $(P_{U_i|Y_i,\overline{U}_i'}, P_{X_i|Y_i,U_i,\overline{U}_i'})_{i\in[N]}$ (where $\overline{U}_{i,\mathcal{S}} := (U_{a_{i,j}})_{j\in\mathcal{S}}$ for $\mathcal{S} \subseteq [d_i]$ and $\overline{U}_i' := \overline{U}_{i,[d_i']}$), which induces the joint distribution of $X^N, Y^N, U^N$ (the "ideal distribution"), there exists a public-randomness coding scheme $(P_W, (f_i)_{i\in[N]})$ such that the joint distribution of $\tilde{X}^N, \tilde{Y}^N$ induced by the scheme (the "actual distribution") satisfies

$$\delta_{\mathrm{TV}}\Big(P_{X^N,Y^N},\, P_{\tilde{X}^N,\tilde{Y}^N}\Big) \le \mathbf{E}\Big[\min\Big\{\sum_{i=1}^{N}\sum_{j=1}^{d_i'} B_{i,j},\, 1\Big\}\Big],$$

where $\gamma_{i,j} := \prod_{k=j+1}^{d_i}(\ln|\mathcal{U}_{a_{i,k}}|+1)$, and

$$B_{i,j} := \gamma_{i,j}\prod_{k=j}^{d_i}\Big(2^{-\iota(\overline{U}_{i,k};\overline{U}_{i,[d_i]\backslash[j..k]},Y_i)+\iota(\overline{U}_{i,k};\overline{U}'_{a_{i,k}},Y_{a_{i,k}})} + \mathbf{1}\{k>j\}\Big).$$

### Techniques

1. **Poisson functional representation [3]**: Let $\mathbf{U} := (Z_u)_{u\in\mathcal{U}}$ be i.i.d. $\mathrm{Exp}(1)$ random variables. Given a distribution $P$ over finite $\mathcal{U}$,
$$\mathbf{U}_P := \mathrm{argmin}_u \frac{Z_u}{P(u)}.$$
2. **Each node is associated with an exponential process.**
3. **Exponential Process Refinement**: For $Q_{V,U}$ over a finite $\mathcal{V}\times\mathcal{U}$, $\forall v\in\mathcal{V}$,
$$\mathbf{E}\left[\frac{1}{Q^{\mathbf{U}}_{V,U}(v,\mathbf{U}_P)}\middle|\mathbf{U}_P\right] \le \frac{\ln|\mathcal{U}|+1}{Q_V(v)}\left(\frac{P(\mathbf{U}_P)}{Q_{U|V}(\mathbf{U}_P|v)}+1\right).$$
$Q_{U,V}$ (prior) $\longrightarrow$ refine by $\mathbf{U}$ (soft decoding) $\longrightarrow$ $Q^{\mathbf{U}}_{V,U}$ (posterior)

### References


[1] Si-Hyeon Lee and Sae-Young Chung. A unified random coding bound. *IEEE Transactions on Information Theory*, 64(10):6779–6802, 2018.

[2] Cheuk Ting Li and Venkat Anantharam. A unified framework for one-shot achievability via the poisson matching lemma. *IEEE Transactions on Information Theory*, 67(5):2624–2651, 2021.

[3] Cheuk Ting Li and Abbas El Gamal. Strong functional representation lemma and applications to coding theorems. *IEEE Transactions on Information Theory*, 64(11):6967–6978, 2018.

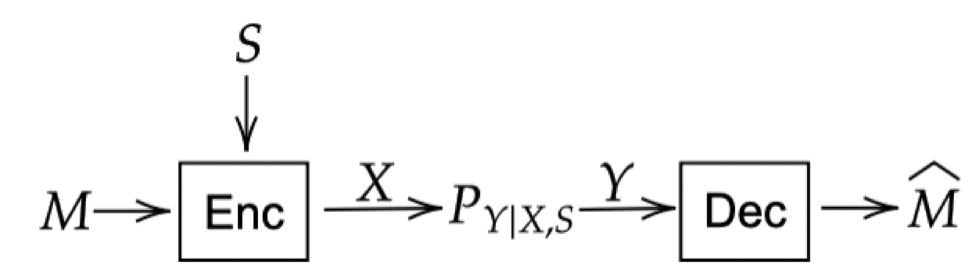
[4] Sung Hoon Lim, Young-Han Kim, Abbas El Gamal, and Sae-Young Chung. Noisy network coding. *IEEE Transactions on Information Theory*, 57(5):3132–3152, 2011.

[5] Yanxiao Liu and Cheuk Ting Li. One-shot coding over general noisy networks. *arXiv preprint arXiv:2402.06021*, 2024.


