One-Shot Coding over General Noisy Networks

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Background

Some key questions of network information theory

1. **Blocklength** in information transmission: asymptoticity & finite blocklength & one-shot achievability?
2. **Noisy network coding**: capacity of noisy networks & coding schemes?
3. **Unified Coding Scheme**: channel coding & source coding?

Our contributions

1. **Part I**: A unified **one-shot** coding framework for communication and compression of messages among multiple nodes across a **general acyclic noisy network**.
   - Proof technique: exponential process refinement lemma
2. **Part II**:
   - Novel one-shot achievability results discovered.
   - Existing one-shot & asymptotic results recovered.

Background: One-shot Information Theory

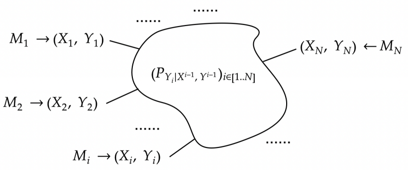
One-shot information theory

1. **Conventional Shannon theory**: **asymptotically optimal** information transmission rates based on the law of large numbers (e.g., **typicality**).
2. **Finite-blocklength regime**[a]: maximal information transmission rate at a given blocklength and error probability?
3. **One-shot** achievability[b]: What if the blocklength can be as short as 1 (each source and channel is only used once)?
   1. Sources and channels can be **arbitrary**: no need to be memoryless or ergodic.
   2. Can recover existing (first-order and second-order) asymptotic results when applied to memoryless sources and channels.

[a] Polyanskiy, Yury, H. Vincent Poor, and Sergio Verdú. "Channel coding rate in the finite blocklength regime."IEEE Transactions on Information Theory 56, no. 5 (2010): 2307-2359.

[b] Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma."IEEE Transactions on Information Theory 67, no. 5 (2021): 2624-2651.

Background: Noisy Network Coding



Noisy Network Coding

- **Noisy network coding**[a]: communicating messages between **multiple** sources and destinations over a general **noisy** network.
- Generalizing:
  1. Noiseless network coding by Ahlswede, Cai, Li and Yeung.
  2. Compress-forward coding for relay channels by Cover and El Gamal.
  3. Coding for relay networks, coding for erasure networks, etc.

[a] Lim, Sung Hoon, Young-Han Kim, Abbas El Gamal, and Sae-Young Chung. "Noisy network coding."IEEE Transactions on Information Theory 57, no. 5 (2011): 3132-3152.

Background: A Unified Random Coding Bound

$$Y_i \longrightarrow \boxed{\text{Node i}} \longrightarrow X_i$$

node operation?

Unified random coding bound

1. Unified random coding bound via blockwose node operation[a]: work for **any** combination of channel coding and source coding
2. **Advantages**:
   1. Unifying and generalizing known relaying strategies, can yield good bounds without error analysis.
   2. Useful for designing automated theorem proving tools[b].

[a] Lee, Si-Hyeon, and Sae-Young Chung. "A unified random coding bound."IEEE Transactions on Information Theory 64, no. 10 (2018): 6779-6802.

[b] Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results."IEEE Transactions on Information Theory (2023).

Overview: Our Contributions

Our contributions

1. A unified one-shot coding scheme
2. over general noisy acyclic discrete networks (ADN)
3. that is applicable to any combination of source coding, channel coding and coding for computing problems.

Special cases

- Novel one-shot achievablity results for:
  1. One-shot relay channels
  2. One-shot primitive relay channels
     - Compress-and-forward bound
     - Partial-decode-and-forward bound
- Recovered one-shot & asymptotic results for:
  1. Source and channel coding
  2. Gelfand-Pinsker, Wyner-Ziv and coding for computing
  3. Multiple access channels
  4. Broadcast channels

Techniques

Poisson functional representation

- For a finite set $\mathcal{U}$, let $\mathbf{U} := (Z_u)_{u\in\mathcal{U}}$ be i.i.d. $\mathrm{Exp}(1)$ random variables.
- Given a distribution P over $\mathcal{U}$, **Poisson functional representation**[a]:

$$\mathbf{U}_P := \mathrm{argmin}_u \frac{Z_u}{P(u)} \tag{1}$$

- We have $\mathbf{U}_P \sim P$.

[a] Li, Cheuk Ting, and Abbas El Gamal. "Strong functional representation lemma and applications to coding theorems."IEEE Transactions on Information Theory 64, no. 11 (2018): 6967-6978.

