THE ATIYAH-SINGER INDEX THEOREM

Contents

1. Overview

1.1. **The Index Theorem.** Let X be a compact Riemannian manifold, E, F be complex smooth vector bundles on X, $P : \Gamma(E) \to \Gamma(F)$ be an **elliptic differential operator**. On the one hand, ellipticity of P implies that P is a **Fredhold operator**, so the **(analytic) index** $\mathrm{ind}(P)$ of P is well-defined. On the other hand, there is a canonical way of associating an element $\boldsymbol{\sigma}(P) \in K_c(TX)$ for P, and there is a **topological index map** $\mathrm{ind}_t : K_c(TX) \to \mathbb{Z}$. The Atiyah-Singer index theorem states that $\mathrm{ind}(P) = \mathrm{ind}_t(\boldsymbol{\sigma}(P))$ for all elliptic operators P.

It is more convenient to rewrite the topological index in terms of cohomological data. If X is oriented, then the index theorem becomes

$$\mathrm{ind}(P) = (-1)^{\frac{n(n+1)}{2}}(\pi_! \mathrm{ch}(\boldsymbol{\sigma}(P)) \cdot \mathbf{Td}(X))[X]$$

where $n = \dim X$ and $\pi_! : H^*(TX;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is the Gysim homomorphism associated to the canonical projection $\pi : TX \to X$, and $[X] \in H_n(X;\mathbb{Q})$ is the orientation of X.

2. Statement of the Index Theorem

2.1. Fredholm Operators. Let H be a separable infinite-dimensional Hilbert space over $\mathbb{C}$. A bounded operator $T \in \mathscr{L}(H)$ on H is called a **Fredholm operator** if the kernel and cokernel of T are both finite-dimensional. In this case, define the **index** of T by

$$\mathrm{ind}(T) = \dim\ker T - \dim \mathrm{coker} T$$

Let $\mathscr{F} \subseteq \mathscr{L}$ be the set of Fredholm operators on H (since H do not play an essential role, we drop it from the notations). We take strong topology on $\mathcal{B}$ and all its subsets unless otherwise specified.

Theorem 2.1.1 (Dieudonné)**.** *$\mathrm{ind} : \mathscr{F} \to \mathbb{Z}$ is locally constant, inducing a bijection of sets $\pi_0(\mathscr{F}) \cong \mathbb{Z}$.*

This means that the index is "homotopic invariant". More precisely, let X be a compact connected topological space, and $T : X \to \mathscr{F}$ a continuous map. Then $\mathrm{ind}(T(x))$ is a constant for all $x \in X$, denoted $\mathrm{ind}(T)$, and homotopy invariance means that this yields a well-defined map

$$\mathrm{ind} : [X, \mathscr{F}] \to \mathbb{Z}$$

Theorem 2.1.2 (Atiyah-Jänich)**.** *$\mathscr{F}$ is the classifying space of the K functor. More precisely, for compact topological spaces X, we have isomorphisms of semi-groups*

$$\mathrm{ind} : [X, \mathscr{F}] \xrightarrow{\sim} K(X) \tag{2.1}$$

functorial in X. Under this identification, the previous index map $\mathrm{ind} : [X, \mathscr{F}] \to \mathbb{Z}$ *is the proper pushforward along the canonical map $X \to pt$ and we use the canonical identification $K(pt) \cong \mathbb{Z}$.*

2.2. Elliptic Operators. Let X be a compact smooth manifold. Let E, F be complex smooth vector bundles on X.

Definition 2.2.1. *A **differential operator** from E to F of order k is an $\mathbb{R}$-linear map $P : \Gamma(E) \to \Gamma(F)$ such that in local coordinates it can be represented as*

$$P = \sum_{|\alpha| \le k} a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha \tag{2.2}$$

for $a_\alpha : E \to F$ homomorphisms of vector bundles. For such an operator, the ***symbol*** *is defined to be the map $\sigma(P) : \pi^*E \to \pi^*F$, where $\pi : T^*X \to X$ is the canonical projection, given by*

$$\sigma(P)(x,\xi) = i^k \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \tag{2.3}$$

for $x \in X$, $\xi \in T^*_xX$ with local coordinates $x_1, \cdots, x_n$ and correspondingly $\xi_1, \cdots, \xi_n$.

It is easy to see that this definition does not depend on the choice of local coordinates. A more conceptual definition is: a differential operator from E to F is an element in $\mathrm{Diff}(E,F) := D(X) \otimes_{C^\infty(X)} \mathrm{Hom}(E,F)$ where $D(X)$ is the algebra of differential operators on X, viewed as a left $C^\infty(X)$-module. The usual filtration on $D(X)$ defines the order of differential operators, denoted $\mathrm{Diff}_k(E,F) = D^k(X) \otimes \mathrm{Hom}(E,F)$ The symbol map is given by

$$\sigma = \mathrm{gr} \otimes \mathrm{id} : D(X) \otimes_{C^\infty(X)} \mathrm{Hom}(E,F) \to C^\infty(T^*X) \otimes_{C^\infty(X)} \mathrm{Hom}(E,F) \tag{2.4}$$

where $\mathrm{gr} : D(X) \to C^\infty(T^*X)$ is given by passing to the associative graded. Clearly

$$C^\infty(T^*X) \otimes_{C^\infty(X)} \mathrm{Hom}(E,F) \cong \mathrm{Hom}_{T^*X}(\pi^*E, \pi^*F)$$

so the symbol map is a map

$$\sigma : \mathrm{Diff}(E,F) \to \mathrm{Hom}(\pi^*E, \pi^*F). \tag{2.5}$$

Definition 2.2.2. *A differential operator $P \in \mathrm{Diff}(E,F)$ is called* ***elliptic*** *if $\sigma(P)(x,\xi) : E_x \to F_x$ are isomorphisms for any $x \in X$ and $\xi \in T^*_xX - \{0\}$. Let $\mathrm{Ell}(E,F) \subseteq \mathrm{Diff}(E,F)$ be the subset of elliptic operators.*

