exercise2 (Score: 13.0 / 13.0)

1. Test cell (Score: 2.0 / 2.0)
2. Test cell (Score: 2.0 / 2.0)
3. Coding free-response (Score: 2.0 / 2.0)
4. Written response (Score: 2.0 / 2.0)
5. Test cell (Score: 1.0 / 1.0)
6. Coding free-response (Score: 2.0 / 2.0)
7. Written response (Score: 2.0 / 2.0)

Lab 3

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-3 (https://yuanyuyuan.github.io/itcm/lab-3.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 10/30(Wed.)**

In [1]:

```
name = "陳彥宇"
student_id = "B05303134"
```

Exercise 2

It is known that when interpolating a function $f(x)$ with a polynomial p_{m+1} of degree m that using x_j for $j = 0, 1, \ldots, m$ as interpolation points the error has the form

$$|f(x) - p_{m+1}(x)| = \frac{\left|f^{(m+1)}(\xi_x)\right|}{(m+1)!} \left|\prod_{k=0}^{m} (x - x_k)\right|,$$

where $\xi_x \in [x_0, x_m]$.

Therefore, the polynomial $\omega_m(t) := \prod_{k=0}^{m} (t - x_k)$ influences the size of the interpolation error.

---

1. Put $m + 1$ *distinct equidistant points* in the interval $[-1, 1]$, and plot $\omega_m(t)$ for $m = 5, 10, 15, 20$.

Part 0. Import libraries.

In [2]:

```
import matplotlib.pyplot as plt
import numpy as np
```

Part 1. Define $\omega_m(t)$ function.

In [3]:

(Top)

```
def omega_m(t, x):

    # ===== 請實做程式 =====
    n = len(x)
    product = 1
    for i in range(n):
        product *= (t-x[i])
    return product
    # ====================
```

In [4]:

omega

(Top)

```
# Test
print('w_5(0.5) =', omega_m(0.5, np.linspace(-1, 1, 6)))

### BEGIN HIDDEN TESTS
from random import random

rd_number = random()
x = np.linspace(-1, 1, 11)

m = len(x)
product = 1

for i in range(m):
    product *= (rd_number - x[i])

assert omega_m(rd_number, np.linspace(-1, 1, 11)) == product, 'omega_m is wrong!'
### END HIDDEN TESTS
```

```
w_5(0.5) = 0.017325000000000007
```

Part 2. Define the equidistant points function.

For example, if $m = 4$, then $m + 1$ distinct equidistant points in the interval $[-1, 1]$ should be $[-1, -0.5, 0, 0.5, 1]$.

So the results of `equidistant_points(4)` will be `[-1.  -0.5  0.   0.5  1. ]`.

In [5]:

(Top)

```
def equidistant_points(m):
    # ===== 請實做程式 =====
    t = 2/m
    return np.arange(-1,1+t,t)
    # ====================
```

In [6]:

```
        points                                                                                    (Top)

# Test
m = 4
print("Equidistant points:", equidistant_points(m))

### BEGIN HIDDEN TESTS
m = 10
assert np.mean(np.array(equidistant_points(m)) - np.linspace(-1, 1, m+1)) < 1e-7, 'equidistant_points is
wrong!'
### END HIDDEN TESTS
```

```
Equidistant points: [-1.  -0.5  0.   0.5  1. ]
```

Part 3. plot $\omega_m(t)$ for $m = 5, 10, 15, 20$.

Please refer parts of plotting in " *lagrange.ipynb* ".

