Universal Exact Compression of Differentially Private Mechanisms

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Introduction

Local differential privacy (DP).

Local randomizer $\mathcal{A}: \mathcal{X} \to \mathcal{Z}$ with distribution $P_{Z|X}$ satisfies (ε, δ)-local DP if for any $x, x' \in \mathcal{X}$ and measurable set $\mathcal{S} \subset \mathcal{Z}$,

$$\Pr(Z \in \mathcal{S}|X = x) \le e^{\varepsilon} \cdot \Pr(Z \in \mathcal{S}|X = x') + \delta.$$

Compression of DP mechanisms.

Objective: Compress DP mechanisms exactly (i.e., $Z \sim P_{Z|X}$) to near-optimal sizes, while ensuring privacy guarantees.

Prior works:

· [1-4]: Compress ε-local DP mechanism **approximately**.

· [5,6]: Dithered quantization tools ensure a correct simulated distribution, but only for additive noise mechanisms.

Poisson Functional Representation (PFR) [7]

Let $(T_i)_i$ be a Poisson process (PP) with rate 1, independent of $Z_i \overset{\text{i.i.d.}}{\sim} Q$. Then $(Z_i, T_i)_i$ is a PP with intensity measure $Q \times \lambda_{[0,\infty)}$. Fix any P over $\mathcal{Z}$ that is absolutely continuous w.r.t Q. Let

$$\tilde{T}_i \triangleq T_i \cdot \left(\frac{\mathrm{d}P}{\mathrm{d}Q}(Z_i)\right)^{-1}.$$

Theorem: $K \triangleq \arg\min_i \tilde{T}_i$ and $Z = Z_K$, then $Z \sim P$.

Our contributions: Poisson private representation, which is:

(a) **Exact**: simulates $P_{Z|X}$ exactly;

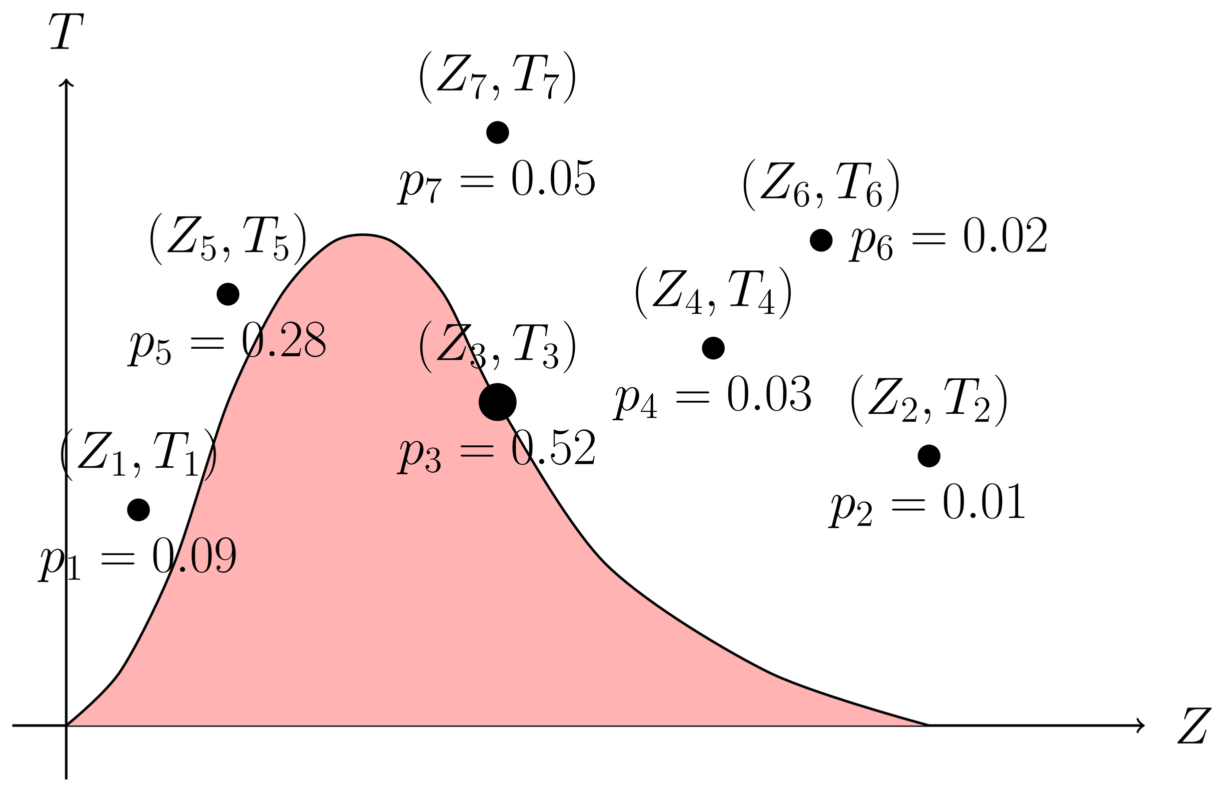
(b) **Universal**: simulates *any* DP mechanism;

(c) **Communication-efficient**: compresses $P_{Z|X}$ to

$$I(X;Z) + \log\left(I(X;Z) + 1\right) + O(1) \text{ bits.}$$

(d) **Private**: ensures both local and central DP.