Note that it only makes sense to talk about ellipticity if E and F have the same rank. The fundamental result is

Theorem 2.2.3. *Let $P \in \mathrm{Ell}(E,F)$. We can take the L^2-norm completion of P to get an operator $P : L^2(E) \to L^2(F)$, where $L^2(-)$ denote the Hilbert space of L^2-sections. Then P is a Fredholm operator.*

2.3. ***K*-theory and the Difference Construction.** For a compact topological space X, let $K(X)$ be the Gröthendieck group of the semi-group of equivalence classes of topological vector bundles on X with respect to Whitney sums. $K(X)$ is actually a ring with multiplication given by tensor product of vector bundles.

For a pointed topological space X with distinguished point $pt \in X$, let $\widetilde{K}(X)$ be the kernel of the map $K(X) \to K(pt) \cong \mathbb{Z}$, called the **reduced *K*-ring** of X. For $Y \subset X$ a closed subset, we define the **relative *K*-ring** by

$$K(X,Y) = \widetilde{K}(X/Y) \tag{2.6}$$

where X/Y is taken to have Y as basepoint. Note that if $Y = \emptyset$, we define $X/Y = X^+ = X \sqcup \{\widetilde{pt}\}$ to be the one-point compactification of X with basepoint $\widetilde{pt}$, so that

$$K(X) \cong \widetilde{K}(X^+) = K(X, \emptyset).$$

Proposition 2.3.1. *For $Y \subseteq X$ closed subset, the natural inclusions*

$$(Y, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, Y)$$

induces a short exact sequence

$$0 \to K(X, Y) \to K(X) \to K(Y) \to 0 \tag{2.7}$$

If X is locally compact, define the K**-ring with compact support** by $K_c(X) = \widetilde{K}(X^+)$.

Next we recall the **difference construction**. Let X, Y be as above. Let $\mathcal{L}_1(X, Y)$ be the set of triples $(V_0, V_1; \sigma)$ where V_0 and V_1 are vector bundles on X and $\sigma : V_0|_Y \xrightarrow{\sim} V_1|_Y$ is an isomorphism of vector bundles over Y. Starting from an element $(V_0, V_1; \sigma) \in \mathcal{L}_1(X, Y)$, we will construct an element $[V_0, V_1; \sigma]$ in $K(X, Y)$.

First let X_0, X_1 be two copies of X, let $Z = X_0 \cup_Y X_1$ be the topological space obtained by gluing X_0 and X_1 along Y. The sequence (0.7) becomes

$$0 \to K(Z, X_1) \xrightarrow{j^*} K(Z) \xrightarrow{i^*} K(X_1) \to 0$$

It is split by the natural retraction $\rho : Z \to X_1$ given by sending both X_0 and X_1 identically to X_1. Moreover, the map $(X, Y) \hookrightarrow (Z, X_1)$ sending X identically to X_0 induces an isomorphism

$$\varphi : K(Z, X_1) \to K(X, Y).$$

Now we can begin the construction. Let W be the vector bundle over Z obtained by gluing V_0 and V_1 along Y via σ, such that $W|_{X_i} \cong V_i$. Let $W_1 = \rho^*(V_1)$, then $i^*[W_1] = V_1 = i^*[W]$, so $[W] - [W_1] \in \ker i^* = \mathrm{Im} j^*$, namely there exists a unique $[U] \in K(Z, X_1)$ such that $j^*[U] = [W] - [W_1]$. Finally we define $[V_0, V_1; \sigma] = \varphi[U]$. This construction is roughly characterized by the following:

Proposition 2.3.2. *There exists a natural equivalence relation on $\mathcal{L}_1(X, Y)$, whose set of equivalence classes is denoted $L_1(X, Y)$, such that the above difference construction induces an isomorphism of abelian groups*

$$L_1(X, Y) \xrightarrow{\sim} K(X, Y)$$

and for $Y = \emptyset$, this construction gives $(V_0, V_1) \mapsto [V_0] - [V_1]$.

In particular, now let X be a Riemannian manifold, so that we can identify the tangent and cotangent bundles of X. Let E, F be vector bundles on X of the same rank, then for any $P \in \mathrm{Ell}(E, F)$, the symbol $\sigma(P)$ is an isomorphism away from the zero sections in π^*E and π^*F, where $\pi : TX \to X$ is the canonical projection. Let DX be the subset of TX given by

$$DX = \{(x, \xi) \in TX : ||\xi|| \leq 1\}$$

It is a D^n-fibre bundle over X. Let ∂DX be the subset of DX given by

$$\partial DX = \{(x, \xi) \in TX : ||\xi|| = 1\}$$

It is a sphere bundle over X with fibre S^{n+1}. After restriction to DX, we can associate an element

$$\boldsymbol{\sigma}(P) = [\pi^* E, \pi^* F; \sigma(P)] \in K(DX, \partial DX) \cong K_c(TX) \tag{2.8}$$

to the elliptic operator P, called the **symbol class** of P.

2.4. **Thom Isomorphisms and the Topological Index.** We first recall the **K-theoretic Thom isomorphism**. Let X be a compact smooth manifold, $\pi : E \to X$ be a complex hermitian vector bundle over X. We define the element

$$\Lambda_{-1}(E) = [\pi^* \wedge_{\mathbb{C}}^{even} E, \pi^* \wedge_{\mathbb{C}}^{odd} E; \sigma] \in K_c(E)$$

where σ is given by

$$\sigma_e(\varphi) = e \wedge \varphi - e^* \lrcorner \varphi,\ e \in E, \varphi \in \wedge_{\mathbb{C}}^{even} E_{\pi(e)} - \{0\}.$$

Here $e^* \in E$ is the dual of e under the hermitian product.

