$\boxed{\text{基本公式}}$

① $b = \dfrac{1}{\int_0^{\omega} e^{-ra} l(a)\,da}$

② $c(a) = b e^{-ra} l(a)$

③ $\int_\alpha^\beta e^{-ra} l(a) \phi(a)\, da = 1 = \psi(a)$

④ $l(a) = e^{-\int_0^a \mu(a)\, da}$

⑤ $e^x = \sum_0^\infty \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n$

⑥ $d\left(\frac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

$\boxed{\text{第1节} \quad \phi(a)}$

一、$\phi(a)$ ~~2节~~ r

I. $\phi'(a) = e^{ka} \phi(a)$

3式 $\int_\alpha^\beta e^{-ra} l(a) \phi'(a) = 1$

$\Rightarrow \int_\alpha^\beta e^{-r'a} l(a) \phi(a) e^{ka} = 1$

$\Rightarrow \int_\alpha^\beta e^{-(r'-k)a} l(a) \phi(a) = 1$

若 $\phi(a)$ 后稳定人口 $\quad -(r'-k) = r$

$r' = r + k$

$\phi'(a) \uparrow \quad k > 0 \quad r' \uparrow$

$\phi(a) \downarrow \quad k < 0 \quad r' \downarrow$

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2. 某一岁 $\Delta m(x)$ 对 r

③式 $$\int_{\alpha}^{\beta} e^{-(r+\Delta r)a} l(a)[m(a)+\Delta m(x)]da = 1$$

$\Delta m(x)$ 有值 仅在 $[x-\frac{1}{2}x, x+\frac{1}{2}x]$，余为0

$$\int_{\alpha}^{\beta} e^{-(r+\Delta r)a} l(a)m(a)da + e^{-(r+\Delta r)x} l(x)\Delta m(x) = 1$$

$$e^{-(r+\Delta r)a} = e^{-ra} \cdot e^{-\Delta ra}$$

泰勒 $\begin{cases} e^{x} \doteq 1+x \\ e^{-\Delta r \cdot a} \doteq (1-\Delta r \cdot a) \end{cases}$

$$= e^{-ra}(1-\Delta ra)$$

$$\int_{\alpha}^{\beta} e^{-ra}(1-\Delta ra)l(a)m(a)da + C = 1$$

μ 平均生育年龄

$$\int_{\alpha}^{\beta} e^{-ra} l(a)m(a)da - \Delta r\int_{\alpha}^{\beta} e^{-ra} l(a)m(a)a + C = 1$$

