Q.1

(a)

$(\vec{A}\cdot\nabla)\vec{B} \rightarrow$ need to calculate

and $\nabla = (\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz})$

Let $\vec{A} = (x_A, y_A, z_A)$

$\vec{B} = (x_B, y_B, z_B)$

Then $(\vec{A}\cdot\nabla)\vec{B} = \left(\frac{dx_A}{dx} + \frac{dy_A}{dy} + \frac{dz_A}{dz}\right)(x_B, y_B, z_B)$

$\Rightarrow$ component

$$x = x_B\left(\frac{dx_A}{dx} + \frac{dy_A}{dy} + \frac{dz_A}{dz}\right)$$

$$y = y_B\left(\frac{dx_A}{dx} + \frac{dy_A}{dy} + \frac{dz_A}{dz}\right)$$

$$z = z_B\left(\frac{dx_A}{dx} + \frac{dy_A}{dy} + \frac{dz_A}{dz}\right)$$

(b)

Rewrite $\hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ for clear

$$\Rightarrow (\hat{r}\cdot\nabla)\hat{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{d}{dx}\hat{i} + y\frac{d}{dy}\hat{j} + z\frac{d}{dz}\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= 0$$

(c)

$\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$\vec{v}_b = xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}$

$(\vec{v}_a\cdot\nabla) = 2x\hat{x} + 0\hat{y} - 2x\hat{z}$

$\Rightarrow (\vec{v}_a\cdot\nabla)\cdot\vec{v}_b = 2x^2y + 0 - 6x^2z$

$= x^2(2y - 6z)$

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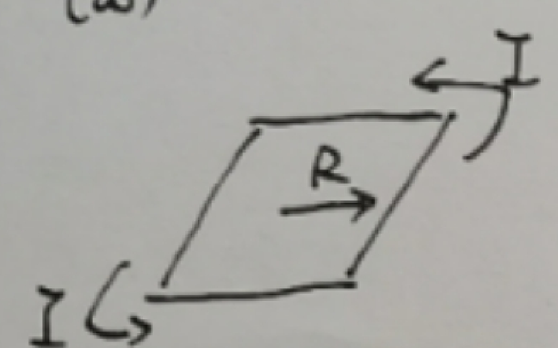
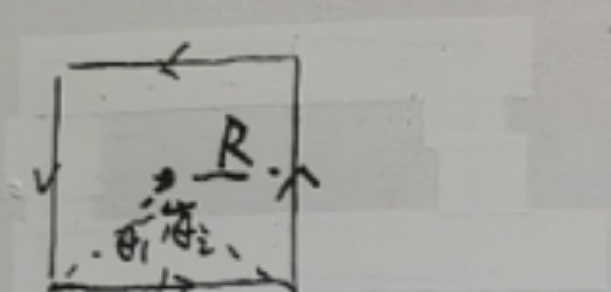
Q2.

(a)



Looking down to the loop:



$$B_1 = \frac{\mu_0 I}{4\pi d}[\sin\theta_1 + \sin\theta_2]$$

$$= \frac{\mu_0 I}{4\pi R}[\sin 45^\circ + \sin 45^\circ]$$

$$= \frac{\sqrt{2}\mu_0 I}{4\pi R}$$

$\Rightarrow B_{total} = 4 \cdot B_1$ (since 4 sides)

$$= \frac{\sqrt{2}\mu_0 I}{\pi R}$$

(b)

Regular n-sided polygon

carrying I $\Rightarrow \theta_1 = \theta_2 = \frac{\pi}{n}$

$$\Rightarrow B_{total} = \frac{n\mu_0 I}{4\pi R}\left[\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right]$$

$$= \frac{n\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)$$

(c)

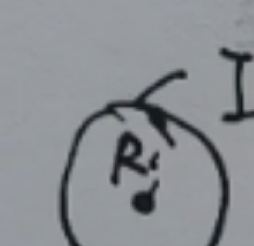
If $n \to \infty \Rightarrow \sin\frac{\pi}{n} \to \frac{\pi}{n}$

$(\sin\theta \to \theta$ if $\theta \to 0)$

$$\Rightarrow B = \frac{n\mu_0 I}{2\pi R}\cdot\frac{\pi}{n}$$

$$= \frac{\mu_0 I}{2R} \quad ①$$

Since the formula of a ring carrying I is exactly ①,



we can conclude it's true.