Theorem 2.4.1 (K-theoretic Thom Isomorphism)**.** *The map*

$$i_! : K(X) \to K_c(E),\ a \mapsto \pi^*(a) \cdot \Lambda_{-1}(E)$$

is an isomorphism of abelian groups.

To continue, we need to define the **K-theoretic Gysin map**, in which we will use the following lemma on **tubular neighborhoods**.

Lemma 2.4.2. *Let $X \subseteq Y$ be a closed submanifold of a smooth manifold X. Then there exists an open neighborhood T of X in Y together with a smooth map $\pi : T \to X$ which is a smooth vector bundle of rank* $\dim Y - \dim X$.

Proof. We quote the idea of proof from [1]. Choose a Riemannian metric on Y, take T to be the set of points in Y with distance $< \epsilon$ to X for some $0 < \epsilon << 1$. Let $N = N_{Y/X}$ be the normal bundle of X in Y, N_ϵ be the subset of elements of length $< \epsilon$ in N. For $(x, v) \in N_\epsilon$, let $\gamma_{x,v}(t)$ be the geodesic near x with initial velocity v, then $(x, v) \mapsto \gamma_{x,v}(1)$ is a diffeomorphism $N_\epsilon \xrightarrow{\sim} T$. $\square$

Now let $f : X \hookrightarrow Y$ be a smooth proper embedding of manifolds, suppose the normal bundle $N = N_{Y/X}$ has a complex structure, then we define the **K-theoretic Gysin map** $f_! : K_c(X) \to K_c(Y)$ to be the composition of the Gysin map $i_! : K_c(X) \to K_c(N)$ with the map $K_c(N) \to K_c(Y)$ obtained by identifying N with a tubular neighborhood T of X in Y (up to homotopy).

Remark 2.4.3. *We recall the cohomological version of the Thom isomorphism and Gysin homomorphism for comparision to the K-theoretic case. Let $f : X \to Y$ be a continuous map between oriented manifolds X and Y*

of dimension n and m respectively. Then f induces $f_* : H_*(X) \to H_*(Y)$ on homology. By Poincaré duality, we have natural isomorphisms

$$H_c^{n-p}(X) \cong H_p(X),\ H_c^{m-p}(Y) \cong H_p(Y)$$

*so f induces a homomorphism $f_! : H_c^*(X) \to H_c^{*+m-n}(Y)$ called the **Gysin homomorphism**.*

In particular, let $\pi : E \to X$ be an oriented real vector bundle over X of rank k, let $i : X \hookrightarrow E$ be the zero section inclusion. Then π and i are inverse of each other up to homotopy, so the induced maps $\pi_* : H_*(E) \to H_*(X)$ and $i_* : H_*(X) \to H_*(E)$ are isomorphisms and mutually inverse of each other. The same is true for the Gysin maps

$$i_! : H_c^*(X) \to H_c^{*+k}(E),\ \pi_! : H_c^*(E) \to H_c^{*-k}(X).$$

*$i_!$ is called the **Thom isomorphism**.*

*If X is compact, let $E_0 = E - i(X)$, then by excision and homotopy, $H^*(E, E_0) \cong H^*(D(E), S(E)) \cong H_c^*(E)$. In this case, there exists $\tau \in H^k(E, E_0)$ called the **Thom class** such that the restriction of τ to each fibre F gives the orientation of F in $H^k(F, F_0) \cong H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong \mathbb{Z}$. The Thom isomorphism $H^*(X) \to H^{*+k}(E, E_0)$ is given by cup product with the Thom class τ.*

In particular, let $f : X \hookrightarrow Y$ be a smooth proper embedding of manifolds, which induces a tangent map $Tf : TX \hookrightarrow TY$. Then the normal bundle $N_{TY/TX}$ is isomorphic to $(Tf)^*(N \oplus N)$, so it has a canonical complex structure. In this way we obtain a canonical Gysin map

$$f_! : K_c(TX) \to K_c(TY)$$

Now we can finish our task of defining the **topological index map**. Let X be a compact Riemannian manifold, $f : X \hookrightarrow \mathbb{R}^N$ is a smooth embedding to some Euclidean space. Then we have a Gysin map $f_! : K_c(TX) \to K_c(T\mathbb{R}^N) = K_c(\mathbb{R}^{2N})$. View $\mathbb{R}^{2N} = \mathbb{C}^N$ as a complex vector bundle over a point, we have a Gysin map $i_! : K(pt) \xrightarrow{\sim} K_c(T\mathbb{R}^N)$, let $q_! : K_c(T\mathbb{R}^N) \to K(pt) \cong \mathbb{Z}$ be the inverse of this map.

