

# One-Shot Coding over General Noisy Networks

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# Background

## Some key questions of network information theory

- ① **Blocklength** in information transmission: asymptoticity & finite blocklength & one-shot achievability?
- ② **Noisy network coding**: capacity of noisy networks & coding schemes?
- ③ **Unified Coding Scheme**: channel coding & source coding?

## Our contributions

- ① **Part I**: A unified **one-shot** coding framework for communication and compression of messages among multiple nodes across a **general acyclic noisy network**.
  - Proof technique: exponential process refinement lemma
- ② **Part II**:
  - Novel one-shot achievability results discovered.
  - Existing one-shot & asymptotic results recovered.



# Background: One-shot Information Theory

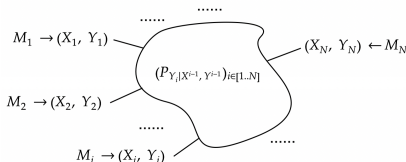
## One-shot information theory

- ① **Conventional Shannon theory:** **asymptotically optimal** information transmission rates based on the law of large numbers (e.g., **typicality**).
- ② **Finite-blocklength regime<sup>a</sup>:** maximal information transmission rate at a given blocklength and error probability?
- ③ **One-shot achievability<sup>b</sup>:** What if the blocklength can be as short as 1 (each source and channel is only used once)?
  - ① Sources and channels can be **arbitrary**: no need to be memoryless or ergodic.
  - ② Can recover existing (first-order and second-order) asymptotic results when applied to memoryless sources and channels.

<sup>a</sup>Polyanskiy, Yury, H. Vincent Poor, and Sergio Verdú. "Channel coding rate in the finite blocklength regime." IEEE Transactions on Information Theory 56, no. 5 (2010): 2307-2359.

<sup>b</sup>Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma." IEEE Transactions on Information Theory 67, no. 5 (2021): 2624-2651.

# Background: Noisy Network Coding



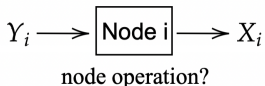
## Noisy Network Coding

- **Noisy network coding**<sup>a</sup>: communicating messages between **multiple** sources and destinations over a general **noisy** network.
- Generalizing:
  - 1 Noiseless network coding by Ahlswede, Cai, Li and Yeung.
  - 2 Compress-forward coding for relay channels by Cover and El Gamal.
  - 3 Coding for relay networks, coding for erasure networks, etc.

<sup>a</sup>Lim, Sung Hoon, Young-Han Kim, Abbas El Gamal, and Sae-Young Chung. "Noisy network coding." IEEE Transactions on Information Theory 57, no. 5 (2011): 3132-3152.



# Background: A Unified Random Coding Bound



## Unified random coding bound

- ① Unified random coding bound via blockwise node operation<sup>a</sup>: work for **any** combination of channel coding and source coding
- ② **Advantages:**
  - ① Unifying and generalizing known relaying strategies, can yield good bounds without error analysis.
  - ② Useful for designing automated theorem proving tools<sup>b</sup>.

<sup>a</sup>Lee, Si-Hyeon, and Sae-Young Chung. "A unified random coding bound." IEEE Transactions on Information Theory 64, no. 10 (2018): 6779-6802.

<sup>b</sup>Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results." IEEE Transactions on Information Theory (2023).



# Overview: Our Contributions

## Our contributions

- ① A unified one-shot coding scheme
- ② over general noisy acyclic discrete networks (ADN)
- ③ that is applicable to any combination of source coding, channel coding and coding for computing problems.

## Special cases

- Novel one-shot achievability results for:
  - ① One-shot relay channels
  - ② One-shot primitive relay channels
    - Compress-and-forward bound
    - Partial-decode-and-forward bound
- Recovered one-shot & asymptotic results on:
  - ① Source and channel coding
  - ② Gelfand-Pinsker, Wyner-Ziv and coding for computing
  - ③ Multiple access channels
  - ④ Broadcast channels



# Techniques

## Poisson functional representation

- For a finite set  $\mathcal{U}$ , let  $\mathbf{U} := (Z_u)_{u \in \mathcal{U}}$  be i.i.d.  $\text{Exp}(1)$  random variables.
- Given a distribution  $P$  over  $\mathcal{U}$ , **Poisson functional representation**<sup>a</sup>:

$$\mathbf{U}_P := \operatorname{argmin}_u \frac{Z_u}{P(u)}$$

- We have  $\mathbf{U}_P \sim P$ .
- It can be viewed as a variant of the  $A^*$  sampling, and have a wide range of applications: minimax learning, neural network compression and differential privacy.