## Examples

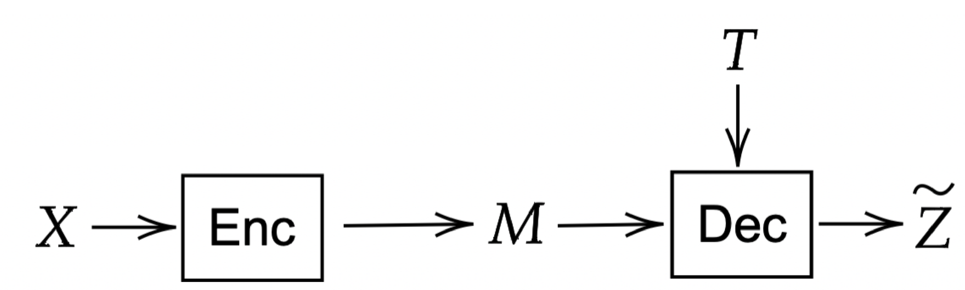
The main theorem can be applied to any combination of source coding, channel coding and coding for computing. Note $\iota(x;y|z) := \log\frac{P(x,y|z)}{(P(x|z)P(y|z))}$.

### Channel Coding with State Info at Encoder




Fix $P_{U|S}$ and $x: \mathcal{U}\times\mathcal{S}\to\mathcal{X}$. For $M\sim\mathrm{Unif}[\mathsf{L}]$, $S\sim P_S$, let $U_1=(U,M)$,
$$P_e := \mathbf{P}(\tilde{X}_2 \ne M) \le \mathbf{E}\Big[\min\Big\{\mathsf{L}2^{-\iota(U;Y)+\iota(U;S)}, 1\Big\}\Big].$$
It recovers asymptotic capacity, attains the best known second-order result.

### Source Coding with Side Info at Decoder

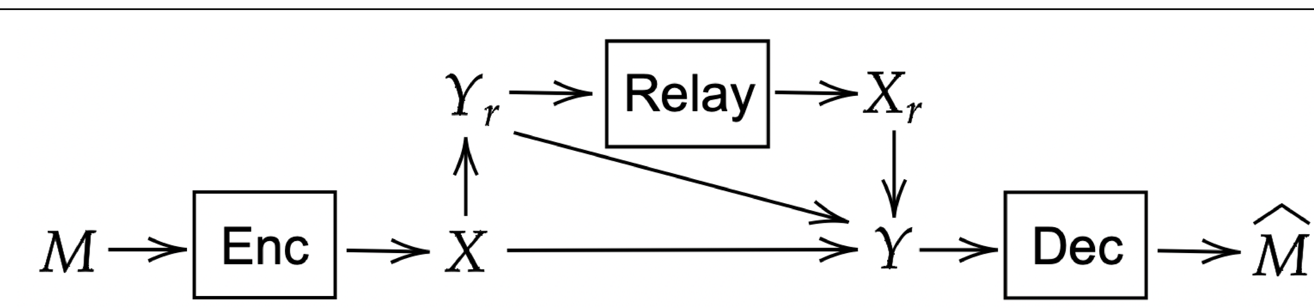



Fix $P_{U|X}$ and $z: \mathcal{U}\times\mathcal{Y}\to\mathcal{Z}$. For $X\sim P_X, T\sim P_{T|X}, M\in[\mathsf{L}]$,
$$P_e := \mathbf{P}\{d(X,\tilde{Z})>\mathsf{D}\} \le \mathbf{E}\Big[\min\Big\{\mathbf{1}\{d(X,Z)>\mathsf{D}\} + \mathsf{L}^{-1}2^{-\iota(U;T)+\iota(U;X)}, 1\Big\}\Big].$$
It recovers asymptotic capacity, and also covers *coding for computing*.

### Multiple Access Channel

For MAC $P_{Y|X_1,X_2}$ and $M_j\sim\mathrm{Unif}[\mathrm{L}_j]$ for $j=1,2$, with $\gamma := \ln(\mathsf{L}_1|\mathcal{X}_1|)+1$,
$$P_e \le \mathbf{E}\Big[\min\Big\{\gamma\mathsf{L}_1\mathsf{L}_2 2^{-\iota(X_1,X_2;Y)} + \gamma\mathsf{L}_2 2^{-\iota(X_2;Y|X_1)} + \mathsf{L}_1 2^{-\iota(X_1;Y|X_2)}, 1\Big\}\Big].$$
It recovers the asymptotic capacity region.

### One-Shot Relay Channel




Let $U_1 := (X,M)$, $U_2 := U$, Main Theorem gives a compress-forward bound:
$$P_e \le \mathbf{E}\Big[\min\Big\{\gamma\mathsf{L}2^{-\iota(X;U,Y)}\Big(2^{-\iota(U;Y)+\iota(U;Y_{\mathrm{r}})}+1\Big), 1\Big\}\Big]$$
where $\gamma = \ln|\mathcal{U}|+1$, $(X, Y_{\mathrm{r}}, U, X_{\mathrm{r}}, Y) \sim P_X P_{Y_{\mathrm{r}}|X} P_{U|Y_{\mathrm{r}}} \delta_{x_{\mathrm{r}}(Y_{\mathrm{r}},U)} P_{Y|X,Y_{\mathrm{r}},X_{\mathrm{r}}}$.

- It is a one-shot version of **relay-with-unlimited-look-ahead**.
- If $Y=(Y',Y'')$ and $P_{Y|X,X_{\mathrm{r}},Y_{\mathrm{r}}} = P_{Y'|X,Y_{\mathrm{r}}} P_{Y''|X_{\mathrm{r}}}$, it is a one-shot version of **primitive relay channel**.
- By message splitting, we can also have a **partial-decode-forward bound**.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