SPM
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Optimal Power Diagrams via Function Approximation

Yanyang Xiao[1], Zhonggui Chen[1,2], Juan Cao[1,2], Jessica Zhang[2], Cheng Wang[1]

1 Xiamen University
2 Carnegie Mellon University

Outline

- Background
- Optimal Power Diagrams
- Optimization Framework
- Results
- Conclusion

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- **Background**
- Optimal Power Diagrams
- Optimization Framework
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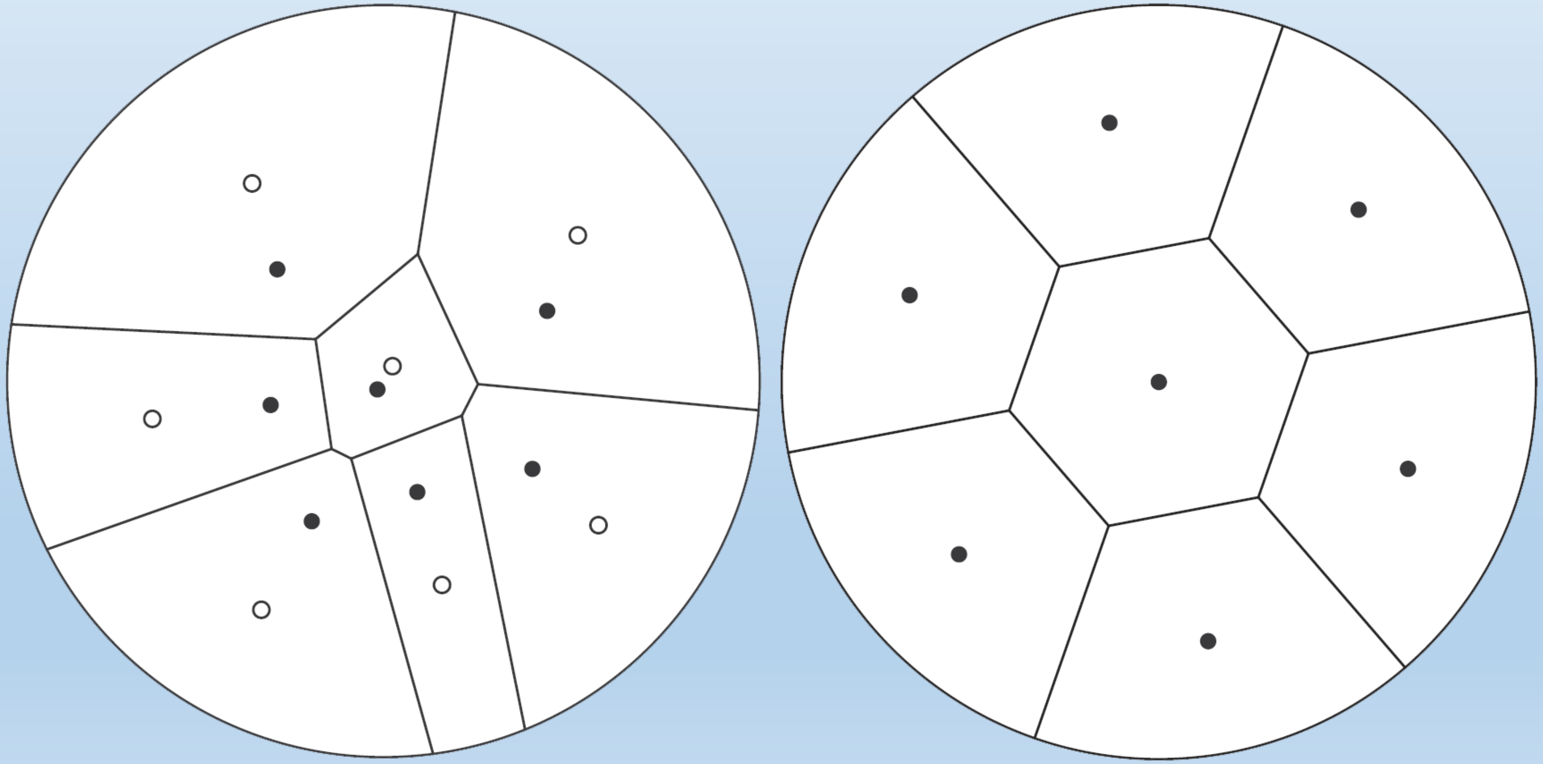
Background

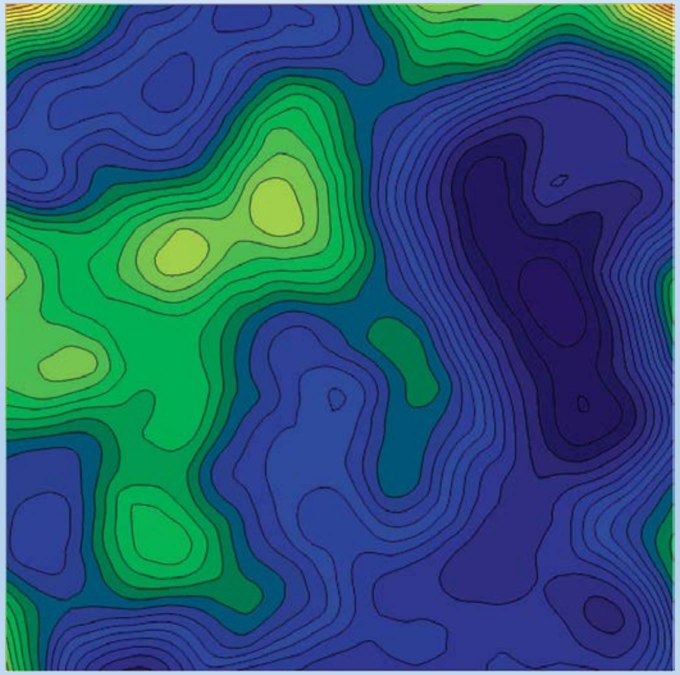
- 1. Centroidal Voronoi Tessellation (CVT) [Qiang Du et.al., SIAM Review 1999]



Background

- 1. Centroidal Voronoi Tessellation (CVT)

$$\mathcal{E}_{CVT}(\mathbf{X}, \mathcal{V}) = \sum_{i=1}^{n} \int_{V_i} \left\| \mathbf{x} - \mathbf{x}_i \right\|^2 \mathrm{d}\mathbf{x} = \sum_{i=1}^{n} \int_{V_i} \left(\mathbf{x}^2 - \left(2\mathbf{x}_i^T \mathbf{x} - \mathbf{x}_i^2 \right) \right) \mathrm{d}\mathbf{x}$$



[Lin Lu et.al., TVCG 2012]

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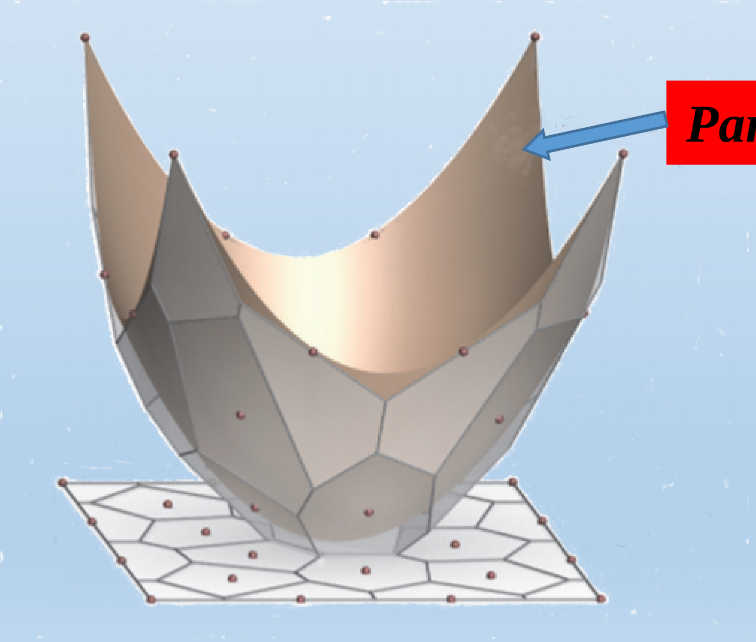

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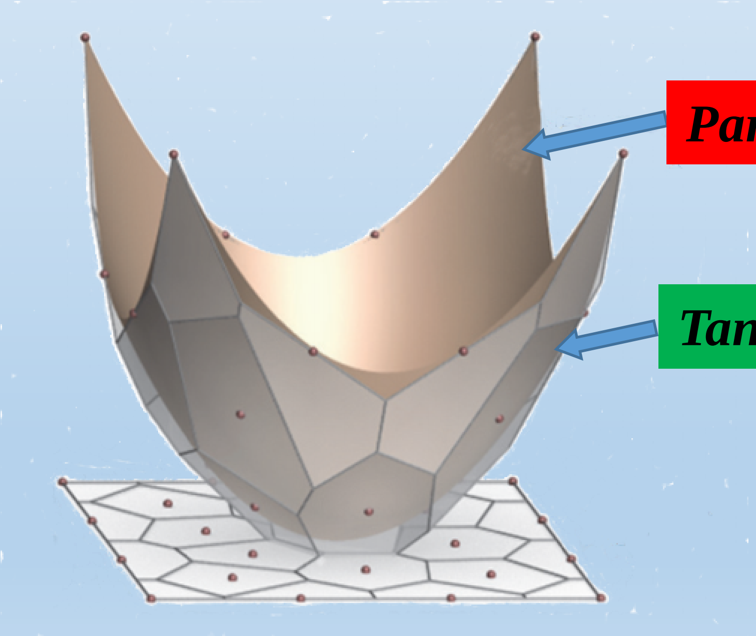

Paraboloid

Tangent hyperplane at a site

Background

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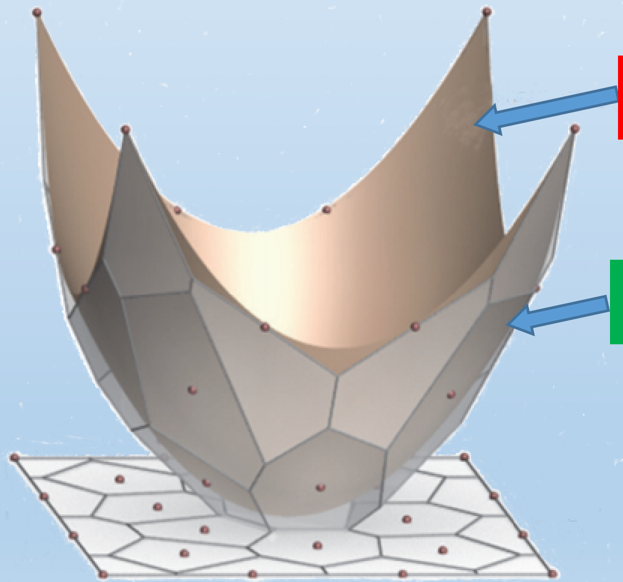

Paraboloid

Tangent hyperplane at a site

Minimizing $\mathcal{E}_{CVT}$

↓

CVT

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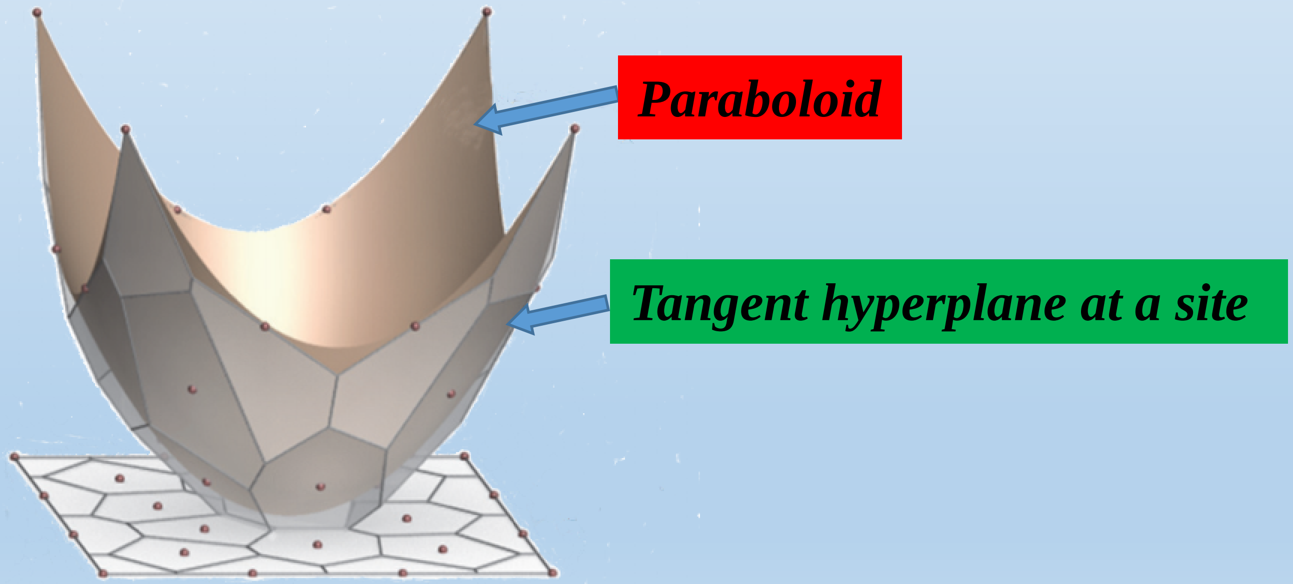