Proposition 2.4.4. *The map $\operatorname{ind}_t : K_c(TX) \to \mathbb{Z}$ defined by $\operatorname{ind}_t = q_! f_!$ is independent of the Euclidean embedding f, called the **topological index map**.*

2.5. **Statement of the Index Theorem.**

Now we can state the Index theorem:

Theorem 2.5.1 (Atiyah-Singer)**.** *Let X be a compact Riemannian manifold, E, F be complex smooth vector bundles on X, $P \in \operatorname{Ell}(E, F)$ be an elliptic differential operator. Then*

$$\operatorname{ind}(P) = \operatorname{ind}_t(\boldsymbol{\sigma}(P)).$$

The index theorem is usually stated in the cohomological form, which is more explicit. To introduce it, we first need to introduce various characteristic classes. We start with the following important lemma:

Lemma 2.5.2 (Splitting Principle)**.** *Let X be a smooth manifold.*
(i) Let E be a complex smooth vector bundle over X of rank n. Then there exists a smooth manifold S_E and a proper smooth fibration $\pi : S_E \to X$ such that $\pi^* : H^*(X) \to H^*(S_E)$ is injective and $\pi^* E \cong \ell_1 \oplus \cdots \oplus \ell_n$ for some complex line bundles $\ell_1, \cdots, \ell_n$ over S_E.
(ii)Let E be a oriented real smooth vector bundle over X of rank $2n$. Then there exists a smooth manifold S_E and a proper smooth fibration $\pi : S_E \to X$ such that $\pi^* : H^*(X) \to H^*(S_E)$ is injective and $\pi^*(E \otimes \mathbb{C}) \cong \ell_1 \oplus \overline{\ell}_1 \oplus \cdots \oplus \ell_n \oplus \overline{\ell}_n$ for some complex line bundles $\ell_1, \cdots, \ell_n$ over S_E.

The splitting principle gives us a way of defining characteristic classes. We take the complex case as an example. Let E be a complex smooth vector bundle over X of rank n over X, let $c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^{2*}(X)$ be the **total Chern class** of E. Let $\pi : S_E \to X$ be a map satisfying the splitting principle. Suppose $\pi^* E \cong \ell_1 \oplus \cdots \oplus \ell_n$ for some complex line bundles $\ell_1, \cdots, \ell_n$ over S_E. Then we have

$$\pi^*(c(E)) = \prod_{i=1}^{n} (1 + x_i)$$

where $x_i = c_1(\ell_i)$, so $\pi^*(c_k(E)) = \sigma_k(x_1, \cdots, x_n)$, where σ_k is the **k-th elementary symmetric polynomial**. Then for any symmetric polynomial $S(x_1, \cdots, x_n)$ in n-variables, there is an assoicated characteristic class defined as follows: we first express S in terms of a polynomial in $\sigma_1, \cdots, \sigma_n$, suppose $S(x_1, \cdots, x_n) = F(\sigma_1, \cdots, \sigma_n)$, then the associated character class is given by $F(c_1(E), \cdots, c_n(E))$. It is well-defined.

Definition 2.5.3 (Chern Character)**.** *Let E be a complex smooth vector bundle over X of rank n over X. Consider the following symmetric power series in $x_1, \cdots, x_n$:*

$$e^{x_1} + \cdots + e^{x_n} = \sum_{k=0}^{\infty} \frac{1}{k!} (x_1^k + \cdots + x_n^k)$$

*The degree k part $\frac{1}{k!}(x_1^k + \cdots + x_n^k)$ is a symmetric polynomial, so there corresponds a characteristic class denoted $\mathrm{ch}^k(E) \in H^{2k}(X; \mathbb{Q})$. The **Chern character** of E is defined to be*

$$\mathrm{ch}(E) := \sum_{k=0}^{n} \mathrm{ch}^k(E) \in H^{2*}(X; \mathbb{Q}). \tag{2.9}$$

It is obvious from definition that

$$\mathrm{ch}(E \oplus E') = \mathrm{ch}(E) + \mathrm{ch}(E'),\ \mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\mathrm{ch}(E').$$

In fact we have

Proposition 2.5.4. *If X is a compact manifold, then the Chern character induces a ring homomorphism*

$$\mathrm{ch} : K(X) \to H^{2*}(X; \mathbb{Q}).$$

A similar result holds for non-compact manifolds, if we replace both sides by the compactly supported version.

We can generalize this construction as follows: let $f \in 1 + x\mathbb{Q}[[x]]$ be a formal power series with constant term 1, then since $f(x_1) \cdots f(x_n)$ is symmetric, we can express

$$f(x_1) \cdots f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + \cdots$$

with F_k **weighted homogeneous** of degree k, namely

$$F_k(t\sigma_1, \cdots, t^k\sigma_k) = t^k F_k(\sigma_1, \cdots, \sigma_k), \forall t \in \mathbb{Q}.$$

Each $F_k(\sigma_1, \cdots, \sigma_k)$ is well-defined and independent of the number of variables n. $\{F_k(\sigma_1, \cdots, \sigma_k\}_{k\geq 0}$ is called the **multiplicative sequence** associated to f. Now for every complex vector bundle E over X and $f \in 1 + x\mathbb{Q}[[x]]$, we define the **total F-class** of E by

$$\mathbf{F}_{\mathbb{C}}(E) := 1 + F_1(c_1(E)) + F_2(c_1(E), c_2(E)) + \cdots \in H^{2*}(X; \mathbb{Q}). \tag{2.10}$$

This definition is universal for vector bundles of all rank. For example, take $f(x) = 1 + x$, the corresponding total F-class is the total Chern class.