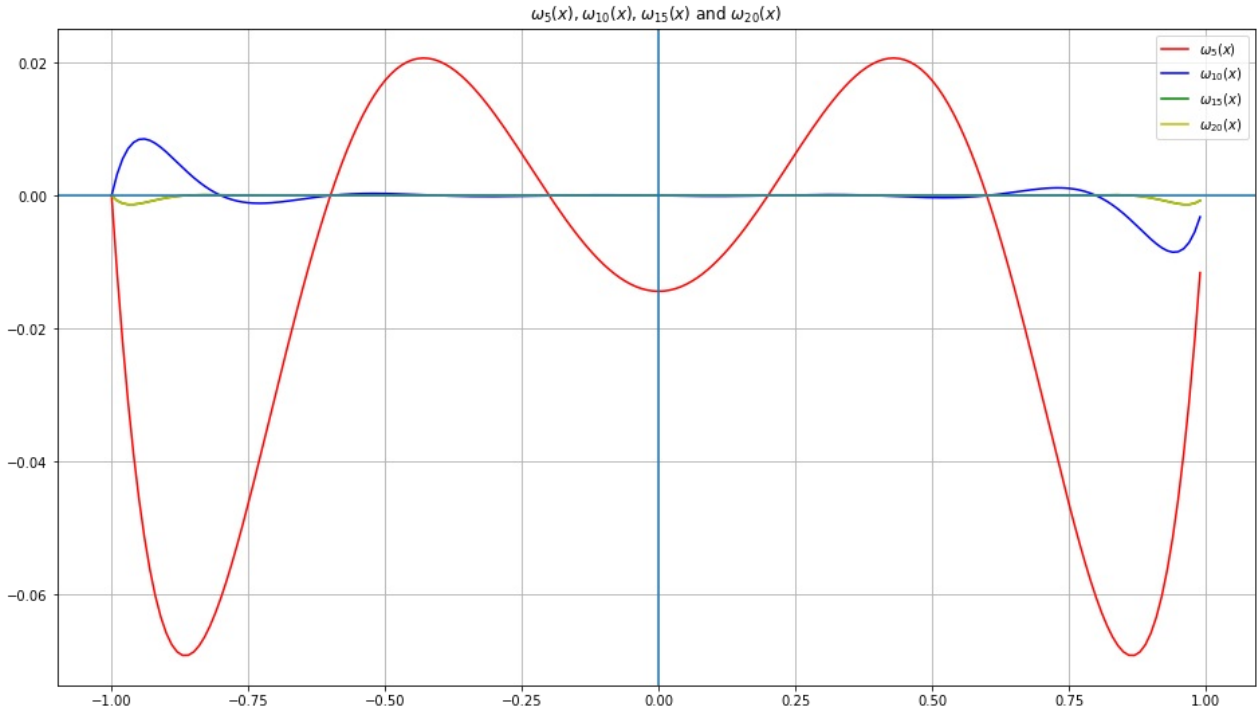
In [7]:

```
                                                                                                  (Top)

x_range = np.arange(-1, 1, 0.01)
fig, ax = plt.subplots(figsize=(16, 9))

# Plot the function w_5(x), w_10(x), w_15(x) and w_20(x)
#
# Hint: ax.plot( x_points, y_points, color='?', label='?')
# ===== 請實做程式 =====
ax.plot( x_range, omega_m(x_range,equidistant_points(5)), color='r', label='$\omega_{5}(x)$')
ax.plot( x_range, omega_m(x_range,equidistant_points(10)), color='b', label='$\omega_{10}(x)$')
ax.plot( x_range, omega_m(x_range,equidistant_points(15)), color='g', label='$\omega_{15}(x)$')
ax.plot( x_range, omega_m(x_range,equidistant_points(15)), color='y', label='$\omega_{20}(x)$')
# ====================

# Add other text and items
ax.set_title(r'$\omega_{5}(x), \omega_{10}(x), \omega_{15}(x)$ and $\omega_{20}(x)$')
plt.legend(loc='upper right')
ax.grid(True)
ax.axhline(y=0)
ax.axvline(x=0)
plt.show()
```



Part 4. What's your observation of the above figure?

(Top)

Let $t = 1 - 2/m$ be the second right-hand side interpolation point at the interval $[-1, 1]$. Then

$$|\omega_{2m}(t)| = \left(\frac{1}{2m}\frac{1}{2m}\frac{2}{2m}\cdots\frac{2m-2}{2m}\frac{2m-1}{2m}\right).$$

Furthermore, if m increases to $2m$, then

$$\begin{aligned}|\omega_{4m}(t)| &= \left(\frac{1}{4m}\frac{2}{4m}\right)\left(\frac{1}{4m}\frac{2}{4m}\right)\cdots\left(\frac{4m-3}{4m}\frac{4m-2}{4m}\right)\\ &\leq \left(1\cdot\frac{2}{4m}\right)\left(1\cdot\frac{2}{4m}\right)\cdots\left(1\cdot\frac{4m-2}{4m}\right) = |\omega_{2m}(t)|.\end{aligned}$$

Hence, $|\omega_{4m}(t)|$ is bounded by $|\omega_{2m}(t)|$. The points near the boundary do not have the Runge phenomenon. That is, thay are well-controlled as m large.

---

2. Redo " *Problem 1.* " using *zeros of the Chebyshev polynomial (Chebyshev nodes)* as the interpolation points.

Part 1. Define Chebyshev nodes.

Please refer the part of Chebyshev nodes in " *lagrange.ipynb* ".

In [8]:

(Top)

```
def chebv_nodes(m):
    # ===== 請實做程式 =====
    return -np.cos(np.linspace(0, np.pi, m+1))
    # ====================
```

In [9]:

chebv_nodes (Top)

```
# Test
m = 5
print("Chebyshev nodes:", chebv_nodes(m))

### BEGIN HIDDEN TESTS
m = 10
assert np.mean(np.array(chebv_nodes(m)) - np.cos(np.linspace(0, np.pi, m+1))) < 1e-7, 'chebv_nodes is wro
ng!'
### END HIDDEN TESTS
```

```
Chebyshev nodes: [-1.         -0.80901699 -0.30901699  0.30901699  0.80901699  1.        ]
```

Part 2. plot $\omega(t)$ for $m = 5, 10, 15, 20$.