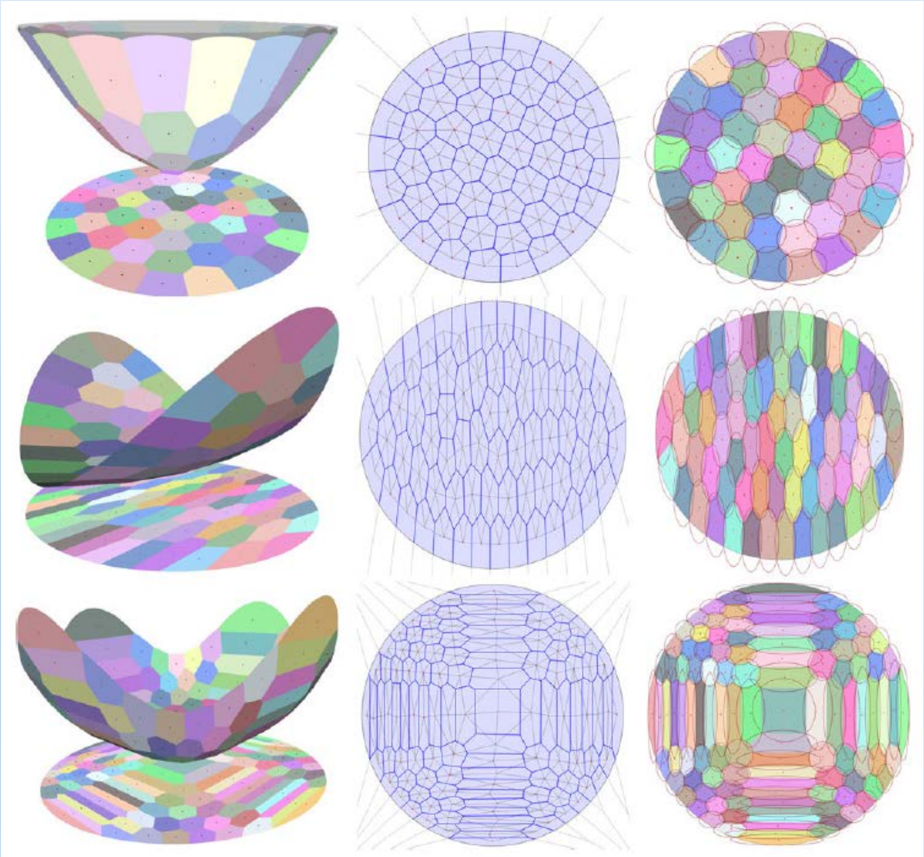
Minimizing $\mathcal{E}_{CVT}$

$\downarrow$ Lloyd & L-BFGS

CVT

Background

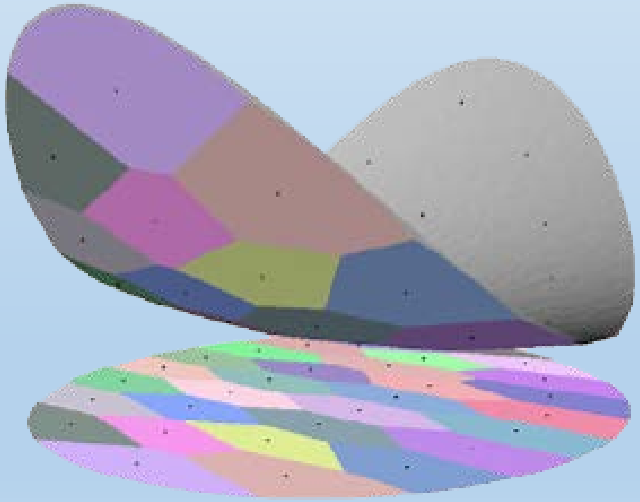
- 2. Optimal Voronoi Tessellation (OVT) [Max Budninskiy et.al., ACM TOG 2016]



Background

- 2. Optimal Voronoi Tessellation (OVT)

$$\mathcal{E}_{OVT}(\mathbf{X},\mathcal{V}) = \| f - f_T \|_{L^1} = \sum_{i=1}^{n} \int_{V_i} (f(\mathbf{x}) - T_i(\mathbf{x}))\mathrm{d}\mathbf{x},$$



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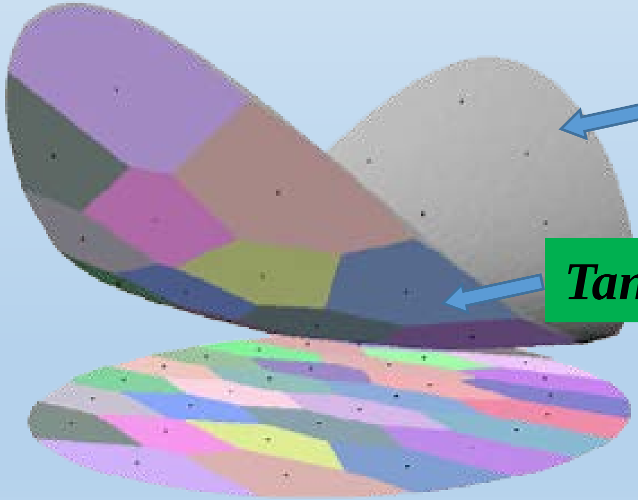

Convex function

Tangent hyperplane at a site

$$T_i(\mathbf{x}) = \nabla f(\mathbf{x}_i) \cdot (\mathbf{x} - \mathbf{x}_i) + f(\mathbf{x}_i)$$

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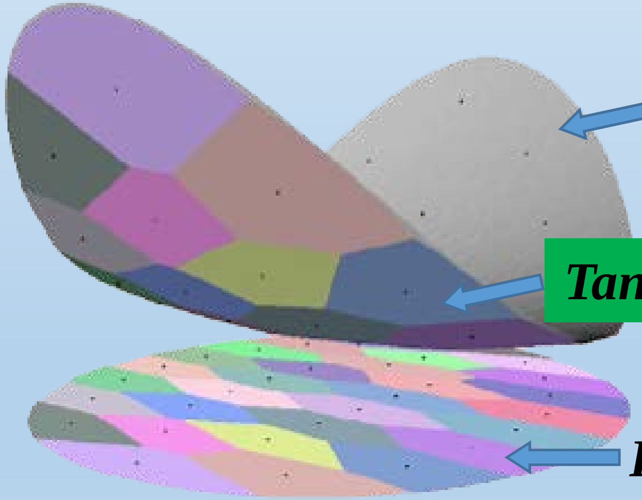

Convex function

Tangent hyperplane at a site

Power diagram defined by

$$\begin{cases} \mathbf{p}_i = \frac{1}{2} \nabla f(\mathbf{x}_i) \\ w_i = \frac{1}{4} \left| \nabla f(\mathbf{x}_i) \right|^2 + f(\mathbf{x}_i) - \nabla f(\mathbf{x}_i) \cdot \mathbf{x}_i \end{cases}$$

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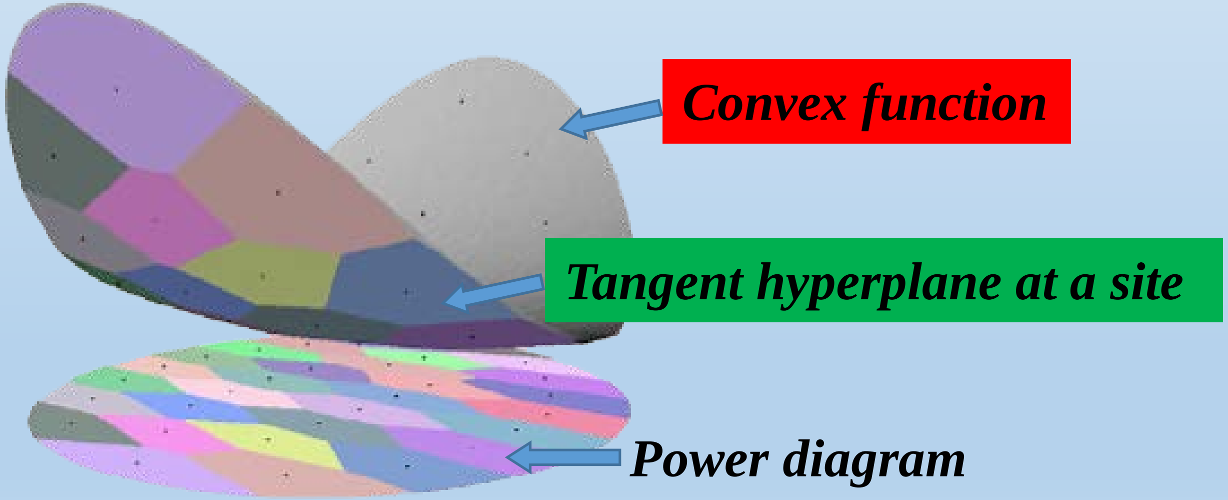

Minimizing $\mathcal{E}_{OVT}$

↓ Lloyd & L-BFGS

OVT

Background

- 2. Optimal Voronoi Tessellation (OVT)

$$f\left(\mathbf{x}\right) = \mathbf{x}^2, \mathcal{E}_{OVT}(\mathbf{X}, \mathcal{V}) \equiv \mathcal{E}_{CVT}(\mathbf{X}, \mathcal{V})$$

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- **Optimal Power Diagrams**
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Optimal Power Diagrams

- 1. Core idea: (1) construct model

Given $f(\mathbf{x}), \mathbf{x} \in \Omega$

and $\mathcal{V} = \left\{ V_i, i = 1, \dots, n \,|\, V_i \in \Omega; \forall i \neq j, V_i \bigcap V_j = \varnothing; \bigcup V_i = \Omega \right\}$

construct $P_i(\mathbf{x}) \approx f(\mathbf{x}), \mathbf{x} \in V_i, i = 1, \dots, n$

Optimal Power Diagrams

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$$\mathcal{E}_{OVT}(\mathbf{X},\mathcal{V}) = \| f - f_T \|_{L^1} = \sum_{i=1}^{n} \int_{V_i} (f(\mathbf{x}) - \boxed{T_i(\mathbf{x})}) \mathrm{d}\mathbf{x},$$

Tangent hyperplane

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Ours $$\mathcal{E}(\mathcal{V},\{P_i(\mathbf{x})\}_{i=1}^{n}) = \| f - f_P \|_{L^1} = \sum_{i=1}^{n} \int_{V_i} \left| f(\mathbf{x}) - P_i(\mathbf{x}) \right| \mathrm{d}\mathbf{x},$$

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Best fitting hyperplane

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Optimal Power Diagrams

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$$\mathcal{E}(\mathcal{V},\{P_i(\mathbf{x})\}_{i=1}^{n}) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \left(f(\mathbf{x}) - P_i(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x},$$

NP-hard

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① Restrict $\mathcal{V}$ to power diagram, determined by $\left(\mathbf{X}, W\right) = \left(\{\mathbf{x}_i\}_{i=1}^{n}, \{w_i\}_{i=1}^{n}\right)$

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② When $\mathcal{V}$ is fixed, best-fit hyperplane on each cell is determined, denote by $P_i^*(\mathbf{x})$

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$$\mathcal{E}_{OPD}(\mathbf{X}, W) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \left(f(\mathbf{x}) - P_i^*(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x},$$

Optimal Power Diagrams

- 2. Derivatives

$$\mathcal{E}_{OPD}(\mathbf{X}, W) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \left(f(\mathbf{x}) - P_i^*(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x},$$

Optimal Power Diagrams

- 2. Derivatives

$$\mathcal{E}_{OPD}(\mathbf{X},W) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \left(f(\mathbf{x}) - P_i^*(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x},$$

$$\frac{\partial \mathcal{E}_{OPD}(\mathbf{X},W)}{\partial \mathbf{x}_i} = \sum_{j \in J_i} \int_{V_{ij}} \left(\left| f(\mathbf{x}) - P_i^*(\mathbf{x}) \right|^2 - \left| f(\mathbf{x}) - P_j^*(\mathbf{x}) \right|^2 \right) \frac{\mathbf{x} - \mathbf{x}_i}{\left| \mathbf{x}_j - \mathbf{x}_i \right|} \mathrm{d}s$$

$$\frac{\partial \mathcal{E}_{OPD}(\mathbf{X},W)}{\partial w_i} = \sum_{j \in J_i} \int_{V_{ij}} \left(\left| f(\mathbf{x}) - P_i^*(\mathbf{x}) \right|^2 - \left| f(\mathbf{x}) - P_j^*(\mathbf{x}) \right|^2 \right) \frac{1}{2\left| \mathbf{x}_j - \mathbf{x}_i \right|} \mathrm{d}s$$

where J_i is the indexes of sites with cells adjacent to V_i,
$V_{ij} = \partial V_i \bigcap \partial V_j$ is the common boundary of V_i and V_j

