

One-Shot Coding over General Noisy Networks

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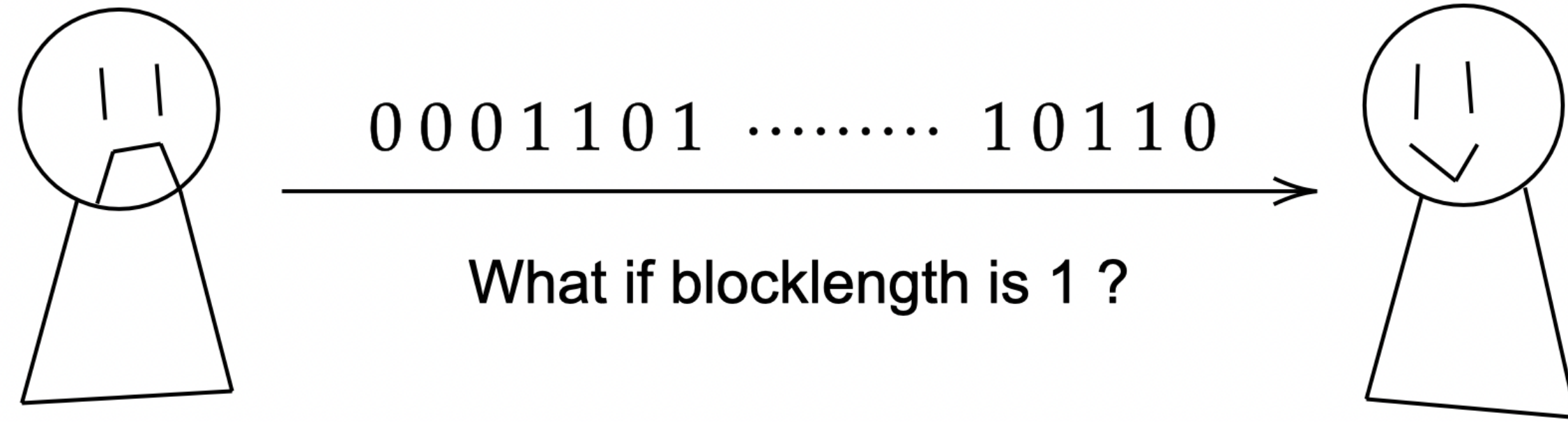
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Key Questions in Information Theory

Some key questions at the heart of information theory.

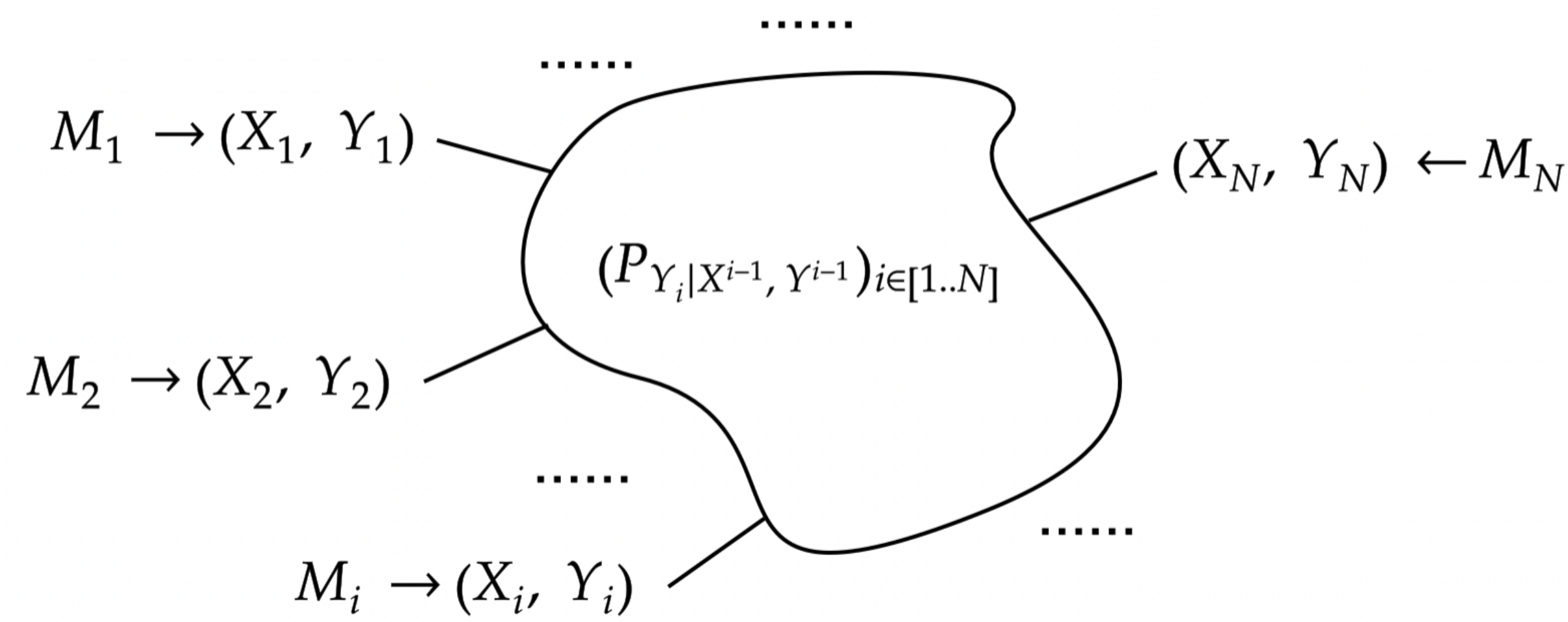
Key Question 1: Blocklength in Information Theory