Techniques

Poisson functional representation

- Given a distribution P over $\mathcal{U}$, **Poisson functional representation**:

$$\mathbf{U}_P := \operatorname{argmin}_u \frac{Z_u}{P(u)} \tag{2}$$

Generalized Poisson matching lemma

- Let $\mathbf{U}_P(1), \ldots, \mathbf{U}_P(|\mathcal{U}|) \in \mathcal{U}$ be the elements of $\mathcal{U}$ sorted in ascending order of $Z_u/P(u)$, let $\mathbf{U}_P^{-1} : \mathcal{U} \to [|\mathcal{U}|]$ for the inverse function of $i \mapsto \mathbf{U}_P(i)$.
- **Generalized Poisson matching lemma**[a]: For distributions P, Q over $\mathcal{U}$, we have the following almost surely:

$$\mathbf{E}\left[\mathbf{U}_Q^{-1}(\mathbf{U}_P) \,\middle|\, \mathbf{U}_P\right] \leq \frac{P(\mathbf{U}_P)}{Q(\mathbf{U}_P)} + 1.$$

[a] Li, Cheuk Ting, and Venkat Anantharam. "A unified framework for one-shot achievability via the Poisson matching lemma."IEEE Transactions on Information Theory 67, no. 5 (2021): 2624-2651.

Techniques

Refining a distribution by an exponential process

- For a joint distribution $Q_{V,U}$ over $\mathcal{V} \times \mathcal{U}$, the refinement of $Q_{V,U}$ by $\mathbf{U}$, denoted as $Q^{\mathbf{U}}_{V,U}$, is a joint distribution

$$Q^{\mathbf{U}}_{V,U}(v,u) := \frac{Q_V(v)}{\left(\mathbf{U}^{-1}_{Q_{U|V}(\cdot|v)}(u) \sum_{i=1}^{|\mathcal{U}|} i^{-1}\right)}$$

  for all (v, u) in the support of $Q_{V,U}$, where Q_V is the V-marginal of $Q_{V,U}$ and $Q_{U|V}$ is the conditional distribution of U given V.
- The refinement $Q^{\mathbf{U}}_U(u)$ is for the **soft decoding** of U, which gives a distribution over U, with $\mathbf{U}_{Q_U}$ having the largest probability.
- Useful in non-unique decoding.
- If the distribution $Q_{V,U}$ represents our "**prior distribution**" of (V, U), then the refinement $Q^{\mathbf{U}}_{V,U}$ is our updated "**posterior distribution**" after taking the exponential process $\mathbf{U}$ into account.

Techniques

Exponential Process Refinement Lemma

- To keep track of the evolution of the "posterior probability" of the correct values of a large number of random variables through the refinement process:
- For a distribution P over $\mathcal{U}$ and a joint distribution $Q_{V,U}$ over a finite $\mathcal{V} \times \mathcal{U}$, for every $v \in \mathcal{V}$, we have, almost surely,

$$\mathbf{E}\left[\frac{1}{Q^{\mathbf{U}}_{V,U}(v, \mathbf{U}_P)} \middle| \mathbf{U}_P \right] \le \frac{\ln |\mathcal{U}| + 1}{Q_V(v)} \left(\frac{P(\mathbf{U}_P)}{Q_{U|V}(\mathbf{U}_P|v)} + 1 \right).$$

Network Model

Acyclic discrete network (ADN)

- N nodes labelled by $1, \ldots, N$.
- Node i observes $Y_i \in \mathcal{Y}_i$ and produces $X_i \in \mathcal{X}_i$.
- Y_i can depend on all previous inputs and outputs X^{i-1}, Y^{i-1}.
- **ADN**: a collection of channels $(P_{Y_i|X^{i-1},Y^{i-1}})_{i\in[N]}$, where $P_{Y_i|X^{i-1},Y^{i-1}}$ is a conditional distribution from $\prod_{j=1}^{i-1} \mathcal{X}_j \times \prod_{j=1}^{i-1} \mathcal{Y}_j$ to $\mathcal{Y}_i$.

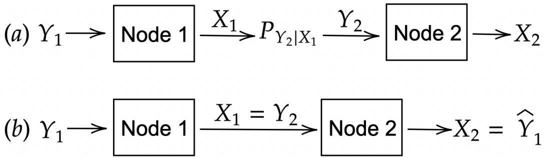


Figura 1: (a) Channel coding. (b) Source coding.