Optimal Power Diagrams

- 3. Goal

$$\mathcal{E}_{OPD}(\mathbf{X},W) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \Big(f(\mathbf{x}) - P_i^*(\mathbf{x}) \Big)^2 \mathrm{d}\mathbf{x},$$

$$\boxed{\text{Minimizing } \mathcal{E}_{OPD}} \longrightarrow \boxed{\text{OPD}}$$

Optimal Power Diagrams

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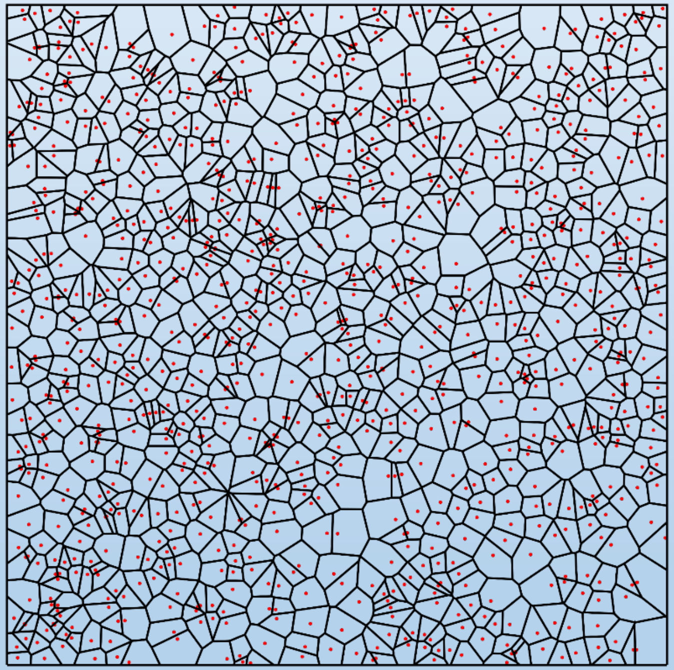
Optimization Framework

- 1. An observation

$$f(x,y) = 100x^2 + y^2, -1 \le x, y \le 1$$



Random initialization

Optimization Framework

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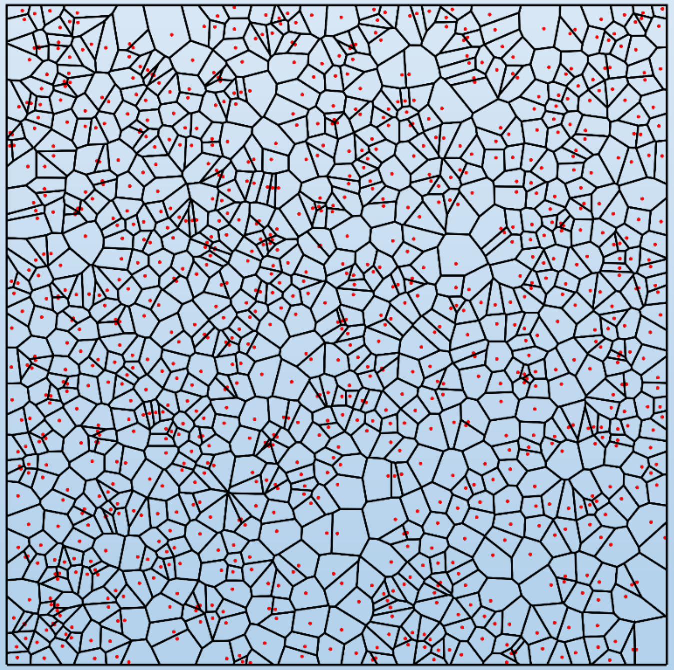

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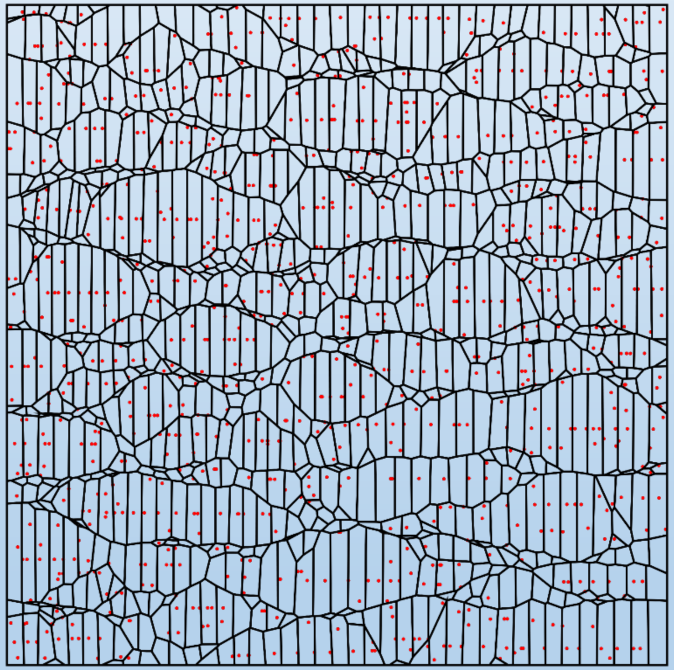

Resulting tessellation after optimizing $\left(\mathbf{X},W\right)$ simultaneously

Optimization Framework

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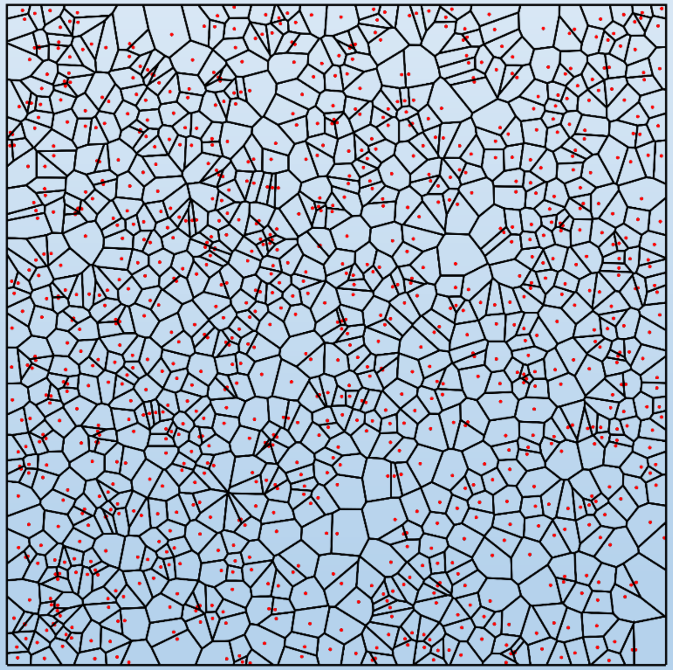

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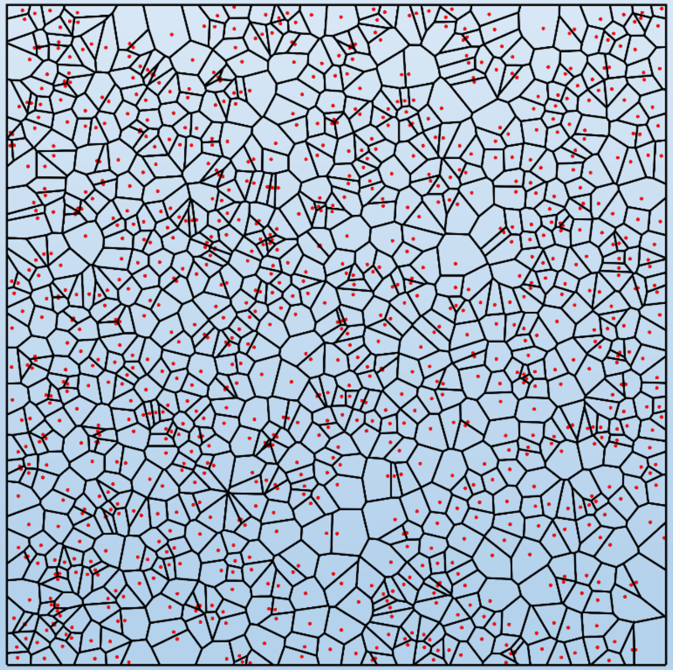

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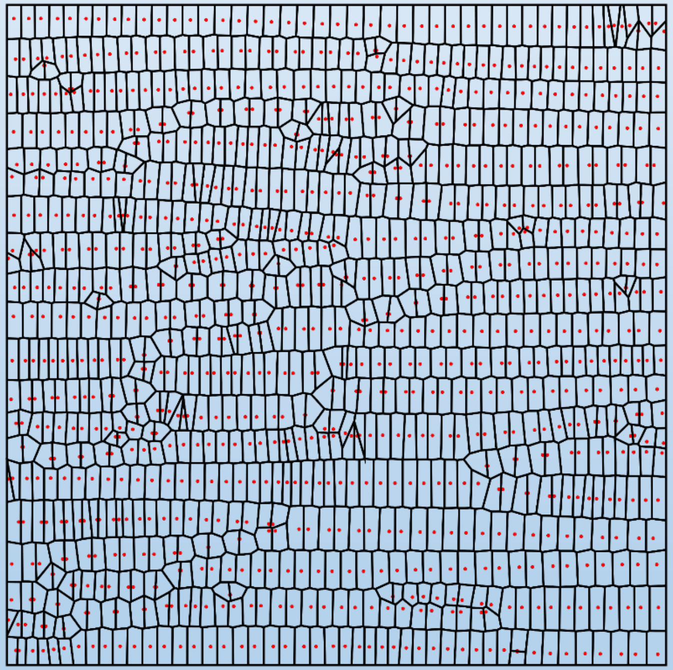

Only position optimization

Optimization Framework

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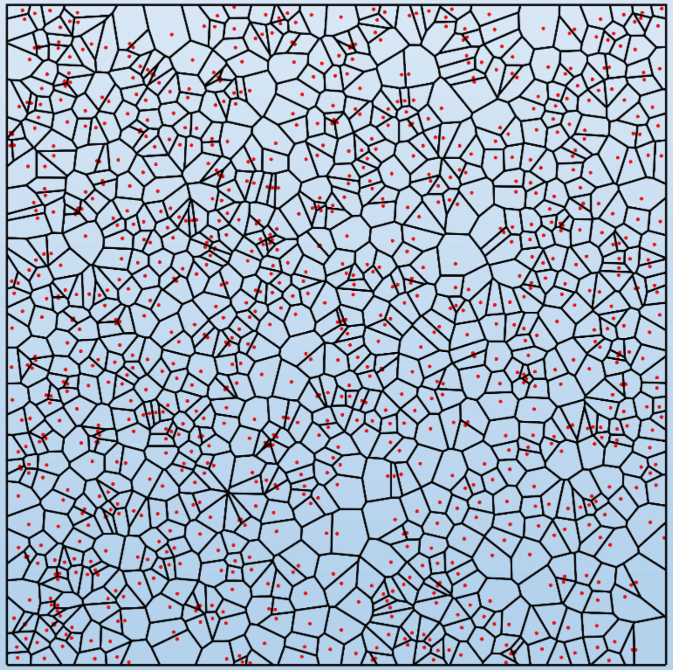

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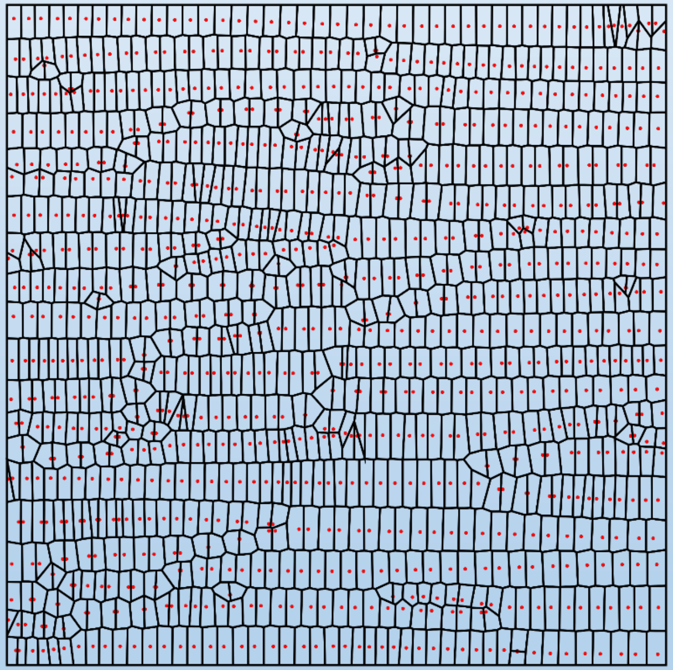

