Stochastic Differential Equations, 6ed. Solution of Exercise Problems

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Abstract

This is a solution manual for the SDE book by Øksendal, *Stochastic Differential Equations, Sixth Edition*, and it is complementary to the book's own solution (in the book's appendix). If you have any comments or find any typos/errors, please email me at quantsummaries@gmail.com. This version omits the problems from the chapters on applications (Chapter 6, 10, 11 and 12).

I thank Saúl Torres for valuable feedback.

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1 Introduction

2 Some Mathematical Preliminaries

2.8. b)

Proof.

$$E[e^{iuB_t}] = \sum_{k=0}^{\infty} \frac{i^k}{k!} E[B_t^k] u^k = \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{t}{2})^k u^{2k}.$$

So

$$E[B_t^{2k}] = \frac{\frac{1}{k!}(-\frac{t}{2})^k}{\frac{(-1)^k}{(2k)!}} = \frac{(2k)!}{k! \cdot 2^k} t^k.$$

□

d)

Proof.

$$\begin{aligned} E^x[|B_t - B_s|^4] &= \sum_{i=1}^{n} E^x[(B_t^{(i)} - B_s^{(i)})^4] + \sum_{i \neq j} E^x[(B_t^{(i)} - B_s^{(i)})^2 (B_t^{(j)} - B_s^{(j)})^2] \\ &= n \cdot \frac{4!}{2! \cdot 4} \cdot (t-s)^2 + n(n-1)(t-s)^2 \\ &= n(n+2)(t-s)^2. \end{aligned}$$

□

2.11.

Proof. Prove that the increments are independent and stationary, with Gaussian distribution. Note for Gaussian random variables, uncorrelatedness=independence. □

2.15.

Proof. Since $B_t - B_s \perp \mathcal{F}_s := \sigma(B_u : u \le s)$, $U(B_t - B_s) \perp \mathcal{F}_s$. Note $U(B_t - B_s) \stackrel{d}{=} N(0, t-s)$. □

3 Itô Integrals

3.2.

Proof. WLOG, we assume $t = 1$, then

$$\begin{aligned} B_1^3 &= \sum_{j=1}^{n} (B_{j/n}^3 - B_{(j-1)/n}^3) \\ &= \sum_{j=1}^{n} [(B_{j/n} - B_{(j-1)/n})^3 + 3B_{(j-1)/n} B_{j/n} (B_{j/n} - B_{(j-1)/n})] \\ &= \sum_{j=1}^{n} (B_{j/n} - B_{(j-1)/n})^3 + \sum_{j=1}^{n} 3B_{(j-1)/n}^2 (B_{j/n} - B_{(j-1)/n}) \\ &\quad + \sum_{j=1}^{n} 3B_{(j-1)/n} (B_{j/n} - B_{(j-1)/n})^2 \\ &:= I + II + III \end{aligned}$$

By Problem EP1-1 and the continuity of Brownian motion.

$$I \leq [\sum_{j=1}^{n}(B_{j/n} - B_{(j-1)/n})^2] \max_{1\leq j\leq n} |B_{j/n} - B_{(j-1)/n}| \to 0 \quad a.s.$$

To argue $II \to 3\int_0^1 B_t^2 dB_t$ as $n \to \infty$, it suffices to show $E[\int_0^1 (B_t^2 - B_t^{(n)})^2 dt] \to 0$, where $B_t^{(n)} = \sum_{j=1}^n B^2_{(j-1)/n} 1_{\{(j-1)/n<t\leq j/n\}}$. Indeed,

$$E[\int_0^1 |B_t^2 - B_t^{(n)}|^2 dt] = \sum_{j=1}^n \int_{(j-1)/n}^{j/n} E[(B_t^2 - B^2_{(j-1)/n})^2]dt$$

We note $(B_t^2 - B^2_{\frac{j-1}{n}})^2$ is equal to

$$(B_t - B_{\frac{j-1}{n}})^4 + 4(B_t - B_{\frac{j-1}{n}})^3 B_{\frac{j-1}{n}} + 4(B_t - B_{\frac{j-1}{n}})^2 B^2_{\frac{j-1}{n}}$$

so $E[(B^2_{(j-1)/n} - B_t^2)^2] = 3(t-(j-1)/n)^2 + 4(t-(j-1)/n)(j-1)/n$, and

$$\int_{\frac{j-1}{n}}^{\frac{j}{n}} E[(B^2_{\frac{j-1}{n}} - B_t^2)^2]dt = \frac{2j+1}{n^3}$$

Hence $E[\int_0^1 (B_t - B_t^{(n)})^2 dt] = \sum_{j=1}^n \frac{2j-1}{n^3} \to 0$ as $n \to \infty$.

To argue $III \to 3\int_0^1 B_t dt$ as $n \to \infty$, it suffices to prove

$$\sum_{j=1}^n B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n})^2 - \sum_{j=1}^n B_{(j-1)/n}(\frac{j}{n} - \frac{j-1}{n}) \to 0 \quad a.s.$$

By looking at a subsequence, we only need to prove the L^2-convergence. Indeed,

$$\begin{aligned}
& E\left(\sum_{j=1}^n B_{(j-1)/n}[(B_{j/n} - B_{(j-1)/n})^2 - \frac{1}{n}]\right)^2 \\
= & \sum_{j=1}^n E\left(B^2_{(j-1)/n}[(B_{j/n} - B_{(j-1)/n})^2 - \frac{1}{n}]^2\right) \\
= & \sum_{j=1}^n \frac{j-1}{n} E\left[(B_{j/n} - B_{(j-1)/n})^4 - \frac{2}{n}(B_{j/n} - B_{(j-1)/n})^2 + \frac{1}{n^2}\right] \\
= & \sum_{j=1}^n \frac{j-1}{n}(3\frac{1}{n^2} - 2\frac{1}{n^2} + \frac{1}{n^2}) \\
= & \sum_{j=1}^n \frac{2(j-1)}{n^3} \to 0
\end{aligned}$$

as $n \to \infty$. This completes our proof.

$\square$

3.9.

Proof. We first note that

$$\begin{aligned}
& \sum_j B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{t_j}) \\
= & \sum_j \left[B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{\frac{t_j+t_{j+1}}{2}}) + B_{t_j}(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j})\right] + \sum_j (B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j})^2.
\end{aligned}$$

The first term converges in $L^2(P)$ to $\int_0^T B_t dB_t$. For the second term, we note

$$\begin{aligned}
& E\left[\left(\sum_j \left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \frac{t}{2}\right)^2\right] \\
= \; & E\left[\left(\sum_j \left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \sum_j \frac{t_{j+1}-t_j}{2}\right)^2\right] \\
= \; & \sum_{j,\,k} E\left[\left(\left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \frac{t_{j+1}-t_j}{2}\right)\left(\left(B_{\frac{t_k+t_{k+1}}{2}} - B_{t_k}\right)^2 - \frac{t_{k+1}-t_k}{2}\right)\right] \\
= \; & \sum_j E\left[\left(B^2_{\frac{t_{j+1}-t_j}{2}} - \frac{t_{j+1}-t_j}{2}\right)^2\right] \\
= \; & \sum_j 2\cdot\left(\frac{t_{j+1}-t_j}{2}\right)^2 \\
\leq \; & \frac{T}{2}\max_{1\leq j\leq n}|t_{j+1}-t_j| \to 0,
\end{aligned}$$

since $E[(B_t^2-t)^2] = E[B_t^4 - 2tB_t^2 + t^2] = 3E[B_t^2]^2 - 2t^2 + t^2 = 2t^2$. So

$$\sum_j B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{t_j}) \to \int_0^T B_t dB_t + \frac{T}{2} = \frac{1}{2}B_T^2 \quad \text{in } L^2(P).$$

□

3.10.

Proof. According to the result of Exercise 3.9., it suffices to show

$$E\left[\left|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j\right|\right] \to 0.$$

Indeed, note

$$\begin{aligned}
& E\left[\left|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j\right|\right] \\
\leq \; & \sum_j E[|f(t_j) - f(t'_j)||\Delta B_j|] \\
\leq \; & \sum_j \sqrt{E[|f(t_j)-f(t'_j)|^2]E[|\Delta B_j|^2]} \\
\leq \; & \sum_j \sqrt{K}|t_j - t'_j|^{\frac{1+\epsilon}{2}}|t_j - t'_j|^{\frac{1}{2}} \\
= \; & \sqrt{K}\sum_j |t_j - t'_j|^{1+\frac{\epsilon}{2}} \\
\leq \; & T\sqrt{K}\max_{1\leq j\leq n}|t_j - t'_j|^{\frac{\epsilon}{2}} \\
\to \; & 0.
\end{aligned}$$

□

3.11.

Proof. Assume W is continuous, then by bounded convergence theorem, $\lim_{s\to t} E[(W_t^{(N)} - W_s^{(N)})^2] = 0$. Since W_s and W_t are independent and identically distributed, so are $W_s^{(N)}$ and $W_t^{(N)}$. Hence

$$E[(W_t^{(N)} - W_s^{(N)})^2] = E[(W_t^{(N)})^2] - 2E[W_t^{(N)}]E[W_s^{(N)}] + E[(W_s^{(N)})^2] = 2E[(W_t^{(N)})^2] - 2E[W_t^{(N)}]^2.$$

Since the RHS=$2Var(W_t^{(N)})$ is independent of s, we must have RHS=0, i.e. $W_t^{(N)} = E[W_t^{(N)}]$ a.s. Let $N \to \infty$ and apply dominated convergence theorem to $E[W_t^{(N)}]$, we get $W_t = 0$. Therefore $W_\cdot \equiv 0$. □

3.18.

Proof. If $t > s$, then

$$E\left[\frac{M_t}{M_s}|\mathcal{F}_s\right] = E\left[e^{\sigma(B_t - B_s) - \frac{1}{2}\sigma^2(t-s)}|\mathcal{F}_s\right] = \frac{E[e^{\sigma B_{t-s}}]}{e^{\frac{1}{2}\sigma^2(t-s)}} = 1$$

The second equality is due to the fact $B_t - B_s$ is independent of $\mathcal{F}_s$. □

4 The Itô Formula and the Martingale Representation Theorem

4.4.

Proof. For part a), set $g(t,x) = e^x$ and use Theorem 4.12. For part b), it comes from the fundamental property of Itô integral, i.e. Itô integral preserves martingale property for integrands in $\mathcal{V}$.

Comments: The power of Itô formula is that it gives martingales, which vanish under expectation. □

4.5.

Proof.

$$B_t^k = \int_0^t kB_s^{k-1}dB_s + \frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds$$

Therefore,

$$\beta_k(t) = \frac{k(k-1)}{2}\int_0^t \beta_{k-2}(s)ds$$

This gives $E[B_t^4]$ and $E[B_t^6]$. For part b), prove by induction.

□

4.6. (b)

Proof. Apply Theorem 4.12 with $g(t,x) = e^x$ and $X_t = ct + \sum_{j=1}^n \alpha_j B_j$. Note $\sum_{j=1}^n \alpha_j B_j$ is a BM, up to a constant coefficient. □

4.7. (a)

Proof. $v \equiv I_{n\times n}$. □

(b)

Proof. Use integration by parts formula (Exercise 4.3.), we have

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX + \int_0^t |v_s|^2 ds = X_0^2 + 2\int_0^t X_s v_s dB_s + \int_0^t |v_s|^2 ds.$$

So $M_t = X_0^2 + 2\int_0^t X_s v_s dB_s$. Let C be a bound for $|v|$, then

$$E\left[\int_0^t |X_s v_s|^2 ds\right] \le C^2 E\left[\int_0^t |X_s|^2 ds\right] = C^2 \int_0^t E\left[\left|\int_0^s v_u dB_u\right|^2\right] ds$$
$$= \quad C^2 \int_0^t E\left[\int_0^s |v_u|^2 du\right] ds \le \frac{C^4 t^2}{2}.$$

So M_t is a martingale. □

4.12.

Proof. Let $Y_t = \int_0^t u(s,\omega)ds$. Then Y is a continuous $\{\mathcal{F}_t^{(n)}\}$-martingale with finite variation. On one hand,

$$\langle Y\rangle_t = \lim_{\Delta t_k \to 0} \sum_{t_k \le t} |Y_{t_{k+1}} - Y_{t_k}|^2 \le \lim_{\Delta t_k \to 0} (\text{total variation of } Y \text{ on } [0,t]) \cdot \max_{t_k} |Y_{t_{k+1}} - Y_{t_k}| = 0.$$

On the other hand, integration by parts formula yields

$$Y_t^2 = 2\int_0^t Y_s dY_s + \langle Y\rangle_t.$$

So Y_t^2 is a local martingale. If $(T_n)_n$ is a localizing sequence of stopping times, by Fatou's lemma,

$$E[Y_t^2] \le \lim_n E[Y_{t\wedge T_n}^2] = E[Y_0^2] = 0.$$

So $Y_\cdot \equiv 0$. Take derivative, we conclude $u = 0$. □

4.16. (a)

Proof. Use Jensen's inequality for conditional expectations. □

(b)

Proof. (i) $Y = 2\int_0^T B_s dB_s$. So $M_t = T + 2\int_0^t B_s dB_s$.

(ii) $B_T^3 = \int_0^T 3B_s^2 dB_s + 3\int_0^T B_s ds = 3\int_0^T B_s^2 dB_s + 3(B_T T - \int_0^T s dB_s)$. So $M_t = 3\int_0^t B_s^2 dB_s + 3TB_t - 3\int_0^t s dB_s = \int_0^t 3(B_s^2 + (T-s_) dB_s$.

(iii)$M_t = E[\exp(\sigma B_T)|\mathcal{F}_t] = E[\exp(\sigma B_T - \frac{1}{2}\sigma^2 T)|\mathcal{F}_t]\exp(\frac{1}{2}\sigma^2 T) = Z_t \exp(\frac{1}{2}\sigma^2 T)$, where $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$. Since Z solves the SDE $dZ_t = Z_t \sigma dB_t$, we have

$$M_t = (1 + \int_0^t Z_s \sigma dB_s)\exp(\frac{1}{2}\sigma^2 T) = \exp(\frac{1}{2}\sigma^2 T) + \int_0^t \sigma \exp(\sigma B_s + \frac{1}{2}\sigma^2(T-s))dB_s.$$

□

5 Stochastic Differential Equations

5.1. (ii)

Proof. Set $f(t,x) = x/(1+t)$, then by Itô's formula, we have

$$dX_t = df(t,B_t) = -\frac{B_t}{(1+t)^2}dt + \frac{dB_t}{1+t} = -\frac{X_t}{1+t}dt + \frac{dB_t}{1+t}$$

□

(iii)

Proof. By Itô's formula, $dX_t = \cos B_t dB_t - \frac{1}{2}\sin B_t dt$. So $X_t = \int_0^t \cos B_s dB_s - \frac{1}{2}\int_0^t X_s ds$. Let $\tau = \inf\{s > 0 : B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. Then

$$\begin{aligned}
X_{t\wedge\tau} &= \int_0^{t\wedge\tau} \cos B_s dB_s - \frac{1}{2}\int_0^{t\wedge\tau} X_s ds \\
&= \int_0^t \cos B_s 1_{\{s\leq\tau\}} dB_s - \frac{1}{2}\int_0^{t\wedge\tau} X_s ds \\
&= \int_0^t \sqrt{1-\sin^2 B_s}1_{\{s\leq\tau\}} dB_s - \frac{1}{2}\int_0^{t\wedge\tau} X_s ds \\
&= \int_0^{t\wedge\tau} \sqrt{1-X_s^2} dB_s - \frac{1}{2}\int_0^{t\wedge\tau} X_s ds.
\end{aligned}$$

So for $t < \tau$, $X_t = \int_0^t \sqrt{1-X_s^2}dB_s - \frac{1}{2}\int_0^t X_s ds$. □

(iv)

Proof. $dX_t^1 = dt$ is obvious. Set $f(t,x) = e^t x$, then

$$dX_t^2 = df(t,B_t) = e^t B_t dt + e^t dB_t = X_t^2 dt + e^t dB_t$$

□

5.3.