Limitation: unable to model cyclic networks, e.g., two-way communication channels, general relay channels that depend on its **past**.

Coding scheme

Deterministic coding scheme

A sequence of encoding functions $(f_i)_{i\in[N]}$, where $f_i : \mathcal{Y}_i \to \mathcal{X}_i$. For $i = 1, \ldots, N$, the following operations are performed:

- **Noisy channel.** The output $\tilde{Y}_i$ is generated conditional on $\tilde{X}^{i-1}, \tilde{Y}^{i-1}$ according to $P_{Y_i|X^{i-1},Y^{i-1}}$. For $i = 1$, $\tilde{Y}_1 \sim P_{Y_1}$ can be regarded as a source or a channel state.
- **Node operation.** Node i observes $\tilde{Y}_i$ and outputs $\tilde{X}_i = f_i(\tilde{Y}_i)$.

Public-randomness coding scheme

A pair $(P_W, (f_i)_{i\in[N]})$, where P_W is the distribution of the public randomness $W \in \mathcal{W}$ available to all nodes and $f_i : \mathcal{Y}_i \times \mathcal{W} \to \mathcal{X}_i$ is the encoding function of node i mapping its observation Y_i and the public randomness W to its output X_i. The operations are as follows. First, generate $W \sim P_W$. For $i = 1, \ldots, N$, generate $\tilde{Y}_i$ conditional on $\tilde{X}^{i-1}, \tilde{Y}^{i-1}$ according to $P_{Y_i|X^{i-1},Y^{i-1}}$, and take $\tilde{X}_i = f_i(\tilde{Y}_i, W)$.

Coding scheme

Achievability

- $\tilde{X}_i, \tilde{Y}_i$ denote the actual random variables from the coding scheme.
- X_i, Y_i denote the random variables following an ideal distribution.
- The goal (the "achievability") is to make the actual joint distribution $P_{\tilde{X}^N,\tilde{Y}^N}$ "approximately as good as" the ideal joint distribution P_{X^N,Y^N}. For an "error set" $\mathcal{E} \subseteq \left(\prod_{i=1}^N \mathcal{X}_i\right) \times \left(\prod_{i=1}^N \mathcal{Y}_i\right)$ that we do not want $(\tilde{X}^N, \tilde{Y}^N)$ to fall into, we want

$$\mathbf{P}\left((\tilde{X}^N, \tilde{Y}^N) \in \mathcal{E}\right) \lesssim \mathbf{P}\left((X^N, Y^N) \in \mathcal{E}\right), \tag{3}$$

which can be guaranteed by $P_{\tilde{X}^N,\tilde{Y}^N}$ being close to P_{X^N,Y^N} in TV distance.

Main Theorem

Theorem

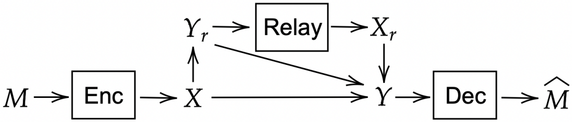
Fix any ADN $(P_{Y_i|X^{i-1},Y^{i-1}})_{i\in[N]}$. *For any collection of indices* $(a_{i,j})_{i\in[N],j\in[d_i]}$ *where* $(a_{i,j})_{j\in[d_i]}$ *is a sequence of distinct indices in* $[i-1]$ *for each* i*, any sequence* $(d'_i)_{i\in[N]}$ *with* $0 \le d'_i \le d_i$ *and any collection of conditional distributions* $(P_{U_i|Y_i,\overline{U}'_i}, P_{X_i|Y_i,U_i,\overline{U}'_i})_{i\in[N]}$ *(where* $\overline{U}_{i,\mathcal{S}} := (U_{a_{i,j}})_{j\in\mathcal{S}}$ *for* $\mathcal{S} \subseteq [d_i]$ *and* $\overline{U}'_i := \overline{U}_{i,[d'_i]}$*), which induces the joint distribution of* X^N, Y^N, U^N *(the "ideal distribution"), there exists a public-randomness coding scheme* $(P_W, (f_i)_{i\in[N]})$ *such that the joint distribution of* $\tilde{X}^N, \tilde{Y}^N$ *induced by the scheme satisfies*