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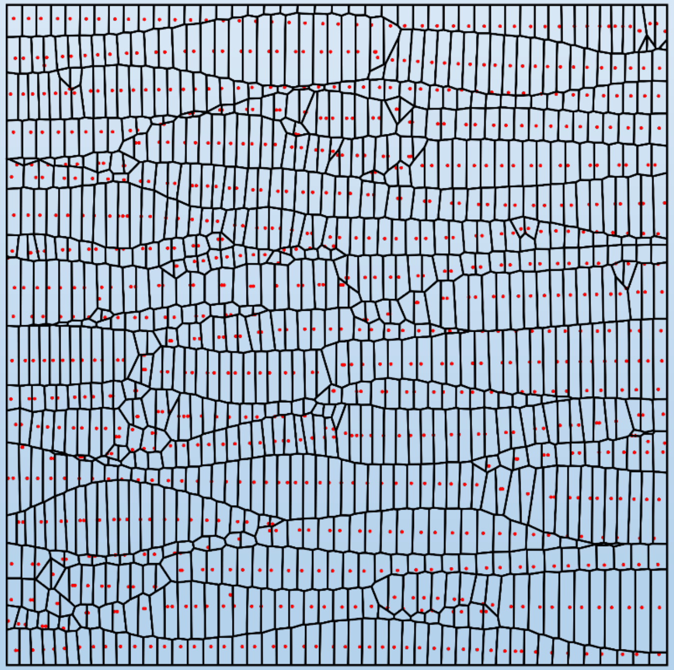

Position-weight optimization
Based on middle tessellation

Optimization Framework

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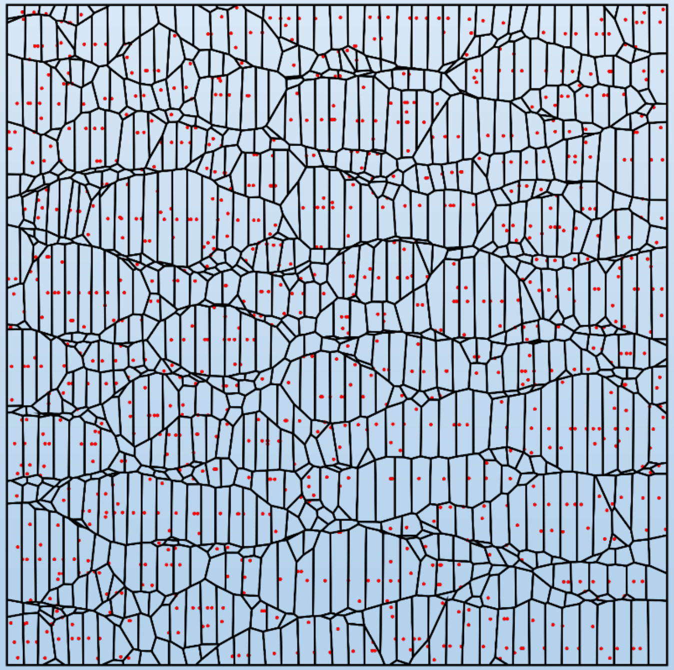
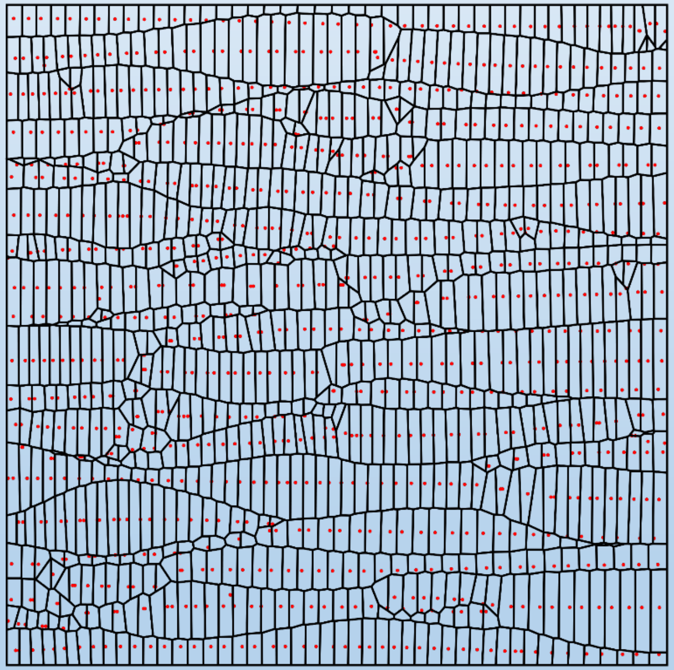


Resulting tessellation with
only position-weight optimization



Resulting tessellation with
firstly position optimization
then position-weight optimization

Optimization Framework

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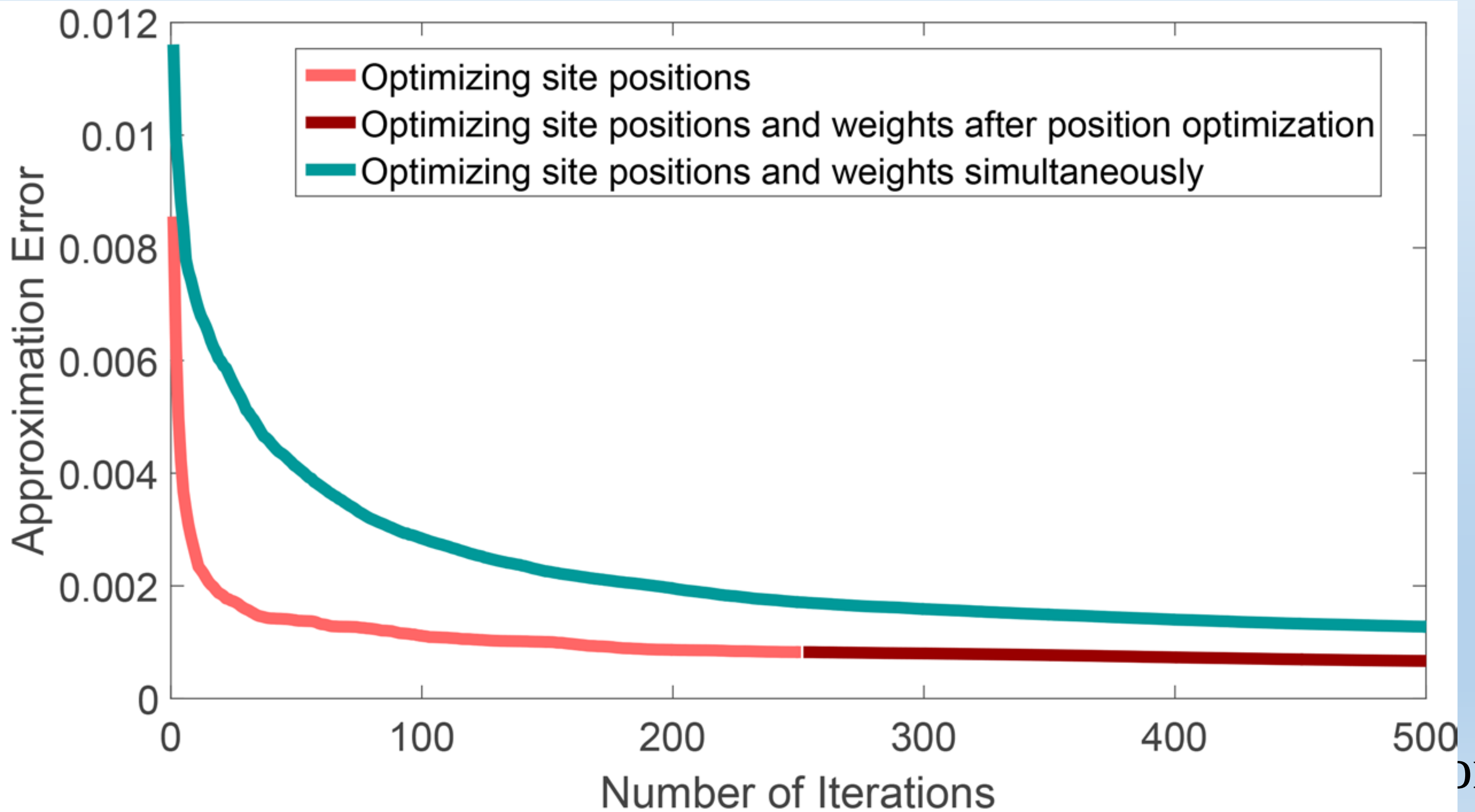
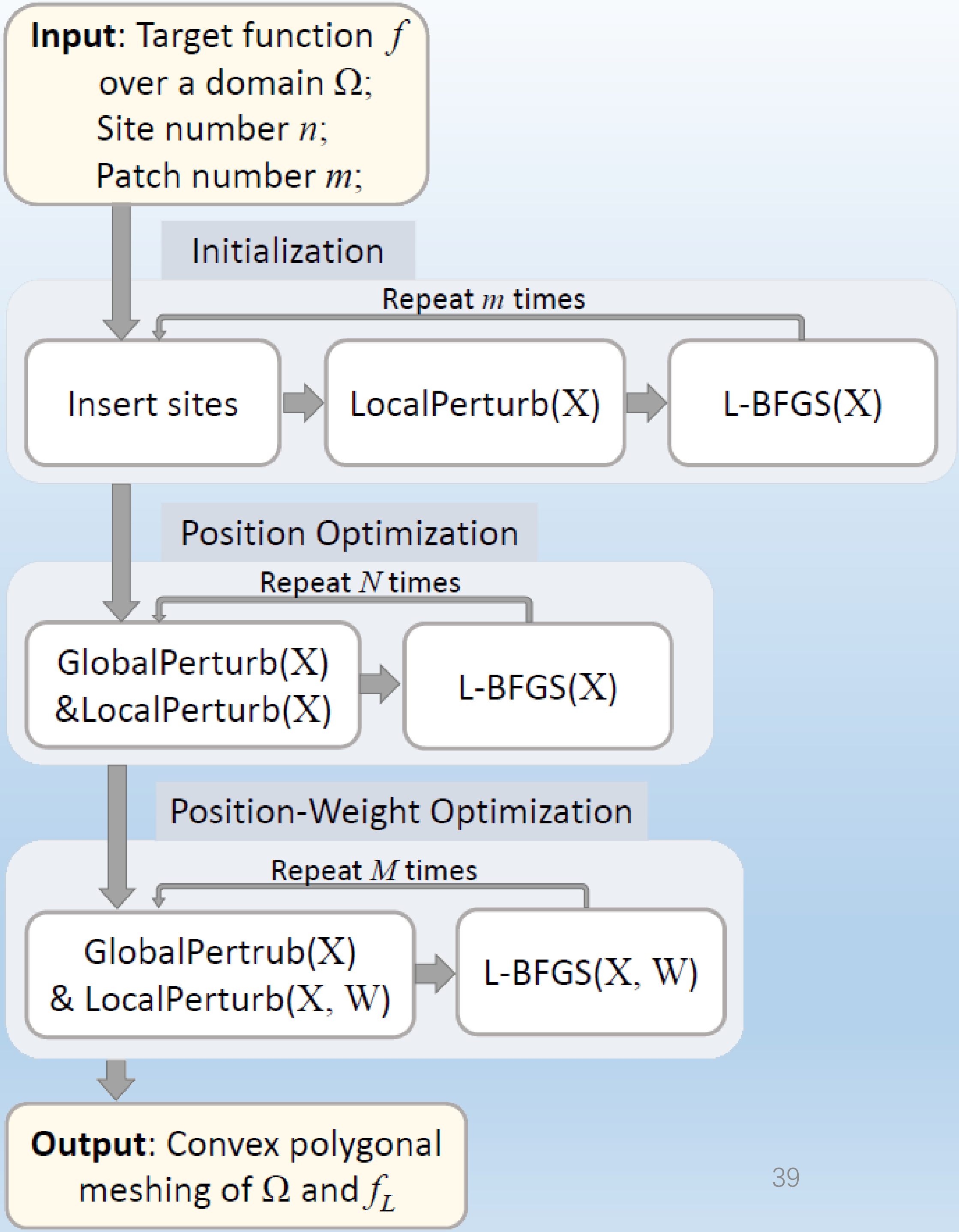