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<sup>a</sup>Li, Cheuk Ting, and Abbas El Gamal. "Strong functional representation lemma and applications to coding theorems." IEEE Transactions on Information Theory 64, no. 11 (2018): 6967-6978.

# Techniques



## Poisson functional representation

- Given a distribution  $P$  over  $\mathcal{U}$ , **Poisson functional representation**:

$$\mathbf{U}_P := \operatorname{argmin}_u \frac{Z_u}{P(u)}$$

## Generalized Poisson matching lemma

- Let  $\mathbf{U}_P(1), \dots, \mathbf{U}_P(|\mathcal{U}|) \in \mathcal{U}$  be the elements of  $\mathcal{U}$  sorted in ascending order of  $Z_u/P(u)$ , let  $\mathbf{U}_P^{-1} : \mathcal{U} \rightarrow [|\mathcal{U}|]$  for the inverse function of  $i \mapsto \mathbf{U}_P(i)$ .
- Generalized Poisson matching lemma**<sup>a</sup>: For distributions  $P, Q$  over  $\mathcal{U}$ , we have the following almost surely:

$$\mathbf{E} \left[ \mathbf{U}_Q^{-1}(\mathbf{U}_P) \mid \mathbf{U}_P \right] \leq \frac{P(\mathbf{U}_P)}{Q(\mathbf{U}_P)} + 1.$$

<sup>a</sup>Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma." IEEE Transactions on Information Theory 67, no. 5 (2021): 2624-2651.



# Techniques



## Refining a distribution by an exponential process

- For a joint distribution  $Q_{V,U}$  over  $\mathcal{V} \times \mathcal{U}$ , the refinement of  $Q_{V,U}$  by  $\mathbf{U}$ , denoted as  $Q_{V,U}^{\mathbf{U}}$ , is a joint distribution

$$Q_{V,U}^{\mathbf{U}}(v, u) := \frac{Q_V(v)}{\left( \mathbf{U}_{Q_{U|V}^{-1}(\cdot|v)}^{-1}(u) \sum_{i=1}^{|\mathcal{U}|} i^{-1} \right)}$$

for all  $(v, u)$  in the support of  $Q_{V,U}$ , where  $Q_V$  is the  $V$ -marginal of  $Q_{V,U}$  and  $Q_{U|V}$  is the conditional distribution of  $U$  given  $V$ .

- The refinement  $Q_U^{\mathbf{U}}(u)$  is for the **soft decoding** of  $U$ , which gives a distribution over  $U$ , with  $\mathbf{U}_{Q_U}$  having the largest probability.
- Useful in non-unique decoding.
- If the distribution  $Q_{V,U}$  represents our “**prior distribution**” of  $(V, U)$ , then the refinement  $Q_{V,U}^{\mathbf{U}}$  is our updated “**posterior distribution**” after taking the exponential process  $\mathbf{U}$  into account.

# Techniques



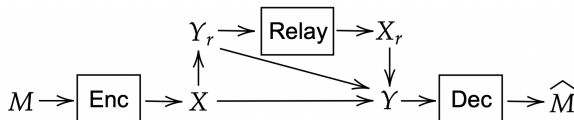
## Exponential Process Refinement Lemma

- To keep track of the evolution of the “posterior probability” of the correct values of a large number of random variables through the refinement process:
- For a distribution  $P$  over  $\mathcal{U}$  and a joint distribution  $Q_{V,U}$  over a finite  $\mathcal{V} \times \mathcal{U}$ , for every  $v \in \mathcal{V}$ , we have, almost surely,

$$\mathbf{E} \left[ \frac{1}{Q_{V,U}^{\mathbf{U}}(v, \mathbf{U}_P)} \middle| \mathbf{U}_P \right] \leq \frac{\ln |\mathcal{U}| + 1}{Q_V(v)} \left( \frac{P(\mathbf{U}_P)}{Q_{U|V}(\mathbf{U}_P|v)} + 1 \right).$$

- We use the following example to demonstrate how to use it.