Definition 2.5.5 (Total Todd Class)**.** *Let $f(x) = \frac{x}{1-e^{-x}}$. The corresponding total class is called the **total Todd class**, denoted $\mathbf{Td}_{\mathbb{C}}$. In particular, let X be a smooth manifold, the **total Todd class** of X is defined by*

$$\mathbf{Td}(X) := \mathbf{Td}_{\mathbb{C}}(TX \otimes \mathbb{C}). \tag{2.11}$$

With these character classes in hand, we can give the cohomological form of the index theorem as follows:

Theorem 2.5.6 (Index Theorem, Cohomological Form)**.** *Let X be a compact Riemannian manifold of dimension n, E, F be complex smooth vector bundles on X, $P \in \mathrm{Ell}(E, F)$ be an elliptic differential operator. Let $\mathrm{ch} : K_c(TX) \to H_c^{2*}(TX; \mathbb{Q})$ be the Chern character of TX, $\pi : TX \to X$ be the canonical projection, $[TX] \in H_{2n}(TX; \mathbb{Q})$ be the canonical orientation of TX. Then*

$$\mathrm{ind}(P) = (-1)^n (\mathrm{ch}(\boldsymbol{\sigma}(P)) \cdot \pi^* \mathbf{Td}(X))[TX].$$

If X is oriented with orientation $[X] \in H_n(X; \mathbb{Q})$, we can pushforward everything to X, giving

$$\mathrm{ind}(P) = (-1)^{\frac{n(n+1)}{2}} (\pi_! \mathrm{ch}(\boldsymbol{\sigma}(P)) \cdot \mathbf{Td}(X))[X]$$

where $\pi_! : H^*(TX; \mathbb{Q}) \to H^{*-n}(X; \mathbb{Q})$ is the inverse of the Thom isomorphism for the tangent bundle $\pi : TX \to X$.

3. Outline of the Proof of the Index Theorem

3.1. The Index Function is Unique.

Definition 3.1.1. *An* ***index function*** *is a collection of homomorphisms* $\mathrm{ind}^X : K_c(TX) \to \mathbb{Z}$ *for all compact smooth manifolds* X*, such that*

(a) ind^X *commutes with Gysin maps, namely for a smooth map* $f : X \to Y$*, let* $f_! : K_c(TX) \to K_c(TY)$ *be the Gysin map defined in Section 2.4, then* $\mathrm{ind}^X = \mathrm{ind}^Y \circ f_!$*, namely the following diagram is commutative:*

$$\begin{tikzcd} K_c(TX) \arrow[rr, "f_!"] \arrow[dr, "\mathrm{ind}^X"'] & & K_c(TY) \arrow[dl, "\mathrm{ind}^Y"] \\ & \mathbb{Z} & \end{tikzcd}$$

(b) $\mathrm{ind}^{pt} : K(pt) \to \mathbb{Z}$ *is the obvious map.*

Proposition 3.1.2. *If* ind^X *is an index function, then it is functorial, namely for a smooth map* $f : X \to Y$*, which induces* $Tf : TX \to TY$*, we have* $\mathrm{ind}^Y = \mathrm{ind}^X \circ (Tf)^*$*, namely the following diagram is commutative:*

$$\begin{tikzcd} K_c(TY) \arrow[rr, "(Tf)^*"] \arrow[dr, "\mathrm{ind}^Y"'] & & K_c(TX) \arrow[dl, "\mathrm{ind}^X"] \\ & \mathbb{Z} & \end{tikzcd}$$

Proposition 3.1.3. *The topological index defined in Proposition 2.4.4 is the unique index function.*

Proof. For any compact smooth manifold X, let $f : X \hookrightarrow \mathbb{R}^N$ be a smooth embedding, $q : \mathbb{R}^N \to pt$ be the map sending everything to the origin, then the map denoted $q_!$ in Section 2.4 is exactly the Gysin map $q_! : K_c(T\mathbb{R}^N) \to K(Tpt) = K(pt)$. $\square$

Thus the idea is to construct an "analytic" index function, namely for every compact smooth manifold X, we define a map $\mathrm{ind}_a : K_c(TX) \to \mathbb{Z}$ as follows: for $u \in K_c(TX)$, we first find some elliptic differential operator P on X such that $u = \boldsymbol{\sigma}(P)$, then define $\mathrm{ind}_a(u) = \mathrm{ind}(P)$. If this function is well-defined and satisfies the axioms (a) and (b) in Definition 3.1.1, then the proof of the index theorem is completed. Unfortunately, there exists elements in $K_c(TX)$ that is not a symbol class of any elliptic differential operator. The way to overcome this problem is to enlarge the class of operators in consideration, namely we consider **pseudo-differential operators** on X.

3.2. Pseudo-Differential Operators.

We start from the discussion on $\mathbb{R}^n$. Let $P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be a differential operator. Define the Fourier transform

$$F(f) = \widehat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} dx$$

for $f \in C_c^\infty(\mathbb{R}^n)$. Then formally $F \circ P \circ F^{-1}$ is given by multiplication by a function $p(x,\xi)$ which smooth in x and polynomial in ξ, because of the formula

$$\widehat{\partial_{x_k} f} = i\xi_k \widehat{f}$$

and Fourier inversion. So we can formally write

$$(Pf)(x) = \int_{\mathbb{R}^n} p(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi \tag{3.1}$$

which at least holds for $f \in C_c^\infty(\mathbb{R}^n)$ or $\mathscr{S}(n)$, the space of **Schwartz functions** on $\mathbb{R}^n$. Psuedo-differential operators on $\mathbb{R}^n$ are defined to be the operators on $\mathscr{S}(\mathbb{R}^n)$ defined by (3.1), but for more general smooth functions $p(x,\xi)$ that is not necessarily polynomial in ξ. Precisely,

Definition 3.2.1. *$P : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a* ***pseudo-differential operator*** *of order k if*

$$(Pf)(x) = \int_{\mathbb{R}^n} p(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi$$

for some smooth function $p(x,\xi)$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{k-|\alpha|}$$

for some $C_{\alpha,\beta}$ for each pair of multi-index (α,β), and for all $x \in \mathbb{R}^n$, $\xi \neq 0$, the limit

$$\sigma_k(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-k} p(x,\lambda\xi)$$

exists, called the ***k-th symbol*** *of P.*

3.3. Defining the Analytic Index.

4. The Index Theorem of Dirac Operators

An important special case of the index theorem is the index formula of a Dirac operator of a Clifford module over a compact Riemannian Manifold. We begin with an exposition of Clifford algebras (there is no need to be coordinate-free, so I choose the most explicit approach).

