Part A: Perturbation theory

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Contents

Lecture 1

Algebraic perturbation theory

1.1 An introductory example

The quadratic equation

$$x^2 - \pi x + 2 = 0\,, \tag{1.1}$$

has the exact solutions

$$x = \tfrac{\pi}{2} \pm \sqrt{\tfrac{\pi^2}{4} - 2} = 2.254464 \qquad \text{and} \qquad 0.887129\,. \tag{1.2}$$

To introduce the main idea of perturbation theory, let's pretend that calculating a square root is a big deal. We notice that if we replace π by 3 in (1.1) then the resulting quadratic equation nicely factors and the roots are just $x = 1$ and $x = 2$. Because π is close to 3, our hope is that the roots of (1.1) are close to 1 and 2. Perturbation theory makes this intuition precise and systematically improves our initial approximations $x \approx 1$ and $x \approx 2$.

A regular perturbation series

We use perturbation theory by writing

$$\pi = 3 + \epsilon\,, \tag{1.3}$$

and assuming that the solutions of

$$x^2 - (3+\epsilon)x + 2 = 0\,, \tag{1.4}$$

are given by a regular perturbation series (RPS):

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{1.5}$$

We are assuming that the x_n's above do not depend on ϵ. Putting (1.5) into (1.4) we have

$$x_0^2 + \epsilon\, 2x_0 x_1 + \epsilon^2 \left(2x_0 x_2 + x_1^2\right) + \epsilon^3 \left(2x_0 x_3 + 2x_1 x_2\right) \tag{1.6}$$

$$- (3+\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + 2 = \mathrm{ord}\left(\epsilon^4\right)\,. \tag{1.7}$$

The notation "ord(ϵ^4)" means that we are suppressing some unnecessary information, but we are indicating that the largest unwritten terms are all proportional to ϵ^4. We're assuming that the expansion (1.5) is unique so we can match up powers of ϵ in (1.7) and obtain a hierarchy of equations for the unknown coefficients x_n in (1.5).

The leading-order terms from (1.7) are

$$\epsilon^0 : \qquad x_0^2 - 3x_0 + 2 = 0\,, \qquad \Rightarrow \qquad x_0 = 1 \quad \text{or} \quad 2\,. \tag{1.8}$$

Quadratic equations have two roots and the important point is that indeed the leading order approximation above delivers two roots: this is a *regular* perturbation problem. We'll soon see examples in which the leading approximation provides only one root: these problems are *singular perturbation problems*.

Let's take $x_0 = 1$. The next three orders are then

$$\epsilon^1 : \quad (2x_0 - 3)x_1 - x_0 = 0\,, \qquad \Rightarrow \qquad x_1 = \frac{x_0}{2x_0 - 3} = -1\,, \tag{1.9}$$

$$\epsilon^2 : \quad (2x_0 - 3)x_2 + x_1^2 - x_1 = 0\,, \qquad \Rightarrow \qquad x_2 = \frac{x_1 - x_1^2}{2x_0 - 3} = 2\,, \tag{1.10}$$

$$\epsilon^3 : \quad (2x_0 - 3)x_3 + 2x_1x_2 - x_2 = 0\,, \qquad \Rightarrow \qquad x_3 = \frac{x_2(1 - 2x_1)}{2x_0 - 3} = -6\,, \tag{1.11}$$

The resulting perturbation expansion is

$$x = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + \mathrm{ord}(\epsilon^4)\,. \tag{1.12}$$

With $\epsilon = \pi - 3$ we find $x = 0.881472 + \mathrm{ord}(\epsilon^4)$.

Exercise: Determine a few terms in the RPS of the root $x_0 = 2$.

This example illustrates the main features of perturbation theory. When faced with a difficult problem one should:

1. Find an easy problem that's close to the difficult problem. It helps if the easier problem has a simple analytic solution.

2. Quantify the difference between the two problems by introducing a small parameter ϵ.

3. Assume that the answer is in the form of a perturbation expansion, such as (1.5).

4. Compute terms in the perturbation expansion.

5. Solve the difficult problem by summing the series with the appropriate value of ϵ.

Step 4 above also deserves some discussion. In this step we are sequentially solving a hierarchy of *linear* equations:

$$(2x_0 - 3)x_{n+1} = R(x_0, x_1, \cdots x_n)\,. \tag{1.13}$$

We can determine x_{n+1} because we can divide by $(2x_0 - 3)$: fortunately $x_0 \neq 3/2$. This structure occurs in other simple perturbation problems: we confront the same linear problem at every level of the hierarchy and we have to solve this problem[1] repeatedly.

1.2 Iteration

Now let's consider the method of iteration — this is an alternative to the RPS. Iteration requires a bit of initial ingenuity. But in cases where the form of the expansion is not obvious, iteration is essential. (One of the strengths of **H** is that it emphasizes the utility of iteration.)

[1] Mathematicians would say that we are inverting the linear operator — this language seems pretentious in this simple example. But we'll soon see examples in which it is appropriate.

Solution of a quadratic equation by iteration

We can rewrite the quadratic equation (1.4) as

$$(x-1)(x-2)=\epsilon x\,. \tag{1.14}$$

If we interested in the effect of ϵ on the root $x=1$ then we rearrange this further as

$$x = 1+\epsilon\underbrace{\frac{x}{x-2}}_{\stackrel{\text{def}}{=}f(x)}\,. \tag{1.15}$$

We iterate by first dropping the ϵ-term on the right — this provides the first guess $x^{(0)}=1$. At the next iteration we keep the ϵ-term with f evaluated at $x^{(0)}$:

$$x^{(1)} = 1+\epsilon f\left(x^{(0)}\right) = 1-\epsilon\,. \tag{1.16}$$

We continue to improve the approximation with more and more iterates:

$$x^{(n+1)} = 1+f\left(x^{(n)}\right)\,. \tag{1.17}$$

So the second iteration is

$$x^{(2)} = 1+\epsilon f\left(x^{(1)}\right) = 1-\epsilon\frac{1-\epsilon}{1+\epsilon}\,. \tag{1.18}$$

This is not an RPS — if we want an answer ordered in powers of ϵ then we must simplify (1.18) further as

$$x^{(2)} = 1-\epsilon\,(1-\epsilon)\left(1-\epsilon+\mathrm{ord}(\epsilon^2)\right) = 1-\epsilon+2\epsilon^2+\mathrm{ord}(\epsilon^3)\,. \tag{1.19}$$

I suspect there is no point in keeping the ϵ^3 in (1.19) because it is probably not correct — I am guessing that we have to iterate one more time to get the correct ϵ^3 term.

Exercise: use iteration to locate the root near $x=2$.

Another example of iteration

Considering the equation

$$4-x^2=\epsilon\ln x\,, \tag{1.20}$$

with $0<\epsilon\ll 1$, we see that there is a positive real solution close to $x=2$. To improve on $x\approx 2$ we rewrite the equation as

$$x = 2-\frac{\epsilon\ln x}{2+x}\,. \tag{1.21}$$

If we drop the ϵ-term we get a first approximation $x^{(1)}=2$, and the next iterate is

$$x^{(2)} = 2-\frac{\epsilon\ln 2}{4}\,, \tag{1.22}$$

and again

$$x^{(3)} = 2-\frac{\epsilon\ln\left(2-\frac{\epsilon\ln 2}{4}\right)}{4-\frac{\epsilon\ln 2}{4}}\,. \tag{1.23}$$

We can develop an RPS by simplifying $x^{(3)}$ as

$$x^{(3)} = 2-\frac{\epsilon}{4}\left(1+\epsilon\frac{\ln 2}{16}\right)\Big[\ln 2+\underbrace{\ln\left(1-\epsilon\frac{\ln 2}{8}\right)}_{=-\epsilon\frac{\ln 2}{8}+\mathrm{ord}(\epsilon^2)}\Big]+\mathrm{ord}(\epsilon^3)\,, \tag{1.24}$$

$$= 2-\frac{\ln 2}{4}\,\epsilon+\frac{\ln 2}{64}\,(2-\ln 2)\,\epsilon^2+\mathrm{ord}(\epsilon^3)\,. \tag{1.25}$$

1.3 Singular perturbation of polynomial equations

Consider the example

$$\epsilon x^2 + x - 1 = 0\,. \tag{1.26}$$

If we simply set $\epsilon = 0$ we find $x = 1$ and we can proceed to nail down this root with an RPS:

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{1.27}$$

Exercise: Find x_1 and x_2 by both RPS and iteration.

But a quadratic equation has two roots: we're missing a root. If we peek at the answer we find that

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots \\ -\epsilon^{-1} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots \end{cases} \tag{1.28}$$

The missing root is going to infinity as $\epsilon \to 0$. Notice that the term we blithely dropped, namely ϵx^2, is therefore $\mathrm{ord}(\epsilon^{-1})$. Dropping a big term is a mistake.

We could have discovered the missing root by looking for *two-term dominant balances* in (1.26):

$$\underbrace{\epsilon x^2 + x}_{\text{dominant balance}} \quad -1 = 0\,. \tag{1.29}$$

The balance above implies that $x = -\epsilon^{-1}$. The balance is consistent because the neglected term in (1.29) (the -1) is smaller than the two retained terms as $\epsilon \to 0$. Once we know that x is varying as ϵ^{-1} we can *rescale* by defining

$$X \stackrel{\text{def}}{=} \epsilon x\,. \tag{1.30}$$

The variable X remains finite as $\epsilon \to 0$, and substituting (1.30) into (1.26) we find that X satisfies the rescaled equation

$$X^2 + X - \epsilon = 0\,. \tag{1.31}$$

Now we can find the big root via an RPS

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots \tag{1.32}$$

This procedure reproduces the expansion that begins with $-\epsilon^{-1}$ in (1.28).

Notice that (1.30) is "only" a change in notation, and (1.31) is equivalent to (1.26). But notation matters: in terms of x the problem is singular while in terms of X the problem is regular.

Example: Find $\epsilon \ll 1$ expansions of the roots of

$$\epsilon x^3 + x - 1 = 0\,. \tag{1.33}$$

One root is obviously obtained via an RPS

$$x = 1 - \epsilon + \mathrm{ord}(\epsilon^2)\,. \tag{1.34}$$

But there are two missing roots. A dominant balance between the first two terms in (1.34),

$$\epsilon x^3 + x \approx 0\,, \tag{1.35}$$

implies that x varies as $\epsilon^{-1/2}$. This balance is consistent, so rescale

$$X \stackrel{\text{def}}{=} \epsilon^{1/2} x\,. \tag{1.36}$$

This definition ensures that X is order unity as $\epsilon \to 0$. The rescaled equation is

$$X^3 + X - \sqrt{\epsilon} = 0\,, \tag{1.37}$$

and there is now a regular perturbation problem with small parameter $\sqrt{\epsilon}$:

$$X = X_0 + \sqrt{\epsilon}\, X_1 + \left(\sqrt{\epsilon}\right)^2 X_2 + \cdots \tag{1.38}$$

The leading order terms are

$$X_0^3 + X_0 = 0 \qquad \Rightarrow \qquad X_0 = \pm\mathrm{i} \qquad \text{and} \qquad X_0 = 0\,. \tag{1.39}$$

The solution $X_0 = 0$ is reproducing the solution back in (1.34). Let's focus on the other two roots, $X_0 = \pm\mathrm{i}$. At next order the problem is

$$\sqrt{\epsilon}\ : \qquad 3X_0^2 X_1 + X_1 - 1 = 0\,, \qquad \Rightarrow \qquad X_1 = \frac{1}{3X_0^2+1} = -\frac{1}{2}\,. \tag{1.40}$$

For good value

$$\left(\sqrt{\epsilon}\right)^2\ : \qquad 3X_0^2 X_2 + X_2 + 3X_0 X_1^2 = 0\,, \qquad \Rightarrow \qquad X_2 = -\frac{3X_0X_1^2}{3X_0^2+1} = \pm\frac{3}{8}\mathrm{i}\,. \tag{1.41}$$

We write the expansion in terms of our original variable as

$$x = \pm\frac{i}{\sqrt{\epsilon}} - \frac{1}{2} \pm \sqrt{\epsilon}\frac{3}{8}\mathrm{i} + \mathrm{ord}(\epsilon^1)\,. \tag{1.42}$$

Example: Find leading-order expressions for all six roots of

$$\epsilon^2 x^6 - \epsilon x^4 - x^3 + 8 = 0\,, \qquad \text{as } \epsilon \to 0. \tag{1.43}$$

This is from **BO** section 7.2.

1.4 Double roots

Now consider

$$\underbrace{x^2 - 2x + 1}_{(x-1)^2} - \epsilon f(x) = 0\,. \tag{1.44}$$

where $f(x)$ is some function of x. Section 1.3 of **H** discusses the case $f(x) = x^2$ — with a surfeit of testosterone we attack the general case.

We try the RPS:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \tag{1.45}$$

We must expand $f(x)$ with a Taylor series:

$$f\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) = f(x_0) + \epsilon x_1 f'(x_0) + \epsilon^2\left(x_2 f'(x_0) + \frac{1}{2}x_1^2 f''(x_0)\right) + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{1.46}$$

This is not as bad as it looks — we'll only need the first term, $f(x_0)$, though that may not be obvious at the outset.

The leading term in (1.44) is

$$x_0^2 - 2x_0 + 1 = 0\,, \qquad \Rightarrow \qquad x_0 = 1\,,\ \text{(twice)}. \tag{1.47}$$

There is a double root. At next order there is a problem:

$$\epsilon^1\ : \qquad \underbrace{2x_1 - 2x_1}_{=0} - f(1) = 0\,. \tag{1.48}$$

Unless $f(1)$ happens to vanish, we're stuck. The problem is that we assumed that the solution has the form in (1.45), and it turns out that this assumption is wrong. The perturbation method kindly tells us this by producing the contradiction in (1.48).

To find the correct form of the expansion we use iteration: rewrite (1.44) as

$$x = 1 \pm \sqrt{\epsilon f(x)} \,. \tag{1.49}$$

and starting with $x^{(0)} = 1$, iterate with

$$x^{(n+1)} = 1 \pm \sqrt{\epsilon f\left(x^{(n-1)}\right)} \,. \tag{1.50}$$

At the first iteration we find

$$x^{(1)} = 1 \pm \sqrt{\epsilon f\,(1)} \,. \tag{1.51}$$

There is a $\sqrt{\epsilon}$ which was not anticipated by the RPS back in (1.45).

Exercise: Go through another iteration cycle to find $x^{(2)}$.

Iteration has shown us the way forward: we proceed assuming that the correct RPS is probably

$$x = x_0 + \epsilon^{1/2} x_1 + \epsilon x_2 + \epsilon^{3/2} x_3 + \cdots \tag{1.52}$$

At leading order we find $x_0 = 1$, and at next order

$$\epsilon^{1/2} : \qquad 2x_1 - 2x_1 = 0 \,. \tag{1.53}$$

This is surprising, but it is not a contradiction: x_1 is not determined at this order. We have to endure some suspense — we go to next order and find

$$\epsilon^1 : \qquad \underbrace{2(x_0-1)x_2}_{=0} + x_1^2 - f(x_0) = 0 \,, \qquad \Rightarrow \qquad x_1 = \pm\sqrt{f(1)} \,. \tag{1.54}$$

The RPS has now managed to reproduce the first iterate $x^{(1)}$. Going to order $\epsilon^{3/2}$, we find that x_3 is undetermined and

$$x_2 = \tfrac{1}{2} f'(1) \,. \tag{1.55}$$

The solution we constructed is

$$x = 1 \pm \sqrt{\epsilon f(1)} + \frac{\epsilon}{2} f'(1) + \mathrm{ord}\left(\epsilon^{3/2}\right) \,. \tag{1.56}$$

This example teaches us that a perturbation "splits" double roots. The splitting is rather large: adding the order ϵ perturbation in (1.44) moves the roots apart by order $\sqrt{\epsilon} \gg \epsilon$. This sensitivity to small perturbations is obvious geometrically — draw a parabola P touching the x-axis at some point, and move P downwards by small distance. The small movement produces two roots separated by a distance that is clearly much greater than the small vertical displacement of P. If P moves upwards (corresponding to $f(1) < 0$ in the example above) then the roots split off along the imaginary axis.

1.5 An example with logarithms

I'll discuss the example[2] from **H** section 1.4:

$$x\mathrm{e}^{-x} = \epsilon \,. \tag{1.57}$$

It is easy to see that if $0 < \epsilon \ll 1$ there is a small solution and a big solution. It is straightforward to find the small solution in terms of ϵ. Here we discuss the more difficult problem of finding the big solution.

[2]This example is related to the Lambert W-function, also known as the omega function and the product logarithm; try help `ProductLog` in MATHEMATICA and `lambertw` in MATLAB.

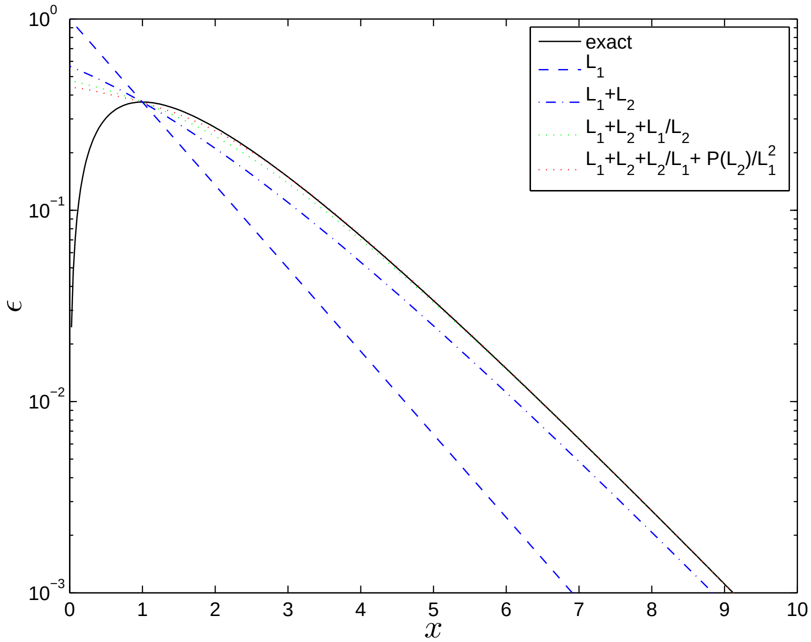

Figure 1.1: Comparison of $\epsilon = x\mathrm{e}^{-x}$ with increasingly accurate small-ϵ approximations to the inverse function $\epsilon(x)$.

Exercise: Show that the small solution is $x(\epsilon) = \epsilon + \epsilon^2 + \frac{3}{2}\epsilon^3 + \mathrm{ord}(\epsilon^4)$.

To get a handle on (1.57), we take the logarithm and write the result as

$$x = L_1 + \ln x\,, \tag{1.58}$$

where

$$L_1 \stackrel{\text{def}}{=} \ln \frac{1}{\epsilon}\,. \tag{1.59}$$

Note if $0 < \epsilon < 1$ then $\ln \epsilon < 0$. To avoid confusion over signs it is best to work with the large positive quantity L_1.

Now observe that if $x \to \infty$ then there is a consistent two-term dominant balance in (1.58): $x \approx L_1$. This is consistent because the neglected term, namely $\ln x$, is much less than x as $x \to \infty$. We can improve on this first approximation using the iterative scheme

$$x^{(n+1)} = L_1 + \ln x^{(n)} \qquad \text{with} \qquad x^{(0)} = L_1\,. \tag{1.60}$$

The first iteration gives

$$x^{(1)} = L_1 + L_2\,, \tag{1.61}$$

where $L_2 \stackrel{\text{def}}{=} \ln L_1$ is the iterated logarithm.

The second iteration[3] is

$$x^{(2)} = L_1 + \ln(L_1 + L_2) \,, \tag{1.62}$$
$$= L_1 + L_2 + \ln\left(1 + \frac{L_2}{L_1}\right) , \tag{1.63}$$
$$= L_1 + L_2 + \frac{L_2}{L_1} - \tfrac{1}{2}\left(\frac{L_2}{L_1}\right)^2 + \cdots \tag{1.64}$$

We don't need L_3.

At the third iteration a pattern starts to emerge

$$\begin{aligned}
x^{(3)} &= L_1 + \ln\left(L_1 + L_2 + \frac{L_2}{L_1} - \tfrac{1}{2}\left(\frac{L_2}{L_1}\right)^2 + \cdots\right) , \\
&= L_1 + L_2 + \ln\left(1 + \frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \tfrac{1}{2}\frac{L_2^2}{L_1^3} + \cdots\right) , \\
&= L_1 + L_2 + \left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \tfrac{1}{2}\frac{L_2^2}{L_1^3} + \cdots\right) - \tfrac{1}{2}\left(\frac{L_2}{L_1} + \frac{L_2}{L_1^2} + \cdots\right)^2 + \tfrac{1}{3}\left(\frac{L_2}{L_1} + \cdots\right)^3 \cdots \\
&= L_1 + L_2 + \frac{L_2}{L_1} + \frac{L_2 - \frac{1}{2}L_2^2}{L_1^2} + \frac{\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2 + \cdots}{L_1^3} + \cdots
\end{aligned} \tag{1.65}$$

The final $\cdots$ above indicates a fraction with L_1^4 in the denominator.

The philosophy is that as one grinds out more terms the earlier terms in the developing expansion stop changing and a stable pattern emerges. In this example the expansion has the form

$$x = L_1 + L_2 + \sum_{n=1}^{\infty} \frac{P_n(L_2)}{L_1^n} \,, \tag{1.66}$$

where P_n is a polynomial of degree n. This was not guessable from (1.57).

1.6 Convergence

Usually we can't prove that an RPS converges. The only way of proving convergence is to have a simple expression for the form of the n'th term. In realistic problems this is not available. One just has to satisfied with consistency and hope for the best.

But with iteration there is a simple result. Suppose that $x = x_*$ is the solution of

$$x = f(x) \,. \tag{1.67}$$

Start with a guess $x = x_0$ and proceed to iterate with $x_{n+1} = f(x_n)$. If an iterate x_n is close to the solution x_* then we have

$$x = x_* + \eta_n \,, \qquad \text{with } \eta_n \ll 1. \tag{1.68}$$

The next iterate is:

$$x_* + \eta_{n+1} = f(x_* + \eta_n) \,, \tag{1.69}$$
$$= x_* + \eta_n f'(x_*) + \mathrm{ord}(\eta_n^2) \,, \tag{1.70}$$

[3] We're using the Taylor series

$$\ln(1+\eta) = \eta - \tfrac{1}{2}\eta^2 + \tfrac{1}{3}\eta^3 + \tfrac{1}{4}\eta^4 + \cdots$$

and therefore

$$\eta_{n+1} = f'(x_*)\eta_n \,. \tag{1.71}$$

The sequence η_n will decrease exponentially if

$$|f'(x_*)| < 1 \,. \tag{1.72}$$

If the condition above is satisfied, and the first guess is good enough, then the iteration converges onto x_*. This is a loose version of the *contraction mapping theorem*

1.7 Problems

Problem 1.1. Because 10 is close to 9 we suspect that $\sqrt{10}$ is close to $\sqrt{9} = 3$. *(i)* Define $x(\epsilon)$ by

$$x(\epsilon)^2 = 9 + \epsilon \,, \tag{1.73}$$

and assume that $x(\epsilon)$ has an RPS like (1.5). Calculate the first four terms, x_0 through x_3. *(ii)* Take $\epsilon = 1$ and compare your estimate of $\sqrt{10}$ with a six decimal place computation. *(iii)* Now solve (1.73) with the binomial expansion and verify that the resulting series is the same as the RPS from part *(ii)*. What is the radius of convergence of the series?

Problem 1.2. Find two-term, $\epsilon \ll 1$ approximations to all roots of

$$x^3 + 5x^2 + 4x + \epsilon = 0 \,, \tag{1.74}$$

and

$$y^3 - y^2 + \epsilon = 0 \,, \tag{1.75}$$

and

$$\epsilon z^4 - z + 1 = 0 \,. \tag{1.76}$$

Problem 1.3. Find rescalings for the roots of

$$\epsilon^2 x^3 - (1 - \epsilon + 3\epsilon^2)x^2 + (3 - 3\epsilon + 2\epsilon^2 - \epsilon^3)x - 2 + 3\epsilon - \epsilon^3 = 0 \tag{1.77}$$

and thence find two non-trivial terms in the approximation for each root using (a) iteration and (b) series expansion.

Problem 1.4. Develop perturbation solutions to

$$x^3 - (6 + \epsilon + \epsilon^2)x^2 + (12 + 3\epsilon + 3\epsilon^2 + 2\epsilon^3)x - 8 - 2\epsilon - 3\epsilon^2 - 2\epsilon^3 - \epsilon^4 = 0 \tag{1.78}$$

finding the the first three terms in the approximation for each root, $x = x_0 + \epsilon^a x_a + \epsilon^{2a} x_{2a}$, and determining a along the way.

Problem 1.5. Find a three-term approximation to the real solutions of

$$\mathrm{e}^{x - x^2} = \epsilon x^2 \,, \qquad \text{as } \epsilon \to 0 \,. \tag{1.79}$$

Problem 1.6. Find a two-term approximation to all positive real roots of $x^2 - 4 = \epsilon \ln x$ as $\epsilon \to 0$.

Problem 1.7. Assume that the Earth is a perfect sphere of radius $R = 6400$km and that it is wrapped tightly at the equator with a rope. Suppose one cuts the rope and splices a length $\ell = 1$cm into the rope. Then the rope is grabbed at a point and hoisted above the surface of the Earth as high as possible. How high is that?

Problem 1.8. Use perturbation theory to solve $(x+1)^7 = \epsilon x$. How rapidly do the n roots vary from $x = -1$ as a function of ϵ? Give the first three terms in the expansion.

Problem 1.9. Here is a medley of algebraic perturbation problems, mostly from **BO** and **H**. Use perturbation theory to find two-term approximations ($\epsilon \to 0$) to all roots of:

$$
\begin{aligned}
&\text{(a)} \qquad x^2 + x + 6\epsilon = 0\,,\\
&\text{(b)} \qquad x^3 - \epsilon x - 1 = 0\,,\\
&\text{(c)} \qquad x^3 + \epsilon x^2 - x - \epsilon = 0\,,\\
&\text{(d)} \qquad \epsilon^2 x^3 + x^2 + 2x + \epsilon = 0\,,\\
&\text{(e)} \qquad \epsilon x^3 + x^2 - 2x + 1 = 0\,,\\
&\text{(f)} \qquad \epsilon x^3 + x^2 + (2+\epsilon)x + 1 = 0\,,\\
&\text{(g)} \qquad \epsilon x^3 + x^2 + (2-\epsilon)x + 1 = 0\,,\\
&\text{(h)} \qquad \epsilon x^4 - x^2 - x + 2 = 0\,,\\
&\text{(i)} \qquad \epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0\,,\\
&\text{(j)} \qquad \epsilon x^8 - \epsilon x^6 + x - 2 = 0\,,\\
&\text{(k)} \qquad \epsilon^2 x^8 - \epsilon x^6 + x - 2 = 0\,,\\
&\text{(l)} \qquad x^3 - x^2 + \epsilon = 0\,,\\
&\text{(m)} \qquad x^{2+\epsilon} = \frac{1}{x + 2\epsilon}\,,\\
&\text{(n)} \qquad \epsilon \mathrm{e}^{x^2} = 1 + \frac{\epsilon}{1+x^2}\,,
\end{aligned}
$$

Problem 1.10. Consider $y(\epsilon, a)$ defined as the solution of

$$\epsilon y^a = \mathrm{e}^{-y}\,. \tag{1.80}$$

Note that $a = -1$ is the example (1.57). Use the method of iteration to find a few terms in the $\epsilon \to 0$ asymptotic solution of (1.80) — "few" means about as many as in (1.65). Consider the case $a = +1$; use MATLAB to compare the exact solution with increasingly accurate asymptotic approximations (e.g., as in Figure 1.1).

Problem 1.11. Let us continue problem 1.10 by considering numerical convergence of iteration in the special case $a = 1$. Figure 1.2 shows numerical iteration of

$$y_{n+1} = \ln \frac{1}{\epsilon} - \ln y_n\,. \tag{1.81}$$

With $\epsilon = 0.25$ everything is hunky-dory. At $\epsilon = 0.35$ the iteration is converging, but it is painfully slow. And at $\epsilon = 0.45$ it all goes horribly wrong. Explain this failure of iteration. To be convincing your explanation should include a calculation of the magic value of ϵ at which numerical iteration fails. That is, if $\epsilon > \epsilon_*$ then the iterates do not converge to the solution of $\epsilon y = \mathrm{e}^{-y}$. Find ϵ_*.

Problem 1.12. The relation

$$xy = \mathrm{e}^{x-y} \tag{1.82}$$

implicitly defines y as a function of x, or vice versa. View y as a function x, and determine the large-x behavior of this function. Calculate enough terms to guess the form of the expansion.

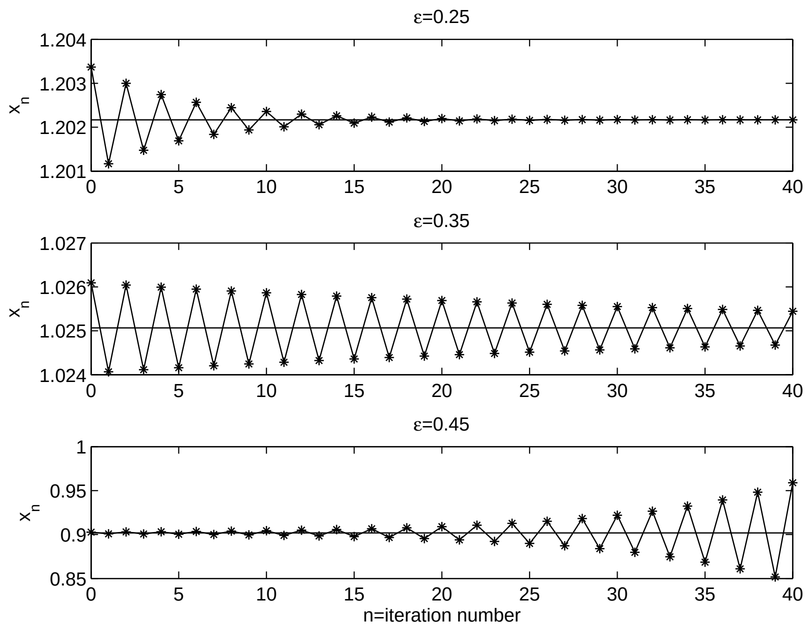

Figure 1.2: Figure for problem 1.11. Numerical iteration of $y_{n+1} = \ln\frac{1}{\epsilon} - \ln y_n$. At $\epsilon = 0.45$ the iteration diverges. In all three cases we start x_0 within 0.1% of the right answer.

Problem 1.13. Consider $z(\epsilon)$ defined as the solution to

$$z^{\frac{1}{\epsilon}} = \mathrm{e}^z\,. \tag{1.83}$$

(i) Use MATLAB to make a graphical analysis of this equation with $\epsilon = 1/5$ and $\epsilon = 1/10$. Convince yourself that as $\epsilon \to 0$ there is one root near $z = 1$, and second, large root that recedes to infinity as $\epsilon \to 0$. *(ii)* Use an iterative method to develop an $\epsilon \to 0$ approximation to the large solution. Calculate a few terms so that you understand the form of the expansion. *(iii)* Use MATLAB to compare the exact answer with approximations of various orders e.g., as in Figure 1.1. *(iv)* Find the dependance of the other root, near $z = 1$, on ϵ as $\epsilon \to 0$.

Problem 1.14. Find the $x \gg 1$ solution of

$$\mathrm{e}^{\mathrm{e}^x} = 10^{10}x^{10}\exp\left(10^{10}x^{10}\right)$$

with one significant figure of accuracy. (I think you can do this without a calculator if you use $\ln 2 \approx 0.69$ and $\ln 10 \approx 2.30$.)

Lecture 2

Regular perturbation of ordinary differential equations

2.1 The projectile problem

If one projects a particle vertically upwards from the surface of the Earth at $z = 0$ with speed u then the projectile reaches a maximum height $h = u^2/2g_0$ and returns to the ground at $t = 2u/g_0$ (ignoring air resistance). At least that's what happens if the gravitational acceleration g_0 is constant. But a better model is that the gravitational acceleration is

$$g(z) = \frac{g_0}{(1 + z/R)^2} \,,$$

where $g_0 = 9.8\text{m s}^{-2}$, $R = 6{,}400$kilometers and z is the altitude. The particle stays aloft longer than $2u/g_0$ because gravity is weaker up there.

Let's use perturbation theory to calculate the correction to the time aloft due to the small decrease in the force of gravity. But first, before the perturbation expansion, we begin with a complete formulation of the problem. We must solve the second-order autonomous differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{g_0}{(1 + z/R)^2} \,, \tag{2.1}$$

with the initial condition

$$t = 0: \qquad z = 0 \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u \,. \tag{2.2}$$

We require the time τ at which $z(\tau) = 0$. Notice that if $R = \infty$ we recover the elementary problem with uniform gravity.

An important part of this problem is *non-dimensionalizing* and identifying the small parameter used to organize a perturbation expansion. We use the elementary problem ($R = \infty$) to motivate the following definition of non-dimensional variables

$$\bar{z} \stackrel{\text{def}}{=} \frac{g_0 z}{u^2} \,, \qquad \text{and} \qquad \bar{t} \stackrel{\text{def}}{=} \frac{g_0 t}{u} \,. \tag{2.3}$$

Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{g_0}{u} \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \,, \qquad \text{and therefore} \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{g_0}{u}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\bar{t}^2} \frac{u^2}{g_0} \bar{z} = g_0 \frac{\mathrm{d}^2 \bar{z}}{\mathrm{d}\bar{t}^2} \tag{2.4}$$

Putting these expressions into (2.1) we obtain the non-dimensional problem

$$\frac{\mathrm{d}^2 \bar{z}}{\mathrm{d}\bar{t}^2} + \frac{1}{(1 + \epsilon \bar{z})^2} = 0 \,, \tag{2.5}$$

where

$$\epsilon \stackrel{\text{def}}{=} \frac{u^2}{Rg_0}\,. \tag{2.6}$$

We must also non-dimensionalize the initial conditions in (2.2):

$$\bar{t}=0: \qquad \bar{z}=0 \qquad \text{and} \qquad \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}}=1\,. \tag{2.7}$$

At this point we have done nothing more than change notation. The original problem was specified by three parameters, g_0, u and u. The non-dimensional problem is specified by a single parameter ϵ, which might be large, small, or in between. If we're interested in balls and bullets fired from the surface of the Earth then $\epsilon \ll 1$.

OK, so assuming that $\epsilon \ll 1$ we try a regular perturbation expansion on (2.5). We also drop all the bars that decorate the non-dimensional variables: we can restore the dimensions at the end of the calculation and it is just too onerous to keep writing all those little bars. The regular perturbation expansion is

$$z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.8}$$

We use the binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \mathrm{ord}\left(x^3\right)\,, \tag{2.9}$$

with $n=-2$ to expand the nonlinear term:

$$(1+\epsilon z)^{-2} = 1 - 2\epsilon z + 3\epsilon z^2 + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.10}$$

Introducing (2.8) into the expansion above gives

$$(1+\epsilon z)^{-2} = 1 - 2\epsilon z_0 + \epsilon^2(3z_0^2 - 2z_1) + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.11}$$

So matching up equal powers of ϵ in (2.5) (and denoting time derivatives by dots) we obtain the first three terms in perturbation hierarchy:

$$\begin{aligned}
\ddot{z}_0 &= -1\,, && \text{with } z_0(0)=0\,, \quad \dot{z}_0(0)=1\,,\\
\ddot{z}_1 &= 2z_0\,, && \text{with } z_1(0)=0\,, \quad \dot{z}_1(0)=0\,,\\
\ddot{z}_2 &= 2z_1 - 3z_0^2\,, && \text{with } z_2(0)=0\,, \quad \dot{z}_2(0)=0\,.
\end{aligned}$$

Above we have the first three terms in a hierarchy of *linear* equations of the form

$$Lz_{n+1} = R(z_0, \cdots z_n)\,, \tag{2.12}$$

where the linear operator is

$$L \stackrel{\text{def}}{=} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,. \tag{2.13}$$

To solve each term in the hirerarchy we must invert this linear operator, being careful to use the correct initial equations that $z_{n+1}(0) = \dot{z}_{n+1}(0) = 0$.

The solution of the first two equations is

$$z_0(t) = t - \frac{t^2}{2}\,, \qquad \text{and} \qquad z_1(t) = \frac{t^3}{3} - \frac{t^4}{12}\,. \tag{2.14}$$

To obtain $z_2(t)$ we integrate

$$\ddot{z}_2 = -3t^2 + \frac{11t^3}{3} - \frac{11t^4}{12}\,, \tag{2.15}$$

to obtain

$$z_2(t) = -\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360}\,. \tag{2.16}$$

Thus the expanded solution is

$$z(t) = t - \frac{t^2}{2} + \epsilon\left(\frac{t^3}{3} - \frac{t^4}{12}\right) + \epsilon^2\left(-\frac{t^4}{4} + \frac{11t^5}{60} - \frac{11t^6}{360}\right) + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.17}$$

We assume that the time aloft, $\tau(\epsilon)$, also has a perturbation expansion

$$\tau(\epsilon) = \tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2 + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.18}$$

The terms in this expansion are determined by solving:

$$z_0\left(\tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2\right) + \epsilon z_1\left(\tau_0 + \epsilon\tau_1\right) + \epsilon^2 z_2\left(\tau_0\right) = \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.19}$$

We have ruthlessly ditched all terms of order ϵ^3 into the garbage heap on the right of (2.19). The left side is a polynomial of order τ^6 so there are six roots. One of these roots is $\tau = 0$ and another root is close to $\tau = 2$. The other four roots are artificial creatures of the perturbation expansion and should be ignored — if we want the time aloft then we focus on the root near $\tau = 2$ by taking $\tau_0 = 2$ in (2.18). Expanding the z_n's in a Taylor series about $\tau_0 = 2$, we have:

$$z_0(2) + (\epsilon\tau_1 + \epsilon^2\tau_2)\dot{z}_0(2) + \tfrac{1}{2}(\epsilon\tau_1)^2\ddot{z}_0(2) + \epsilon z_1(2) + \epsilon^2\tau_1\dot{z}_1(2) + \epsilon^2 z_2(2) = \mathrm{ord}\left(\epsilon^3\right)\,. \tag{2.20}$$

Now we can match up powers of ϵ:

$$\begin{aligned} z_0(2) &= 0\,,\\ \tau_1\dot{z}_0(2) + z_1(2) &= 0\,,\\ \tau_2\dot{z}_0(2) + \tfrac{1}{2}\tau_1^2\ddot{z}_0(2) + \tau_1\dot{z}_1(2) + z_2(2) &= 0\,. \end{aligned}$$

Solving[1] these equations, one finds

$$\tau = 2 + \frac{4}{3}\epsilon + \frac{4}{5}\epsilon^2 + O(\epsilon^3)\,.$$

The Taylor series above is another procedure for generating the expansion of a regularly perturbed root of a polynomial.

Attempted solution of the projectile problem by iteration

We're considering the differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{1}{(1+\epsilon z)^2} = 0\,, \tag{2.21}$$

again. Our first iterate is

$$z^{(0)} = t - \frac{t^2}{2}\,, \tag{2.22}$$

[1]Some intermediate results $\dot{z}_0(2) = -1$, $z_1(2) = 4/3$, $\dot{z}_1(2) = 4/3$ and $z_2(2) = -4/45$.

which is the same as the first term in the earlier RPS. To obtain the next iterate, $z^{(1)}(t)$, we try to solve

$$\frac{\mathrm{d}^2 z^{(1)}}{\mathrm{d}t^2} + \frac{1}{\left(1 + \epsilon\left(t - \frac{t^2}{2}\right)\right)^2} = 0\,, \tag{2.23}$$

with the initial condition

$$z^{(1)} = 0\,, \qquad \dot{z}^{(1)}(0) = 1\,. \tag{2.24}$$

We could assault this problem with Mathematica or Maple:

```
DSolve[{z''[t] + 1/(1 + (t - t^2/2))^2 == 0 , z[0] == 0 , z'[0] == 1}, z[t], t]
```

However the answer is not presentable in polite company. In this example, the RPS back in (2.17) is definitely superior to iteration.

2.2 A boundary value problem: belligerent drunks

Imagine a continuum of drunks random-walking along a stretch of sidewalk, the x-axis, that lies between bars at $x = 0$ and $x = \ell$. When a drunk collides with another drunk they have a certain probability of mutual destruction: a fight breaks out that may result in the death of one or both participants. We desire the density of drunks on the stretch of sidewalk between $x = 0$ and $x = \ell$. The mathematical description of this problem is based on the density (drunks per meter) $u(x,t)$, which is governed by the partial differential equation

$$u_t = \kappa u_{xx} - \mu u^2\,, \tag{2.25}$$

with boundary conditions at the bars

$$u(0,t) = u(\ell,t) = U\,. \tag{2.26}$$

We're modeling the bars using a Dirichlet boundary condition — there is constant density at each bar and drunks spill out onto the sidewalk. The parameter μ models the lethality of the interaction between pairs of drunks.

Exercise: How would the formulation change if the drunks are not belligerent? They peacefully ignore each other. But instead, drunks have a constant probability per unit time of dropping dead. How does the formulation of the continuum model change? (In this case you might prefer to think of $u(x,t)$ as the concentration of a radioactive element, rather than drunken walkers.)

If we integrate (2.25) from $x = 0$ to ℓ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell u\,\mathrm{d}x = \left[\kappa u_x\right]_0^\ell - \int_0^\ell \mu u^2\,\mathrm{d}x\,. \tag{2.27}$$

You should be able to interpret each term in this budget.

First order of business is to non-dimensionalize the problem. How many control parameters are there? With the definitions

$$\bar{t} \stackrel{\text{def}}{=} \frac{\kappa t}{\ell^2}\,, \qquad \bar{x} \stackrel{\text{def}}{=} \frac{x}{\ell} - \frac{1}{2}\,, \qquad \text{and} \qquad \bar{u} \stackrel{\text{def}}{=} \frac{u}{U}\,, \tag{2.28}$$

we quickly find that the scaled boundary value problem is

$$u_t = u_{xx} - \beta u^2\,, \qquad \text{with BCs} \qquad u(\pm 1/2) = 1\,. \tag{2.29}$$

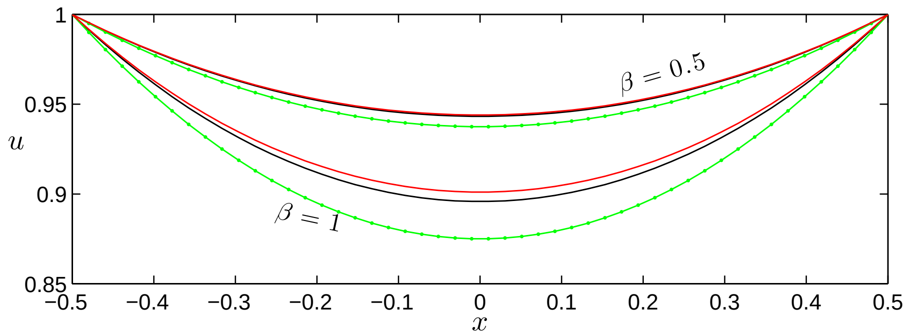

Figure 2.1: Comparison of the perturbation solution for $u(x, \beta)$ with numerical solution by `bvp4c`. The numerical solution is the black curve. The approximation $1 + \beta u_1$ is green, and $1 + \beta u_1 + \beta^2 u_2$ is red.