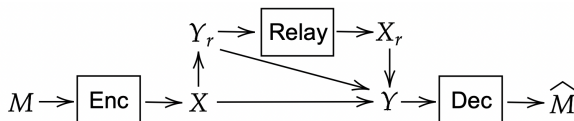
# One-Shot Relay Channel



## One-Shot Relay Channel

- ① Encoder observes  $M \sim \text{Unif}[L]$  and outputs  $X$ , which is passed through the channel  $P_{Y_r|X}$ .
- ② Relay observes  $Y_r$  and outputs  $X_r$ .
- ③  $(X, X_r, Y_r)$  is passed through the channel  $P_{Y|X, X_r, Y_r}$ .
  - $Y$  depends on all of  $X, X_r, Y_r$  and  $X_r$  may interfere with  $(X, Y_r)$ :
  - Possible when the relay outputs  $X_r$  instantaneously or the channel has a long memory, or it is a storage device.
- ④ Decoder observes  $Y$  and recovers  $\hat{M}$ .

# One-Shot Relay Channel

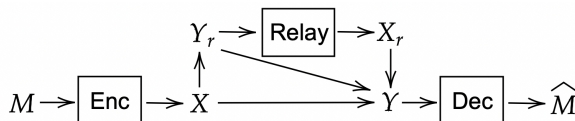


## One-Shot Relay Channel

- Inherent limitation of one-shot setting: unable to model cyclic networks, e.g., two-way communication channels, general relay channels that depend on its **past**.
- Above: One-shot version of **relay-with-unlimited-look-ahead**<sup>a</sup>
- “Best one-shot approximation” of the conventional relay channel.

<sup>a</sup>El Gamal, Abbas, Navid Hassanpour, and James Mammen. "Relay networks with delays." IEEE Transactions on Information Theory 53, no. 10 (2007): 3413-3431.

# One-Shot Relay Channel



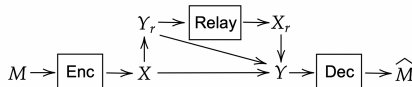
## Corollary

For any  $P_X$ ,  $P_{U|Y_r}$ , function  $x_r(y_r, u)$ , there is a deterministic coding scheme for the one-shot relay channel such that the error probability satisfies

$$P_e \leq \mathbf{E} \left[ \min \left\{ \gamma L 2^{-\iota(X;U,Y)} \left( 2^{-\iota(U;Y) + \iota(U;Y_r)} + 1 \right), 1 \right\} \right],$$

where  $(X, Y_r, U, X_r, Y) \sim P_X P_{Y_r|X} P_{U|Y_r} \delta_{x_r(Y_r, U)} P_{Y|X, Y_r, X_r}$ , and  $\gamma := \ln |\mathcal{U}| + 1$ .

# One-Shot Relay Channel



## Proof

- ① Auxiliaries:  $U_1 := (X, M)$ ,  $U_2 := U$ .
- ② “Random codebooks”  $\mathbf{U}_1$ ,  $\mathbf{U}_2$ : independent exponential processes.
- ③ Encoder:  $U_1 = (\mathbf{U}_1)_{P_{U_1} \times \delta_M}$  (PFR), then outputs  $X$ -component of  $U_1$ .
- ④ Relay:  $U_2 = (\mathbf{U}_2)_{P_{U_2|Y_r}(\cdot|Y_r)}$ , then outputs  $X_r = x_r(Y_r, U_2)$ .
- ⑤ Decoder observes  $Y$ , and:
  - Refine  $P_{U_2|Y}(\cdot|Y)$  to  $Q_{U_2} := P_{U_2|Y}^{U_2}$ . By Exponential Process Refinement Lemma:

$$\mathbf{E} \left[ \frac{1}{Q_{U_2}(U_2)} \middle| U_2, Y, Y_r \right] \leq (\ln |\mathcal{U}_2| + 1) \left( \frac{P_{U_2|Y_r}(U_2)}{P_{U_2|Y}(U_2)} + 1 \right).$$

- Compute  $Q_{U_2} P_{U_1|U_2, Y}$  over  $\mathcal{U}_1 \times \mathcal{U}_2$ , and let its  $U_1$ -marginal be  $\tilde{Q}_{U_1}$ .
- Let  $\tilde{U}_1 = (\mathbf{U}_1)_{\tilde{Q}_{U_1} \times P_M}$ , and output its  $M$ -component.