Proof. Apply Itô's formula to $e^{-rt}X_t$. □

5.5. (a)

Proof. $d(e^{-\mu t}X_t) = -\mu e^{-\mu t}X_t dt + e^{-\mu t}dX_t = \sigma e^{-\mu t}dB_t$. So $X_t = e^{\mu t}X_0 + \int_0^t \sigma e^{\mu(t-s)}dB_s$. □

(b)

Proof. $E[X_t] = e^{\mu t}E[X_0]$ and

$$X_t^2 = e^{2\mu t}X_0^2 + \sigma^2 e^{2\mu t}(\int_0^t e^{-\mu s}dB_s)^2 + 2\sigma e^{2\mu t}X_0\int_0^t e^{-\mu s}dB_s.$$

So

$$\begin{aligned}
E[X_t^2] &= e^{2\mu t}E[X_0^2] + \sigma^2 e^{2\mu t}\int_0^t e^{-2\mu s}ds \\
&\quad \text{since } \int_0^t e^{-\mu s}dB_s \text{ is a martingale vanishing at time 0} \\
&= e^{2\mu t}E[X_0^2] + \sigma^2 e^{2\mu t}\frac{e^{-2\mu t}-1}{-2\mu} \\
&= e^{2\mu t}E[X_0^2] + \sigma^2\frac{e^{2\mu t}-1}{2\mu}.
\end{aligned}$$

So $Var[X_t] = E[X_t^2] - (E[X_t])^2 = e^{2\mu t}Var[X_0] + \sigma^2 \frac{e^{2\mu t}-1}{2\mu}$. $\square$

5.6.

Proof. We find the integrating factor F_t by the follows. Suppose F_t satisfies the SDE $dF_t = \theta_t dt + \gamma_t dB_t$. Then

$$\begin{aligned} d(F_tY_t) &= F_tdY_t + Y_tdF_t + dY_tdF_t \\ &= F_t(rdt + \alpha Y_t dB_t) + Y_t(\theta_t dt + \gamma_t dB_t) + \alpha\gamma_t Y_t dt \\ &= (rF_t + \theta_t Y_t + \alpha\gamma_t Y_t)dt + (\alpha F_t Y_t + \gamma_t Y_t)dB_t. \end{aligned} \tag{1}$$

Solve the equation system

$$\begin{cases} \theta_t + \alpha\gamma_t = 0 \\ \alpha F_t + \gamma_t = 0, \end{cases}$$

we get $\gamma_t = -\alpha F_t$ and $\theta_t = \alpha^2 F_t$. So $dF_t = \alpha^2 F_t dt - \alpha F_t dB_t$. To find F_t, set $Z_t = e^{-\alpha^2 t}F_t$, then

$$dZ_t = -\alpha^2 e^{-\alpha^2 t}F_t dt + e^{-\alpha^2 t}dF_t = e^{-\alpha^2 t}(-\alpha)F_t dB_t = -\alpha Z_t dB_t.$$

Hence $Z_t = Z_0 \exp(-\alpha B_t - \alpha^2 t/2)$. So

$$F_t = e^{\alpha^2 t}F_0 e^{-\alpha B_t - \frac{1}{2}\alpha^2 t} = F_0 e^{-\alpha B_t + \frac{1}{2}\alpha^2 t}.$$

Choose $F_0 = 1$ and plug it back into equation (1), we have $d(F_tY_t) = rF_t dt$. So

$$Y_t = F_t^{-1}(F_0Y_0 + r\int_0^t F_s ds) = Y_0 e^{\alpha B_t - \frac{1}{2}\alpha^2 t} + r\int_0^t e^{\alpha(B_t - B_s) - \frac{1}{2}\alpha^2(t-s)}ds.$$

$\square$

5.7. (a)

Proof. $d(e^t X_t) = e^t(X_t dt + dX_t) = e^t(mdt + \sigma dB_t)$. So

$$X_t = e^{-t}X_0 + m(1 - e^{-t}) + \sigma e^{-t}\int_0^t e^s dB_s.$$

$\square$

(b)

Proof. $E[X_t] = e^{-t}E[X_0] + m(1 - e^{-t})$ and

$$\begin{aligned} E[X_t^2] &= E[(e^{-t}X_0 + m(1-e^{-t}))^2] + \sigma^2 e^{-2t}E[\int_0^t e^{2s}ds] \\ &= e^{-2t}E[X_0^2] + 2m(1-e^{-t})e^{-t}E[X_0] + m^2(1-e^{-t})^2 + \frac{1}{2}\sigma^2(1-e^{-2t}). \end{aligned}$$

Hence $Var[X_t] = E[X_t^2] - (E[X_t])^2 = e^{-2t}Var[X_0] + \frac{1}{2}\sigma^2(1 - e^{-2t})$. $\square$

5.9.

Proof. Let $b(t,x) = \log(1+x^2)$ and $\sigma(t,x) = 1_{\{x>0\}}x$, then

$$|b(t,x)| + |\sigma(t,x)| \leq \log(1+x^2) + |x|$$

Note $\log(1+x^2)/|x|$ is continuous on $\mathbb{R} - \{0\}$, has limit 0 as $x \to 0$ and $x \to \infty$. So it's bounded on $\mathbb{R}$. Therefore, there exists a constant C, such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|)$$

Also,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq \frac{2|\xi|}{1+\xi^2}|x-y| + |1_{\{x>0\}}x - 1_{\{y>0\}}y|$$

for some ξ between x and y. So

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq |x-y| + |x-y|$$

Conditions in Theorem 5.2.1 are satisfied and we have existence and uniqueness of a strong solution. $\square$

5.10.

Proof. $X_t = Z + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s$. Since Jensen's inequality implies $(a_1 + \cdots + a_n)^p \leq n^{p-1}(a_1^p + \cdots + a_n^p)$ $(p \geq 1, a_1, \cdots, a_n \geq 0)$, we have

$$\begin{aligned}
E[|X_t|^2] &\leq 3\left(E[|Z|^2] + E\left[\left|\int_0^t b(s,X_s)ds\right|^2\right] + E\left[\left|\int_0^t \sigma(s,X_s)dB_s\right|^2\right]\right) \\
&\leq 3\left(E[|Z|^2] + E[\int_0^t |b(s,X_s)|^2 ds] + E[\int_0^t |\sigma(s,X_s)|^2 ds]\right) \\
&\leq 3(E[|Z|^2] + C^2E[\int_0^t (1+|X_s|)^2 ds] + C^2E[\int_0^t (1+|X_s|)^2 ds]) \\
&= 3(E[|Z|^2] + 2C^2E[\int_0^t (1+|X_s|)^2 ds]) \\
&\leq 3(E[|Z|^2] + 4C^2E[\int_0^t (1+|X_s|^2) ds]) \\
&\leq 3E[|Z|^2] + 12C^2T + 12C^2\int_0^t E[|X_s|^2]ds \\
&= K_1 + K_2\int_0^t E[|X_s|^2]ds,
\end{aligned}$$

where $K_1 = 3E[|Z|^2] + 12C^2T$ and $K_2 = 12C^2$. By Gronwall's inequality, $E[|X_t|^2] \leq K_1 e^{K_2 t}$. $\square$

5.11.

Proof. First, we check by integration-by-parts formula,

$$dY_t = \left(-a + b - \int_0^t \frac{dB_s}{1-s}\right)dt + (1-t)\frac{dB_t}{1-t} = \frac{b - Y_t}{1-t}dt + dB_t$$

Set $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$, then X_t is centered Gaussian, with variance

$$E[X_t^2] = (1-t)^2\int_0^t \frac{ds}{(1-s)^2} = (1-t) - (1-t)^2$$

So X_t converges in L^2 to 0 as $t \to 1$. Since X_t is continuous a.s. for $t \in [0,1)$, we conclude 0 is the unique a.s. limit of X_t as $t \to 1$. $\square$

5.14. (i)

Proof.

$$\begin{aligned}
dZ_t &= d(u(B_1(t),B_2(t)) + iv(B_1(t),B_2(t))) \\
&= \bigtriangledown u\cdot(dB_1(t),dB_2(t)) + \frac{1}{2}\Delta u dt + i\bigtriangledown v\cdot(dB_1(t),dB_2(t)) + \frac{i}{2}\Delta v dt \\
&= (\bigtriangledown u + i\bigtriangledown v)\cdot(dB_1(t),dB_2(t)) \\
&= \frac{\partial u}{\partial x}(\mathbf{B}(t))dB_1(t) - \frac{\partial v}{\partial x}(\mathbf{B}(t))dB_2(t) + i(\frac{\partial v}{\partial x}(\mathbf{B}(t))dB_1(t) + \frac{\partial u}{\partial x}(\mathbf{B}(t))dB_2(t)) \\
&= (\frac{\partial u}{\partial x}(\mathbf{B}(t)) + i\frac{\partial v}{\partial x}(\mathbf{B}(t)))dB_1(t) + (i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}(\mathbf{B}(t)))dB_2(t) \\
&= F'(\mathbf{B}(t))d\mathbf{B}(t).
\end{aligned}$$

□

(ii)

Proof. By result of (i), we have $de^{\alpha\mathbf{B}(t)} = \alpha e^{\alpha\mathbf{B}(t)}d\mathbf{B}(t)$. So $Z_t = e^{\alpha\mathbf{B}(t)} + Z_0$ solves the complex SDE $dZ_t = \alpha Z_t d\mathbf{B}(t)$. □

5.15.

Proof. The deterministic analog of this SDE is a Bernoulli equation $\frac{dy_t}{dt} = rKy_t - ry_t^2$. The correct substitution is to multiply $-y_t^{-2}$ on both sides and set $z_t = y_t^{-1}$. Then we'll have a linear equation $dz_t = -rKz_t + r$.

Similarly, we multiply $-X_t^{-2}$ on both sides of the SDE and set $Z_t = X_t^{-1}$. Then

$$-\frac{dX_t}{X_t^2} = -\frac{rKdt}{X_t} + rdt - \beta\frac{dB_t}{X_t}$$

and

$$dZ_t = -\frac{dX_t}{X_t^2} + \frac{dX_t\cdot dX_t}{X_t^3} = -rKZ_tdt + rdt - \beta Z_tdB_t + \frac{1}{X_t^3}\beta^2X_t^2dt = rdt - rKZ_tdt + \beta^2Z_tdt - \beta Z_tdB_t.$$

Define $Y_t = e^{(rK-\beta^2)t}Z_t$, then

$$dY_t = e^{(rK-\beta^2)t}(dZ_t + (rK-\beta^2)Z_tdt) = e^{(rK-\beta^2)t}(rdt - \beta Z_tdB_t) = re^{(rK-\beta^2)t}dt - \beta Y_tdB_t.$$

Now we imitate the solution of Exercise 5.6. Consider an integrating factor N_t, such that $dN_t = \theta_tdt + \gamma_tdB_t$ and

$$d(Y_tN_t) = N_tdY_t + Y_tdN_t + dN_t\cdot dY_t = N_tre^{(rK-\beta^2)t}dt - \beta N_tY_tdB_t + Y_t\theta_tdt + Y_t\gamma_tdB_t - \beta\gamma_tY_tdt.$$

Solve the equation

$$\begin{cases}\theta_t = \beta\gamma_t \\ \gamma_t = \beta N_t,\end{cases}$$

we get $dN_t = \beta^2N_tdt + \beta N_tdB_t$. So $N_t = N_0e^{\beta B_t + \frac{1}{2}\beta^2t}$ and

$$d(Y_tN_t) = N_tre^{(rK-\beta^2)t}dt = N_0re^{(rK-\frac{1}{2}\beta^2)t+\beta B_t}dt.$$

Choose $N_0 = 1$, we have $N_tY_t = Y_0 + \int_0^t re^{(rK-\frac{\beta^2}{2})s+\beta B_s}ds$ with $Y_0 = Z_0 = X_0^{-1}$. So

$$X_t = Z_t^{-1} = e^{(rK-\beta^2)t}Y_t^{-1} = \frac{e^{(rK-\beta^2)t}N_t}{Y_0 + \int_0^t re^{(rK-\frac{1}{2}\beta^2)s+\beta B_s}ds} = \frac{e^{(rK-\frac{1}{2}\beta^2)t+\beta B_t}}{x^{-1} + \int_0^t re^{(rK-\frac{1}{2}\beta^2)s+\beta B_s}ds}.$$

□

5.15. *(Another solution)*

Proof. We can also use the method in Exercise 5.16. Then $f(t,x) = rKx - rx^2$ and $c(t) \equiv \beta$. So $F_t = e^{-\beta B_t + \frac{1}{2}\beta^2 t}$ and Y_t satisfies

$$dY_t = F_t(rKF_t^{-1}Y_t - rF_t^{-2}Y_t^2)dt.$$

Divide $-Y_t^2$ on both sides, we have

$$-\frac{dY_t}{Y_t^2} = \left(-\frac{rK}{Y_t} + rF_t^{-1}\right)dt.$$

So $dY_t^{-1} = -Y_t^{-2}dY_t = (-rKY_t^{-1} + rF_t^{-1})dt$, and

$$d(e^{rKt}Y_t^{-1}) = e^{rKt}(rKY_t^{-1}dt + dY_t^{-1}) = e^{rKt}rF_t^{-1}dt.$$

Hence $e^{rKt}Y_t^{-1} = Y_0^{-1} + r\int_0^t e^{rKs}e^{\beta B_s - \frac{1}{2}\beta^2 s}ds$ and

$$X_t = F_t^{-1}Y_t = e^{\beta B_t - \frac{1}{2}\beta^2 t}\frac{e^{rKt}}{Y_0^{-1} + r\int_0^t e^{\beta B_s + (rK - \frac{1}{2}\beta^2)s}ds} = \frac{e^{(rK-\frac{1}{2}\beta^2)t + \beta B_t}}{x^{-1} + r\int_0^t e^{(rK-\frac{1}{2}\beta^2)s + \beta B_s}ds}.$$

□

5.16. (a) and (b)

Proof. Suppose F_t is a process satisfying the SDE $dF_t = \theta_t dt + \gamma_t dB_t$, then

$$\begin{aligned} d(F_tX_t) &= F_t(f(t,X_t)dt + c(t)X_t dB_t) + X_t\theta_t dt + X_t\gamma_t dB_t + c(t)\gamma_t X_t dt \\ &= (F_tf(t,X_t) + c(t)\gamma_t X_t + X_t\theta_t)dt + (c(t)F_tX_t + \gamma_t X_t)dB_t. \end{aligned}$$