One-shot information theory [2, 3]: network is only used **once**!

- Error probability cannot be driven to zero!
- No law of large number \rightarrow no typicality!
- No time-sharing!
- No memoryless/ergodic assumption!
- **Objective:** One-shot achievabilities that can imply existing (first/second order) asymptotic/finite-blocklength bounds?

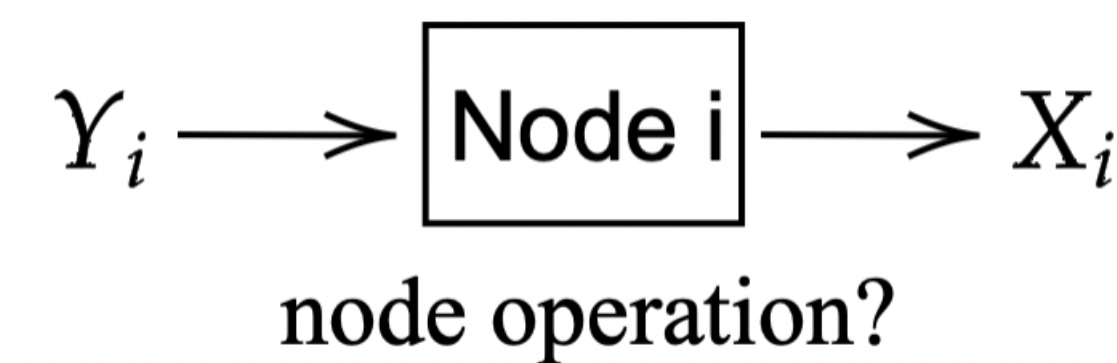
Key Question 2: Noisy network coding



Noisy network coding [4]:

- What is the capacity of a **noisy network**?
- What coding scheme can achieve the capacity?

Key Question 3: Unified Coding Scheme



A unified coding schemes [1]:

- A unified **node operation** in networks?
- Unify channel coding, source coding, and coding for computing?
- Automated machine-proving tools?