Only

on

Optimization Framework

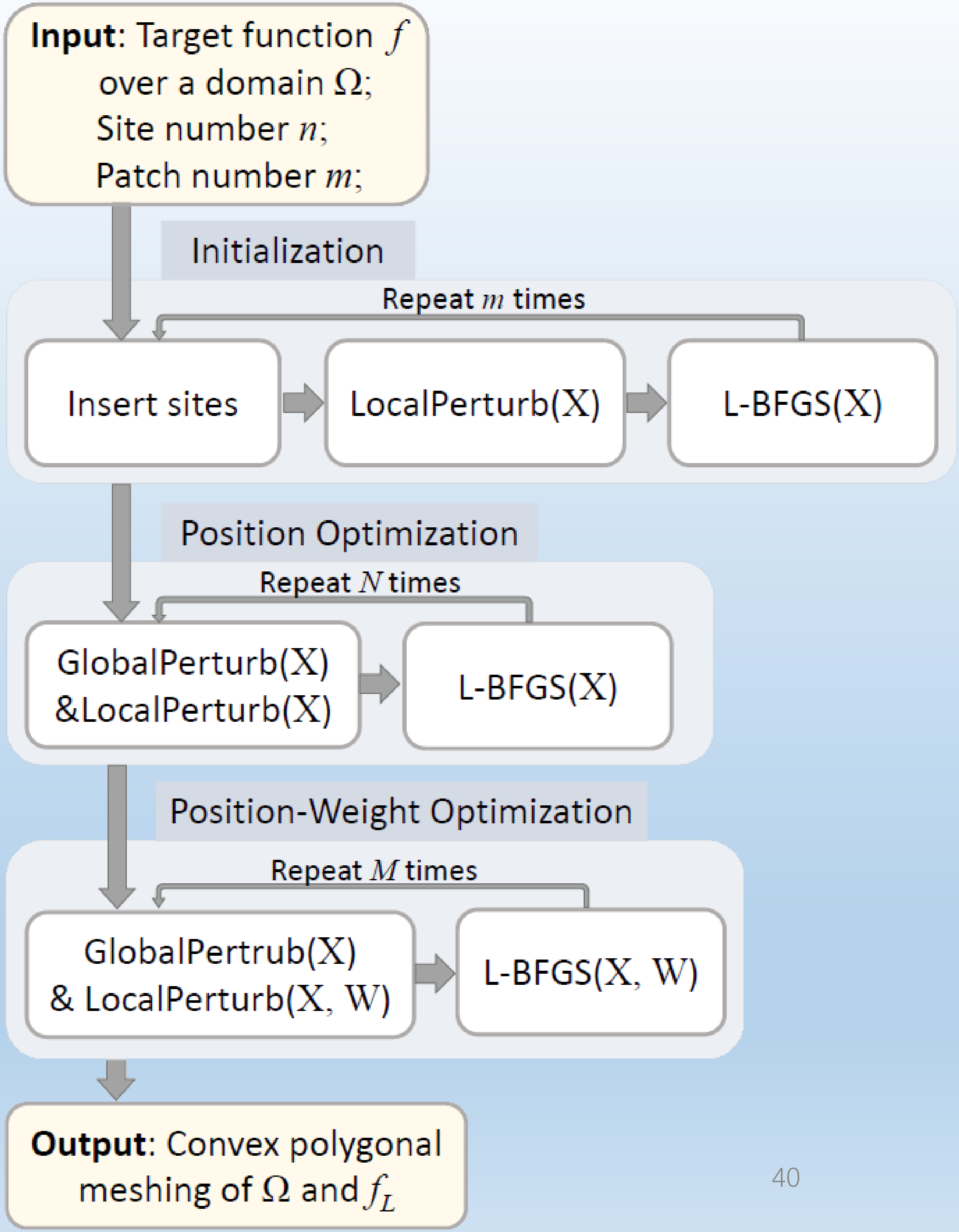
- 2. Overview



Optimization Framework

- 2. Overview

(1) Initialization:
inserting-optimizing site positions

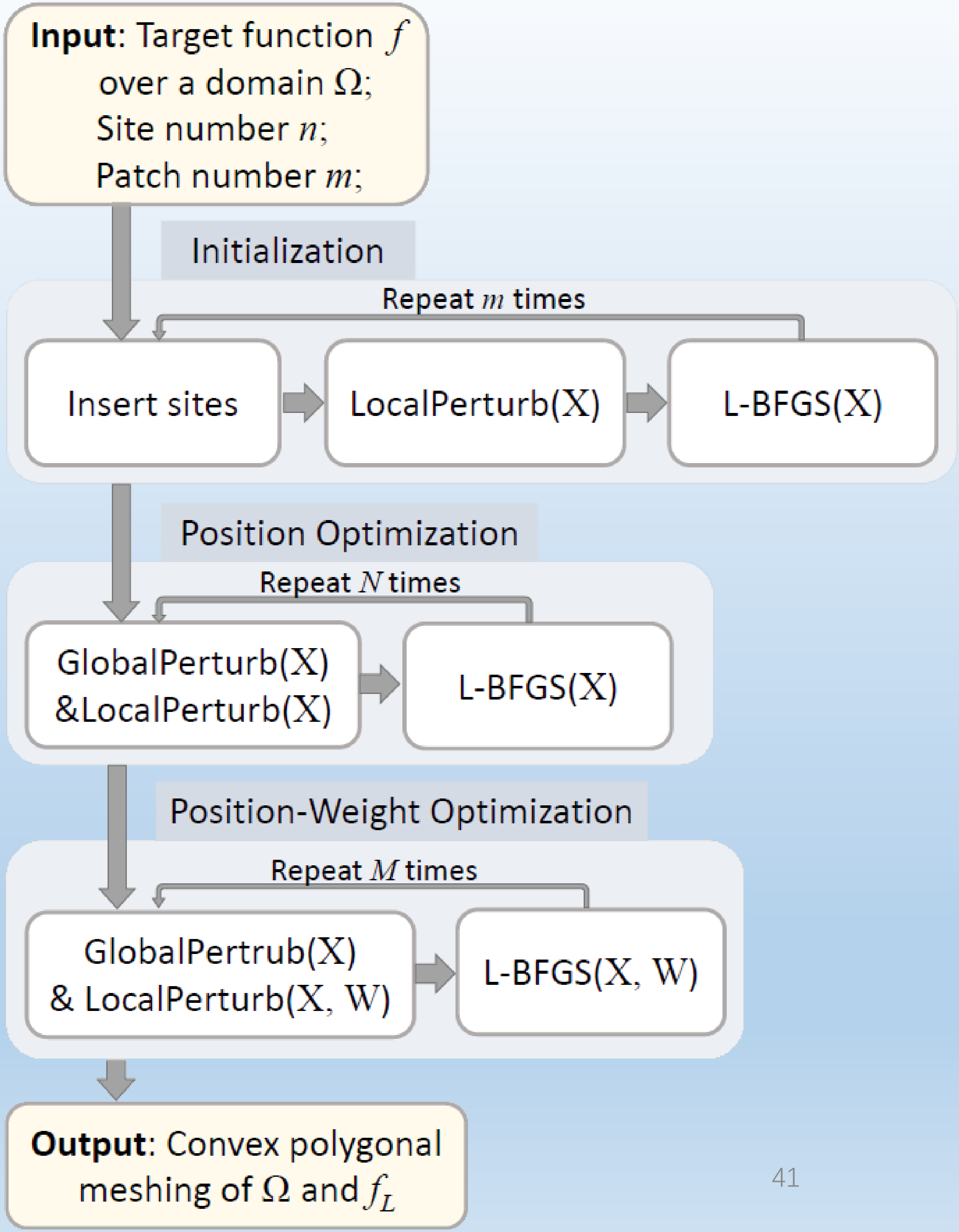


Optimization Framework

- 2. Overview

(1) Initialization:
inserting-optimizing site positions

(2) Position optimization:
main stage of the framework



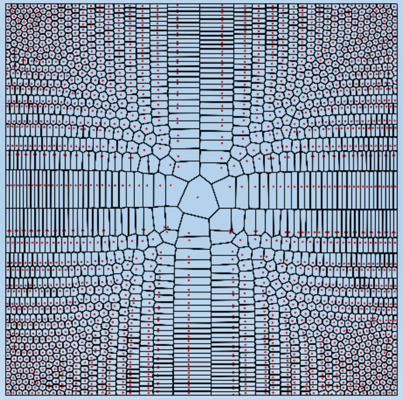
Optimization Framework

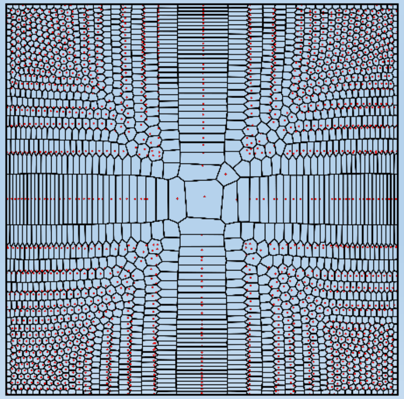
- 2. Overview

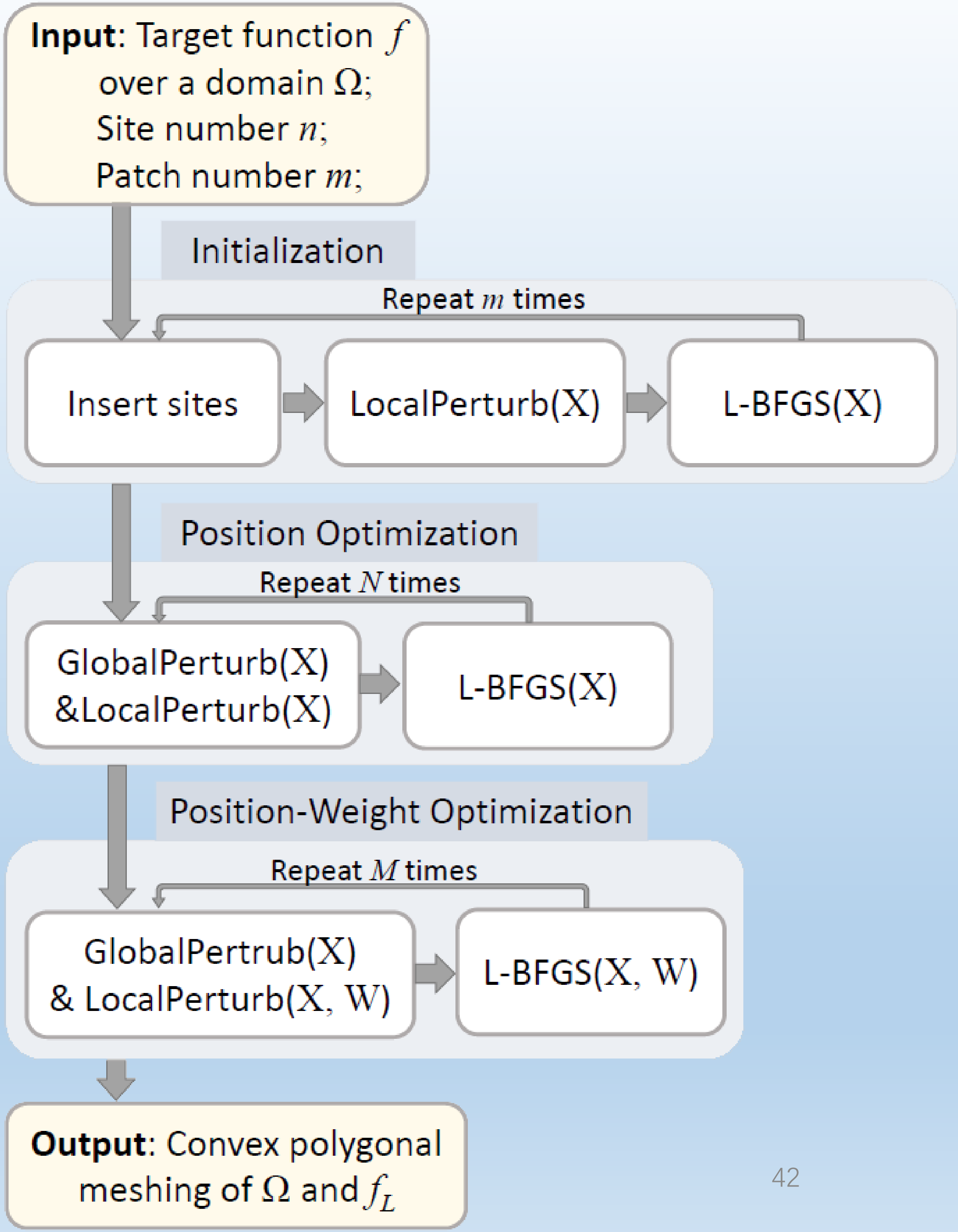
(1) Initialization:
inserting-optimizing site positions