Please refer parts of plotting in " *lagrange.ipynb* ".

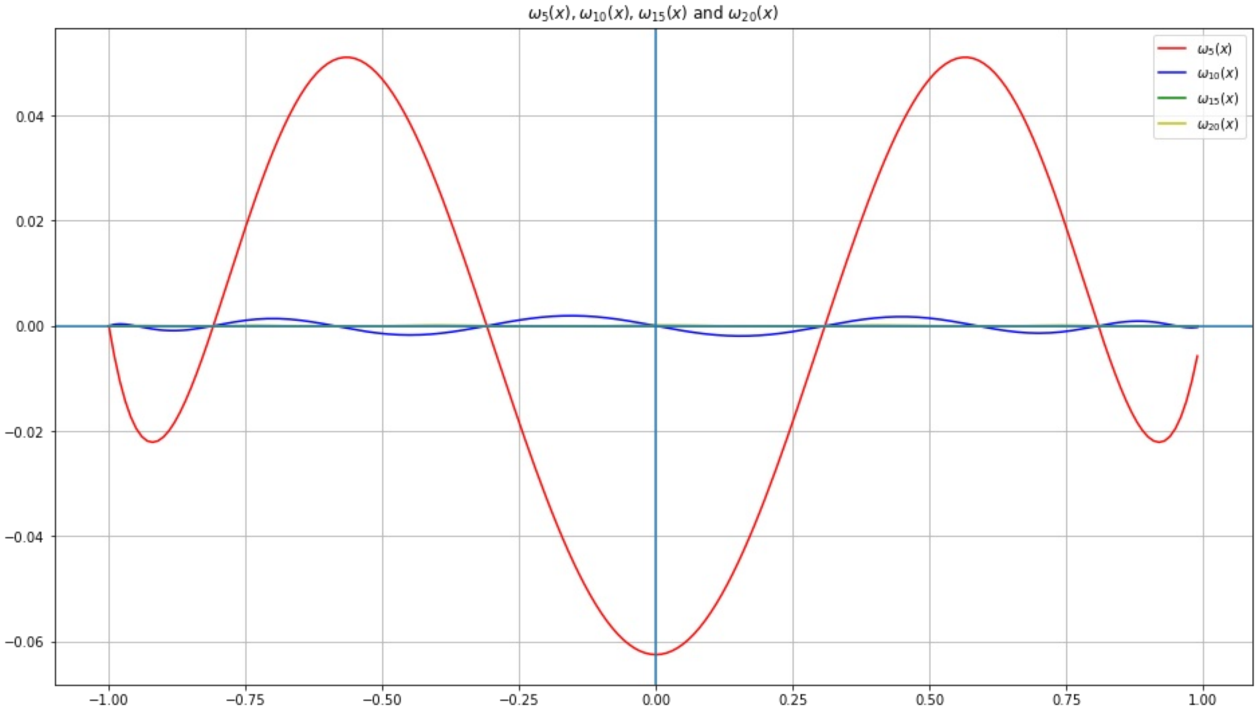
In [10]:

(Top)

```
x_range = np.arange(-1, 1, 0.01)
fig, ax = plt.subplots(figsize=(16, 9))

# Plot the function w_5(x), w_10(x), w_15(x) and w_20(x)
#
# Hint: ax.plot( x_points, y_points, color='?', label='?')
# ===== 請實做程式 =====
ax.plot( x_range, omega_m(x_range,chebv_nodes(5)), color='r', label='$\omega_{5}(x)$')
ax.plot( x_range, omega_m(x_range,chebv_nodes(10)), color='b', label='$\omega_{10}(x)$')
ax.plot( x_range, omega_m(x_range,chebv_nodes(15)), color='g', label='$\omega_{15}(x)$')
ax.plot( x_range, omega_m(x_range,chebv_nodes(15)), color='y', label='$\omega_{20}(x)$')
# ====================

# Add other text and items
ax.set_title(r'$\omega_{5}(x), \omega_{10}(x), \omega_{15}(x)$ and $\omega_{20}(x)$')
plt.legend(loc='upper right')
ax.grid(True)
ax.axhline(y=0)
ax.axvline(x=0)
plt.show()
```



Part 3. What's your observation of the above figure?

(Top)

Since we let our interpolation points become denser near the boundary to avoid the Runge phenomenon, it is expected that we will received a worse behavior at the middle when m is fixed. However, in this case, it has no Runge phenomenon, and, thus, the overall behavior of $\omega_m(t)$ under Chebyshev nodes seems worse than that under equidistant points. But, as m large, both cases have similar behavior, converging to zero.