A Unified One-Shot Coding Scheme

To answer the key questions on the left, our scheme [5] combines:

1. One-shot/finite-blocklength network information theory
2. Noisy network coding
3. Unified scheme (source coding/channel coding/coding for computing)

Main Theorem

For any acyclic discrete network $(P_{Y_i|X^{i-1}, Y^{i-1}})_{i \in [N]}$, we provide a one-shot achievability result: For any collection of indices $(a_{i,j})_{i \in [N], j \in [d_i]}$ where $(a_{i,j})_{j \in [d_i]}$ is a sequence of distinct indices in $[i-1]$ for each i , any sequence $(d'_i)_{i \in [N]}$ with $0 \leq d'_i \leq d_i$ and any collection of conditional distributions $(P_{U_i|Y_i, \bar{U}'_i}, P_{X_i|Y_i, U_i, \bar{U}'_i})_{i \in [N]}$ (where $\bar{U}_{i,S} := (U_{a_{i,j}})_{j \in S}$ for $S \subseteq [d_i]$ and $\bar{U}'_i := \bar{U}_{i,[d'_i]}$), which induces the joint distribution of X^N, Y^N, U^N (the “ideal distribution”), there exists a public-randomness coding scheme $(P_W, (f_i)_{i \in [N]})$ such that the joint distribution of \tilde{X}^N, \tilde{Y}^N induced by the scheme (the “actual distribution”) satisfies

$$\delta_{TV}(P_{X^N, Y^N}, P_{\tilde{X}^N, \tilde{Y}^N}) \leq \mathbf{E} \left[\min \left\{ \sum_{i=1}^N \sum_{j=1}^{d'_i} B_{i,j}, 1 \right\} \right],$$

where $\gamma_{i,j} := \prod_{k=j+1}^{d_i} (\ln |\mathcal{U}_{a_{i,k}}| + 1)$, and

$$B_{i,j} := \gamma_{i,j} \prod_{k=j}^{d_i} \left(2^{-\iota(\bar{U}_{i,k}; \bar{U}_{i,[d_i] \setminus [j,k]}, Y_i)} + \iota(\bar{U}_{i,k}; \bar{U}'_{a_{i,k}}, Y_{a_{i,k}}) + \mathbf{1}\{k > j\} \right).$$

Techniques

1. **Poisson functional representation [3]:** Let $\mathbf{U} := (Z_u)_{u \in \mathcal{U}}$ be i.i.d. $\text{Exp}(1)$ random variables. Given a distribution P over finite \mathcal{U} ,

$$\mathbf{U}_P := \arg \min_u \frac{Z_u}{P(u)}.$$

2. **Each node is associated with an exponential process.**
3. **Exponential Process Refinement:** For $Q_{V,U}$ over a finite $\mathcal{V} \times \mathcal{U}$, $\forall v \in \mathcal{V}$,

$$\mathbf{E} \left[\frac{1}{Q_{V,U}(v, \mathbf{U}_P)} \middle| \mathbf{U}_P \right] \leq \frac{\ln |\mathcal{U}| + 1}{Q_V(v)} \left(\frac{P(\mathbf{U}_P)}{Q_{U|V}(\mathbf{U}_P|v)} + 1 \right).$$

$Q_{U,V}$ (prior) \rightarrow refine by \mathbf{U} (soft decoding) $\rightarrow Q_{V,U}^{\mathbf{U}}$ (posterior)