Q3. New equation: $\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} (1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} \hat{r}$

(a)

$$\hat{\rho} = \rho (1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} \hat{r}$$

$\Rightarrow$ electric field due to

$$\oint \varepsilon \cdot ds = \int \frac{\hat{\rho} \cdot dv}{\varepsilon_0}$$

where $dv = r^2 \sin\theta \, dr \cos\varphi$

$$\Rightarrow \varepsilon = 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^r r^2 \rho (1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} dr \int_0^\theta \sin\theta d\theta \int_0^{2\lambda} d\varphi$$

$$\Leftrightarrow \varepsilon = \frac{\rho}{\varepsilon_0 r_0^2} [8\lambda^3 \cdot e^{-\frac{r}{\lambda}} - 8\lambda^3]$$

$$\Rightarrow E = \frac{\rho}{\varepsilon_0 r^2} [8\lambda^3 \cdot e^{-\frac{r}{\lambda}} - 8\lambda^3]$$

(b) & (c)

Original electric field due to ρ is : $\vec{E} = \frac{\vec{F}}{q}$

$$= \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{r}}{r^2} \rho \, dt$$

$$\Rightarrow \text{new } \vec{E} = \frac{1}{4\pi\varepsilon_0} \int \hat{r}/r^2 \rho (1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} dt$$

since λ is very large $\Rightarrow e^{-\frac{r}{\lambda}} \to 1$

The new law can be regarded as another point charge with a slightly large force.

The reason E admits a scalar potential is that $\nabla \times \vec{E} = 0$

This is due to the point charge at the origin being radical and symmetric.

(d)

$\oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V V d\tau$ is what we consider

$\Rightarrow$ need to analyze the flux

By taking electric field we get in (a):

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q}{\varepsilon_0} \int_0^r r e^{-\frac{r}{\lambda}} dr$$

$$= \frac{\lambda^2 q}{\varepsilon_0} \left(- e^{(-\frac{r}{\lambda})} \cdot \frac{1+r}{\lambda}\right)\Big|_{r=0}^{r}$$

$$= \frac{\lambda^2 q}{\varepsilon_0} \left(-(1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} + 1\right) \quad ①$$

Plugging ① into the following equation

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V V d\tau = \frac{q}{\varepsilon_0}\left[(1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} - (1+\frac{r}{\lambda}) e^{-\frac{r}{\lambda}} + 1\right]$$

$$= \frac{q}{\varepsilon_0}$$ Hence proved

(e) assume n point charge

$$\sum_{i=1}^{n} \left(\oint \vec{E}_i \, d\vec{a}_i + \frac{1}{\lambda^2} \int_V \vec{V} d\tau\right) = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} Q_i$$ Hence proved

Q4

(a)

$$\vec{F} = \vec{B}\vec{v}q \Rightarrow \vec{a} = \frac{\frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}\, q_e \times \vec{v}}{m}$$

(b)

since $\vec{a} = \underbrace{c}_{\text{constant}} \cdot \hat{r} \times \vec{v}$

$\Rightarrow$ velocity is always vertical to acceleration

$\Rightarrow$ accerelation (force) just changes the moving direction $\Rightarrow$ $|\vec{v}|$ is always a constant

(c)

$$\frac{d\vec{Q}}{dt} = m\frac{d(\vec{r}\times\vec{v})}{dt} - \frac{\mu_0 q_e q_m}{4\pi}\frac{d\hat{r}}{dt}$$

$$\frac{1}{mr^2}\frac{d\vec{Q}}{dt} = \frac{1}{r^2}\frac{d(\vec{r}\times\vec{v})}{dt} - \frac{\mu_0 q_e q_m}{4\pi m r^2}\cdot\frac{d\hat{r}}{dt}$$

assume the angle between $\vec{r}$ and $\vec{v}$ is θ

$$\text{right} = \frac{|v|}{|r|}\cdot\frac{d\sin\theta\cdot\hat{j}}{dt} - \frac{|\vec{a}|}{\sin\theta|\vec{v}||\vec{r}|}\cdot\frac{d\hat{r}}{dt} = 0$$

$\Rightarrow \vec{Q}$ is constant

(b)

$$E(x,y,z) = E_0 e^{-jk_x x - jk_y y - jk_z z}$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon$$

$\nabla^2 E + k^2 E = 0 \rightarrow$ homogeneous Helmholtz's equation