Solve the equation

$$\begin{cases} c(t)\gamma_t + \theta_t = 0 \\ c(t)F_t + \gamma_t = 0, \end{cases}$$

we have

$$\begin{cases} \gamma_t = -c(t)F_t \\ \theta_t = c^2(t)F(t). \end{cases}$$

So $dF_t = c^2(t)F_tdt - c(t)F_tdB_t$. Hence $F_t = F_0 e^{\frac{1}{2}\int_0^t c^2(s)ds - \int_0^t c(s)dB_s}$. Choose $F_0 = 1$, we get desired integrating factor F_t and $d(F_tX_t) = F_tf(t,X_t)dt$. □

(c)

Proof. In this case, $f(t,x) = \frac{1}{x}$ and $c(t) \equiv \alpha$. So F_t satisfies $F_t = e^{-\alpha B_t + \frac{1}{2}\alpha^2 t}$ and Y_t satisfies $dY_t = F_t \cdot \frac{1}{F_t^{-1}Y_t}dt = F_t^2 Y_t^{-1}dt$. Since $dY_t^2 = 2Y_tdY_t + dY_t \cdot dY_t = 2F_t^2 dt = 2e^{-2\alpha B_t + \alpha^2 t}dt$, we have $Y_t^2 = 2\int_0^t e^{-2\alpha B_s + \alpha^2 s}ds + Y_0^2$, where $Y_0 = F_0X_0 = X_0 = x$. So

$$X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}\sqrt{x^2 + 2\int_0^t e^{-2\alpha B_s + \alpha^2 s}ds}.$$

□

(d)

Proof. $f(t,x) = x^\gamma$ and $c(t) \equiv \alpha$. So $F_t = e^{-\alpha B_t + \frac{1}{2}\alpha^2 t}$ and Y_t satisfies the SDE

$$dY_t = F_t(F_t^{-1}Y_t)^\gamma dt = F_t^{1-\gamma}Y_t^\gamma dt.$$

Note $dY_t^{1-\gamma} = (1-\gamma)Y_t^{-\gamma}dY_t = (1-\gamma)F_t^{1-\gamma}dt$, we conclude $Y_t^{1-\gamma} = Y_0^{1-\gamma} + (1-\gamma)\int_0^t F_s^{1-\gamma}ds$ with $Y_0 = F_0X_0 = X_0 = x$. So

$$Y_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}(x^{1-\gamma} + (1-\gamma)\int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)}{2}s}ds)^{\frac{1}{1-\gamma}}.$$

□

5.17.

Proof. Assume $A \neq 0$ and define $\omega(t) = \int_0^t v(s)ds$, then $\omega'(t) \leq C + A\omega(t)$ and

$$\frac{d}{dt}(e^{-At}\omega(t)) = e^{-At}(\omega'(t) - A\omega(t)) \leq Ce^{-At}.$$

So $e^{-At}\omega(t) - \omega(0) \leq \frac{C}{A}(1-e^{-At})$, i.e. $\omega(t) \leq \frac{C}{A}(e^{At}-1)$. So $v(t) = \omega'(t) \leq C + A \cdot \frac{C}{A}(e^{At}-1) = Ce^{At}$. □

5.18. (a)

Proof. Let $Y_t = \log X_t$, then

$$dY_t = \frac{dX_t}{X_t} - \frac{(dX_t)^2}{2X_t^2} = \kappa(\alpha - Y_t)dt + \sigma dB_t - \frac{\sigma^2 X_t^2 dt}{2X_t^2} = (\kappa\alpha - \frac{1}{2}\alpha^2)dt - \kappa Y_t dt + \sigma dB_t.$$

So

$$d(e^{\kappa t}Y_t) = \kappa Y_t e^{\kappa t}dt + e^{\kappa t}dY_t = e^{\kappa t}[(\kappa\alpha - \frac{1}{2}\sigma^2)dt + \sigma dB_t]$$

and $e^{\kappa t}Y_t - Y_0 = (\kappa\alpha - \frac{1}{2}\sigma^2)\frac{e^{\kappa t}-1}{\kappa} + \sigma\int_0^t e^{\kappa s}dB_s$. Therefore

$$X_t = \exp\{e^{-\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1-e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}dB_s\}.$$

□

(b)

Proof. $E[X_t] = \exp\{e^{-\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1-e^{-\kappa t})\}E[\exp\{\sigma e^{-\kappa t}\int_0^t e^{\kappa s}dB_s\}]$. Note $\int_0^t e^{\kappa s}dB_s \sim N(0, \frac{e^{2\kappa t}-1}{2\kappa})$, so

$$E[\exp\{\sigma e^{-\kappa t}\int_0^t e^{\kappa s}dB_s\}] = \exp\left\{\frac{1}{2}\sigma^2 e^{-2\kappa t}\frac{e^{2\kappa t}-1}{2\kappa}\right\} = \exp\left\{\frac{\sigma^2(1-e^{-2\kappa t})}{4\kappa}\right\}.$$

□

5.19.

Proof. We follow the hint.

$$\begin{aligned}
& P\left[\int_0^T \left|b(s,Y_s^{(K)})-b(s,Y_s^{(K-1)})\right| ds > 2^{-K-1}\right] \\
\leq\ & P\left[\int_0^T D\left|Y_s^{(K)}-Y_s^{(K-1)}\right| ds > 2^{-K-1}\right] \\
\leq\ & 2^{2K+2}E\left[\left(\int_0^T D\left|Y_s^{(K)}-Y_s^{(K-1)}\right| ds\right)^2\right] \\
\leq\ & 2^{2K+2}E\left[D^2\int_0^T \left|Y_s^{(K)}-Y_s^{(K-1)}\right|^2 dsT\right] \\
\leq\ & 2^{2K+2}D^2TE\left[\int_0^T \left|Y_s^{(K)}-Y_s^{(K-1)}\right|^2 ds\right] \\
\leq\ & D^2T2^{2K+2}\int_0^T \frac{A_2^K t^K}{K!}ds \\
=\ & \frac{D^2T2^{2K+2}A_2^K}{(K+1)!}T^{K+1}.
\end{aligned}$$

$$\begin{aligned}
& P\left[\sup_{0\leq t\leq T}\left|\int_0^t \left(\sigma(s,Y_s^{(K)})-\sigma(s,Y_s^{(K-1)})\right)dB_s\right| > 2^{-K-1}\right] \\
\leq\ & 2^{2K+2}E\left[\left|\int_0^t \left(\sigma(s,Y_s^{(K)})-\sigma(s,Y_s^{(K-1)})\right)dB_s\right|^2\right] \\
\leq\ & 2^{2K+2}E\left[\int_0^t \left(\sigma(s,Y_s^{(K)})-\sigma(s,Y_s^{(K-1)})\right)^2 ds\right] \\
\leq\ & 2^{2K+2}E\left[\int_0^t D^2|Y_s^{(K)}-Y_s^{(K-1)}|^2 ds\right] \\
\leq\ & 2^{2K+2}D^2\int_0^T \frac{A_2^K t^K}{K!}dt \\
=\ & \frac{2^{2K+2}D^2A_2^K}{(K+1)!}T^{K+1}.
\end{aligned}$$

So

$$P[\sup_{0\leq t\leq T}|Y_t^{(K+1)}-Y_t^{(K)}| > 2^{-K}] \leq D^2T\frac{2^{2K+2}A_2^K}{(K+1)!}T^{K+1}+D^2\frac{2^{2K+2}A_2^K}{(K+1)!}T^{K+1} \leq \frac{(A_3T)^{K+1}}{(K+1)!},$$

where $A_3 = 4(A_2+1)(D^2+1)(T+1)$. □

6 The Filtering Problem

7 Diffusions: Basic Properties

7.2. *Remark:* When an Itô diffusion is explicitly given, it's usually straightforward to find its infinitesimal generator, by Theorem 7.3.3. The converse is not so trivial, as we're faced with double difficulties: first, the desired n-dimensional Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ involves an m-dimensional BM B_t, where m is unknown *a priori*; second, even if m can be determined, we only know $\sigma\sigma^T$, which is the product of an $n \times m$ and an $m \times n$ matrix. In general, it's hard to find σ according to $\sigma\sigma^T$. This suggests maybe there's more than one diffusion that has the given generator. Indeed, when restricted to $C_0^2(\mathbb{R}_+)$, BM, BM killed at 0 and reflected BM all have Laplacian operator as generator. What differentiate them is the *domain* of generators: domain is part of the definition of a generator!

With the above theoretical background, it should be OK if we find more than one Itô diffusion process with given generator. A basic way to find an Itô diffusion with given generator can be trial-and-error. To tackle the first problem, we try $m = 1$, $m = 2$, $\cdots$. To tackle the second problem, note $\sigma\sigma^T$ is symmetric, so we can write $\sigma\sigma^T$ as AMA^T where M is the diagonalization of $\sigma\sigma^T$, and then set $\sigma = AM^{1/2}$. In general, to deal directly with $\sigma^T\sigma$ instead of σ, we should use the martingale problem approach of Stoock and Varadhan. See the preface of their classical book for details.

a)

Proof. $dX_t = dt + \sqrt{2}dB_t$. □

b)

Proof.

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ cX_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha X_2(t) \end{pmatrix} dB_t.$$

□

c)

Proof. $\sigma\sigma^T = \begin{pmatrix} 1 + x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix}$. If

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 2X_2(t) \\ \log(1 + X_1^2(t) + X_2^2(t)) \end{pmatrix} dt + \begin{pmatrix} a \\ b \end{pmatrix} dB_t,$$

then $\sigma\sigma^T$ has the form $\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$, which is impossible since $x_1^2 \neq (1 + x_1^2) \cdot 1$. So we try 2-dim. BM as the driving process. Linear algebra yields $\sigma\sigma^T = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}$. So we can choose

$$dX_t = \begin{pmatrix} 2X_2(t) \\ \log(1 + X_1^2(t) + X_2^2(t)) \end{pmatrix} dt + \begin{pmatrix} 1 & X_1(t) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dB_t(t) \\ dB_2(t) \end{pmatrix}.$$

□

7.3.

Proof. Set $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ and $\mathcal{F}_t^B = \sigma(B_s : s \leq t)$. Since $\sigma(X_t) = \sigma(B_t)$, we have, for any bounded Borel function $f(x)$,

$$E[f(X_{t+s})|\mathcal{F}_t^X] = E[f(xe^{c(t+s)+\alpha B_{t+s}})|\mathcal{F}_t^B] = E^{B_t}[f(xe^{c(t+s)+\alpha B_s})] \in \sigma(B_t) = \sigma(X_t).$$

So $E[f(X_{t+s})|\mathcal{F}_t^X] = E[f(X_{t+s})|X_t]$.

□

7.4. a)

Proof. Choose $b \in \mathbb{R}_+$, so that $0 < x < b$. Define $\tau_0 = \inf\{t > 0 : B_t = 0\}$, $\tau_b = \inf\{t > 0 : B_t = b\}$ and $\tau_{0b} = \tau_0 \wedge \tau_b$. Clearly, $\lim_{b\to\infty} \tau_b = \infty$ a.s. by the continuity of Brownian motion. Consequently, $\{\tau_0 < \tau_b\} \uparrow \{\tau_0 < \infty\}$ as $b \uparrow \infty$. Note $(B_t^2 - t)_{t\geq 0}$ is a martingale, by Doob's optional stopping theorem, we have $E^x[B_{t\wedge\tau_{0b}}^2] = E^x[t \wedge \tau_{0b}]$. Apply bounded convergence theorem to the LHS and monotone convergence theorem to the RHS, we get $E^x[\tau_{0b}] = E^x[B^2_{\tau_{0b}}] < \infty$. In particular, $\tau_{0b} < \infty$ a.s. Moreover, by considering the martingale $(B_t)_{t\geq 0}$ and similar argument, we have $E^x[B_{\tau_{0b}}] = E^x[B_0] = x$. This leads to the equation

$$\begin{cases} P^x(\tau_0 < \tau_b)\cdot 0 + P^x(\tau_0 > \tau_b)\cdot b = x \\ P^x(\tau_0 < \tau_b) + P^x(\tau_0 > \tau_b) = 1. \end{cases}$$

Solving it gives $P^x(\tau_0 < \tau_b) = 1 - \frac{x}{b}$. So $P^x(\tau_0 < \infty) = \lim_{b\to\infty} P^x(\tau_0 < \tau_b) = 1$.

□

b)

Proof. $E^x[\tau] = \lim_{b\to\infty} E^x[\tau_{0b}] = \lim_{b\to\infty} E^x[B^2_{\tau_{0b}}] = \lim_{b\to\infty} b^2 \cdot \frac{x}{b} = \infty$. □

Remark: (1) Another *easy* proof is based on the following result, which can be proved independently and via elementary method: let $W = (W_t)_{t\geq 0}$ be a Wiener process, and T be a stopping time such that $E[T] < \infty$. Then $E[W_T] = 0$ and $E[W_T^2] = E[T]$ ([6]).

(2) The solution in the book is not quite right, since Dynkin's formula assumes $E^x[\tau_K] < \infty$, which needs proof in this problem.

7.5.

Proof. The hint is detailed enough. But if we want to be really rigorous, note Theorem 7.4.1. (Dynkin's formula) studies Itô diffusions, not Itô processes, to which standard form semi-group theory (in particular, the notion of generator) doesn't apply. So we start from scratch, and re-deduce Dynkin's formula for Itô processes.

First of all, we note $b(t,x)$, $\sigma(t,x)$ are bounded in a bounded domain of x, uniformly in t. This suffices to give us martingales, not just local martingales. Indeed, Itô's formula says

$$\begin{aligned} & |X(t)|^2 \\ = \; & |X(0)|^2 + \int_0^t \sum_i 2X_i(s)dX_i(s) + \int_0^t \sum_i \langle dX_i(s)\rangle \\ = \; & |X(0)|^2 + 2\sum_i \int_0^t X_i(s)b_i(s,X(s))ds + 2\sum_{ij}\int_0^t X_i(s)\sigma_{ij}(s,X(s))dB_j(s) + \sum_i \int_0^t \sigma_{ii}^2(s,X_s)ds. \end{aligned}$$

Let $\tau = t\wedge\tau_R$ where $\tau_R = \inf\{t>0 : |X_t| \geq R\}$. Then by previous remark on the boundedness of σ and b, $\int_0^{t\wedge\tau_R} X_i(s)\sigma_{ij}(s,X(s))dB_j(s)$ is a martingale. Take expectation, we get

$$\begin{aligned} & E[|X(\tau)|^2] \\ = \; & E[|X(0)|^2] + 2\sum_i E[\int_0^\tau X_i(s)b_i(s,X(s))ds] + \sum_i \int_0^t E[\sigma_{ii}^2(s,X(s))]ds \\ \leq \; & E[|X(0)|^2] + 2C\sum_i E[\int_0^\tau |X_i(s)|(1+|X(s)|)ds] + \int_0^t C^2 E[(1+|X(s)|)^2]ds. \end{aligned}$$

Let $R\to\infty$ and use Fatou's Lemma, we have

$$\begin{aligned} & E[|X(t)|^2] \\ \leq \; & E[|X(0)|^2] + 2C\sum_i E[\int_0^t |X_i(s)|(1+|X(s)|)ds] + C^2\int_0^t E[(1+|X(s)|)^2]ds \\ \leq \; & E[|X(0)|^2] + K\int_0^t (1+E[|X(s)|^2])ds, \end{aligned}$$

for some K dependent on C only. To apply Gronwall's inequality, note for $v(t) = 1 + E[|X(t)|^2]$, we have $v(t) \leq v(0) + K\int_0^t v(s)ds$. So $v(t) \leq v(0)e^{Kt}$, which is the desired inequality.