There is a single control parameter

$$\beta \stackrel{\text{def}}{=} \frac{\ell^2 \mu U}{\kappa} . \tag{2.30}$$

We made an aesthetic decision to put the boundaries at $x = \pm 1/2$.

Looking for a steady solution ($u_t = 0$) to the partial differential equation, we are lead to consider the nonlinear boundary value problem

$$u_{xx} = \beta u^2 , \qquad \text{with BCs} \qquad u(\pm 1) = 1. \tag{2.31}$$

The weakly interacting limit $\beta \ll 1$

If $\beta \ll 1$ — the weakly interacting limit — we can use an RPS

$$u = u_0(x) + \beta u_1(x) + \cdots \tag{2.32}$$

The leading-order problem

$$u_{0xx} = 0 , \quad \text{with BCs} \quad u_0(\pm 1/2) = 1 , \tag{2.33}$$

and solution

$$u_0(x) = 1 . \tag{2.34}$$

At subsequent orders, the BCs are homogeneous. For example, the first-order problem is

$$u_{1xx} = \underbrace{u_0^2}_{=1} , \qquad \text{with BCs} \qquad u_1(\pm 1/2) = 0 . \tag{2.35}$$

The solution is

$$u_1(x) = \frac{4x^2 - 1}{8} . \tag{2.36}$$

At second order, β^2,

$$u_{2xx} = 2u_0 u_1 = x^2 - \frac{1}{4} , \qquad \text{with} \qquad u_2(\pm 1/2) = 0 , \tag{2.37}$$

and solution

$$u_2(x) = \frac{x^4}{12} - \frac{x^2}{8} + \frac{5}{192} = \frac{(4x^2 - 1)(4x^2 - 5)}{192} . \tag{2.38}$$

The concentration at the center of the domain is

$$u(0,\beta) = 1 - \frac{\beta}{8} + \frac{5\beta^2}{192} + \mathrm{ord}(\beta^3)\,. \tag{2.39}$$

Figure 2.1 compares the perturbation solution with a numerical solution obtained using the MATLAB routine `bvp4c`. The three-term expansion is not bad, even at $\beta = 1$. We resist the temptation to compute u_3.......

At every step of the perturbation hierarchy we are inverting the linear operator $\mathrm{d}^2/\mathrm{d}x^2$ with homogeneous boundary conditions. You should recognize that all the regular perturbation problems we've seen have this structure. There is a general result, called the implicit function theorem, which assures us that if we know how to solve these reduced linear problems, with invertible linear operators, then the original problem has a solution for some sufficiently small value of the expansion parameter (β in the problem above).

2.3 Failure of RPS — examples of singular perturbation problems

Let's close by giving a few examples of differential equation which do not obligingly yield to regular perturbation methods.

Boundary layers

First, consider the boundary value problem (2.31) with $\beta = \epsilon^{-1} \gg 1$. In terms of ϵ, the problem is

$$\epsilon u_{xx} = u^2\,, \qquad \text{with BCs} \qquad u(\pm 1) = 1. \tag{2.40}$$

We try the RPS

$$u = u_0(x) + \epsilon u_1(x) + \cdots \tag{2.41}$$

The leading order is

$$0 = u_0^2\,, \qquad \text{with BCs} \qquad u_0(\pm 1) = 1\,. \tag{2.42}$$

Immediately we see that there is no solution to the leading-order problem.

What's gone wrong? Let's consider a linear problem with the same issues:

$$\epsilon v_{xx} = v\,, \qquad \text{with BCs} \qquad v(\pm 1) = 1\,. \tag{2.43}$$

Again the RPS fails because the leading-order problem,

$$0 = v_0\,, \qquad \text{with BCs} \qquad v_0(\pm 1) = 1\,, \tag{2.44}$$

has no solution. The advantage of a linear example is that we can exhibit the exact solution:

$$v = \frac{\cosh(x/\sqrt{\epsilon})}{\cosh(1/\sqrt{\epsilon})}\,, \tag{2.45}$$

see figure **??**. The exact solution has *boundary layers* near $x = -1$ and $x = +1$. In these regions v varies rapidly so that the term ϵv_{xx} in (2.43) is not small relative to v. Note that the leading order interior solution, $v_0 = 0$ is a good approximation to the correct solution *outside the boundary layers*. In this interior region the exact solution is exponentially small e.g.,

$$v(0,\epsilon) = \frac{1}{\cosh(1/\sqrt{\epsilon})} \sim 2\mathrm{e}^{-1/\sqrt{\epsilon}} \quad \text{as } \epsilon \to 0. \tag{2.46}$$

Our attempted RPS is using ϵ^n as gauge functions and as $\epsilon \to 0$

$$2\mathrm{e}^{-1/\sqrt{\epsilon}} = O(\epsilon^n)\,, \quad \text{for all } n \geq 0. \tag{2.47}$$

As far as the ϵ^n gauge is concerned, $\mathrm{e}^{-1/\sqrt{\epsilon}}$ is indistinguishable from zero.

The problem in both the examples above is that the small parameter ϵ multiplies the term with the most derivatives. Thus the leading-order problem in the RPS is of lower order than the exact problem. In fact, in the examples above, the leading-order problem is not even a differential equation.

Rapid oscillations

Another linear problem that defeats a regular perturbation expansion is

$$\epsilon w_{xx} = -w\,, \qquad \text{with BCs} \qquad w(\pm 1) = 1\,. \tag{2.48}$$

The exact solution, shown in figure **??**, is

$$w = \frac{\cos(x/\sqrt{\epsilon})}{\cos(1/\sqrt{\epsilon})}\,. \tag{2.49}$$

In this case the solution is rapidly varying throughout the domain. The term ϵw_{xx} is *never* smaller than w.

Secular errors

Let's consider a more subtle problem:

$$\ddot{x} + (1+\epsilon)x = 0\,, \qquad \text{with ICs} \quad x(0) = 1\,, \quad \text{and} \quad \dot{x}(0) = 0\,. \tag{2.50}$$

The exact solution of this oscillator problem is

$$x(t,\epsilon) = \cos\left(\sqrt{1+\epsilon}\, t\right)\,. \tag{2.51}$$

In this case it looks like the RPS

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{2.52}$$

might work. The leading-order problem is

$$\ddot{x}_0 + x_0 = 0\,, \qquad \text{with ICs} \quad x_0(0) = 1\,, \quad \text{and} \quad \dot{x}_0(0) = 0\,, \tag{2.53}$$

with solution

$$x_0 = \cos t\,. \tag{2.54}$$

In fact, this RPS does work for some time — see figure **??**. But eventually the exact solution (2.51) and the leading-order approximation in (2.54) have different signs. That's a bad error if $x_0(t)$ is a clock.

Maybe we can improve the approximation by calculating the next term? The order ϵ^1 problem is

$$\ddot{x}_1 + x_1 = -\cos t\,, \qquad \text{with homogeneous ICs} \quad x_1(0) = 0\,, \quad \text{and} \quad \dot{x}_1(0) = 0\,. \tag{2.55}$$

I hope you recognize a resonantly forced oscillator when you see it: the solution of (2.55) is

$$x_1 = -\tfrac{1}{2} t \sin t\,. \tag{2.56}$$

Thus the perturbation solution is now

$$x = \cos t - \epsilon \tfrac{1}{2} t \, \sin t + \mathrm{ord}(\epsilon^2) \, . \tag{2.57}$$

This first-order "correction" makes matters worse — see figure **??**. The RPS in (2.57) is "disordered" once $\epsilon t = \mathrm{ord}(1)$: we don't expect an RPS to work if the higher order terms are larger than the earlier terms. Clearly there is a problem with this direct perturbative solution of an elementary problem.

In this example the term ϵx is small relative to the other two terms in differential equation at all time. Yet the small error slowly accumulates over long times $\sim \epsilon^{-1}$. Astronomers call this a *secular* error[2]. We did not face secular errors in the projectile problem because we were solving the differential equation only for the time aloft, which was always much less than $1/\epsilon$.

[2]From Latin *saecula*, meaning a long period of time. Saecula saeculorum is translated literally as "in a century of centuries", or more poetically as "forever and ever", or "world without end".

2.4 Problems

Problem 2.1. *(i)* Consider the projectile problem with linear drag:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \mu \frac{\mathrm{d}z}{\mathrm{d}t} = -g_0 \,, \tag{2.58}$$

and the initial conditions $z(0) = 0$ and $\mathrm{d}z/\mathrm{d}t = u$. Find the solution with no drag, $\mu = 0$, and calculate the time aloft, τ. *(ii)* Suppose that the drag is small — make this precise by non-dimensionalizing the equation of motion and exhibiting the relevant small parameter ϵ. *(iii)* Use a regular perturbation expansion to determine the first correction to τ associated with non-zero drag. *(iv)* Integrate the non-dimensional differential equation exactly and obtain a transcendental equation for $\tau(\epsilon)$. Asymptotically solve this transcendental equation approximately in the limits $\epsilon \to 0$ and $\epsilon \to \infty$. Make sure the $\epsilon \to 0$ solution agrees with the earlier RPS.

Problem 2.2. Consider the projectile problem with quadratic drag:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \nu \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| \frac{\mathrm{d}z}{\mathrm{d}t} = -g_0 \,, \tag{2.59}$$

and the initial conditions $z(0) = 0$ and $\mathrm{d}z/\mathrm{d}t = u$. *(i)* Explain why the absolute value $|\dot{z}|$ in (2.59) is necessary if this term is to model air resistance. *(ii)* What are the dimensions of the coefficient ν? Nondimensionalize the problem so there is only one control parameter. *(iii)* Suppose that ν is small. Use a regular perturbation expansion to determine the first correction to the time aloft. *(iv)* Solve the nonlinear problem exactly and obtain a transcendental equation for the time aloft. (This is complicated.)

Problem 2.3. *(i)* Solve the problem

$$\ddot{x} + \left(1 + \epsilon \mathrm{e}^{\alpha t}\right) x = 0 \,, \qquad \text{with IC} \qquad x(0, \epsilon, \alpha) = 1 \,, \qquad \dot{x}(0, \epsilon, \alpha) = 0 \,, \tag{2.60}$$

with the RPS

$$x(t, \epsilon, \alpha) = x_0(t, \alpha) + \epsilon x_1(t, \alpha) + \cdots \tag{2.61}$$

Calculate x_0 and x_1. *(ii)* Bearing in mind that α might be positive or negative, discuss the utility of the RPS when t is large.

Problem 2.4. Consider a partial differential equation analog to the boundary value problem in (2.31). The domain is the disc $r = \sqrt{x^2 + y^2} < a$ in the (x, y)-plane and the problem is

$$u_{xx} + u_{yy} = \alpha u^2 \,, \qquad \text{with BC:} \qquad u(a, \theta) = U \,. \tag{2.62}$$

Following the discussion in section 2.2, compute three terms in the RPS.

Problem 2.5. Let's make a small change to the formulation of the belligerent-drunks example in (2.25) and (2.26). Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and ℓ: the boundary condition in (2.26) is changed to

$$\kappa u_x(0, t) = -F \,, \qquad \text{and} \qquad \kappa u_x(\ell, t) = F \,, \tag{2.63}$$

where F, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat *all calculations* in section 2.2, including the analog of the $\beta \ll 1$ perturbation expansion. You'll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.

Problem 2.6. First read the section entitled **Boundary layers**. Inspired by the example in that section, find an approximate solution of the boundary value problem:

$$10^{-12} v_{xx} = \mathrm{e}^{x^4}\, v \,, \qquad \text{with BCs} \qquad v(\pm 1) = 1 \,. \tag{2.64}$$

If you can do this, you'll be on on your way to understanding boundary layer theory.

Problem 2.7. Consider the non-dimensional oscillator problem

$$\ddot{x} + \beta \dot{x} + x = 0 \tag{2.65}$$

with the initial conditions

$$x(0) = 0 \,, \qquad \text{and} \qquad \dot{x}(0) = 1 \,. \tag{2.66}$$

(i) Supposing that $\beta > 2$, solve the problem exactly. *(ii)* Show that if $\beta \gg 1$ then the long-time behaviour of your exact solution is

$$x \propto \mathrm{e}^{-t/\beta} \,, \tag{2.67}$$

i.e., the displacement very slowly decays to zero. *(iii)* Motivated by this exact solution, "rescale" the problem (and the initial condition) by defining the slow time

$$\tau \stackrel{\text{def}}{=} \frac{t}{\beta} \,, \tag{2.68}$$

and $X(\tau) = ?x(t)$. Show that with a suitable choice of ?, the rescaled problem is

$$\epsilon X_{\tau\tau} + X_\tau + X = 0 \qquad \text{with the IC:} \quad X(0) = 0 \,, \quad X_\tau(0) = 1 \,. \tag{2.69}$$

Make sure you give the definition of $X(\tau)$ and $\epsilon \ll 1$ in terms of the parameter $\beta \gg 1$ and the original variable $x(t)$. *(iv)* Try to solve the rescaled problem (2.69) using an RPS

$$X(\tau, \epsilon) = X_0(\tau) + \epsilon X_1(\tau) + \cdots \tag{2.70}$$

Discuss the miserable failure of this approach by analyzing the dependence of the exact solution from part *(i)* on β. That is, simplify the exact solution to deduce a useful $\beta \to \infty$ approximation, and explain why the RPS (2.70) cannot provide this useful approximation.

Lecture 3

Autonomous differential equations

This long lecture has too much material. But a lot of it is stuff you should have learnt in school e.g., how to solve the simple harmonic oscillator. What's not covered in lectures is assigned as reading.

3.1 The phase line

As an application of algebraic perturbation theory we'll discuss the "phase line" analysis of first-order autonomous differential equations. That is, equations of the form:

$$\dot{x} = f(x)\,. \tag{3.1}$$

These equations are separable: see chapter 14 of **RHB**, or chapter 1 of **BO**.

Separation of variables followed by integration often leads to an opaque solution with x given only as an implicit function of t. A typical example is

$$\dot{x} = \sin x \qquad \text{with initial condition} \qquad x(0) = x_0\,. \tag{3.2}$$

We can separate variables and integrate

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x'}{\sin x'}\,, \qquad \Rightarrow \qquad \tan\left(\frac{x}{2}\right) = \mathrm{e}^t \tan\left(\frac{x_0}{2}\right)\,. \tag{3.3}$$

You can check by substitution that the solution above satisfies the differential equation and the initial condition. Suppose the initial condition that $x(0) = 17\pi/4$. Can you use the solution in (3.3) to find $\lim_{t\to\infty} x(t)$? It's not so easy because the inverse of tan is multivalued.

Fortunately it is much simpler to analyze (3.1) on the *phase line*: see Figure 3.1. With this construction it is very easy to see that

$$x_0 = \frac{9}{4}\pi\,, \qquad \Rightarrow \qquad \lim_{t\to\infty} x(t) = 3\pi\,. \tag{3.4}$$

The solution of (3.2) moves monotonically along the x-axis and, in the case above, approaches the *fixed point* at $x = 3\pi$, where $\dot{x} = 0$.

If we consider (3.1) with a moderately complicated $f(x)$ given graphically — for example in Figure 3.2 — then we can predict the long-time behaviour of all initial conditions with no effort at all. The solutions either trek off to $+\infty$, or to $-\infty$, or evolve towards *fixed points* defined by $f(x) = 0$. Moreover the evolution of $x(t)$ is *monotonic*.

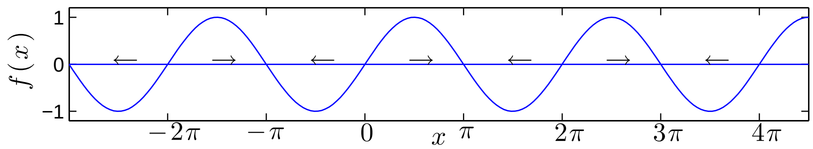

Figure 3.1: The arrows indicate the direction of the motion along the line produced by $\dot{x} = \sin x$.

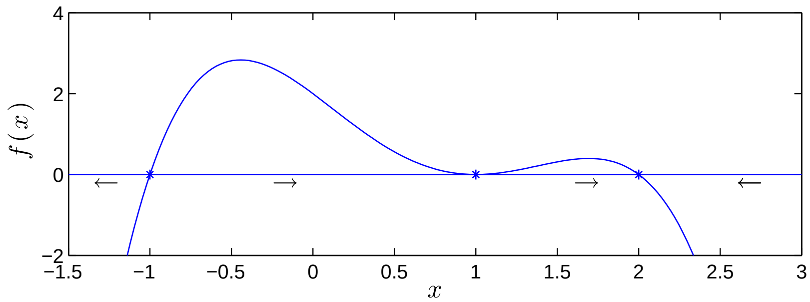

Figure 3.2: There are three fixed points indicated by the $*$'s on the x-axis. The fixed point at $x = -1$ is unstable and point at $x = 2$ is stable. The point at $x = 1$ is stable to negative displacements and unstable to positive displacements.

Example: Sketch the 1D vector field corresponding to

$$\dot{x} = \mathrm{e}^{-x^2} - x\,. \tag{3.5}$$

The main point of this example is that it is easier to separately draw the graphs of e^{-x^2} and x (rather than the difference of the two functions). This makes it clear that there is one stable fixed point at $x \approx 0.65$, and that this fixed point attracts all initial conditions.

3.2 Population growth — the logistic equation

Malthus (1798) in *An essay on the principle of population* argued that human populations increase according to

$$\dot{N} = rN\,. \tag{3.6}$$

If $r > 0$ then the population increases without bound. Verhulst (1838) argued that Malthusian growth must be limited by a nonlinear saturation mechanism, and proposed the simplest model of this saturation:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)\,. \tag{3.7}$$

A phase-line analysis of the Verhulst equation (3.7) quickly shows that for all $N(0) > 0$:

$$\lim_{t\to\infty} N(t) = K\,. \tag{3.8}$$

In ecology the Verhulst equation (3.7) is known as the r-K model; K is the "carrying capacity" and r is the growth rate. Yet another name for (3.7) is the "logistic equation".

To solve (3.7) we could use separation of variables, or alternatively we might recognize a Bernoulli equation[1]. For a change of pace, let's use the trick for solving Bernoulli equations:

[1] That is, an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + b(x)y^n\,.$$

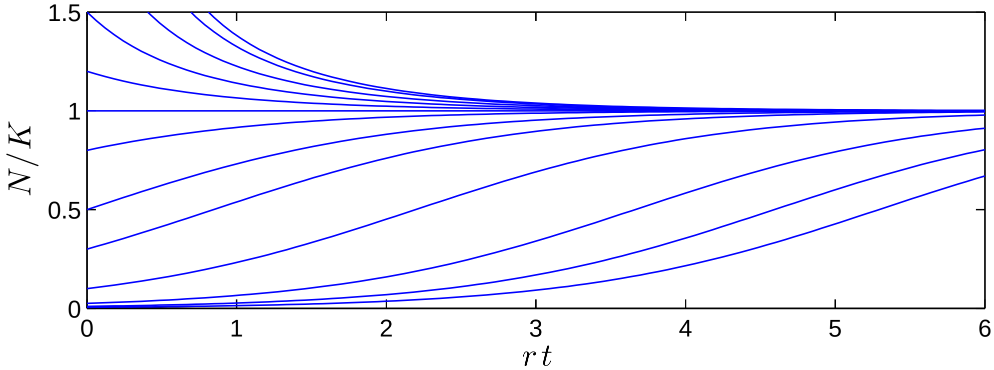

Figure 3.3: Solutions of the logistic equation (3.7). The curves which start with small $N(0)$ are S-shaped ("sigmoid"). Can you show that the inflection point ($\ddot{N}=0$) is at time t_* defined by $N(t_*)=K/2$?

divide (3.7) by $-N^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N} = -\frac{r}{N} + \frac{r}{K}\,. \tag{3.9}$$

This is a linear differential equation for $X \stackrel{\text{def}}{=} 1/N$, with integrating factor e^{rt}, and solution

$$N(t) = \frac{N_0 K}{(K-N_0)\mathrm{e}^{-rt} + N_0}\,. \tag{3.10}$$

Above, the initial condition is $N_0 = N(0)$. This solution with various values of N_0/K produces the "sigmoid curves" shown in Figure 3.3.

The logistic equation is notable because the exact solution is *not* an opaque implicit formula like (3.3) — the solution in (3.10) exhibits N as an explicit function of t. This is one of the few cases in which the explicit solution is useful.

Exercise: *(i)* Solve the logistic equation by separation of variables. *(ii)* Show that the population is increasing most rapidly when $N=K/2$. (Hint: only a very small calculation is required in *(ii)*.)

3.3 The phase plane

A two-dimensional autonomous system has the form

$$\dot{x} = u(x,y)\,, \qquad\qquad \dot{y} = v(x,y)\,. \tag{3.11}$$

(The dot indicates a time derivative.) The phase plane, (x,y), is the two-dimensional analog of the phase line. The state of the system at some time t_0 is specified by giving the location of a point (x,y) and at every point there is an arrow indicating the instantaneous direction in which the system moves. The collection of all these arrows is a "quiver". The set of arrows is also called a *direction field*, but `quiver` is the relevant MATLAB command.

The simplest example is the harmonic oscillator

$$\ddot{x} + x = 0\,. \tag{3.12}$$

We begin by writing this second-order equation as a system with the form in (3.11):

$$\dot{x} = y \qquad \dot{y} = -x\,. \tag{3.13}$$

Thus at each point in the (x, y)-plane there is a velocity vector,

$$\boldsymbol{q} = y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}\,, \tag{3.14}$$

and in a small time δt the system moves along this vector through a distance $\delta t\boldsymbol{q}$ to the next point in the plane. Thus the system moves along an *orbit* in the phase plane; the vector $\boldsymbol{q}$ is tangent to every point on the orbit.

The harmonic oscillator example is so simple that you should be able to draw the sketch vector field without the aid of MATLAB. The orbits are just circles centered on the origin,

$$x^2 + y^2 = \tfrac{1}{2}E\,, \tag{3.15}$$

where the energy of the oscillator, E, is constant.

Here is a list of things we should do with this example, and with other phase-plane differential equations

1. Locate the fixed points;
2. Perform a linear stability analysis of the fixed points;
3. Admire the "nullclines": $\dot{x} = 0$ or $\dot{y} = 0$.
4. Calculate the divergence of the two-dimensional phase fluid i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$.

This linear example is too simple to illustrate the power of this technique so let's move on.

Example: As a slightly more complicated example of phase-plane analysis we consider the Volterra equations:

$$\dot{r} = \underbrace{r - fr}_{u}\,, \qquad \dot{f} = \underbrace{-f + fr}_{v}\,. \tag{3.16}$$

This is a simple "predator-prey" model in which $f(t)$ is the population of foxes and $r(t)$ is the population of rabbits. In the absence of foxes, and with unlimited grass, the rabbit population grows exponentially. The growth of the fox population requires rabbits, else foxes starve.

The state of the system is specified by giving the location of a point in the *phase plane* (r, f). In this example the arrow at the point (r, f) is

$$\boldsymbol{q}(r, f) \stackrel{\text{def}}{=} (r - fr)\hat{\boldsymbol{r}} + (-f + fr)\hat{\boldsymbol{f}}\,. \tag{3.17}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{f}}$ are unit vectors along the rabbit axis and the fox axis respectively. The collection of all solutions is visualized as a collection of phase-plane orbits, with the vector $\boldsymbol{q}$ tangent to every point of the orbit. Figure 3.4 shows three phase-space orbits, and the associated quiver.

We can easily locate the fixed points. There are just two:

$$(r, f) = (0, 0)\,, \qquad \text{and} \qquad (r, f) = (1, 1)\,. \tag{3.18}$$

First consider the linear stability analysis of the fixed point at the origin. We're interested in small displacements away from the origin, so we simply drop the nonlinear terms in (3.16) to obtain the *associated linear system*

$$\dot{r} = r\,, \qquad \dot{f} = -f\,. \tag{3.19}$$

The solution is

$$r = r_0 \mathrm{e}^{t} \qquad f = f_0 \mathrm{e}^{-t}\,. \tag{3.20}$$

The origin is an unstable fixed point: a small rabbit population grows exponentially (e.g., the invasion of Australia by 24 rabbits released in 1859). Moreover, we can eliminate t between r and f in (3.20) to obtain

$$rf = r_0 f_0\,. \tag{3.21}$$

Thus near the origin the phase space orbits are hyperbolas. This type of fixed point, with exponential-time-time growth in one direction and exponential-in-time decay in another direction, is called a *saddle point*, or an x-point.

Now turn to the fixed point at $(1,1)$. To look at displacements from $(1,1)$ we introduce new variables (a,b) defined by

$$r = 1 + a\,, \qquad f = 1 + b\,. \tag{3.22}$$

In this simple example we can rewrite the system exactly in terms of (a,b):

$$\dot{a} = -b - ab\,, \qquad \dot{b} = a + ab\,. \tag{3.23}$$

Neglecting the quadratic term ab, the associated linear system is

$$\dot{a} = -b\,, \qquad \dot{b} = a \tag{3.24}$$

We could solve (3.24) by eliminating b or a to obtain

$$\ddot{a} + a = 0\,, \qquad \text{or} \qquad \ddot{b} + b = 0\,. \tag{3.25}$$

The general solution is a linear combination of $\cos t$ and $\sin t$, and the constants of integration are determined by the initial conditions (a_0, b_0). This is simple, but there is an alternative based on a trick that will come in handy later: introduce

$$z \stackrel{\text{def}}{=} a + \mathrm{i}b\,. \tag{3.26}$$

With this "complexification", the system (3.24) is

$$\dot{z} = \mathrm{i}z \qquad \text{with solution} \qquad \underbrace{a + \mathrm{i}b}_{z} = \underbrace{(a_0 + \mathrm{i}b_0)}_{z_0}\,\underbrace{(\cos t + \mathrm{i}\sin t)}_{\mathrm{e}^{\mathrm{i}t}}\,. \tag{3.27}$$

Notice that

$$|z|^2 = a^2 + b^2 = a_0^2 + b_0^2\,. \tag{3.28}$$

Thus, according to the linear approximation[2], if the system is slightly displaced from the fixed point $(1,1)$ it simply orbits around at a fixed distance from $(1,1)$ — this type of fixed point is called a *center* or an *o*-point.

[2]In this case we have to be concerned that the neglected nonlinear terms have a long-term impact e.g., the radius of the circle could grow slowly as a result of weak nonlinearity.

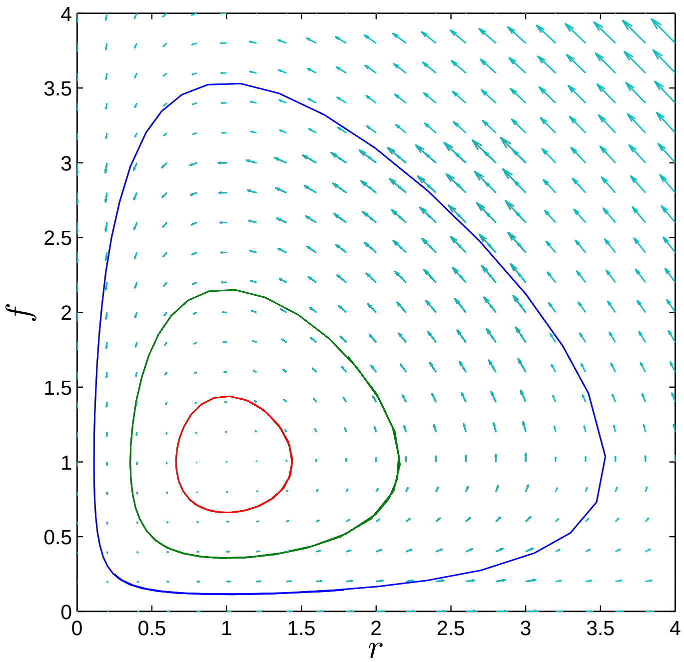

Figure 3.4: Three solutions of the Volterra system (3.16). The vector field is tangent to these phase space orbits. The MATLAB code is the box below. Notice the kinks in the blue trajectory.

```
function foxRabbit
%phase portrait of the Volterra predator-prey system
tspan = [0  10];
aZero = [ 0.25, 0.25 ]; bZero = [ 0.5, 0.5 ]; cZero = [ 0.75, 0.75 ];
[ta, xa] = ode45(@dfr,tspan,aZero);
[tb, xb] = ode45(@dfr,tspan,bZero);
[tc, xc] = ode45(@dfr,tspan,cZero);
plot(xa(:,1), xa(:,2),xb(:,1),xb(:,2),xc(:,1),xc(:,2))
axis equal
hold on
xlabel('$r$','interpreter','latex','fontsize',20)
ylabel('$f$','interpreter','latex','fontsize',20)
axis([0 4 0 4])
% now the quiver
[R F]= meshgrid(0:0.2:4);
U = R - F.*R;
V = -F + F.*R;
quiver(R,F,U,V)
%------- nested function --------%
    function dxdt = dfr(t,x)
    dxdt = [ x(1) - x(1)*x(2); - x(2)+x(1)*x(2)];
    end
end
```

3.4 Matlab ODE tools

The MATLAB code `foxRabbit` that produces figure 3.4 is shown in the associated verbatim box. The code is encapsulated as a function `foxRabbit`, with neither input nor output arguments. This construction enables the function `dfr` — which is called by `ode45` with the handle `@dfr` — to be included inline. The command `axis equal` is used so that circles look like circles.

One problem with figure 3.4 is that solution curves are not smooth. There are kinks in the biggest orbit — the one that corresponds to initial condition `a`. The problem is that `ode45` aggressively uses large time steps if possible. The command

$$[\mathtt{ta}, \mathtt{xa}] = \mathtt{ode45}(@\mathtt{dfr}, \mathtt{tspan}, \mathtt{aZero})$$

outputs the solution at times determined by the internal logic of `ode45` and those times are too coarsely spaced to make a smooth plot of the solution.

To get a smooth solution curve, at closely spaced times controlled by you, rather than by `ode45`, there are several modifications of the script, indicated in the code `smoothFoxRabbit` in the verbatim box below figure 3.5. First, create a vector that contains the desired output times:

$$\mathtt{t} = \mathtt{linspace}(0, \mathtt{max}(\mathtt{tspan}), 200).$$

Next, `ode45` is called with a single output argument:

$$\mathtt{sola} = \mathtt{ode45}(@\mathtt{dfr}, \mathtt{tspan}, \mathtt{aZero}, \mathtt{options});$$

This creates a MATLAB structure, called `sola` in this example. The structure `sola` contains all the information required to interpolate the solution between the times determined by `ode45`. The MATLAB function `deval` performs that interpolation. We access the solution at the times specified in `t` via the command `xa = deval(sola,t)`. This creates a matrix `xa` with two columns and `length(t)` rows. The first column is the `dfr` rabbit variable, `x(1)`, and the second column of `xa` is the foxes `x(2)`.

Note that in the upper panel Figure 3.5 the the rotation of ordinate label created by `ylabel` is set to zero. More importantly perhaps, the tolerances for `ode45` are set with the MATLAB command `odeset`. The command

$$\mathtt{options} = \mathtt{odeset}('\mathtt{AbsTol}', 1\mathtt{e} - 7, '\mathtt{RelTol}', 1\mathtt{e} - 4);$$

creates a MATLAB structure called `options`. `ode45` will accept this structure as an optional input argument. I must confess that I don't understand how these tolerances work. You'll note that if you use the default tolerances then the phase space orbit computed by `smoothFoxRabbit` doesn't close. This is a numerical error: the orbits really are closed — see problem 3.10. When I saw this problem I decreased the tolerances using `odeset` and the picture improved. This adventure shows that numerical solutions are not the same as exact solutions.

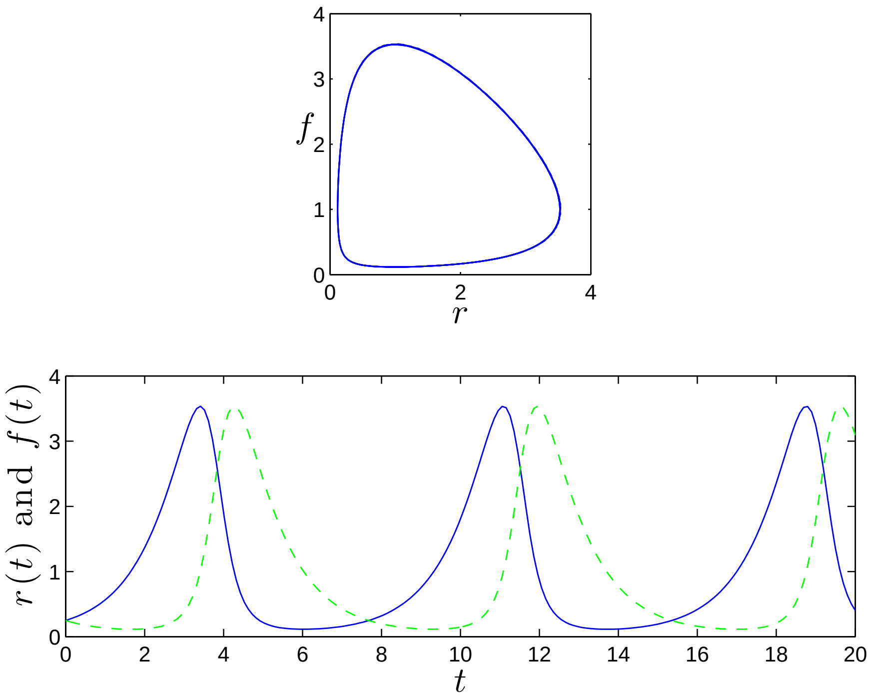

Figure 3.5: Another version of figure 3.4. The trajectory in the top panel is evaluated at densely sampled times so that the plot is smoother than in figure 3.4. The lower panel shows the two populations as functions of time. Which is the fox and which is the rabbit?

```
function smoothFoxRabbit
%phase portrait of the Volterra predator-prey system
tspan = [0  20];    t = linspace(0,max(tspan),200);
options = odeset('AbsTol',1e-7, 'RelTol',1e-4);
aZero = [ 0.25, 0.25 ];
sola = ode45(@dfr,tspan,aZero,options);
xa = deval(sola,t);
subplot(2,1,1)
plot(xa(1,:), xa(2,:))
axis equal
hold on
xlabel('$r$','interpreter','latex','fontsize',20)
ylabel('$f$','interpreter','latex','fontsize',20,'rotation',0)
axis([0 4 0 4])
subplot(2,1,2)
plot(t,xa(1,:),t,xa(2,:),'g--')
xlabel('$t$','interpreter','latex','fontsize',20)
ylabel('$r(t)$ and $f(t)$','interpreter','latex','fontsize',20)

%------- nested function --------%
    function dxdt = dfr(t,x)
    dxdt = [ x(1) - x(1)*x(2); - x(2)+x(1)*x(2)];
    end
end
```

3.5 The linear oscillator

Consider the damped and forced oscillator equation,

$$m\ddot{x} + \alpha\dot{x} + kx = f\,, \tag{3.29}$$

with an initial condition such as

$$x(0) = x_0\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = u_0\,. \tag{3.30}$$

You can think of this as the mass-spring system in the Figure 3.6, with damping provided by low-Reynolds number air resistance so that the drag is linearly proportional to the velocity.

We can obtain the energy equation if we multiply (3.29) by $\dot{x}$ and write the result as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\right) = -\alpha\dot{x}^2 + \dot{x}f\,. \tag{3.31}$$

This expresses the rate of change of energy as the difference between the rate at which the force f does work, $\dot{x}f$, and the dissipation of energy by drag $-\alpha\dot{x}^2$.

Resonance

Begin by considering an harmonically forced oscillator with no damping:

$$\ddot{x} + \omega^2 x = \cos\sigma t\,. \tag{3.32}$$

Suppose that the oscillator is at rest at $t = 0$:

$$x(0) = 0\,, \qquad \dot{x}(0) = 0\,. \tag{3.33}$$

The solution is

$$x = \frac{\cos\sigma t - \cos\omega t}{\omega^2 - \sigma^2}\,. \tag{3.34}$$

We can check this answer by taking $t \to 0$, and showing that

$$x \to \frac{t^2}{2} \tag{3.35}$$

both by expanding the solution in (3.34) or by identifying a small-t dominant balance between two of the three terms in (3.32).

There is a problem if the oscillator is resonantly forced i.e., if the forcing frequency σ is equal to the natural frequency ω. Then the solution is

$$x(t) = \lim_{\omega\to\sigma} \frac{\cos\sigma t - \cos\omega t}{\omega^2 - \sigma^2} = \frac{t}{2\sigma}\sin\omega t\,. \tag{3.36}$$

(You can use l'Hôpital's rule to evaluate the limit.) If the oscillator is resonantly forced, then the displacement grows linearly with time. We'll use this basic result later many times in the sequel.

Exercise: Solve the initial value problem

$$\ddot{x} + \omega^2 x = \sin\sigma t\,, \qquad x(0) = \dot{x}(0) = 0\,. \tag{3.37}$$

What happens if $\omega = \sigma$?

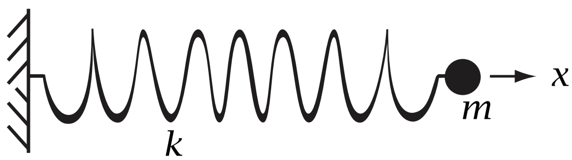

Figure 3.6: A mass-spring oscillator. The spring constant is k, and the heavy particle at the end of the spring has mass m so that the "natural frequency" of the oscillator is $\sqrt{k/m}$.

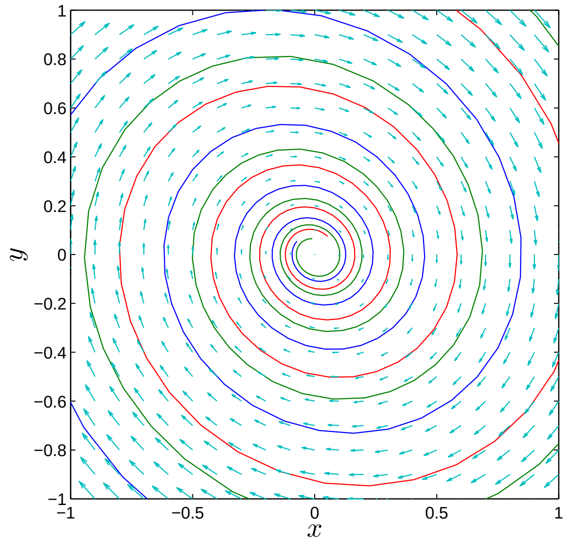

Figure 3.7: Three solutions of the damped oscillator equation (3.40) with $\beta = 0.2$.

An initial value problem for a damped oscillator

Now consider an unforced oscillator ($f = 0$) with initial conditions $x(0) = 0$ and $\dot{x}(0) = u_0$. The "natural" frequency of the undamped ($\alpha = 0$) and unforced ($f = 0$) oscillator is

$$\omega \stackrel{\text{def}}{=} \sqrt{\frac{k}{m}}\,. \tag{3.38}$$

This suggests a non-dimensionalization

$$\bar{t} \stackrel{\text{def}}{=} \omega t\,, \qquad \text{and} \qquad \bar{x} = \frac{u_0}{\omega} x\,. \tag{3.39}$$

The scaled problem is

$$\frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\bar{t}^2} + \underbrace{\frac{\alpha}{m\omega}}_{\stackrel{\text{def}}{=}\beta} \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} + \bar{x} = 0\,, \tag{3.40}$$

with initial conditions

$$\bar{x}(0) = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}}(0) = 1\,. \tag{3.41}$$

We've also taken $x_0 = 0$ so that there is a single non-dimensional control parameter, β. We proceed dropping the bars.

If $\beta < 2$, then the exact solution of the initial value problem posed above is

$$x = \nu^{-1}\mathrm{e}^{-\beta t/2}\sin\nu t\,, \qquad \text{with} \qquad \nu \stackrel{\text{def}}{=} \sqrt{1 - \frac{\beta^2}{4}}\,. \tag{3.42}$$

Figure 3.7 shows the phase-space portrait of the damped oscillator. Because of damping, all trajectories spiral into the origin. If the damping is weak the spiral is wound tightly i.e., it takes many periods for the energy to decay to half of its initial value.

The main effect of small damping is to reduce the amplitude of the oscillation exponentially in time, with an e-folding time $2/\beta$. Damping also slightly shifts the frequency of the oscillation:

$$\nu = 1 - \frac{\beta^2}{8} + \mathrm{ord}\left(\beta^4\right) . \tag{3.43}$$

The frequency shift is only important once $\beta^2 t \sim 1$, and on that long time the amplitude of the residual oscillation is exponentially small ($\sim \mathrm{e}^{-1/2\beta}$). So we don't worry too much about the frequency shift. A good $\beta \ll 1$ approximation to the exact solution in (3.42) is

$$x \approx \mathrm{e}^{-\beta t/2} \sin t \,. \tag{3.44}$$

Exercise: When does the approximation in (3.44) first differ in sign from the exact $x(t)$?

The method of averaging

If $\beta = 0$ then the solution of the oscillator equation (3.40) is

$$x = a \sin(t + \chi) \,, \tag{3.45}$$

where the amplitude a and the phase χ are constants set by initial conditions. The undamped oscillator conserves energy

$$E \stackrel{\text{def}}{=} \underbrace{\tfrac{1}{2}\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\tfrac{1}{2}x^2}_{\text{potential energy}} . \tag{3.46}$$

Moreover, E is "equipartitioned" between kinetic and potential. Thus, averaged over a cycle,

$$E = \langle \dot{x}^2 \rangle = \langle x^2 \rangle = \frac{a^2}{2} \,. \tag{3.47}$$

These results are exact if $\beta = 0$.

But because of dissipation the energy decays:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\beta \dot{x}^2 \,. \tag{3.48}$$

If β is non-zero, but small, then we might guess that the solution has the form in (3.45) except that the amplitude a is slowly deceasing i.e., a is a function of *slow time*. We also assume that the phase χ is slowly changing. Thus

$$\dot{x} = a\cos(t + \chi) + \underbrace{\dot{a}\sin(t + \chi) + \dot{\chi} a \cos(t + \chi)}_{\text{small}} . \tag{3.49}$$

Because only a little energy is lost in each cycle we can average (3.45) to obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\beta \langle \dot{x}^2 \rangle \,, \tag{3.50}$$

$$= -\beta \langle E \rangle \,. \tag{3.51}$$

Thus

$$E = E_0 \mathrm{e}^{-\beta t} \,. \tag{3.52}$$

The amplitude therefore varies as $a = a_0 \mathrm{e}^{-\beta t/2}$, which is in agreement with the exact solution (3.44). This argument does not determine the evolution of the phase χ.

3.6 Nonlinear oscillators

The nonlinear oscillator equation for $x(t)$ is

$$\ddot{x} = -U_x\,, \tag{3.53}$$

where $U(x)$ is the potential. The linear oscillator is the special case $U = \omega^2 x/2$.

