RC2

1 Vector Calculus

1.1 Divergence Theorem

$$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

1.2 Stokes's Theorem

$$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

1.3 Null Identities

$$\nabla \times (\nabla V) \equiv 0$$

- The curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.

$$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

- The divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field.

1.4 Exercise

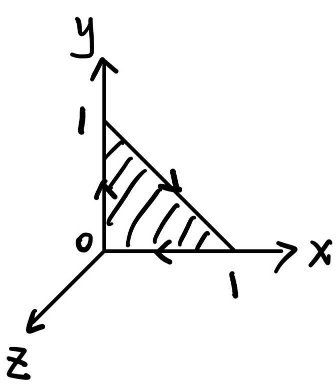
- (HW1-4) Find the divergence of the following radial vector fields:

  a) $f_1(\mathbf{R}) = \mathbf{a_R} R^n$,

b) $f_2(\mathbf{R}) = \mathbf{a_R}\frac{k}{R^2}$.

- (HW1-5) For vector function $\mathbf{A} = \mathbf{a_r}r^2 + \mathbf{a_z}2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r = 5$, $z = 0$, and $z = 4$.

- Given $\mathbf{F} = \mathbf{a_x}xy - \mathbf{a_y}2x$, verify Stokes's theorem using the triangle in the following picture.



2 Electrostatics in Free Space

Static electric charges (source) in free space → electric field

2.1 Electric field intensity

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

2.2 Fundamental Postulates of Electrostatics

- Differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (divergence)$$
$$\nabla \times \mathbf{E} = 0 \quad (curl)$$

where ρ is the volume charge density of free charges (C/m^3), ϵ_0 is the permittivity of free space, a universal constant.

- Integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$
$$\oint_C \mathbf{E} \cdot d\ell = 0$$

where Q is the total charge contained in volume V bounded by surface S. Also, the scalar line integral of the static electric field intensity around any closed path vanishes.

$\mathbf{E}$ is **not solenoidal** (unless $\rho = 0$), but **irrotational (conservative)**

3 Coulomb's Law

3.1 Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

$$\mathbf{E} = \mathbf{a_R} E_R = \mathbf{a_R} \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V}/\mathrm{m})$$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\left(\mathbf{R} - \mathbf{R}'\right)}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R}'\right|^3} \quad (\mathbf{V}/\mathrm{m})$$

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force $\mathbf{F_{12}}$ is experienced by q_2 due to the electric field intensity $\mathbf{E_{12}}$ of q_1 at q_2. Then we have:

$$\mathbf{F_{12}} = q_2 \mathbf{E_{12}} = \mathbf{a_R} \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

- **several point charges:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k\left(\mathbf{R}-\mathbf{R}_k'\right)}{\left|\mathbf{R}-\mathbf{R}_k'\right|^3}$$

3.2 Electric Dipole

- **Electric Field** general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3}-\frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

- **Electric Dipole Moment**
Definition:

$$\mathbf{p} = q\mathbf{d}$$

,where q is the charge, vector $\mathbf{d}$ goes from $-q$ to $+q$.

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R\cos\theta-\mathbf{a}_\theta\sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

- **Electric Field:** (spherical coordinate)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta+\mathbf{a}_\theta\sin\theta\right)\quad(\mathrm{V/m})$$

3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R\frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

, where dv' is the differential volume element.

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'}\mathbf{a}_R\frac{\rho_\ell}{R^2}d\ell'\quad(\mathbf{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'}\mathbf{a}_R\frac{\rho_s}{R^2}ds'\quad(\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'}\mathbf{a}_R\frac{\rho}{R^2}dv'\quad(\mathbf{V/m})$$

3.4 Exercise

- (HW2-1) A line charge of uniform density ρ_l in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

- (HW2-2) Three uniform line charges——ρ_{l1}, ρ_{l2}, and ρ_{l3}, each of length L——form an equilateral triangle. Assuming that $\rho_{l1} = 2\rho_{l2} = 2\rho_{l3}$, determine the electric field intensity at the center of the triangle.

4 Gauss's Law and Application

4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

4.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

4.3 Example

- Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ.

- Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.

4.4 Exercise

- (HW2-3) Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ $(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb}, respectively.

  a) Determine **E** everywhere.

  b) What must be the relation between a and b in order that **E** vanishes for $r > b$?

4.5 Several Useful Models (paste on your ctpp!)

Note: The charge distribution should be **uniform**.

different models	E (magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r \epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0 \ \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ \ (r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi\epsilon_0 R^3} \ \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ \ (r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} \ \ (r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0} \ \ (r > R) \end{cases}$