(2) Position optimization:
main stage of the framework

(3) Position-weight optimization:
generate better result







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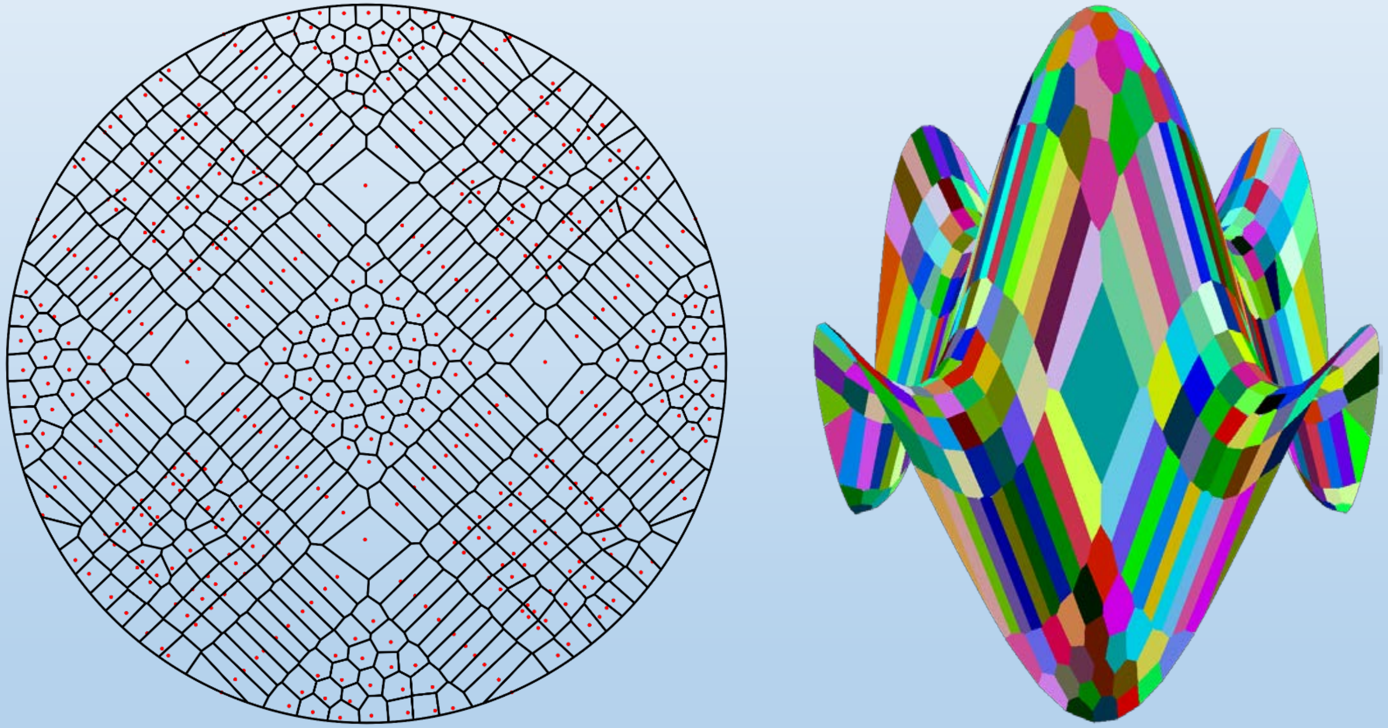
Results

- 1. Non-convex function approximation



Resulting tessellation (left) and piecewise linear fit (right) of a non-convex target function $f\left(x,y\right)=\sin\left(\pi\left(x+0.5\right)\right)\cos\left(\pi y\right), x^2+y^2\leq 1$ with 500 sites

Results

- 2. Density control

$$\mathcal{E}_{OPD}(\mathbf{X},W) = \| f - f_P \|_{L^2} = \sum_{i=1}^{n} \int_{V_i} \boxed{\rho(\mathbf{x})} \left(f(\mathbf{x}) - P_i^*(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x},$$

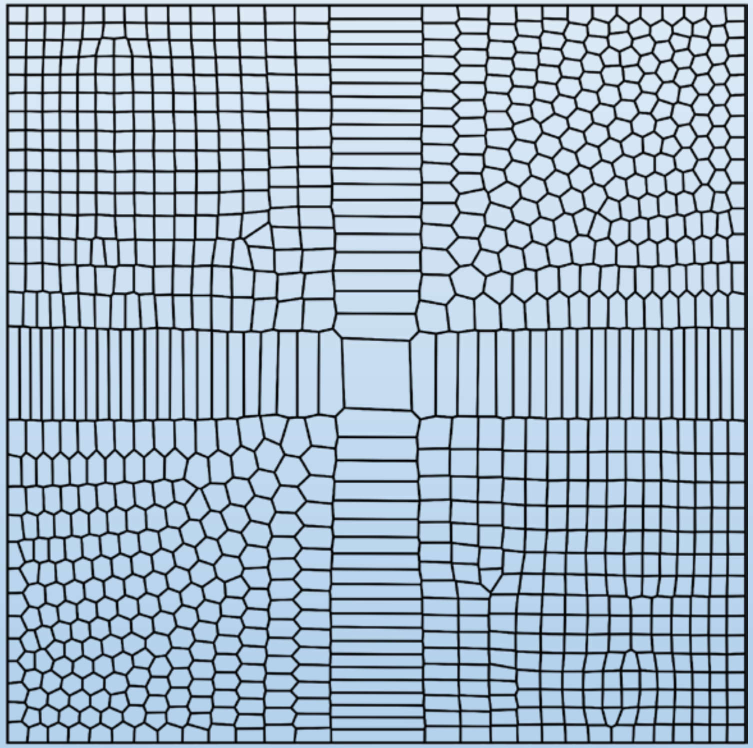
$$\frac{\partial \mathcal{E}_{OPD}(\mathbf{X},W)}{\partial \mathbf{x}_i} = \sum_{j \in J_i} \int_{V_{ij}} \rho(\mathbf{x}) \left(\left| f(\mathbf{x}) - P_i^*(\mathbf{x}) \right|^2 - \left| f(\mathbf{x}) - P_j^*(\mathbf{x}) \right|^2 \right) \frac{\mathbf{x} - \mathbf{x}_i}{\left| \mathbf{x}_j - \mathbf{x}_i \right|} \mathrm{d}s$$

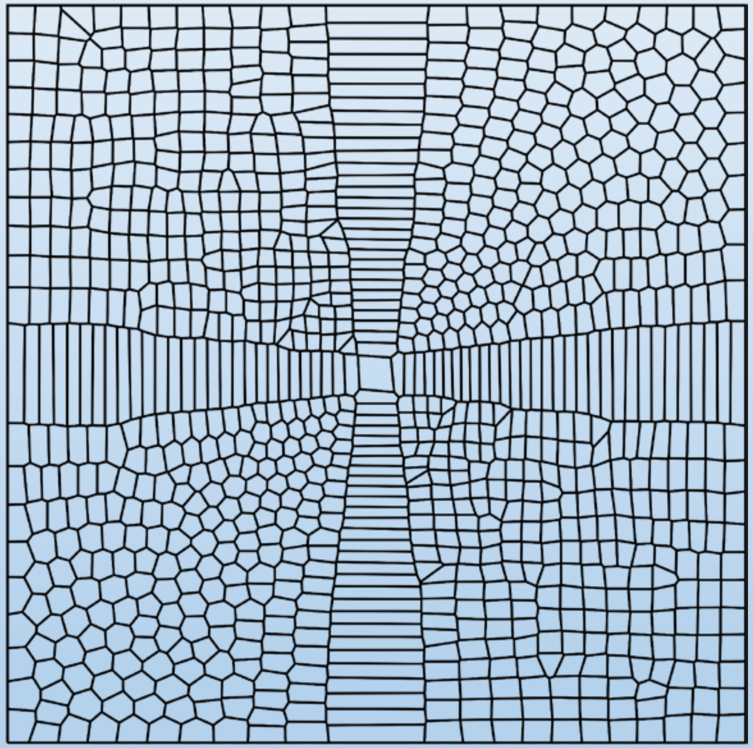
$$\frac{\partial \mathcal{E}_{OPD}(\mathbf{X},W)}{\partial w_i} = \sum_{j \in J_i} \int_{V_{ij}} \rho(\mathbf{x}) \left(\left| f(\mathbf{x}) - P_i^*(\mathbf{x}) \right|^2 - \left| f(\mathbf{x}) - P_j^*(\mathbf{x}) \right|^2 \right) \frac{1}{2\left| \mathbf{x}_j - \mathbf{x}_i \right|} \mathrm{d}s$$

Results

- 2. Density control

$$\rho(x, y) = 1.0 \Big/ \left(\left(x^2 + y^2 \right)^2 + 0.001 \right)$$

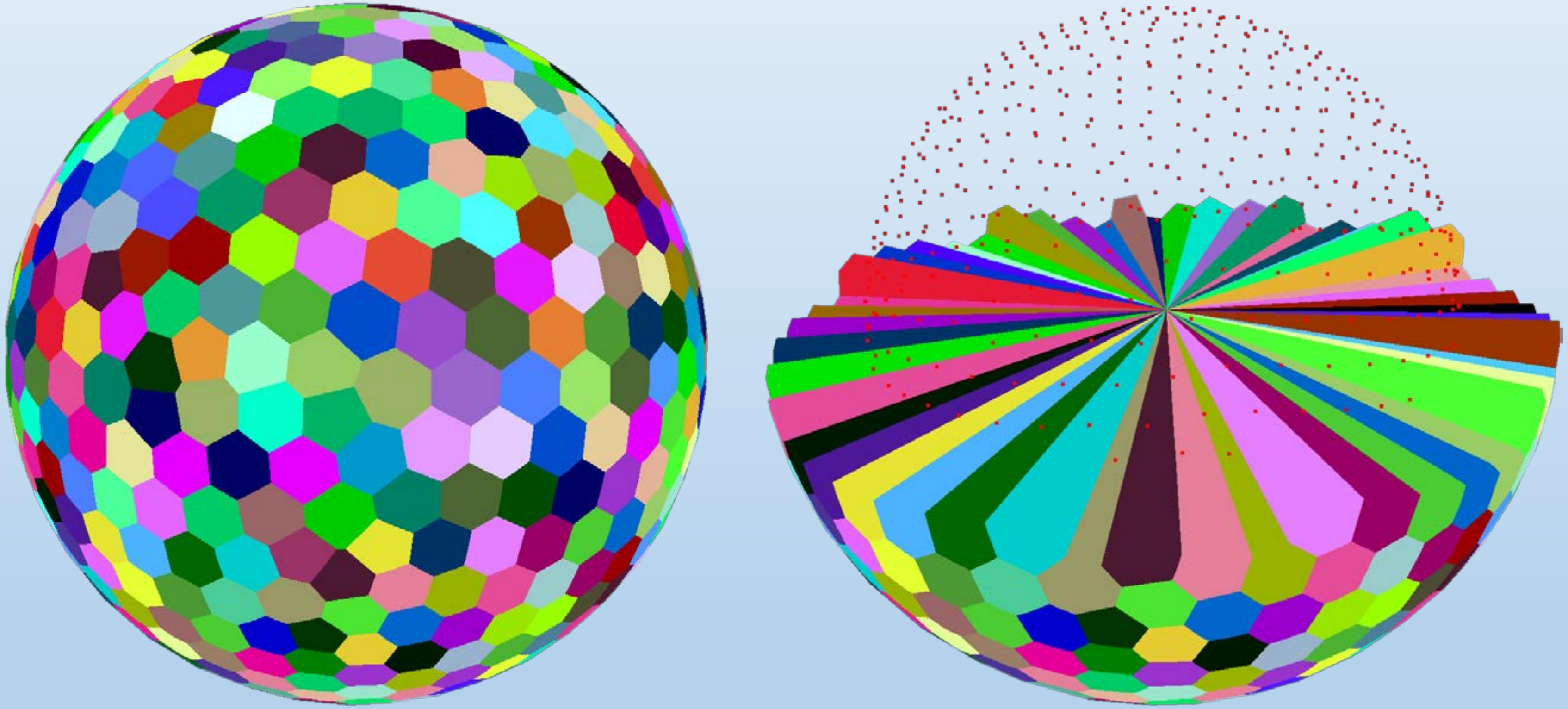




Resulting tessellations for a non-convex target function $f(x, y) = x^3 + y^3, -1 \le x, y \le 1$ with a constant density (left) and a non-uniform density function (right)

Results

- 3. 3D results



Tessellation of sphere for a non-smooth target function $f\left(x,y,z\right)=\sqrt{x^2+y^2+z^2}$ with 800 sites