We can obtain a good characterization of the solutions of (3.53) using conservation of energy: multiply (3.53) by $\dot{x}$ and integrate to obtain

$$\tfrac{1}{2}\dot{x}^2 + U(x) = E\,, \tag{3.54}$$

where the constant energy E is determined by the initial condition

$$E = \left[\tfrac{1}{2}\dot{x}^2 + U(x)\right|_{@t=0}. \tag{3.55}$$

Let's consider the mass-spring system in Figure 3.6 as an example. Suppose that the spring gets stronger as the extension x increases. We can model this "stiff" spring by adding nonlinear terms to Hooke's law:

$$\text{spring force} = -k_1 x - k_3 x^3 + \cdots \tag{3.56}$$

where the $\cdots$ indicate the possible presence of additional terms as the displacement x increases further. If the spring is stiff then $k_3 > 0$ i.e., the first non-Hookean term increases the restoring force above Hooke's law.

Note that in (3.56) are assuming that the force depends symmetrically on the displacement x i.e., the series in (3.56) contains only odd terms. Don't worry too much about that assumption — the problems offer plenty of scope to investigate asymmetric restoring forces.

The equation of motion of the mass m on a non-Hookean spring is therefore

$$m\ddot{x} = -k_1 x - k_3 x^3 + \cdots \tag{3.57}$$

This is equivalent to (3.54) with

$$U = \frac{k_1}{m}\frac{x^2}{2} + \frac{k_3}{m}\frac{x^4}{4} + \cdots \tag{3.58}$$

Now we can simply contour the energy E in the phase plane $(x, \dot{x})$. We don't need to draw the quiver of direction field arrows because we know that the orbits are confined to curves of constant energy. The arrows are tangent to the curves of constant E and you can easily visualize them if so inclined.

The Duffing oscillator

For example, suppose we truncate the series in (3.57) after the $k_3 x^3$. Then, after some scaling, we have the *Duffing oscillator*

$$\ddot{x} + x \pm x^3 = 0\,, \tag{3.59}$$

where $\pm$ depends on the sign of k_3. Please make sure you understand how all the coefficients have been normalized to either 1 or -1 without loss of generality.

The energy of the Duffing oscillator is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 \pm \tfrac{1}{4}x^4\,. \tag{3.60}$$

Figure 3.8 shows the curves of constant energy drawn with the MATLAB routines `meshgrid` and `contour`.

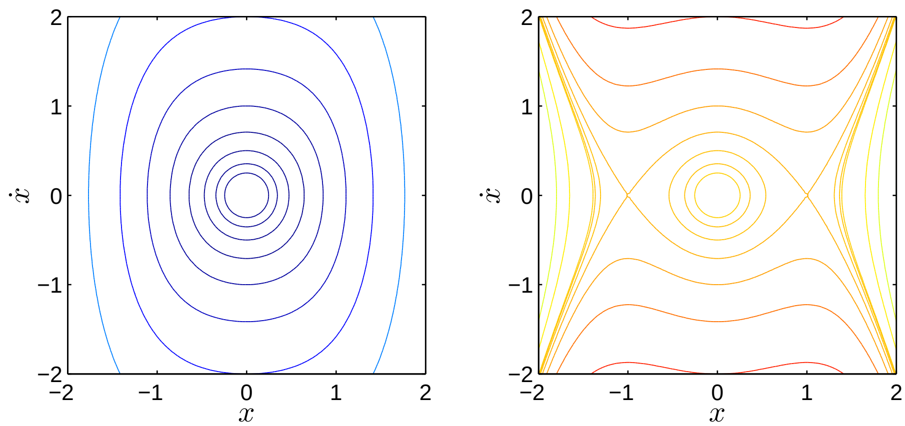

Figure 3.8: The phase plane of the Duffing oscillator. Can you tell which panel corresponds to the + sign in (3.60)? Does a low-energy solution orbit the origin in a clockwise or a counter clockwise direction?

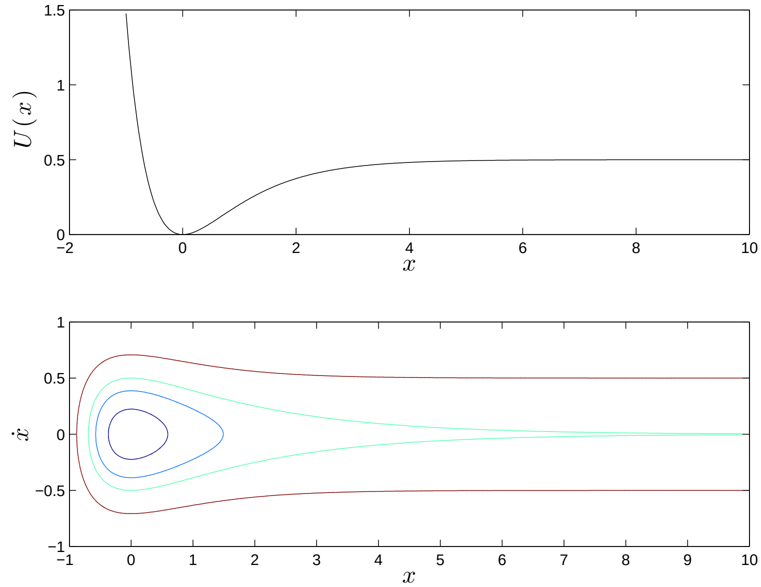

Figure 3.9: The top panel shows the Morse potential and the bottom panel shows three phase space trajectories corresponding to $E = 0.1$, 0.3, 0.5 and 1.

The Morse oscillator - turning points

Although it is easy to draw energy contours with MATLAB there is an educational construction that enables one to sketch the energy curves by hand. I'll explain this construction using the Morse potential,

$$U = \frac{1}{2}\left(1 - \mathrm{e}^{-x}\right)^2 , \tag{3.61}$$

as an example. The top panel of figure 3.9 shows the Morse potential and the bottom panel shows three phase space trajectories corresponding to $E = 0.1$, 0.3 and 1. The construction involves:

1. Drawing an energy level E in the top panel;
2. Locating the turning points, $x(E)$'s defined by $E = U(x)$, in the top panel;
3. Dropping down to the bottom panel, and locating the turning points in the phase plane;
4. Sketching the curve of constant energy E — keep in mind that it is symmetric about the x-axis.

This is best explained on a blackboard.

3.7 Problems

Problem 3.1. *(a)* Find $\lim_{t\to\infty} x(t)$, where $x(t)$ is the solution of

$$\dot{x} = (x-1)^2 - \frac{x^3}{100} , \qquad x(0) = 1 .$$

(b) Find the $t \to \infty$ limit if the initial condition is changed to $x(0) = 1.2$. In both cases give a numerical answer with two significant figures.

Problem 3.2. Consider

$$\dot{x} = x^p \,, \qquad \text{with initial condition } x(0) = 1. \tag{3.62}$$

If $p = 1$, the solution $x(t)$ grows exponentially and takes an infinite time to reach $x = \infty$. On the other hand, if $p = 2$, then $x(t)$ reaches ∞ in finite time. Draw a graph of the time to ∞ as a function of p.

Problem 3.3. Back in the day, students were taught to evaluate trigonometric integrals like (3.3) with the substitution $\theta = \tan x'/2$. Show that $\mathrm{d}x'/\sin x' = \mathrm{d}\theta/\theta$ and do the integral.

Problem 3.4. The velocity of a skydiver falling to the ground is given by

$$m\dot{v} = mg - kv^2 \,, \tag{3.63}$$

where m is the mass, $g = 32.2$ feet/(second)^2 is gravity and k is an empirical constant related to air resistance. *(a)* Obtain an analytic solution assuming that $v(0) = 0$. *(b)* Use your solution to find the terminal velocity in terms of m, g and k. *(c)* Check your answer by analyzing the problem on the phase line. *(d)* An experimental study with skydivers in 1942 was conducted by dropping men from $31,400$ feet to an altitude of $2,100$ feet at which point the skydivers opened their chutes. This long freefall took 116 seconds on average and the average weight of the men plus their equipment was 261.2 pounds. Calculate the average velocity. *(e)* Use the data above to estimate the terminal velocity and the drag constant k. A straightfoward approach requires solving a transcendental equation either graphically or numerically. But you can avoid this labor by making an approximation that the average velocity is close to the terminal velocity. If you do make this approximation, then you should check it carefully and identify the non-dimensional parameter that controls the validity of the approximation.

Problem 3.5. Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \,, \qquad \text{with} \qquad K = K_0 + K_1 \cos\omega t \,. \tag{3.64}$$

The initial condition is $N(0) = N_0$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $K_1/K_0 \ll 1$ and that $N(t) \approx K_0$. Find the periodic-in-time solution of the perturbed problem (e.g., see Figure 3.10). *(iii)* Discuss the phase lag between the population, $N(t)$, and the carrying capacity $K(t)$ e.g., in figure 3.10 which curve is the carrying capacity?

Problem 3.6. As a model of combustion triggered by a small perturbation, consider

$$\dot{x} = x^2(1 - x) \,, \qquad x(0) = \epsilon \,. \tag{3.65}$$

(i) Start with the simpler problem

$$\dot{y} = y^2 \,, \qquad y(0) = \epsilon \,. \tag{3.66}$$

Explain why problem (3.66) is a small-time approximation to problem (3.65). *(ii)* Use separation of variables to find the exact solution of (3.66) and show that $y(t)$ reaches ∞ in a finite time. Let's call this the "blow-up" time, $t_\star(\epsilon)$. Determine the function $t_\star(\epsilon)$. *(iii)* Use a phase-line analysis to show that the solution of (3.65) never reaches ∞ — in fact:

$$\lim_{t\to\infty} x(t;\epsilon) = 1 \,. \tag{3.67}$$

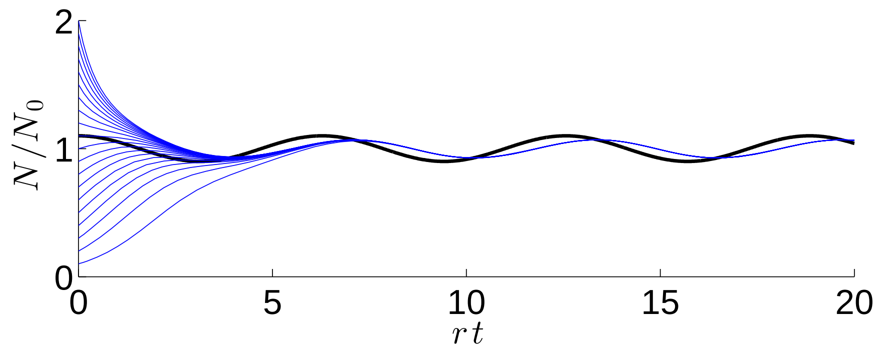

Figure 3.10: Numerical solution of (3.64) with various initial conditions. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity.

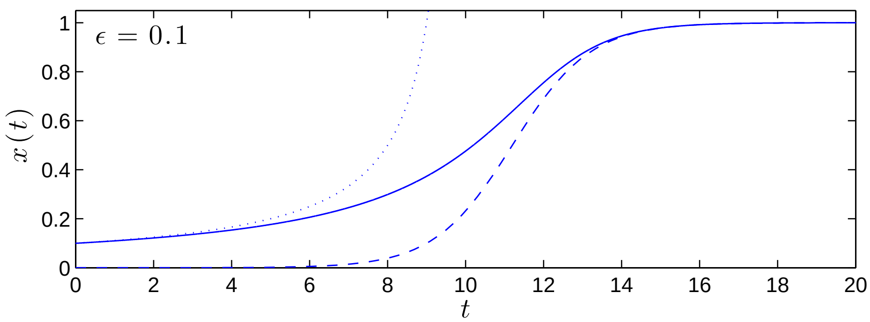

Figure 3.11: The exact solution of (3.65) (the solid curve) compared with large and small time approximations.

(iv) Use separation of variables to find the exact solution of (3.65); make sure your solution satisfies the initial condition. (I encourage you to do the integral with Mathematica or Maple.) *(v)* At large times $x(t, \epsilon)$, is somewhere close to 1. Simplify the exact solution from *(iv)* to obtain an explicit (i.e., exhibit x as a function of t) large-time solution. Make sure sure you explain how large t must be to ensure that this approximate solution is valid. *(vi)* Summarize your investigation with a figure such as 3.11.

Problem 3.7. Consider the differential equation

$$\dot{x} = r - x - \mathrm{e}^{-x} . \tag{3.68}$$

Sketch all the qualitatively different vector fields on the x- axis that occur as the parameter r is varied between $-\infty$ and $+\infty$. Show that something interesting happens as r passes through one. Suppose $r = 1 + \epsilon$, with $0 < \epsilon \ll 1$. Determine the location of the fixed points as a function of ϵ and decide their stability. Obtain an approximation to the differential equation (3.68), valid in the limit $\epsilon \to 0$ and $x = \mathrm{ord}(\sqrt{\epsilon})$. (Make sure you explain why $x = \mathrm{ord}(\sqrt{\epsilon})$ is interesting.)

Problem 3.8. Kermack & McKendrick [*Proc. Roy. Soc. A* **115** A, 700 (1927)] proposed a model for the evolution of an epidemic. The population is divided into three classes:

$$\begin{aligned} x(t) &= \text{number of healthy people}\,, \\ y(t) &= \text{number of infected people}\,, \\ z(t) &= \text{ number of dead people}\,. \end{aligned}$$

Assume that the epidemic evolves very rapidly so that slow changes due to births, emigration, and the 'background death rate', are negligible. (Kermack & McKendrick argue that bubonic plague is so virulent that this assumption is valid.) The other model assumptions are that healthy people get sick at a rate proportional to the product of x and y. This is plausible if healthy people and sick people encounter each other at a rate proportional to their numbers, and if there is a constant probability of transmission. Sick people die at a constant rate. Thus, the model is

$$\dot{x} = -\alpha x y, \qquad \dot{y} = \alpha x y - \beta y, \qquad \dot{z} = \beta y\,.$$

(i) Show that $N = x + y + z$ is constant. *(ii)* Use the $\dot{x}$ and $\dot{z}$ equations to express $x(t)$ in terms of $z(t)$. *(iii)* Show that $z(t)$ satisfies first order equation:

$$\dot{z} = \beta\left[N - z - x_0 \exp(-\alpha z/\beta)\right]$$

where $x_0 = x(0)$. Use non-dimensionalization to put the equation above into the form:

$$u_\tau = a - bu - e^{-u},$$

and show that $a \geq 1$ and $b > 0$. *(iv)* Determine the number of fixed points and decide their stability. *(v)* Show that if $b < 1$, then the death rate, $\dot{z} \propto u_\tau$, is increasing at $t = 0$ and reaches its maximum at some time $0 < t_* < \infty$. Show that the number of infectives, $y(t)$, reaches its maximum at the same time, t_*, that the death rate peaks. The term *epidemic* is reserved for this case in which things get worse before they get better. *(vi)* Show that if $b > 1$ then the maximum value of the death rate is at $t = 0$. Thus, there is no epidemic if $b > 1$. *(vii)* The condition that $b = 1$ is the threshold for the epidemic. Can you give a biological interpretation of this condition? That is, does the dependence of b on α, β and x_0 seem 'reasonable'?

Problem 3.9. How is the Kermack-McKendrick model modified if the infected people are flesh-eating zombies?

Problem 3.10. Integrate the system (3.4) and show that the closed orbits in Figure 3.4 are given by

$$r + f - \ln(rf) = \text{constant} \tag{3.69}$$

Problem 3.11. As a model of competition (for grass) between rabbits and sheep consider the autonomous system

$$\dot{r} = r(3 - r - 2s) \qquad \text{and} \qquad \dot{s} = s(2 - r - s)\,. \tag{3.70}$$

In the absence of one species, the population of the other species is governed by a logistic model. The competition is interesting because rabbits reproduce faster than sheep. But sheep can gently nudge rabbits out of the way, so the negative sheep-feedback is stronger on the rabbits than vice versa. Find the four fixed points of this system and analyze their stability. Compute some solutions and draw a phase-space figure analogous to Figure 3.4.

Problem 3.12. The red army, with strength $R(t)$, fights the green army, with strength $G(t)$. The conflict starts from an initial condition $G(0) = 2R(0)$ and proceeds according to

$$\dot{R} = -G\,, \qquad \dot{G} = -3R\,. \tag{3.71}$$

The war stops when one army is extinct. Which army wins, and how many soldiers are left at this time? (You can solve this problem without solving a differential equation.)

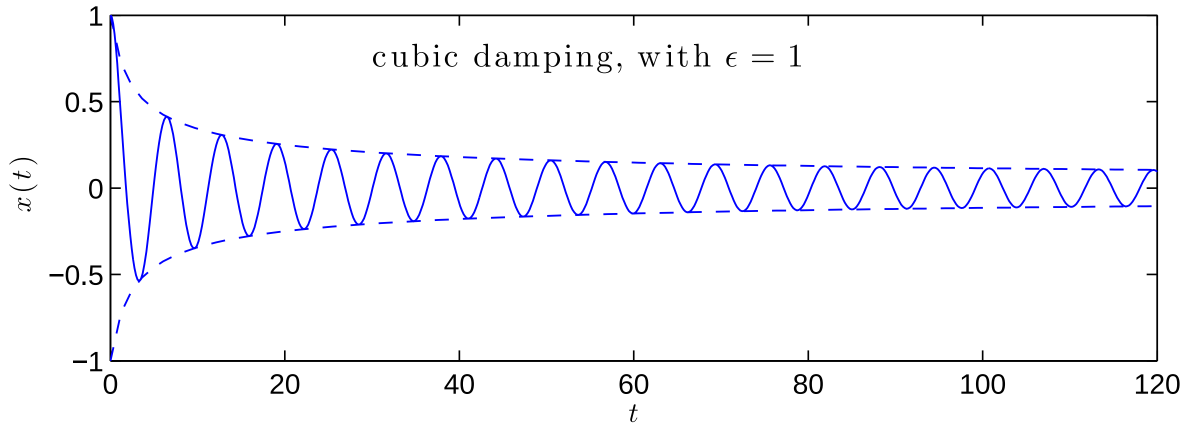

Figure 3.12: Solution of problem 3.15. The dashed curve is the envelope predicted by multiple scale theory and the solid curve is the `ode45` solution.

Problem 3.13. Consider the system

$$\dot{x} = -x + y^2 \,, \qquad \dot{y} = x - 2y + y^2 \tag{3.72}$$

Use MATLAB to compute a few orbits visualize the direction field. Locate the fixed points and analyze their stability. Sketch the orbits near the fixed points. Show that $x = y$ is an orbit and that $|x - y| \to 0$ as $t \to \infty$ for all other orbits.

Problem 3.14. A theoretically inclined vandal wants to break a steam radiator away from its foundation. She steadily applies a force of $F = 100$ Newtons and discovers that the top of the radiator is displaced by 2 cm. Unfortunately this is only one tenth of the displacement required. But the vandal can apply an unsteady force $f(t)$ according to the schedule

$$f(t) = \tfrac{1}{2}F(1 - \cos\omega t) \,, \qquad F = 100\text{N} \,.$$

The mass of the radiator is 50 kilograms and the foundation resists movement with a force proportional to displacement. At what frequency and for how long must the vandal exert the force above to succeed?

Problem 3.15. Consider the nonlinearly damped oscillator

$$\ddot{x} + \epsilon\dot{x}^3 + x = 0 \,, \quad \text{with ICs} \quad x(0) = 1 \,, \quad \dot{x}(0) = 0 \,. \tag{3.73}$$

Assuming that $\epsilon \ll 1$, use the energy equation and the method of averaging to determine the slow evolution of the amplitude a in the approximate solution (3.45). Take $\epsilon = 1$ and use `ode45` to compare a numerical solution of the cubically damped oscillator with the method of averaging (see Figure 3.12).

Problem 3.16. Consider a medium $-\ell < x < \ell$ in which the temperature $\theta(x,t)$ is determined by

$$\theta_t - \kappa\theta_{xx} = \alpha \mathrm{e}^{\beta\theta} \,, \tag{3.74}$$

with boundary conditions $\theta(\pm\ell, t) = 0$. The right hand side is a heat source due to an exothermic chemical reaction. The simple form in (3.74) is obtained by linearizing the Arrhenius law. The medium is cooled by the cold walls at $x = \pm\ell$. *(i)* Put the problem into the non-dimensional form

$$\Theta_T - \Theta_{XX} = \epsilon \mathrm{e}^{\Theta} \qquad \text{with BCs} \qquad \Theta(\pm 1, \epsilon) = 0 \,. \tag{3.75}$$

Your answer should include a definition of the dimensionless control parameter ϵ in terms of κ, α, β and ℓ. *(ii)* Assuming that $\epsilon \ll 1$, calculate the *steady* solution $\Theta(X,\epsilon)$ using a regular perturbation expansion. Obtain two or three non-zero terms and check your answer by showing that the "central temperature" is

$$
\begin{aligned}
C(\epsilon) &\stackrel{\text{def}}{=} \Theta(0,\epsilon)\,, && (3.76)\\
&= \frac{\epsilon}{2} + \frac{5\epsilon^2}{24} + \frac{47\epsilon^3}{360} + \operatorname{ord}\left(\epsilon^4\right)\,. && (3.77)
\end{aligned}
$$

(iii) Develop an approximate solution with iteration. *(iv)* Integrate the steady version of (3.75) exactly and deduce that:

$$
\underbrace{\mathrm{e}^{-C/2}\tanh^{-1}\sqrt{1-\mathrm{e}^{-C}}}_{\stackrel{\text{def}}{=}F(C)} = \sqrt{\frac{\epsilon}{2}}\,. \tag{3.78}
$$

(Use MATHEMATICA to do the integral.) Plot the function $F(C)$ and show that there is no steady solution if $\epsilon > 0.878$. *(v)* Based on the graph of $F(C)$, if $\epsilon < 0.878$ then there are *two* solutions. There is the "cold solution", calculated perturbatively in (3.77), and there is a second "hot solution" with a large central temperature. Find an asymptotic expression for the hot central temperature as $\epsilon \to 0$.

Problem 3.17. In this problem we use energy conservation to obtain a solution to the projectile problem which is superior to (2.17). (a) From the non-dimensional equation of motion (2.5), show that

$$
\tfrac{1}{2}\dot{z}^2 - \frac{1}{\epsilon}\frac{1}{1+\epsilon z} = \tfrac{1}{2} - \frac{1}{\epsilon}\,. \tag{3.79}
$$

(b) Find the maximum height reached by the projectile, $z_{\max}$, in terms of ϵ. (c) Show that the time aloft is given exactly by

$$
\tau = 2z_{\max}\int_0^1 \sqrt{\frac{1+a\xi}{1-\xi}}\,\mathrm{d}\xi\,, \qquad \text{with} \qquad a(\epsilon) \stackrel{\text{def}}{=} \frac{\epsilon}{2-\epsilon}\,. \tag{3.80}
$$

(d) Evaluate the integral above exactly. (e) Use MATHEMATICA or some other tool to obtain the $\epsilon \ll 1$ expansions

$$
\tau = \frac{4}{2-\epsilon}\left(1 + \frac{a}{3} - \frac{a^2}{15} + \frac{a^3}{35} - \frac{a^4}{63} + \frac{a^5}{99} - \frac{a^6}{143} + \operatorname{ord}\left(a^7\right)\right)\,. \tag{3.81}
$$

and

$$
\tau = 2 + \frac{4\epsilon}{3} + \frac{4\epsilon^2}{5} + \frac{16\epsilon^3}{35} + \frac{16\epsilon^4}{63} + \frac{32\epsilon^5}{231} + \frac{32\epsilon^6}{429} + \operatorname{ord}\left(\epsilon^7\right)\,. \tag{3.82}
$$

Which series is superior at $\epsilon = 0.5$?

Problem 3.18. In section 2.2 we encountered the concentration $u(x,\beta)$ defined by boundary value problem

$$
u_{xx} = \beta u^2\,, \qquad \text{with BCs} \qquad u(\pm 1/2) = 1\,. \tag{3.83}
$$

Let $c(\beta) \stackrel{\text{def}}{=} u(0,\beta)$ be the concentration at the center of the domain. *(i)* Show that

$$
\sqrt{\frac{2\beta c}{3}} = \int_1^{c^{-1}} \frac{\mathrm{d}u}{\sqrt{u^3-1}}\,. \tag{3.84}
$$

(ii) Show that the expression above for $c(\beta)$ reproduces our earlier $\beta \to 0$ perturbation series in (2.39). *(iii)* Find the leading order behaviour of c as $\beta \to \infty$.

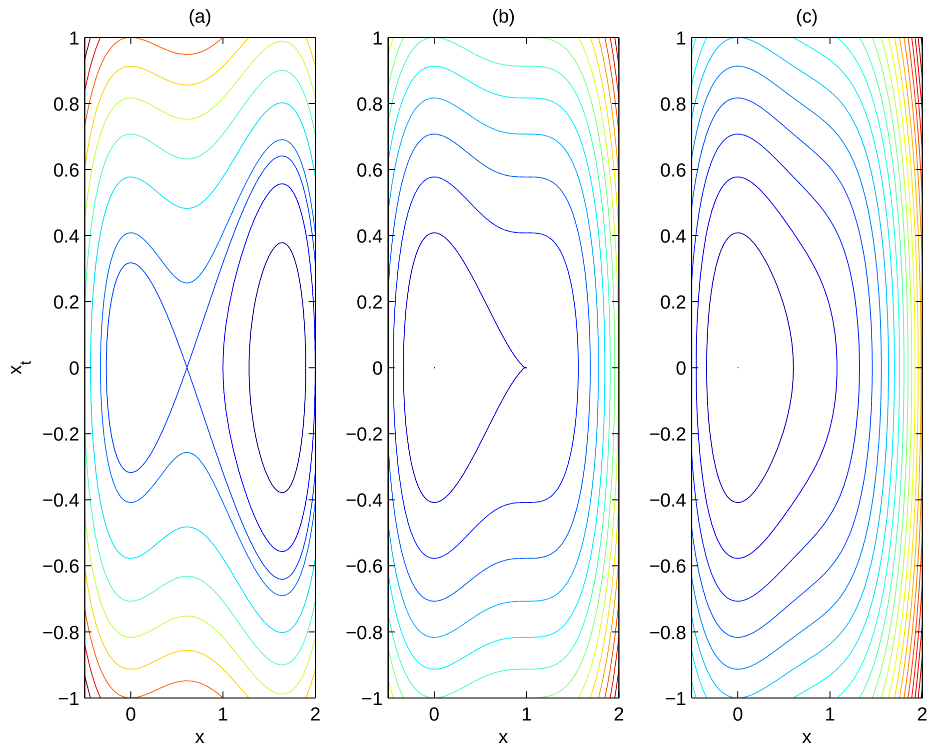

Figure 3.13: Which phase-plane corresponds to (3.85)?

Problem 3.19. The nonlinear oscillator

$$\ddot{x} + x - 2x^2 + x^3 = 0\,, \tag{3.85}$$

has an energy integral of the form

$$E = \tfrac{1}{2}\dot{x}^2 + V(x)\,. \tag{3.86}$$

(a) Find the potential function $V(x)$ and sketch this function on the range $-\frac{1}{2} < x < 2$. Label your axes so that your sketch of $V(x)$ is quantitative. *(b)* Figure 3.13 shows three possible phase plane diagrams. In ten or twenty words explain which diagram corresponds to the oscillator in (3.85).

Problem 3.20. The top panel in figure 3.14 shows a potential and the bottom panel shows four constant energy curves in the phase plane. Match the curves in the bottom panel to the indicated energy levels.

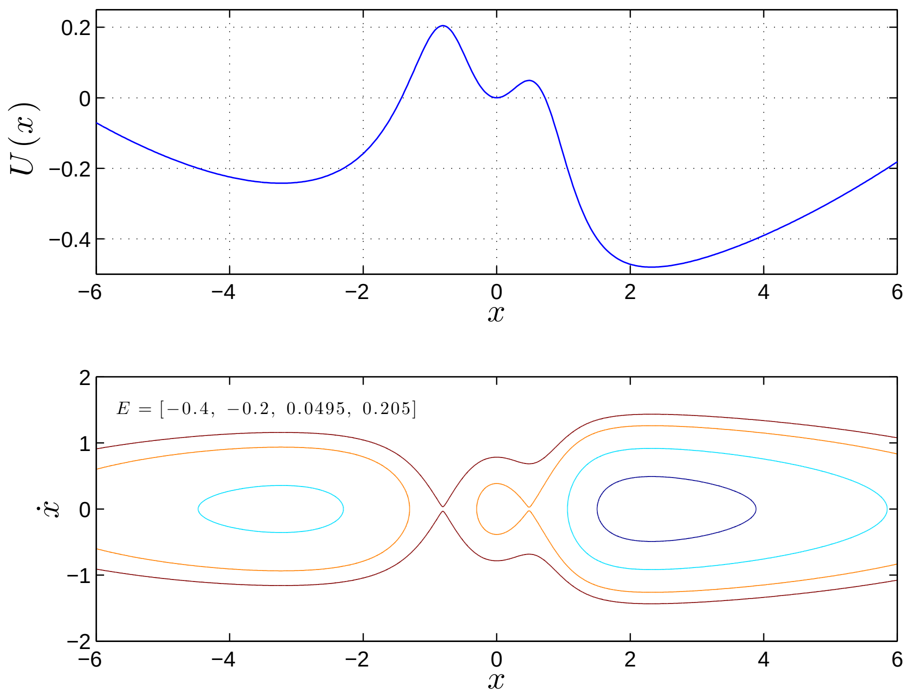

Figure 3.14: Match the curves to the energy level.

Lecture 4

Regular perturbation of partial differential equations

4.1 Potential flow round a slightly distorted cylinder

4.2 Gravitational field of a slightly distorted sphere

4.3 Problems

Problem 4.1. Consider Laplace's equation,

$$\phi_{xx} + \phi_{yy} = 0\,, \tag{4.1}$$

in a domain which is a periodic-in-x channel with walls at $y = \pm(1 + \epsilon \cos kx)$. The boundary condition on the walls is

$$(\boldsymbol{\nabla}\phi + \hat{\boldsymbol{\imath}})\boldsymbol{\cdot}\hat{\boldsymbol{n}} = 0\,, \tag{4.2}$$

where $\hat{\boldsymbol{n}}$ is the outward normal and $\hat{\boldsymbol{\imath}}$ is the unit vector in the x-direction. Obtain two terms in the expansion of

$$J(\epsilon) \stackrel{\text{def}}{=} \iint \phi_x \,\mathrm{d}x\mathrm{d}y\,. \tag{4.3}$$

Lecture 5

Boundary Layers

5.1 Stommel's dirt pile

Consider a pile of dirt formed by a rain of sediment falling onto a conveyor belt. The belt stretches between $x = 0$ and $x = \ell$ and moves to the left with speed $-c$: see the figure. If $h(x,t)$ denotes the height of a sandpile, then a very simple model is

$$h_t - c h_x = s + \kappa h_{xx}\,, \tag{5.1}$$

with boundary conditions

$$h(0,t) = 0\,, \qquad \text{and} \qquad h(\ell,t) = 0\,. \tag{5.2}$$

The term $s(x)$ on the right of (5.1) is the rate (meters per second) at which sand is accumulating on the belt.

We can make a sanity check by integrating (5.1) from $x = 0$ to $x = \ell$:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell h(x,t)\,\mathrm{d}x}_{\text{rate of accumulation}} = \underbrace{\int_0^\ell s(x,t)\,\mathrm{d}x}_{\text{sedimentation from above}} + \underbrace{\kappa h_x(\ell,t) - \kappa h_x(0,t)}_{\text{loss of dirt by falling over the edges}}\,. \tag{5.3}$$

Notice that the advective term, $c h_x$, does not contribute to the budget above — advection is moving dirt but because $h = 0$ at the boundaries advection is not directly contributing to the fall of dirt over the edges.

The steady solution with a uniform source

If the sedimentation rate, $s(x,t)$, is a constant then we can easily solve the steady state ($t \to \infty$) problem exactly:

$$h(x,\infty) = \frac{s\ell}{c}\frac{1 - \mathrm{e}^{-cx/\kappa}}{1 - \mathrm{e}^{-c\ell/\kappa}} - \frac{sx}{c}\,. \tag{5.4}$$

If the diffusion is very weak, meaning that

$$\epsilon \stackrel{\text{def}}{=} \frac{\kappa}{c\ell} \ll 1\,, \tag{5.5}$$

then there is a region of rapid variation, the *boundary layer*, at $x = 0$. This is where all the sand accumulated on the conveyor belt is pushed over the edge. Obviously if we reverse the direction of the belt, then the boundary layer will move to $x = \ell$. Note we're assuming that $c > 0$ so that $\epsilon > 0$.

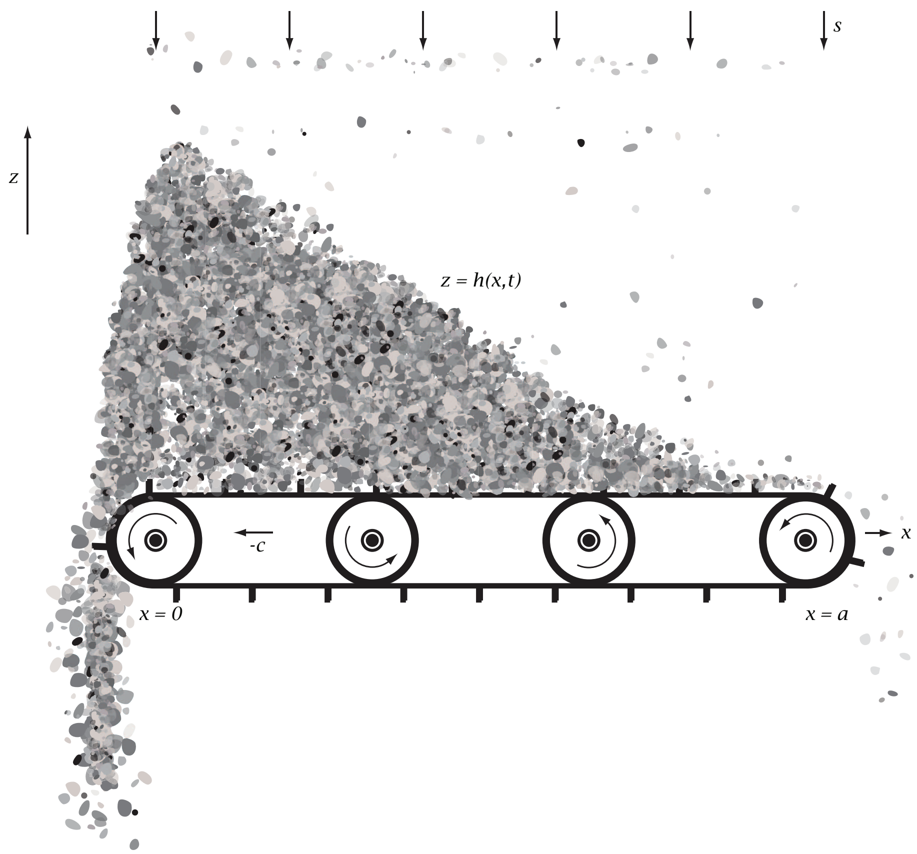

Figure 5.1: Stommel's boundary-layer problem.

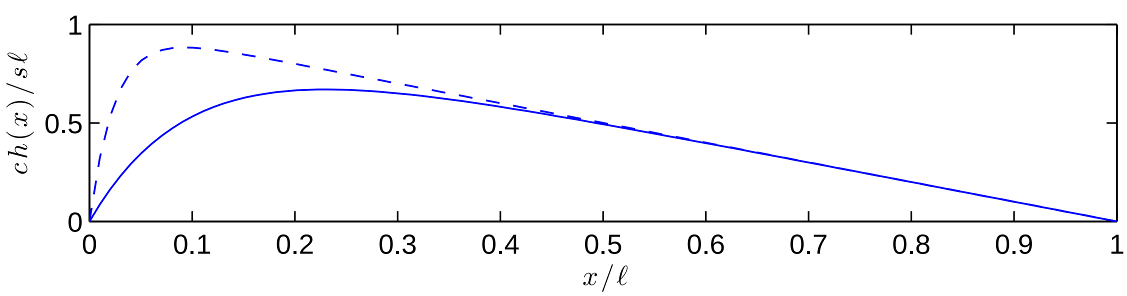

Figure 5.2: The solution in (5.7). The solid curve is $\epsilon = 0.1$ and the dashed curve is $\epsilon = 0.025$.

If we introduce non-dimensional variables

$$\bar{x} \stackrel{\text{def}}{=} \frac{x}{\ell}, \qquad \text{and} \qquad \bar{h} = \frac{ch}{s\ell}, \tag{5.6}$$

then the solution in (5.4) is

$$h(x,\epsilon) = \frac{1 - \mathrm{e}^{-x/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}} - x\,. \tag{5.7}$$

This solution is shown in figure 5.2. We can consider two different limiting processes

1. *The outer limit:* $\epsilon \to 0$, with x fixed. Under this limit, the exact solution in (5.7) is $h \to 1-x$. The outer limit produces a good approximation to the exact $h(x,\epsilon)$, except close to $x = 0$ where the boundary condition is not satisfied.

2. *The inner limit:* $\epsilon \to 0$ with $X \stackrel{\text{def}}{=} x/\epsilon$ fixed. Under this limit the exact solution in (5.7) is $h \to 1 - \mathrm{e}^{-X}$. The inner limit produces a good approximation to the solution within the *boundary layer*. This is a small region in which x is order ϵ. It is vital to understand that the term ϵh_{xx} is leading order within the boundary layer, and enables the solution to satisfy the boundary condition at $x = 0$.

Thus the function in (5.7) has two different asymptotic expansions. Each expansion is limited by non-uniformity as $\epsilon \to 0$.

5.2 Leading-order solution of the dirt-pile model

We want to take the inner and outer limits directly in the differential equation, *before* we have a solution. Understanding how to do this in the Stommel problem is one goal this lecture.

To make the problem a little more interesting, suppose that the sedimentation rate is some function of x:

$$s = s_{\max}\bar{s}\left(\frac{x}{\ell}\right). \tag{5.8}$$

We use $s_{\max}$ to define the non-dimensional $\bar{h}$ back in (5.6). Dropping the bars, the non-dimensional problem is

$$\epsilon h_{xx} + h_x = -s\,, \qquad \text{with BCs:} \qquad h(0) = h(1) = 0\,. \tag{5.9}$$

We're going to use boundary layer theory to obtain a quick and dirty leading-order solution of this problem. We'll return later to a more systematic discussion.

The outer expansion

Start the attack on (5.9) with a regular perturbation expansion

$$h(x,\epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + \cdots \tag{5.10}$$

We're assuming that as $\epsilon \to 0$ with fixed x — the **outer limit** — that the solution has the structure in (5.17). Note that in the outer limit the h_n's in (5.10) are independent of ϵ.

Exercise: Consider the special case $s = 1$, with the exact solution in (5.7). Does the outer limit of that exact solution agree with the assumption in (5.10)?

The leading order is

$$h_{0x} = -s\,, \tag{5.11}$$

and we can solve this problem as

$$\underbrace{h_0(x) = \int_x^1 s(x')\,\mathrm{d}x'}_{\text{correct}}, \qquad \text{or perhaps as} \qquad \underbrace{h_0(x) = -\int_0^x s(x')\,\mathrm{d}x'}_{\text{incorrect}}. \tag{5.12}$$

Looking at the exact solution we know that the correct choice satisfies the BC at $x = 1$. We proceed with this inner solution and return later to show why the alternative ends in tears.

The inner expansion, and a quick-and-dirty matching argument

Now we can also define

$$X \stackrel{\text{def}}{=} \frac{x}{\delta}, \qquad \text{so that} \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}X}. \tag{5.13}$$

δ is the boundary layer thickness — we're pretending that δ is unknown. Using the inner variable X, the problem (5.9) becomes is

$$\underbrace{\epsilon\delta^{-2}h_{XX} + \delta^{-1}h_X}_{\text{two term balance}} = -s(\delta X). \tag{5.14}$$

We get a nice two-term balance if

$$\delta = \epsilon. \tag{5.15}$$

With this definition of δ we have the rescaled problem

$$h_{XX} + h_X = -\epsilon s(\epsilon X). \tag{5.16}$$

Now attack (5.16) with a a regular perturbation expansion

$$h(x,\epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{5.17}$$

Notice that in (5.17) we're assuming that the H_n's are independent of ϵ.

At leading order

$$H_{0XX} + H_{0X} = 0, \qquad \text{with solution} \qquad H_0 = A_0\left(1 - \mathrm{e}^{-X}\right). \tag{5.18}$$

We've satisfied the BC at $X = 0$. But we still have an unknown constant A_0.

To determine A_0 we insist that "the inner limit of the outer solution is equal to the outer limit of the inner limit solution". This means that there is a region of overlap in which

$$\underbrace{A_0\left(1 - \mathrm{e}^{-X}\right)}_{\text{Inner solution}} \approx \underbrace{\int_x^1 s(x')\,\mathrm{d}x'}_{\text{Outer solution}}. \tag{5.19}$$

For instance, if $x = \mathrm{ord}(\epsilon^{1/2}) \ll 1$ then $X = \mathrm{ord}(\epsilon^{-1/2}) \gg 1$, and (5.19) tells us that

$$A_0 = \int_0^1 s(x')\,\mathrm{d}x'. \tag{5.20}$$

Construction of a uniformly valid solution

With A_0 determined by (5.20) we have completed the leading-order solution. We can combine our two asymptotic expansions into a single *uniformly valid* solutions using the recipe

$$\begin{aligned}\text{uniformly valid} &= \text{outer} + \text{inner} - \text{match}\,, &(5.21)\\ &= \int_x^1 s(x')\,\mathrm{d}x' + \int_0^1 s(x')\,\mathrm{d}x'\,\left(1-\mathrm{e}^{-X}\right) - \int_0^1 s(x')\,\mathrm{d}x'\,, &(5.22)\\ &= \int_x^1 s(x')\,\mathrm{d}x' - \int_0^1 s(x')\,\mathrm{d}x'\mathrm{e}^{-x/\epsilon}\,. &(5.23)\end{aligned}$$

This is also known as the *composite expansion.*

Why can't we have a boundary layer at $x = 1$?

Now we return to (5.12) and discuss what happens if we make the incorrect choice

$$h_0(x) \stackrel{?}{=} -\int_0^x s(x')\,\mathrm{d}x'\,. \tag{5.24}$$

This outer solution satisfies the BC at $x = 0$. So we try to put a boundary layer at $x = 1$. Again we introduce a boundary-layer coordinate:

$$X \stackrel{\text{def}}{=} \frac{x-1}{\delta}\,, \qquad \text{so that } \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}X}\,. \tag{5.25}$$

The dominant balance argument convinces us that $\delta = \epsilon$, and using (5.17) we find exactly the same leading-order solution as before:

$$H_0 = A_0\left(1-\mathrm{e}^{-X}\right)\,, \qquad \text{except that now} \qquad X = \frac{x-1}{\delta}\,. \tag{5.26}$$

$H_0(X)$ above satisfies the BC at $X = 0$, which is the same as $x = 1$. But now when we attempt to match the outer solution in (5.24) it all goes horribly wrong: we take the limit $X \to -\infty$ and the exponential explodes. It is impossible to match the outer solution (5.24) with the inner solution in (5.26).

5.3 Stommel's problem at infinite order

Example: To save chalk, in the lecture we use the particular source function $s = \mathrm{e}^{a(x-1)}$ and assign the general $s(x)$ as reading.