# One-Shot Relay Channel

## Proof

$$\begin{aligned}
 & \mathbf{P}(\tilde{U}_1 \neq U_1 \mid X, Y_r, U_2, X_r, Y, M) \\
 & \stackrel{(a)}{\leq} \mathbf{E} \left[ \min \left\{ \frac{P_{U_1}(U_1) \delta_M(M)}{P_{U_1|U_2,Y}(U_1|U_2, Y) Q_{U_2}(U_2) P_M(M)}, 1 \right\} \mid X, Y_r, U_2, X_r, Y, M \right] \\
 & \stackrel{(b)}{=} \mathbf{E} \left[ \min \left\{ L \frac{P_{U_1}(U_1)}{P_{U_1|U_2,Y}(U_1|U_2, Y) Q_{U_2}(U_2)}, 1 \right\} \mid X, Y_r, U_2, X_r, Y, M \right] \\
 & \stackrel{(c)}{\leq} \min \left\{ L \frac{P_{U_1}(U_1)}{P_{U_1|U_2,Y}(U_1|U_2, Y)} (\ln |U_2| + 1) \left( \frac{P_{U_2|Y_r}(U_2)}{P_{U_2|Y}(U_2)} + 1 \right), 1 \right\} \\
 & = \min \left\{ (\ln |U_2| + 1) L 2^{-\iota(X; U_2, Y)} (2^{-\iota(U_2; Y) + \iota(U_2; Y_r)} + 1), 1 \right\}.
 \end{aligned}$$

(a) is by the Poisson matching lemma; (b) is by  $\delta_M(M) = 1$ ,  $P_M(M) = 1/L$ ; (c) is by the refinement step (previous page) and Jensen's inequality.

For some  $P_{U|Y_r}$  and function  $x_r(y_r, u_2)$ , it yields the asymptotic achievable rate:

$$R \leq I(X; U, Y) - \max \{ I(U; Y_r) - I(U; Y), 0 \}.$$



# One-Shot Primitive Relay Channel

When  $Y = (Y', Y'')$  consists of two components and the channel  $P_{Y|X, X_r, Y_r} = P_{Y'|X, Y_r} P_{Y''|X_r}$  can be decomposed into two orthogonal components: one-shot version of the **primitive relay channel**.

## Corollary

*For any  $P_X, P_{X_r}, P_{U'|Y_r}$ , there is a deterministic coding scheme for the one-shot primitive relay channel with  $M \sim \text{Unif}[L]$  such that*

$$P_e \leq \mathbf{E} \left[ \min \left\{ (\ln(|\mathcal{U}'||\mathcal{X}_r|) + 1) L 2^{-\iota(X; U', Y')} (2^{-\iota(X_r; Y'')} + \iota(U'; Y_r | Y')) + 1, 1 \right\} \right],$$

$(X, Y_r, U', Y'') \sim P_X P_{Y_r|X} P_{U'|Y_r} P_{Y''|X_r}$  independent of  $(X_r, Y'') \sim P_{X_r} P_{Y''|X_r}$ .

- $C_r = \max_{P_{X_r}} I(X_r; Y'')$  is the capacity of channel  $P_{Y''|X_r}$ , then the asymptotic achievable rate is  $R \leq I(X; U', Y') - \max\{I(U'; Y_r | Y') - C_r, 0\}$ .
- It implies the **compress-forward bound**<sup>a</sup>.

<sup>a</sup>Kim, Young-Han. "Coding techniques for primitive relay channels." In Proc. Forty-Fifth Annual Allerton Conf. Commun., Contr. Comput., p. 2007. 2007.





# One-Shot Relay Channel

## Partial-Decode-and-Forward Bound

- Split the message:  $M \sim \text{Unif}[\mathcal{L}] \Rightarrow M_1 \sim \text{Unif}[\mathcal{J}], M_2 \sim \text{Unif}[\mathcal{L}/\mathcal{J}]$ .
- Node 1 sees  $Y_1 = M_1$ , outputs  $X_1 = V$ , and has an auxiliary  $U_1 = (M_1, V)$
- Node 2 sees  $Y_2 = (M_1, M_2, V)$ , outputs  $X_2 = X$ , and  $U_2 = (M_1, M_2, X)$ .
- Relay sees  $Y_3 = Y_r$ , decodes  $U_1$ , outputs  $X_3 = X_r$ , and  $U_3 = (M_1, U)$ .
- Decoder sees  $Y_4 = Y$  and uses the decoding order “ $U_2, U_3?, U_1?$ ”.

