exercise3 (Score: 12.0 / 12.0)

1. Task (Score: 12.0 / 12.0)

Lab 5

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-5 (https://yuanyuyuan.github.io/itcm/lab-5.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 12/11(Wed.)**

In [1]:

```
name = "陳彥宇"
student_id = "B05303134"
```

(Top)

Exercise 3

Analyse the convergence properties of the Jacobi and Gauss-Seidel methods for the solution of a linear system whose matrix is

$$\left[\begin{matrix}

```
\alpha &&0 &&1\\
0 &&\alpha &&0\\
1 &&0 &&\alpha
\end{matrix}\right],
\quad \quad
\alpha \in \mathbb{R}.$$
```

Please write down your analysis in detail with LaTeX/Markdown at here. And if you need to do some numerical experiments, you can add more blocks to test your codes at below.

Jacobi Method

Let A be the matrix $\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & \alpha \end{bmatrix}$ for $\alpha \in \mathbb{R}$. The matrix A can be decomposed into the three matrix, L, D, and U, such that

$A = L + D + U$, where $L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, and $U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The Jacobi iteration is thus followed by

$$x^{(n)} = D^{-1}(b - (L+U)x_{n-1}) = \begin{bmatrix} b_1/\alpha \\ b_2/\alpha \\ b_3/\alpha \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1/\alpha \\ 0 & 0 & 0 \\ 1/\alpha & 0 & 0 \end{bmatrix} x^{(n-1)} = \begin{bmatrix} (b_1 - x_3^{(n-1)})/\alpha \\ b_2/\alpha \\ (b_3 - x_1^{(n-1)})/\alpha \end{bmatrix},$$

provided by $\alpha \neq 0$ for $n \in \mathbb{N}$. Furthermore, if we defined the iteration matrix $G = -D^{-1}(L+U) = I - D^{-1}A$, then the error $e^{(n)}$ of the nth iteration can be computed by

$$e^{(n)} = Ge^{(n-1)} = \begin{bmatrix} 0 & 0 & -1/\alpha \\ 0 & 0 & 0 \\ -1/\alpha & 0 & 0 \end{bmatrix} e^{(n-1)} = G^n e^{(0)} = \begin{cases} \begin{bmatrix} 0 & 0 & -1/\alpha^n \\ 0 & 0 & 0 \\ -1/\alpha^n & 0 & 0 \end{bmatrix} e^{(0)} & \text{if } n \text{ is odd.} \\ \begin{bmatrix} 1/\alpha^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\alpha^n \end{bmatrix} e^{(0)} & \text{if } n \text{ is even.} \end{cases}$$

Or simply $e^{(n)} = \begin{cases} \begin{bmatrix} -e_3^{(0)}/\alpha^n \\ 0 \\ -e_1^{(0)}/\alpha^n \end{bmatrix} & \text{if } n \text{ is odd.} \\ \begin{bmatrix} e_1^{(0)}/\alpha^n \\ 0 \\ e_3^{(0)}/\alpha^n \end{bmatrix} & \text{if } n \text{ is even.} \end{cases}$

From the explicit form of $e^{(n)}$, it is clear that $e^{(n)} \to 0$ as $n \to \infty$ if $|\alpha| > 1$. Alternatively, since the eignvales of G are $-1/\alpha$, 0, and $1/\alpha$, $e^{(n)}$ converges to zero if and only if $\rho(G) = \max\{\lambda : \lambda \text{ is an eigenvalue of } G\} = 1/|\alpha| < 1$, namely $|\alpha| > 1$.

The rate of convergence can also be derived by $\|e^{(n)}\| \leq \|G\|\|e^{(n-1)}\| = \alpha^{-1}\|e^{(n-1)}\|$. That is, in this case, the Jacobi method is linear convergence.

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt
from numpy import linalg as LA
```

In [3]:

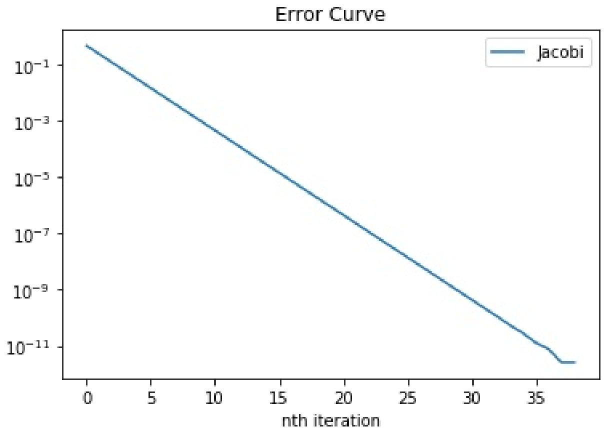
```
# For alpha = 2
A = np.array([[2, 0 , 1], [0, 2, 0], [1, 0, 2]])
b = np.array([1,1,1])