The problem is

$$\epsilon h_{xx} + h_x = -\mathrm{e}^{a(x-1)}\,, \qquad \text{with BCs} \qquad h(0) = h(1) = 0\,. \tag{5.27}$$

The interior solution, to infinite order, is

$$h = \underbrace{\left(1 + \epsilon a + \epsilon^2 a^2 + \epsilon^3 a^3 + \cdots\right)}_{\frac{1}{1-\epsilon a}} \frac{1}{a}\left[1 - \mathrm{e}^{a(x-1)}\right]\,. \tag{5.28}$$

No matter how many terms we calculate, we will never satisfy the $x = 0$ boundary condition.

To expose the complete structure of higher-order boundary-layer problems let us discuss the form of the high-order terms in Stommel's problem. Recall our model for the steady state sandpile is

$$\epsilon h_{xx} + h_x = -s\,. \tag{5.29}$$

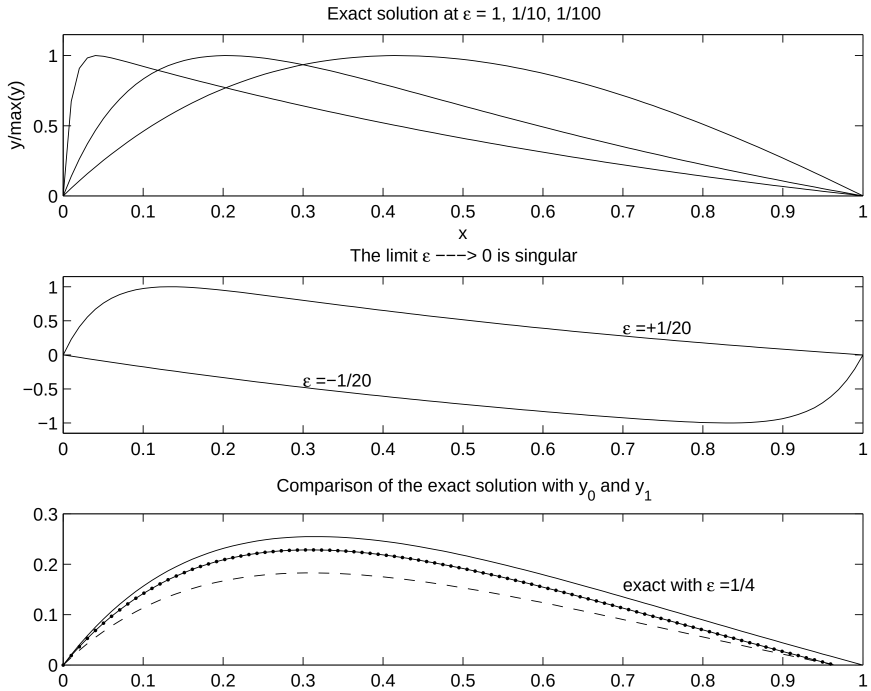

Figure 5.3: Solution with $a = -1$.

We assume that the source $s(x)$ has the Taylor series expansion around $x = 0$:

$$s(x) = s_0 + s_0' x + \tfrac{1}{2} x^2 s_0'' + \cdots , \tag{5.30}$$

and around $x = 1$:

$$s(x) = s_1 + (x-1)s_1' + \tfrac{1}{2}(x-1)^2 s_1'' + \cdots \tag{5.31}$$

The outer solution

The leading-order outer problem is

$$h_{0x} = -s\,, \qquad \Rightarrow \qquad h_0 = \int_x^1 s(x')\,\mathrm{d}x'\,, \tag{5.32}$$

and the following orders are

$$h_{1x} = -h_{0xx} = +s_x\,, \qquad \Rightarrow \qquad h_1 = s(x) - s_1\,, \tag{5.33}$$

$$h_{2x} = -h_{1xx} = -s_{xx}\,, \qquad \Rightarrow \qquad h_2 = s_1' - s_x(x)\,, \tag{5.34}$$

$$h_{3x} = -h_{2xx} = +s_{xxx}\,, \qquad \Rightarrow \qquad h_3 = s_{xx}(x) - s_1''\,. \tag{5.35}$$

Notice that $h_n(1) = 0$. It is clear how this series continues to higher order. We can assemble the first three terms of the outer solution as

$$h = \int_0^1 s(x')\,\mathrm{d}x' - \int_0^x s(x')\,\mathrm{d}x' + \epsilon\,[s(x) - s_1] + \epsilon^2\left[s_1' - s_x(x)\right] + O(\epsilon^3)\,. \tag{5.36}$$

The boundary-layer solution

In the boundary layer, we *must* expand the source in a Taylor series

$$s(\epsilon X) = s_0 + \epsilon X s_0' + \tfrac{1}{2}\epsilon^2 X^2 s_0'' + \cdots \tag{5.37}$$

If we don't expand the source then there is no way to collect powers of ϵ and maintain our assumption that the H_n's in

$$h(x,\epsilon) = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{5.38}$$

are independent of ϵ. The RPS above leads to

$$H_{0XX} + H_{0X} = 0\,, \qquad \Rightarrow \qquad H_0 = A_0\left(1 - \mathrm{e}^{-X}\right)\,, \tag{5.39}$$

$$H_{1XX} + H_{1X} = -s_0\,, \qquad \Rightarrow \qquad H_1 = A_1\left(1 - \mathrm{e}^{-X}\right) - s_0\left(X - 1 + \mathrm{e}^{-X}\right)\,, \tag{5.40}$$

$$H_{2XX} + H_{2X} = -s_0' X\,, \qquad \Rightarrow \qquad H_2 = A_2\left(1 - \mathrm{e}^{-X}\right) - s_0'\left(\tfrac{1}{2}X^2 - X + 1 - \mathrm{e}^{-X}\right)\,. \tag{5.41}$$

At every order we've satisfied the boundary condition $H_n(0) = 0$. Matching determines the constants A_n.

Matching

In the matching region $X \gg 1$ and we simplify the boundary layer solution by neglecting all the exponentially small terms involving e^{-X}. This gives

$$h \sim \underbrace{A_0}_{H_0} + \underbrace{\epsilon A_1 - \epsilon s_0 (X - 1)}_{\epsilon H_1} + \underbrace{\epsilon^2 A_2 - \epsilon^2 s_0'\left(\tfrac{1}{2}X^2 - X + 1\right)}_{\epsilon^2 H_2} + O(\epsilon^3)\,. \tag{5.42}$$

We rewrite the outer solution in terms of $X = x/\epsilon$ and take the inner limit, keeping terms of order ϵ^2:

$$h \sim \underbrace{\int_0^1 s(x')\,\mathrm{d}x' - \epsilon s_0 X - \tfrac{1}{2}\epsilon^2 s_0' X^2}_{h_0} + \epsilon \underbrace{\left[s_0 + \epsilon X s_0' - s_1\right]}_{h_2} + \epsilon^2 \underbrace{\left[s_1' - s_0'\right]}_{h_2} + O(\epsilon^3)\,. \tag{5.43}$$

The inner limit of $h_0(x)$ produces terms of all order in ϵ — above we've explicitly written only terms up to $\mathrm{ord}(\epsilon^2)$.

A shot-gun marriage between these different expansions (5.42) and (5.43) of the *same function* $h(x,\epsilon)$ implies that

$$A_0 = \int_0^1 s(x')\,\mathrm{d}x'\,, \qquad A_1 = -s_1\,, \qquad A_2 = s_1'\,. \tag{5.44}$$

All the other terms in (5.42) and (5.43) match. Notice that terms from h_0 match terms from H_1 and H_2, and from H_3 if we continue to higher order. It is interesting that the boundary layer constants A_1 and A_2 are determined by properties of the source at $x = 1$ i.e., the BL near $x = 0$ is affected by the source near $x = 1$.

The special case $s(x) = 1$

This special case is very simple: the infinite-order, the boundary-layer solution is

$$H = \underbrace{1 - \mathrm{e}^{-X}}_{H_0} - \epsilon \underbrace{X}_{H_1}\,. \tag{5.45}$$

And the infinite-order outer solution is simply

$$h = \underbrace{1 - x}_{h_0} . \tag{5.46}$$

All the higher-order terms are zero. With the recipe

$$\text{uniform} = \text{outer} + \text{inner} - \text{match} , \tag{5.47}$$

we assemble an infinite-order uniform approximation:

$$h_{\text{uni}}(x) = 1 - x - \mathrm{e}^{-x/\epsilon} . \tag{5.48}$$

Notice that the exact solution is

$$h(x) = \frac{1 - \mathrm{e}^{-x/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}} - x ; \tag{5.49}$$

this differs from the infinite-order approximation by the exponentially small $\mathrm{e}^{-1/\epsilon}$.

5.4 A nonlinear Stommel problem

Consider the Stommel model with nonlinear diffusivity:

$$\left(\tfrac{1}{2}h^2\right)_{xx} + h_x = -1 , \qquad h(0) = h(1) = 0 . \tag{5.50}$$

Where the pile is deeper there is more height for diffusion to move dirt around.

If we assume that the boundary layer is at $x = 0$ then easy calculations show that the leading-order interior solution is

$$h_0 = 1 - x , \tag{5.51}$$

and that the series continues as

$$h = \left(1 + \epsilon + 2\epsilon^2 + \cdots\right)(1 - x) . \tag{5.52}$$

This perturbation series indicates that there is a simple exact solution that satisfies the $x = 1$ boundary condition:

$$h = A(\epsilon)\,(1 - x) , \qquad \text{where} \qquad \epsilon A^2 - A + 1 = 0 . \tag{5.53}$$

This is pleasant, but it does not help with the boundary condition at $x = 0$.

Introducing the boundary layer variable

$$X \stackrel{\text{def}}{=} x/\epsilon , \tag{5.54}$$

we have the re-scaled equation

$$\left(\tfrac{1}{2}h^2\right)_{XX} + h_X = -\epsilon . \tag{5.55}$$

We try for a solution with $h = H_0(X) + \epsilon H_1(X) + \cdots$ The leading-order equation is

$$\left(\tfrac{1}{2}H_0^2\right)_{XX} + H_{0X} = 0 , \tag{5.56}$$

which integrates to

$$H_0 H_{0X} + H_0 = C . \tag{5.57}$$

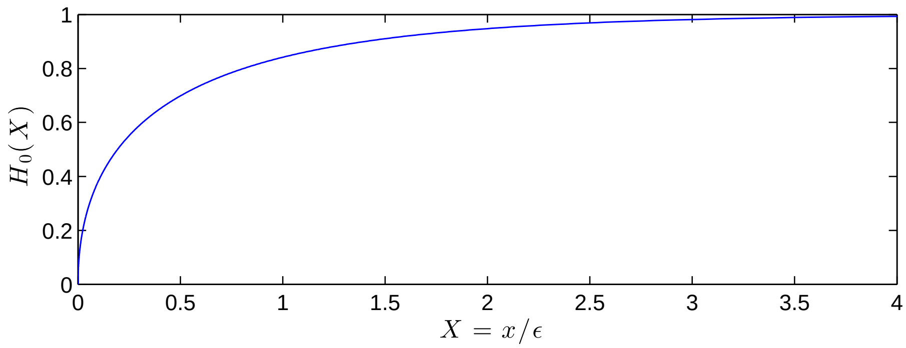

Figure 5.4: The boundary-layer solution in (5.63) of the nonlinear Stommel problem in (5.50).

This leading-order solution must satisfy both the $X = 0$ boundary condition and the matching condition

$$H_0(0) = 0 \,, \qquad \text{and} \qquad \lim_{X\to\infty} H(X) = 1 \,. \tag{5.58}$$

If we apply the $x = 0$ boundary condition to (5.57), and assume that

$$\lim_{X\to 0} H_0 H_{0X} \stackrel{?}{=} 0 \,, \tag{5.59}$$

then we conclude that $C = 0$. But $C = 0$ in (5.57) quickly leads to $H_0 = -X$. This satisfies the boundary condition at $x = 0$, but not the matching condition. We are forced to consider that the limit above is non-zero. In that case we can determine the constant C in (5.57) by matching to the interior. Thus $C = 1$ and

$$H_{0X} = \frac{1}{H_0} - 1 \,. \tag{5.60}$$

We solve (5.60) via separation of variables

$$\frac{H_0 \, \mathrm{d}H_0}{1 - H_0} = \mathrm{d}X \,, \tag{5.61}$$

integrating to

$$-H_0 + \ln \frac{1}{1 - H_0} = X + A \,. \tag{5.62}$$

Applying the boundary condition at $X = 0$ shows that $A = 0$, and thus $H_0(X)$ is determined implicitly by

$$H_0 = 1 - \mathrm{e}^{-X - H_0} \,. \tag{5.63}$$

This implicit solution is shown in figure 5.4. As $X \to \infty$ we use iteration to obtain the large-X behaviour of the boundary layer solution

$$H_0(X) \sim 1 - \mathrm{e}^{-X-1} + \mathrm{e}^{-2X-2} + \cdots \quad \text{as } X \to \infty \,. \tag{5.64}$$

This demonstrates matching to the leading-order interior solution.

Might we find another solution of (5.50) with a boundary layer at $x = 1$? The answer is **yes**: (5.50) has both the reflection symmetry

$$h \to -h \,, \qquad \text{and} \qquad x \to -x \,, \tag{5.65}$$

and the translation symmetry

$$x \to x + a \,. \tag{5.66}$$

Thus we can define $x_1 = x - 1/2$ so that the boundary conditions are applied at $x_1 = \pm 1/2$. The reflection symmetry then implies that if $h(x_1)$ is a solution then so is $-h(-x_1)$. With this trickery the solution we've just described is transformed into a perfectly acceptable solution but with a boundary layer at the other end of the domain.

Exercise: assume that the boundary layer is at $x = 1$, so that the leading-order outer solution is now $h_0 = -x$. Construct the boundary-layer solution using the inner variable $X = (x-1)/\epsilon$ — you'll be able to satisfy both the $x = 1$ boundary condition and match onto the inner limit of the outer solution. Notice that this solution has $h(x) \le 0$.

Reformulation of the nonlinear diffusion model

As a solution of the dirt-pile model the second solution above makes no sense: dirt piles can't have negative height. And the physical intuition that put the boundary layer at $x = 0$ can't be wrong simply because we use a more complicated model of diffusion. The problem is that the nonlinear diffusion equation in (5.50) should be

$$\left(\tfrac{1}{2}|h|h\right)_{xx} + h_x = -1\,, \qquad h(0) = h(1) = 0\,. \tag{5.67}$$

In other words, the diffusivity should vary with $|h|$, not h. Back in (5.50), our translation of the physical problem into mathematics was faulty. Changing h to $|h|$ in destroys the symmetry in (5.65).

Now let's use the correct model in (5.67) and show that the boundary layer cannot be at $x = 1$. If we try to put the boundary layer at $x = 1$ then the leading-order interior solution is

$$h_0 = -x\,. \tag{5.68}$$

Using the boundary layer coordinate

$$X \stackrel{\text{def}}{=} \frac{x-1}{\epsilon}\,, \tag{5.69}$$

the leading-order boundary layer equation is

$$-\left(\tfrac{1}{2}H_0^2\right)_{XX} + H_{0X} = 0\,, \tag{5.70}$$

Above we have assumed that $H_0(X) < 0$ so that $|H_0| = -H_0$. The differential equation in (5.73) must be solved with boundary and matching conditions

$$H_0(0) = 0\,, \qquad \text{and} \qquad \lim_{X\to-\infty} H_0 = -1\,. \tag{5.71}$$

The second condition above is matching onto the inner limit of the outer solution. We can integrate (5.73) and apply the matching condition to obtain

$$\frac{\mathrm{d}H_0}{\mathrm{d}X} = \frac{H_0 + 1}{H_0}\,. \tag{5.72}$$

Now if $-1 < H_0 < 0$ then the equation above implies that

$$\frac{\mathrm{d}H_0}{\mathrm{d}X} < 0\,. \tag{5.73}$$

The sign in (5.73) is not consistent with a solution that increases monotonically from $H_0(-\infty) = -1$ to $H_0(0) = 1$. Moreover if we integrate (5.72) with separation of variables we obtain an implicit solution

$$X = H_0 - \ln(1 + H_0)\,, \qquad \text{or equivalently} \qquad H_0 = -1 + \mathrm{e}^{-X + H_0}\,. \tag{5.74}$$

But as $X \to -\infty$ we do not get a match — the boundary layer cannot be at $x = 1$. Thus we cannot construct a solution of the $|h|$-model in (5.67) with a boundary layer at $x = 1$

5.5 Problems

Problem 5.1. *(i)* Find a leading order uniformly valid solution of

$$-h_x = \epsilon h_{xx} + x\,, \qquad h(0) = h(1) = 0\,.$$

(ii) Solve the BVP above exactly and compare the exact solution to the boundary layer approximation with $\epsilon = 0.1$.

Problem 5.2. *(i)* Solve the boundary value problem

$$h_x = \epsilon h_{xx} + \sin x\,, \qquad h(0) = h(\pi) = 0\,,$$

exactly. To assist communication, please use the notation

$$X \stackrel{\text{def}}{=} \frac{x-\pi}{\epsilon}\,, \qquad \text{and} \qquad E \stackrel{\text{def}}{=} \mathrm{e}^{-\pi/\epsilon}\,.$$

This should enable you to write the exact solution in a compact form. *(ii)* Find the first three terms in the regular perturbation expansion of the outer solution

$$h(x) = h_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + O\left(\epsilon^3\right)\,.$$

(iii) There is a boundary layer at $x = \pi$. "Rescale" the equation using X above as the independent variable and denote the solution in the boundary layer by $H(X)$. Find the first three terms in the regular perturbation expansion of the boundary-layer equation:

$$H = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + O\left(\epsilon^3\right)\,.$$

(iv) The H_n's above will each contain an unknown constant. Determine the three constants by matching to the interior solution. *(v)* Construct a uniformly valid solution, up to an including terms of order ϵ^2. You can check your algebra by comparing your boundary layer solution with the expansion of the exact solution from part *(i)*. *(vi)* With $\epsilon = 0.2$ and 0.5, use MATLAB to compare the exact solution from part *(i)* with the approximation in part *(v)*.

Problem 5.3. Find a leading-order boundary layer solution to the forced Burgers equation

$$\epsilon h_{xx} + \left(\tfrac{1}{2}h^2\right)_x = -1\,, \qquad h(0) = h(1) = 0\,.$$

Use `bvp4c` to solve this problem numerically, and compare your leading order solution to the numerical solution: see figure 5.5.

Problem 5.4. The result of problem 5.3 is disappointing: even though $\epsilon = 0.05$ seems rather small, the approximation in Figure 5.5 is only so-so. Calculate the next correction and compare the new improved solution with the `bvp4c` solution. (The numerical solution seems to have finite slope at $x = 1$, while the leading-order outer solution has infinite slope as $x \to 1$: perhaps there a higher-order boundary layer at $x = 1$ is required to heal this singularity?)

Problem 5.5. Use boundary layer theory to find leading order solution of

$$h_x = \epsilon\left(\tfrac{1}{3}h^3\right)_{xx} + 1\,, \tag{5.75}$$

on the domain $0 < x < 1$ with boundary conditions $h(0) = h(1) = 0$. You can check your answer by showing that $h = 1/2$ at $x \approx 1 - 0.057\epsilon$.

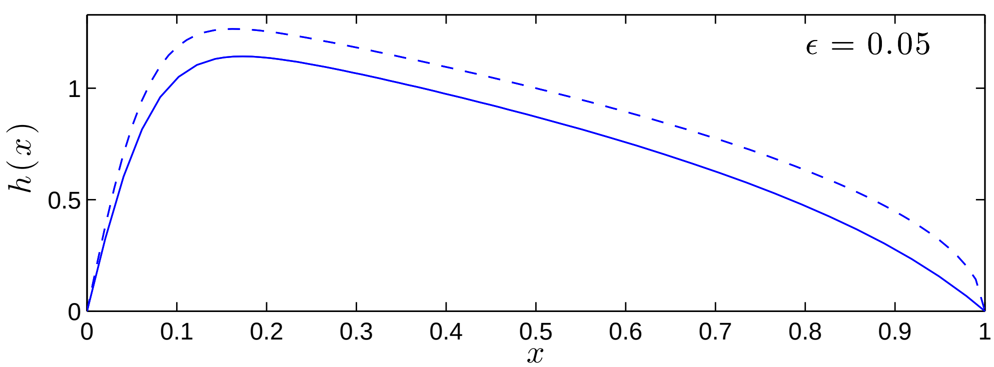

Figure 5.5: Figure for problem 5.3.

Lecture 6

More boundary layer theory

6.1 Variable speed

Suppose the conveyor belt is a stretchy membrane which moves with non-uniform speed $-c(x)$. With non-constant c, the dirt conservation equation in (5.1) generalizes to

$$h_t - (ch)_x = s + \kappa h_{xx}\,, \tag{6.1}$$

with boundary conditions unchanged: $h(0,t) = 0$ and $h(\ell,t) = 0$.

Exercise: Make sure you understand why it is $(ch)_x$, rather than ch_x, in (6.1). Nondimensionalize (6.1) so that the steady state problem is

$$\epsilon h_{xx} - (ch)_x = -s\,, \qquad h(0) = h(1) = 0\,, \tag{6.2}$$

with $\max c(x) = 1$ and $\max s(x)$=1.

Example: A slow down

Suppose that the belt slows near $x = 0$. Specifically, let's assume that the belt speed is

$$c = \sqrt{x}\,. \tag{6.3}$$

The speed is zero at $x = 0$, so we expect that dirt will start to pile up. If the source is uniform then the steady-state problem is

$$\epsilon h_{xx} + \left(\sqrt{x}h\right)_x = -1\,, \qquad \text{with BCs } h(0) = h(1) = 0. \tag{6.4}$$

Exercise: Show that a particle starting at $x = 1$ and moving with $\dot{x} = -x^{\beta}$, with $\beta < 1$, reaches $x = 0$ in a finite time. What happens if $\beta \geq 1$?

The first two terms in the interior solution are

$$h(x,\epsilon) = \left(x^{-1/2} - x^{1/2}\right) + \epsilon\left(\frac{1}{2}x^{-2} + \frac{1}{2}x^{-1} - x^{-1/2}\right) + \mathrm{ord}(\epsilon^2)\,. \tag{6.5}$$

We've satisfied the BC at $x = 1$ and the pile-up at $x = 0$ is evident via the divergence of the outer solution as $x \to 0$. Notice that this divergence is stronger at higher orders, and the RPS above is disordered as $x \to 0$.

Turning to the boundary layer at $x = 0$, we introduce

$$X \stackrel{\text{def}}{=} \frac{x}{\delta} \tag{6.6}$$

so that

$$\frac{\epsilon}{\delta^2}h_{XX}+\frac{1}{\delta^{1/2}}\left(\sqrt{X}h\right)_X=-1\,. \tag{6.7}$$

A dominant balance between the first two terms is achieved with $\epsilon=\delta^{3/2}$, or

$$\delta=\epsilon^{2/3}\,. \tag{6.8}$$

With this definition of δ, and

$$h=H(X,\epsilon)\,, \tag{6.9}$$

the boundary layer equation is

$$H_{XX}+\left(\sqrt{X}H\right)_X=-\epsilon^{1/3}\,. \tag{6.10}$$

We attack with an RPS: $h=H_0(X)+\epsilon^{1/3}H_1(x)+\cdots$

At leading order

$$H_{0XX}+\left(\sqrt{X}H_0\right)_X=0\,, \tag{6.11}$$

with solution

$$H_0(X)=A_0\mathrm{e}^{-2X^{3/2}/3}\int_0^X \mathrm{e}^{2t^{3/2}/3}\,\mathrm{d}t\,. \tag{6.12}$$

We've satisfied the boundary condition at $x=0$, and we must determine the remaining constant of integration A_0 by matching to the interior solution.

To match the interior, we need the asymptotic expansion of (6.12) as $X\to\infty$: this can be obtained by following our earlier discussion of Dawson's integral:

$$H_0(X)\sim\frac{A_0}{\sqrt{X}}\,,\qquad \text{as } X\to\infty, \tag{6.13}$$

$$=\frac{\epsilon^{1/3}A_0}{x^{1/2}}\,. \tag{6.14}$$

On the other hand the inner expansion of the outer solution in (6.5) is

$$h=\frac{1}{x^{1/2}}+O\left(x^{1/2},\epsilon x^{-2}\right)\,. \tag{6.15}$$

We *almost* have a match — it seems we should take $A_0=\epsilon^{-1/3}$ in (6.14) so that both functions are equal to $x^{-1/2}$ in the matching region. But remember that we assumed that $H_0(x)$ is independent of ϵ, so A_0 *cannot* depend on ϵ. Our expansion has failed.

Exercise: How would you gear so that the term ϵx^{-2} in (6.14) is asymptotically negligible relative to $x^{-1/2}$ in the matching region?

Fortunately there is a simple cure: the correct definition of the boundary layer solution — which replaces (6.9) — is

$$h=\epsilon^{-1/3}\mathcal{H}(X,\epsilon)\,. \tag{6.16}$$

In retrospect perhaps the rescaling in (6.16) is obvious — the interior RPS in (6.5) is becoming disordered as $x\to 0$. The problem is acute once the second term in the expansion is comparable to the first term, which happens once

$$x^{-1/2}\sim\epsilon x^{-2}\qquad\text{or}\qquad x\sim\epsilon^{2/3}=\delta\,. \tag{6.17}$$

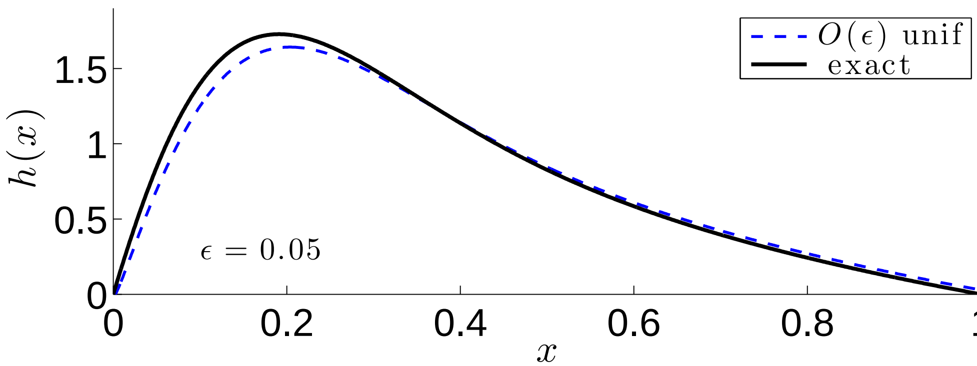

Figure 6.1: Comparison of (6.21) with the exact solution of (6.4).

This the boundary layer scale, and as we enter this region the interior solution is of order $x^{-1/2} \sim \epsilon^{-1/3}$ — this is why the rescaling in (6.16) is required. If we'd been smart we would have made this argument immediately after (6.5) and avoided the mis-steps in (6.9) and (6.10).

Using the rescaled variable in (6.14), the boundary layer equation that replaces (6.10) is

$$\mathcal{H}_{XX} + \left(\sqrt{X}\mathcal{H}\right)_X = -\epsilon^{2/3} \,. \tag{6.18}$$

Now we can try the RPS

$$\mathcal{H}(X,\epsilon) = \mathcal{H}_0(X) + \epsilon^{2/3}\mathcal{H}_1(X) + \cdots \tag{6.19}$$

We quickly find the leading order solution

$$\mathcal{H}_0 = \mathrm{e}^{-2X^{3/2}/3} \int_0^X \mathrm{e}^{2t^{3/2}/3} \, \mathrm{d}t \,. \tag{6.20}$$

This satisfies the $x = 0$ boundary condition and also matches the $x^{-1/2}$ from the interior.

We can now construct a leading-order uniformly valid solution as

$$h_{\mathrm{uni}}(x,\epsilon) = \epsilon^{-1/3}\mathrm{e}^{-2X^{3/2}/3} \int_0^X \mathrm{e}^{2t^{3/2}/3} \, \mathrm{d}t - x^{1/2} \,. \tag{6.21}$$

Figure 6.1 compares the uniformly valid approximation (6.21) with an exact solution of (6.4).

Exercise: evaluate the integral $\int_0^1 h(x,\epsilon)\,\mathrm{d}x$ to leading order as $\epsilon \to 0$.

Example: higher-order corrections

To illustrate how to bash out higher order corrections let's calculate the first two terms in the BL solution of the BVP

$$\epsilon h_{xx} + [\mathrm{e}^x h]_x = -2\mathrm{e}^{2x} \,, \quad \text{with BCs} \quad h(0) = h(1) = 0 \,. \tag{6.22}$$

We suspect there is a BL at $x = 0$. So we first develop the interior solution

$$h(x,\epsilon) = h_0(x) + \epsilon h_1(x) + \epsilon h_2(x) + \cdots \tag{6.23}$$

by satisfying the boundary condition at $x = 1$ at every order.

The leading-order term is

$$[\mathrm{e}^x h_0]_x = -2\mathrm{e}^{2x} \,, \qquad \Rightarrow \qquad h_0 = \mathrm{e}^{2-x} - \mathrm{e}^x \,. \tag{6.24}$$

The next two orders are

$$[\mathrm{e}^x h_1]_x = -h_{0xx}\,, \qquad \Rightarrow \qquad h_1 = 1 - 2\mathrm{e}^{1-x} + \mathrm{e}^{2-2x}\,, \tag{6.25}$$
$$[\mathrm{e}^x h_2]_x = -h_{1xx}\,, \qquad \Rightarrow \qquad h_2 = 2\left(\mathrm{e}^{2-3x} - \mathrm{e}^{1-2x}\right)\,. \tag{6.26}$$

Later, to perform the match, we will need the inner limit of this outer solution. So in preparation for that, as $x \to 0$,

$$\begin{aligned} h_0 + \epsilon h_1 + \epsilon^2 h_2 = (\mathrm{e}^2 - 1) - (\mathrm{e}^2 + 1)x + \tfrac{1}{2}(\mathrm{e}^2 - 1)x^2 \\ + \epsilon(1-\mathrm{e})^2 - \epsilon x 2(\mathrm{e}^2 - \mathrm{e}) \\ + \epsilon^2 2(\mathrm{e}^2 - \mathrm{e}) + \mathrm{ord}(x^3, \epsilon x^2, \epsilon^2 x)\,. \end{aligned} \tag{6.27}$$

Turning to the boundary layer, we use the inner variable $X = x/\epsilon$ so that the rescaled differential equation is

$$h_{XX} + \left[\mathrm{e}^{\epsilon X} h\right]_X = -2\epsilon \mathrm{e}^{2\epsilon X}\,. \tag{6.28}$$

We substitute the inner expansion

$$h = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + \cdots \tag{6.29}$$

into the differential equation and collect powers of ϵ. The first three orders of the boundary-layer problem are

$$H_{0XX} + H_{0X} = 0\,, \tag{6.30}$$
$$H_{1XX} + [H_1 + XH_0]_X = -2\,, \tag{6.31}$$
$$H_{2XX} + \left[H_2 + XH_1 + \tfrac{1}{2}X^2 H_0\right]_X = -4X\,. \tag{6.32}$$

Note that it is necessary to expand the exponentials within the boundary layer, otherwise we cannot ensure that the H_n's do not depend on ϵ.

The solution for H_0 that satisfies the boundary condition at $x = 0$, and also matches the first term on the right of (6.27), is

$$H_0 = (\mathrm{e}^2 - 1)\left(1 - \mathrm{e}^{-X}\right)\,. \tag{6.33}$$

The solution for H_1 that satisfies the boundary condition at $x = 0$ is

$$H_1 = A_1(1 - \mathrm{e}^{-X}) + (\mathrm{e}^2 + 1)\left(1 - X - \mathrm{e}^{-X}\right) + \tfrac{1}{2}(\mathrm{e}^2 - 1)X^2 \mathrm{e}^{-X}\,. \tag{6.34}$$

The constant A_1 is determined by matching to the interior solution. We can do this by taking the limit as $X \to \infty$ in the boundary layer solution $H_0 + \epsilon H_1$. Effectively this means that all terms involving e^{-X} are exponentially small and therefore negligible in the matching. To help with pattern recognition we rewrite the outer limit of the boundary-layer solution in terms of the outer variable x. Thus, in the matching region where $X \gg 1$ and $x \ll 1$, the boundary-layer solution in (6.33) and (6.34) is:

$$H_0 + \epsilon H_1 \to (\mathrm{e}^2 - 1) + \epsilon A_1 + \epsilon(1 + \mathrm{e}^2) - (1 + \mathrm{e}^2)x\,. \tag{6.35}$$

To match the first term on the second line of (6.27) with (6.35) we require

$$\epsilon A_1 + \epsilon(1 + \mathrm{e}^2) = \epsilon(1 - \mathrm{e})^2\,, \qquad \Rightarrow \qquad A_1 = -2\mathrm{e}\,. \tag{6.36}$$

The final term in (6.35), namely $-(1 + \mathrm{e}^2)x$, matches against a term on the first line of (6.27). That's interesting, because $-(1 + \mathrm{e}^2)x$ comes from H_1 and matches against h_0.

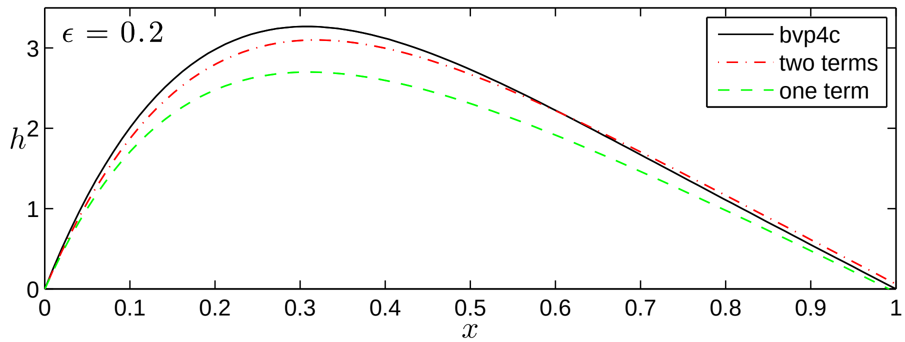

Figure 6.2: Comparison of the one and two term uniform approximations in (6.38) with the numerical solution of (6.22).

There are many remaining ummatched terms in (6.27) e..g, $\frac{1}{2}(\mathrm{e}^2-1)x^2$ on the first line. This term will match against terms from H_2 i.e., it will require an infinite number of terms in the boundary layer expansion just to match terms arising from the expansion of the leading-order interior solution.

Now we construct a uniformly valid approximation using the recipe

$$\text{uniform} = \text{outer} + \text{inner} - \text{match}\,. \tag{6.37}$$

This gives

$$\begin{aligned} h^{\text{uni}} &= \mathrm{e}^{2-x} - \mathrm{e}^{x} - (\mathrm{e}^2-1)\mathrm{e}^{-X} \\ &\qquad + \epsilon\left[1 - 2\mathrm{e}^{1-x} + \mathrm{e}^{2-2x} + \mathrm{e}^{-X}\left(\tfrac{1}{2}X^2(\mathrm{e}^2-1) - (\mathrm{e}-1)^2\right)\right]\,. \end{aligned} \tag{6.38}$$

This construction satisfies the $x = 0$ boundary condition exactly. But there is an exponentially small embarrassment at $x = 1$. Figure 6.2 compares the numerical solution of (6.22) with the approximation in (6.38). At $\epsilon = 0.2$ the two-term approximation is significantly better than just the leading-order term. We don't get line-width agreement — the ϵ^2 term would help.

```
function StommelBL
% Solution of epsilon h_{xx} +(exp(x) h)_x = - 2 \exp(2 x)
epsilon = 0.2;
solinit = bvpinit(linspace(0 , 1 , 10) , @guess);
sol     = bvp4c(@odez,@bcs,solinit);
% My fine mesh
xx = linspace(0,1,100);    hh = deval(sol,xx);
figure;                    subplot(2,1,1)
plot(xx , hh(1,:),'k')
hold on
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$h$','interpreter','latex','fontsize',16,'rotation',0)
axis([0 1 0 3.5])
% The BL solution
XX = xx/epsilon;         EE =exp(-XX);
hZero = exp(2-xx) - exp(xx) - (exp(2) - 1).*EE;
hOne = 1 - 2*exp(1-xx) + exp(2-2*xx)...
               + EE.*( 0.5*XX.^2*(exp(2) - 1) - (exp(1) - 1)^2);
plot(xx, hZero+epsilon*hOne,'-.r' , xx,hZero,'--g')
legend('bvp4c','two terms' , 'one term')
text(0.02,3.2,'$\epsilon =0.2$','interpreter','latex','fontsize',16)

%% Inline functions
    %The differential equations
    function dhdx = odez(x,h)
        dhdx = [h(2)/epsilon ; ...
                    - exp(x).*h(2)/epsilon - exp(x).*h(1) - 2*exp(2*x)];
    end

    %residual in the boundary condition
    function res = bcs(um,up)
        res = [um(1) ; up(1) ];
    end

    % Inital guess at the solution
    function hinit = guess(x)
        hinit = [(1-x^2) ; 2*x];
    end
end
```

6.2 A second-order BVP with a boundary layer

At the risk of repetition, let's discuss another elementary example of boundary layer theory, focussing on intuitive concepts and on finding the leading-order uniformly valid solution. We use the BVP

$$\epsilon y'' + a(x)y' + b(x)y = 0\,, \tag{6.39}$$

with BCs

$$y(0) = p\,, \qquad y(1) = q\,, \tag{6.40}$$

as our model.

The case $a(x) > 0$

In the outer region we can look for a solution with the expansion

$$y(x,\epsilon) = y_0(x) + \epsilon y_1(x) + \cdots \tag{6.41}$$

The leading-order term satisfies

$$a y_0' + b y_0 = 0\,. \tag{6.42}$$

Solutions of this first order differential equation cannot satisfy two boundary conditions. With the construction

$$y_0(x) = q \exp\left(\int_x^1 \frac{b(v)}{a(v)} \mathrm{d}v\right) \tag{6.43}$$

we have satisfied the boundary condition at $x = 1$. We return later to discuss why this is the correct choice if $a(x) > 0$. (If $a(x) < 0$ the boundary layer is at $x = 1$ and the outer solution should satisfy the boundary condition at $x = 0$.)

Unless we're very lucky, (6.43) will not satisfy the boundary condition at $x = 0$. Let's try to fix this problem by building a boundary layer at $x = 0$. Begin by introducing a boundary layer coordinate

$$X \stackrel{\text{def}}{=} \frac{x}{\epsilon}\,, \tag{6.44}$$

and writing

$$y(x,\epsilon) = Y(X,\epsilon)\,. \tag{6.45}$$

Then "re-scale" the differential equation (6.39) using the boundary-layer variables:

$$Y_{XX} + a(\epsilon X)Y_X + \epsilon b(\epsilon X)Y = 0\,. \tag{6.46}$$

Notice that within the boundary layer, where $X = \mathrm{ord}(1)$,

$$a(\epsilon X) = a(0) + \epsilon X a'(0) + \tfrac{1}{2}\epsilon^2 X^2 a''(0) + \mathrm{ord}\left(\epsilon^3\right)\,. \tag{6.47}$$

There is an analogous expansion for $b(\epsilon X)$.

In the boundary layer we use the "inner expansion":

$$Y(X,\epsilon) = Y_0(X) + \epsilon Y_1(X) + \cdots \tag{6.48}$$

The leading-order term is

$$\epsilon^0: \qquad Y_{0XX} + a(0)Y_{0X} = 0\,, \tag{6.49}$$

and, for good measure, the next term is

$$\epsilon^1: \qquad Y_{1XX} + a(0)Y_{1X} + a'(0)XY_{0X} + b(0)Y_0 = 0\,. \tag{6.50}$$

Terms in the Taylor series (6.47) will impact the higher orders.

The solution of (6.49) that satisfies the boundary condition at $X = 0$ is

$$Y_0 = p + A_0 \left(1 - \mathrm{e}^{-a(0)X}\right) , \tag{6.51}$$

where A_0 is a constant of integration. We are assuming that $a(0) > 0$ so that the exponential in (6.51) decays to zero as $X \to \infty$. This is why the boundary layer *must* be at $x = 0$. The constant A_0 can then be determined by demanding that in the outer solution (6.43) agrees with the inner solution (6.51) in the *matching region* where $X \gg 1$ and $x \ll 1$. This requirement determines A_0:

$$p + A_0 = q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v\right) . \tag{6.52}$$

Hence the leading order boundary-layer solution is

$$Y_0 = p + \left[q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v\right) - p\right] \left(1 - \mathrm{e}^{-a(0)X}\right) , \tag{6.53}$$

$$= p\mathrm{e}^{-a(0)X} + q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v\right) \left(1 - \mathrm{e}^{-a(0)X}\right) . \tag{6.54}$$

We construct a uniformly valid solutions using the earlier recipe

$$\text{uniformly valid} = \text{outer} + \text{inner} - \text{match} . \tag{6.55}$$

In this case we obtain

$$y_{\text{uni}} = q \exp\left(\int_x^1 \frac{b(v)}{a(v)} \mathrm{d}v\right) + \left[p - q \exp\left(\int_0^1 \frac{b(v)}{a(v)} \mathrm{d}v\right)\right] \mathrm{e}^{-a(0)X} . \tag{6.56}$$

6.3 Other BL examples

Not all boundary layers have thickness ϵ. Let's quickly consider some examples.

Example:

$$\epsilon y'' - y = -f(x) , \qquad y(-1) = y(1) = 0 , \tag{6.57}$$

If we solve the simple case with $f(x) = 1$ exactly we quickly see that $y \approx 1$, except that there are boundary layers with thickness $\sqrt{\epsilon}$ at *both* $x = 0$ and $x = 1$.

Thus we might hope to construct the outer solution of (6.57) via the RPS

$$y = f + \epsilon f'' + \epsilon^2 f'''' + \text{ord}(\epsilon^3) . \tag{6.58}$$

The outer solution above doesn't satisfy either boundary condition: we need boundary layers at $x = -1$, and at $x = +1$.

Turning to the boundary layer at $x = -1$ we introduce

$$X \stackrel{\text{def}}{=} \frac{x + 1}{\sqrt{\epsilon}} , \qquad \text{and} \qquad y(x, \epsilon) = Y(X, \sqrt{\epsilon}) . \tag{6.59}$$

The re-scaled differential equation is

$$Y_{XX} - Y = f\left(-1 + \sqrt{\epsilon}X\right) , \tag{6.60}$$

and we look for a solution with

$$Y = Y_0(X) + \sqrt{\epsilon}Y_1(X) + \epsilon Y_2(X) + \cdots \tag{6.61}$$

The leading-order problem is

$$Y_{0XX} - Y_0 = -f(-1) , \tag{6.62}$$

with solution

$$Y_0 = f(-1) + A_0 e^{-X} + \underbrace{B_0}_{=0} e^X . \tag{6.63}$$

We quickly set the constant of integration B_0 to zero — the alternative would prevent matching with the interior solution. Then the other constant of integration A_0 is determined so that the boundary condition at $X = 0$ is satisfied:

$$Y_0 = f(-1)\left(1 - e^{-X}\right) . \tag{6.64}$$

The boundary condition at $x = +1$ is satisfied with an analogous construction using the coordinate $X \stackrel{\text{def}}{=} (x-1)/\sqrt{\epsilon}$. One finds

$$Y_0 = f(1)\left(1 - e^{X}\right) . \tag{6.65}$$

Notice that the outer limit of this boundary layer is obtained by taking $X \to -\infty$.