$$\cancel{\psi(a)}\ 1 - \Delta r \cdot \mu + C = 1$$

$$C = \Delta r \cdot \mu$$

$$e^{-(r+\Delta r)x} l(x)\Delta m(x) = \Delta r \cdot \mu$$

$$e^{-rx}(1-\Delta rx)l(x)\Delta m(x) = \Delta r \cdot \mu$$

$\because$ $\Delta m(x)$ 无穷小

$\therefore$ $$\Delta r \cdot \mu = e^{-rx} l(x)\Delta m(x)$$

$$\Delta r = \underbrace{\frac{e^{-rx} l(x)}{\mu}}_{C} \Delta m(x)$$

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二、中 $l(a)$ 对 $C(a)$

②式 $c'(a) = b' e^{-r'a} l(a)$

同理 $r' = r + k$

$$c'(a) = b' e^{-(r+k)a} l(a)$$

$$= b' e^{-ka} e^{-ra} l(a)$$

$$= b' e^{-ka} \cdot \frac{c(a)}{b}$$

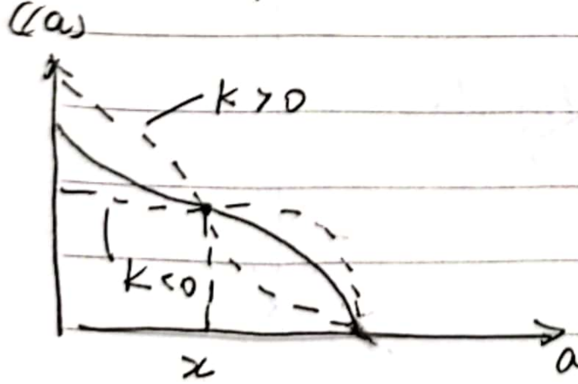
C_1 C_2

$$= \frac{b'}{b} e^{-ka} c(a)$$

$$\frac{c'(a)}{c(a)} = \frac{b'}{b} e^{-ka}$$

讨论
- $k = 0$　$e^{-\mu} = 1$　$(e^{-k})^a = 1$
- $k > 0$　$e^{-\mu} < 1$　$(e^{-k})^{x+1} < (e^{-k})^x$
- $k < 0$　$e^{-\mu} > 1$　$(e^{-k})^{x+1} > (e^{-k})^x$

- $k > 0$　$a \uparrow$　$e^{-ka} \downarrow$　$\frac{b'}{b} c(a) \cdot e^{-ka} \downarrow$
  $a \uparrow$ 比重变小 年轻化
- $k < 0$　$a \uparrow$　$e^{-ka} \uparrow$　$\frac{b'}{b} c(a) e^{-ka} \uparrow$
  $a \uparrow$ 比重变大 年老化



$\because$ $\frac{c'(x)}{c(x)} = \frac{b'}{b} e^{-kx}$

令 $c'(x) = c(x)$

$$\frac{b'}{b} e^{-kx} = 1$$

$$\ln(e^{-kx}) = \ln\frac{b'}{b}$$

即 $x = \frac{1}{k} \ln\frac{b'}{b}$

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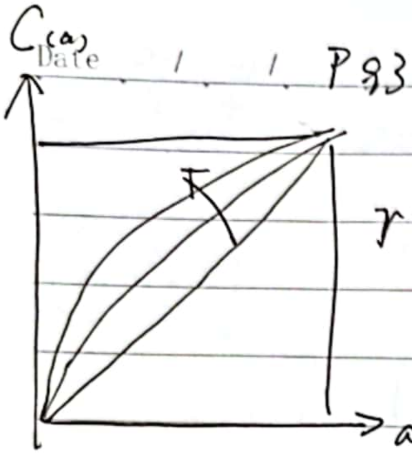

$r \uparrow$

试2

$$C(x) = \int_0^x c(a)\,da$$

$$c(a) = b e^{-ra} l(a)$$

$$C(x) = \int_0^x b e^{-ra} l(a)\,da$$

$$= b \int_0^x e^{-ra} l(a)\,da$$

$$= \frac{\int_0^x e^{-ra} l(a)\,da}{\int_0^\omega e^{-ra} l(a)\,da}$$

试6

$$\frac{dC(x)}{dr} = \left[\int_0^\omega e^{-ra} l(a)\,da \cdot \frac{d\int_0^x e^{-ra} l(a)\,da}{dr} - \int_0^x e^{-ra} l(a)\,da \cdot \frac{d\int_0^\omega e^{-ra} l(a)\,da}{dr}\right] \Big/ \left[\int_0^\omega e^{-ra} l(a)\,da\right]^2$$

$$= \frac{\int_0^x (-a) e^{-ra} l(a)\,da}{\int_0^\omega e^{-ra} l(a)\,da} - \int_0^x e^{-ra} l(a)\,da \cdot \frac{\int_0^\omega (-a) e^{-ra} l(a)\,da}{\left[\int_0^\omega e^{-ra} l(a)\,da\right]^2}$$

$$= -\frac{1}{\int_0^\omega e^{-ra} l(a)\,da} \cdot \frac{\int_0^x a e^{-ra} l(a)\,da}{\int_0^x e^{-ra} l(a)\,da} \cdot \cancel{\int_0^x e^{-ra} l(a)\,da}$$

$$= -C(x) \cdot \bar{a}_x$$

x岁下 $\bar{a}$

$$= -\frac{\int_0^x e^{-ra} l(a)\,da}{\int_0^\omega e^{-ra} l(a)\,da} \cdot \frac{\int_0^\omega (-a) e^{-ra} l(a)\,da}{\int_0^\omega e^{-ra} l(a)\,da}$$

$$= +C(x) \cdot \bar{a}_s$$ 全体 $\bar{a}$

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$$\frac{d\,\varepsilon(x)}{dr} = -\varepsilon(x)\,\bar{a}_x - \varepsilon(x)\,\bar{a}_s$$