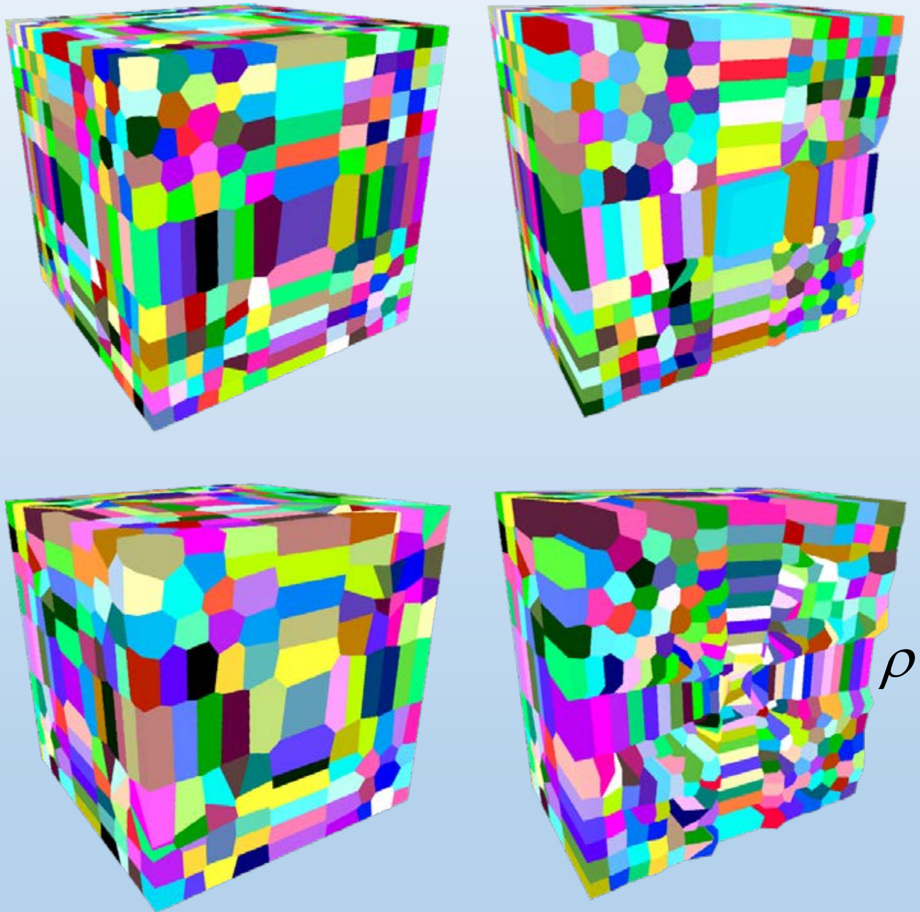
Results

- 3. 3D results

3D optimal power diagrams with increasing anisotropy
Left: isotropic, middle: 2:1:1, right: 8:1:1

Results

- 3. 3D results



$$\rho(x,y,z) = 1.0 \Big/ \left(\left(x^2 + y^2 + z^2 \right)^2 + 0.001 \right)$$

Exterior and cutaway views of tessellations for a non-convex target function with a constant density (top) and non-uniform density (bottom)

$$f(x,y,z) = x^3 + y^3 + z^3, -1 \le x, y, z \le 1$$

Results

- 4. Comparisons with OVT

Three measures of cell V_i: [Max Budninsky et.al, ACM TOG 2016]

(1) Hessian variation: $\max_{\mathbf{x},\mathbf{y}\in V_i} \| Hess[f](\mathbf{x}) - Hess[f](\mathbf{y}) \|_F$

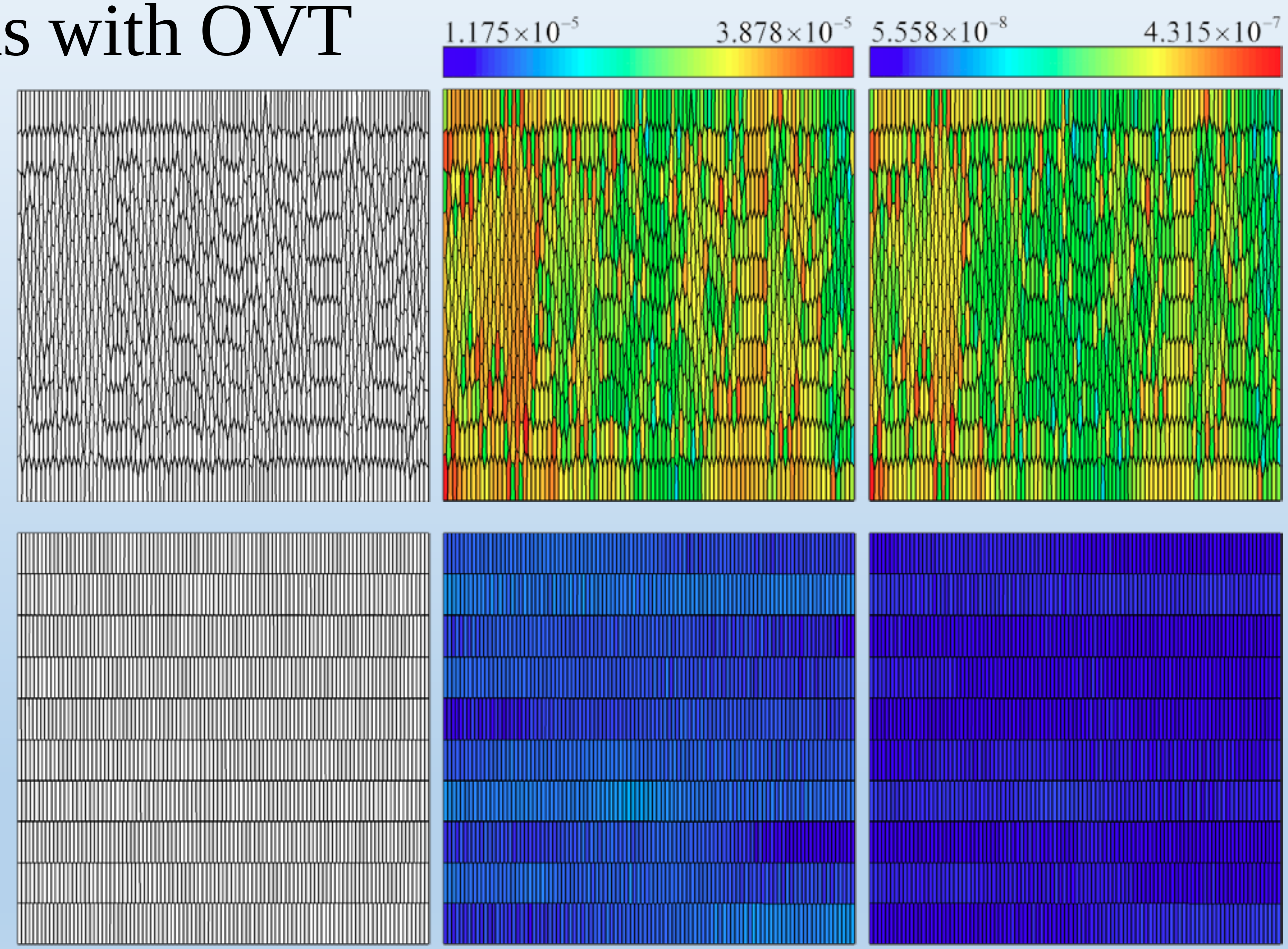
(2) Shape ratio: $\max_{\mathbf{x},\mathbf{y}\in V_i} \left[\sqrt{(\mathbf{x}-\mathbf{y})^t \,\overline{H}_{V_i}\, (\mathbf{x}-\mathbf{y})} \right] \left[|V_i| \sqrt{\det \overline{H}_{V_i}} \right]^{-\frac{1}{d}}$

(3) Modified area: $\left(\overline{\rho}_{V_i}^{d} \det \overline{H}_{V_i} \right)^{\frac{1}{d+2}} |V_i|$