Remark: Compared with Exercise 5.10, the power of this problem's method comes from application of Itô formula, or more precisely, martingale theory, while Exercise 5.10 only resorts to Hölder inequality. □

7.7. a)

Proof. Let U be an orthogonal matrix, then $B' = U \cdot B$ is again a Brownian motion. For any $G \in \partial D$, $\mu_D^X(G) = P^x(B_{\tau_D} \in G) = P^x(U \cdot B_{\tau_D} \in U \cdot G) = P^x(B'_{\tau_D} \in U \cdot G) = \mu_D^x(U \cdot G)$. So μ_D^x is rotation invariant. □

b)

Proof.

$$
\begin{aligned}
u(x) &= E^x[\phi(B_{\tau_W})] = E^x[E^x[\phi(B_{\tau_W})|B_{\tau_D}]] = E^x[E^x[\phi(B_{\tau_W} \circ \theta_{\tau_D})|B_{\tau_D}]] \\
&= E^x[E^{B_{\tau_D}}[\phi(B_{\tau_W}]] = E^x[u(B_{\tau_D})] = \int_{\partial D} u(y)\mu_D^x(dy) = \int_{\partial D} u(y)\sigma(dy).
\end{aligned}
$$

□

c)

Proof. See, for example, Evans: *Partial Differential Equations*, page 26. □

7.8. a)

Proof. $\{\tau_1 \wedge \tau_2 \leq t\} = \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{N}_t$. And since $\{\tau_i \geq t\} = \{\tau_i < t\}^c \in \mathcal{N}_t$, $\{\tau_1 \vee \tau_2 \geq t\} = \{\tau_1 \geq t\} \cup \{\tau_2 \geq t\} \in \mathcal{N}_t$. □

b)

Proof. $\{\tau < t\} = \cup_n \{\tau_n < t\} \in \mathcal{N}_t$. □

c)

Proof. By b) and the hint, it suffices to show for any open set G, $\tau_G = \inf\{t > 0 : X_t \notin G\}$ is an $\mathcal{M}_t$-stopping time. This is Example 7.2.2. □

► 7.9. Let X_t be a geometric Brownian motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t,\ X_0 = x > 0$$

where $B_t \in \mathbb{R}$; r, α are constants.
a) find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$; $x > 0$, γ cnstant.

Solution. By Theorem 7.3.3, A restricted to $C_0^2(\mathbb{R})$ is $rx\frac{d}{dx} + \frac{\alpha^2 x^2}{2}\frac{d^2}{dx^2}$. For $f(x) = x^\gamma$, Af can be calculated by definition. Indeed, $X_t = xe^{(r-\frac{\alpha^2}{2})t + \alpha B_t}$, and $E^x[f(X_t)] = x^\gamma e^{(r-\frac{\alpha^2}{2}+\frac{\alpha^2\gamma}{2})\gamma t}$. So

$$\lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} = \left[r\gamma + \frac{\alpha^2}{2}\gamma(\gamma - 1)\right] x^\gamma$$

So $f \in D_A$ and $Af(x) = \left[r\gamma + \frac{\alpha^2}{2}\gamma(\gamma - 1)\right] x^\gamma$. □

b) If $r < \frac{1}{2}\alpha^2$ then $X_t \to 0$ as $t \to \infty$, a.s. Q^x (Example 5.5.1). But what is the probability p that X_t, when starting from $x < R$, ever hits the value R? Use Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R}\right)^{\gamma_1}$$

Proof. We choose ρ such that $0 < \rho < x < R$. We choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f$ on (ρ, R). Define $\tau_{(\rho,R)} = \inf\{t > 0 : X_t \notin (\rho, R)\}$. Then by Dynkin's formula, and the fact $Af_0(x) = Af(x) = \gamma_1 x^{\gamma_1}\left[r + \frac{\alpha^2}{2}(\gamma_1 - 1)\right] = 0$ on (ρ, R), we get

$$E^x[f_0(X_{\tau_{(\rho,R)}\wedge k})] = f_0(x)$$

As far as $r \neq \frac{\alpha^2}{2}$, X_t will exit the interval (ρ, R) a.s. (Example 5.1.1) so that $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, by bounded convergence theorem and the fact $\tau_{(\rho,R)} < \infty$, we conclude

$$f_0(\rho)[1 - p(\rho)] + f_0(R)p(\rho) = f_0(x)$$

where $p(\rho) = P^x\{X_t \text{ exits } (\rho, R) \text{ by hitting R first}\}$. Then

$$p(\rho) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

Note under the condition $r < \frac{1}{2}\alpha^2$, we have $\gamma_1 > 0$. So by letting $\rho \downarrow 0$, we can get the desired result. □

c) If $r > \frac{1}{2}\alpha^2$ then $X_t \to \infty$ as $t \to \infty$, a.s. Q^x. Put

$$\tau = \inf\{t > 0; X_t \geq R\}.$$

Use Dynkin's formula with $f(x) = \ln x$, $x > 0$ to prove that

$$E^x[\tau] = \frac{\ln \frac{R}{x}}{r - \frac{1}{2}\alpha^2}.$$

(Hint: First consider exit times from (ρ, R), $\rho > 0$ and then let $\rho \to 0$. You need estimates for

$$(1 - p(\rho)) \ln \rho,$$

where

$$p(\rho) = Q^x[X_t \text{ reaches the value } R \text{ before } \rho],$$

which you can get from the calculations in a), b).)

Proof. We consider $\rho > 0$ such that $\rho < x < R$. Let $\tau_{(\rho,R)}$ denote the first exit time of X from (ρ, R). Choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f = \ln x$ on (ρ, R) and apply Dynkin's formula to $f_0(x)$, we have $Af_0(x) = Af(x) = r - \frac{\alpha^2}{2}$ for $x \in (\rho, R)$ and

$$E^x[f_0(X_{\tau_{(\rho,R)}\wedge k})] = f_0(x) + (r - \frac{\alpha^2}{2})E^x[\tau_{(\rho,R)} \wedge k]$$

As far as $r \neq \frac{\alpha^2}{2}$, X_t will exit the interval (ρ, R) a.s. (Example 5.1.1) so that $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, we get

$$E^x[\tau_{(\rho,R)}] = \frac{f_0(R)p(\rho) + f_0(\rho)[1 - p(\rho)] - f_0(x)}{r - \frac{\alpha^2}{2}}$$

where $p(\rho) = P^x(X_t \text{ exits } (\rho, R) \text{ by hitting R first})$. To get the desired formula, we only need to show $\lim_{\rho\to 0} p(\rho) = 1$ and $\lim_{\rho\to 0}[1 - p(\rho)] \ln \rho = 0$.

Indeed, recall as far as $r \neq \frac{\alpha^2}{2}$, we have obtained in b)

$$p(\rho) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

Under the condition $r > \frac{1}{2}\alpha^2$, we have $\gamma_1 = 1 - \frac{2r}{\alpha^2} < 0$ and

$$\lim_{\rho\to 0} p(\rho) = \frac{\rho^{-\gamma_1}x^{\gamma_1} - 1}{\rho^{-\gamma_1}R^{\gamma_1} - 1} = 0.$$

Furthermore, we have

$$\lim_{\rho\to 0}[1-p(\rho)]\ln\rho = \lim_{\rho\to 0}\frac{R^{\gamma_1}-x^{\gamma_1}}{R^{\gamma_1}-\rho^{\gamma_1}}\ln\rho = \lim_{\rho\to 0}\frac{R^{\gamma_1}-x^{\gamma_1}}{\rho^{-\gamma_1}R^{\gamma_1}-1}\cdot\rho^{-\gamma_1}\ln\rho = 0.$$

□

7.10. a)

Proof. $E^x[X_T|\mathcal{F}_t] = E^{X_t}[X_{T-t}]$. By Exercise 5.10. or 7.5., $\int_0^t X_s dB_s$ is a martingale. So $E^x[X_t] = x + r\int_0^t E^x[X_s]ds$. Set $E^x[X_t] = v(t)$, we get $v(t) = x + r\int_0^t v(s)ds$ or equivalently, the initial value problem $\begin{cases} v'(t) = rv(t) \\ v(0) = x \end{cases}$. So $v(t) = xe^{rt}$. Hence $E^x[X_T|\mathcal{F}_t] = X_t e^{r(T-t)}$. □

b)

Proof. Since M_t is a martingale, $E^x[X_T|\mathcal{F}_t] = xe^{rT}E^x[M_T|\mathcal{F}_t] = xe^{rT}M_t = X_t e^{r(T-t)}$. □

► 7.11. Let X_t be an Itô diffusion in $\mathbf{R}^n$ and let $f : \mathbf{R}^n \to \mathbf{R}$ be a function such that

$$E^x\left[\int_0^\infty |f(X_t)|dt\right] < \infty \text{ for all } x \in \mathbf{R}^n.$$

Let τ be a stopping time. Use the strong Markov property to prove that

$$E^x\left[\int_\tau^\infty f(X_t)dt\right] = E^x[g(X_\tau)],$$

where

$$g(y) = E^y\left[\int_0^\infty f(X_t)dt\right].$$

Proof. By the change-of-variable formula and the definition of shift operator, we have

$$\int_\tau^\infty f(X_t)dt = \int_0^\infty f(X_{\tau+t})dt = \int_0^\infty f(X_t\circ\theta_\tau)dt.$$

By the general version of strong Markov property (equation (7.2.5) of Øksendal [7, page 120], with $\eta = \int_0^\infty f(X_t)dt$), we have

$$E^x\left[\int_0^\infty f(X_t\circ\theta_\tau)dt\middle|\mathcal{F}_\tau\right] = E^{X_\tau}\left[\int_0^\infty f(X_t)dt\right] = g(X_\tau),$$

where $g(y) = E^y\left[\int_0^\infty f(X_t)dt\right]$. Combined, we have

$$E^x\left[\int_\tau^\infty f(X_t)dt\right] = E^x\left[\int_0^\infty f(X_t)\circ\theta_\tau dt\right] = E^x\left[E^x\left[\int_0^\infty f(X_t)\circ\theta_\tau dt\middle|\mathcal{F}_\tau\right]\right] = E^x[g(X_\tau)].$$

□

7.12. a)

Proof. For any t, s with $0 \le s < t \le T$ and τ_K, we have $E[Z_{t\wedge\tau_K}|\mathcal{F}_s] = Z_{s\wedge\tau_K}$. Let $K\to\infty$, then $Z_{s\wedge\tau_K}\to Z_s$ a.s. and $Z_{t\wedge\tau_K}\to Z_t$ a.s. Since $(Z_\tau)_{\tau\le T}$ is uniformly integrable, $Z_{s\wedge\tau_K}\to Z_s$ and $Z_{t\wedge\tau_K}\to Z_t$ in L^1 as well. So $E[Z_t|\mathcal{F}_s] = \lim_{K\to\infty}E[Z_{t\wedge\tau_K}|\mathcal{F}_s] = \lim_{K\to\infty}Z_{s\wedge\tau_K} = Z_s$. Hence $(Z_t)_{t\le T}$ is a martingale. □

b)

Proof. The given condition implies $(Z_\tau)_{\tau\le T}$ is uniformly integrable. □

c)

Proof. Without loss of generality, we assume $Z \geq 0$. Then by Fatou's lemma, for $t > s \geq 0$,

$$E[Z_t|\mathcal{F}_s] \leq \lim_{k\to\infty} E[Z_{t\wedge\tau_k}|\mathcal{F}_s] = \lim_{k\to\infty} Z_{s\wedge\tau_k} = Z_s.$$

□

d)

Proof. Define $\tau_k = \inf\{t > 0 : \int_0^t \phi^2(s,\omega)ds \geq k\}$, then

$$Z_{t\wedge\tau_k} = \int_0^{t\wedge\tau_k} \phi(s,\omega)dB_s = \int_0^t \phi(s,\omega)1_{\{s\leq\tau_k\}}dB_s$$

is a martingale, since $E[\int_0^T \phi^2(s,\omega)1_{\{s\leq\tau_k\}}ds] = E[\int_0^{T\wedge\tau_k} \phi^2(s,\omega)ds] \leq k$. □

7.13. a)

Proof. Take $f \in C_0^2(\mathbb{R}_+^2)$ so that $f(x) = \ln|x|$ on $\{x : \epsilon \leq |x| \leq R\}$. Then

$$\begin{aligned}
df(B(t)) &= \sum_{i=1}^2 \frac{B_i(t)}{|B(t)|^2}dB_i(t) + \frac{1}{2}\frac{B_2^2(t)-B_1^2(t)}{|B(t)|^4}dt + \frac{1}{2}\frac{B_1^2(t)-B_2^2(t)}{|B(t)|^4}dt \\
&= \sum_{i=1}^2 \frac{B_i(t)}{|B(t)|^2}dB_i(t) \\
&= \frac{B(t)\cdot dB(t)}{|B(t)|^2}.
\end{aligned}$$

Since $\frac{B(t)}{|B(t)|^2}1_{\{t\leq\tau\}} \in \mathcal{V}(0,T)$, we conclude $f(B(t\wedge\tau)) = \ln|B(t\wedge\tau)|$ is a martingale. To show $\ln|B(t)|$ is a local martingale, it suffices to show $\tau \to \infty$ as $\epsilon \downarrow 0$ and $R \uparrow \infty$. Indeed, by optional stopping theorem, $\ln|x| = E^x[\ln|B(t\wedge\tau)|] = P^x(\tau_\epsilon < \tau_R)\ln\epsilon + P^x(\tau_\epsilon > \tau_R)\ln R$, where $\tau_\epsilon = \inf\{t > 0 : |B(t)| \leq \epsilon\}$ and $\tau_R = \inf\{t > 0 : |B(t)| \geq R\}$. So $P^x(\tau_\epsilon < \tau_R) = \frac{\ln R - \ln|x|}{\ln R - \ln\epsilon}$. By continuity of B, $\lim_{R\to\infty}\tau_R = \infty$. If we define $\tau_0 = \inf\{t > 0 : |B(t)| = 0\}$, then $\tau_0 = \lim_{\epsilon\downarrow 0}\tau_\epsilon$. So $P^x(\tau_0 < \infty) = \lim_{R\uparrow\infty} P^x(\tau_0 < \tau_R) = \lim_{R\uparrow\infty}\lim_{\epsilon\downarrow 0} P^x(\tau_\epsilon < \tau_R) = 0$. This shows $\lim_{\epsilon\downarrow 0}\tau_\epsilon = \tau_0 = \infty$ a.s. □

b)

Proof. Similar to part a). □

Remark: Note neither example is a martingale, as they don't have finite expectation.