L = np.array([[0, 0 , 0], [0, 0, 0], [1, 0, 0]])
D = np.array([[2, 0 , 0], [0, 2, 0], [0, 0, 2]])
U = np.array([[0, 0 , 1], [0, 0, 0], [0, 0, 0]])
x_0 = np.array([1,1,1])

D_inv = LA.inv(D)
tmp = L+U
x = [D_inv @ (b-tmp @ x_0)]
er = []
for i in range(39):
    x.append(D_inv @ (b-tmp @ x[i]))
for i in range(39):
    er.append(LA.norm(x[-1]-x[i]))
```

In [4]:

```
# plot of linear convergence
plt.plot(er, label = 'Jacobi')
plt.xlabel('nth iteration')
plt.title(r'Error Curve')
plt.yscale('log')
plt.legend()
plt.show()
```



Gauss-Seidel Method

Using the previous notations, the Gauss-seidel method is followed by

$$x^{(n)} = (L+D)^{-1}(b - Ux_{n-1}) = \begin{bmatrix} b_1/\alpha \\ b_2/\alpha \\ b_3/\alpha - b_1/\alpha^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1/\alpha \\ 0 & 0 & 0 \\ 0 & 0 & -1/\alpha^2 \end{bmatrix} x^{(n-1)} = \begin{bmatrix} (b_1 - x_3^{(n-1)})/\alpha \\ b_2/\alpha \\ b_3/\alpha - (b_1 - x_3^{(n-1)})/\alpha^2 \end{bmatrix},$$

provided by $\alpha \neq 0$ for $n \in \mathbb{N}$. Also, if we defined the iteration matrix $G = -(L+D)^{-1}U = I - (L+U)^{-1}A$, then the error $e^{(n)}$ of the n th iteration can be computed by

$$e^{(n)} = Ge^{(n-1)} = \begin{bmatrix} 0 & 0 & -1/\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 1/\alpha^2 \end{bmatrix} e^{(n-1)} = G^n e^{(0)} = \begin{bmatrix} 0 & 0 & -1/\alpha^{2n-1} \\ 0 & 0 & 0 \\ 0 & 0 & 1/\alpha^{2n} \end{bmatrix} e^{(0)} = \begin{bmatrix} -e_3^{(0)}/\alpha^{2n-1} \\ 0 \\ e_3^{(0)}/\alpha^{2n} \end{bmatrix}.$$

Similarly, from the explicit form of $e^{(n)}$, it is clear that $e^{(n)} \to 0$ as $n \to \infty$ if $|\alpha| > 1$. Alternatively, since the eignvales of G are $0, 0$, and $1/\alpha^2$, $e^{(n)}$ converges to zero if and only if $\rho(G) = 1/\alpha^2 < 1$, namely $|\alpha| > 1$. Notice that for fixed $|\alpha| > 1$, $\rho(G_{\mathrm{GS}}) < \rho(G_{\mathrm{J}})$, which implies that $\|G_{\mathrm{GS}}\|_\infty < \|G_{\mathrm{J}}\|_\infty$ in this case.

The rate of convergence can also be derived by $\|e^{(n)}\| \leq \|G\|\|e^{(n-1)}\| = \alpha^{-2}\|e^{(n-1)}\|$. That is, the linear convergence. Moreover, since $\|G_{\mathrm{GS}}\|_\infty < \|G_{\mathrm{J}}\|_\infty$ the slope of Gauss-Seidel method should be less than that of Jacobi method.

In [5]:

```
LD_inv = LA.inv(L+D)
y = [LD_inv @ (b-U @ x_0)]
err = []
for i in range(39):
    y.append(LD_inv @ (b-U @ y[i]))
for i in range(39):
    err.append(LA.norm(y[-1]-y[i]))
```

In [6]:

```
# Slope of Gauss-Seidel method is less than that of Jacobi method.
plt.plot(er,label = 'Jacobi')
plt.plot(err,label = 'Gauss-Seidel')
plt.xlabel('nth iteration')
plt.title(r'Error Curve')
plt.yscale('log')
plt.legend()
plt.show()
```