Results

$$f(x,y) = 100x^2 + y^2, -1 \le x, y \le 1$$

- 4. Comparisons with OVT



OVT

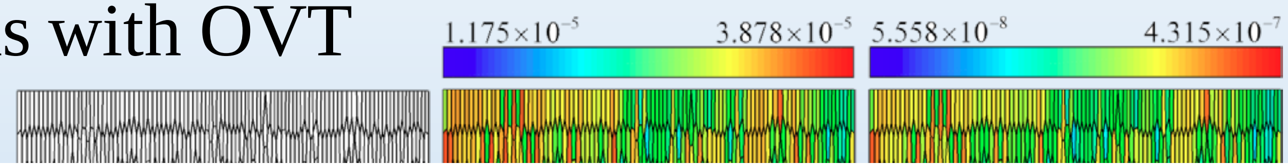
OPD

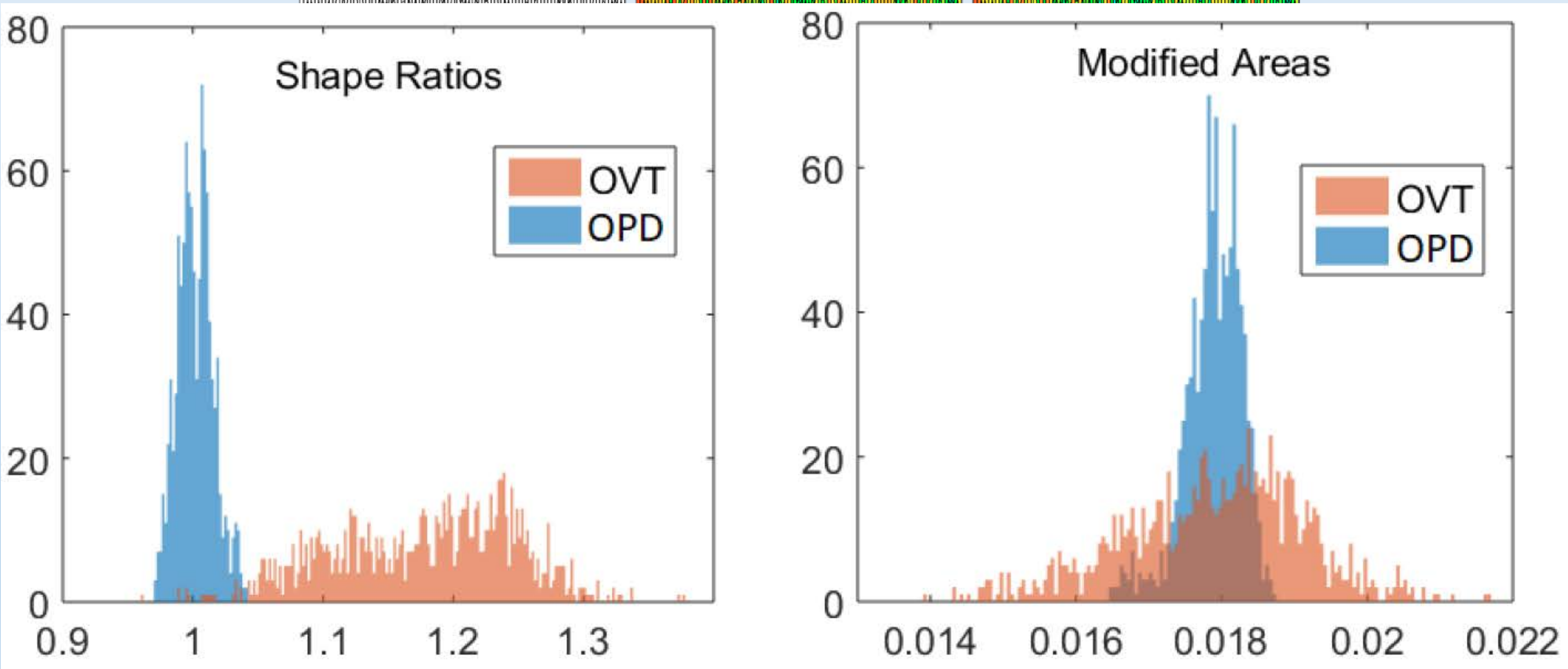
tessellation L^1 error L^2 error

Results

$$f(x,y) = 100x^2 + y^2, -1 \le x, y \le 1$$

- 4. Comparisons with OVT



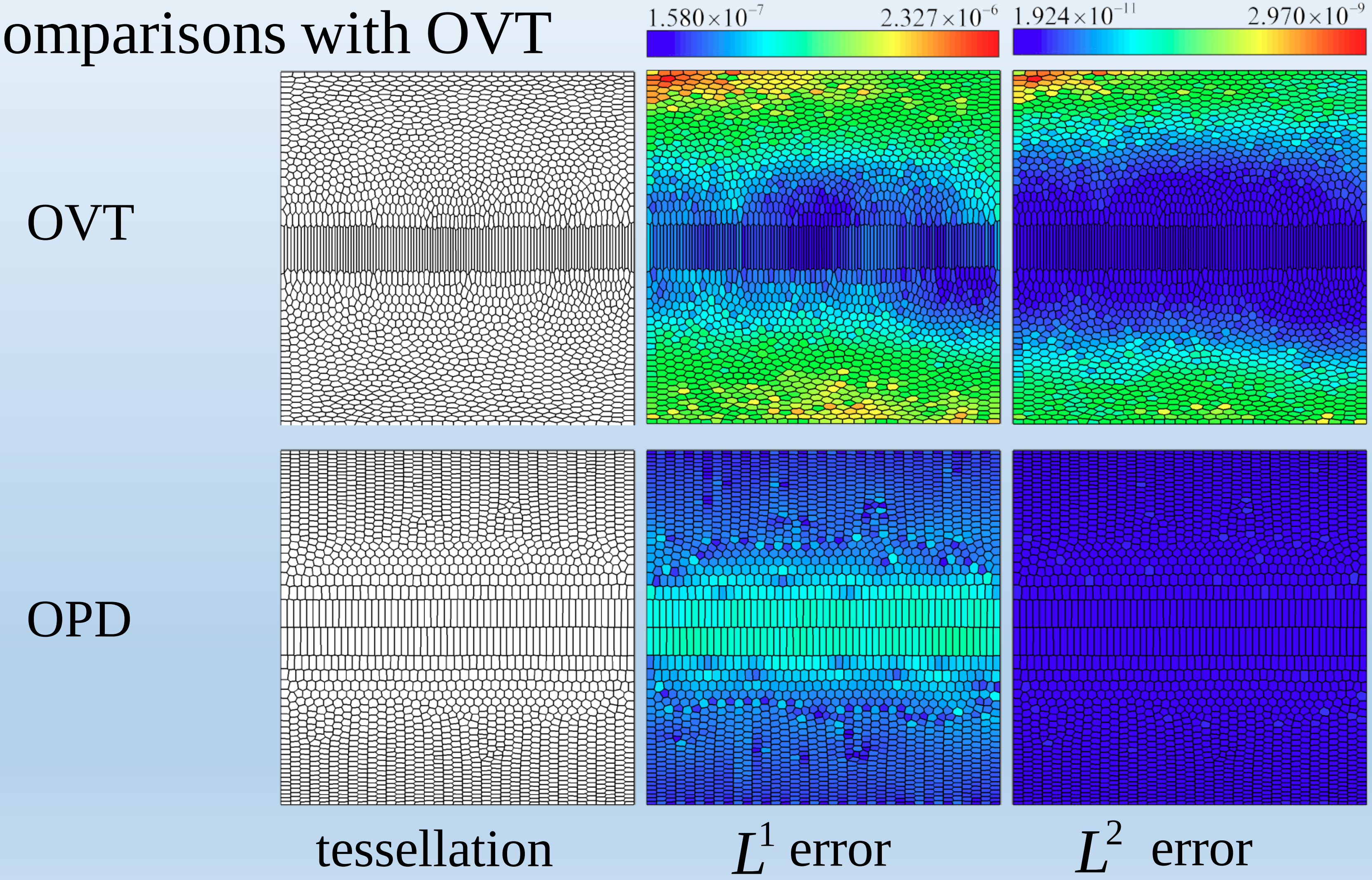


tessellation L^1 error L^2 error

Results

$$f\left(x,y\right)=x^2+10^{-5}y^2+y^4,-1\le x,y\le 1$$

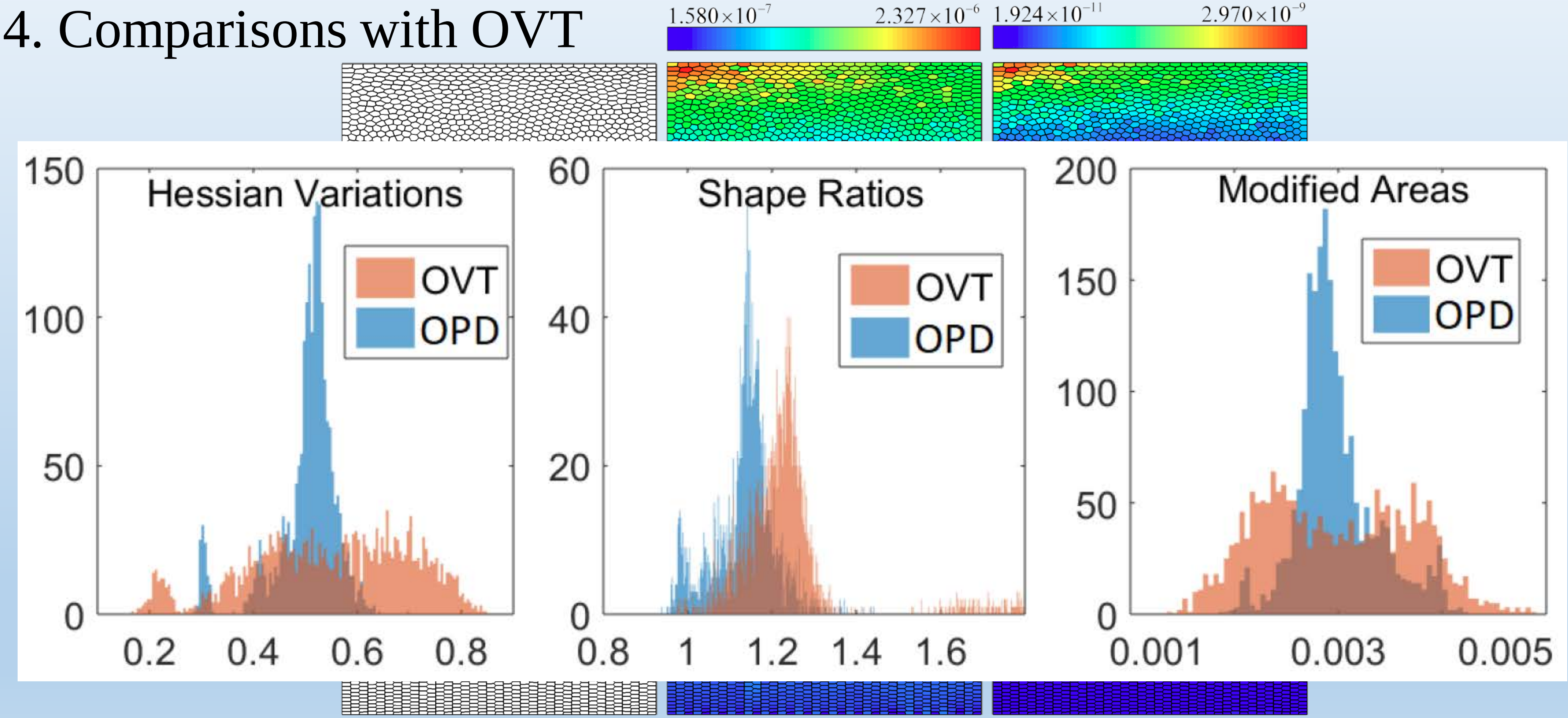
- 4. Comparisons with OVT



Results

$$f\left(x,y\right)=x^2+10^{-5}y^2+y^4,-1\le x,y\le 1$$

- 4. Comparisons with OVT



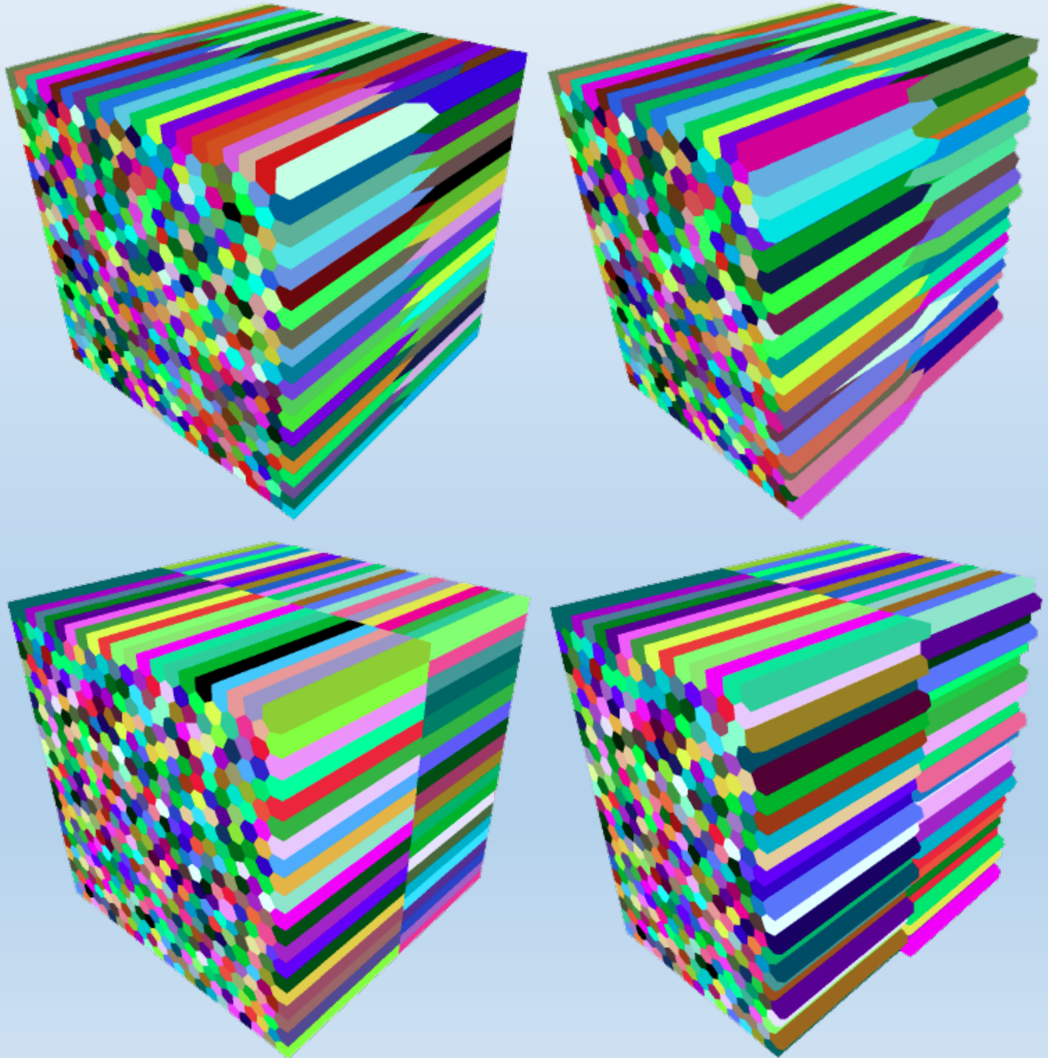
tessellation L^1 error L^2 error

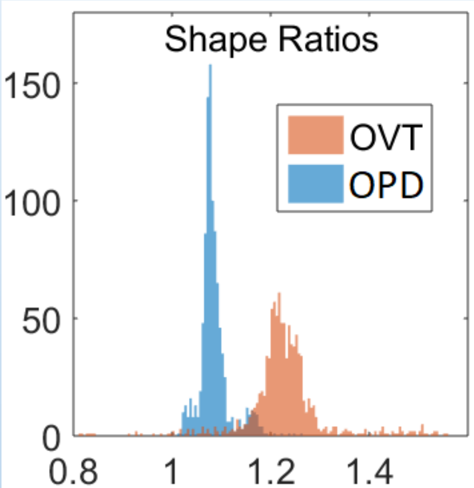
Results

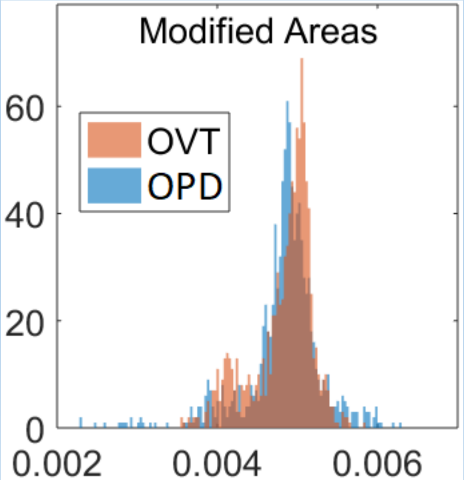
- 4. Comparisons with OVT

OVT

OPD







Outline

- Background
- Optimal Power Diagrams
- Optimization Framework
- Results
- Conclusion

Conclusion

- 1. Contributions

(1) We extend the OVT method for generating anisotropic meshes that the target function is not necessarily convex;

(2) The anisotropy of the resulting power cells conforms to the Hessian of an arbitrarily given function;

(3) A modified Monte Carlo minimization method with a local search strategy is tailored for effective optimization.

Conclusion

- 2. Limitations and feature work

(1) High computational cost;

Figure	Site Number	Initialization (second)	Position Optimization (second)	Position-Weight Optimization (second)	Total Time (second)
1(a)	50	0.1	1.6	2.4	4.1
1(b)	200	49	65	72	186
1(c)	800	32	18.8	17.6	68.4
7	2,000	270	213	109	592
8	500	56	27.8	44.8	128.6
9	1,000	57	59.7	43.3	160
10	800	780	246	1,245	2,271
11	500	860	414	283	1,557
12(a)	2,000	1,731	3,615	615	5,961
12(b)	2,000	1,694	3,521	354	5,579
13	1,000	63	67	23	153
14	2,000	98	200	36	334
15	1,000	598	1,028	286	1,912

Conclusion

- 2. Limitations and feature work

(2) Cannot directly used for the situation that generating meshes adapted to a given tensor field instead of Hessian matrix;

(3) Integrating more constraints on the geometry shape of cells and apply it to analysis and simulation tasks.

Acknowledgements

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