$$\delta_{\mathrm{TV}}\left(P_{X^N,Y^N},\, P_{\tilde{X}^N,\tilde{Y}^N}\right) \le \mathbf{E}\left[\min\left\{\sum_{i=1}^{N}\sum_{j=1}^{d'_i} B_{i,j},\, 1\right\}\right],$$

where $\gamma_{i,j} := \prod_{k=j+1}^{d_i}\left(\ln|\mathcal{U}_{a_{i,k}}| + 1\right)$ *and*

$$B_{i,j} := \gamma_{i,j}\prod_{k=j}^{d_i}\left(2^{-\iota(\overline{U}_{i,k};\overline{U}_{i,[d_i]\setminus[j..k]},Y_i)+\iota(\overline{U}_{i,k};\overline{U}'_{a_{i,k}},Y_{a_{i,k}})} + \mathbf{1}\{k>j\}\right).$$

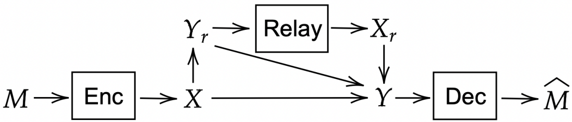
One-Shot Relay Channel



One-Shot Relay Channel

1. Encoder observes $M \sim \mathrm{Unif}[L]$ and outputs X, which is passed through the channel $P_{Y_\mathrm{r}|X}$.
2. Relay observes Y_r and outputs X_r.
3. $(X, X_\mathrm{r}, Y_\mathrm{r})$ is passed through the channel $P_{Y|X,X_\mathrm{r},Y_\mathrm{r}}$.
   - Y depends on all of $X, X_\mathrm{r}, Y_\mathrm{r}$ and X_r may interfere with (X, Y_r): possible when the relay outputs X_r instantaneously or the channel has a long memory, or it is a storage device.
4. Decoder observes Y and recovers $\hat{M}$.

One-Shot Relay Channel

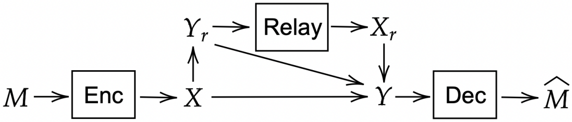


One-Shot Relay Channel

- One-shot version of **relay-with-unlimited-look-ahead** [a]
- When $Y = (Y', Y'')$ consists of two components and the channel $P_{Y|X,X_r,Y_r} = P_{Y'|X,Y_r} P_{Y''|X_r}$ can be decomposed into two orthogonal components: one-shot version of the **primitive relay channel**
- “Best one-shot approximation” of the conventional relay channel.

[a] El Gamal, Abbas, Navid Hassanpour, and James Mammen. "Relay networks with delays."IEEE Transactions on Information Theory 53, no. 10 (2007): 3413-3431.

One-Shot Relay Channel



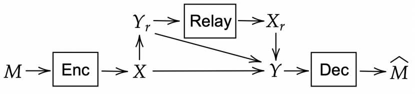
Corollary

For any P_X, $P_{U|Y_\mathrm{r}}$, function $x_\mathrm{r}(y_\mathrm{r}, u)$, there is a deterministic coding scheme for the one-shot relay channel such that the error probability satisfies

$$P_e \le \mathbf{E}\left[\min\left\{\gamma L 2^{-\iota(X;U,Y)}\left(2^{-\iota(U;Y)+\iota(U;Y_\mathrm{r})}+1\right), 1\right\}\right], \tag{4}$$

where $(X, Y_\mathrm{r}, U, X_\mathrm{r}, Y) \sim P_X P_{Y_\mathrm{r}|X} P_{U|Y_\mathrm{r}} \delta_{x_\mathrm{r}(Y_\mathrm{r},U)} P_{Y|X,Y_\mathrm{r},X_\mathrm{r}}$, *and* $\gamma := \ln|\mathcal{U}| + 1$.

