## GAUSSIAN MIXTURE MODEL (CONTINUED)

**MACHINE LEARNING 1, SS21** 

Ceca Kraišniković Dipl.-Ing. Dipl. inž. elektr. i računar. Institute of Theoretical Computer Science Graz University of Technology

# GAUSSIAN MIXTURE MODEL: ESTIMATION OF PARAMETERS $\Sigma_k, \pi_k$

### Derivation for $\Sigma_k$

Log-likelihood function

$$\ln p(X \mid \pi, \mu, \Sigma) = \sum_{n=1}^{N} \ln \sum_{k=1}^{K} \pi_k \mathcal{N}(x_n \mid \mu_k, \Sigma_k),$$

$$\Theta = \{\pi, \mu, \Sigma\}$$

Set the derivative to zero:

$$\frac{\partial \ln p(X \mid \Theta)}{\partial \Sigma_k} \stackrel{!}{=} 0$$

• Solution (similar steps as in the derivation for  $\mu_k$  from the previous lecture):

$$\Sigma_{k} = \frac{1}{N_{k}} \sum_{n=1}^{N} \gamma_{nk} (x_{n} - \mu_{k}) (x_{n} - \mu_{k})^{T},$$

$$N_{k} = \sum_{n=1}^{N} \gamma_{nk}$$

#### Derivation for $\pi_k$

• Set the derivative of the log-likelihood function w.r.t.  $\pi_k$  to zero:

$$\frac{\partial \ln p(X \mid \Theta)}{\partial \pi_k} \stackrel{!}{=} 0$$

Optimization with a constraint:

$$\sum_{k=1}^{K} \pi_k = 1$$

Use Lagrange multipliers, optimize the following objective function:

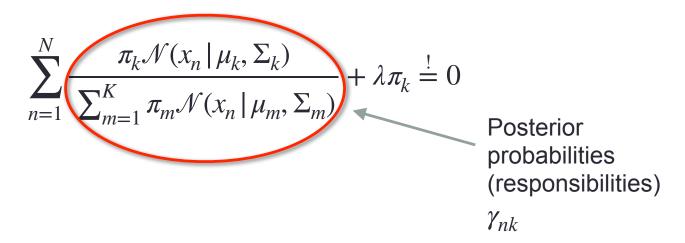
$$J(k) = \ln p(X \mid \Theta) + \lambda \left(\sum_{k=1}^{K} \pi_k - 1\right)$$

and set  $\frac{\partial J(k)}{\partial \pi_k}$  to zero, which yields:

$$\sum_{n=1}^{N} \frac{\mathcal{N}(x_n | \mu_k, \Sigma_k)}{\sum_{m=1}^{K} \pi_m \mathcal{N}(x_n | \mu_m, \Sigma_m)} + \lambda \stackrel{!}{=} 0$$

#### Derivation for $\pi_k$

• Multiply everything with  $\pi_k$ :



#### Derivation for $\pi_k$

• This gives:

$$\sum_{n=1}^{N} \gamma_{nk} + \lambda \pi_k \stackrel{!}{=} 0$$

• We set  $\sum_{k=1}^{N} \gamma_{nk} = N_k$ , and then sum over all k components:

$$\sum_{k=1}^K N_k + \sum_{k=1}^K \lambda \pi_k = 0$$

• By making use of the constraint  $\sum_{k=1}^K \pi_k = 1$ , i.e.,  $\lambda \sum_{k=1}^K \pi_k = \lambda$ , and  $\sum_{k=1}^K N_k = N$ , we obtain  $\lambda = -N$ .

By substituting in the first equation (of this slide):  $N_k - N\pi_k = 0$ ,

$$\pi_k = \frac{N_k}{N}.$$