7.14. a)

Proof. According to Theorem 7.3.3, for any $f \in C_0^2$,

$$\mathcal{A}f(x) = \sum_i \frac{1}{h(x)}\frac{\partial h(x)}{\partial x_i}\frac{\partial f(x)}{\partial x_i} + \frac{1}{2}\Delta f(x) = \frac{2\bigtriangledown h\cdot\bigtriangledown f + h\Delta f}{2h} = \frac{\Delta(hf)}{2h},$$

where the last equation is due to the harmonicity of h. □

7.15.

Proof. If we assume formula (7.5.5), then (7.5.6) is straightforward from Markov property. As another solution, we derive (7.5.6) directly.

We define $M_t = E^x[F|\mathcal{F}_t]$ $(t \le T)$, then $M_t = E[F] + \int_0^t \phi(s)dB_s$. Set $f(z,u) = E^z[(B_u - K)^+]$, then $M_t = E^x[(B_T - K)^+|\mathcal{F}_t] = E^{B_t}[(B_{T-t} - K)^+] = f(B_t, T-t)$. By Itô's formula,

$$dM_t = f_z'(B_t, T-t)dB_t + f_u'(B_t, T-t)(-dt) + \frac{1}{2}f_{zz}''(B_t, T-t)dt.$$

So $\phi(t,\omega) = f_z'(B_t, T-t)$. Note by elementary calculus,

$$f(z,u) = \int_{-\infty}^{\infty} (z + x - K)^+ \frac{e^{-x^2/2u}}{\sqrt{2\pi u}} dx = \sqrt{u}N'(\frac{K-z}{\sqrt{u}}) - (K-z) + (K-z)N(\frac{K-z}{\sqrt{u}}),$$

where $N(\cdot)$ is the distribution function of standard normal random variable. So it's easy to see $f_z'(z,u) = 1 - N(\frac{K-z}{\sqrt{u}})$. Hence $\phi(t,\omega) = 1 - N(\frac{K-B_t}{\sqrt{T-t}}) = \frac{1}{\sqrt{2\pi(T-t)}}\int_K^\infty e^{-\frac{(x-B_t)^2}{2(T-t)}}dx$. □

7.17.

Proof. If $t \le \tau$, then Y clearly satisfies the integral equation corresponding to (7.5.8), since

$$Y_t = X_t = X_0 + \int_0^t \frac{1}{3}X_s^{\frac{1}{3}}ds + \int_0^t X_s^{\frac{2}{3}}dB_s = Y_0 + \int_0^t \frac{1}{3}Y_s^{\frac{1}{3}}ds + \int_0^t Y_s^{\frac{2}{3}}dB_s.$$

If $t > \tau$, then $Y_t = 0 = X_\tau = \int_0^\tau \frac{1}{3}X_s^{\frac{1}{3}}ds + \int_0^\tau X_s^{\frac{2}{3}}dB_s + X_0 = Y_0 + \int_0^\tau \frac{1}{3}Y_s^{\frac{1}{3}}ds + \int_0^\tau X_s^{\frac{2}{3}}dB_s = Y_0 + \int_0^t \frac{1}{3}Y_s^{\frac{1}{3}}ds + \int_0^t Y_s^{\frac{2}{3}}dB_s$. So Y is also a strong solution of (7.5.8).

If we write (7.5.8) in the form of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, then $b(x) = \frac{1}{3}x^{\frac{1}{3}}$ and $\sigma(x) = x^{\frac{2}{3}}$. Neither of them satisfies the Lipschiz condition (5.2.2). So this does not conflict with Theorem 5.2.1. □

7.18. a)

Proof. The line of reasoning is exactly what we have done for 7.9 b). Just replace x^γ with a general function $f(x)$ satisfying certain conditions. □

b)

Proof. The characteristic operator $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$ and $f(x) = x$ are such that $\mathcal{A}f(x) = 0$. By formula (7.5.10), we are done. □

c)

Proof. $\mathcal{A} = \mu\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}$. So we can choose $f(x) = e^{-\frac{2\mu}{\sigma^2}x}$. Therefore

$$p = \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu b}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}$$

□

7.19. a)

Proof. Following the hint, and by Doob's optional sampling thoerem, $E^x[e^{-\sqrt{2\lambda}B_{t\wedge\tau} - \lambda t\wedge\tau}] = E^x[M_{t\wedge\tau}] = E^x[M_0] = e^{-\sqrt{2\lambda}x}$. Let $t \uparrow \infty$ and apply bounded convergence theorem, we get $E^x[e^{-\lambda\tau}] = e^{-\sqrt{2\lambda}x}$. □

b)

Proof. Define $g(x) = \int_0^\infty e^{-\lambda t}\frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}dt$, and we can show $g(x)$ satisfies the ODE $g''(x) = 2\lambda g(x)$. For details, see [11], Problem 3.9. □

8 Other Topics in Diffusion Theory

8.1. a)

Proof. $g(t,x) = E^x[\phi(B_t)]$, where B is a Brownian motion. □

b)

Proof. Note the equation to be solved has the form $(\alpha - \mathcal{A})u = \psi$ with $\mathcal{A} = \frac{1}{2}\Delta$, so we should apply Theorem 8.1.5. More precisely, since $\psi \in C_b(\mathbb{R}^n)$, by Theorem 8.1.5. b), we know $(\alpha - \frac{1}{2}\Delta)R_\alpha\psi = \psi$, where R_α is the α-resolvent corresponding to Brownian motion. So $R_\alpha\psi(x) = E^x[\int_0^\infty e^{-\alpha t}\psi(B_t)dt]$ is a bounded solution of the equation $(\alpha - \frac{1}{2}\Delta)u = \psi$ in $\mathbb{R}^n$. To see the uniqueness, it suffices to show $(\alpha - \frac{1}{2}\Delta)u = 0$ has only zero solution. Indeed, if $u \not\equiv 0$, we can find $u_n \in C_0^2(\mathbb{R}^n)$ such that $u_n = u$ in $B(0,n)$. Then $(\alpha - \frac{1}{2}\Delta)u_n = 0$ in $B(0,n)$. Applying Theorem 8.1.5.a), $u_n = R_\alpha(\alpha - \frac{1}{2}\Delta)u_n = 0$. So $u \equiv 0$ in $B(0,n)$. Let $n \uparrow \infty$, we are done. □

8.2.

Proof. By Kolmogorov's backward equation (Theorem 8.1.1), it suffices to solve the SDE $dX_t = \alpha X_t dt + \beta X_t dB_t$. This is the geometric Brownian motion $X_t = X_0 e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$. Then

$$u(t,x) = E^x[f(X_t)] = \int_{-\infty}^{\infty} f(xe^{(\alpha - \frac{\beta^2}{2})t + \beta y})\frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}dy.$$

□

8.3.

Proof. By (8.6.34) and Dynkin's formula, we have

$$\begin{aligned} E^x[f(X_t)] &= \int_{\mathbb{R}^n} f(y)p_t(x,y)dy \\ &= f(x) + E^x[\int_0^t \mathcal{A}f(X_s)ds] \\ &= f(x) + \int_0^t P_s\mathcal{A}f(x)ds \\ &= f(x) + \int_0^t \int_{\mathbb{R}^n} p_s(x,y)\mathcal{A}_y f(y)dyds. \end{aligned}$$

Differentiate w.r.t. t, we have

$$\int_{\mathbb{R}^n} f(y)\frac{\partial p_t(x,y)}{\partial t}dy = \int_{\mathbb{R}^n} p_t(x,y)\mathcal{A}_y f(y)dy = \int_{\mathbb{R}^n} \mathcal{A}_y^* p_t(x,y)f(y)dy,$$

where the second equality comes from integration by parts. Since f is arbitrary, we must have $\frac{\partial p_t(x,y)}{\partial t} = \mathcal{A}_y^* p_t(x,y)$. □

8.4.

Proof. The expected total length of time that $B.$ stays in F is

$$T = E[\int_0^\infty 1_F(B_t)dt] = \int_0^\infty \int_F \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dxdt.$$

(Sufficiency) If $m(F) = 0$, then $\int_F \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx = 0$ for every $t > 0$, hence $T = 0$.

(Necessity) If $T = 0$, then for a.s. t, $\int_F \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx = 0$. For such a $t > 0$, since $e^{-\frac{x^2}{2t}} > 0$ everywhere in $\mathbb{R}^n$, we must have $m(F) = 0$. □

8.5.

Proof. Apply the Feynman-Kac formula, we have

$$u(t,x) = E^x[e^{\int_0^t \rho ds} f(B_t)] = e^{\rho t}(2\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2t}} f(y)dy.$$

□

8.6.

Proof. The major difficulty is to make legitimate using Feynman-Kac formula while $(x-K)^+ \notin C_0^2$. For the conditions under which we can indeed apply Feynman-Kac formula to $(x-K)^+ \notin C_0^2$, c f. the book of Karatzas & Shreve, page 366. □

8.7.

Proof. Let $\alpha_t = \inf\{s > 0 : \beta_s > t\}$, then X_{α_t} is a Brownian motion. Since $\beta.$ is continuous and $\lim_{t\to\infty} \beta_t = \infty$ a.s., by the law of iterated logarithm for Brownian motion, we have

$$\limsup_{t\to\infty} \frac{X_{\alpha_{\beta_t}}}{\sqrt{2\beta_t \log\log \beta_t}} = 1, \text{ a.s.}$$

Assume $\alpha_{\beta_t} = t$ (this is true when, for example, *beta.* is strictly increasing), then we are done. □

8.8.

Proof. Since $dN_t = (u(t) - E[u(t)|\mathcal{G}_t])dt + dB_t = dZ_t - E[u(t)|\mathcal{G}_t]dt$, $\mathcal{N}_t = \sigma(N_s : s \leq t) \subset \mathcal{G}_t$. So $E[u(t) - E[u(t)|\mathcal{G}_t]|\mathcal{N}_t] = 0$. By Corollary 8.4.5, N is a Brownian motion. □

8.9.

Proof. By Theorem 8.5.7, $\int_0^{\alpha_t} e^s dB_s = \int_0^t e^{\alpha_s}\sqrt{\alpha_s'} d\tilde{B}_s$, where $\tilde{B}_t$ is a Brownian motion. Note $e^{\alpha_t} = \sqrt{1+\frac{2}{3}t^3}$ and $\alpha_t' = \frac{t^2}{1+\frac{2}{3}t^3}$, we have $e^{\alpha_t}\sqrt{\alpha_t'} = t$. □

8.10.

Proof. By Itô's formula, $dX_t = 2B_t dB_t + dt$. By Theorem 8.4.3, and $4B_t^2 = 4|X_t|$, we are done. □

8.11. a)

Proof. Let $Z_t = \exp\{-B_t - \frac{t^2}{2}\}$, then it's easy to see Z is a martingale. Define Q_T by $dQ_T = Z_T dP$, then Q_T is a probability measure on $\mathcal{F}_T$ and $Q_T \sim P$. By Girsanov's theorem (Theorem 8.6.6), $(Y_t)_{t\geq 0}$ is a Brownian motion under Q_T. Since Z is a martingale, $dQ|_{\mathcal{F}_t} = Z_T dP|_{\mathcal{F}_t} = Z_t dP = dQ_t$ for any $t \leq T$. This allows us to define a measure Q on $\mathcal{F}_\infty$ by setting $Q|_{\mathcal{F}_T} = Q_T$, for all $T > 0$. □

b)

Proof. By the law of iterated logarithm, if $\hat{B}$ is a Brownian motion, then

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \text{ a.s. and } \liminf_{t\to\infty} \frac{B_t}{2t\log\log t} = -1, \text{ a.s.}$$

So under P,

$$\limsup_{t\to\infty} Y_t = \limsup_{t\to\infty} \left(\frac{B_t}{2t\log\log t} + \frac{t}{\sqrt{2t\log\log t}} \right) \sqrt{2t\log\log t} = \infty, \text{ a.s.}$$

Similarly, $\liminf_{t\to\infty} Y_t = \infty$ a.s. Hence $P(\lim_{t\to\infty} Y_t = \infty) = 1$. Under Q, Y is a Brownian motion. The law of iterated logarithm implies $\lim_{t\to\infty} Y_t$ does'nt exist. So $Q(\lim_{t\to\infty} Y_t = \infty) = 0$. This is not a contradiction, since Girsanov's theorem only requires $Q \sim P$ on $\mathcal{F}_T$ for any $T > 0$, but not necessarily on $\mathcal{F}_\infty$. □

8.12.

Proof. $dY_t = \beta dt + \theta dB_t$ where $\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\theta = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$. We solve the equation $\theta u = \beta$ and get $u = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. Put $M_t = \exp\{-\int_0^t u dB_s - \frac{1}{2}\int_0^t u^2 ds\} = \exp\{3B_1(t) - B_2(t) - 5t\}$ and $dQ = M_T dP$ on $\mathcal{F}_T$, then by Theorem 8.6.6, $dY_t = \theta d\tilde{B}_t$ with $\tilde{B}_t = \begin{pmatrix} -3t \\ t \end{pmatrix} + B(t)$ a Brownian motion w.r.t. Q. □

8.13. a)

Proof. $\{X_t^x \geq M\} \in \mathcal{F}_t$, so it suffices to show $Q(X_t^x \geq M) > 0$ for any probability measure Q which is equivalent to P on $\mathcal{F}_t$. By Girsanov's theorem, we can find such a Q so that X_t is a Brownian motion w.r.t. Q. So $Q(X_t^x \geq M) > 0$, which implies $P(X_t^x \geq M) > 0$. □

b)

Proof. Use the law of iterated logarithm and the proof is similar to that of Exercise 8.11.b). □

► **8.15.** Let $f \in C_0^2(\mathbf{R}^n)$ and $\alpha(x) = (\alpha_1(x), \cdots, \alpha_n(x))$ with $\alpha_i \in C_0^2(\mathbf{R}^n)$ be given functions and consider the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i=1}^n \alpha_i(x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}; & t > 0, x \in \mathbf{R}^n \\ u(0,x) = f(x); & x \in \mathbf{R}^n. \end{cases}$$

a) Use the Girsanov theorem to show that the unique bounded solution $u(t,x)$ of this equation can be expressed by

$$u(t,x) = E^x\left[\exp\left(\int_0^t \alpha(B_s)dB_s - \frac{1}{2}\int_0^t \alpha^2(B_s)ds\right) f(B_t)\right]$$

where E^x is the expectation w.r.t. P^x.