One-Shot Relay Channel



Proof

1. Auxiliaries: $U_1 := (X, M)$, $U_2 := U$.
2. "Random codebooks" $\mathbf{U}_1$, $\mathbf{U}_2$: independent exponential processes.
3. Encoder: $U_1 = (\mathbf{U}_1)_{P_{U_1} \times \delta_M}$ (PFR), then outputs X-component of U_1.
4. Relay: $U_2 = (\mathbf{U}_2)_{P_{U_2|Y_\mathrm{r}}(\cdot|Y_\mathrm{r})}$, then outputs $X_\mathrm{r} = x_\mathrm{r}(Y_\mathrm{r}, U_2)$.
5. Decoder observes Y, and:
   - Refine $P_{U_2|Y}(\cdot|Y)$ to $Q_{U_2} := P^{\mathbf{U}_2}_{U_2|Y}$. By Exponential Process Refinement Lemma:
   $$\mathbf{E}\left[\frac{1}{Q_{U_2}(U_2)} \,\middle|\, U_2, Y, Y_\mathrm{r}\right] \le (\ln|\mathcal{U}_2| + 1)\left(\frac{P_{U_2|Y_\mathrm{r}}(U_2)}{P_{U_2|Y}(U_2)} + 1\right).$$
   - Compute $Q_{U_2} P_{U_1|U_2,Y}$ over $\mathcal{U}_1 \times \mathcal{U}_2$, and let its U_1-marginal be $\tilde{Q}_{U_1}$.
   - Let $\tilde{U}_1 = (\mathbf{U}_1)_{\tilde{Q}_{U_1} \times P_M}$, and output its M-component.

One-Shot Relay Channel

Proof

$$
\begin{aligned}
&\mathbf{P}(\tilde{U}_1 \neq U_1 \,|\, X, Y_{\mathrm{r}}, U_2, X_{\mathrm{r}}, Y, M) \\
&\overset{(a)}{\leq} \mathbf{E}\left[\min\left\{\frac{P_{U_1}(U_1)\delta_M(M)}{P_{U_1|U_2,Y}(U_1|U_2,Y)Q_{U_2}(U_2)P_M(M)}, 1\right\} \,\middle|\, X, Y_{\mathrm{r}}, U_2, X_{\mathrm{r}}, Y, M\right] \\
&\overset{(b)}{=} \mathbf{E}\left[\min\left\{\mathsf{L}\frac{P_{U_1}(U_1)}{P_{U_1|U_2,Y}(U_1|U_2,Y)Q_{U_2}(U_2)}, 1\right\} \,\middle|\, X, Y_{\mathrm{r}}, U_2, X_{\mathrm{r}}, Y, M\right] \\
&\overset{(c)}{\leq} \min\left\{\mathsf{L}\frac{P_{U_1}(U_1)}{P_{U_1|U_2,Y}(U_1|U_2,Y)}(\ln|\mathcal{U}_2|+1)\left(\frac{P_{U_2|Y_{\mathrm{r}}}(U_2)}{P_{U_2|Y}(U_2)}+1\right), 1\right\} \\
&= \min\left\{(\ln|\mathcal{U}_2|+1)\mathsf{L}2^{-\iota(X;U_2,Y)}\left(2^{-\iota(U_2;Y)+\iota(U_2;Y_{\mathrm{r}})}+1\right), 1\right\}.
\end{aligned}
$$

(a) is by the Poisson matching lemma; (b) is by $\delta_M(M) = 1$, $P_M(M) = 1/\mathsf{L}$; (c) is by the refinement step (previous page) and Jensen's inequality.

For some $P_{U|Y_{\mathrm{r}}}$ and function $x_{\mathrm{r}}(y_{\mathrm{r}}, u_2)$, it yields the asymptotic achievable rate:

$$R \leq I(X; U, Y) - \max\left\{I(U; Y_{\mathrm{r}}) - I(U; Y), 0\right\}.$$

One-Shot Primitive Relay Channel

For the one-shot primitive relay channel ($P_{Y|X,X_{\mathrm{r}},Y_{\mathrm{r}}} = P_{Y'|X,Y_{\mathrm{r}}} P_{Y''|X_{\mathrm{r}}}$), consider (X, Y_{r}, Y') independent of (X_{r}, Y'') in the ideal distribution and take $U = (U', X_{\mathrm{r}})$ where U' follows $P_{U'|Y_{\mathrm{r}}}$.