Finally we can construct a uniformly valid solutions via

$$y^{\text{uni}}(x) = f(x) - f(-1)e^{-(x+1)/\sqrt{\epsilon}} - f(+1)e^{(x-1)/\sqrt{\epsilon}} . \tag{6.66}$$

Example:

$$\epsilon y'' + y = f(x) , \qquad y(0) = y(1) = 0 , \tag{6.67}$$

If we solve the simple case with $f(x) = 1$ exactly we quickly see that this is not a boundary layer problem. This belongs in the WKB lecture.

Example: Find the leading order BL approximation to

$$\epsilon u'' - u = -\frac{1}{\sqrt{1-x^2}} , \qquad \text{with BCs} \qquad u(\pm 1) = 0 . \tag{6.68}$$

The leading-order outer solution is

$$u_0 = \frac{1}{\sqrt{1-x^2}} . \tag{6.69}$$

Obviously this singular solution doesn't satisfy the boundary conditions. We suspect that there are boundary layers of thickness $\sqrt{\epsilon}$ at $x = \pm 1$. Notice that the interior solution (6.69) is $\sim \epsilon^{-1/4}$ as x moves into this BL. Moreover, considering the BL at $x = -1$, we use $X = (1+x)/\sqrt{\epsilon}$ as the boundary layer coordinate, so that

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\epsilon^{1/4}\sqrt{X(2-\sqrt{\epsilon}X)}} . \tag{6.70}$$

Hence we try a boundary-layer expansion with the form

$$u(x,\epsilon) = \epsilon^{-1/4}\left[U_0(X) + \sqrt{\epsilon}U_1(X) + \text{ord}(\epsilon)\right] . \tag{6.71}$$

A main point of this example is that it is necessary to include the factor $\epsilon^{-1/4}$ above.

The leading-order term in the boundary layer expansion is then

$$U_0'' - U_0 = -\frac{1}{\sqrt{X}} , \tag{6.72}$$

which we solve using variation of parameters

$$U_0(X) = \tfrac{1}{2}\underbrace{e^{-X}\int_0^X \frac{e^v}{\sqrt{v}}\,dv}_{\sim e^X/\sqrt{X}} - \tfrac{1}{2}\underbrace{e^{X}\int_0^X \frac{e^{-v}}{\sqrt{v}}\,dv}_{\sim\sqrt{\pi}-(e^{-X}/\sqrt{X})} + Pe^X + Qe^{-X} . \tag{6.73}$$

The boundary condition at $X = 0$ requires

$$P + Q = 0 . \tag{6.74}$$

To match the outer solution as $X \to \infty$, we must use the $X \to \infty$ asymptotic expansion of the integrals in (6.73), indicated via the underbrace. We determine P so that the exponentially growing terms are eliminated, which requires that $P = \sqrt{\pi}/2$. Thus the boundary layer solution is

$$U_0(X) = \sqrt{\pi}\sinh X - \int_0^X \frac{\sinh(X-v)}{\sqrt{v}}\,dv . \tag{6.75}$$

(Must check this, and then construct the uniformly valid solution!)

Example: Find the leading order BL approximation to

$$\epsilon y'' + xy' + x^2 y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{6.76}$$

We divide and conquer by writing the solutions as

$$y = pf(x,\epsilon) + qg(x,\epsilon)\,, \tag{6.77}$$

where

$$\epsilon f'' + xf' + x^2 f = 0\,, \qquad \text{with BCs} \qquad f(0) = 1\,, \qquad f(1) = 0\,, \tag{6.78}$$

and

$$\epsilon g'' + xg' + x^2 g = 0\,, \qquad \text{with BCs} \qquad g(0) = 0\,, \qquad g(1) = 1\,, \tag{6.79}$$

The outer solution of the g-problem is

$$g = \mathrm{e}^{(1-x^2)/2} + \epsilon g_1 + \cdots \tag{6.80}$$

We need a BL at $x = 0$. A dominant balance argument shows that the correct BL variable is

$$X = \frac{x}{\sqrt{\epsilon}}\,, \tag{6.81}$$

and if $g(x,\epsilon) = G(X,\sqrt{\epsilon})$ then the rescaled problem is

$$G_{XX} + XG_X + \epsilon X^2 G = 0\,. \tag{6.82}$$

The leading-order problem is

$$G_{0XX} + XG_{0X} = 0\,, \tag{6.83}$$

with general solution

$$G_0 = \underbrace{P}_{=0} + \underbrace{Q}_{\sqrt{\mathrm{e}/2\pi}} \int_0^X \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{6.84}$$

To satisfy the $X = 0$ boundary condition we take $P = 0$, and to match the outer solution we require

$$Q\int_0^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v = \sqrt{\mathrm{e}}\,. \tag{6.85}$$

The uniformly valid solution is

$$g^{\mathrm{uni}}(x,\epsilon) = \mathrm{e}^{(1-x^2)/2} + \sqrt{\frac{\mathrm{e}}{2\pi}}\int_0^{x/\sqrt{\epsilon}} \mathrm{e}^{-v^2/2}\,\mathrm{d}v - \sqrt{\mathrm{e}}\,, \tag{6.86}$$

$$= \mathrm{e}^{(1-x^2)/2} - \sqrt{\frac{\mathrm{e}}{2\pi}}\int_{x/\sqrt{\epsilon}}^{\infty} \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{6.87}$$

Now turn to the f-problem. The outer solution is $f_n(x) = 0$ at all orders. The solution of the leading-order boundary-layer problem is

$$F_0(X) = \frac{1}{\sqrt{2\pi}}\int_X^\infty \mathrm{e}^{-v^2/2}\,\mathrm{d}v\,. \tag{6.88}$$

This is a stand-alone boundary layer.

Example: Let's analyze the higher-order terms in the BL solution of our earlier example

$$\epsilon y'' - y = -f(x)\,, \qquad y(-1) = y(1) = 0\,. \tag{6.89}$$

Our provisional outer solution is

$$y(x) = f(x) + \epsilon f''(x) + \epsilon^2 f''''(x) + \mathrm{ord}(\epsilon^3)\,. \tag{6.90}$$

Let's rewrite this outer solution in terms of the inner variable $X \stackrel{\mathrm{def}}{=} (x-1)/\sqrt{\epsilon}$

$$y(x) = f(1+\sqrt{\epsilon}X) + \epsilon f''(1+\sqrt{\epsilon}X) + \epsilon^2 f''''(1+\sqrt{\epsilon}X) + \mathrm{ord}(\epsilon^3)\,. \tag{6.91}$$

Assuming that $\sqrt{\epsilon}X$ is small in the matching region, we expand the outer solution:

$$\begin{aligned} y(x) = f(1) + \sqrt{\epsilon}Xf'(1) + \epsilon\left(\tfrac{1}{2}X^2 + 1\right)f''(1) + \epsilon^{3/2}\left(X + \frac{1}{6}X^3\right)f'''(1) \\ + \epsilon^2\left(1 + \frac{1}{2}X^2 + \frac{1}{24}X^4\right)f''''(1) + \mathrm{ord}\left(\epsilon^{5/2}\right)\,. \end{aligned} \tag{6.92}$$

We hope that the outer expansion of the inner solution at $x = 1$ will match the series above. The rescaled inner problem is

$$
\begin{aligned}
Y_{XX} - Y &= -f(1 + \sqrt{\epsilon}X)\,, && (6.93)\\
&= -f(1) - \sqrt{\epsilon}Xf'(1) - \epsilon\frac{1}{2}X^2 f''(1) + \mathrm{ord}(\epsilon^{3/2})\,. && (6.94)
\end{aligned}
$$

The RPS is

$$
Y = f(1)\left(1 - \mathrm{e}^X\right) + \sqrt{\epsilon}Y_1(X) + \epsilon Y_2(X) + \epsilon^{3/2}Y_3(X)\mathrm{ord}(\epsilon^2)\,, \tag{6.95}
$$

with

$$
\begin{aligned}
Y_1'' - Y_1 &= -Xf'(1)\,, && (6.96)\\
Y_2'' - Y_2 &= -\tfrac{1}{2}X^2 f''(1)\,, && (6.97)\\
Y_3'' - Y_3 &= -\frac{1}{6}X^3 f'''(1)\,. && (6.98)
\end{aligned}
$$

We solve the equations above, applying the boundary condition $Y_n(0) = 0$, to obtain

$$
Y_1(X) = Xf'(1)\,, \qquad Y_2(X) = \left(1 + \tfrac{1}{2}X^2 - \mathrm{e}^X\right) f''(1)\,, \tag{6.99}
$$

$$
\text{and} \qquad Y_3(X) = \left(X + \frac{1}{6}X^3\right) f'''(1)\,. \tag{6.100}
$$

Notice how the inner limit of the leading-order outer solution, $y_0(x) = f(x)$, produces terms at all orders in the matching region. In order to match all of $y_0(x)$ one requires *all* the $Y_n(X)$'s.

6.4 Problems

Problem 6.1. Find the leading-order uniformly valid boundary-layer solution to the Stommel problem

$$
-\left(\mathrm{e}^x g\right)_x = \epsilon g_{xx} + 1\,, \qquad \text{with BCs} \quad g(0) = g(1) = 0\,. \tag{6.101}
$$

Do the same for

$$
\left(\mathrm{e}^x f\right)_x = \epsilon f_{xx} + 1\,, \qquad \text{with BCs} \quad f(0) = f(1) = 0\,. \tag{6.102}
$$

Problem 6.2. Analyze the variable-speed Stommel problem

$$
\epsilon h'' + \left(x^a h\right)_x = -1\,, \qquad \text{with BCs} \quad h(0) = h(1) = 0\,, \tag{6.103}
$$

using boundary layer theory. (The case $a = 1/2$ was discussed in the lecture.) How thick is the boundary layer at $x = 0$, and how large is the solution in the boundary layer? Check your reasoning by constructing the leading-order uniformly valid solution when $a = -1$, $a = 1$ and $a = 2$.

Problem 6.3. Find the leading-order, uniformly valid solution of

$$
\epsilon h'' + \left(\sin x\, h\right)_x = -1\,, \qquad \text{with BCs} \quad h(0) = h\left(\frac{\pi}{2}\right) = 0\,. \tag{6.104}
$$

Problem 6.4. Find a leading-order boundary layer solution to

$$
\epsilon h'' + (\sin x\, h)_x = -1, \qquad \text{with BCs} \quad h(0) = h(\pi) = 0\,. \tag{6.105}
$$

(I think there are boundary layers at both $x = 0$ and $x = 1$.)

Problem 6.5. Considering the slow-down example (6.4), find the next term in the boundary-layer solution of this problem. Make sure you explain how the term ϵx^{-2} in the outer expansion is matched as $x \to 0$.

Problem 6.6. Assuming that $a(x) < 0$, construct the uniformly valid leading-order approximation to the solution of

$$\epsilon y'' + ay' + by = 0\,, \qquad \text{with BCs} \quad y'(0) = p\,, \quad y'(1) = q\,. \tag{6.106}$$

(Consider using linear superposition by first taking $(p, q) = (1, 0)$, and then $(p, q) = (0, 1)$.)

Problem 6.7. Consider

$$\epsilon y'' + \sqrt{x} y' + y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{6.107}$$

(i) Find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading-order outer approximation and match to determine all constants of integration. *(ii)* Repeat for

$$\epsilon y'' - \sqrt{x} y' + y = 0\,, \qquad \text{with BCs} \qquad y(0) = p\,, \qquad y(1) = q\,. \tag{6.108}$$

Problem 6.8. Find a leading order, uniformly valid solution of

$$\epsilon y'' + \sqrt{x} y' + y^2 = 0\,, \qquad \text{with BCs} \qquad y(0, \epsilon) = 2\,, \qquad y(1; \epsilon) = \frac{1}{3}\,. \tag{6.109}$$

Problem 6.9. Find a leading order, uniformly valid solution of

$$\epsilon y'' - (1 + 3x^2) y = x\,, \qquad \text{with BCs} \qquad y(0, \epsilon) = y(1; \epsilon) = 1\,. \tag{6.110}$$

Problem 6.10. Find a leading-order, uniformly valid solution of

$$\epsilon y'' - \frac{y'}{1 + 2x} - \frac{1}{y} = 0\,, \qquad \text{with BCs} \qquad y(0, \epsilon) = y(1; \epsilon) = 3\,. \tag{6.111}$$

Problem 6.11. In an earlier problem you were asked to construct a leading order, uniformly valid solution of

$$\epsilon y'' - (1 + 3x^2) y = x\,, \qquad \text{with BCs} \qquad y(0, \epsilon) = y(1; \epsilon) = 1\,. \tag{6.112}$$

Now construct the uniformly valid two-term boundary layer approximation.

Problem 6.12. Consider

$$\epsilon y'' + (1 + \epsilon) y' + y = 0\,, \qquad y(0) = 0\,, \qquad y(1) = e^{-1}\,, \tag{6.113}$$

$$m' = y\,, \qquad m(1) = 0\,. \tag{6.114}$$

Find two terms in the outer expansion of $y(x)$ and $m(x)$, applying only boundary conditions at $x = 1$. Next find two terms in the inner approximation at $x = 0$, applying the boundary condition at $x = 0$. Determine the constants of integration by matching. Calculate $m(0)$ correct to order ϵ.

Lecture 7

Multiple scale theory

7.1 Introduction to two-timing

In a previous lecture we solved the damped oscillator equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \tag{7.1}$$

with initial conditions

$$x(0) = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 1\,. \tag{7.2}$$

You should recall that the exact solution is

$$x = \nu^{-1} \mathrm{e}^{-\beta t/2} \sin \nu t\,, \qquad \text{with} \qquad \nu \stackrel{\text{def}}{=} \sqrt{1 - \frac{\beta^2}{4}}\,. \tag{7.3}$$

A good or useful $\beta \ll 1$ approximation to this exact solution is

$$x \approx \mathrm{e}^{-\beta t/2} \sin t\,. \tag{7.4}$$

Let's use this example to motivate the multiple-scale method.

Failure of the regular perturbation expansion

If $\beta \ll 1$ we might be tempted to try an RPS on (7.1):

$$x(t, \beta) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t) + \cdots \tag{7.5}$$

A reasonable goal is to produce the good approximation (7.4). The RPS will not be successful and this failure will drive us towards the method of multiple time a scales, also know as "two timing".

The leading-order problem is

$$\ddot{x}_0 + x_0 = 0\,, \qquad \text{with IC} \qquad x_0 = 0\,, \qquad \dot{x}_0(0) = 1\,. \tag{7.6}$$

The solution is

$$x_0 = \sin t\,. \tag{7.7}$$

The first-order problem is

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + x_1 = -\cos t\,, \qquad \text{with IC} \qquad \bar{x}_1(0) = 0\,, \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t}(0) = 0\,. \tag{7.8}$$

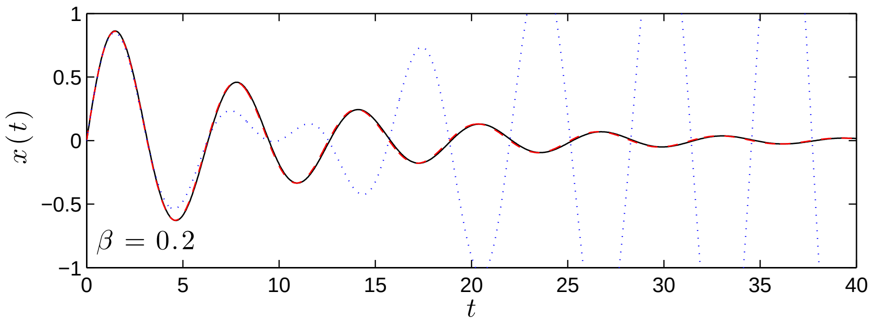

Figure 7.1: Comparison of the exact solution in (7.3) (the solid black curve), with the two-term RPS in (7.10) (the blue dotted curve) and the two-time approximation in (7.23) (the dashed red curve). It is difficult to distinguish the two-time approximation from the exact result.

This is a resonantly forced oscillator equation, with solution

$$x_1 = -\frac{t}{2}\sin t\,. \tag{7.9}$$

Thus the developing RPS is

$$x(t,\beta) = \sin t - \frac{\beta t}{2}\sin t + \beta^2 x_2(t) + \cdots \tag{7.10}$$

At this point we recognize that the RPS is misleading: the exact solution damps to zero on a time scale $2/\beta$, while the RPS suggests that the solution is growing linearly with time. With hindsight we realize that the RPS is producing the Taylor series expansion of the exact solution in (3.42) about $\beta = 0$. Using MATHEMATICA, this series is

$$x(t,\beta) = \sin t - \frac{\beta t}{2}\sin t + \frac{\beta^2}{8}\left[t^2\sin t + \sin t - t\cos t\right] + O(\beta^3)\,. \tag{7.11}$$

Calculating more terms in the RPS will not move us closer to the useful approximation in (7.4): instead we'll grind out the useless approximation in (7.11). In this example the small term in (3.40) is small relative to the other terms at all times. Yet the small error slowly accumulates over long times $\sim \beta^{-1}$. This is a *secular error*.

Two-timing

Looking at the good approximation in (7.4) we are inspired to introduce a *slow time*:

$$s \stackrel{\text{def}}{=} \beta t\,. \tag{7.12}$$

We assume that $x(t,\beta)$ has a perturbation expansion of the form

$$x(t,\beta) = x_0(t,s) + \beta x_1(t,s) + \beta^2 x_3(t,s) + \cdots \tag{7.13}$$

Notice how this differs from the RPS in (7.5).

At each order x_n is a function of both s and t a function of both t and s. To keep track of all the terms we use the rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \beta\partial_s\,, \tag{7.14}$$

and the equation of motion is

$$(\partial_t + \beta\partial_s)^2 x + \beta(\partial_t + \beta\partial_s) x + x = 0\,. \tag{7.15}$$

At leading order

$$\beta^0: \qquad \partial_t^2 x_0 + x_0 = 0\,, \qquad \text{with general solution} \qquad x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,. \tag{7.16}$$

Notice that the "constant of integration" is actually a function of the slow time s. We determine the evolution of this function $A(s)$ at next order[1].

At next order

$$\beta^1: \qquad \partial_t^2 x_1 + x_1 = -2x_{0ts} - x_{0t}\,, \tag{7.18}$$

$$= -2\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A\mathrm{e}^{\mathrm{i}t} + c.c. \tag{7.19}$$

Again we have a resonantly forced oscillator. but this time we can prevent the secular growth of x_1 on the fast time scale by requiring that

$$2A_s + A = 0\,. \tag{7.20}$$

Thus the leading-order solution is

$$x_0(s,t) = A_0\mathrm{e}^{-s/2}\mathrm{e}^{\mathrm{i}t} + A_0^*\mathrm{e}^{-s/2}\mathrm{e}^{-\mathrm{i}t}\,. \tag{7.21}$$

The constant of integration A_0 is determined to satisfy the initial conditions. This requires

$$0 = A_0 + A_0^*\,, \qquad 1 = \mathrm{i}A_0 - \mathrm{i}A_0^*\,, \qquad \Rightarrow \qquad A_0 = \tfrac{1}{2}\mathrm{i}\,. \tag{7.22}$$

Thus we have obtained the good approximation

$$x_0 = \mathrm{e}^{-\beta t/2}\sin t\,. \tag{7.23}$$

7.2 The Duffing oscillator

We consider an oscillator with a nonlinear spring

$$m\ddot{x} + k_1 x + k_3 x^3 = 0\,, \tag{7.24}$$

and an initial condition

$$x(0) = x_0 \qquad \dot{x}(0) = 0\,. \tag{7.25}$$

If $k_3 > 0$ then the restoring force is stronger than linear — this is a *stiff spring*.

We can non-dimensionalize this problem into the form

$$\ddot{x} + x + \epsilon x^3 = 0\,, \tag{7.26}$$

with the initial condition

$$x(0) = 1 \qquad \dot{x}(0) = 0\,. \tag{7.27}$$

[1] We could alternatively write the general solution of the leading order problem as

$$x_0 = R\cos(t+\phi)\,, \tag{7.17}$$

where the amplitude R and the phase ϕ are as yet undetermined functions of s. I think the complex notation in (7.16) is a little simpler.

We use this *Duffing oscillator* as an introductory example of the multiple time scale method.

Energy conservation,

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{1}{4}\epsilon x^4 = \underbrace{E}_{=\frac{1}{2}+\frac{1}{4}\epsilon}, \tag{7.28}$$

immediately provides a phase-plane visualization of the solution and shows that the oscillations are bounded.

Exercise: Show that in (7.26), $\epsilon = k_3 x_0^2/k_1$.

Exercise: Derive (7.28).

The naive RPS

$$x = x_0(t) + \epsilon x_1(t) + \cdots \tag{7.29}$$

leads to

$$\ddot{x}_0 + x_0 = 0\,, \qquad \Rightarrow \qquad x_0 = \cos t\,, \tag{7.30}$$

and at next order

$$\ddot{x}_1 + x_1 = -\cos^3 t\,, \tag{7.31}$$

$$= -\tfrac{1}{8}\left(\mathrm{e}^{3\mathrm{i}t} + 3\mathrm{e}^{\mathrm{i}t} + c.c.\right)\,, \tag{7.32}$$

$$= -\tfrac{1}{4}\cos 3t - \tfrac{3}{4}\cos t\,. \tag{7.33}$$

The x_1-oscillator problem is resonantly forced and the solution will grow secularly, with $x_1 \propto t \sin t$. Thus the RPS fails once $t \sim \epsilon^{-1}$.

Two-timing

Instead of an RPS we use the two-time expansion

$$x = x_0(s,t) + \epsilon x_1(s,t) + \cdots \tag{7.34}$$

where $s = \epsilon t$ is the slow time. Thus the expanded version of (7.26) is

$$(\partial_t + \epsilon\partial_s)^2 \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^3 = 0 \tag{7.35}$$

The leading order is

$$\partial_t^2 x_0 + x_0 = 0\,, \tag{7.36}$$

with general solution

$$x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,. \tag{7.37}$$

The amplitude A is a function of the slow time s. At next order, ϵ^1, we have

$$\partial_t^2 x_1 + x_1 = -2\partial_t\partial_s x_0 - x_0^3\,, \tag{7.38}$$

$$= -2\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - A^3\mathrm{e}^{3\mathrm{i}t} - 3A^2A^*\mathrm{e}^{\mathrm{i}t} + c.c. \tag{7.39}$$

To prevent the secular growth of x_1 we must remove the resonant terms, $\mathrm{e}^{\pm\mathrm{i}t}$ on the right of (7.39) — this prescription determines the evolution of the slow time:

$$2\mathrm{i}A_s + 3|A|^2A = 0\,. \tag{7.40}$$

The remaining terms in (7.39) are

$$\partial_t^2 x_1 + x_1 = -A^3\mathrm{e}^{3\mathrm{i}t} + c.c. \tag{7.41}$$

The solution will be $x_1 \propto \mathrm{e}^{\pm 3\mathrm{i}t}$ and will remain bounded.

Polar coordinates

To solve (7.40) it is best to transform to polar coordinates

$$A = r(s)\mathrm{e}^{\mathrm{i}\theta(s)} \qquad \text{and} \qquad A_s = (r_s + \mathrm{i} r\theta_s)\,\mathrm{e}^{\mathrm{i}\theta}\,; \tag{7.42}$$

substituting into the amplitude equation (7.40)

$$r_s = 0\,, \qquad \text{and} \qquad \theta_s = \frac{3}{2}r^2\,. \tag{7.43}$$

The energy of this nonlinear oscillator is constant and thus r is constant, $r(s) = r_0$. The phase $\theta(s)$ therefore evolves as $\theta = \theta_0 + 3r_0^2 s/2$.

The reconstituted solution is

$$x = r_0 \exp\left[\mathrm{i}\left(1 + \tfrac{3}{2}\epsilon r_0^2\right)t + \mathrm{i}\theta_0\right] + c.c. + \mathrm{ord}(\epsilon)\,. \tag{7.44}$$

The velocity of the oscillator is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i} r_0 \exp\left[\mathrm{i}\left(1 + \tfrac{3}{2}\epsilon r_0^2\right)t + \mathrm{i}\theta_0\right] + c.c. + \mathrm{ord}(\epsilon)\,. \tag{7.45}$$

To satisfy the initial condition in (7.27) at leading order, we take $\theta_0 = 0$ and $r_0 = 1/2$. Thus, with this particular initial condition,

$$x = \cos\left[\left(1 + \frac{3\epsilon}{8}\right)t\right] + \mathrm{ord}(\epsilon)\,. \tag{7.46}$$

The frequency of the oscillator in (7.44),

$$\nu = 1 + \tfrac{3}{2}\epsilon r_0^2\,, \tag{7.47}$$

depends on the amplitude r_0 and the sign of ϵ. If the spring is stiff (i.e., $k_3 > 0$) then ϵ is positive and bigger oscillations have higher frequency.

Higher-order corrections to the frequency

7.3 The quadratic oscillator

The quadratic oscillator is

$$\ddot{x} + x + \epsilon x^2 = 0\,. \tag{7.48}$$

The conserved energy is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{1}{3}\epsilon x^3\,, \tag{7.49}$$

and the curves of constant energy in the phase plane are shown in Figure 7.2.

Following our experience with the Duffing oscillator we try the two-time expansion

$$\begin{aligned}(\partial_t + \epsilon\partial_s)^2\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)&\\ + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^2 = 0\,.&\end{aligned} \tag{7.50}$$

The leading order solution is again

$$x_0 = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\,, \tag{7.51}$$

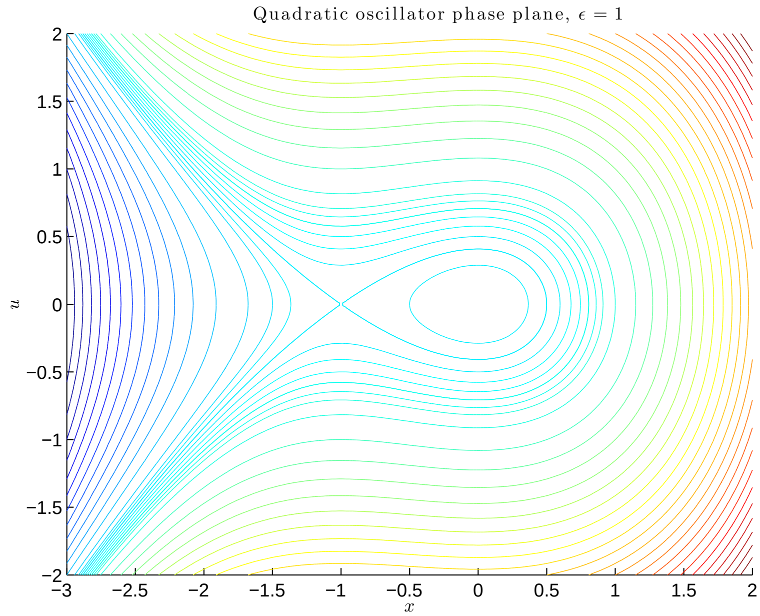

Figure 7.2: Quadratic oscillator phase plane.

and at next order

$$\partial_t^2 x_1 + x_1 = -2\left(\mathrm{i}A_s \mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_s \mathrm{e}^{\mathrm{i}t}\right) - \underbrace{\left(A^2\mathrm{e}^{2\mathrm{i}t} + 2|A|^2 + A^{*2}\mathrm{e}^{-2\mathrm{i}t}\right)}_{x_0^2} . \tag{7.52}$$

Elimination of the resonant terms $\mathrm{e}^{\pm\mathrm{i}t}$ requires simply $A_s = 0$, and then the solution of the remaining equation is

$$x_1 = \tfrac{1}{3}A^2\mathrm{e}^{2\mathrm{i}t} - 2|A|^2 + \tfrac{1}{3}A^{*2}\mathrm{e}^{-2\mathrm{i}t} . \tag{7.53}$$

This is why the quadratic oscillator is not used as an introductory example: there is no secular forcing at order ϵ.

To see the effects of nonlinearity in the quadratic oscillator, we must press on to higher orders, and use a slower slow time:

$$s = \epsilon^2 t . \tag{7.54}$$

Thus we revise (7.50) to

$$\begin{aligned}\big(\partial_t + \underbrace{\epsilon^2\partial_s}_{NB}\big)^2 \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) + \left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right) \\ + \epsilon\left(x_0(s,t) + \epsilon x_1(s,t) + \cdots\right)^2 = 0 .\end{aligned} \tag{7.55}$$

The solutions at the first two orders are the same as (7.51) and (7.53). At order ϵ^2 we have

$$\begin{aligned}\partial_t^2 x_2 + x_2 &= -2\partial_t\partial_s x_0 - 2x_0 x_1 , && (7.56)\\ &= -2\left(\mathrm{i}A_s\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_s^*\mathrm{e}^{-\mathrm{i}t}\right) - 2\underbrace{\left(A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t}\right)\left(\tfrac{1}{3}A^2\mathrm{e}^{2\mathrm{i}t} - 2|A|^2 + \tfrac{1}{3}A^{*2}\mathrm{e}^{-2\mathrm{i}t}\right)}_{x_0x_1} . && (7.57)\end{aligned}$$

Eliminating the $\mathrm{e}^{\pm\mathrm{i}t}$ resonant terms produces the amplitude equation

$$\mathrm{i}A_s = \frac{5}{3}|A|^2 A . \tag{7.58}$$

Although the nonlinearity in (7.48) is quadratic, the final amplitude equation in (7.58) is cubic. In fact, despite the difference in the original nonlinear term, the amplitude equation in (7.58) is essentially the same as that of the Duffing oscillator in (7.40).

Example: the Morse oscillator. Using dimensional variables, the Morse oscillator is

$$\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0 \qquad \text{with the potential} \qquad V = \frac{\nu}{2}\left(1 - \mathrm{e}^{-\alpha x}\right)^2 . \tag{7.59}$$

The phase plane is shown in figure **??** — the orbits are curves of constant energy

$$E = \frac{1}{2}\dot{x}^2 + \frac{\nu}{2}\left(1 - \mathrm{e}^{-\alpha x}\right)^2 . \tag{7.60}$$

There is a turning point at $x = \infty$ corresponding to the "energy of escape" $E_{\text{escape}} = \nu/2$.
A "natural" choice of non-dimensional variables is

$$\bar{x} \stackrel{\text{def}}{=} \alpha x\,, \qquad \text{and} \qquad \bar{t} = \alpha\sqrt{\nu}t\,. \tag{7.61}$$

In these variables, $\nu \to 1$ and $\alpha \to 1$ in the barred equations. Thus, the non-dimensional equation of motion is

$$\ddot{x} + \mathrm{e}^{-x}\left(1 - \mathrm{e}^{-x}\right) = 0\,. \tag{7.62}$$

If we're interested in small oscillations around the minimum of the potential at $x = 0$, then the small parameter is supplied by an initial condition such as

$$x(0) = \epsilon\,, \qquad \text{and} \qquad \dot{x}(0) = 0\,. \tag{7.63}$$

We rescale with

$$x = \epsilon X\,, \tag{7.64}$$

so that the equation is

$$\epsilon\ddot{X} + \mathrm{e}^{-\epsilon X}\left(1 - \mathrm{e}^{-\epsilon X}\right) = 0\,, \tag{7.65}$$

or

$$\ddot{X} + X - \epsilon\frac{3}{2}X^2 + \frac{7}{6}\epsilon^2 X^3 = \mathrm{ord}\left(\epsilon^3\right) . \tag{7.66}$$

The multiple time scale expansion is now

$$X = X_0(s,t) + \epsilon X_1(s,t) + \epsilon^2 X_2(s,t) + \cdots \tag{7.67}$$

with slow time $s = \epsilon^2 t$.
The main point of this example is that it is necessary to proceed to order ϵ^2, and therefore to retain the term $7\epsilon^2 X^3/6$, to obtain the amplitude equation. One finds

$$\mathrm{i}A_s = \tag{7.68}$$

7.4 Symmetry and the universality of the Landau equation

So far the two-time expansion always leads to the Landau equation

$$A_s = pA + q|A|^2 A\,. \tag{7.69}$$

If you dutifully solve some of the early problems in this lecture you'll obtain (7.69) again and again. Why is that? If we simply list all the terms up to cubic order that *might* occur in an amplitude equation we have

$$\underline{A_s} = \underline{?A} + ?A^* + ?A^2 + ?|A|^2 + ?A^{*2} + ?A^3 + \underline{?|A|^2 A} + ?|A|^2 A^* + ?A^{*3} + \cdots \tag{7.70}$$

The coefficients are denoted by "?" and we're not interested in the precise value of these numbers, except in so far as most of them turn out to be zero. The answer in (7.69) is simple because we have two terms on the right, instead of the nine in (7.70). We've been down in the weeds calculating, but we have not asked the big question why do we have to calculate only the two coefficients p and q?

We have been considering only *autonomous* differential equations, such as

$$\frac{\mathrm{d}x}{\mathrm{d}t} + x + \epsilon x^5 = 0\,. \tag{7.71}$$

This means that if $x(t)$ is a solution of (7.71) then so is $x(t-\alpha)$, where α is *any* constant. In other words, the equations we've been considering are unchanged ("invariant") if

$$t \to t + \alpha\,. \tag{7.72}$$

Now if we try to solve (7.69) with a solution of the form

$$x(t) = A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t} + \epsilon x_1(t,s) + \cdots \tag{7.73}$$

then

$$x(t+\alpha) = A\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{e}^{-\mathrm{i}t} + \epsilon x_1 + \cdots \tag{7.74}$$

Thus the time-translation symmetry of the original differential equation implies that the amplitude equation should be invariant under the rotation

$$A \to A\mathrm{e}^{\mathrm{i}\alpha}\,, \tag{7.75}$$

where α is any constant. Only the underlined terms in (7.70) respect this symmetry and therefore only the underlined terms can appear in the amplitude equation.

Exercise: many of our examples have time reversal symmetry i.e., the equation is invariant under $t \to -t$. For example, the nonlinear oscillator (with no damping) is invariant under $t \to -t$. Show that this implies that p and q in (7.69) must be pure imaginary.

7.5 The resonantly forced Duffing oscillator

The linear oscillator

First consider the forced linear oscillator

$$\ddot{x} + \mu\dot{x} + \omega^2 x = f\cos\sigma t\,. \tag{7.76}$$

We can find the "steady solution" with

$$x = X\mathrm{e}^{\mathrm{i}\sigma t} + X^*\mathrm{e}^{-\mathrm{i}\sigma t} \tag{7.77}$$

After some easy algebra

$$X = \frac{f}{2}\frac{1}{\omega^2 - \sigma^2 + \mathrm{i}\mu\omega}\,, \tag{7.78}$$

and the squared amplitude of the response is

$$|X|^2 = \frac{f^2}{4}\frac{1}{(\omega^2-\sigma^2)^2 + \mu^2\omega^2}\,. \tag{7.79}$$

We view $|X|^2$ as a function of the forcing frequency σ and notice there is a maximum at $\sigma = \omega$ i.e., when the oscillator is resonantly forced. The maximum response, namely

$$\max_{\forall\sigma} |X| = \frac{f}{2\mu\omega}\,, \tag{7.80}$$

is limited by the damping μ.

In the neighbourhood of this peak the amplitude in (7.79) can approximated by the Lorentzian

$$|X|^2 \approx \frac{f^2}{4\omega^2} \frac{1}{4(\omega - \sigma)^2 + \mu^2} . \tag{7.81}$$

The difference between ω and σ is *de-tuning*.

Nondimensionalization of the nonlinear oscillator

Now consider the forced and damped Duffing oscillator:

$$\ddot{x} + \mu \dot{x} + \omega^2 x + \beta x^3 = f \cos \sigma t . \tag{7.82}$$

We're interested in the weakly damped and nearly resonant problem. That is μ/ω is small and σ is close to ω. Inspired by the linear solution we define non-dimensional variables

$$\bar{t} = \omega t , \qquad \text{and} \qquad \bar{x} = \frac{\mu \omega x}{f} . \tag{7.83}$$

The non-dimensional equation is then

$$\bar{x}_{\bar{t}\bar{t}} + \epsilon \bar{x}_{\bar{t}} + x + \epsilon \beta_1 x^3 = \epsilon \cos \left[(1 + \epsilon \sigma_1) \, \bar{t} \right] . \tag{7.84}$$

The non-dimensional parameters above are

$$\epsilon \stackrel{\text{def}}{=} \frac{\mu}{\omega} \ll 1 , \tag{7.85}$$

and

$$\beta_1 \stackrel{\text{def}}{=} \frac{\beta f^2}{\mu^2 \omega^3} , \qquad \sigma_1 \stackrel{\text{def}}{=} \frac{1}{\epsilon} \left(\frac{\sigma}{\omega} - 1 \right) = \frac{\sigma - \omega}{\mu} . \tag{7.86}$$

We refer to σ_1 as the "detuning".

Notice how we have used the solution of the linear problem to make a non-obvious definition of non-dimensional variables. We now take the *distinguished limit* $\epsilon \to 0$ with β_1 and σ_1 fixed.

The amplitude equation and its solution

We attack (7.84) with our multiple-scale expansion

$$x = A(s) \mathrm{e}^{\mathrm{i}t} + A^*(s) \mathrm{e}^{-\mathrm{i}t} + \epsilon x_1(t, s) + \cdots \tag{7.87}$$

The amplitude equation that emerges at order ϵ^1 is

$$A_s + \frac{1}{2} A - \frac{3\mathrm{i}}{2} \beta_1 |A|^2 A = -\frac{\mathrm{i}}{4} \mathrm{e}^{\mathrm{i}\sigma_1 s} . \tag{7.88}$$

We can remove the s-dependence by going to a rotating frame

$$A(s) = B(s) \mathrm{e}^{\mathrm{i}\sigma_1 s} . \tag{7.89}$$

In terms of B we have

$$B_s + \frac{1}{2} B + \mathrm{i} \left[\sigma_1 - \frac{3}{2} \beta_1 |B|^2 \right] B = -\frac{\mathrm{i}}{4} . \tag{7.90}$$

The term in the square brackets is an amplitude-dependent frequency.

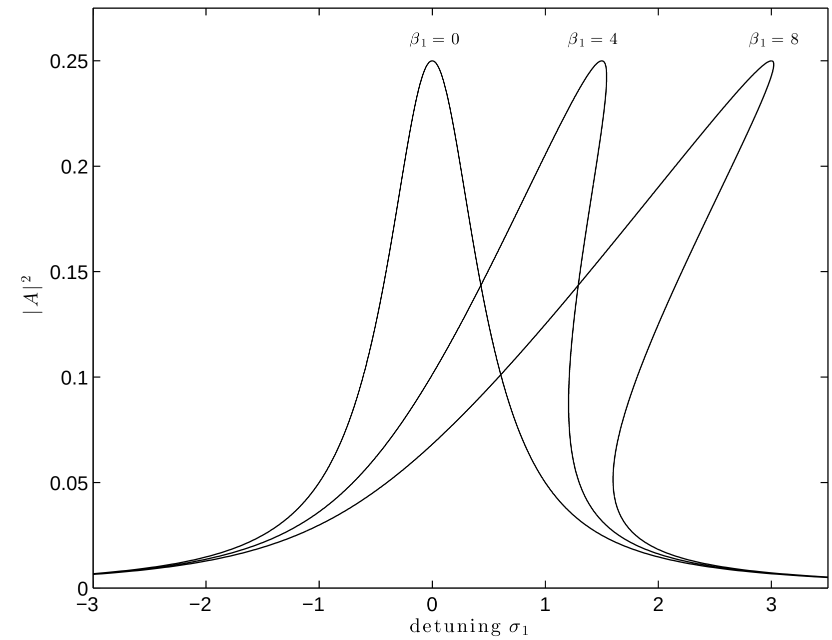

Figure 7.3: The amplitude as a function of detuning obtained from (7.91). Notice one can solve for σ_1 as a single-valued function of $|B|^2$ — it is not necessary to solve a cubic equation to draw this figure.

Now look for steady solutions — we find

$$|B|^2 = \frac{1}{4}\frac{1}{1+(2\sigma_1 - 3\beta_1|B|^2)^2}\,. \tag{7.91}$$

If we set $\beta_1 = 0$ we recover a non-dimensional version of our earlier Lorentzian approximation to the response curve of a linear oscillator. With non-zero β_1 we might have to solve a cubic equation to determine the steady state amplitude: see figure 7.3. There are "multiple solutions" i.e., for the same detuning σ_1 there as many as three solutions for $|B|^2$. The middle branch is unstable — the system ends up on either the lower or upper branch, depending on initial conditions. Figure 7.4 illustrates the two different attracting solutions.

Notice that the scaled problem has three non-dimensional parameters, ϵ, β_1 and σ_1. But in (7.88) only β_1 and σ_1 appear. (Of course ϵ is hidden in the definition of the slow time s.) These perturbation expansions are called *reductive* because they reduce the number of non-dimensional parameters by taking a distinguished limit.

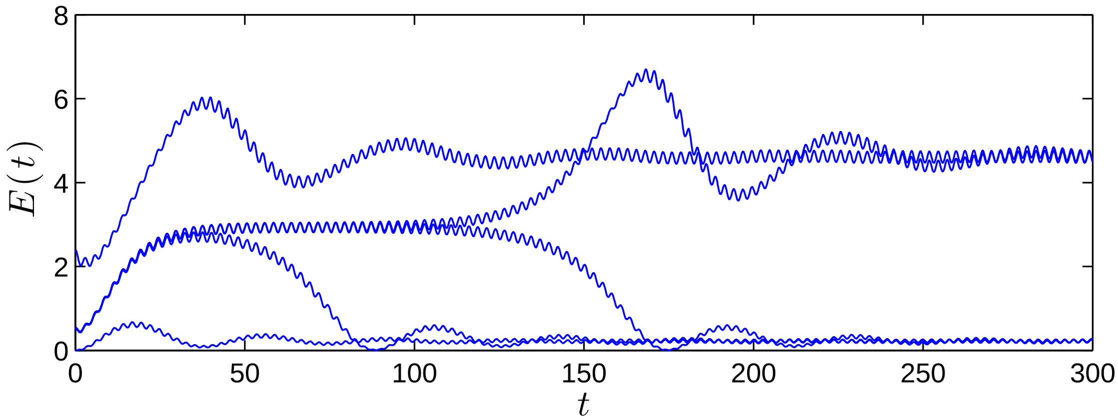

Figure 7.4: Energy $E = (\dot{x}^2 + \omega^2 x^2)/2 + \beta x^4/4$ as a function of time for five `ode45` solutions of the forced Duffing equation (7.82) differing only in initial conditions. There is a high energy attractor that collects two of the solutions, and a low energy attractor that gets the other three solutions. The MATLAB code is below. Note how the differential equation is defined in the nested function `oscill` so that the parameters `om`, `mu` defined in the main fundtion `ForcedDuffing` are passed.

```
function ForcedDuffing
% Multiple solutions of the forced Duffing equation
% Slightly different initial conditions fall on different limit cycles
tspan = [0 300];   om =1;  mu =0.05;   beta = 0.1;   f = 0.25;
sig = 1.2*om;       yinit = [0 1 1.0188 1.0189  2];
for n=1:1:length(yinit)
    yZero=[yinit(n) 0];
    [t,y] = ode45(@oscill,tspan,yZero);
    %Use the linear energy E as an index of amplitude
    E = 0.5*( om*om* y(:,1).^2 + 0.5*beta*y(:,1).^4 + y(:,2).^2 );
    subplot(2,1,1)          plot(t,E(:))
    xlabel('$t$','interpreter','latex','fontsize',16)
    ylabel('$E(t)$','interpreter','latex','fontsize',16)
    hold on
end
%------------- nested function --------------------%
    function dydt = oscill(t,y)
        dydt = zeros(2,1);
        dydt(1) = y(2);
        dydt(2) = - mu*y(2) - om^2*y(1) - beta*y(1)^3 + f*cos( sig*t );
    end
end
```

7.6 Problems

Problem 7.1. In an early lecture we compared the exact solution of the initial value problem

$$\ddot{f} + (1+\epsilon)f = 0\,, \qquad \text{with ICs} \quad f(0) = 1\,, \quad \text{and} \quad \dot{f}(0) = 0\,, \tag{7.92}$$

with an approximation based on a regular perturbation expansion — see the discussion surrounding (2.50). Redo this problem with a two-time expansion. Compare your answer with the exact solution and explain the limitations of the two-time expansion.