4.1. Clifford Algebras, Spin Groups, and Spinor Representations.

Let $\mathrm{Cl}(n)$ be the Clifford algebra of $\mathbb{R}^n$ with the usual quadratic form. Let $\mathrm{Cl}_{\mathbb{C}}(n) = \mathrm{Cl}(n) \otimes \mathbb{C}$ be the complexification of $\mathrm{Cl}(n)$, which is the Clifford algebra of the complex quadratic space $\mathbb{C}^n$. The Clifford algebra is naturally $\mathbb{Z}/2$-graded. For simplicity, from now on we suppose $n = 2r$ is even.

There is a natural adjoint action of $\mathrm{Cl}^\times(n)$ on $\mathrm{Cl}(n)$ given by $\mathrm{Ad}_y(x) := yxy^{-1}$ for $x \in \mathrm{Cl}(n)$, $y \in \mathrm{Cl}^\times(n)$. For any non-zero $v \in \mathbb{R}^n$, we have $v^{-1} = -(v,v)v$ in $\mathrm{Cl}(n)$, so for any $w \in \mathbb{R}^n \subseteq \mathrm{Cl}(n)$ we have

$$\mathrm{Ad}_v(w) = -(v,v)^{-1}vwv = -(w - \frac{2(v,w)}{(v,v)}v).$$

Namely the adjoint action Ad_v on $\mathrm{Cl}(n)$ restricts to the reflection with respect to v on the subspace $\mathbb{R}^n \subseteq \mathrm{Cl}(n)$ followed by an antipode. So we let

$\mathrm{Pin}(n)$ be the subgroup of $\mathrm{Cl}^\times(n)$ generated by all $v \in \mathbb{R}^n$ with $(v, v) = 1$, $\mathrm{Spin}(n)$ be the intersection of $\mathrm{Pin}(n)$ with the even part of $\mathrm{Cl}(n)$ (recall that $\mathrm{Cl}(n)$ is naturally $\mathbb{Z}/2$-graded). The adjoint action on generators induces a group homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, which is a central extension of $\mathrm{SO}(n)$ by $\mathbb{Z}/2$.

Next we introduce the spinor representation. Let V be a complex vector space of dimension r, there is a canonical complex bilinear form on $V \oplus V^*$, and $S = \wedge^* V$ is an irreducible $\mathrm{Cl}(V \oplus V^*)$-module, called the **spin representation**. If we take a basis of V and take dual basis on V^*, then the spin representation is an algebra homomorphism $\Delta_n : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}(\mathbb{C}^{2^r})$. It also restricts to a representation of $\mathrm{Cl}(n)$, also called the spin representation and denoted Δ_n. Note that the spin representation is actually a super-representation of the super-algebra $\mathrm{Cl}_{\mathbb{C}}(n)$.

There is another $\mathbb{Z}/2$-grading on $S = \wedge^* V$ given as follows: let $e_1, \cdots, e_r$ be an orthonormal basis of V, $e_1^*, \cdot, e_r^*$ the corresponding dual basis of V^*, consider the element $\omega_{\mathbb{C}} := i^r e_1 \cdots e_r e_1^* \cdots e_r^* \in \mathrm{Cl}(V \oplus V^*)$, then $\omega_{\mathbb{C}}^2 = 1$, so it gives a $\mathbb{Z}/2$-grading on S, denoted $S = S^+ \oplus S^-$. Clearly S^+ and S^- are both invariant under the action of $\mathrm{Spin}(n) \subseteq \mathrm{Cl}_{\mathbb{C}}(n)$, so we have representations $\Delta_n^\pm : \mathrm{Spin}(n) \to \mathrm{End}(S^\pm)$ of $\mathrm{Spin}(n)$ which are irreducible, called the **half-spin representations** of $\mathrm{Spin}(n)$.

4.2. **Clifford Modules and Dirac Operators.** Let X be an oriented Riemannian manifold of dimension n. For any $x \in X$, the Riemannian metric makes $T_x X$ a real quadratic space, and the corresponding Clifford algebra $\mathrm{Cl}_x(X) := \mathrm{Cl}(T_x X)$ is isomorphic to $\mathrm{Cl}(n)$. Let $\mathrm{Cl}(X)$ be the **Clifford bundle** on X, namely the bundle over X whose fibre over $x \in X$ is the Clifford algebra $\mathrm{Cl}_x(X)$. This construction can be generalized to any **Riemannian vector bundle** E over X, namely a real vector bundle E on X whose fibres are equipped continuously with positive-definite inner products. For such E, let $\mathrm{Cl}(E)$ be the associated Clifford bundle.

A **Clifford module** over X is a vector bundle S over X together with a bundle homomorphism $\mathrm{Cl}(X) \otimes S \to S$ such that for any $x \in X$, the induced linear map on the fibre $\mathrm{Cl}_x(X) \otimes S_x \to S_x$ is a Clifford action of $\mathrm{Cl}_x(X)$ on S_x. Also we can consider $\mathbb{Z}/2$-graded Clifford modules, namely $\mathbb{Z}/2$-graded vector bundles $S = S^+ \oplus S^-$ on X such that $S_x^+ \oplus S_x^-$ is a graded representation of $\mathrm{Cl}_x(X)$ for any $x \in X$.