Corollary

For any P_X, $P_{X_{\mathrm{r}}}$, $P_{U'|Y_{\mathrm{r}}}$, there is a deterministic coding scheme for the one-shot primitive relay channel with $M \sim \mathrm{Unif}[\mathsf{L}]$ such that

$$P_e \leq \mathbf{E}\left[\min\left\{ \left(\ln(|\mathcal{U}'||\mathcal{X}_{\mathrm{r}}|) + 1\right) \mathsf{L}2^{-\iota(X;U',Y')}\left(2^{-\iota(X_{\mathrm{r}};Y'')+\iota(U';Y_{\mathrm{r}}|Y')} + 1\right), 1\right\}\right],$$

$(X, Y_{\mathrm{r}}, U', Y') \sim P_X P_{Y_{\mathrm{r}}|X} P_{U'|Y_{\mathrm{r}}} P_{Y'|X,Y_{\mathrm{r}}}$ *independent of* $(X_{\mathrm{r}}, Y'') \sim P_{X_{\mathrm{r}}} P_{Y''|X_{\mathrm{r}}}$.

- $C_{\mathrm{r}} = \max_{P_{X_{\mathrm{r}}}} I(X_{\mathrm{r}}; Y'')$ is the capacity of channel $P_{Y''|X_{\mathrm{r}}}$, then the asymptotic achievable rate is $R \leq I(X; U', Y') - \max\{I(U'; Y_{\mathrm{r}}|Y') - C_{\mathrm{r}}, 0\}$.
- It implies the compress-forward bound[a].

[a] Kim, Young-Han. "Coding techniques for primitive relay channels."In Proc. Forty-Fifth Annual Allerton Conf. Commun., Contr. Comput, p. 2007. 2007.

One-Shot Relay Channel

Partial-Decode-and-Forward Bound

- Split the message: $M \sim \mathrm{Unif}[\mathsf{L}] \Rightarrow M_1 \sim \mathrm{Unif}[\mathsf{J}], M_2 \sim \mathrm{Unif}[\mathsf{L}/\mathsf{J}]$.
- Node 1 sees $Y_1 = M_1$, outputs $X_1 = V$, and has an auxiliary $U_1 = (M_1, V)$
- Node 2 sees $Y_2 = (M_1, M_2, V)$, outputs $X_2 = X$, and $U_2 = (M_1, M_2, X)$.
- Relay sees $Y_3 = Y_r$, decodes U_1, outputs $X_3 = X_r$, and $U_3 = (M_1, U)$.
- Decoder sees $Y_4 = Y$ and uses the decoding order "$U_2, U_3?, U_1?$".

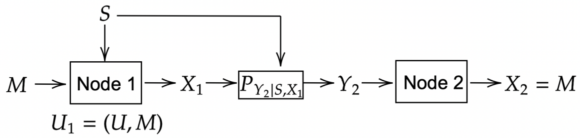
Corollary

Fix any $P_{X,V}$, $P_{U|Y_\mathrm{r},V}$, function $x_\mathrm{r}(y_\mathrm{r}, u, v)$, and J which is a factor of L. There exists a deterministic coding scheme for the one-shot relay channel with

$$P_e \le \mathbf{E}\Big[\min\Big\{\mathsf{J}2^{-\iota(V;Y_\mathrm{r})} + (\ln(\mathsf{J}|\mathcal{U}|)+1)(\ln(\mathsf{J}|\mathcal{V}|)+1)\mathsf{L}\mathsf{J}^{-1}2^{-\iota(X;U,Y|V)} \cdot \big(2^{-\iota(U;V,Y)+\iota(U;V,Y_\mathrm{r})}+1\big)\big(\mathsf{J}2^{-\iota(V;Y)}+1\big), 1\Big\}\Big],$$

where $(X, V, Y_\mathrm{r}, U, X_\mathrm{r}, Y) \sim P_{X,V}P_{Y_\mathrm{r}|X,V}P_{U|Y_\mathrm{r},V}\delta_{x_\mathrm{r}(Y_\mathrm{r},U,V)}P_{Y|X,Y_\mathrm{r},X_\mathrm{r}}$.

ADN: Gelfand-Pinsker Problem



Gelfand-Pinsker Problem

- $Y_1 := (M, S)$, $Y_2 := Y$, $P_{Y_2|Y_1,X_1}$ be $P_{Y|S,X}$, and $X_2 := M$.
- The auxiliary of node 1 is $U_1 = (U, M)$ for some U following $P_{U|S}$ given S.
- The decoding order of node 2 is "U_1" (i.e., it only wants U_1).