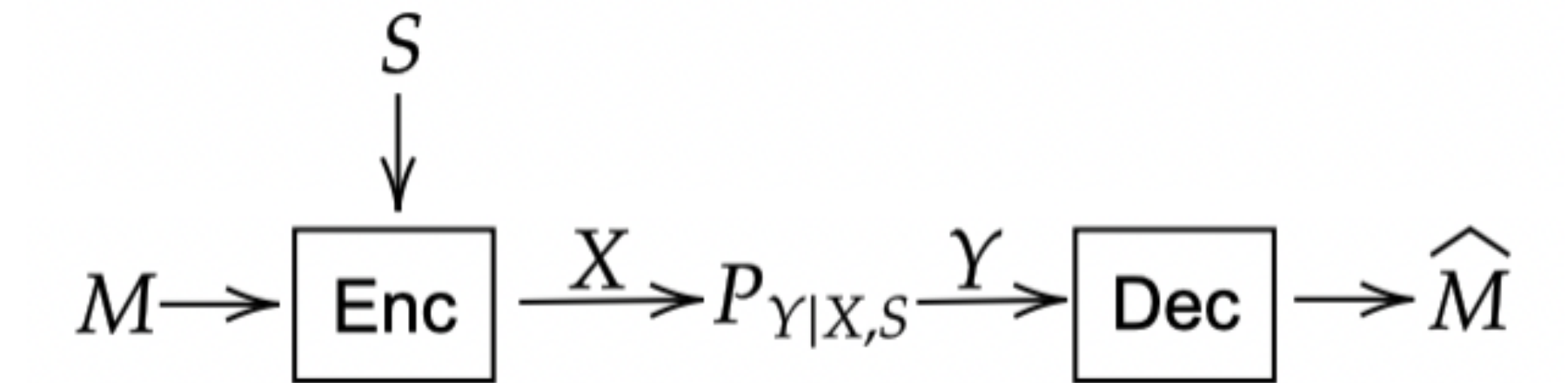
References

- [1] Si-Hyeon Lee and Sae-Young Chung. A unified random coding bound. *IEEE Transactions on Information Theory*, 64(10):6779–6802, 2018.
- [2] Cheuk Ting Li and Venkat Anantharam. A unified framework for one-shot achievability via the poisson matching lemma. *IEEE Transactions on Information Theory*, 67(5):2624–2651, 2021.
- [3] Cheuk Ting Li and Abbas El Gamal. Strong functional representation lemma and applications to coding theorems. *IEEE Transactions on Information Theory*, 64(11):6967–6978, 2018.
- [4] Sung Hoon Lim, Young-Han Kim, Abbas El Gamal, and Sae-Young Chung. Noisy network coding. *IEEE Transactions on Information Theory*, 57(5):3132–3152, 2011.
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Examples

The main theorem can be applied to any combination of source coding, channel coding and coding for computing. Note $\iota(x; y|z) := \log \frac{P(x,y|z)}{(P(x|z)P(y|z))}$.

Channel Coding with State Info at Encoder

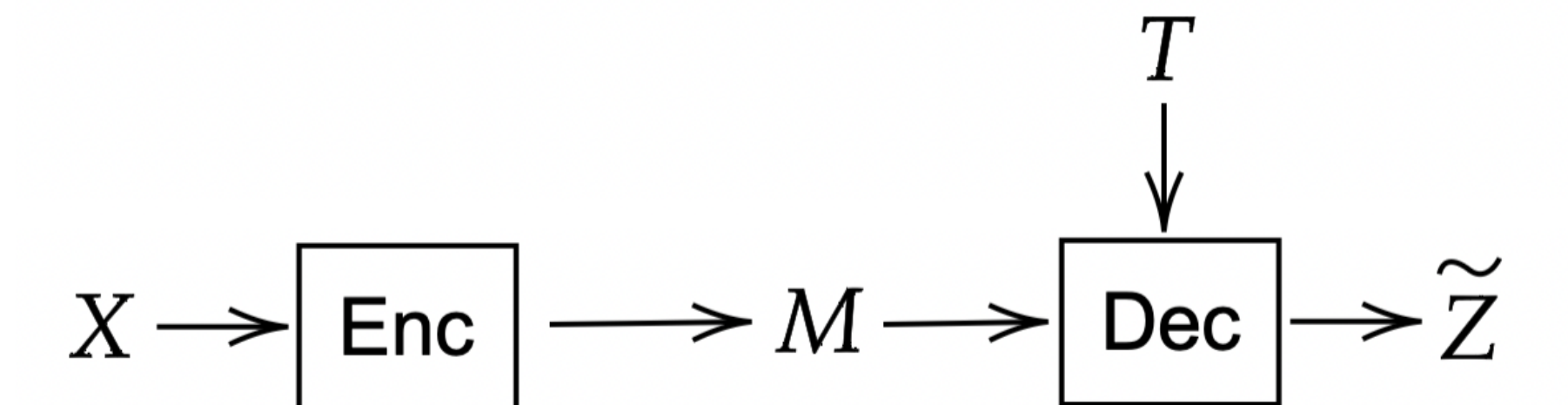


Fix $P_{U|S}$ and $x : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{X}$. For $M \sim \text{Unif}[L]$, $S \sim P_S$, let $U_1 = (U, M)$,

$$P_e := \mathbf{P}(\tilde{X}_2 \neq M) \leq \mathbf{E} \left[\min \left\{ L 2^{-\iota(U; Y) + \iota(U; S)}, 1 \right\} \right].$$

It recovers asymptotic capacity, attains the best known second-order result.