Definition 4.2.1 (Dirac Operators)**.** *Let S be a Clifford module over X, together with a connection $\nabla : \Gamma(S) \to \Gamma(S \otimes T^* X)$. The **Dirac operator** associated with S is the map $D : \Gamma(S) \to \Gamma(S)$ given by*

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(S \otimes T^* X) \xrightarrow{\sim} \Gamma(S \otimes TX) \hookrightarrow \Gamma(\mathrm{Cl}(X)) \otimes \Gamma(S) \xrightarrow{\text{Clifford action}} \Gamma(S)$$

where the middle arrow labelled by $\sim$ comes from the identification of tangent and cotangent bundles of X via the Riemannian metric.

In local coordinates, let $e_1, \cdots, e_n$ be a local orthonormal frame of TX, then

$$D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}$$

where $\cdot$ is the Clifford action.

Example 4.2.2. *Let $X = \mathbb{R}^n$, then $\mathrm{Cl}(X)$ is the trivial bundle $\mathbb{R}^n \times \mathrm{Cl}(n)$, so the trivial bundle $\mathbb{R}^n \times \mathbb{C}^{2^r}$ is a Clifford module over $\mathbb{R}^n$ with fibrewise Clifford action given by the spin representation Δ_n. Let $e_1, \cdots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, let $E_i = \Delta_n(e_i) \in M_{2^r}(\mathbb{C})$ be the corresponding matrices under the spin representation, then the Dirac operator associated to the trivial bundle $\mathbb{R}^n \times \mathbb{C}^{2^r}$ with the Levi-Civita connection is given by*

$$D = \sum_{i=1}^{n} E_i \frac{\partial}{\partial x_i}$$

It has the following main properties:

- *D is a first order elliptic differential operator.*
- *Recall that $\mathbb{C}^{2^r}$ is a graded representation of $\mathrm{Cl}(n)$ whose grading operator is $\epsilon = \begin{pmatrix} I_{2^{r-1}} & \\ & -I_{2^{r-1}} \end{pmatrix} \in M_{2^r}(\mathbb{C})$. Then $D\epsilon + \epsilon D = 0$, namely we can write D in terms of matrices as*
$$D = \begin{pmatrix} & D_- \\ D_+ & \end{pmatrix}$$
- *$D^2 = \Delta \otimes I_{2^r}$, where $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial^2 x_i}$ is the Laplace operator on $\mathbb{R}^n$. In fact, the origin of Dirac operator is from the work of Dirac on finding a "square root" of the Euclidean Laplacian.*

The properties listed in the above can be generalized to any Dirac operator coming from a Clifford module with connection:

Proposition 4.2.3. *Let (S, ∇) be a Clifford module over X with connection.*

(i) D is a first order elliptic differential operator. In fact the symbol of D is given as follows: for $x \in X$, $v \in T_xX$, $s_x \in S_x$, we have
$$\sigma(D)_{x,v}(s_x) = iv \cdot s_x$$
here as usual, $\cdot$ is the Clifford action, and we identified the tangent and cotangent bundles of X.

(ii) If $S = S^+ \oplus S^-$ is $\mathbb{Z}/2$-graded, then D has the form $D = \begin{pmatrix} & D_- \\ D_+ & \end{pmatrix}$ for some $D_- : S_- \to S_+$, $D_+ : S_+ \to S_-$.

(iii) D^2 is a Laplacian-type operator on X.

Example 4.2.4. *Let X be a Riemannian manifold. The most natural Clifford module over X is $\wedge^*(T^*X)$, on which the Clifford action by $\mathrm{Cl}(X)$ is given by pointwise contraction by the tangent vectors. It is also naturally*

$\mathbb{Z}/2$-graded. If we take the lifting of the Levi-Civita connection on X, then the resulting Dirac operator is

$$D = d + d^* : \Omega^*(X) \to \Omega^*(X)$$

whose square is the Hodge Laplacian $\Delta = dd^* + d^*d$.

4.3. **Spin Manifolds.** Let E be an oriented n-dimensional Riemannian vector bundle over X. A **Spin structure** on E is a principal $\mathrm{Spin}(n)$-bundle P over X such that $E \cong P \times_{\mathrm{Spin}(n)} \mathbb{R}^n$. A **Spin manifold** is an oriented Riemannian manifold X together with a spin structure on TX.

Definition 4.3.1 (Complex Spinor Bundle)**.** *Let X be a spin manifold of even dimension $n = 2r$, with spin structure given by a principal* $\mathrm{Spin}(n)$*-bundle P. The* ***complex spinor bundle*** *on X is the $\mathbb{Z}/2$-graded complex vector bundle $\$ = \$^+ \oplus \$^-$ over X given by*

$$\$^{\pm} := P \times_{\mathrm{Spin}(n)} \Delta_n^{\pm}$$

where $\Delta_n^{\pm}$ are the two half-spin representations for $\mathrm{Spin}(n)$*.*

Clearly $\$^{\pm}$ is a Clifford module over X, let $\not{D} = \begin{pmatrix} & \not{D}^- \\ \not{D}^+ & \end{pmatrix}$ be the corresponding Dirac operator on S.

5. The Heat Kernel Approach

5.1. **Heat Kernel on $\mathbb{R}^n$.** We review the construction of heat kernel on $\mathbb{R}^n$ completely heuristically.

Recall the **Heat equation** or **diffusion equation** on $\mathbb{R}^n$ is

$$\partial_t u = \Delta u, u|_{t=0} = \phi \tag{5.1}$$

for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and $\phi : \mathbb{R}^n \to \mathbb{C}$ sufficiently regular. Here $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the spatial Laplacian.