Poisson Private Representation ($p_i \triangleq \Pr(K = k)$):



Poisson Private Representation (PPR)

Algorithm 1 (PPR).

Input: private $x \in \mathcal{X}$, (ε, δ)-local DP mechanism $P_{Z|X}$, reference distribution $Q(\cdot)$, parameter $\alpha > 1$.

(a) Generate public random variables

$$(Z_i)_{i=1,2,\ldots} \overset{\text{i.i.d.}}{\sim} Q(\cdot).$$

(b) The user knows $(Z_i)_i$, x, $P_{Z|X}$ and performs:

(1) Generate the Poisson process $(T_i)_i$ with rate 1.

(2) Compute $\tilde{T}_i \triangleq T_i \cdot \left(\frac{dP_{Z|X}(\cdot|x)}{dQ}(Z_i)\right)^{-1}$.

(3) Generate $K \in \mathbb{Z}_+$ with

$$\Pr(K = k) = \tilde{T}_k^{-\alpha} \Big/ \left(\sum_{i=1}^{\infty} \tilde{T}_i^{-\alpha}\right).$$

(4) Compress and send K (e.g., by Elias delta code).

(c) The server, which knows $(Z_i)_i, K$, outputs $Z = Z_K$.

Privacy guarantees

Thm 4.5: If the mechanism $P_{Z|X}$ is ε-DP, then PPR $P_{(Z_i)_i,K|X}$ with $\alpha > 1$ is $2\alpha\varepsilon$-DP.

Thm 4.8: If $P_{Z|X}$ is (ε, δ)-DP, then PPR $P_{(Z_i)_i,K|X}$ is $(\alpha\varepsilon + \tilde{\varepsilon}, 2(\delta + \tilde{\delta}))$-DP, for $\alpha > 1$, $\tilde{\varepsilon} \in (0, 1]$ and $\tilde{\delta} \in (0, 1/3]$ s.t.

$$\alpha \le e^{-4.2}\tilde{\delta}\tilde{\varepsilon}^2/(-\ln\tilde{\delta}) + 1.$$

Exactness

The output Z of PPR follows $P_{Z|X}$ exactly.

Communication Efficiency

Thm 4.3: For PPR with $\alpha > 1$, message K satisfies

$$\mathbb{E}[\log_2 K] \le D_{\mathsf{KL}}(P(\cdot|x)\|Q(\cdot)) + \log_2(3.56)/\min((\alpha - 1)/2, 1).$$

K can be encoded by a prefix-free code with expected length $\approx D_{\mathsf{KL}}(P(\cdot|x)\|Q(\cdot))$ bits within a $\log$ gap. If $X \sim P_X$ is random, take $Q = P_Z$ and the expected length $\approx I(X;Z)$ (near-optimal).

Corollary 4.4: For $P_{Z|X}$ with ε-local DP, the compression size

$$\le \ell + \log_2(\ell + 1) + 2 \text{ (bits),}$$

where $\ell \triangleq \varepsilon \log_2 e + \log_2(3.56)/\min((\alpha - 1)/2, 1)$.

Remarks

- The exactness of PPR follows from the PFR [7].
- While the algorithm requires infinite samples, it can be reparameterized and terminates in finite steps.
- When $\alpha = \infty$, PPR reduces to PFR.

Application: Metric Privacy and Laplace Mechanism

For a mechanism $\mathcal{A}$ with $P_{Z|X}$ and a metric $d_{\mathcal{X}}$ over $\mathcal{X}$, it satisfies $\varepsilon \cdot d_{\mathcal{X}}$-privacy if $\forall x, x' \in \mathcal{X}$, $\mathcal{S} \subseteq \mathcal{Z}$, we have