Proof. We define a probability measure Q by $dQ|_{\mathcal{F}_t} = M_t dP|_{\mathcal{F}_t}$, where

$$M_t = \exp\{\int_0^t \alpha(B_s)dB_s - \frac{1}{2}\int_0^t \alpha^2(B_s)ds\}.$$

Then by Girsanov's theorem, $\hat{B}_t \triangleq B_t - \int_0^t \alpha(B_s)ds$ is a Brownian motion. So B_t satisfies the SDE $dB_t = \alpha(B_t)dt + d\hat{B}_t$. By Theorem 8.1.4, the solution can be represented as

$$E_Q^x[f(B_t)] = E^x[\exp(\int_0^t \alpha(B_s)dB_s - \frac{1}{2}\int_0^t \alpha^2(B_s)ds)f(B_t)].$$

□

Remark 1. *To see the advantage of this approach, we note the given PDE is like Kolmogorov's backward equation. So directly applying Theorem 8.1.1, we get the solution $E^x[f(X_t)]$ where X solves the SDE $dX_t = \alpha(X_t)dt + dBt$. However, the formula $E^x[f(X_t)]$ is not sufficiently explicit if α is non-trivial and the expression of X is hard to obtain. Resorting to Girsanov' s theorem makes the formula more explicit.*

b) Now assume that α is a gradient, i.e. that there exists $\gamma \in C^1(\mathbf{R}^n)$ such that

$$\nabla\gamma = \alpha.$$

Assume for simplicity that $\gamma \in C_0^2(\mathbf{R}^n)$. Use Itô's formula to prove that (see Exercise 4.8)

$$u(t,x) = \exp(-\gamma(x))E^x\left[\exp\left\{-\frac{1}{2}\int_0^t (\nabla\gamma^2(B_s) + \Delta\gamma(B_s))ds\right\}\exp(\gamma(B_t))f(B_t)\right].$$

Proof.

$$e^{\int_0^t \alpha(B_s)dB_s - \frac{1}{2}\int_0^t \alpha^2(B_s)ds} = e^{\int_0^t \triangledown\gamma(B_s)dB_s - \frac{1}{2}\int_0^t \triangledown\gamma^2(B_s)ds} = e^{\gamma(B_t)-\gamma(B_0)-\frac{1}{2}\int_0^t \Delta\gamma(B_s)ds - \frac{1}{2}\int_0^t \triangledown\gamma^2(B_s)ds}$$

So

$$u(t,x) = e^{-\gamma(x)}E^x\left[e^{\gamma(B_t)}f(B_t)e^{-\frac{1}{2}\int_0^t(\triangledown\gamma^2(B_s)+\Delta\gamma(B_s))ds}\right].$$

□

c) Put $v(t,x) = \exp(\gamma(x))u(t,x)$. Use the Feynman-Kac formula to show that $v(t,x)$ satisfies the partial differential equation

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{1}{2}(\nabla\gamma^2 + \Delta\gamma)\cdot v + \frac{1}{2}\Delta v; & t > 0, x \in \mathbf{R}^n \\ v(0,x) = exp(\gamma(x))f(x); & x \in \mathbf{R}^n. \end{cases}$$

(See also Exercise 8.16.)

Proof. By Feynman-Kac formula and part b),

$$v(t,x) = E^x\left[e^{\gamma(B_t)}f(B_t)e^{-\frac{1}{2}\int_0^t(\triangledown\gamma^2+\Delta\gamma)(B_s)ds}\right] = e^{\gamma(x)}u(t,x).$$

□

8.16 a)

Proof. Let $L_t = -\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dB_s^i$. Then L is a square-integrable martingale. Furthermore, $\langle L\rangle_T = \int_0^T |\triangledown h(X_s)|^2 ds$ is bounded, since $h \in C_0^1(\mathbb{R}^n)$. By Novikov's condition, $M_t = \exp\{L_t - \frac{1}{2}\langle L\rangle_t\}$ is a martingale. We define $\bar{P}$ on $\mathcal{F}_T$ by $d\bar{P} = M_T dP$. Then

$$dX_t = \triangledown h(X_t)dt + dB_t$$

defines a BM under $\bar{P}$.

$$\begin{aligned} & E^x[f(X_t)] \\ = \;& \bar{E}^x[M_t^{-1}f(X_t)] \\ = \;& \bar{E}^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dX_s^i - \frac{1}{2}\int_0^t |\triangledown h(X_s)|^2 ds}f(X_t)] \\ = \;& E^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i - \frac{1}{2}\int_0^t |\triangledown h(B_s)|^2 ds}f(B_t)] \end{aligned}$$

Apply Itô's formula to $Z_t = h(B_t)$, we get

$$h(B_t) - h(B_0) = \int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i + \frac{1}{2}\int_0^t \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(B_s)ds$$

So

$$E^x[f(X_t)] = E^x[e^{h(B_t)-h(B_0)}e^{-\int_0^t V(B_s)ds}f(B_t)]$$

□

b)

Proof. If Y is the process obtained by killing B_t at a certain rate V, then it has transition operator

$$T_t^Y(g,x) = E^x[e^{-\int_0^t V(B_s)ds}g(B_t)]$$

So the equality in part a) can be written as

$$T_t^X(f,x) = e^{-h(x)}T_t^Y(fe^h, x)$$

□

8.17.

Proof.

$$dY(t) = \begin{pmatrix} dY_1(t) \\ dY_2(t) \end{pmatrix} = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} dt + \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{pmatrix}.$$

So equation (8.6.17) has the form

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}.$$

The general solution is $u_1 = -2u_2 + \beta_1 - 3(\beta_1 - \beta_2) = -2u_2 - 2\beta_1 + 3\beta_2$ and $u_3 = \beta_1 - \beta_2$. Define Q by (8.6.19), then there are infinitely many equivalent martingale measure Q, as u_2 varies. □

9 Applications to Boundary Value Problems

9.2. (i)

Proof. The book's solution is detailed enough. We only comment that for any bounded or positive $g \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$E^{s,x}[g(X_t)] = E[g(s+t, B_t^x)],$$

where the left hand side is expectation under the measure induced by $X_t^{s,x}$ on $\mathbb{R}^2$, while the right hand side is expectation under the original given probability measure P.

Remark: The adding-one-dimension trick in the solution is quite typical and useful. Often in applications, the SDE of our interest may not be homogeneous and the coefficients are functions of both X and t. However, to obtain (strong) Markov property, it is necessary that the SDE is homogeneous. If we augment the original SDE with an additional equation $dX'_t = dt$ or $dX'_t = -dt$, then the SDE system is an $(n+1)$-dimension SDE driven by an m-dimensional BM. The solution $Y_t^{s,x} = (X'_t, X_t)$ ($X'_0 = s$ and $X_0 = x$) can be identified with a probability measure $P^{s,x}$ on $\mathbb{R}^{n+1}$, with $P^{s,x} = Y^{s,x}(P)$, where $Y^{s,x}(P)$ means the distribution function of $Y^{s,x}$. With this perspective, we have $E^{s,x}[g(X_t)] = E[g(t+s, B_t^x)]$.

Abstractly speaking, the (strong) Markov property of SDE solution can be formulated precisely as follows. Suppose we have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$, on which an m-dimensional continuous semimartingale Z is defined. Then we can consider an n-dimensional SDE driven by Z, $dX_t = f(t, X_t)dZ_t$. If X^x is a solution with $X_0 = x$, the distribution $X^x(P)$ of X^x, denoted by P^x, induces a probability measure on $C(\mathbb{R}_+, \mathbb{R}^n)$. The (strong) Markov property then means the coordinate process defined on $C(\mathbb{R}_+, \mathbb{R}^n)$ is a (strong) Markov process under the family of measures $(P^x)_{x\in\mathbb{R}^n}$. Usually, we need the SDE $dX_t = f(t, X_t)dZ_t$ is homogenous, i.e. $f(t,x) = f(x)$, and the driving process Z is itself a Markov process. When Z is a BM, we emphasize that it is a *standard* BM (cf. [8] Chapter IX, Definition 1.2)

□

9.5. a)

Proof. If $\frac{1}{2}\Delta u = -\lambda u$ in D, then by integration by parts formula, we have $-\lambda\langle u, u\rangle = -\lambda\int_D u^2(x)dx = \frac{1}{2}\int_D u(x)\Delta u(x)dx = -\frac{1}{2}\int_D \triangledown u(x)\cdot\triangledown u(x)dx \le 0$. So $\lambda \ge 0$. Because u is not identically zero, we must have $\lambda > 0$. □

b)

Proof. We follow the hint. Let u be a solution of (9.3.31) with $\lambda = \rho$. Applying Dynkin's formula to the process $dY_t = (dt, dB_t)$ and the function $f(t,x) = e^{\rho t}u(x)$, we get

$$E^{(t,x)}[f(Y_{\tau\wedge n})] = f(t,x) + E^{(t,x)}\left[\int_0^{\tau\wedge n} Lf(Y_s)ds\right].$$

Since $Lf(t,x) = \rho e^{\rho t}u(x) + \frac{1}{2}e^{\rho t}\Delta u(x) = 0$, we have $E^{(t,x)}[e^{\rho\tau\wedge n}u(B_{\tau\wedge n})] = e^{\rho t}u(x)$. Let $t = 0$ and $n \uparrow \infty$, we are done. Note $\forall \xi \in b\mathcal{F}_\infty$, $E^{(t,x)}[\xi] = E^x[\xi]$ (cf. (7.1.7)). □

c)

Proof. This is straightforward from b). □

9.6.

Proof. Suppose $f \in C_0^2(\mathbb{R}^n)$ and let $g(t,x) = e^{-\alpha t} f(x)$. If τ satisfies the condition $E^x[\tau] < \infty$, then by Dynkin's formula applied to Y and y, we have

$$E^{(t,x)}[e^{-\alpha\tau} f(X_\tau)] = e^{-\alpha t} f(x) + E^{(t,x)}\left[\int_0^\tau (\frac{\partial}{\partial s} + \mathcal{A}) g(s, X_s) ds\right].$$

That is,

$$E^x[e^{-\alpha\tau} f(X_\tau)] = e^{-\alpha\tau} f(x) + E^x[\int_0^\tau e^{-\alpha s}(-\alpha + \mathcal{A}) f(X_s) ds].$$

Let $t = 0$, we get

$$E^x[e^{-\alpha\tau} f(X_\tau)] = f(x) + E^x[\int_0^\tau e^{-\alpha s}(\mathcal{A} - \alpha) f(X_s) ds].$$

If $\alpha > 0$, then for any stopping time τ, we have

$$E^x[e^{-\alpha\tau\wedge n} f(X_{\tau\wedge n})] = f(x) + E^x[\int_0^{\tau\wedge n} e^{-\alpha s}(\mathcal{A} - \alpha) f(X_s) ds].$$

Let $n \uparrow \infty$ and apply dominated convergence theorem, we are done. □

9.7. a)

Proof. Without loss of generality, assume $y = 0$. First, we consider the case $x \neq 0$. Following the hint and note $\ln|x|$ is harmonic in $\mathbb{R}^2 \backslash \{0\}$, we have $E^x[f(B_\tau)] = f(x)$, since $E^x[\tau] = \frac{1}{2} E^x[|B_\tau|^2] < \infty$. If we define $\tau_\rho = \inf\{t > 0 : |B_t| \leq \rho\}$ and $\tau_R = \inf\{t > 0 : |B_t| \geq R\}$, then

$$\begin{cases} P^x(\tau_\rho < \tau_R)\ln\rho + P^x(\tau_\rho > \tau_R)\ln R = \ln|x|, \\ P^x(\tau_\rho < \tau_R) + P^x(\tau_\rho > \tau_R) = 1. \end{cases}$$

So $P^x(\tau_\rho < \tau_R) = \frac{\ln R - \ln|x|}{\ln R - \ln\rho}$. Hence $P^x(\tau_0 < \infty) = \lim_{R\to\infty} P^x(\tau_\rho < \tau_R) = \lim_{R\to\infty}\lim_{\rho\to 0} P^x(\tau_\rho < \tau_R) = \lim_{R\to\infty}\lim_{\rho\to 0} \frac{\ln R - \ln|x|}{\ln R - \ln\rho} = 0$.

For the case $x = 0$, we have

$$\begin{aligned}
& P^0(\exists\, t > 0,\ B_t = 0) \\
= \ & P^0(\exists\, \epsilon > 0,\ \tau_0 \circ \theta_\epsilon < \infty) \\
= \ & P^0(\cup_{\epsilon > 0,\ \epsilon \in \mathbb{Q}^+} \{\tau_0 \circ \theta_\epsilon < \infty\}) \\
= \ & \lim_{\epsilon \to 0} P^0(\tau_0 \circ \theta_\epsilon < \infty) \\
= \ & \lim_{\epsilon \to 0} E^0[P^{B_\epsilon}(\tau_0 < \infty)] \\
= \ & \lim_{\epsilon \to 0} \int \frac{e^{-\frac{z^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} P^z(\tau_0 < \infty) dz \\
= \ & 0.
\end{aligned}$$

□

b)

Proof. $\tilde{B}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B_t$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is orthogonal, so $\tilde{B}$ is also a Brownian motion. □

c)

Proof. $P^0(\tau_D = 0) = \lim_{\epsilon\to 0} P^0(\tau_D \le \epsilon) \ge \lim_{\epsilon\to 0} P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \ge 0,\ B_t^{(2)} = 0)$. Part a) implies

$$\begin{aligned} & P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \ge 0,\ B_t^{(2)} = 0) + P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \le 0,\ B_t^{(2)} = 0) \\ = \ & P^0(\exists\, t \in (0,\epsilon],\ B_t^{(2)} = 0) + P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} = 0,\ B_t^{(2)} = 0) \\ = \ & 1. \end{aligned}$$

And part b) implies $P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \ge 0,\ B_t^{(2)} = 0) = P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \le 0,\ B_t^{(2)} = 0)$. So $P^0(\exists\, t \in (0,\epsilon],\ B_t^{(1)} \ge 0,\ B_t^{(2)} = 0) = \frac{1}{2}$. Hence $P^0(\tau_D = 0) \ge \frac{1}{2}$. By Blumenthal's 0-1 law, $P^0(\tau_D = 0) = 1$, i.e. 0 is a regular boundary point. □

d)

Proof. $P^0(\tau_D = 0) \le P^0(\exists\, t > 0,\ B_t = 0) \le P^0(\exists\, t > 0,\ B_t^{(2)} = B_t^{(3)} = 0) = 0$. So 0 is an irregular boundary point. □

9.9. a)

Proof. Assume g has a local maximum at $x \in G$. Let $U \subset\subset G$ be an open set that contains x, then $g(x) = E^x[g(X_{\tau_U})]$ and $g(x) \ge g(X_{\tau_U})$ on $\{\tau_U < \infty\}$. When X is non-degenerate, $P^x(\tau_U < \infty) = 1$. So we must have $g(x) = g(X_{\tau_U})$ a.s.. This implies g is locally a constant. Since G is connected, g is identically a constant. □

9.10.