$$= (\bar{a}_s - \bar{a}_x)\,\varepsilon(x)$$

$\because$ $\bar{a}_s - \bar{a}_x > 0$ $\varepsilon(x) > 0$

$$\frac{d\,\varepsilon(x)}{dr} > 0$$

仅当 $x=0$ 或 $x=\omega$ 时 为0

$r\uparrow$ $\varepsilon(x)\uparrow$ 同向变化 即累积比值$\uparrow$

第二章 l(x) 变动

一、$l(x)$ 对 r

(4式) $l(x) = e^{-\int_0^x \mu(a)\,da}$

若 $\mu'(a) = \mu(a) - k$

$$l'(x) = e^{-\int_0^x (\mu(a) - k)\,da}$$

$$= e^{-\int_0^x \mu(a)\,da} \cdot e^{kx}$$

$$= l(x) \cdot e^{kx}$$

(3式) $\int_\alpha^\beta e^{-ra}\, l(a)\, \phi(a)\, da = 1$

$$\int_\alpha^\beta e^{-r'a}\, l'(a)\, \phi(a)\, da = 1$$

$$\int_\alpha^\beta l(a) \cdot e^{ka} \cdot e^{-r'a}\, \phi(a)\, da = 1$$

$$-ra = (-r' + k)a$$

$$r' = r + k$$

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二. $l(x)$ 对 $C(a)$

式1式2 $$C'(a)=\frac{e^{-r'a}l'(a)}{\int_0^\omega e^{-r'a}l'(a)da}$$

$\because r'=r+k,\ l'(a)=l(a)e^{ka}$

$$c'(a)=\frac{e^{-(r+k)a}l(a)\cancel{e^{ka}}}{\int_0^\omega e^{-(r+k)a}l(a)\cancel{e^{ka}}da}$$

$$=C(a)$$

why? $\Longleftrightarrow \int_0^\omega$

$$\mu'(\omega)=\mu(\omega)-k=1-k<1$$

三. $l(x)=1$ 对 $C(a)$ 死亡消失

$\mu(a)=0\quad l(a)=1\quad b=r$

$NRR=R_0=\int_\alpha^\beta l(a)\phi(a)da$
净再生率

$=\int_\alpha^\beta \phi(a)da=GRR$
粗再生率

$$C(a)=\frac{e^{-ra}\cancel{l(a)}}{\int_0^\omega e^{-ra}\cancel{l(a)}da}=b\cdot e^{-ra}=r\cdot e^{-ra}$$

稳定人口中 $a\uparrow$ $e^{-ra}\downarrow$ $l(a)\downarrow$ $C(a)\_old\downarrow$

死亡消失中 仅 e^{-ra} 作用 $C(a)\_old\uparrow$ $\bar{a}_x\uparrow$

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第3节 NRR 变动

基础公式
$$R_0 = \int_\alpha^\beta l(a)\phi(a)da \quad \triangle 4$$
$$R_0 = e^{rT} \quad \ln R_0 = rT \quad \triangle 8$$

式3 $\int_\alpha^\beta e^{-ra} l(a)\phi(a)da = 1$

式7 解法

当满足 $\phi(a)=1$

定性认识

$R_0>1 \quad e^{-ra}\downarrow$ 趋于0 $\quad r>0$

$R_0=1 \quad e^{-ra}=1 \quad r=0$

$R_0<1 \quad e^{-ra}$ 趋于 $+\infty \quad r<0$

R_0 与 r 同向变化

为求 R_0 对 r 影响 引入 $\phi(r)=1$ 代入

参见旅游书 直书略

$$\frac{\phi(r)}{R_0} = \int_\alpha^\beta e^{-ra}\frac{l(a)\phi(a)}{R_0}da$$

式5 泰勒
$$= \int_\alpha^\beta \left(1-ra+\frac{r^2}{2!}a^2+\cdots\right)\frac{l(a)\phi(a)}{R_0}da$$
$$= \int_\alpha^\beta \frac{l(a)\phi(a)}{R_0}da + \int_\alpha^\beta (-ra)\frac{l(a)\phi(a)}{R_0}da + \cdots$$
$$= \frac{R_0}{R_0} + (-r)\frac{R_1}{R_0} + \frac{r^2}{2!}\left[\frac{R_2}{R_0}-\left(\frac{R_1}{R_0}\right)^2\right]+\cdots$$