Source Coding with Side Info at Decoder



Fix $P_{U|X}$ and $z : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}$. For $X \sim P_X$, $T \sim P_{T|X}$, $M \in [L]$,

$$P_e := \mathbf{P}\{d(X, \tilde{Z}) > D\} \leq \mathbf{E} \left[\min \left\{ \mathbf{1}\{d(X, Z) > D\} + L^{-1} 2^{-\iota(U; T) + \iota(U; X)}, 1 \right\} \right].$$

It recovers asymptotic capacity, and also covers *coding for computing*.

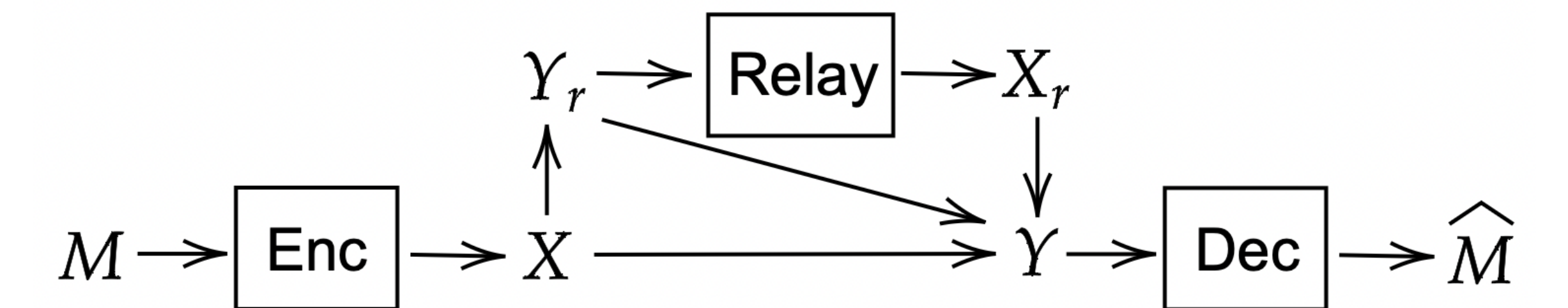
Multiple Access Channel

For MAC $P_{Y|X_1, X_2}$ and $M_j \sim \text{Unif}[L_j]$ for $j = 1, 2$, with $\gamma := \ln(L_1 |\mathcal{X}_1|) + 1$,

$$P_e \leq \mathbf{E} \left[\min \left\{ \gamma L_1 L_2 2^{-\iota(X_1, X_2; Y)} + \gamma L_2 2^{-\iota(X_2; Y|X_1)} + L_1 2^{-\iota(X_1; Y|X_2)}, 1 \right\} \right].$$

It recovers the asymptotic capacity region.

One-Shot Relay Channel



Let $U_1 := (X, M)$, $U_2 := U$, Main Theorem gives a compress-forward bound:

$$P_e \leq \mathbf{E} \left[\min \left\{ \gamma L 2^{-\iota(X; U, Y)} \left(2^{-\iota(U; Y) + \iota(U; Y_r)} + 1 \right), 1 \right\} \right]$$

where $\gamma = \ln |\mathcal{U}| + 1$, $(X, Y_r, U, X_r, Y) \sim P_X P_{Y_r|X} P_{U|Y_r} \delta_{x_r(Y_r, U)} P_{Y|X, Y_r, X_r}$.

- It is a one-shot version of **relay-with-unlimited-look-ahead**.
- If $Y = (Y', Y'')$ and $P_{Y|X, X_r, Y_r} = P_{Y'|X, X_r} P_{Y''|X_r}$, it is a one-shot version of **primitive relay channel**.
- By message splitting, we can also have a **partial-decode-forward bound**.