Proof. Consider the diffusion process Y that satisfies

$$dY_t = \begin{pmatrix} dt \\ dX_t \end{pmatrix} = \begin{pmatrix} dt \\ \alpha X_t dt + \beta X_t dB_t \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha X_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \beta X_t \end{pmatrix} dB_t.$$

Let $\tau = \inf\{t > 0 : Y_t \notin (0,T) \times (0,\infty)\}$, then by Theorem 9.3.3,

$$\begin{aligned} f(t,x) &= E^{(t,x)}[e^{-\rho\tau}\phi(X_\tau)] + E^{(t,x)}[\int_0^\tau K(X_s)e^{-\rho s}ds] \\ &= E[e^{-\rho(T-t)}\phi(X_{T-t}^x)] + E[\int_0^{T-t} K(X_s^x)e^{-\rho(s+t)}ds], \end{aligned}$$

where $X_t^x = xe^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$. Then it's easy to calculate

$$f(t,x) = e^{-\rho(T-t)}E[\phi(X_{T-t}^x)] + \int_0^{T-t} e^{-\rho(s+t)}E[K(X_s^x)]ds.$$

□

9.11. a)

Proof. First assume F is closed. Let $\{\phi_n\}_{n\ge 1}$ be a sequence of bounded continuous functions defined on ∂D such that $\phi_n \to 1_F$ boundedly. This is possible due to Tietze extension theorem. Let $h_n(x) = E^x[\phi_n(B_\tau)]$. Then by Theorem 9.2.14, $h_n \in C(\bar{D})$ and $\Delta h_n(x) = 0$ in D. So by Poisson formula, for $z = re^{i\theta} \in D$,

$$h_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)h_n(e^{it})dt$$

Let $n \to \infty$, $h_n(z) \to E^x[1_F(B_\tau)] = P^x(B_\tau \in F)$ by bounded convergence theorem, and $RHS \to \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$ by dominated convergence theorem. Hence

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

Then by $\pi - \lambda$ theorem and the fact Borel σ-field is generated by closed sets, we conclude

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

for any Borel subset of ∂D. □

b)

Proof. Let B be a BM starting at 0. By example 8.5.9, $\phi(B_t)$ is, after a change of time scale $\alpha(t)$ and under the original probability measure P, a BM in the plane. $\forall F \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned} & P(B \text{ exits } D \text{ from } \psi(F)) \\ = \ & P(\phi(B) \text{ exits upper half plane from } F) \\ = \ & P(\phi(B)_{\alpha(t)} \text{ exits upper half plane from } F) \\ = \ & \text{Probability of BM starting at i that exits from } F \\ = \ & \mu(F) \end{aligned}$$

So by part a), $\mu(F) = \frac{1}{2\pi}\int_0^{2\pi} 1_{\psi(F)}(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} 1_F(\phi(e^{it}))dt$. This implies

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi(e^{it}))dt = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\phi(z))}{z}dz$$

□

c)

Proof. By change-of-variable formula,

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{\pi}\int_{\partial H} f(\omega)\frac{d\omega}{|\omega - i|^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{dx}{x^2+1}$$

□

d)

Proof. Let $g(z) = u + vz$, then g is a conformal mapping that maps i to $u + vi$ and keeps upper half plane invariant. Use the harmonic measure on x-axis of a BM starting from i, and argue as above in part a)-c), we can get the harmonic measure on x-axis of a BM starting from $u + iv$. □

9.12.

Proof. We consider the diffusion $dY_t = \begin{pmatrix} dX_t \\ q(X_t)dt \end{pmatrix}$, then the generator of Y is $\mathcal{A}\phi(y_1, y_2) = L_{y_1}\phi(y) + q(y_1)\frac{\partial}{\partial y_2}\phi(y)$, for any $\phi \in C_0^2(\mathbb{R}^n \times \mathbb{R})$. Choose a sequence $(U_n)_{n\geq 1}$ of open sets so that $U_n \subset\subset D$ and $U_n \uparrow D$. Define $\tau_n = \inf\{t > 0 : Y_t \notin U_n \times (-n, n)\}$. Then for a bounded solution h, Dynkin's formula applied to $h(y_1)e^{-y_2}$ (more precisely, to a C_0^2-function which coincides with $h(y_1)e^{-y_2}$ on $U_n \times (-n, n)$) yields

$$E^y[h(Y^{(1)}_{\tau_n\wedge n})e^{-Y^{(2)}_{\tau_n\wedge n}}] = h(y_1)e^{-y_2} - E^y\left[\int_0^{\tau_n\wedge n} g(Y_s^{(1)})e^{-Y_s^{(2)}}ds\right],$$

since $\mathcal{A}(h(y_1)e^{-y_2}) = -g(y_1)e^{-y_2}$. Let $y_2 = 0$, we have

$$h(y_1) = E^{(y_1,0)}[h(Y^{(1)}_{\tau_n\wedge n})e^{-Y^{(2)}_{\tau_n\wedge n}}] + E^{(y_1,0)}\left[\int_0^{\tau_n\wedge n} g(Y_s^{(1)})e^{-Y_s^{(2)}}ds\right].$$

Note $Y_t^{(2)} = y_2 + \int_0^t q(X_s)ds \geq y_2$, let $n \to \infty$, by dominated convergence theorem, we have

$$\begin{aligned} h(y_1) &= E^{(y_1,0)}[h(Y^{(1)}_{\tau_D})e^{-Y^{(2)}_{\tau_D}}] + E^{(y_1,0)}\left[\int_0^{\tau_D} g(Y_s^{(1)})e^{-Y_s^{(2)}}ds\right] \\ &= E[e^{-\int_0^{\tau_D} q(X_s)ds}\phi(X^{y_1}_{\tau_D})] + E\left[\int_0^{\tau_D} g(X_s^{y_1})e^{-\int_0^s q(X_u^{y_1})du}ds\right]. \end{aligned}$$

Hence

$$h(x) = E^x[e^{-\int_0^{\tau_D} q(X_s)ds}\phi(X_{\tau_D})] + E^x\left[\int_0^{\tau_D} g(X_s)e^{-\int_0^s q(X_u)du}ds\right].$$

□

Remark: An important application of this result is when $g = 0$, $\phi = 1$ and q is a constant, the Laplace transform of first exit time $E^x[e^{-q\tau_D}]$ is the solution of

$$\begin{cases} Ah(x) - qh(x) = 0 & \text{on } D \\ \lim_{x\to y} h(x) = 1 & y \in \partial D. \end{cases}$$

In the one-dimensional case, the ODE can be solved by separation of variables and gives explicit formula for $E^x[e^{-q\tau_D}]$. For details, see Exercise 9.15 and Durrett [3], page 170.

9.13. a)

Proof. $w(x)$ solves the ODE

$$\begin{cases} \mu w'(x) + \frac{\sigma^2}{2}w''(x) = -g(x), & a < x < b; \\ w(x) = \phi(x), & x = a \text{ or } b. \end{cases}$$

The first equation gives $w''(x) + \frac{2\mu}{\sigma^2}w'(x) = -\frac{2g(x)}{\sigma^2}$. Multiply $e^{\frac{2\mu}{\sigma^2}x}$ on both sides, we get

$$(e^{\frac{2\mu}{\sigma^2}x}w'(x))' = -e^{\frac{2\mu}{\sigma^2}x}\frac{2g(x)}{\sigma^2}.$$

So $w'(x) = C_1e^{-\frac{2\mu}{\sigma^2}x} - e^{-\frac{2\mu}{\sigma^2}x}\int_a^x e^{\frac{2\mu}{\sigma^2}\xi}\frac{2g(\xi)}{\sigma^2}d\xi$. Hence

$$w(x) = C_2 - \frac{\sigma^2}{2\mu}C_1e^{-\frac{2\mu}{\sigma^2}x} - \int_a^x e^{-\frac{2\mu}{\sigma^2}y}\int_a^y e^{\frac{2\mu}{\sigma^2}\xi}\frac{2g(\xi)}{\sigma^2}d\xi dy.$$

By boundary condition,

$$\begin{cases} \phi(a) = C_2 - \frac{\sigma^2}{2\mu}C_1e^{-\frac{2\mu}{\sigma^2}a} \\ \phi(b) = C_2 - \frac{\sigma^2}{2\mu}C_1e^{-\frac{2\mu}{\sigma^2}b} - \int_a^b e^{-\frac{2\mu}{\sigma^2}y}\int_a^y e^{\frac{2\mu}{\sigma^2}\xi}\frac{2g(\xi)}{\sigma^2}d\xi dy. \end{cases} \tag{2}$$

Let $\frac{2\mu}{\sigma^2} = \theta$ and solve the above equation, we have

$$\begin{aligned} C_1 &= \frac{\theta[\phi(b)-\phi(a)] + \frac{\theta^2}{\mu}\int_a^b\int_a^y e^{\theta(\xi-y)}g(\xi)d\xi dy}{e^{-\theta a} - e^{-\theta b}}, \\ C_2 &= \phi(a) + \frac{C_1}{\theta}e^{-\theta a}. \end{aligned}$$

□

b)

Proof. $\int_a^b g(y)G(x,dy) = E^x[\int_0^{\tau_D} g(X_t)dt] = w(x)$ in part a), when $\phi \equiv 0$. In this case, we have

$$\begin{aligned} C_1 &= \frac{\theta^2}{\mu(e^{-\theta a}-e^{-\theta b})}\int_a^b\int_a^y e^{\theta(\xi-y)}g(\xi)d\xi dy \\ &= \frac{\theta^2}{\mu(e^{-\theta a}-e^{-\theta b})}\int_a^b e^{\theta\xi}g(\xi)\int_\xi^b e^{-\theta y}dyd\xi \\ &= \frac{\theta^2}{\mu(e^{-\theta a}-e^{-\theta b})}\int_a^b e^{\theta\xi}g(\xi)\frac{e^{-\theta\xi}-e^{-\theta b}}{\theta}d\xi \\ &= \int_a^b g(\xi)\frac{\theta}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(\xi-b)})d\xi, \end{aligned}$$

and

$$C_2 = \int_a^b g(\xi)\frac{e^{-\theta a}}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(\xi-b)})d\xi.$$

So

$$\begin{aligned} &\int_a^b g(y)G(x,dy) \\ =\ & C_2 - \frac{1}{\theta}C_1e^{-\theta x} - \int_a^x\int_a^y e^{\theta(\xi-y)}\frac{\theta}{\mu}g(\xi)d\xi dy \\ =\ & \frac{1}{\theta}C_1(e^{-\theta a}-e^{-\theta x}) - \int_a^b\int_a^b 1_{\{a<y\leq x\}}1_{\{a<\xi\leq y\}}e^{\theta(\xi-y)}\frac{\theta}{\mu}g(\xi)dyd\xi \\ =\ & \int_a^b g(\xi)\frac{e^{-\theta a}-e^{-\theta x}}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(\xi-b)})d\xi - \frac{\theta}{\mu}\int_a^b g(\xi)e^{\theta\xi}1_{\{a<\xi\leq x\}}\int_a^b 1_{\{\xi<y\leq x\}}e^{-\theta y}dyd\xi \\ =\ & \int_a^b g(\xi)\frac{e^{-\theta a}-e^{-\theta x}}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(\xi-b)})d\xi - \frac{\theta}{\mu}\int_a^x g(\xi)e^{\theta\xi}\frac{e^{-\theta\xi}-e^{-\theta x}}{\theta}d\xi \\ =\ & \int_a^b g(\xi)\left[\frac{e^{-\theta a}-e^{-\theta x}}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(\xi-b)}) - \frac{1-e^{\theta(\xi-x)}}{\mu}1_{\{a<y\leq x\}}\right]d\xi. \end{aligned}$$

Therefore

$$G(x,dy) = \left(\frac{e^{-\theta a}-e^{-\theta x}}{\mu(e^{-\theta a}-e^{-\theta b})}(1-e^{\theta(y-b)}) - \frac{1-e^{\theta(y-x)}}{\mu}1_{\{a<y\leq x\}}\right)dy.$$

□

9.14.

Proof. By Corollary 9.1.2, $w(x) = E^x[\phi(X_{\tau_D})] + E^x[\int_0^{\tau_D} g(X_t)dt]$ solves the ODE

$$\begin{cases} rxw'(x) + \frac{1}{2}\alpha^2x^2w''(x) = -g(x) \\ w(a) = \phi(a), w(b) = \phi(b). \end{cases}$$

Choose $g \equiv 0$ and $\phi(a) = 0$, $\phi(b) = 1$, we have $w(x) = P^x(X_{\tau_D} = b)$. So it's enough if we can solve the ODE for general g and ϕ. Assume $w(x) = h(\ln x)$, then the ODE becomes ($t = \ln x$)

$$\begin{cases} \frac{1}{2}\alpha^2h''(t) + (r - \frac{1}{2}\alpha^2)h'(t) = -g(e^t) \\ w(a) = h(\ln a) = \phi(a), w(b) = h(\ln b) = \phi(b). \end{cases}$$

Let $\theta = \frac{2r-\alpha^2}{\alpha^2}$, then the equation becomes $h''(t)+\theta h'(t) = -\frac{2g(e^t)}{\alpha^2}$. So

$$h(t) = C_2 - \frac{C_1 e^{-\theta t}}{\theta} - \frac{2}{\alpha^2}\int_a^t e^{-\theta y}\int_a^y e^{\theta s} g(e^s)dsdy,$$

$$\phi(a) = h(\ln a) = C_2 - \frac{C_1 a^{-\theta}}{\theta} - \frac{2}{\alpha^2}\int_a^{\ln a}\int_a^y e^{\theta(s-y)} g(e^s)dsdy,$$

and $\phi(b) = h(\ln b) = C_2 - \frac{C_1 b^{-\theta}}{\theta} - \frac{2}{\alpha^2}\int_a^{\ln b}\int_a^y e^{\theta(s-y)}g(e^s)dsdy$. So

$$\phi(b)-\phi(a) = \frac{C_1}{\theta}(a^{-\theta}-b^{-\theta}) - \frac{2}{\alpha^2}\int_{\ln a}^{\ln b}\int_a^y e^{\theta(s-y)}g(e^s)dsdy,$$

$$C_1 = \frac{\theta}{a^{-\theta}-b^{-\theta}}\left[\phi(b)-\phi(a)+\frac{2}{\alpha^2}\int_{\ln a}^{\ln b}\int_a^y e^{\theta(s-y)}g(e^s)dsdy\right],$$

and

$$C_2 = \phi(b) + \frac{2}{\alpha^2}\int_a^{\ln b}\int_a^y e^{\theta(s-y)}g(e^s)dsdy + \frac{b^{-\theta}}{a^{-\theta}-b^{-\theta}}\left[\phi(b)-\phi(a)+\frac{2}{\alpha^2}\int_{\ln a}^{\ln b}\int_a^y e^{\theta(s-y)}g(e^s)dsdy\right].$$

In particular, $P^x(X_{\tau_D}=b) = h(\ln x) = C_2 - \frac{C_1}{\theta}x^{-\theta} = 1 + \frac{b^{-\theta}}{a^{-\theta}-b^{-\theta}} - \frac{x^{-\theta}\theta}{\theta(a^{-\theta}-b^{-\theta})} = \frac{a^{-\theta}-a^{-\theta}}{a^{-\theta}-b^{-\theta}}$. (Compare with Exercise 7.9.b).) □

9.16. a)

Proof. Consider the diffusion $dY_t = \begin{pmatrix} dt \\ dX_t \end{pmatrix} = \begin{pmatrix} dt \\ rX_t dt + \sigma X_t dB_t \end{pmatrix} = \begin{pmatrix} 1 \\ rX_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma X_t \end{pmatrix} dB_t$. Then Y has generator $Lf(t,x) = \frac{\partial}{\partial t}f(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x)$ and the original Black-Scholes PDE becomes

$$\begin{cases} Lw - rw = 0 & \text{in D} \\ w(T,x) = (x-K)^+. \end{cases}$$

By the Feynman-Kac formula for boundary value problem (Exercise 9.12), we have

$$w(s,x) = E^{(s,x)}[e^{-\int_0^{\tau_D} rds}(X_{\tau_D}-K)^+] = E^x[e^{-r(T-s)}(X_{T-s}-K)^+].$$

□

Another solution:

Proof. Set $u(t,x) = w(T-t,x)$, then u satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = rx\frac{\partial}{\partial x}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) - ru(t,x), & (t,x) in D \\ u(0,x) = (x-K)^+; & x \geq 0 \end{cases}$$