## Corollary

Fix any  $P_{X,V}$ ,  $P_{U|Y_r,V}$ , function  $x_r(y_r, u, v)$ , and  $\mathcal{J}$  which is a factor of  $\mathcal{L}$ . There exists a deterministic coding scheme for the one-shot relay channel with

$$P_e \leq \mathbf{E} \left[ \min \left\{ \mathcal{J} 2^{-\iota(V; Y_r)} + (\ln(\mathcal{J}|\mathcal{U}|) + 1)(\ln(\mathcal{J}|\mathcal{V}|) + 1) \mathcal{L} \mathcal{J}^{-1} 2^{-\iota(X; U, Y|V)} \right. \right. \\ \left. \left. \cdot (2^{-\iota(U; V, Y) + \iota(U; V, Y_r)} + 1) (\mathcal{J} 2^{-\iota(V; Y)} + 1), 1 \right\} \right],$$

where  $(X, V, Y_r, U, X_r, Y) \sim P_{X,V} P_{Y_r|X,V} P_{U|Y_r,V} \delta_{x_r(Y_r, U, V)} P_{Y|X, Y_r, X_r}$ .

# General Acyclic Discrete Network



## Acyclic discrete network (ADN)

- $N$  nodes labelled by  $1, \dots, N$ .
- Node  $i$  observes  $Y_i \in \mathcal{Y}_i$  and produces  $X_i \in \mathcal{X}_i$ .
- $Y_i$  can depend on all previous inputs and outputs  $X^{i-1}, Y^{i-1}$ .
- **ADN**: a collection of channels  $(P_{Y_i|X^{i-1}, Y^{i-1}})_{i \in [N]}$ , where  $P_{Y_i|X^{i-1}, Y^{i-1}}$  is a conditional distribution from  $\prod_{j=1}^{i-1} \mathcal{X}_j \times \prod_{j=1}^{i-1} \mathcal{Y}_j$  to  $\mathcal{Y}_i$ .

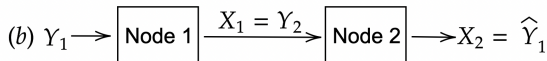
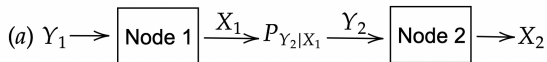


Figura 1: (a) Channel coding. (b) Source coding.

# Coding scheme



## Public-randomness coding scheme

A pair  $(P_W, (f_i)_{i \in [N]})$ , where  $P_W$  is the distribution of the public randomness  $W \in \mathcal{W}$  available to all nodes and  $f_i : \mathcal{Y}_i \times \mathcal{W} \rightarrow \mathcal{X}_i$  is the encoding function of node  $i$  mapping its observation  $Y_i$  and the public randomness  $W$  to its output  $X_i$ . The operations are as follows. First, generate  $W \sim P_W$ . For  $i = 1, \dots, N$ , generate  $\tilde{Y}_i$  conditional on  $\tilde{X}^{i-1}, \tilde{Y}^{i-1}$  according to  $P_{Y_i|X^{i-1}, Y^{i-1}}$ , and take  $\tilde{X}_i = f_i(\tilde{Y}_i, W)$ .

## Deterministic coding scheme

A sequence of encoding functions  $(f_i)_{i \in [N]}$ , where  $f_i : \mathcal{Y}_i \rightarrow \mathcal{X}_i$ . For  $i = 1, \dots, N$ , the following operations are performed:

- **Noisy channel.** The output  $\tilde{Y}_i$  is generated conditional on  $\tilde{X}^{i-1}, \tilde{Y}^{i-1}$  according to  $P_{Y_i|X^{i-1}, Y^{i-1}}$ . For  $i = 1$ ,  $\tilde{Y}_1 \sim P_{Y_1}$  can be regarded as a source or a channel state.
- **Node operation.** Node  $i$  observes  $\tilde{Y}_i$  and outputs  $\tilde{X}_i = f_i(\tilde{Y}_i)$ .

