$$\boxed{p(X|\Theta) = \prod_{n=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

likelihood function

$/\ln$

$= \frac{1}{\sqrt{2\pi\sigma^2}}$

$$\boxed{\ln p(X|\Theta)} = \sum_{n=1}^{N} \ln\left(\overbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}^{p_1} \cdot \overbrace{e^{-\frac{(x_n-\mu)^2}{2\sigma^2}}}^{p_2} \right) =$$

log-likelihood function

$\boxed{\ln(p_1 \cdot p_2) = \ln p_1 + \ln p_2}$

$$= \sum_{n=1}^{N} \left(\overbrace{\ln \frac{1}{\sqrt{2\pi\sigma^2}}}^{\ln p_1} + \overbrace{\ln e^{-\frac{(x_n-\mu)^2}{2\sigma^2}}}^{\ln p_2} \right) =$$

$\boxed{\ln \frac{a}{b} = \ln a - \ln b}$

$$= \sum_{n=1}^{N} \left(\underbrace{\ln 1}_{=0} - \ln\left(\sqrt{2\pi\sigma^2}\right) - \frac{(x_n-\mu)^2}{2\sigma^2} \right) =$$

$\boxed{\sqrt{x} = x^{\frac{1}{2}}}$

$$= \sum_{n=1}^{N} \left(-\ln(2\pi\sigma^2)^{\frac{1}{2}} - \frac{1}{2} \frac{(x_n-\mu)^2}{\sigma^2} \right) =$$

$\boxed{\ln a^b = b \cdot \ln a}$

$$= \sum_{n=1}^{N} \left(\underbrace{-\frac{1}{2} \ln(2\pi\sigma^2)}_{\text{const, does not depend on } n} - \frac{1}{2} \frac{(x_n-\mu)^2}{\sigma^2} \right) =$$

$\boxed{\sum_{n=1}^{N} c = c \cdot N, \quad c = \text{const}}$

$$= -\frac{1}{2} \ln(2\pi\sigma^2) \cdot N + \sum_{n=1}^{N} \left(-\frac{1}{2} \frac{(x_n-\mu)^2}{\sigma^2} \right) =$$

$\boxed{\sum_{n=1}^{N} a \cdot x_n = a \cdot \sum_{n=1}^{N} x_n}$

$$= \boxed{-\frac{N}{2} \cdot \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n-\mu)^2}$$

$$\frac{\partial \ln p(X|\Theta)}{\partial \mu} = 0 - \frac{1}{2\sigma^2} \cdot \sum_{n=1}^{N} \frac{\partial}{\partial \mu} (x_n-\mu)^2 \overset{!}{=} 0 \quad / \cdot \left(\frac{-2\sigma^2}{1}\right)$$

$$\sum_{n=1}^{N} 2(x_n-\mu) \cdot (-1) = 0 \quad / \cdot (-1)$$

$$2 \sum_{n=1}^{N} (x_n - \mu) = 0 \quad / \cdot \frac{1}{2}$$

$$\sum_{n=1}^{N} x_n - \underbrace{\sum_{n=1}^{N} \mu}_{= N \cdot \mu} = 0$$

$$N \cdot \mu = \sum_{n=1}^{N} x_n \implies \boxed{\mu = \frac{1}{N} \sum_{n=1}^{N} x_n}$$

$$\frac{\partial}{\partial \sigma^2}\left(\ln p(X|\Theta)\right) = -\frac{N}{2}\frac{\partial}{\partial \sigma^2}\left(\ln(2\pi\sigma^2)\right) + \frac{\partial}{\partial \sigma^2}\left(\sum_{n=1}^{N} - \frac{(x_n-\mu)^2}{2\sigma^2}\right) \stackrel{!}{=} 0$$

$\Rightarrow$ Replace σ^2 with v (for easier derivations)

$$\frac{\partial \ln p(X|\Theta)}{\partial v} = -\frac{N}{2}\frac{\partial}{\partial v}\ln(2\pi v) - \frac{\partial}{\partial v}\left(\frac{1}{2v}\cdot\sum_{n=1}^{N}(x_n-\mu)^2\right) =$$

$$-\frac{N}{2}\cdot\frac{1}{\cancel{2\pi} v}\cdot\cancel{(2\pi)} - \frac{1}{2}\cdot\left(\frac{\partial}{\partial v}\cdot\underbrace{\left(\frac{1}{v}\right)}_{v^{-1}}\right)\cdot\sum_{n=1}^{N}(x_n-\mu)^2 =$$

$\left[(\ln x)' = \frac{1}{x}\right]$

$$= -\frac{N}{2\cdot v} - \frac{1}{2}\cdot(-1)\cdot v^{-2}\cdot\sum_{n=1}^{N}(x_n-\mu)^2 =$$

$\left[\frac{1}{x} = x^{-1}\right.$

$\left.(x^n)' = n\,x^{n-1}\right]$

$$= -\frac{N}{2v} + \frac{1}{2v^2}\cdot\sum_{n=1}^{N}(x_n-\mu)^2 \stackrel{!}{=} 0$$

$\Rightarrow$ $(v := \sigma^2)$, i.e, substitute back σ^2 and express σ^2

$$-\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\cdot\sum_{n=1}^{N}(x_n-\mu)^2 \stackrel{!}{=} 0$$

$$\frac{1}{2\sigma^2}\left(-N + \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right) = 0 \quad \Big/ \cdot \frac{1}{2\sigma^2}$$

$$N = \frac{1}{\sigma^2}\sum(x_n-\mu)^2 \quad \Big/ \cdot \frac{1}{N}\cdot\sigma^2$$

$$\boxed{\sigma^2 = \frac{1}{N}\cdot\sum_{n=1}^{N}(x_n-\mu)^2}$$