This is reduced to Exercise 8.6, where we can apply Feynman-Kc formula. □

b)

Proof.

$$
\begin{aligned}
w(0,x) &= E^x[e^{-rT}(X_T-K)^+] = e^{-rT}E[(xe^{(r-\frac{\sigma^2}{2})T+\sigma B_T}-K)^+] \\
&= e^{-rT}\int_{-\infty}^{\infty}(xe^{(r-\frac{\sigma^2}{2})T+\sigma z}-K)^+\frac{e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}}dz \\
&= e^{-rT}\int_{\frac{\ln K-\ln x-(r-\frac{\sigma^2}{2})T}{\sigma}}^{\infty}(xe^{(r-\frac{\sigma^2}{2})T+\sigma z}-K)\frac{e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}}dz \\
&= \int_{\frac{\ln K-\ln x-(r-\frac{\sigma^2}{2})T}{\sigma}}^{\infty}\frac{xe^{-\frac{1}{2}\sigma^2T+\sigma z}e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}}dz - Ke^{-rT}\int_{\frac{\ln K-\ln x-(r-\frac{\sigma^2}{2})T}{\sigma}}^{\infty}\frac{e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}}dz \\
&= \int_{\frac{\ln K-\ln x-(r-\frac{\sigma^2}{2})T}{\sigma}}^{\infty}\frac{xe^{-\frac{(z-\sigma T)^2}{2T}}}{\sqrt{2\pi T}}dz - Ke^{-rT}\int_{\frac{\ln K-\ln x-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz \\
&= \int_{\frac{\ln\frac{K}{x}-rT}{\sigma}+\frac{1}{2}\sigma T}^{\infty}\frac{xe^{-\frac{(z-\sigma T)^2}{2T}}}{\sqrt{2\pi T}}dz - Ke^{-rT}\Phi(\frac{rT+\ln\frac{x}{K}}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}) \\
&= \int_{\frac{\ln\frac{K}{x}-rT}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}}^{\infty}\frac{xe^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz - Ke^{-rT}\Phi(\eta-\frac{1}{2}\sigma\sqrt{T}) \\
&= x\Phi(\eta+\frac{1}{2}\sigma\sqrt{T}) - Ke^{-rT}\Phi(\eta-\frac{1}{2}\sigma\sqrt{T}).
\end{aligned}
$$

□

10 Application to Optimal Stopping

11 Application to Stochastic Control

12 Application to Mathematical Finance

12.1 a)

Proof. Let θ be an arbitrage for the market $\{X_t\}_{t\in[0,T]}$. Then for the market $\{\bar{X}_t\}_{t\in[0,T]}$:
(1) θ is self-financing, i.e. $d\bar{V}_t^\theta = \theta_t d\bar{X}_t$. This is (12.1.14).
(2) θ is admissible. This is clear by the fact $\bar{V}_t^\theta = e^{-\int_0^t \rho_s ds}V_t^\theta$ and ρ being bounded.
(3) θ is an arbitrage. This is clear by the fact $V_t^\theta > 0$ if and only if $\bar{V}_t^\theta > 0$.

So $\{\bar{X}_t\}_{t\in[0,T]}$ has an arbitrage if $\{X_t\}_{t\in[0,T]}$ has an arbitrage. Conversely, if we replace ρ with $-\rho$, we can calculate X has an arbitrage from the assumption that $\bar{X}$ has an aribitrage. □

12.2

Proof. By $V_t = \sum_{i=0}^n \theta_i X_i(t)$, we have $dV_t = \theta\cdot dX_t$. So θ is self-financing. □

12.6 (e)

Proof. Arbitrage exists, and one hedging strategy could be $\theta = (0, B_1+B_2, B_1-B_2+\frac{1-3B_1+B_2}{5}, \frac{1-3B_1+B_2}{5})$. The final value would then become $B_1(T)^2+B_2(T)^2$. □

12.10

Proof. Becasue we want to represent the contingent claim in terms of original BM B, the measure Q is the same as P. Solving SDE $dX_t = \alpha X_t dt + \beta X_t dB_t$ gives us $X_t = X_0 e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$. So

$$\begin{aligned}
& E^y[h(X_{T-t})] \\
= \; & E^y[X_{T-t}] \\
= \; & y e^{(\alpha - \frac{\beta^2}{2})(T-t)} e^{\frac{\beta^2}{2}(T-t)} \\
= \; & y e^{\alpha(T-t)}
\end{aligned}$$

Hence $\phi = e^{\alpha(T-t)} \beta X_t = \beta X_0 e^{\alpha T - \frac{\beta^2}{2} t + \beta B_t}$. □

12.11 a)

Proof. According to (12.2.12), $\sigma(t,\omega) = \sigma$, $\mu(t,\omega) = m - X_1(t)$. So $u(t,\omega) = \frac{1}{\sigma}(m - X_1(t) - \rho X_1(t))$. By (12.2.2), we should define Q by setting

$$dQ|_{\mathcal{F}_t} = e^{-\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds} dP$$

Under Q, $\tilde{B}_t = B_t + \frac{1}{\sigma}\int_0^t (m - X_1(s) - \rho X_1(s)) ds$ is a BM. Then under Q,

$$dX_1(t) = \sigma d\tilde{B}_t + \rho X_1(t) dt$$

So $X_1(T) e^{-\rho T} = X_1(0) + \int_0^T \sigma e^{-\rho t} d\tilde{B}_t$ and $E_Q[\xi(T) F] = E_Q[e^{-\rho T} X_1(T)] = x_1$. □

b)

Proof. We use Theorem 12.3.5. From part a), $\phi(t,\omega) = e^{-\rho t}\sigma$. We therefore should choose $\theta_1(t)$ such that $\theta_1(t) e^{-\rho t}\sigma = \sigma e^{-\rho t}$. So $\theta_1 = 1$ and θ_0 can then be chosen as 0. □

A Probabilistic solutions of PDEs

This section is based on [7].

1. *Resolvent equation.* Suppose X is a diffusion with generator $\mathcal{A}$, and for $\alpha > 0$, the resolvent operator $\mathcal{R}_\alpha$ is defined by

$$R_\alpha g(x) = E^x\Big[\int_0^\infty e^{-\alpha t} g(X_t) dt\Big],\ g \in C_b(\mathbb{R}^n).$$

Then we have

$$\mathcal{R}_\alpha(\alpha - \mathcal{A})|_{C_c^2(\mathbb{R}^n)} = id,\ (\alpha - \mathcal{A})\mathcal{R}_\alpha|_{C_b(\mathbb{R}^n)} = id.$$

Note the former equation is a special case of resolvent equation (see, for example, [4] for the semigroup theory involving resolvent equation), since $C_c^2(\mathbb{R}^n) \subset \mathcal{D}(\mathcal{A})$. But the latter is not necessarily a special case, since we don't necessarily have $C_b(\mathbb{R}^n) \subset \mathcal{B}_0(\mathbb{R}^n)$.

2. *Parabolic equation: heat equation via Kolmogorov's backward equation* ($dP_t f/dt = P_t \mathcal{A} f = \mathcal{A} P_t f$). If X is a diffusion with generator $\mathcal{A}$, then for $f \in C_c^2(\mathbb{R}^n)$, $E^x[f(X_t)] := E[f(X_t^x)]$ solves the initial value problem of parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{A} u, & t > 0,\ x \in \mathbb{R}^n \\ u(0,x) = f(x); & x \in \mathbb{R}^n. \end{cases}$$

Remark:

(i) If X satisfies $dX_t = \mu(X_t) dt + \sigma dB_t$, one way to explicitly calculate $E^x[f(X_t)]$ without solving the SDE is via Girsanov's theorem (cf. [7], Exercise 8.15).

(ii) If we let $v(t,x) = u(T-t,x)$, then on $(0,T)$, v satisfies the equation

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{A}v = 0, & 0 < t < T,\ x \in \mathbb{R}^n \\ v(T,x) = f(x); & x \in \mathbb{R}^n. \end{cases}$$

3. *Parabolic equation: Schrödinger equation via Feynman-Kac formula.* Suppose X is a diffusion with generator $\mathcal{A}$. If $f \in C_c^2(\mathbb{R}^n)$, $q \in C(\mathbb{R}^n)$ and q is lower bounded, then

$$v(t,x) = E^x\left[e^{-\int_0^t q(X_s)ds} f(X_t)\right]$$

solves the initial value problem of parabolic PDE

$$\begin{cases} \frac{\partial v}{\partial t} = \mathcal{A}v - qv, & t > 0,\ x \in \mathbb{R}^n \\ v(0,x) = f(x); & x \in \mathbb{R}^n. \end{cases}$$

Remark: (i) The Feynman-Kac formula can be seen as a special case of the heat equation. If we kill X according to a terminal time τ such that $\sup_x |\frac{1}{t}P^x(\tau \le t) - q(x)| \to 0$ as $t \downarrow 0$, then the killed process $\widetilde{X}_t = X_t 1_{\{t<\tau\}} + \partial 1_{\{t \ge \tau\}}$ has infinitesimal generator $\mathcal{A} - q$ and transition semigroup $S_t f(x) = E^x[f(\widetilde{X}_t)] = E^x[e^{-\int_0^t q(X_s)ds} f(X_t)] = E[e^{-\int_0^t q(X_s^x)ds} f(X_t^x)]$.

(ii) The Feyman-Kac formula also helps to solve Black-Scholes PDE after we replace t by $T-t$ and transform the PDE into the form $\frac{\partial u}{\partial t} = Au - \rho t$.

4. *Elliptic equation: the combined Dirichlet-Poisson problem via Dynkin's formula.* Suppose X is a diffusion with generator $\mathcal{A}$. Set $\tau_D = \inf\{t > 0 : X_t \notin D\}$, then $E^x[\phi(X_{\tau_D})1_{\{\tau_D<\infty\}}] + E^x[\int_0^{\tau_D} g(X_t)dt]$ is a candidate for the solution of the equation

$$\begin{cases} \mathcal{A}\omega = -g & \text{in } D \\ \lim_{\substack{x\to y \\ x\in D}} \omega(x) = \phi(y) & \text{for all } y \in \partial D. \end{cases}$$

Remark:

(i) Connection with parabolic equations. The parabolic operator $\frac{\partial}{\partial t} + \mathcal{A}$ (or $-\frac{\partial}{\partial t} + \mathcal{A}$) is the generator of the diffusion $Y_t = (t, X_t)$ (or $Y_t = (-t, X_t)$), where X has generator $\mathcal{A}$. So, if we let $D = (0,T) \times \mathbb{R}^n$ and regard f as a function defined on $\partial D = \{T\} \times \mathbb{R}^n$, then $E^{t,x}[f(Y_{\tau_D})] = E[f(X_{T-t}^x)]$ solves the parabolic equation

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{A}v = 0, & 0 < t < T,\ x \in \mathbb{R}^n \\ v(T,x) = f(x); & x \in \mathbb{R}^n. \end{cases}$$

By setting $u(t,x) = v(T-t,x) = E[f(X_t^x)]$, u solves the heat equation on $(0,T) \times \mathbb{R}^n$. Since T is arbitrary, u is a solution on $(0,\infty) \times \mathbb{R}^n$. This reproduces the result for heat equation via the Kolmogorov's backward equation. More generally, this method can solve the generalized heat equation

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}u = -g, & 0 < t < T,\ x \in \mathbb{R}^n \\ u(T,x) = f(x); & x \in \mathbb{R}^n. \end{cases} \quad \text{or equivalently,} \quad \begin{cases} -\frac{\partial u}{\partial t} + \mathcal{A}u = -g, & t > 0,\ x \in \mathbb{R}^n \\ u(0,x) = f(x); & x \in \mathbb{R}^n. \end{cases}$$

Also important is that we can use either (t, X_t) or $(T-t, X_t)$. The effect of the latter is the combined effects of the first and the transformation $v(t,x) \to u(t,x) = v(T-t,x)$.

(ii) A Feynman-Kac formula for boundary value problem is

$$E^x\left[\int_0^{\tau_D} e^{-\int_0^t q(X_s)ds} g(X_t)dt + e^{-\int_0^{\tau_D} q(X_s)ds}\phi(X_{\tau_D})\right].$$

For details, see [7], Exercise 9.12.

(iii) Basic steps of solution.

(a) Formulation of stochastic Dirichlet/Poisson problem: $\mathcal{A}$ is replaced by the characteristic operator A and the boundary condition is replaced by a pathwise one.

(b) Formulation of generalized Dirichlet/Poisson problem: boundary condition only holds for regular points.

(c) Relating stochastic problems to original problems.

(iiii) Summary of results.

(a) If ϕ is just bounded measurable, then $E^x[\phi(X_{\tau_D})]$ solves the stochastic Dirichlet problem. If in addition, L is uniformly elliptic and ϕ is bounded continuous, $E^x[\phi(X_{\tau_D})]$ solves the generalized Dirichlet problem.

(b) If g is continuous with $E^x[\int_0^{\tau_D}|g(X_s)|ds] < \infty$ for all $x \in D$, $E^x[\int_0^{\tau_D} g(X_s)ds]$ solves the stochastic Poisson problem. If in addition, $\tau_D < \infty$ a.s. Q^x for all x, then $E^x[\int_0^{\tau_D} g(X_s)ds]$ solves the original Poisson problem.

(c) Put together, conditions for the existence of the original problem are: $\phi \in C_b(\partial D)$, $g \in C(D)$ with $E^x[\int_0^{\tau_D}|g(X_s)|ds] < \infty$ for all $x \in D$, and $\tau_D < \infty$ a.s. Q^x for all x. Then $E^x[\phi(X_{\tau_D})] + E^x[\int_0^{\tau_D} g(X_s)ds]$ solves the original problem.

(v) If $g \in C(D)$ with $E^x[\int_0^{\tau_D}|g(X_s)|ds] < \infty$ for all $x \in D$, then $(A-\alpha)\mathcal{R}_\alpha g = -g$ for $\alpha \geq 0$. Here $\mathcal{R}_\alpha g(x) = E^x[\int_0^{\tau_D} e^{-\alpha s} g(X_s)ds]$.

If $E^x[\tau_K] < \infty$ ($\tau_K := \inf\{t > 0 : X_t \notin K\}$) for all compacts $K \subset D$ and all $x \in D$, then $-\mathcal{R}_\alpha$ ($\alpha \geq 0$) is the inverse of characteristic operator A on $C_c^2(D)$:

$$(A-\alpha)(\mathcal{R}_\alpha f) = \mathcal{R}_\alpha(A-\alpha)f = -f, \ \forall f \in C_c^2(D).$$

Note when $D = \mathbb{R}^n$, we get back to the resolvent equation in 1.

B Application of diffusions to obtaining formulas

The following is a table of computation tricks used to obtain formulas:

	BM w/o drift	general diffusion, esp. BM with drift
Distribution of first passage time	reflection principle	Girsanov’s theorme
Exit probability $P(\tau_a < \tau_b)$, $P(\tau_b < \tau_a)$	BM as a martingale	Dynkin’s formula / boundary value problems
Expectation of exit time	$W_t^2 - t$ is a martingale	Dynkin’s formula / boundary value problems
Laplace transform of first passage time	exponential martingale	Girsanov’s theorem
Laplace transform of first exit time	exponential martingale	FK formula for boundary value problems

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