# Coding scheme



## Achievability

- $\tilde{X}_i, \tilde{Y}_i$  denote the **actual** random variables from the coding scheme.
- $X_i, Y_i$  denote the random variables following an **ideal** distribution.
- The goal (the “**achievability**”) is to make the actual joint distribution  $P_{\tilde{X}^N, \tilde{Y}^N}$  “approximately as good as” the ideal joint distribution  $P_{X^N, Y^N}$ . For an “error set”  $\mathcal{E} \subseteq (\prod_{i=1}^N \mathcal{X}_i) \times (\prod_{i=1}^N \mathcal{Y}_i)$  that we do not want  $(\tilde{X}^N, \tilde{Y}^N)$  to fall into, we want

$$\mathbf{P}((\tilde{X}^N, \tilde{Y}^N) \in \mathcal{E}) \lesssim \mathbf{P}((X^N, Y^N) \in \mathcal{E}),$$

which can be guaranteed by  $P_{\tilde{X}^N, \tilde{Y}^N}$  being close to  $P_{X^N, Y^N}$  in TV distance.



# Main Theorem

## Theorem

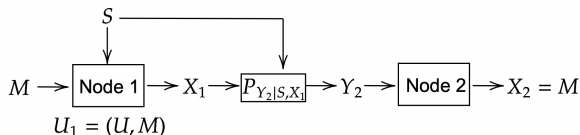
Fix any ADN  $(P_{Y_i|X^{i-1}, Y^{i-1}})_{i \in [M]}$ . For any collection of indices  $(a_{i,j})_{i \in [M], j \in [d_i]}$  where  $(a_{i,j})_{j \in [d_i]}$  is a sequence of distinct indices in  $[i-1]$  for each  $i$ , any sequence  $(d'_i)_{i \in [M]}$  with  $0 \leq d'_i \leq d_i$  and any collection of conditional distributions  $(P_{U_i|Y_i, \bar{U}'_i}, P_{X_i|Y_i, U_i, \bar{U}'_i})_{i \in [M]}$  (where  $\bar{U}_{i,S} := (U_{a_{i,j}})_{j \in S}$  for  $S \subseteq [d_i]$  and  $\bar{U}'_i := \bar{U}_{i,[d'_i]}$ ), which induces the joint distribution of  $X^N, Y^N, U^N$  (the “ideal distribution”), there exists a public-randomness coding scheme  $(P_W, (f_i)_{i \in [M]})$  such that the joint distribution of  $\tilde{X}^N, \tilde{Y}^N$  induced by the scheme satisfies

$$\delta_{\text{TV}}(P_{X^N, Y^N}, P_{\tilde{X}^N, \tilde{Y}^N}) \leq \mathbf{E} \left[ \min \left\{ \sum_{i=1}^N \sum_{j=1}^{d'_i} B_{i,j}, 1 \right\} \right],$$

where  $\gamma_{i,j} := \prod_{k=j+1}^{d_i} (\ln |\mathcal{U}_{a_{i,k}}| + 1)$  and

$$B_{i,j} := \gamma_{i,j} \prod_{k=j}^{d_i} (2^{-\iota(\bar{U}_{i,k}; \bar{U}_{i,[d_i] \setminus [j..k]}, Y_i) + \iota(\bar{U}_{i,k}; \bar{U}'_{a_{i,k}}, Y_{a_{i,k}})} + \mathbf{1}\{k > j\}).$$

# ADN: Gelfand-Pinsker Problem



## Gelfand-Pinsker Problem

- $Y_1 := (M, S)$ ,  $Y_2 := Y$ ,  $P_{Y_2|Y_1,X_1}$  be  $P_{Y|S,X}$ , and  $X_2 := M$ .
- The auxiliary of node 1 is  $U_1 = (U, M)$  for some  $U$  following  $P_{U|S}$  given  $S$ .
- The decoding order of node 2 is “ $U_1$ ” (i.e., it only wants  $U_1$ ).