Problem 7.2. Consider

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} + \left[1 + \epsilon\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2\right] g = 0\,, \qquad \text{with ICs} \qquad g(0) = 1\,, \quad \text{and} \quad \frac{\mathrm{d}g}{\mathrm{d}t}(0) = 0\,. \tag{7.93}$$

(i) Show that a RPS fails once $t \sim \epsilon^{-1}$. *(ii)* Use the two-timing method to obtain the solution on the long time scale.

Problem 7.3. Consider the initial value problem:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u = 2 + 2\epsilon u^2\,, \qquad \text{with ICs} \qquad u(0) = \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0\,. \tag{7.94}$$

(i) Supposing that $\epsilon \ll 1$, use the method of multiple time scales ($s = \epsilon t$) to obtain an approximate solution valid on times of order ϵ^{-1}. *(ii)* Consider

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + v = u\,, \qquad \text{with ICs} \qquad v(0) = \frac{\mathrm{d}v}{\mathrm{d}t}(0) = 0\,, \tag{7.95}$$

where $u(t,\epsilon)$ on the right is the solution from part *(i)*. Find a leading-order approximation to $v(t,\epsilon)$, valid on the long time scale $t \sim \epsilon^{-1}$.

Problem 7.4. Consider the initial value problem:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + w = 2\cos(\epsilon t) + 2\epsilon w^2\,, \qquad \text{with ICs} \qquad w(0) = \frac{\mathrm{d}w}{\mathrm{d}t}(0) = 0\,. \tag{7.96}$$

Supposing that $\epsilon \ll 1$, use the method of multiple time scales ($s = \epsilon t$) to obtain an approximate solution valid on times of order ϵ^{-1}.

Problem 7.5. Use multiple scale theory to find an approximate solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = \mathrm{e}^{\epsilon^2 t} + \epsilon \mathrm{e}^{-\epsilon t} x^2\,, \qquad \text{with ICs} \qquad x(0) = \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \tag{7.97}$$

valid on the time scale $t \sim \epsilon^{-1} \ll \epsilon^{-2}$.

Problem 7.6. A multiple scale ($0 < \epsilon \ll 1$) reduction of the system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon y \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{1}{2}\epsilon x^2\,, \tag{7.98}$$

begins with

$$x = \left[A(s)\mathrm{e}^{\mathrm{i}t} + A^*(s)\mathrm{e}^{-\mathrm{i}t}\right] + \epsilon x_1(t,s) + \cdots\,, \qquad y = B(s) + \epsilon y_1(t,s) + \cdots \tag{7.99}$$

where $s = \epsilon t$ is the slow time. *(i)* Find coupled evolution equations for $A(s)$ and $B(s)$. *(ii)* Show that the system from part *(i)* can be reduced to

$$B_s = E - B^2\,, \tag{7.100}$$

where E is a constant of integration. *(iii)* Determine E if the initial conditions are

$$x(0) = 2\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \qquad y(0) = 0\,. \tag{7.101}$$

(iv) With the initial condition above, find

$$\lim_{s\to\infty} B(s)\,. \tag{7.102}$$

Note: even if you can't do part *(ii)*, you can still answer parts *(iii)* and *(iv)* by assuming (7.100).

Problem 7.7. *(a)* Use multiple scales to derive a set of amplitude equations for the two coupled, linear oscillators:

$$\begin{aligned}\ddot{x} + 2\epsilon\alpha\dot{x} + (1 + k\epsilon)x &= 2\epsilon\mu(x - y)\,,\\ \ddot{y} + 2\epsilon\beta\dot{y} + (1 - k\epsilon)y &= 2\epsilon\mu(y - x)\,.\end{aligned} \tag{7.103}$$

(b) Consider the special case $\alpha = \beta = k = 0$. Solve both the amplitude equations and the exact equation with the initial condition $x(0) = 1$, $y(0) = \dot{y}(0) = \dot{x}(0) = 0$. Show that both methods give

$$x(t) \approx \cos\left[(1 - \epsilon\mu)t\right]\cos(\epsilon\mu t)\,. \tag{7.104}$$

Problem 7.8. Consider two nonlinearly coupled oscillators:

$$\ddot{x} + 4x = \epsilon y^2\,, \qquad \ddot{y} + y = -\epsilon\alpha x y\,, \tag{7.105}$$

where $\epsilon \ll 1$. *(a)* Show that the nonlinearly coupled oscillators in (1) have an energy conservation law. *(b)* The multiple scale method begins with

$$x(t) = A(s)e^{2it} + \text{c.c.}\,, \qquad y(t) = B(s)e^{it} + \text{c.c.}\,, \tag{7.106}$$

where $s \stackrel{\text{def}}{=} \epsilon t$ is the "slow time" and A and B are "amplitudes". Find the coupled evolution equations for A and B using the method of multiple scales. *(c)* Show that the amplitude equations have a conservation law

$$|B|^2 - 2\alpha|A|^2 = E\,, \tag{7.107}$$

and use this result to show that

$$4A_{ss} - \alpha E A - 2\alpha^2|A|^2A = 0\,. \tag{7.108}$$

Obtain the analogous equation for $B(s)$. *(d)* Describe the solutions of (7.108) in qualitative terms. Does the sign of α have a qualitative impact on the solution?

Problem 7.9. The equation of motion of a pendulum with length ℓ in a gravitational field g is

$$\ddot{\theta} + \omega^2\sin\theta = 0\,, \qquad \text{with} \qquad \omega^2 \stackrel{\text{def}}{=} \frac{g}{\ell}\,. \tag{7.109}$$

Suppose that the maximum displacement is $\theta_{\max} = \phi$. *(a)* Show that the period P of the oscillation is

$$\omega P = 2\sqrt{2}\int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\phi}}\,.$$

(b) Suppose that $\phi \ll 1$. By approximating the integral above, obtain the coefficient of ϕ^2 in the expansion:

$$\omega P = 2\pi \left[1 + ?\phi^2 + O(\phi^3)\right]$$

(c) Check this result by re-derving it via a multiple scale expansion applied to (7.109). *(d)* A grandfather clock swings to a maximum angle $\phi = 5°$ from the vertical. How many seconds does the clock lose or gain each day if the clock is adjusted to keep perfect time when the swing is $\phi = 2°$?

Problem 7.10. (**H**) Find a leading order approximation to the general solution $x(t, \epsilon)$ and $y(t, \epsilon)$ of the system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon y \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{1}{2}\epsilon \ln x^2 , \tag{7.110}$$

which is valid for $t = \mathrm{ord}(\epsilon^{-1})$. You can quote the result

$$\frac{1}{2\pi} \int_0^{2\pi} \ln \cos^2 \theta \, \mathrm{d}\theta = -\ln 4 . \tag{7.111}$$

Problem 7.11. (**H**) Find the leading order approximation, valid for times of order ϵ^{-1}, to the solution $x(t, \epsilon)$ and $y(t, \epsilon)$ of the system

$$\ddot{x} + \epsilon y \dot{x} + x = y^2 , \qquad \text{and} \qquad \dot{y} = \epsilon(1 + x - y - y^2) , \tag{7.112}$$

with initial conditions $x = 1$, $\dot{x} = 0$ and $y = 0$.

Lecture 8

Rapid fluctuations

In this lecture we consider some unusual examples of the two-timing method.

8.1 A Lotka-Volterra Example

As another example of multiple scale theory, let's consider the Lotka-Volterra equation with a sinusoidally varying carrying capacity:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = n\left(1 - \frac{n}{1+\kappa\cos\omega t}\right) . \tag{8.1}$$

Problem 3.5 asked you to analyze this equation with $\kappa \ll 1$ i.e., small fluctuations in the carrying capacity. Here we consider the case of rapid fluctuations: $\omega \to \infty$, with κ fixed and order unity. In this limit the small parameter is

$$\epsilon \stackrel{\text{def}}{=} \frac{1}{\omega} . \tag{8.2}$$

Figure 8.1 shows a numerical solution: the population fluctuates about an average value which is close to $\sqrt{9/16} = 0.6614$. This average population is quite different from the average carrying capacity, namely 1.

Define a fast time

$$\tau \stackrel{\text{def}}{=} \omega t = \frac{t}{\epsilon} , \tag{8.3}$$

and assume that the solution depends on both t and τ so that

$$\left(\frac{1}{\epsilon}\partial_\tau + \partial_t\right) n = n\left(1 - \frac{n}{1+\kappa\cos\tau}\right) . \tag{8.4}$$

Now we attempt to solve this equation with the multiple time scale expansion

$$n = n_0(t,\tau) + \epsilon n_1(t,\tau) + \cdots \tag{8.5}$$

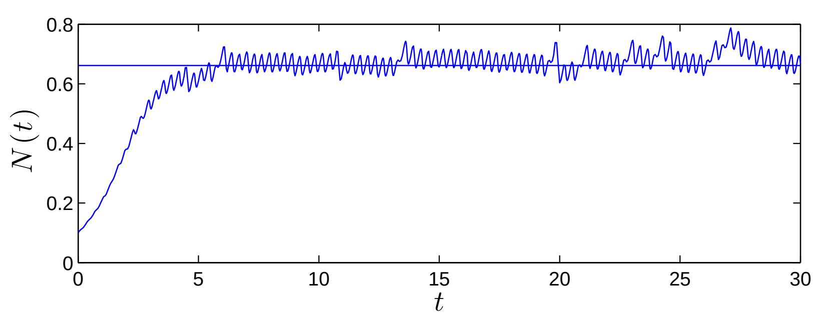

Figure 8.1: An `ode45` solution of the Lotka-Volterra equation (8.1) with $\omega = 20$ and $\kappa = 3/4$.

The leading order is

$$\frac{1}{\epsilon}: \qquad \partial_\tau n_0 = 0\,, \qquad \text{with solution} \qquad n_0 = f(t)\,. \tag{8.6}$$

Although the carrying capacity is varying rapidly, the leading-order population $f(t)$ does not vary on the fast time scale τ. The environment is fluctuating so rapidly that the population does not react e.g., clouds blowing overhead lead to modulations in sunlight on the scale of minutes. But plants don't die when the sun is momentarily obscured by a cloud.

At next order

$$\epsilon^0: \qquad \partial_\tau n_1 + \frac{\mathrm{d}f}{\mathrm{d}t} = f\left(1 - \frac{f}{1+\kappa\cos\tau}\right)\,. \tag{8.7}$$

We average the equation above over the fast time scale[1] to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t} = f\left(1 - \frac{f}{\sqrt{1-\kappa^2}}\right)\,. \tag{8.8}$$

Thus the long time limit is $f \to \sqrt{1-\kappa^2}$. This is in agreement with the MATLAB solution in Figure 8.1. To determine the fluctuations about this average we can subtract (8.8) from (8.7) to obtain

$$\partial_\tau n_1 = \underbrace{\left(\frac{1}{\sqrt{1-\kappa^2}} - \frac{1}{1+\kappa\cos\tau}\right)}_{\text{expand this in a Fourier series}} \underbrace{f^2}_{=1-\kappa^2} \tag{8.9}$$

8.2 Stokes drift

Consider the motion along the x-axis of a fluid particle in a simple compressive wave e.g., a sound wave. The position of the particle is determined by solving the nonlinear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\cos(kx - \omega t)\,, \tag{8.10}$$

with an initial condition $x(0) = a$. We non-dimensionalize this problem by defining

$$\bar{x} \stackrel{\text{def}}{=} kx \qquad \text{and} \qquad \bar{t} \stackrel{\text{def}}{=} \omega t\,. \tag{8.11}$$

The non-dimensional problem is

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \epsilon\cos(\bar{x} - \bar{t})\,, \qquad \text{with IC} \quad \bar{x}(0) = \bar{a}\,. \tag{8.12}$$

The non-dimensional wave amplitude,

$$\epsilon \stackrel{\text{def}}{=} \frac{uk}{\omega}\,, \tag{8.13}$$

is the ratio of the maximum particle speed u to the phase speed ω/k. We proceed dropping all bars.

Figure 8.2 shows some numerical solutions of (8.12) with $\epsilon = 0.3$. Even though the time-average velocity at a fixed point is zero there is a slow motion of the particles along the x-axis with constant average velocity. If one waits long enough then a particle will move very far from its initial position and travel through many wavelengths.

[1]The time average of the reciprocal carrying capacity is computed using:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\tau}{1+\kappa\cos\tau} = \frac{1}{\sqrt{1-\kappa^2}}\,.$$

This integral is a favorite textbook example of the residue theorem.

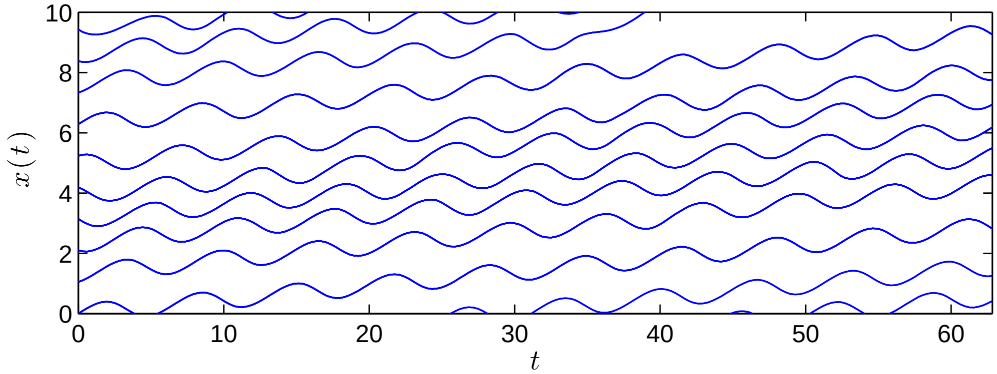

Figure 8.2: Solutions of $\dot{x} = 0.3\cos(x - t)$.

Method 1: RPS to obtain a one-period map

Fortunately the velocity is a periodic function of time. Given the initial condition $x(0)$, we use a straightforward RPS to determine the *one-period map* F:

$$x(2\pi) = F\left[x(0), \epsilon\right] . \tag{8.14}$$

The position of the particle over many periods follows by iteration of this map. That is

$$x(4\pi) = F\left[x(2\pi), \epsilon\right] , \qquad x(6\pi) = F\left[x(4\pi), \epsilon\right] , \qquad \text{etc.} \tag{8.15}$$

We don't have to worry about secular errors because F is determined by solving the differential period over a *finite* time $0 < t < 2\pi$. The second period is just the same as the first.

Method 2: two-timing

To analyze this problem with multiple scale theory we introduce

$$s \stackrel{\text{def}}{=} \epsilon^2 t \,. \tag{8.16}$$

Why ϵ^2 above? Because we tried ϵ^1 and found that there were no secular terms on this time scale.

Exercise: Assume that $s = \epsilon t$ and repeat the following calculation. Does it work?

With the slow time s, the dressed-up problem is

$$\epsilon^2 x_s + x_t = \epsilon \cos(x - t) \,. \tag{8.17}$$

We now go to town with the RPS:

$$x = x_0(s,t) + \epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots \tag{8.18}$$

Notice that

$$\begin{aligned} \cos(x - t) = \cos(x_0 - t) &- \sin(x_0 - t)\left[\epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots\right] \\ &- \cos(x_0 - t)\left[\epsilon x_1(s,t) + \epsilon^2 x_2(s,t) + \cdots\right]^2 + \cdots \end{aligned} \tag{8.19}$$

We cannot assume that x_0 is smaller than one, so must keep $\cos(x_0 - t)$ and $\sin(x_0 - t)$. We are assuming the higher order x_n's are bounded, and since $\epsilon \ll 1$ we can expand the sinusoids as above.

At leading order, ϵ^0:

$$x_{0t} = 0\,, \qquad \Rightarrow \qquad x_0 = f(s)\,. \tag{8.20}$$

The function $f(s)$ is the slow drift. At next order, ϵ^1:

$$x_{1t} = \cos(f - t) \qquad \Rightarrow \qquad x_1 = \sin f - \sin(f - t)\,. \tag{8.21}$$

We determined the constant of integration above so that x_1 is zero initially i.e., we are saying that $f(0)$ is equal to the initial position of the particle.

At ϵ^2

$$f_s + x_{2t} = -\sin(f - t)\,\underbrace{[\sin f - \sin(f - t)]}_{=x_1}\,. \tag{8.22}$$

Averaging over the fast time t we obtain

$$f_s = \big\langle \sin^2(f - t) \big\rangle = \frac{1}{2}\,. \tag{8.23}$$

Thus the average position of the particle is

$$f = \frac{s}{2} + a = \frac{\epsilon^2}{2} t + a\,. \tag{8.24}$$

The prediction is that the averaged velocity in figure 8.2 is $(0.3)^2/2 = 0.045$. You can check this by noting that the final time is 20π.

Subtracting (8.23) from (8.22) we have the remaining oscillatory terms:

$$x_{2t} = -\sin(f - t)\sin f - \sin(2f - 2t)\,. \tag{8.25}$$

Integrating and applying the initial condition we have

$$x_2 = -\cos(f - t)\sin f + \tfrac{1}{4}\cos(2f - 2t) + \cos f \sin f - \tfrac{1}{4}\cos 2f\,. \tag{8.26}$$

This is bounded and all is well.

The solution we've constructed consists of a slow drift and a rapid oscillation about this slowly evolving mean position. Note however that the mean position of the particle is

$$\langle x \rangle = f + \epsilon \underbrace{\sin f}_{\langle x_1 \rangle} + \epsilon^2 \underbrace{\left[\tfrac{1}{2}\sin 2f - \tfrac{1}{4}\cos 2f\right]}_{\langle x_2 \rangle} + \mathrm{ord}(\epsilon^2) \tag{8.27}$$

In other words, the mean position is not the same as the leading-order term.

Method 3: two-timing and the "guiding center"

In this variant we use the two-timing but insist that the leading-order term is the mean position of the particle. This means that the leading-order solution no longer satisfies the initial condition, and that constants of integration at higher orders are determined by insisting that

$$\forall n \geq 1: \qquad \langle x_n \rangle = 0\,. \tag{8.28}$$

OK, let's do it, starting with the scaled two-time equation in (8.17). The leading order is

$$x_{0t} = 0\,, \qquad \Rightarrow \qquad x_0 = g(s)\,. \tag{8.29}$$

The function $g(s)$ is the "guiding center" — it's different from $f(s)$ in the previous method.

At next order, ϵ^1:

$$x_{1t} = \cos(f - t) \qquad \Rightarrow \qquad x_1 = -\sin(g - t)\,. \tag{8.30}$$

This is not the same as the first-order term in (8.21): in (8.30) we have determined the constant of integration so that $\langle x_1 \rangle = 0$.

At order ϵ^2 we have

$$g_s + x_{2t} = \sin^2(g - t) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2g - 2t)\,. \tag{8.31}$$

The average of (8.31) is the motion of the guiding center:

$$g_s = \tfrac{1}{2} \qquad \Rightarrow \qquad g = \frac{\epsilon^2}{2} t + g(0)\,. \tag{8.32}$$

The oscillatory part of the solution, with zero time average, is

$$x_2 = \tfrac{1}{4}\sin(2g - 2t)\,. \tag{8.33}$$

Now we must satisfy the initial conditions by requiring that

$$a = g(0) - \epsilon \sin\left(g(0)\right) + \epsilon^2 \tfrac{1}{4} \sin\left(2g(0)\right) + \cdots \tag{8.34}$$

We can invert this series to obtain

$$g(0) = a + \epsilon \sin a + \cdots \tag{8.35}$$

I prefer this guiding center method. But in either case the essential point is that the leading-order drift velocity is $\epsilon^2/2$.

8.3 Problems

Problem 8.1. Investigate Stokes drift in the two-dimensional incompressible velocity field with streamfunction $\psi(x, y, t)$. The velocity is obtained as

$$\dot{x} = -\psi_y\,, \qquad \dot{y} = \psi_x\,, \tag{8.36}$$

Supposing that the streamfunction has the form

$$\psi = a(x, y)\cos(t/\epsilon) + b(x, y)\sin(t/\epsilon)\,, \tag{8.37}$$

obtain a an expression for the streamfunction of the Stokes flow. Study the special case $a = \cos x$ and $b = \cos y$ numerically.

Problem 8.2. Consider an oscillator with a slowly changing frequency $\omega(\epsilon t)$:

$$\ddot{x} + \omega^2 x = 0\,. \tag{8.38}$$

Use the method of averaging to show that the action $A \overset{\text{def}}{=} E/\omega$ is approximately constant. Test this result with `ode45` using the frequency

$$\omega(t) = 3 + 2\tanh\left(\epsilon t\right)\,, \tag{8.39}$$

and the initial condition $x(-40) = 0$ and $\dot{x}(-40) = 1$ e.g., see Figure 8.3. Use several values of ϵ to test action conservation e.g., try to break the constant-action approximation with large ϵ.

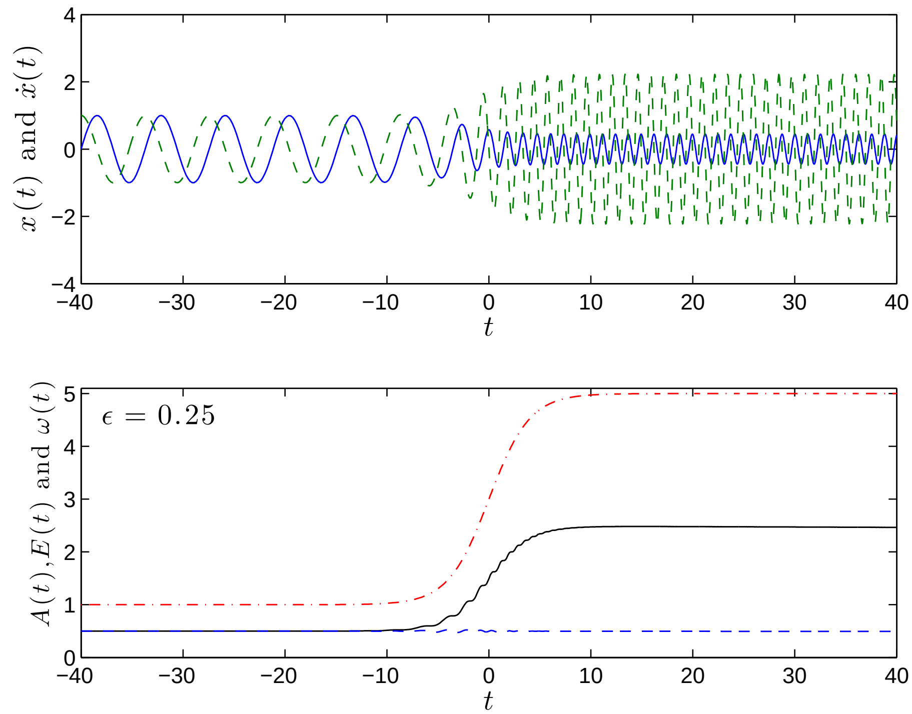

Figure 8.3: Solution of (8.38) and (8.39) with `ode45`. In the lower panel the almost constant action is the blue dashed curve.

Lecture 9

Eigenvalue problems

9.1 Regular Sturm-Liouville problems

The second-order differential equation associated with a Sturm-Liouville eigenproblem has the form

$$(p\phi')' - q\phi + \lambda w\phi = 0\,, \tag{9.1}$$

where $p(x)$, $q(x)$ and $r(x)$ are real functions. The associated boundary value problem is posed on $a < x < b$ with BCs which we write as

$$\alpha\phi(a) - \alpha'\phi'(a) = 0\,, \qquad \text{and} \qquad \beta\phi(b) + \beta'\phi'(b) = 0\,, \tag{9.2}$$

where $[\alpha, \alpha', \beta, \beta']$ are real constants.

The sign convention in (9.2) is so that (9.27) below looks pretty. In physical problems, such as radiation of heat through the boundaries at $x = a$ and b, the mixed boundary condition is

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} + (\text{a positive constant}) \times \phi = 0\,, \tag{9.3}$$

where n is the *outwards* normal at the boundary. At $x = a < b$ the outwards normal implies $n = -x$. Thus if $[\alpha, \alpha', \beta, \beta']$ are all positive then the boundary conditions in (9.2) conform to (9.3) at $x = a$ and b.

Example: The archetypical SL eigenproblem with Dirichlet boundary conditions is:

$$-\phi'' = \lambda\phi \qquad \text{with BCs} \qquad \phi(0) = \phi(\pi) = 0\,. \tag{9.4}$$

Prove that $\lambda \geq 0$ before attempting a solution.

Example: The archetypical SL eigenproblem with Neuman boundary conditions is:

$$-\phi'' = \lambda\phi\,, \qquad \text{with BCs} \qquad \phi'(0) = \phi'(\pi) = 0\,. \tag{9.5}$$

Prove that $\lambda \geq 0$ before attempting a solution.

Example: A soluble example with non-constant coefficients:

$$-\phi'' = \lambda x^{-2}\phi\,, \qquad \text{with BCs} \qquad \phi(1) = \phi(\ell) = 0\,. \tag{9.6}$$

First we prove that $\lambda \geq 0$. The differential equation is an Euler equation which can be solved with $\phi = x^\nu$. Thus we find that

$$\nu^2 - \nu + \lambda = 0\,, \qquad \text{or} \qquad \nu = \tfrac{1}{2} \pm \sqrt{\lambda - \tfrac{1}{4}}\,. \tag{9.7}$$

We have to consider two cases: (a) $\lambda < 1/4$ and (b) $\lambda > 1/4$. In case (a) we have two real ν's, but there is no way to combine these into a solution that satisfies the boundary conditions. So we turn to case (b), for which

$$\lambda = \tfrac{1}{2} \pm \mathrm{i}\sqrt{\lambda - \tfrac{1}{4}} \tag{9.8}$$

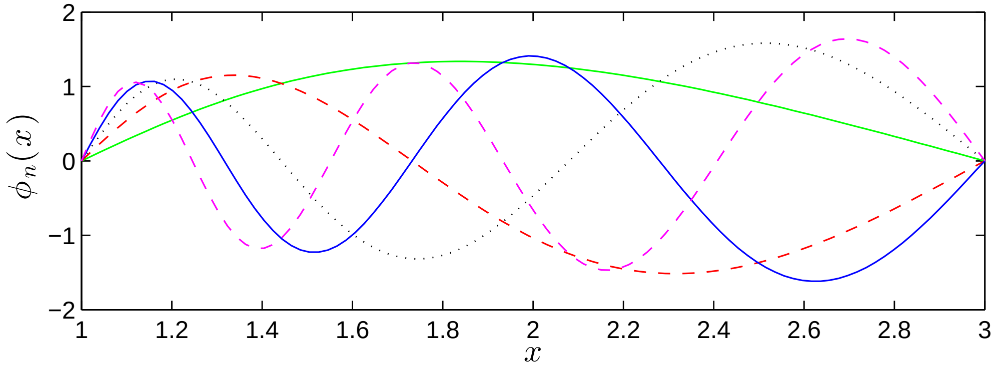

Figure 9.1: The first five Sturm-Liouville eigenfunctions in (9.12) with $\ell = 3$.

and

$$x^\nu = x^{1/2}\left[\cos\left(\sqrt{\lambda - \tfrac{1}{4}}\,\ln x\right) \pm \mathrm{i}\sin\left(\sqrt{\lambda - \tfrac{1}{4}}\,\ln x\right)\right] \tag{9.9}$$

We can linearly combine the two solutions above to obtain a solution satisfying the BC at $x = 1$

$$\phi = x^{1/2}\sin\left(\sqrt{\lambda - \tfrac{1}{4}}\,\ln x\right) \tag{9.10}$$

The second boundary condition at $x = \ell > 1$ determines the eigenvalue

$$\lambda = \tfrac{1}{4} + \left(\frac{n\pi}{\ln \ell}\right)^2 , \tag{9.11}$$

and therefore

$$\phi = x^{1/2}\sin\left(n\pi\frac{\ln x}{\ln \ell}\right) . \tag{9.12}$$

The first five eigenfunctions are shown in figure 9.1. Notice that the n'th eigenfunction has $n-1$ interior zeros.

Example: We'll use the SL problem

$$-\phi'' = \lambda x \phi \tag{9.13}$$

as an Airy-function reminder. The problem is posed on the interval $0 < x < \ell$ with Dirichlet boundary conditions

$$\phi(0) = \phi(\ell) = 0 . \tag{9.14}$$

This SL problem is a little irregular because the weight function is zero at $x = 0$. But, once again, we can show that λ is positive. The analytic solution of (9.13) is

$$\phi = \frac{\mathrm{Ai}\left(-x\lambda^{1/3}\right)}{\mathrm{Ai}(0)} - \frac{\mathrm{Bi}\left(-x\lambda^{1/3}\right)}{\mathrm{Bi}(0)} . \tag{9.15}$$

It is certainly reassuring to know that $\lambda^{1/3}$ in the equation above is a real positive number. The construction above satisfies both the differential equation and the boundary condition at $x = 0$. The boundary condition at $x = \ell$ produces the eigenrelation

$$\frac{\mathrm{Ai}\,(-\eta)}{\mathrm{Ai}(0)} = \frac{\mathrm{Bi}\,(-\eta)}{\mathrm{Bi}(0)} , \tag{9.16}$$

where $\eta \overset{\text{def}}{=} \ell\lambda^{1/3}$. If you look at the graphs of the Airy and Bairy function in the upper panel of Figure 9.2 then you can anticipate that this eigenrelation has an infinite number of solutions. The first five are

$$\ell\lambda^{1/3} = [2.6664\,, 4.3425\,, 5.7410\,, 6.9861\,, 8.1288]\ , \tag{9.17}$$

and the lower panel of Figure 9.2 shows the corresponding eigenfunctions. Again, notice that the n'th eigenfunction has $n-1$ interior zeros.

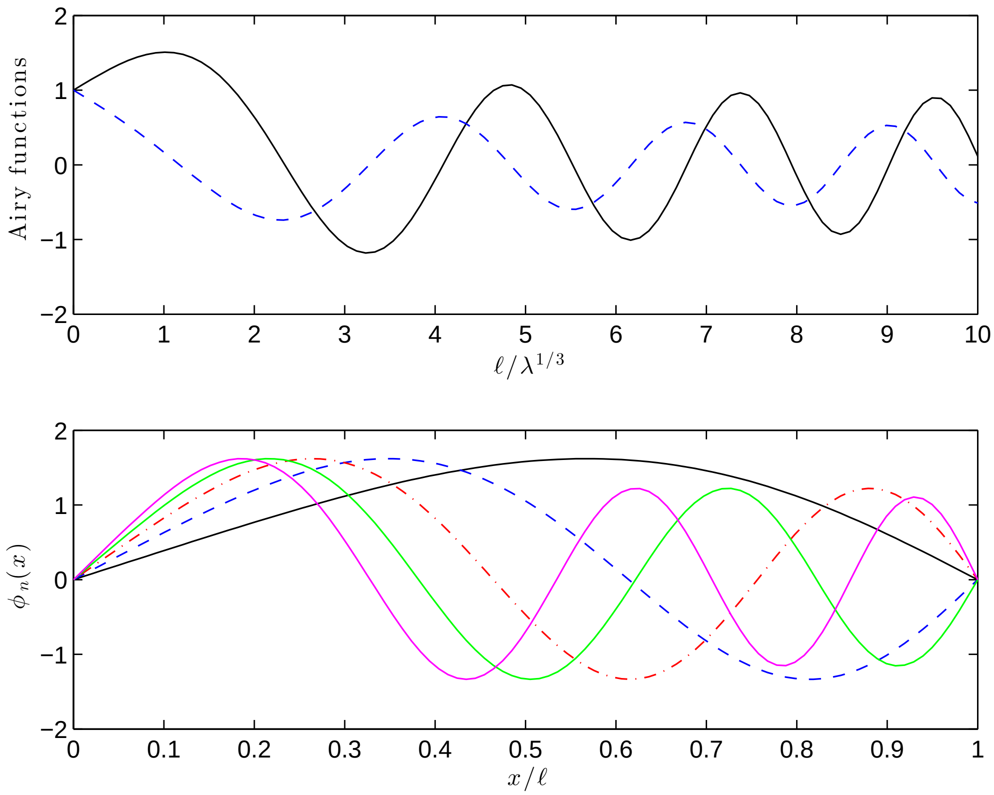

Figure 9.2: The upper panel shows the two Airy functions on the right of (9.16); each interesection corresponds to an eignevalue. The lower panel shows the first five eigenfunctions corresponding to the eigenvalues in (9.17).

9.2 Properties of Sturm-Liouville eigenproblems

We write the differential equation in (9.1) as

$$\mathsf{L}\phi = \lambda w \phi\,, \tag{9.18}$$

where L is the second-order differential operator

$$\mathsf{L} \stackrel{\text{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}x} p \frac{\mathrm{d}}{\mathrm{d}x} + q\,. \tag{9.19}$$

L is defined with the minus sign so that if p is a positive function then L is a positive operator.

The operator L is hungry, looking for a function to differentiate. If $p(x) > 0$ and $w(x) > 0$ on $[a, b]$ and p, q and w are all free from bad singularities, then the problem is *regular*.

The main results for regular problems are:

1. The eigenvalues are real, countable and ordered so that there is a smallest one

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{9.20}$$

   Moreover $\lim_{n\to\infty} \lambda_n = \infty$.

2. The zeroth eigenfunction (also known as the gravest mode, or the ground state) has no interior zeroes. Eigenfunction $\phi_{n+1}(x)$ has one more interior zero than $\phi_n(x)$.

3. Eigenfunctions with different eigenvalues are orthogonal with respect to the weight function w:

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,\mathrm{d}x = \xi_n \delta_{mn}\,. \tag{9.21}$$

   Often, but not always, we normalize the eigenfunctions so that $\xi_n = 1$.

4. *Within* (a, b), any square integrable[1] function $f(x)$ can be represented as

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)\,, \tag{9.22}$$

   with

$$f_n = \xi_n^{-1} \int_a^b \phi_n(x) f(x) w(x)\,\mathrm{d}x\,. \tag{9.23}$$

5. Considering the truncated sum

$$\hat{f}_N(x) = \sum_{n=1}^{N} f_n \phi_n(x)\,, \tag{9.24}$$

   as an approximation to the target function $f(x)$, we define the error in $\hat{f}_N(x)$ as

$$e(f_1, f_2, \cdots f_N) \stackrel{\text{def}}{=} \int_a^b \left[f - \sum_{n=1}^{N} f_n \phi_n \right]^2 w\,\mathrm{d}x\,. \tag{9.25}$$

   If we adjust f_1 through f_N to minimize the error e then we recover the expression for f_n in (9.23). Moreover

$$\lim_{N\to\infty} e = 0\,. \tag{9.26}$$

[1] Square integrable means with respect to the weight function. That is

$$\int_a^b f^2 w \mathrm{d}x < \infty\,.$$

Eigenvalue are real

To prove that all eigenvalues are real, multiply (9.1) by $\phi^*(x)$ and integrate over the interval. With IP we find

$$\lambda = \frac{\int_a^b p|\phi'|^2 + q|\phi|^2 \mathrm{d}x + \hat\beta p(b)|\phi'(b)|^2 + \hat\alpha p(a)|\phi'(a)|^2}{\int_a^b |\phi|^2 w \, \mathrm{d}x} \,, \tag{9.27}$$

where $\hat\beta \stackrel{\text{def}}{=} \beta'/\beta$ and $\hat\alpha \stackrel{\text{def}}{=} \alpha'/\alpha$. The right hand side — which is manifestly real — is known as the Rayleigh quotient. If $q(x)$, $\hat\alpha$ and $\hat\beta$ are all non-negative then the Rayleigh quotient also shows that the eigenvalues are positive.

Exercise: how does (9.27) change if $\beta = 0$ and/or $\alpha = 0$?

Eigenfunctions are orthogonal

To prove orthogonality above we first prove

$$u\mathsf{L}v - v\mathsf{L}u = \left[p(vu' - uv')\right]' \,, \tag{9.28}$$

and therefore

$$\int_a^b u\mathsf{L}v \, \mathrm{d}x - \int_a^b v\mathsf{L}u \, \mathrm{d}x = \left[p(vu' - v'u)\right]_a^b . \tag{9.29}$$

Exercise: use the identity above to prove (9.21)

The eigenfunction expansion minimizes the squared error

Exercise: prove that f_n in (9.23) minimizes $e(f_1, f_2, \cdots f_N)$ in (9.25).

Example: expand $f(x) = 1$ in terms of $\sin nx$ (again).

9.3 Trouble with BVPs

Not all BVPs have solutions. An example is the thermal diffusion problem

$$\psi_t = \psi_{xx} + 1 \,, \tag{9.30}$$

posed on $0 < x < 1$ with insulating boundary conditions

$$\psi_x(0, t) = \psi_x(1, t) = 0 \,. \tag{9.31}$$

For obvious reasons there are no steady solutions to this partial differential equation: the BVP

$$-\psi_{xx} = 1 \,, \qquad \text{with} \qquad \psi_x(0) = \psi_x(1) = 0 \,, \tag{9.32}$$

has no solutions. To see this mathematically, integrate the differential equation in (9.32) from $x = 0$ to $x = 1$:

$$-\underbrace{[\psi_x]_0^1}_{=0} = 1 \,. \tag{9.33}$$

The contradiction tells us not to bother looking for a steady solution to (9.30). Instead, we should solve the diffusion equation (9.30) with the ansatz $\psi = t + \hat\psi(x, t)$, where $\hat\psi$ is required to satisfy the initial conditions.

Exercise: show that $\lim_{t\to\infty} \hat\psi = 0$.

In other examples, the non-existence of solutions is less obvious. Consider the example

$$y'' + y = \ln x\,, \qquad \text{with} \qquad y(0) = y(\pi) = 0\,. \tag{9.34}$$

If we multiply the differential equation by $\sin x$ and integrate from $x = 0$ to π then, with some IP, we quickly obtain

$$\underbrace{\int_0^\pi \sin x\,(y'' + y)\,\mathrm{d}x}_{=0} = \int_0^\pi \sin x \ln x\,\mathrm{d}x = 0.641182\,. \tag{9.35}$$

Again, the contradiction tells us that there is no point in looking for a solution.

We can write (9.34) using fancy operator notation as

$$\mathsf{L}y = \ln\frac{1}{x}\,, \qquad \text{where } \mathsf{L} \overset{\text{def}}{=} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1. \tag{9.36}$$

Now let's do the calculation using some Sturm-Liouville machinery: Multiply by $\sin x$, integrate over the interval, use the identity (9.28) and use the fact that $\sin x$ and y are zero on both boundaries. Thus again we obtain the contradiction

$$\int y\,\underbrace{\mathsf{L}\sin x}_{=0}\,\mathrm{d}x = \int_0^\pi \sin x \ln x\,\mathrm{d}x \tag{9.37}$$

It's the same calculation as before. You should get used to seeing that

$$\int u\mathsf{L}v\mathrm{d}x = \int v\mathsf{L}u\mathrm{d}x\,. \tag{9.38}$$

If you use the identity above be sure to check that the boundary conditions imply that all the terms falling outside the integral are zero! Go back to (9.28) and see what's required for that to happen.....

More generally we might confront

$$y'' + y = f(x)\,, \qquad \text{with} \qquad y(0) = y(\pi) = 0\,, \tag{9.39}$$

where $f(x)$ is some function. If we multiply the differential equation by $\sin x$ and integrate from $x = 0$ to π we have

$$\int_0^\pi \sin x f(x)\,\mathrm{d}x = 0\,. \tag{9.40}$$

In order for a solution of (9.39) to exist, the *solvability condition* in (9.40) must be satisfied. And if (9.40) is satisfied then (9.39) has an infinity of solutions: if $y(x)$ is a solution of (9.39) then $y(x) + a\sin x$ is also a solution for every value of the constant a.

To understand what's going on here, we turn to the eigenfunction expansion method.

9.4 The eigenfunction expansion method

Suppose that somehow we have solved the eigenvalue problem

$$\mathsf{L}\phi = \lambda w\phi\,, \tag{9.41}$$

with BCs $\phi_n(a) = \phi_n(b) = 0$. This means we possess the full spectrum λ_n and $\phi_n(x)$ with $n = 0$, $1 \cdots$

The solution of the boundary value problem

$$\mathsf{L}y = f\,, \tag{9.42}$$

with BCs $y(a) = 0$ and $y(b) = 0$, is represented as

$$y(x) = \sum_{n=0}^{\infty} y_n \phi_n(x)\,, \qquad \text{with} \qquad y_m = \xi_m^{-1} \int_a^b \phi_m y\, w\, \mathrm{d}x\,. \tag{9.43}$$

Multiply (9.42) by $\phi_m(x)$ and integrate over the interval. Using the identity in (9.28), and the boundary conditions, the left hand side of (9.42) produces

$$\begin{aligned}\int_a^b \phi_m \mathsf{L}y\, \mathrm{d}x &= \int_a^b y \mathsf{L}\phi_m\, \mathrm{d}x\,,\\ &= \lambda_m \int_a^b y\phi_m w\, \mathrm{d}x\,,\\ &= \lambda_m \xi_m y_m\,.\end{aligned} \tag{9.44}$$

Thus

$$y_m = \frac{\int_a^b \phi_m f\, \mathrm{d}x}{\lambda_m \xi_m} \tag{9.45}$$

This is OK provided that *all the eigenvalues are non-zero*. If there is a zero eigenvalue then this problem has no solution.

Example: Solve

$$y'' = \ln x \tag{9.46}$$

with $y(0) = y(\pi) = 0$. In this example the weight function is $w(x) = 1$ and

$$\mathsf{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \phi_n = \sin nx\,, \qquad \lambda_n = n^2\,, \tag{9.47}$$

with $n = 1,\ 2 \cdots$ There are no zero modes, and therefore there is no problem with existence of the solution. The expansion coefficients are

$$y_n = -\frac{2}{\pi n^2} \int_0^{\pi} \sin nx\ \ln x\, \mathrm{d}x\,. \tag{9.48}$$

Evaluating the integral in (9.48) numerically, we find that the first six terms in the series solution are

$$\begin{aligned}y_1 = -0.408189 \sin x + 0.193982 \sin 2x + 0.0122721 \sin 3x&\\ + 0.0309791 \sin 4x + 0.0052863 \sin 5x + 0.0103642 \sin 6x + \cdots&\end{aligned} \tag{9.49}$$

Figure 9.3 compares the exact solution of this BVP, namely

$$y = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\pi x \ln \pi + \frac{3}{4}\left(\pi x - x^2\right)\,, \tag{9.50}$$

with (9.49) truncated with 2, 4 and 6 terms.