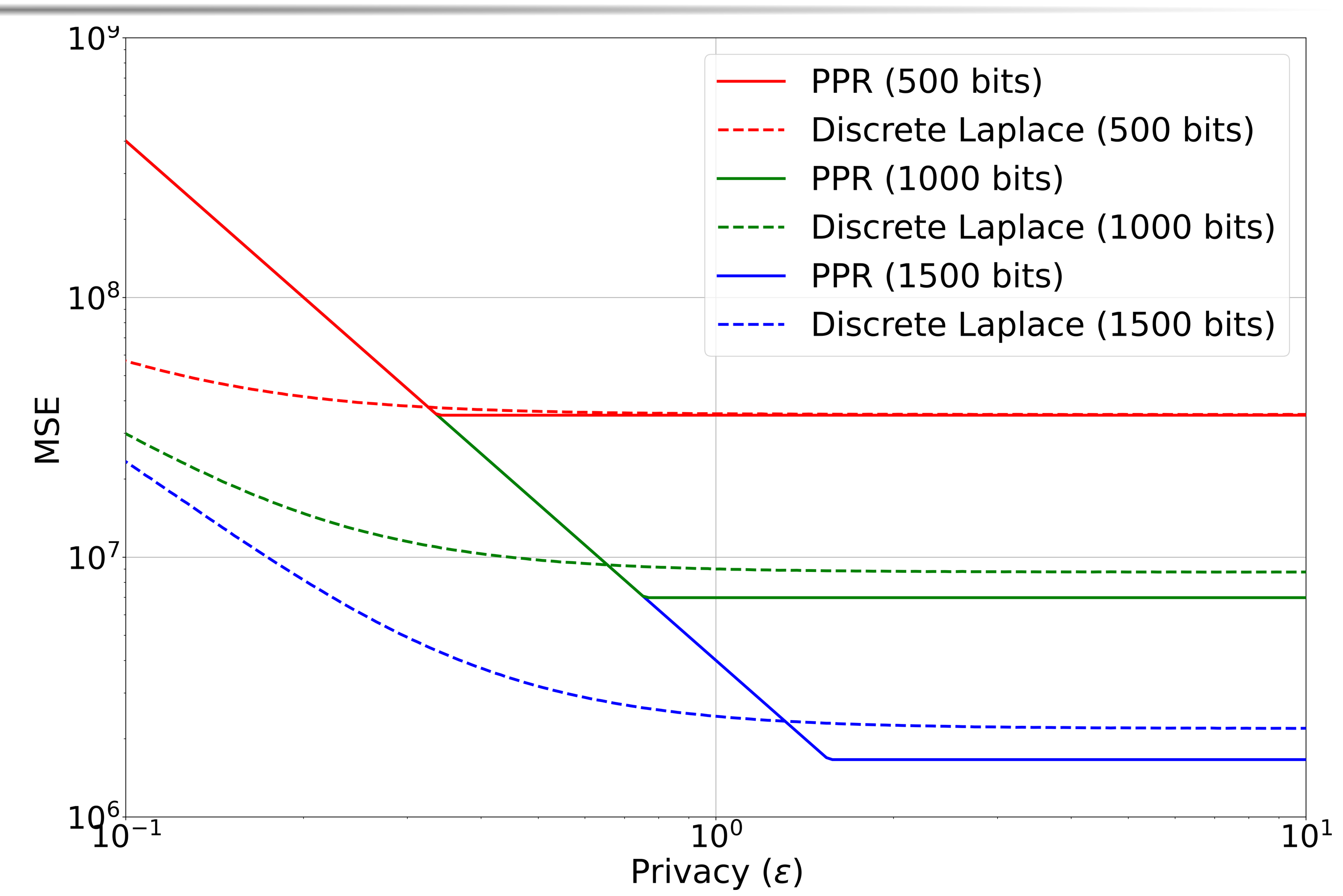
$$\Pr(Z \in \mathcal{S} \,|\, X = x) \le e^{\varepsilon \cdot d_{\mathcal{X}}(x,x')} \Pr(Z \in \mathcal{S} \,|\, X = x').$$

PPR-compressed Laplace mechanism:

For Laplace mechanism $P_{Z|X}$ with $X \in \{x \in \mathbb{R}^d \,|\, \|x\|_2 \le C\}$ and proposal distribution $Q = \mathcal{N}(0, (\frac{C^2}{d} + \frac{d+1}{\varepsilon^2})\mathbb{I}_d)$, the output of PPR has MSE $\frac{d(d+1)}{\varepsilon^2}$, $2\alpha\epsilon \cdot d_{\mathcal{X}}$-privacy and compression size $\le \ell + \log_2(\ell + 1) + 2$ bits, where $\ell \triangleq$

$$\frac{d}{2}\log_2\left(\frac{2}{e}\left(\frac{C^2\varepsilon^2}{d} + d + 1\right)\right) - \log_2\frac{\Gamma(d+1)}{\Gamma(\frac{d}{2}+1)} + \frac{\log_2(3.56)}{\min\{\frac{\alpha-1}{2}, 1\}}.$$

We compare with the discrete Laplace mechanism [8], $d = 500$.