## Corollary

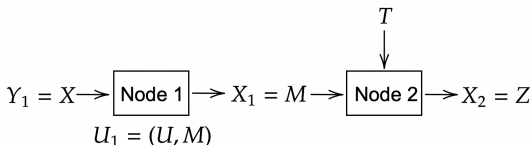
Fix  $P_{U|S}$  and function  $x : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{X}$ . There exists a deterministic coding scheme for the channel  $P_{Y|X,S}$  with  $S \sim P_S$ ,  $M \sim \text{Unif}[\mathcal{L}]$  such that

$$P_e \leq \mathbf{E} \left[ \min \left\{ \mathcal{L} 2^{-\iota(U;Y) + \iota(U;S)}, 1 \right\} \right],$$

where  $S, U, X, Y \sim P_S P_{U|S} \delta_{x(U,S)} P_{Y|X,S}$ .



# ADN: Wyner-Ziv Problem and Coding for Computing



## Corollary

Fix  $P_{U|X}$  and function  $z : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}$ . There exists a coding scheme s.t.

$$P_e \leq \mathbf{E} \left[ \min \left\{ \mathbf{1} \{d(X, Z) > D\} + L^{-1} 2^{-\iota(U; T) + \iota(U; X)}, 1 \right\} \right],$$

where  $X, Y, U, Z \sim P_X P_{Y|X} P_{U|X} \delta_Z(u, y)$ .

- Reduced to lossy source coding by  $T = \emptyset$ : let  $U = Z$ , then
 
$$P_e \leq \mathbf{P}(d(X, Z) > D) + \mathbf{E} \left[ \min \left\{ L^{-1} 2^{\iota(Z; X)}, 1 \right\} \right].$$
- Coding for computing: node 2 recovers a function  $f(X, T)$ ,
 
$$P_e \leq \mathbf{E} [\min \{ \mathbf{1} \{d(f(X, T), Z) > D\} + L^{-1} 2^{-\iota(U; T) + \iota(U; X)}, 1 \}].$$



# ADN: Multiple Access Channel

## Multiple Access Channel

- For  $j = 1, 2$ , two independent messages  $M_j \sim \text{Unif}[L_j]$  are encoded to  $X_j$ . The decoder observes the output of  $P_{Y|X_1, X_2}$  and produces  $(\hat{M}_1, \hat{M}_2)$ .
- ADN:  $Y_1 := M_1$ ,  $Y_2 := M_2$ ,  $Y_3 := Y$  and  $X_3 := (M_1, M_2)$ .
- Auxiliaries:  $U_1 := (X_1, M_1)$  and  $U_2 := (X_2, M_2)$ .
- Decoding order of node 3: “ $U_2, U_1$ ” (i.e., decode  $U_1$  (soft), and then  $U_2$  (unique), and then  $U_1$  (unique)).

## Corollary

Fix  $P_{X_1}, P_{X_2}$ . There exists a deterministic coding scheme for the multiple access channel  $P_{Y|X_1, X_2}$  with

$$P_e \leq \mathbf{E} \left[ \min \left\{ \gamma L_1 L_2 2^{-\iota(X_1, X_2; Y)} + \gamma L_2 2^{-\iota(X_2; Y|X_1)} + L_1 2^{-\iota(X_1; Y|X_2)}, 1 \right\} \right],$$

where  $\gamma := \ln(L_1 | \mathcal{X}_1 |) + 1$ ,  $(X_1, X_2, Y) \sim P_{X_1} P_{X_2} P_{Y|X_1, X_2}$ .

Asymptotic region:  $R_1 < I(X_1; Y|X_2)$ ,  $R_2 < I(X_2; Y|X_1)$ ,  $R_1 + R_2 < I(X_1, X_2; Y)$ .





# Summary

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- We provide a **unified one-shot coding framework** for communication and compression over general noisy networks.
- We design a proof technique “**exponential process refinement lemma**” that can keep track of a large number of auxiliary random variables.
- We provide **novel one-shot results** for various multi-hop settings.
- We recover most of the best-known one-shot results of different settings.

## Future Directions

- Continuous case of the current framework.
- One-shot versions of other asymptotic bounds for relay channels.
- A unified coding scheme is useful to design automated theorem proving tools, e.g., PSITIP<sup>a</sup>. Extensions to one-shot results is left for future study.

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<sup>a</sup>Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results." IEEE Transactions on Information Theory (2023).