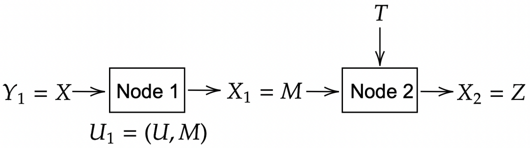
Corollary

Fix $P_{U|S}$ and function $x : \mathcal{U} \times \mathcal{S} \to \mathcal{X}$. There exists a deterministic coding scheme for the channel $P_{Y|X,S}$ with $S \sim P_S$, $M \sim \mathrm{Unif}[\mathsf{L}]$ such that

$$P_e \leq \mathbf{E}\Big[\min\big\{\mathsf{L}2^{-\iota(U;Y)+\iota(U;S)}, 1\big\}\Big],$$

where $S, U, X, Y \sim P_S P_{U|S} \delta_{x(U,S)} P_{Y|X,S}$.

ADN: Wyner-Ziv Problem and Coding for Computing



Corollary

Fix $P_{U|X}$ and function $z : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$. There exists a coding scheme s.t.

$$P_e \le \mathbf{E}\left[\min\left\{\mathbf{1}\{d(X,Z) > \mathsf{D}\} + \mathsf{L}^{-1} 2^{-\iota(U;T)+\iota(U;X)}, 1\right\}\right], \tag{5}$$

where $X, Y, U, Z \sim P_X P_{Y|X} P_{U|X} \delta_{z(U,Y)}$.

- Reduced to lossy source coding by $T = \emptyset$: let $U = Z$, then $P_e \le \mathbf{P}(d(X,Z) > \mathsf{D}) + \mathbf{E}\left[\min\left\{\mathsf{L}^{-1} 2^{\iota(Z;X)}, 1\right\}\right]$.
- Coding for computing: node 2 recovers a function $f(X,T)$, $P_e \le \mathbf{E}[\min\{\mathbf{1}\{d(f(X,T),Z) > \mathsf{D}\} + \mathsf{L}^{-1} 2^{-\iota(U;T)+\iota(U;X)}, 1\}]$.

ADN: Multiple Access Channel

Multiple Access Channel

- For $j = 1, 2$, two independent messages $M_j \sim \mathrm{Unif}[\mathsf{L}_j]$ are encoded to X_j. The decoder observes the output of $P_{Y|X_1,X_2}$ and produces $(\hat{M}_1, \hat{M}_2)$.
- ADN: $Y_1 := M_1$, $Y_2 := M_2$, $Y_3 := Y$ and $X_3 := (M_1, M_2)$.
- Auxiliaries: $U_1 := (X_1, M_1)$ and $U_2 := (X_2, M_2)$.
- Decoding order of node 3: "U_2, U_1" (i.e., decode U_1 (soft), and then U_2 (unique), and then U_1 (unique)).

Corollary

Fix P_{X_1}, P_{X_2}. There exists a deterministic coding scheme for the multiple access channel $P_{Y|X_1,X_2}$ with

$$P_e \le \mathbf{E}\left[\min\left\{\gamma \mathsf{L}_1\mathsf{L}_2 2^{-\iota(X_1,X_2;Y)} + \gamma \mathsf{L}_2 2^{-\iota(X_2;Y|X_1)} + \mathsf{L}_1 2^{-\iota(X_1;Y|X_2)}, 1\right\}\right],$$

where $\gamma := \ln(\mathsf{L}_1|\mathcal{X}_1|) + 1$, $(X_1, X_2, Y) \sim P_{X_1}P_{X_2}P_{Y|X_1,X_2}$.

Asymptotic region: $R_1 < I(X_1; Y|X_2)$, $R_2 < I(X_2; Y|X_1)$, $R_1 + R_2 < I(X_1, X_2; Y)$.

Summary

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- We provide a **unified one-shot coding framework** for communication and compression over general noisy networks.
- We design a proof technique "**exponential process refinement lemma**" that can keep track of a large number of auxiliary random variables.
- We provide **novel one-shot results** for various multi-hop settings.
- We recover most of the best-known one-shot results of different settings.

Future Directions

- Continuous case of the current framework.
- One-shot versions of other asymptotic bounds for relay channels.
- A unified coding scheme is useful to design automated theorem proving tools, e.g., PSITIP[a]. Extensions to one-shot results is left for future study.

[a] Li, Cheuk Ting. "An automated theorem proving framework for information-theoretic results."IEEE Transactions on Information Theory (2023).