Heuristically solutions of (5.1) can be written as $u(t;x) = (e^{t\Delta}\phi)(x)$, as if we regard Δ as a constant. The operator $e^{t\Delta}$ is called the **heat operator**. The **heat kernel** $K_t(x,y)$ is the kernel of the heat operator $e^{t\Delta}$, that is to say, we have

$$(e^{t\Delta}\phi)(x) = \int_{\mathbb{R}^n} K_t(x,y)\phi(y)dy \tag{5.2}$$

for ϕ in $\mathscr{S}(\mathbb{R}^n)$. We only seek for a heat kernel of the form $K_t(x,y) = K_t(x-y)$ for some one-variable function K_t (with the obvious abuse of notations). Then (5.2) can be written as

$$(e^{t\Delta}\phi)(x) = \int_{\mathbb{R}^n} K_t(x-y)\phi(y)dy$$

or more elegantly,

$$e^{t\Delta}\phi = K_t * \phi$$

where $*$ is the convolution product. Take Fourier transform of both sides gives

$$e^{-t||x||^2}\widehat{\phi} = \widehat{K}_t\widehat{\phi}$$

so if ϕ is not identically zero, this gives $\widehat{K}_t = e^{-t||x||^2}$. Finally take Fourier inverse transform, we get the formula

$$K_t(x,y) = K_t(x-y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{||x-y||^2}{4t}}$$

for the heat kernel for Euclidean space.

6. Noncommutative Geometry

6.1. The Ell **theory of Atiyah and K-homology of Manifolds.** In this subsection we follow [2]. Let X be a compact riemannian manifold. If $P : C^\infty(X) \to C^\infty(X)$ is a differential operator of order k, then for any $f \in C^\infty(X)$, $[P, f] = Pf - fP$ is a differential operator of order $k-1$ on X, where f also denote the operator of multiplication by f. A similar result for psuedo-differential operator is

Proposition 6.1.1. *Let $P : C^\infty(X) \to C^\infty(X)$ be a pseudo-differential operator of order* 0*, $f \in C^\infty(X)$, then the operator $[P, f]$ is a compact operator on $L^2(X)$.*

This proposition is correct even for *continuous* $f \in C(X)$. This leads to the following definition:

Definition 6.1.2 (Atiyah)**.** *Let X be a compact Hausdorff space, $C(X)$ be the C^*-algebra of continuous functions on X. Let* $\mathrm{Ell}(X)$ *be the set of pairs (H, P) where H is a complex Hilbert spaces which is a* ***$*$-modules*** *over $C(X)$ (namely there is a $*$-homomorphisms $\rho : C(X) \to \mathscr{L}(H)$) and $P \in \mathscr{F}(H)$ is a Fredholm operator on H such that for any $f \in C(X)$, $[P, \rho(f)] \in \mathscr{K}(H)$ is a compact operator on H. Elements in* $\mathrm{Ell}(X)$ *are called* ***elliptic operators on** X.*

There is a natural map $\mathrm{ind} : \mathrm{Ell}(X) \to \mathbb{Z}$ by sending (H, P) to the index of P. In fact, a continuous map $f : X \to Y$ between compact Hausdorff spaces naturally induces $f_* : \mathrm{Ell}(X) \to \mathrm{Ell}(Y)$. If Y is a point, clearly $\mathrm{Ell}(pt) \cong \mathscr{F}$ is the set of Fredholm operators, so there is a natural index map $\mathrm{ind} : \mathrm{Ell}(pt) \to \mathbb{Z}$. Then the index map on $\mathrm{Ell}(X)$ agrees with the map $\mathrm{Ell}(X) \to \mathrm{Ell}(pt) \to \mathbb{Z}$.

The main feature of Atiyah's Ell-theory is that there is a natural "index pairing"

$$\mathrm{Ell}(X) \times K^0(X) \to \mathbb{Z}$$

Here we denote the topological K-theory groups by K^0because, as also mentioned in [2], there is a topological way to define the **K-homology** groups $K_0(X)$ for "good" topological spaces, which is somehow in duality with the K^0-theory, and the above pairing actually comes from a map $\mathrm{Ell}(X) \to K_0(X)$ to the K-homology.

6.2. **Fredholm Modules and K-homology for C^*-algebras.** The Ell-theory above is defined purely algebraically, so we can generalize it to arbitrary C^*-algebras, which leads to the following

Definition 6.2.1. *Let A be a C^*-algebra. An* ***(odd) Fredholm module*** *over A is a pair (H, F) where H is a $*$-module over A and $F \in \mathscr{F}(H)$ is a Fredholm operator on H such that for any $a \in A$, the operators*

$$(F^2 - 1)a,\ (F - F^*)a,\ [F, a]$$

are all compact on H.

An ***even Fredholm module*** *over A is a triple (H, γ, F) where H is a $*$-module over A, $\gamma \in \mathscr{L}(H)$ a $*$-automorphism of H of order 2, $F \in \mathscr{F}(H)$ a Fredholm operator such that the operators*

$$(F^2 - 1)a,\ (F - F^*)a,\ [F, a]$$

are all compact on H and $\gamma F + F\gamma = 0$.

Remark 6.2.2. *The operator γ in the definition of an even Fredholm module actually gives a $\mathbb{Z}/2$-grading of the Hilbert space, so we can rewrite this definition as follows:*

An ***even Fredholm module*** *over A is a pair (H, F) where $H = H_+ \oplus H_-$ is a $\mathbb{Z}/2$-graded $*$-module over A, and $F : H \to H$ is a Fredholm operator of the form $F = \begin{pmatrix} & F_- \\ F_+ & \end{pmatrix}$ such that $F_- - (F_+)^*$ is a compact operator.*

It follows immediately that F_+ is a Fredholm operator. The ***index*** *of an even Fredholm module (H, F) is defined to be* $\operatorname{ind}(F_+)$.

References

[1] D. Arapura, *Complex algebraic geometry.*
[2] M. Atiyah, *Global theory of elliptic operators*
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