Example: Reconsider our earlier example

$$\underbrace{y'' + y}_{-\mathsf{L}y} = \ln x\,, \qquad \text{with BCs} \qquad y(0) = y(\pi) = 0\,. \tag{9.51}$$

We first find the eigenspectrum by solving

$$\mathsf{L}\phi = \lambda\phi \qquad \phi(0) = \phi(\pi) = 0\,. \tag{9.52}$$

The eigenfunctions are

$$\phi_n(x) = \sin nx\,, \qquad n = 1\,,\ 2\,, 3\,, \cdots \tag{9.53}$$

with eigenvalues

$$\lambda_n = n^2 - 1 = 0,\ 1, 3,\ 8 \cdots \tag{9.54}$$

Oh oh, there is a zero eigenvalue. So there is no solution.

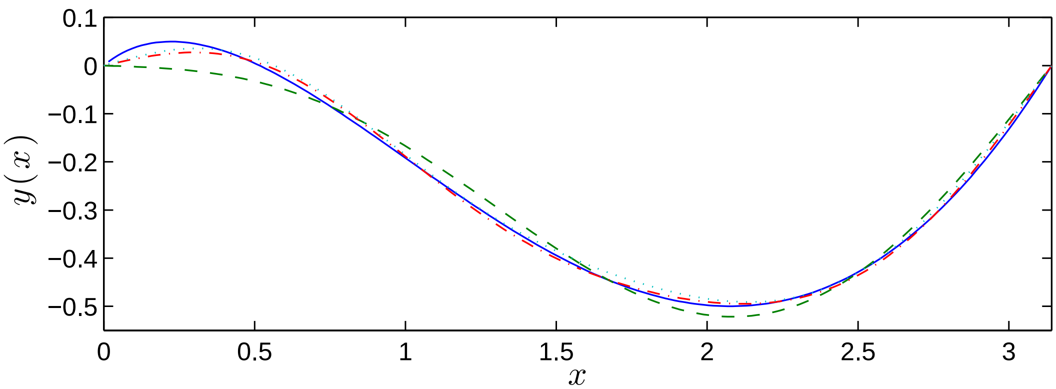

Figure 9.3: Comparison of the exact solution (9.50) (the solid curve) with truncations of the series (9.49). The two-term truncation is the green dashed curve, the four-term truncation is the red dash-dot curve and the six term truncation is the dotted cyan curve.

Example: Find a so that the problem

$$y'' + y = \ln x + a\,, \qquad \text{with BCs} \qquad y(0) = y(\pi) = 0\,, \tag{9.55}$$

has a solution. Construct the most general solution as an eigenfunction expansion.
We know that $\sin x$ is a zero-eigenmode, and therefore the solvability condition is

$$0 = \int_0^\pi (\ln x + a)\sin x\,\mathrm{d}x \qquad \text{implying that} \qquad a = \frac{\int_0^\pi \sin x\,\ln\frac{1}{x}\,\mathrm{d}x}{\int_0^\pi \sin x\,\mathrm{d}x} = -0.320591\,. \tag{9.56}$$

Now we can solve the problem using the eigenmodes defined by the SL problem

$$\underbrace{-\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1\right)}_{\mathsf{L}}\phi = \lambda\phi\,, \qquad \text{with} \qquad \phi(0) = \phi(\pi) = 0\,. \tag{9.57}$$

The eigenvalues are

$$\lambda_n = n^2 - 1\,, \qquad \text{with eigenfunction} \qquad \phi_n(x) = \sin nx\,. \tag{9.58}$$

Operationally, we multiply (9.55) by $\sin mx$ and integrate over the interval from $x = 0$ to π. Thus

$$(1 - m^2)\underbrace{\int_0^\pi y\sin mx\,\mathrm{d}x}_{\frac{\pi}{2}y_m} = \int_0^\pi (\ln x + a)\sin mx\,\mathrm{d}x\,, \qquad \text{with } m = 2, 3, \cdots \tag{9.59}$$

The solution of the BVP is

$$y = A\sin x - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\int_0^\pi (\ln x + a)\sin nx\,\mathrm{d}x}{n^2 - 1}\sin nx\,, \tag{9.60}$$

where A is an arbitrary constant.

Exercise: Show that the BVP

$$y'' = \ln x\,, \qquad \text{with BCs} \qquad y'(0) = y'(\pi) = 0\,, \tag{9.61}$$

has no solution. Determine a so that the related BVP

$$y'' = \ln x + a\,, \qquad \text{with BCs} \qquad y'(0) = y'(\pi) = 0\,, \tag{9.62}$$

has a solution. Find the most general solution of this problem by direct integration, and by the eigenfunction expansion method.

9.5 Eigenvalue perturbations

Consider a string with slightly non-uniform mass density

$$\rho(x) = \rho_0 \left[1 + \epsilon b(x)\right] . \tag{9.63}$$

We compute the change in the eigenfrequencies induced by the small non-uniformity in density.

Using nondimensional variables, the eigenproblem is

$$\psi'' + \lambda(1 + \epsilon b)\psi = 0 , \tag{9.64}$$

with $\psi(0) = \psi(\pi) = 0$.

First suppose that $\epsilon = 0$. We quickly see that the solution of the $\epsilon = 0$ eigenproblem is

$$\lambda = k^2 , \qquad \text{with corresponding eigenfunction} \qquad s_k(x) \stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}} \sin kx , \tag{9.65}$$

where $k = 1,\ 2 \cdots$ is the mode number.

Let's focus on mode number m and ask how the eigenvalue m^2 changes when $\epsilon \neq 0$? We use the RPS

$$\psi = \underbrace{s_m}_{=\psi^{(0)}} + \epsilon\psi^{(1)} + \cdots \qquad \text{and} \qquad \lambda = \underbrace{m^2}_{\lambda^{(0)}} + \epsilon\lambda^{(1)} + \cdots \tag{9.66}$$

Although it is not strictly necessary, it helps to normalize the perturbed eigenfunction so that

$$\int_0^\pi \psi^2 \, \mathrm{d}x = 1 . \tag{9.67}$$

Expansion of the normalization above implies

$$\int_0^\pi s_m^2 \, \mathrm{d}x = 1 , \qquad \int_0^\pi s_m \psi^{(1)} \, \mathrm{d}x = 0 , \qquad \int_0^\pi 2 s_m \psi^{(2)} + \left(\psi^{(1)}\right)^2 \mathrm{d}x , \qquad \text{etc.} \tag{9.68}$$

The expressions above will simplify the appearance of subsequent formulas.

The first three orders of the perturbation hierarchy are

$$\mathsf{L} s_m = 0 , \tag{9.69}$$

$$\mathsf{L}\psi^{(1)} = \left(\lambda^{(1)} + m^2 b\right) s_m , \tag{9.70}$$

$$\mathsf{L}\psi^{(2)} = \left(\lambda^{(2)} + \lambda^{(1)} b\right) s_m + \left(\lambda^{(1)} + m^2 b\right) \psi^{(1)} . \tag{9.71}$$

Above,

$$\mathsf{L} \stackrel{\text{def}}{=} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - m^2 , \tag{9.72}$$

is the "unperturbed operator".

Notice that (9.69) says that L has a zero eigenvalue with the zero-mode s_m. Thus there is a problem with solving boundary value problems of the form

$$\mathsf{L} f = \text{some function of } x , \tag{9.73}$$

with $f(0) = f(\pi) = 0$. For a solution to exist, the right hand side must be orthogonal to $\phi^{(0)}$. Thus to ensure that (9.70) and (9.71) have solutions, one requires

$$\lambda^{(1)} = -m^2 \int b s_m^2 \, \mathrm{d}x , \tag{9.74}$$

$$\lambda^{(2)} = -\lambda^{(1)} \int b s_m^2 \, \mathrm{d}x - m^2 \int b s_m \psi^{(1)} \, \mathrm{d}x , \tag{9.75}$$

and so on. (All integrals are from 0 to π.) Notice how the expansion of the eigenvalue is obtained by enforcing the solvability condition order by order.

The first-order shift in the m'th eigenvalue is given by (9.74). If the density in the middle of the string is increased then the eigenfrequency is decreased.

Exercise: At one point in the calculation above (9.68) was used to simplify an expression. Where was that?

The second-order terms

The second-order correction from (9.75) can be written as

$$\lambda^{(2)} = \frac{\left(\lambda^{(1)}\right)^2}{m^2} - m^2 \int b s_m \psi^{(1)} \, \mathrm{d}x \,. \tag{9.76}$$

To get $\lambda^{(2)}$, we have to solve (9.70) for $\psi^{(1)}$ and then substitute into (9.76). Using the modal projection method, the solution of (9.70) is

$$\psi^{(1)}(x) = m^2 \sum_{n=1}^{\infty} \frac{J_{mn} s_n(x)}{n^2 - m^2} \,, \tag{9.77}$$

where the sum above does not include the singular term, $m = n$, and

$$J_{mn} \stackrel{\text{def}}{=} \int_0^{\pi} b s_m s_n \, \mathrm{d}x \,. \tag{9.78}$$

Thus the second-order shift in the eigenvalue is

$$\lambda^{(2)} = \frac{\left(\lambda^{(1)}\right)^2}{m^2} - m^4 \sum_{n=1}^{\infty} \frac{J_{mn}^2}{n^2 - m^2} \,. \tag{9.79}$$

9.6 The vibrating string

A physical example is a string with uniform tension T (in Newtons) and mass density $\rho(x)$ (kilograms per meter) stretched along the axis of x. The wave speed is

$$c \stackrel{\text{def}}{=} \sqrt{\frac{T}{\rho}} \,. \tag{9.80}$$

After linearization the transverse displacement $\eta(x,t)$ satisfies

$$\rho \eta_{tt} - T \eta_{xx} = -\rho g + f \,. \tag{9.81}$$

Above $f(x,t)$ is an externally imposed force (in addition to gravity g). If the string is stretched between two supports at $x = 0$ and ℓ then the BCs are

$$\eta(0) = 0 \,, \qquad \text{and} \qquad \eta(\ell) = 0 \,. \tag{9.82}$$

We remove the "static sag" via

$$\eta(x,t) = \eta_s(x) + u(x,t) \,, \tag{9.83}$$

where the static solution $\eta_s(x)$ is determined by solving

$$\eta_{sxx} = \frac{\rho g}{T} \tag{9.84}$$

with $\eta_s = 0$ at the boundaries.

Exercise: Show that if ρ is constant

$$\eta_s(x) = -\frac{\rho g}{2T}x(\ell - x)\,. \tag{9.85}$$

Thus the disturbance, $u(x,t)$, satisfies

$$\rho u_{tt} - T u_{xx} = f\,. \tag{9.86}$$

If $f = 0$ there is a special class of "eigensolutions" which we find with the substitution

$$u(x,t) = \phi(x)\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{9.87}$$

This produces a typical SL eigenproblem

$$T\phi_{xx} + \omega^2 \rho\phi = 0\,, \tag{9.88}$$

with BCs $\phi(0) = \phi(\ell) = 0$ inherited from (9.82).

Example: Suppose that ρ is uniform so that the wave speed, $c = \sqrt{T/\rho}$, is a constant. the spectrum is therefore

$$\phi_n(x) = \sin k_n x\,, \qquad \text{with eigenfrequency} \qquad \omega_n = \frac{n\pi c}{\ell}\,. \tag{9.89}$$

The wavenumber above is $k_n = \omega_n/c = n\pi/\ell$.

Exercise: Solve the problem above with a free boundary at $x = \ell$: i.e., the boundary condition is changed to $u_x(\ell) = 0$.

After we solve (9.88) we possess a set of complete orthogonal functions, $\phi_n(x)$ each with a corresponding eigenfrequency. The orthogonality condition is

$$\int_0^\ell \phi_p\phi_q\rho\,\mathrm{d}x = \beta_p\delta_{pq}\,, \tag{9.90}$$

where β_p is a normalization constant. (Often, but not always, we make $\beta_p = 1$.)

Returning to the forced problem in (9.86) we build the solution of this partial differential equation as linear superposition of the eigenmodes

$$u(x,t) = \sum_{m=1}^{\infty} \hat{u}_m(t)\phi_m(x)\,. \tag{9.91}$$

Using orthogonality, the modal amplitude is given by

$$\beta_n\hat{u}_n(t) = \int_0^\ell u(x,t)\phi_n(x)\rho(x)\,\mathrm{d}x\,. \tag{9.92}$$

Evolution equations for the modal amplitudes follow via projection of the equation (9.86) onto the modes i.e., multiply (9.86) by $\phi_n(x)$ and integrate from $x = 0$ to $x = \ell$. After some integration by parts one finds

$$\frac{\mathrm{d}^2\hat{u}_n}{\mathrm{d}t} + \omega_n^2\hat{u}_n = \beta_n^{-1}\int_0^\ell \phi_n(x)f(x,t)\mathrm{d}x\,. \tag{9.93}$$

If we also represent the forcing f as

$$f(x,t) = \rho(x)\sum_{m=0}^{\infty}\hat{f}(t)\phi_m(x)\,, \qquad \text{with} \qquad \beta_n\hat{f}_n(t) = \int_0^\ell \phi_n(x)f(x,t)\,\mathrm{d}x\,, \tag{9.94}$$

then we have

$$\frac{\mathrm{d}^2\hat{u}_n}{\mathrm{d}t^2} + \omega_n^2\hat{u}_n = \hat{f}_n\,. \tag{9.95}$$

Each modal amplitude, $\hat{u}_n(t)$, satisfies a forced harmonic oscillator equation.

We declare victory after reducing the partial differential equation (9.86) to a big set of *uncoupled* ordinary differential equations in (9.95). This victory is contingent on solving the eigenproblem (9.88) and understanding basic properties of SL problems such as the orthogonality condition in (9.90)

Example: In some problems the tension T is non-uniform and the wave equation is then

$$\rho u_{tt} - (T u_x)_x = f\,. \tag{9.96}$$

A nice example is a dangling chain of length ℓ suspended from the point $x = 0$, with x positive downwards. The tension is then

$$T(x) = \rho g \int_x^\ell \rho(x')\,\mathrm{d}x' \tag{9.97}$$

The BCs $\eta(0) = 0$ and $\eta_x(\ell) = 0$.

9.7 Problems

Problem 9.1. *(i)* Transform

$$a(x)y'' + b(x)y' + c(x)y + d(x)Ey(x) = 0\,, \tag{9.98}$$

with boundary conditions $y(\alpha) = y(\beta) = 0$, into the Sturm-Liouville form

$$\left[p(x)y'\right]' + \left[q(x) + Ew(x)\right]y = 0\,. \tag{9.99}$$

Hint: multiply by an integrating factor $I(x)$; determine $I(x)$ by matching up terms. Your answer should include clear expressions for $[p, q, w]$ in terms of $[a, b, c, d]$. *(ii)* Write Bessel's equation

$$y'' + x^{-1}y' + \left[E - \nu^2 x^{-2}\right]y = 0 \tag{9.100}$$

in Sturm-Liouville form. *(iii)* Prove that the eigenfunctions $y_n(x)$ associated with (9.99) satisfy

$$(E_n - E_m)\int_\alpha^\beta y_n(x)y_m(x)w(x)\,\mathrm{d}x = 0\,. \tag{9.101}$$

Thus if $E_m \neq E_n$ then the eigenfunctions are orthogonal with respect to the *weight function* $r(x)$.

Problem 9.2. Consider the eigenvalue problem

$$y'' = -\lambda y\,, \qquad \text{with BCs:} \qquad y(0) = 0\,, \qquad y'(1) = y(1)\,. \tag{9.102}$$

(i) Prove that all the eigenvalues are real. *(ii)* Find the transcendental equation whose solutions determine the eigenvalues λ_n. *(iii)* Find an explicit expression for the smallest eigenvalue λ_0 and the associated eigenfunction $y_0(x)$. *(iv)* Show that the eigenfunctions are orthogonal with respect to an appropriately defined inner product. *(v)* Attempt to solve the inhomogeneous boundary value problem

$$y'' = a(x), \qquad y(0) = 0\,, \qquad \text{with BCs:} \qquad y'(1) = y(1)\,, \tag{9.103}$$

via an expansion using the eigenmodes. Show that this expansion fails because the problem has no solution for an arbitrary $a(x)$. *(iv)* Find the solvability condition on $a(x)$ which ensures that the problem (9.103) does have a solution, and then obtain the solution using a modal expansion.

Problem 9.3. Consider the eigenproblem

$$-\phi'' = \lambda x^{-2+\epsilon}\phi\,, \qquad \text{with BCs} \qquad \phi(1) = \phi(\ell) = 0\,. \tag{9.104}$$

In the lecture we solved the $\epsilon = 0$ problem and showed that the smallest eigenvalue is

$$\lambda_1 = \tfrac{1}{4} + \frac{\pi}{\ln \ell}\,. \tag{9.105}$$

Find the change in λ_1 induced by the perturbation $\epsilon \ll 1$.

Problem 9.4. There is a special value of α for which the boundary value problem

$$y'' = \sin x - \alpha\,, \qquad y'(0) = y'(\pi) = 0 \tag{9.106}$$

has a solution. Find the special value of α and in that case solve the boundary value problem by: *(i)* expansion using a set of eigenfunctions; *(ii)* explicit solution using a combination of homogeneous and particular solutions. *(iii)* Use MATLAB to compare the explicit solution with a three-term truncation of the series solution.

Problem 9.5. Consider the eigenproblem

$$\big(a(x)y'\big)' + \lambda y = 0\,, \qquad y'(0) = y'(1) = 0\,, \tag{9.107}$$

where $a(x) > 0$ for $0 \le x \le 1$. *(i)* Verify that $y(x) = 1$ and $\lambda = 0$ is an eigensolution. *(ii)* Show that $\lambda = 0$ is the smallest eigenvalue. *(iii)* Now consider the perturbed problem in which the boundary condition at $x = 1$ is changed to

$$\epsilon y(1) + y'(1) = 0\,,$$

where ϵ is a positive real number. With $0 < \epsilon \ll 1$, use perturbation theory to determine the $O(\epsilon)$ shift of the $\lambda = 0$ eigenvalue. *(iii)* Obtain the $O(\epsilon^2)$ term in the expansion of the $\lambda = 0$ eigenvalue.

Problem 9.6. *(i)* Verify that $y(x) = 1$ and $\lambda = 0$ is an eigensolution of

$$\big(xy'\big)' + \lambda y = 0\,, \qquad \lim_{x\to 0} xy' = 0\,, \qquad y'(1) = 0\,. \tag{9.108}$$

(ii) Show that $\lambda = 0$ is the smallest eigenvalue. *(iii)* Now consider the perturbed problem in which the boundary condition at $x = 1$ is changed to

$$\epsilon y(1) + y'(1) = 0\,,$$

where ϵ is a positive real number. With $0 < \epsilon \ll 1$, use perturbation theory to determine the $O(\epsilon)$ shift of the $\lambda = 0$ eigenvalue. *(iii)* Obtain the $O(\epsilon^2)$ term in the expansion of the $\lambda = 0$ eigenvalue.

Problem 9.7. When $\epsilon = 0$ the eigenproblem

$$y'' + \epsilon y y' + \lambda y = 0\,, \qquad y(0) = y(\pi) = 0\,,$$

has the solution $\lambda = 1$ and $y = a \sin x$. Use perturbation theory ($\epsilon \ll 1$) to investigate the dependence of the eigenvalue λ on a and ϵ.

Problem 9.8. Consider a diffusion problem defined on the interval $0 \le x \le \ell$:

$$u_t = \kappa u_{xx}\,, \qquad u(0,t) = 0\,, \qquad u_x(\ell,t) = 0\,, \tag{9.109}$$

with initial condition $u(x,t) = f(x)$. *(i)* If you use separation of variables then it is easy to anticipate that you'll find a Sturm-Liouville eigenproblem with sinusoidal solutions. Sketch the first two eigenfunctions *before* doing this algebra. Explain why you are motivated to nondimensionalize so that $0 \le x \le \pi/2$. *(ii)* With $\ell \to \pi/2$ and $\kappa \to 1$, work out the Sturm-Liouville algebra and find the eigenfunctions and eigenvalues. *(iii)* With $f(x) = 1$ find the solution as a Fourier series and use MATLAB to visualize the answer.

Problem 9.9. A rod occupies $1 \leq x \leq 2$ and the thermal conductivity depends on x so that diffusion equation is

$$u_t = (x^2 u_x)_x \,.$$

The boundary and initial conditions are

$$u(1,t) = u(2,t) = 0\,, \qquad u(x,0) = 1\,.$$

(i) The total amount of heat in the rod is

$$H(t) = \int_1^2 u(x,t)\,\mathrm{d}x\,.$$

Show that $H(0) = 1$ and

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 4u_x(2,t) - u_x(1,t)\,.$$

Physically interpret the two terms on the right hand side above. What is the sign of the $u_x(2,t)$ and the sign $u_x(1,t)$? *(ii)* Before solving the PDE, show that roughly 61% of the heat escapes through $x = 2$. (There is a simple analytic expression for the fraction $0.61371\cdots$ which you should find.) *(iii)* Use separation of variables to show that the eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)\,.$$

Find the eigenvalue which is associated with the n'th eigenfunction. *(iv)* Use modal orthogonality to find the series expansion of the initial value problem.

Problem 9.10. Solve the BVP

$$y'' + y + \epsilon f(x) y = 0\,, \qquad y(0) = 1\,, \qquad y(\pi) = 0\,, \tag{9.110}$$

with a perturbation expansion in ϵ; $f(x)$ is some $O(1)$ function. *Hint:* solve the problem exactly in the special case $f(x) = 1$ to infer the form of the perturbation expansion. Is this a regular or singular problem?

Problem 9.11. Consider the inhomogeneous diffusion equation

$$u_t - u_{xx} = \frac{1}{\sqrt{x(\pi - x)}}\,, \qquad u(0,t) = u(\pi,t) = 0\,.$$

(i) Using a modal expansion

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx\,,$$

find the ordinary differential equations satisfied by the amplitudes $u_n(t)$. *(ii)* Your answer to *(i)* will involve the integral

$$h_n \equiv \frac{2}{\pi} \int_0^{\pi} \frac{\sin nx}{\sqrt{x(\pi - x)}}\,\mathrm{d}x\,,$$

which you probably can't evaluate off-the-cuff. However you should deduce that h_n is zero if n is even and

$$h_n \equiv \frac{4}{\pi} \int_0^{\pi/2} \frac{\sin nx}{\sqrt{x(\pi - x)}} \mathrm{d}x\,, \qquad \text{if } n \text{ is odd.}$$

Find a leading-order $n \to \infty$ asymptotic approximation of h_n. *(iii)* I believe that $u_n \sim n^{-q}$ as $n \to \infty$. Find q.

Problem 9.12. Find the eigenfunction expansion of $f(t) = 1 - \mathrm{e}^{-\alpha x}$ in $0 < x < \pi$ in terms of the eigenfunctions defined by

$$u'' + \lambda u = 0\,, \qquad \text{with BCs} \qquad u(0) = u'(\pi) = 0\,. \tag{9.111}$$

For which values of α is this eigenfunction expansion most effective?

Problem 9.13. Solve the eigenproblem

$$(xu')' + \frac{3+\lambda}{x} u = 0\,, \qquad \text{with BCs} \qquad y(1) = y(2) = 0\,. \tag{9.112}$$

Use these eigenfunctions to solve the inhomogeneous probem

$$(xy')' + \frac{3+\lambda}{x} y = \sin x \qquad \text{with BCs} \qquad y(1) = y(2) = 0\,. \tag{9.113}$$

Problem 9.14. Use an eigenfunction expansion to solve

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} = \delta(x - x')\,, \qquad \text{with BCs} \qquad g(0, x') = g(1, x') = 0\,, \tag{9.114}$$

on the interval $0 < x < 1$ for the Green's function $g(x, x')$. You can also solve this problem using the patching method. **Remark:** considering the inhomogeneous BVP on the interval $0 < x < 1$ for $y(x)$:

$$y'' = f\,, \qquad \text{with BCs} \qquad y(0) = y(1) = 0\,, \tag{9.115}$$

we now have two different solution methods: Green's functions and eigenfunction expansions. The eigenfunction solution of (9.114) is the connection between these two methods. Can you extend this to the general SL problem

$$(pg')' + qg = \delta(x - x') \qquad \text{with our usual homogenous BCs?} \tag{9.116}$$

Problem 9.15. Use separation of variables to solve the eigenproblem

$$\underbrace{\phi_{xx} + \phi_{yy}}_{\nabla^2\phi} = -\lambda\phi\,. \tag{9.117}$$

The domain is the $\pi \times \pi$ square, $0 < (x, y) < \pi$, with the Dirichlet boundary condition $u = 0$. The two-dimensional Lagrange identity for the operator ∇^2 (aka $\mathcal{L}$) is a well-known vector identity. Use this to prove that the eigenfunctions are orthogonal. Use Galerkin projection and an eigenfunction expansion to solve the inhomogeneous problem

$$u_{xx} + u_{yy} = 1\,, \qquad \text{with BCs } u = 0 \text{ on the square.} \tag{9.118}$$

Lecture 10

WKB

10.1 The WKB approximation

Suppose we need to solve

$$\epsilon^2 y'' = Q(x) y \,, \quad \text{as } \epsilon \to 0. \tag{10.1}$$

The approximate WKB solution to this singular perturbation problem is

$$y \approx A Q^{-1/4} \exp\left[\frac{1}{\epsilon}\int^x \sqrt{Q(t)}\,\mathrm{d}t\right] + B Q^{-1/4} \exp\left[-\frac{1}{\epsilon}\int^x \sqrt{Q(t)}\,\mathrm{d}t\right] . \tag{10.2}$$

The approximation above fails in the neighbourhood of x_* where $Q(x_*) = 0$. The point x_* is called a *turning point*. We'll need a different approximation in the vicinity of a turning point. But everywhere else (10.2) is a spectacular approximation.

Exercise: Check the special cases $Q = 1$ and $Q = -1$ and commit (10.2) to memory.

Following **BO**, the most efficient route to (10.2) is to make the exponential substitution

$$y = \mathrm{e}^{\frac{S}{\epsilon}} \tag{10.3}$$

in (10.1). One finds that the phase function $S(x)$ satisfies the Ricatti equation

$$\epsilon S'' + S'^2 = Q \,. \tag{10.4}$$

We've "nonlinearized" the linear equation (10.1). The advantage is that (10.4) can be solved using a regular perturbation series

$$S = S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots \tag{10.5}$$

The first three terms are

$$S_0'^2 = Q^2 \,, \tag{10.6}$$

$$2S_0' S_1' + S_0'' = 0 \,, \tag{10.7}$$

$$2S_0' S_2' + S_1'^2 + S_1'' = 0 \,. \tag{10.8}$$

The solutions are

$$S_0 = \pm \int^x \sqrt{Q(t)}\,\mathrm{d}t \,, \tag{10.9}$$

$$S_1 = -\tfrac{1}{4} \ln Q \,, \tag{10.10}$$

$$S_2 = \pm \int^x \frac{Q''}{8Q^{3/2}} - \frac{5}{32}\frac{Q'^2}{Q^{5/2}}\,\mathrm{d}t \,. \tag{10.11}$$

Example: Solve

$$y'' + \sqrt{x}y = 0\,, \qquad \text{with } y(1) = 0 \text{ and } y'(1) = 1. \tag{10.12}$$

In this example $\epsilon = 1$ and $Q = -\sqrt{x}$. Therefore

$$\pm \int_1^x \sqrt{Q(t)}\,\mathrm{d}t = \pm i\frac{4}{5}\left(x^{5/4} - 1\right) \tag{10.13}$$

We're going to apply the initial conditions at $x = 1$ so it is wise use $t = 1$ as the lower limit in the integral above. Thus the solution satisfying $y(1) = 0$ is

$$y \approx \frac{A}{x^{1/8}} \sin\left[\frac{1}{\epsilon}\frac{4}{5}\left(x^{5/4} - 1\right)\right]. \tag{10.14}$$

The cosine is eliminated by the requirement that $y(1) = 0$. We've included the factor ϵ^{-1} because when we take the derivative of the we have

$$y' \approx \frac{A}{\epsilon} x^{1/8} \cos\left[\frac{1}{\epsilon}\frac{4}{5}\left(x^{5/4} - 1\right)\right]. \tag{10.15}$$

When we take the derivative we only differentiate the cosine — the derivative of the amplitude $x^{-1/8}$ is order ϵ smaller. Requiring that $y'(1) = 1$ we see that $A = \epsilon = 1$. Thus the WKB approximation is

$$y^{WKB}(x) = x^{-1/8} \sin\left[\frac{4}{5}\left(x^{5/4} - 1\right)\right]. \tag{10.16}$$

Figure **??** shows that this is an excellent approximation to the solution to the initial value problem — unless we get too close to the turning point at $x = 0$.

10.2 Applications of the WKB approximation

Example: Let's assess the accuracy of WKB applied to Airy's equation

$$y'' = xy \tag{10.17}$$

with $x \gg 1$. There is no explicit small parameter. But if we're interested in $x \approx 10$ we might write $x = 10X$ where $X \sim 1$. With this rescaling, Airy's equation acquires a small WKB parameter:

$$10^{-3} y_{XX} = Xy\,. \tag{10.18}$$

This argument correctly suggests that the WKB approximation works if $x \gg 1$.

Example: Consider

$$\epsilon^2 y'' - x^{-1} y = 0\,. \tag{10.19}$$

How small must ϵ be in order for the physical optics approximation to within 5% when $x \geq 1$?

Example: Consider

$$y'' + kx^{-\alpha} y = 0\,, \qquad y(1) = 0\,, \qquad y'(1) = 1\,. \tag{10.20}$$

Is WKB valid as $x \to \infty$?

With $k = 1$, I found

$$S_0 = \pm\frac{2}{2-\alpha}\left(x^{1-\frac{\alpha}{2}} - 1\right), \qquad \text{and} \qquad S_1 = \frac{\alpha}{4}\ln x\,, \qquad \text{and} \qquad S_2 = \frac{\alpha(\alpha-4)}{16(\alpha-2)}\left(x^{\frac{\alpha}{2}-1} - 1\right). \tag{10.21}$$

The calculation of S_2 should be checked (and should do general k). But the tentative conclusion is that WKB works if $\alpha < 2$. Note $\alpha = 2$ is a special case with an elementary solution.

Example: Let's use the WKB approximation to estimate the eigenvalues of the Sturm-Liouville eigenproblem

$$y'' + \lambda \underbrace{\left(x + x^{-1}\right)}_{w(x)} y = 0\,, \quad \text{with BCs} \qquad y'(1) = 0\,, \quad y(L) = 0\,. \tag{10.22}$$

The physical optics approximation is

$$y = w^{-1/4} \sin\Big(\lambda^{1/2} \underbrace{\int_x^L \sqrt{w(x')}\,\mathrm{d}x'}_{\text{phase}}\Big)\,, \tag{10.23}$$

and the leading-order derivative is

$$y' = -\lambda^{1/2} w^{+1/4} \cos\left(\lambda^{1/2}\int_x^L \sqrt{w(x')}\,\mathrm{d}x'\right) . \tag{10.24}$$

Notice how the phase in (10.23) has been constructed so that the boundary condition at $x = L$ is already satisfied. To apply the derivative boundary condition at $x = 1$ we have from (10.24)

$$\sqrt{\lambda_n^{\mathrm{WKB}}}\, J(L) = \pi\left(n + \frac{1}{2}\right) , \qquad n = 0, 1, \cdots \tag{10.25}$$

where

$$J(L) \stackrel{\text{def}}{=} \int_1^L \sqrt{x + x^{-1}}\,\mathrm{d}x . \tag{10.26}$$

Notice that we have only used the geometric optics approximation to obtain the eigencondition in (10.25).

In Figure 10.1 we take $L = 5$ and compare the WKB eigenvalue with those obtained from the MATLAB routine `bvp4c`. It is not easy to analytically evaluate $J(L)$, so instead we calculate $J(L)$ using `quad`. Figure 10.1 shows the relative percentage error,

$$e \equiv 100 \times \frac{\lambda_{\mathtt{bvp4c}} - \lambda_{\mathrm{WKB}}}{\lambda_{\mathtt{bvp4c}}} , \tag{10.27}$$

as a function of $n = 0, 2, \cdots 5$. The WKB approximation has about 18% error for λ_0, but the higher eigenvalues are accurate.

To use `bvp4c` we let $y_1(x) = y(x)$ and write the eigenproblem as the first-order system

$$y_1' = y_2 , \tag{10.28}$$

$$y_2' = -\lambda\left(x + x^{-1}\right) y_1 , \tag{10.29}$$

$$y_3' = \left(x + x^{-1}\right) y_1^2 . \tag{10.30}$$

This Sturm-Liouville boundary value problem always has a trivial solution viz., $y(x) = 0$ and λ arbitrary. We realize that this is trivial, but perhaps `bvp4c` isn't that smart. So with (10.30) we force `bvp4c` to look for a nontrivial solution by adding an extra equation with the boundary conditions

$$y_3(0) = 0 , \qquad \text{and} \qquad y_3(L) = 1 . \tag{10.31}$$

We also have $y_2(1) = 0$ and $y_1(L) = 0$, so there are four boundary conditions on a third-order problem. This is OK because we also have the unknown parameter λ. The addition of $y_3(x)$ also ensures that `bvp4c` returns a normalized solution:

$$\int_1^L y^2\left(x + x^{-1}\right)\,\mathrm{d}x = 1 . \tag{10.32}$$

An alternative that avoids the introduction of $y_3(x)$ is to use $y_1(1) = 1$ as a normalization, and as an additional boundary condition. However the normalization in (10.32) is standard.

In summary, the system for $[y_1, y_2, y_3]$ now only has nontrivial solutions at special values of the eigenvalue λ.

The function `billzWKBeig`, with neither input nor output arguments, solves the eigenproblem with $L = 5$. The code is written as an argumentless function so that three nested functions can be embedded. This is particularly convenient for passing the parameter L — avoid global variables. Notice that all functions are concluded with `end`. In this relatively simple application of `bvp4c` there are only three arguments:

1. a function `odez` that evaluates the right of (10.28) through (10.30);
2. a function `bcz` for evaluating the residual error in the boundary conditions;
3. a MATLAB structure `solinit` that provides a guess for the mesh and the solution on this mesh.

`solinit` is set-up with the utility function `bvpinit`, which calls the nested function `initz`. `bvp4c` returns a MATLAB structure that I've imaginatively called `sol`. In this structure, `sol.x` contains the mesh and `sol.y` contains the solution on that mesh. `bvp4c` uses the smallest number of mesh points it can. So, if you want to make a smooth plot of the solution, as in the lower panel of Figure 10.1, then you need the solution on a finer mesh, called `xx` in this example. Fortunately `sol` contains all the information needed to compute the smooth solution on the fine mesh, which is done with the auxiliary function `deval`.

Example: Compute the next WKB correction to the $n = 0$ eigenvalue and compare both (10.25) and the improved eigenvalue to the numerical solution for $1 \leq L \leq 10$.

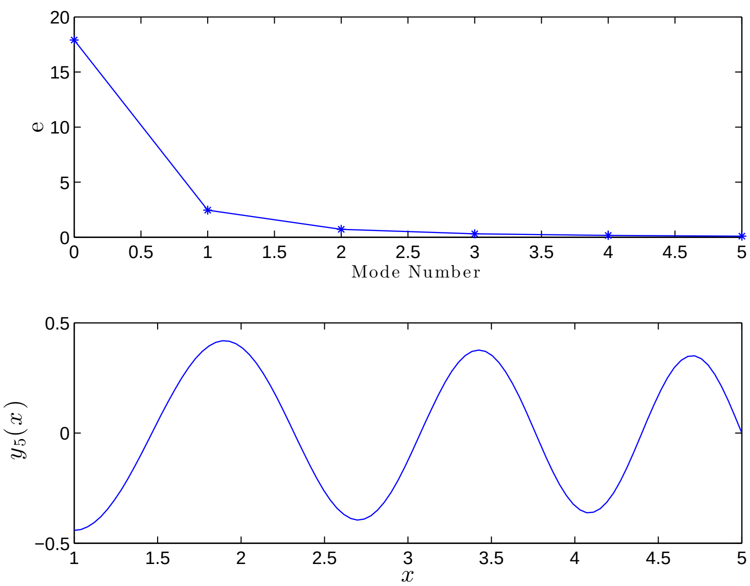

Figure 10.1: Solution with $L = 5$. Notice there are five interior zeros.

```
function billzWKBeig
L = 5;          J = quad(@(x)sqrt(x+x.^(-1)),1,L);
%The first 6 eigenvalues; n = 0 is the ground state.
nEig = [0 1 2 3 4 5];                 lamWKB = (nEig+0.5).^2*(pi/J)^2;
lamNum = zeros(1,length(lamWKB));

    for N = 1:1:length(nEig)
        lamGuess = lamWKB(N);
        x = linspace(1,L,10);
        solinit = bvpinit(x,@initz,lamGuess);
        sol = bvp4c(@odez,@bcz,solinit);
        lambda = sol.parameters;
        lamNum(N) = lambda;
     end

err = 100*(lamNum - lamWKB)./lamNum;
figure
subplot(2,1,1)
plot(nEig,err,'*-')
xlabel('Mode Number','interpreter','latex')
ylabel('$e$','interpreter','latex','fontsize',16)
% Plot the last eigenfunction
xx = linspace(1,L);                ssol = deval(sol,xx);
subplot(2,1,2)
plot(xx,ssol(1,:))
xlabel('$x$','interpreter','latex','fontsize',16)
ylabel('$y_5(x)$','interpreter','latex','fontsize',16)

        %--------------- Nested Functions ----------------%
   function dydx = odez(x,y,lambda)
        %ODEZ evalates the derivatives
         dydx = [ y(2); -lambda*(x+x^(-1))*y(1);
                               (x+x^(-1))*y(1)*y(1)];
    end

    %% BCs applied
    function res = bcz(ya, yb, lambda)
         res = [ ya(2) ; yb(1); ya(3) ; yb(3) - 1];
         %Four BCs: solve three first-order
         %equations and also determine lambda.
    end

    %% Use a simple guess for the Nth eigenmode
    function yinit = initz(x)
        alpha = (N + 1/2)*pi/(L-1);
        yinit = [ sin(alpha*(L - x))
                  alpha*cos(alpha*(L - x))
                  (x - 1)/(L - 1)     ];
    end
end
```

10.3 An eigenproblem with a turning point

Let's apply the WKB approximation to estimate the large eigenvalues of the Sturm-Liouville eigenproblem

$$\phi'' + \lambda \sin x \phi = 0\,, \qquad \phi(0) = \phi\left(\frac{\pi}{2}\right) = 0\,. \tag{10.33}$$

There is a turning point at $x = 0$ so the WKB approximation does not apply close to the boundary.

Hope is eternal and we begin by ignoring the turning point and constructing a physical optics approximation:

$$\phi^{\text{hope}} = x^{-1/4} \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\sin v}\,\mathrm{d}v\right)\,. \tag{10.34}$$

The construction above satisfies the boundary condition at $x = 0$ and then the other boundary condition at $\pi/2$ determines our hopeful approximation to the eigenvalue. To ensure that $\phi^{\text{hope}}(\pi/2) = 0$, the argument of the sin must be $n\pi$ and thus the approximate eigenvalue is

$$\lambda_n^{\text{hope}} = \left(\frac{n\pi}{J}\right)^2\,, \quad n = 1\,,\ 2\,,\cdots \tag{10.35}$$

In the expression above the integral of the phase function is

$$J \stackrel{\text{def}}{=} \int_0^{\pi/2} \sqrt{\sin v}\,\mathrm{d}v = \sqrt{\frac{2}{\pi}}\Gamma^2\left(\frac{3}{4}\right) = 1.19814\cdots \tag{10.36}$$

We'll see later that the approximation in (10.35) is not very accurate — we can't just ignore the turning point and hope for the best.

Let's use a combination of WKB and asymptotic matching to account for the turning point and obtain a better approximation to the eigenvalues.

The outer solution — use WKB

We apply the WKB approximation where it is guaranteed to work. This is in the outer region defined by $\lambda^{1/3}x \gg 1$. The construction that satisfies the boundary condition at $x = \pi/2$ is

$$\phi^{WKB} = x^{1/4} \sin\left(\sqrt{\lambda}\int_x^{\pi/2} \sqrt{\sin t}\,\mathrm{d}t\right)\,. \tag{10.37}$$

To perform the match we will need the "inner limit" of the approximation above. In the region where

$$\lambda^{-1/3} \ll x \ll 1 \tag{10.38}$$

the WKB approximation is valid *and* we can simplify the phase function in (10.37):

$$\phi^{WKB} = x^{-1/4} \sin\left(\sqrt{\lambda}J - \int_0^x \sqrt{\sin v}\,\mathrm{d}v\right)\,, \tag{10.39}$$

$$\sim x^{-1/4} \sin\left(\sqrt{\lambda}J - \tfrac{2}{3}\sqrt{\lambda}x^{3/2} + \mathrm{ord}\left(x^{7/2}\right)\right)\,. \tag{10.40}$$

The inner solution — an Airy approximation

Close to $x = 0$ — specifically in the region where $x\lambda^{1/3}$ is order unity — we can approximate the differential equation by

$$\phi_{xx} + \lambda\left(x + \mathrm{ord}(x^3)\right)\phi = 0 \tag{10.41}$$

As an inner variable we use

$$X = \lambda^{1/3} x \,, \tag{10.42}$$

so that the leading-order inner approximation is a variant of Airy's equation

$$\Phi_{XX} + X\Phi = 0 \,. \tag{10.43}$$

The solution that satisfies the boundary condition at $X = 0$ is

$$\Phi = Q \left[\frac{\mathrm{Ai}(-X)}{\mathrm{Ai}(0)} - \frac{\mathrm{Bi}(-X)}{\mathrm{Bi}(0)} \right] . \tag{10.44}$$

Matching

To take the outer limit of the inner solution in (10.44) we look up the relevant asymptotic expansions[1] of the Airy functions. Then we write the outer limit of (10.44) as

$$\Phi \sim \frac{2Q}{\sqrt{3\pi}\mathrm{Ai}(0)} \frac{1}{X^{1/4}} \Big[\underbrace{\tfrac{\sqrt{3}}{2}}_{\cos\frac{\pi}{6}} \sin\left(\frac{2}{3}X^{3/2} + \frac{\pi}{4}\right) - \underbrace{\tfrac{1}{2}}_{\sin\frac{\pi}{6}} \cos\left(\frac{2}{3}X^{3/2} + \frac{\pi}{4}\right) \Big] , \tag{10.45}$$

$$= \frac{2Q}{\sqrt{3\pi}\mathrm{Ai}(0)} \frac{1}{X^{1/4}} \sin\left(\frac{2}{3}X^{3/2} + \frac{\pi}{12}\right) . \tag{10.46}$$

We now match the phase in (10.40) with that in (10.46). this requires

$$\sqrt{\lambda} J - n\pi = \frac{\pi}{12} , \tag{10.47}$$

or

$$\lambda^{\mathrm{WKB}} = \left(\frac{(12n-1)\pi}{12J} \right)^2 , \qquad n = 1\,,\ 2\,, 3 \cdots \tag{10.48}$$

Notice that with $n = 1$ the hopeful approximation in (10.36) is about 18% larger than the correct WKB-Airy approximation in (10.48). The numerical comparison below shows that (10.48) is good even for $n = 1$:

$\lambda_{\mathtt{bvp4c}}$	5.7414	25.2094	58.4349	105.4114	166.1422	240.6232
λ_{WKB}	5.7771	25.2568	58.4341	105.4673	166.1456	240.6793

Note that numerical results above fluctuate in the final decimal place as I change the resolution and the initial guess in `bvp4c`.