$\because \phi(r)=1$

~~$\ln\left(\frac{1}{R_0}\right) =$~~

$$\therefore -\ln R_0 = -r\mu + \frac{r^2}{2!}\sigma^2+\cdots$$
$$\ln R_0 \doteq r\mu - \frac{\sigma^2}{2}r^2$$

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Date / $\ln R_0 = \mu r - \frac{\sigma^2}{2} r^2$

① 对~~R~~求导

$$\frac{1}{R_0} = \mu \frac{dr}{dR_0} - \frac{\sigma^2}{2} \cdot 2r \cdot \frac{dr}{dR_0}$$

$$\frac{dr}{dR_0} = \frac{1}{R_0(\mu - \sigma^2 r)}$$

时 $\mu - \sigma^2 r > 0$

$$\frac{dr}{dR_0} > 0$$

∴ r 随 R_0 ↑而↑

反之 R_0 越高 r 影响越小

题目 $R_0 = 2$ $\mu = 27$ $\sigma^2 = 40$ 求 r

$$\ln R_0 = \mu r - \frac{\sigma^2}{2} r^2$$

$$r = \frac{2\mu \pm \sqrt{4\mu^2 - 4\sigma^2 \cdot 2\ln R}}{2\sigma^2}$$

$= 0.102618$. （12名）

② 对 μ 求导

$$\frac{dr}{d\mu} = \frac{-r}{\mu - \sigma^2 r}$$

③ 对 σ^2 求导

$$\frac{dr}{d\sigma^2} = \frac{r^2}{2(\mu - \sigma^2 r)}$$

$$\frac{\Delta r}{\Delta \sigma^2} \doteq \frac{dr}{d\sigma^2} \doteq \frac{r^2}{2(\mu - \sigma^2 r)} \quad \Delta r = \frac{1}{2}\left(\frac{r^2}{\mu - \sigma^2 r}\right)\Delta\sigma^2$$

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式8 解法

$$r = \frac{\ln R_0}{T}$$

$$r' = \frac{\ln R_0'}{T}$$

$$\Delta r = r' - r = \frac{1}{T}\ln\frac{R_0'}{R_0}$$

$$= \frac{1}{T}\ln\left(\frac{R_0+\Delta R_0}{R_0}\right)$$

$$= \frac{1}{T}\ln\left(\frac{\Delta R_0}{R_0}\right)$$

$$\approx \frac{1}{T}\cdot\frac{\Delta R_0}{R_0}$$ （泰勒约等）

~~第4节 T变动~~

~~式(7) SRB 解法~~

$$r = \frac{\ln R_0}{T} = \frac{\ln\left(\int_\alpha^\beta l(a)\phi(a)da\right)}{T}$$ 引入 Preston节

$$= \frac{\ln(tfr\cdot srb\cdot l(\alpha))}{T}$$

$$= \frac{\ln tfr + \ln srb + \ln l(\alpha)}{T}$$

$$\Delta r = \ln\frac{tfr_1}{tfr_2}/T$$

第4节 T变动

变形：

式8 解法（泰勒展开解法 思路）

式6 $$\frac{dr}{dT} = \frac{T\cdot\frac{d\ln R_0}{dT} - \ln R_0\frac{dT}{dT}}{T^2} = -\frac{\ln R_0}{T^2}$$

$dC=0$

$$dr = -\frac{\ln R_0 dT}{T^2}$$

$$\frac{dr}{r} = \frac{\frac{-\ln R_0 dT}{T^2}}{\frac{\ln R_0}{T}} = \frac{-\cancel{\ln R_0}\, dT}{T\cancel{\ln R_0}}$$

$$\frac{dr}{r} = -\frac{dT}{T}$$

$$\frac{dr}{dT} = \frac{\Delta r}{\Delta T} \approx -\frac{r}{T}$$ r与T方向相反 相关