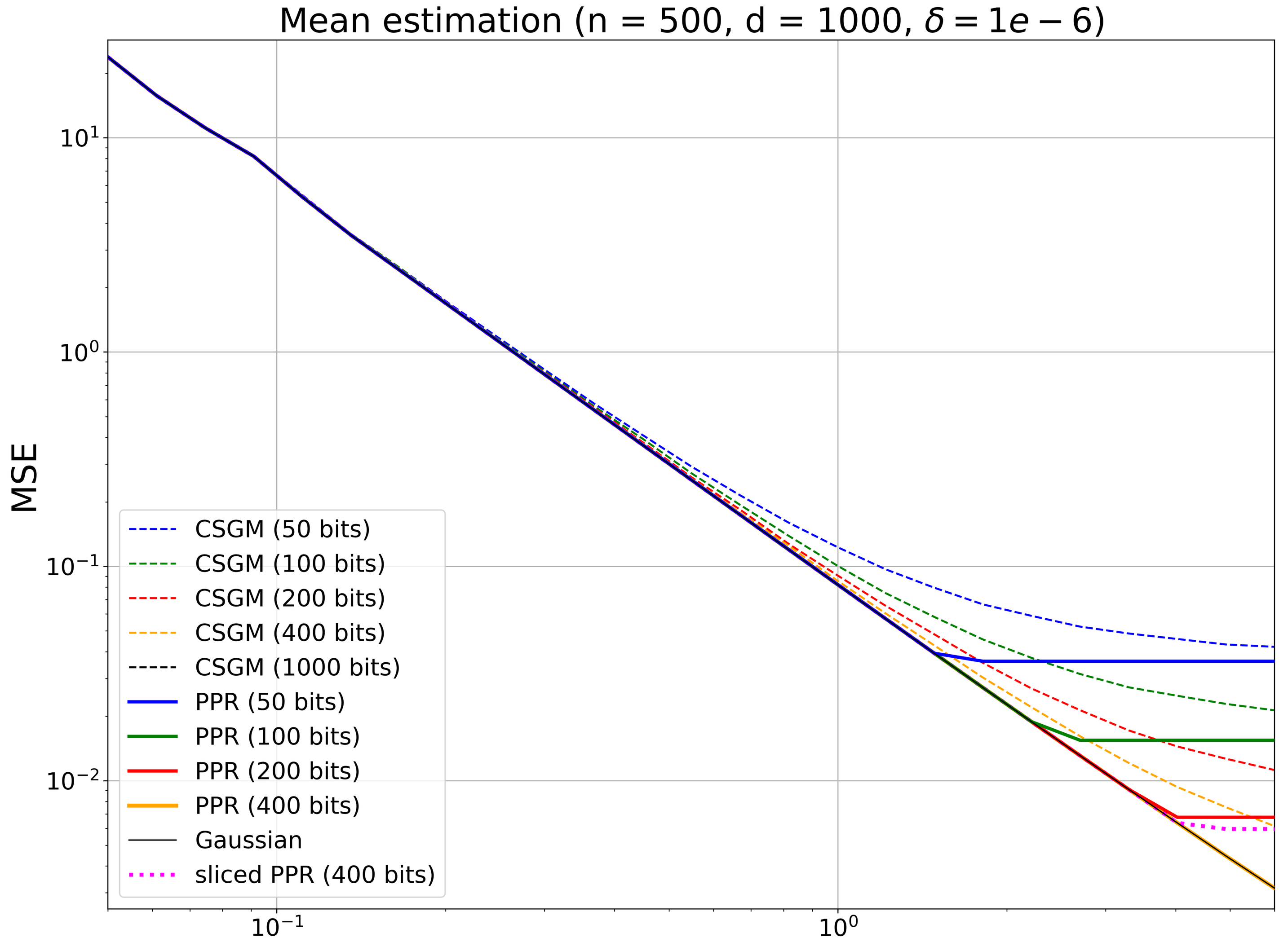
Distributed Mean Estimation

Consider n users, each with data $X_i \in \mathbb{R}^d$. They use **Gaussian mechanism** and send $Z_i \sim \mathcal{N}(X_i, \frac{\sigma^2}{n}\mathbb{I}_d)$ to server, where $\sigma \ge C\sqrt{2\ln(1.25/\delta)}/\varepsilon$. Server estimates mean as $\hat{\mu}(Z^n) = \frac{1}{n}\sum_i Z_i$. Using PPR to compress the Gaussian mechanism:

- $\hat{\mu}(Z^n) = \frac{1}{n}\sum_i Z_i$ is unbiased, has (ε, δ)-central DP.
- PPR satisfies $(2\alpha\sqrt{n}\varepsilon, 2\delta)$-local DP for $\epsilon < 1/\sqrt{n}$.
- The average per-client communication cost is at most

$$\frac{d}{2}\log\left(\frac{n\varepsilon^2}{2d\log(1.25/\delta)} + 1\right) + \frac{\log_2(3.56)}{\min\{(\alpha-1)/2, 1\}} \text{ bits.}$$

Compare to CSGM [9] on distributed mean estimation:



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