[1] Airy factoids used here are

as $t \to -\infty$: $\quad \mathrm{Ai}(t) \sim \dfrac{1}{\sqrt{\pi}(-t)^{1/4}} \sin\left(\dfrac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right) , \quad$ and $\quad \mathrm{Bi}(t) \sim \dfrac{1}{\sqrt{\pi}(-t)^{1/4}} \cos\left(\dfrac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right) .$

And

$$\mathrm{Ai}(0) = \frac{\mathrm{Bi}(0)}{\sqrt{3}} = \frac{1}{3^{2/3}\Gamma(2/3)} = 0.355028 \,.$$

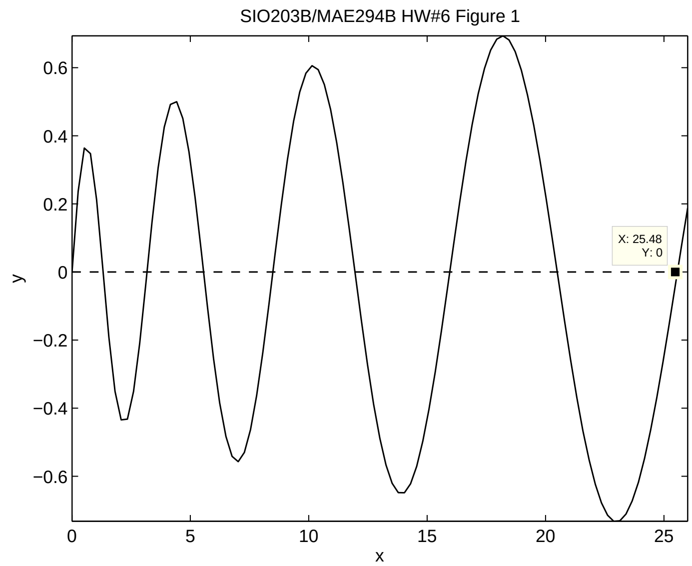

Figure 10.2: A figure for problem 10.1.

10.4 Problems

Problem 10.1. Figure 10.2 shows the solution of one of the four initial value problems:

$$\begin{aligned}
\epsilon^2 y_1'' + (1+x) y_1 &= 0\,, \qquad y_1(0) = 0\,, \quad y_1'(0) = 1\,, && (10.49)\\
\epsilon^2 y_2'' + (1+x)^{-1} y_2 &= 0\,, \qquad y_2(0) = 0\,, \quad y_2'(0) = 1\,, && (10.50)\\
\epsilon^2 y_3'' - (1+x) y_3 &= 0\,, \qquad y_3(0) = 0\,, \quad y_3'(0) = 1\,, && (10.51)\\
\epsilon^2 y_4'' - (1+x)^{-1} y_4 &= 0\,, \qquad y_4(0) = 0\,, \quad y_4'(0) = 1\,. && (10.52)
\end{aligned}$$

(i) Which $y_n(x)$ is shown in Figure 10.2? *(ii)* Use the WKB approximation to estimate the value of ϵ used to draw figure 10.2.

Problem 10.2. Consider the IVP

$$\ddot{x} + 256 \mathrm{e}^{4t} x = 0\,, \qquad x(0) = 0\,, \qquad \dot{x}(0) = 1\,. \tag{10.53}$$

Estimate the position and magnitude of the first positive maximum of $y(t)$. Compare the WKB approximation with a numerical solution on the interval $0 < t \leq 1$.

Problem 10.3. Consider the differential equation

$$y'' + \underbrace{\frac{400}{400 + x^2}}_{Q(x)} y = 0\,. \tag{10.54}$$

How can we apply the WKB approximation to this equation? Compare the physical optics approximation to a numerical solution with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Problem 10.4. Consider the differential equation

$$y'' + x^2 y = 0\,, \tag{10.55}$$

with $x \gg 1$. Solve the differential equation using the WKB approximation and following our discussion of Airy's equation, assess the accuracy of the large-x WKB approximation to (10.55). Go to Chapter 10 of the DLMF (or some other reference, such as the appendix of **BO**) and find the exact solution of the differential equation

$$y'' + a^2 x^q y = 0 \tag{10.56}$$

in terms of Bessel functions. Compare the asymptotic expansion of the exact Bessel function solution with your WKB approximation.

Problem 10.5. Use the exponential substitution $y = \exp(S/\epsilon)$ to construct a WKB approximation to the differential equation

$$\epsilon^2 \left(py'\right)' + qy = 0\,. \tag{10.57}$$

Above $p(x)$ and $q(x)$ are coefficient functions, independent of the small parameter ϵ.

Problem 10.6. Consider the eigenproblem

$$\phi'' + \lambda w \phi = 0\,, \qquad \phi(0) = 0\,, \qquad \phi'(1) + \phi(1) = 0\,. \tag{10.58}$$

The weight function, $w(x)$ above, is positive for $0 \le x \le 1$. *(i)* Show that the eigenvalues λ_n are real and positive. *(ii)* Show that eigenfunctions with distinct eigenvalues are orthogonal

$$(\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m w \,\mathrm{d}x = 0\,. \tag{10.59}$$

(iii) With $w = 1$, find the first five eigenvalues and plot the first five eigenfunctions. You should obtain transcendental equation for λ, and then solve that equation with MATLAB. *(iv)* Next, with non-constant $w(x)$, use the WKB approximation to obtain a formula for λ_n. *(v)* Consider

$$w = (a + x)^2\,. \tag{10.60}$$

Take $a = 1$ and use `bvp4c` to calculate the first five eigenvalues and compare λ^{WKB} with $\lambda^{\texttt{bvp4c}}$. *(vi)* Is the WKB approximation better or worse if a increases?

Problem 10.7. Consider the Sturm-Liouville problem

$$(wy')' + \lambda y = 0\,, \tag{10.61}$$

with boundary conditions

$$\lim_{x \to 0} wy' = 0\,, \qquad y(1) = 0\,. \tag{10.62}$$

Assume that $w(x)$ increases monotonically with $w(0) = 0$ and $w(1) = 1$ e.g., $w(x) = \sin \pi x/2$. Further, suppose that if $x \ll 1$ then

$$w(x) = w_1 x + \frac{w_2}{2} x^2 + \frac{w_3}{6} x^3 + \cdots\,. \tag{10.63}$$

There is a regular singular point at $x = 0$, and thus we require only that $y(0)$ is not infinite.

Show that the transformation $y = w^{-1/2} Y$ puts the equation into the Schrödinger form

$$Y'' + \left[\frac{\lambda}{w} - \frac{w''}{2w} + \frac{w'^2}{4w^2}\right] Y = 0\,. \tag{10.64}$$

Use the WKB method and matching to find an approximation for the large eigenvalues ($\lambda = \epsilon^{-2} \gg 1$) in terms of the w_n's and the constant

$$q \equiv \int_0^1 \frac{\mathrm{d}x}{\sqrt{w(x)}} \,. \tag{10.65}$$

Some useful information from DLMF: The solution of

$$u'' + \left[\frac{a^2}{4z} + \frac{1-\nu^2}{4z^2}\right] u = 0 \tag{10.66}$$

is

$$u(z) = A\sqrt{z}J_\nu(a\sqrt{z}) + B\sqrt{z}Y_\nu(a\sqrt{z}) \,, \tag{10.67}$$

where J_ν and Y_ν are Bessel functions. You will need to look up basic properties of Bessel functions.

Problem 10.8. Consider the epsilonless Schrödinger equation

$$y'' + p^2 y = 0 \,, \tag{10.68}$$

where $p(x) > 0$. *(i)* Try to solve the equation by substituting

$$Y \equiv \exp\left(\pm\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \,. \tag{10.69}$$

Unfortunately this doesn't work: $Y(x)$ is not an exact solution of (10.68) unless p is constant. Instead, show that Y satisfies

$$Y'' + \left(p^2 \mp \mathrm{i}p'\right) Y = 0 \,. \tag{10.70}$$

(ii) Compare (10.70) with (10.68), and explain why $Y(x)$ is an approximate solution of (10.68) if

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{p}\right| \ll 1 \,. \tag{10.71}$$

(iii) Prove that if y_1 and y_2 are two linearly independent solutions of (10.68) then the Wronskian

$$W \equiv y_1 y_2' - y_1' y_2 \tag{10.72}$$

is constant. *(iv)* Show that the Wronskian of

$$Y_1 \equiv \exp\left(+\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \qquad \text{and} \qquad Y_2 \equiv \exp\left(-\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \tag{10.73}$$

is equal to $2\mathrm{i}p$. This suggests that if we modify the amplitude of $Y(x)$ like this:

$$Y_3 \equiv \frac{1}{\sqrt{p}}\exp\left(+\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \qquad \text{and} \qquad Y_4 \equiv \frac{1}{\sqrt{p}}\exp\left(-\mathrm{i}\int_0^x p(t)\,\mathrm{d}t\right) \,, \tag{10.74}$$

then we might have a better approximation. *(v)* Show that the Wronskian of Y_3 and Y_4 is a constant. *(vi)* Find a Schrödinger equation satisfied by Y_3 and Y_4 and discuss the circumstances in which this equation is close to (10.68).

Problem 10.9. Consider

$$y'' + xy = 0 \,, \tag{10.75}$$

and suppose that

$$y(x) \sim x^{-1/4}\cos(2x^{3/2}/3) \qquad \text{as} \qquad x \to +\infty \,. \tag{10.76}$$

Solve this problem exactly in terms of well known special functions. Find the asymptotic behaviour of $y(x)$ as $x \to -\infty$. Check your answer with MATLAB (see Figure 10.9)

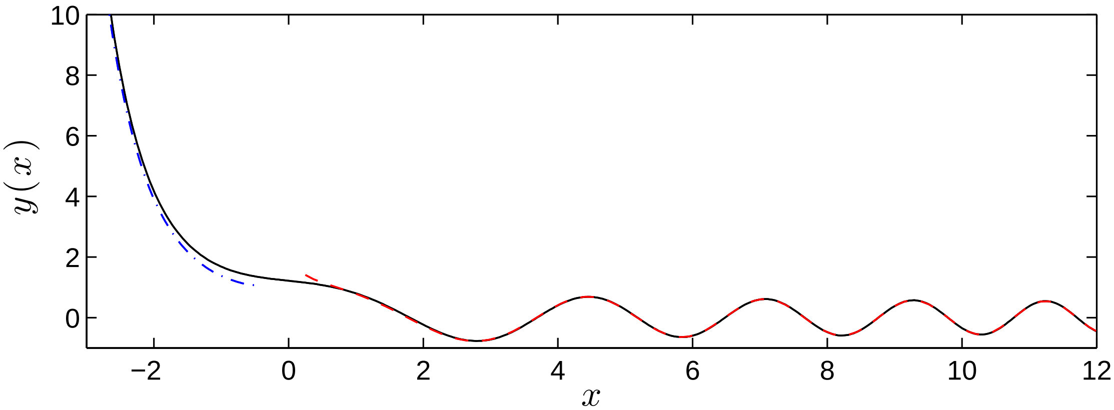

Figure 10.3: Figure for the problem 10.9 showing a comparison of the exact solution (the solid black curve) with the asymptotic expansions as $x \to -\infty$ (the dot-dash blue curve) and (10.76) as $x \to +\infty$ (the dashed red curve).

Problem 10.10. *(i)* Consider the eigenproblem

$$y'' + E\,(1+\eta x)\,y = 0\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0\,, \tag{10.77}$$

where η is a parameter and E is an eigenvalue. With $\eta = 0$ the gravest mode is

$$y = \sin x\,, \qquad E = 1\,. \tag{10.78}$$

(i) Suppose $|\eta| \ll 1$. Find the $O(\eta)$ shift in the eigenvalue using perturbation theory. If you're energetic, calculate the $O(\eta^2)$ term for good measure (optional). *(ii)* In equation (10.1.31) of BO, there is a WKB approximation to the eigenvalue $E(\eta)$. Take $n = 1$, and expand this formula for E up to and including terms of order η^2; compare this with your answer to part *(i)*. *(iii)* Use `bvp4c` in MATLAB to calculate $E(\eta)$, with $0 < \eta < 2$, numerically. Compare the WKB approximation in (10.1.31) with your numerical answer by plotting $E_{\texttt{bvp4c}}(\eta)$ and $E_{\mathrm{WKB}}(\eta)$ in the same figure.

Problem 10.11. Consider the Sturm-Liouville eigenproblem

$$y'' + \lambda\,(1 + a\sin x)^2\,y = 0\,, \quad y(0) = y(\pi) = 0\,. \tag{10.79}$$

(a) Using `bvp4c`, compute the first two eigenvalues, λ_1 and λ_2, as a functions of a in the range $-3/4 < a < 3$. *(b)* Estimate $\lambda_1(a)$ and $\lambda_2(a)$ using the WKB approximation. *(c)* Assuming $|a| \ll 1$ use perturbation theory to compute the first two nonzero terms in the expansion of $\lambda_1(a)$ and $\lambda_2(a)$ about $a = 0$. Compare these approximations with the WKB solution — do they agree? *(d)* Compare the WKB approximation to those from `bvp4c` by plotting the various results for $\lambda_n(a)/n^2$ on the interval $-3/4 < a < 3$.

Remark: If $a = -1$ the differential equation has a turning point at $x = \pi/2$. This requires special analysis — so we're staying well away from this ticklish situation by taking $a > -3/4$.

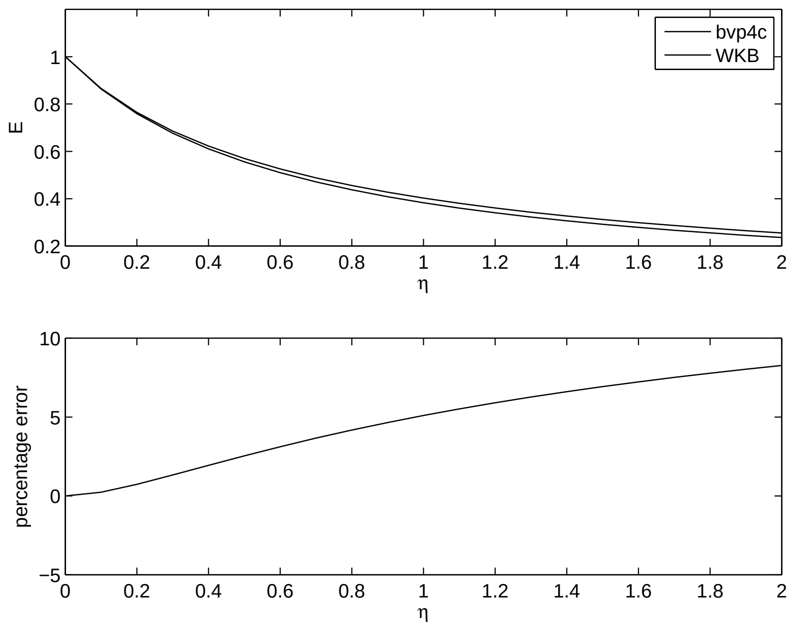

Figure 10.4: Figure for the problem with (10.77).

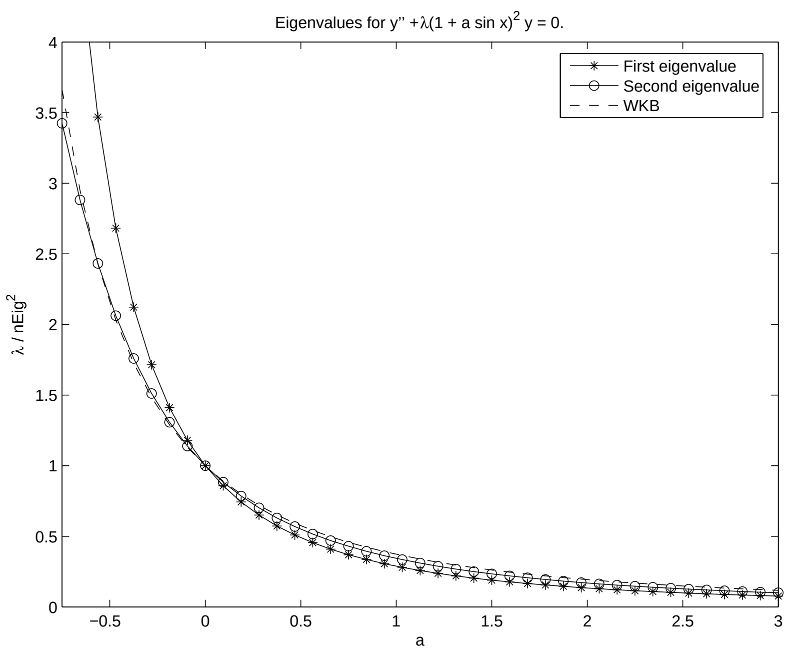

Figure 10.5: Figure for the problem containing (10.79).

Lecture 11

Internal boundary layers

11.1 A linear example

BO section 9.6 has an extensive discussion of the boundary layer problem

$$y'' + ay' + by = 0\,, \tag{11.1}$$

in which $a(x)$ has an internal zero. Suppose that we re-define the coordinate so that the domain is $-1 \le x \le 1$ and the boundary conditions in (6.40) are

$$y(-1) = p\,, \qquad y(+1) = q\,. \tag{11.2}$$

Further, suppose that $a(x)$ has a simple zero at $x = 0$. The differential equation

$$\epsilon y'' + \frac{\alpha x}{1+x^2} y + \beta y = 0\,, \tag{11.3}$$

is a typical example — figure 11.1 shows some `bvp4c` solutions.

Case 1: $\alpha > 0$

Let's consider (11.3) with $\alpha > 0$ and boundary conditions

$$y(-1) = 1\,, \qquad y(+1) = 0\,. \tag{11.4}$$

This example will reveal all the main features of the general case. Our earlier arguments indicate that boundary layers not possible at either end of the domain. Thus there is a left interior solution, $u(x)$, satisfying the boundary condition at $x = -1$:

$$y = u_0 + \epsilon u_1 + \cdots \tag{11.5}$$

with leading order

$$\frac{\alpha x}{1+x^2} u_{0x} + \beta u_0 = 0\,, \qquad \Rightarrow \qquad u_0 = |x|^{-\beta/\alpha} \exp\left[-\frac{\beta}{2\alpha}\left(x^2 - 1\right)\right]. \tag{11.6}$$

There is also a right interior solution $v(x)$, satisfying the boundary condition at $x = +1$:

$$y = v_0 + \epsilon v_1 + \cdots \tag{11.7}$$

In this case, with the homogeneous $x = +1$ boundary conditions in (11.4), the right interior solution is zero at all orders

$$v_n = 0\,. \tag{11.8}$$

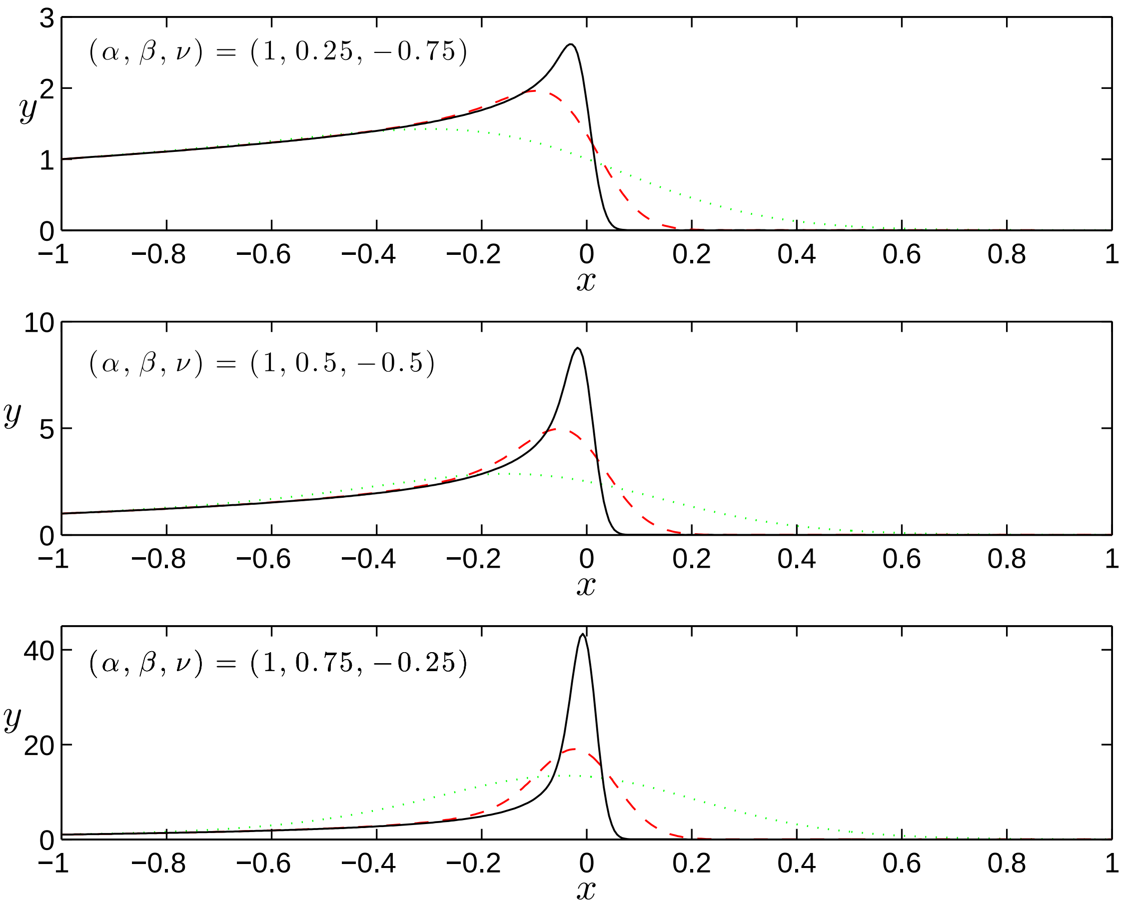

Figure 11.1: Internal boundary layer solution of (11.3) with $p = 1$ and $q = 0$, and $\epsilon = 0.05$ (green dotted) and 0.005 (red dashed) and 0.0005 (solid black).

We need a boundary layer at $x = 0$ to heal the $x \to 0$ singularity in (11.6) and to connect the right interior solution to the left interior solution. A distinguished-limit shows that the correct boundary-layer coordinate is

$$X \stackrel{\text{def}}{=} \frac{x}{\sqrt{\epsilon}}\,. \tag{11.9}$$

We must also re-scale the solution:

$$y = \epsilon^{-\beta/2\alpha} Y(X)\,. \tag{11.10}$$

The scaling above is indicated because the interior solution in (11.6) is order $\epsilon^{-\beta/2\alpha}$ once $x \sim \sqrt{\epsilon}$. Without much work we have now determined the boundary layer thickness and the amplitude of the solution within the boundary layer. This is valuable information in interpreting the numerical solution in figure 11.1 — we now understand how the vertical axis must be rescaled if we reduce ϵ further.

Using the boundary-layer variables, the BL equation is

$$Y_{XX} + \frac{\alpha X}{1 + \epsilon X^2} Y_X + \beta Y = 0\,. \tag{11.11}$$

We solve (11.11) with the RPS

$$Y = Y_0(X) + \epsilon Y_1(X) + \cdots \tag{11.12}$$

Leading order is the three-term balance

$$Y_{0XX} + \alpha X Y_{0X} + \beta Y_0 = 0\,, \tag{11.13}$$

with matching conditions

$$Y_0 \to |X|^{-\beta/\alpha} \mathrm{e}^{\beta/2\alpha}\,, \qquad \text{as } X \to -\infty, \tag{11.14}$$

$$Y_0 \to 0\,, \qquad \text{as } X \to +\infty. \tag{11.15}$$

We have to solve (11.13) exactly. When confronted with a second-order differential equation it is always a good idea to remove the first derivative term with the standard multiplicative substitution. In this case the substitution

$$Y_0 = W \mathrm{e}^{-\alpha X^2/4} \tag{11.16}$$

into (11.13) results in

$$W_{XX} + \left(\beta - \tfrac{1}{2}\alpha - \tfrac{1}{4}\alpha^2 X^2\right) W = 0\,. \tag{11.17}$$

Then, with $Z \stackrel{\text{def}}{=} \sqrt{\alpha} X$, we obtain the parabolic cylinder equation

$$W_{ZZ} + \Bigg(\underbrace{\frac{\beta}{\alpha} - 1}_{\nu + \frac{1}{2}} - \tfrac{1}{4} Z^2\Bigg) W = 0\,, \tag{11.18}$$

of order

$$\nu \stackrel{\text{def}}{=} \frac{\beta}{\alpha} - 1\,. \tag{11.19}$$

Provided that

$$\frac{\beta}{\alpha} \neq 1,\ 2,\ 3, \cdots \tag{11.20}$$

the general solution of (11.13) is

$$Y_0 = \mathrm{e}^{-\alpha X^2/4} \left[A D_\nu\left(\sqrt{\alpha} X\right) + B D_\nu\left(-\sqrt{\alpha} X\right)\right]\,. \tag{11.21}$$

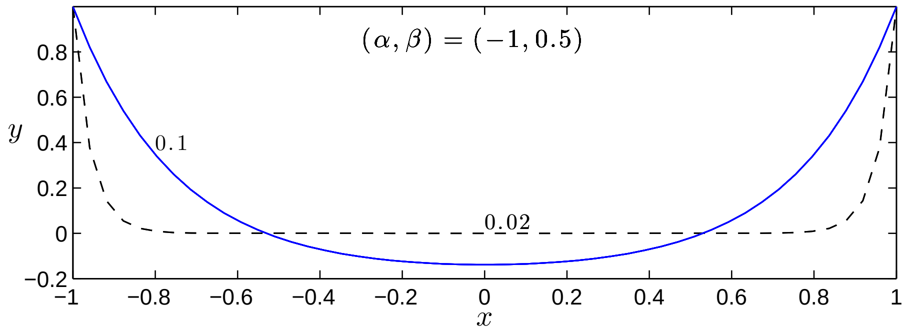

Figure 11.2: Solutions of (11.3) with $\alpha = -1 < 0$ and $\epsilon = 0.1$ (solid blue) and 0.02 (dashed black). There are boundary layers at $x = \pm 1$. The interior solution is zero to all orders in ϵ. There is no internal boundary layer at $x = 0$.

We return to the exceptional case, in which $\nu = 0,\ 1,\ 2\ \cdots$, later.

To take the outer limits, $X \to \pm\infty$, of the internal boundary layer solution in (11.21) we look up the asymptotic expansion of the parabolic cylinder functions e.g., in the appendix of **BO**, or in the **DLMF**:

$$D_\nu(t) \sim t^\nu \mathrm{e}^{-t^2/4}\,, \qquad \text{as } t \to \infty, \tag{11.22}$$

$$D_\nu(-t) \sim \tfrac{\sqrt{2\pi}}{\Gamma(-\nu)}\, t^{-\nu-1}\mathrm{e}^{t^2/4}\,, \qquad \text{as } t \to \infty. \tag{11.23}$$

Matching in the right-hand outer limit, $X \to +\infty$, implies that $B = 0$. Matching in the left-hand outer limit $X \to -\infty$ requires that

$$A = (\alpha \mathrm{e})^{(\nu+1)/2}\,\frac{\Gamma(-\nu)}{\sqrt{2\pi}}\,. \tag{11.24}$$

Case 2: $\alpha < 0$

There is no internal boundary layer. Instead there are boundary layers at *both* $x = 1$ and $x = -1$. — see figure 11.2

Lecture 12

Initial layers

12.1 The over-damped oscillator

With our knowledge of boundary layer theory, let's reconsider the over-damped harmonic oscillator from problem 2.7. With a change of notation, the problem in (2.69) is:

$$\epsilon x_{tt} + x_t + x = 0\,, \qquad \text{with the IC:} \quad x(0) = 0\,, \quad x_\tau(0) = 1\,. \tag{12.1}$$

This scaling is convenient for the long-time solution, but not for satisfying the two initial conditions.

We are going to use a boundary-layer-in-time, also known as an initial layer, to solve this problem. To address the initial layer we introduce

$$T \stackrel{\text{def}}{=} t/\epsilon\,, \qquad \text{and} \qquad X(T,\epsilon) = x(t,\epsilon)\,. \tag{12.2}$$

The rescaled problem is

$$X_{TT} + X_T + \epsilon X = 0\,, \qquad \text{with the IC:} \quad X(0) = 0\,, \quad X_\tau(0) = \epsilon\,. \tag{12.3}$$

Because X satisfies both the initial conditions it is convenient to attack this problem by first solving the initial-layer equation with

$$X(T,\epsilon) = \epsilon X_1(T) + \epsilon^2 X_2(T) + \cdots \tag{12.4}$$

One finds

$$X_{1TT} + X_{1T} = 0\,, \qquad \Rightarrow \qquad X_1 = 1 - \mathrm{e}^{-T}\,, \tag{12.5}$$

$$X_{2TT} + X_{2T} = -X_1\,, \qquad \Rightarrow \qquad X_1 = 2(1 - \mathrm{e}^{-T}) - T - T\mathrm{e}^{-T}\,. \tag{12.6}$$

All the constants of integration are determined because the initial-layer solution satisfies both initial conditions. Once $T \gg 1$, the initial-layer solution is

$$X \to \epsilon + \epsilon^2\,(2 - T) + \mathrm{ord}(\epsilon^3)\,, \tag{12.7}$$

$$= \epsilon(1 - t) + 2\epsilon^2 + \mathrm{ord}(\epsilon^3)\,. \tag{12.8}$$

We must match the outer solution onto this function — to facilitate the match, we've written the solution in terms of the outer time t. Notice how the term $\epsilon^2 T$ switched orders. We can anticipate that there are further switchbacks from the $\mathrm{ord}(\epsilon^3)$ terms.

We obtain the outer solution by solving (12.1) (without the initial conditions!) with the RPS

$$x(t,\epsilon) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{12.9}$$

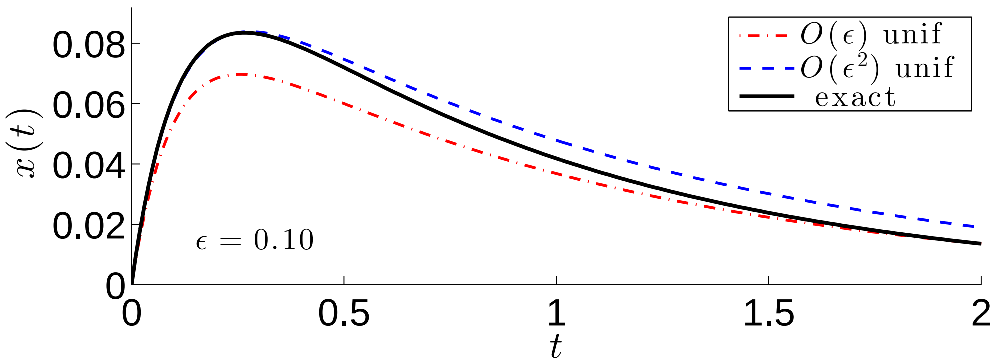

Figure 12.1: Comparison of (12.15) with the exact solution of (12.1).

The first two terms are

$$x_{1t} + x_1 = 0\,, \qquad \Rightarrow \qquad x_1 = A_1 \mathrm{e}^{-t}\,, \tag{12.10}$$

$$x_{2t} + x_2 = -x_{1tt}\,, \qquad \Rightarrow \qquad x_1 = A_1 t\mathrm{e}^{-t} + A_2 \mathrm{e}^{-t}\,, \tag{12.11}$$

and the reconstituted outer solution is

$$x = \epsilon A_1 \mathrm{e}^{-t} + \epsilon^2 \left(A_1 t\mathrm{e}^{-t} + A_2 \mathrm{e}^{-t} \right) . \tag{12.12}$$

In the matching region, $\epsilon \ll t \ll 1$, (12.12) is

$$X \to \epsilon A_1 (1-t) + \underbrace{\epsilon^2 A_1 t}_{\text{orphan}} + \epsilon^2 A_2 + \mathrm{ord}(\epsilon^3, \epsilon t^2, \epsilon^2 t^2) \tag{12.13}$$

Comparing (12.13) with (12.8) we see that

$$A_1 = 1\,, \qquad \text{and} \qquad A_2 = 2\,. \tag{12.14}$$

The term $\epsilon^2 A_1 t$ is orphaned because there is nothing in (12.8) to match it. We expect that switchbacks from the $\mathrm{ord}(\epsilon^3)$ terms in (12.8) will take care of the orphan. Notice that if we geared by taking $t \sim \epsilon^{1/2}$ then the orphan is the smallest term in (12.13) — we can move it into the ord.

Finally we can construct a uniformly valid solution as

$$x^{\mathrm{uni}} = \epsilon \left(\mathrm{e}^{-t} - \mathrm{e}^{-T} \right) + \epsilon^2 \left(t\mathrm{e}^{-t} - T\mathrm{e}^{-T} + 2\mathrm{e}^{-t} - 2\mathrm{e}^{-T} \right) + \mathrm{ord}(\epsilon^3)\,. \tag{12.15}$$

Figure 12.1 compares (12.15) with the exact solution

$$x = \frac{2\epsilon}{\sqrt{1-4\epsilon}}\, \mathrm{e}^{-t/2\epsilon} \sinh\left(\frac{\sqrt{1-4\epsilon}\, t}{2\epsilon} \right) . \tag{12.16}$$

Example: Consider

$$\dot{x} = -x - xy + \epsilon\kappa y\,, \qquad \epsilon\dot{y} = x - xy - \epsilon\kappa y\,. \tag{12.17}$$

12.2 Problems

Problem 12.1. Use boundary-layer theory to construct a leading-order solution of the IVP

$$\epsilon x_{tt} + x_t + x = t\mathrm{e}^{-t}\,, \qquad \text{with} \qquad x(0) = \dot{x}(0) = 0\,, \qquad \text{as } \epsilon \to 0. \tag{12.18}$$

Problem 12.2. Find the leading order $\epsilon \to 0$ solution of

$$\dot{u} = v\,, \qquad \epsilon \dot{v} = -v - u^2\,, \tag{12.19}$$

for $t > 0$ with initial conditions $u(0) = 1$ and $v(0) = 0$.

Problem 12.3. Find the leading order $\epsilon \to 0$ solution of

$$\epsilon \ddot{u} + (1+t)\dot{u} + u = 1\,, \tag{12.20}$$

for $t > 0$ with initial conditions $u(0) = 1$ and $\dot{u}(0) = -\epsilon^{-1}$.

Problem 12.4. A function $y(t,x)$ satisfies the integro-differential equation

$$\epsilon y_t = -y + f(t) + Y(t)\,, \tag{12.21}$$

where

$$Y(t) \stackrel{\text{def}}{=} \int_0^\infty y(t,x) \mathrm{e}^{-\beta x}\, \mathrm{d}x\,, \tag{12.22}$$

with $\beta > 1$. The initial condition is $y(0,x) = a(x)$. (This is the Grodsky model for insulin release.) Use boundary layer theory to find the composite solution on the interval $0 < t < \infty$. Compare this approximate solution with the exact solution of the model. To assist communication, use the notation

$$\alpha \stackrel{\text{def}}{=} 1 - \beta^{-1} \qquad \text{and} \qquad A \stackrel{\text{def}}{=} Y(0)\,, \qquad \text{and} \qquad \tau \stackrel{\text{def}}{=} \frac{t}{\epsilon}\,. \tag{12.23}$$

Problem 12.5. Solve the previous problem with $\beta = 1$.

Problem 12.6. The Michaelis-Menten model for an enzyme catalyzed reaction is

$$\dot{s} = -s + (s + k - 1)c\,, \qquad \epsilon \dot{c} = s - (s+k)c\,, \tag{12.24}$$

where $s(t)$ is the concentration of the substrate and $c(t)$ is the concentration of the catalyst. The initial conditions are

$$s(0) = 1\,, \qquad c(0) = 0\,. \tag{12.25}$$

Find the first term in the: *(i)* outer solution; *(ii)* the "initial layer" ($\tau \stackrel{\text{def}}{=} t/\epsilon$); *(iii)* the composite expansion.

Lecture 13

Boundary layers in fourth-order problems

13.1 A fourth-order differential equation

Let us consider a fourth-order boundary value problem which is similar to problems occurring in the theory of elasticity:

$$-\epsilon^2 u_{xxxx} + u_{xx} = 1\,, \tag{13.1}$$

with boundary conditions

$$u(-1) = u'(-1) = u(1) = u'(1) = 0\,. \tag{13.2}$$

The outer solution might be obtained with the RPS such as

$$u(x,\epsilon) = u_0(x) + \epsilon^{?} u_1(x) + \cdots \tag{13.3}$$

At leading order

$$u_{0xx} = 1\,, \qquad \Rightarrow \qquad u_0 = \frac{x^2-1}{2}\,. \tag{13.4}$$

We've applied only two of the four boundary conditions above.

Before worrying about higher order terms in (13.3), let's turn to the boundary layer at $x = -1$. We assume that the solution is an even function of x so the boundary layer at $x = +1$ can be constructed by symmetry.

If we look for a dominant balance with $X = (x+1)/\delta$ we find that $\delta = \epsilon$. Thus we consider a boundary layer rescaling

$$u(x,\epsilon) = U(X,\epsilon)\,, \qquad \text{where} \qquad X \stackrel{\text{def}}{=} \frac{x+1}{\epsilon}\,. \tag{13.5}$$

The boundary layer problem is then

$$-U_{XXXX} + U_{XX} = \epsilon^2\,. \tag{13.6}$$

Writing the leading-order outer solution in (13.4) in terms of X, we have

$$u_0(x,\epsilon) = -\epsilon X + \frac{1}{2}\epsilon^2 X^2\,. \tag{13.7}$$

Anticipating that we'll ultimately need to match the term $-\epsilon X$ in (13.7), we pose the boundary-layer expansion

$$U(X,\epsilon) = \epsilon U_1(X) + \epsilon^2 U_2(X) + \epsilon^3 U_3(X) + \cdots \tag{13.8}$$

There is no term $U_0(X)$ because the outer solution is $\mathrm{ord}(\epsilon)$ in the matching region.

Thus we have the hierarchy

$$-U_{1XXXX} + U_{1XX} = 0\,, \tag{13.9}$$

$$-U_{2XXXX} + U_{2XX} = 1\,, \tag{13.10}$$

$$-U_{3XXXX} + U_{3XX} = 0\,, \tag{13.11}$$

and so on.

The general solution of (13.9) is

$$U_1 = A_1 + B_1 X + C_1 \mathrm{e}^{-X} + \underbrace{D_1}_{=0} \mathrm{e}^{X}\,. \tag{13.12}$$

Above we've anticipated that $D_0 = 0$ to remove the exponentially growing solution. Then applying the boundary conditions at $X = 0$ we find

$$U_1 = A_1 \left(1 - X - \mathrm{e}^{-X}\right)\,. \tag{13.13}$$

To match (13.13) against the term $-\epsilon X$ in the interior solution in (13.7) we take

$$A_1 = 1\,. \tag{13.14}$$

Now we can construct a leading-order solution that is uniformly valid in the region near $x = -1$:

$$u_{\mathrm{uni}}(x) = \frac{x^2 - 1}{2} + \epsilon\left(1 - \mathrm{e}^{-(x+1)/\epsilon}\right)\,. \tag{13.15}$$

The derivative is

$$u_{\mathrm{uni}x}(x) = x + \mathrm{e}^{-(x+1)/\epsilon}\,, \tag{13.16}$$

which is indeed zero at $x = -1$.

Higher order terms

The equation for U_2, (13.10), has a solution

$$U_2(X) = \frac{X^2}{2} + A_2\left(1 - X - \mathrm{e}^{-X}\right)\,. \tag{13.17}$$

Above, we've satisfied both boundary conditions at $X = 0$. We've also matched the term $\epsilon^2 X^2/2$ in (13.7). To summarize, our boundary layer solution is

$$U(X) = \epsilon\Big(\underbrace{1}_{\text{orphan}} - X - \mathrm{e}^{-X}\Big) + \epsilon^2\frac{X^2}{2} + \epsilon^2 A_2\left(1 - X - \mathrm{e}^{-X}\right) + \mathrm{ord}(\epsilon^3)\,. \tag{13.18}$$

But we have unfinished business: we have not matched the orphan above with any term in the leading-order outer solution $u_0(x)$.

To take care of the orphan we must go to next order in the interior expansion:

$$u(x,\epsilon) = \frac{x^2 - 1}{2} + \epsilon u_1(x) + \mathrm{ord}\left(\epsilon^2\right)\,. \tag{13.19}$$

Thus

$$u_{1xx} = 0\,, \qquad \Rightarrow \qquad u_1(x) = \underbrace{P_1}_{=1} + \underbrace{Q_1}_{=0} x \tag{13.20}$$

We take $Q_1 = 0$ because the solution is even, and $P_1 = 1$ to take care of the orphan. The solution $u_1(x)$ does not satisfy any of the four boundary conditions. To summarize, the outer solution is

$$u(x, \epsilon) = \frac{x^2 - 1}{2} + \epsilon + \mathrm{ord}(\epsilon^2)\,. \tag{13.21}$$

The ord(ϵ) term above was accidently included in the uniform solution (13.16): in the outer region the expansion of (13.15) already agrees with all terms in (13.21).

Because $u_{0xxxx} = 0$, there are now no more non-zero terms in the outer region i.e., $u_2 = 0$, and therefore $A_2 = 0$ in (13.18). Moreover, all terms U_3, U_4 etcetera are also zero. Thus we have constructed an infinite-order asymptotic expansion. Using symmetry we can construct a uniformly valid solution throughout the whole domain

$$u_{\mathrm{uni}}(x) = \frac{x^2 - 1}{2} + \epsilon\left(1 - \mathrm{e}^{-(x+1)/\epsilon} - \mathrm{e}^{(x-1)/\epsilon}\right)\,. \tag{13.22}$$

13.2 Problems

Problem 13.1. Solve (13.1) exactly and use MATLAB to compare the exact solution with the asymptotic solution in (13.22).

Problem 13.2. Find two terms in ϵ in the outer region and match to the inner solution at both boundaries for

$$\epsilon^2 u'''' - u'' = \mathrm{e}^{ax}\,. \tag{13.23}$$

The domain is $-1 \le x \le 1$ with BCs

$$u(-1) = u'(-1) = 0\,, \qquad \text{and} \qquad u(1) = u'(1) = 0\,. \tag{13.24}$$

Problem 13.3. Find two terms in ϵ in the outer region and match to the inner solution at both boundaries for

$$\epsilon^2 u'''' - u'' = 0\,. \tag{13.25}$$

The domain is $0 \le x \le 1$ with BCs

$$u(0) = 0\,, \quad u'(0) = 1\,, \qquad \text{and} \qquad u(1) = u'(1) = 0\,. \tag{13.26}$$

Problem 13.4. Considering the eigenproblem

$$-\epsilon^2 u'''' + u'' = \lambda u\,, \tag{13.27}$$

on the domain is $0 \le x \le \pi$ with BCs

$$u(0) = u'(0) = 0\,, \qquad \text{and} \qquad u(\pi) = u'(\pi) = 0\,. \tag{13.28}$$

(i) Prove that all eigenvalues are real and positive. *(ii)* Show that with a suitable definition of inner product, that eigenfunctions with different eigenvalues are orthogonal. *(iii)* Use boundary layer theory to find the shift in the unperturbed spectrum, $\lambda = 1,\, 2,\, 3 \cdots$, induced by ϵ.