

Steady Magnetic Fields

1 Fundamental Postulates

differential form	integral form	Comment
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	\mathbf{B} is solenoidal
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$	Ampere's circuital law

where μ_0 is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$.

Conversion of magnetic flux: no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves.

Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \mathbf{J} = \frac{\nabla \cdot (\nabla \times \mathbf{B})}{\mu_0} = 0$$

which is consistent with the formula

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} = 0$$

for steady current.

Ex6.1

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I . Determine the magnetic flux density both inside and outside the conductor.

2 Vector Magnetic Potential & Biot-Savart Law

As $\nabla \cdot \mathbf{B} = 0$, \mathbf{B} is solenoidal, thus could be expressed as:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (T) \quad (1)$$

where \mathbf{A} is called the **vector magnetic potential**.

Magnetic flux Φ :

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \mathbf{A} = 0$, we have

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= -\nabla^2 \mathbf{A} \end{aligned}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

The solution is then

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv' \quad (2)$$

For a thin wire with cross-sectional area S , $dv' = S dl'$, current flow is entirely along the wire, we then have

$$\mathbf{J} dv' = JS dl' = I dl'$$

Thus, Eq 2 becomes:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R}$$

Based on this form and properties of differentiation, we can get **Biot-Savart law**:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2}$$

The formula for Biot-Savart law could also be written as:

$$\mathbf{B} = \oint_{C'} d\mathbf{B}$$

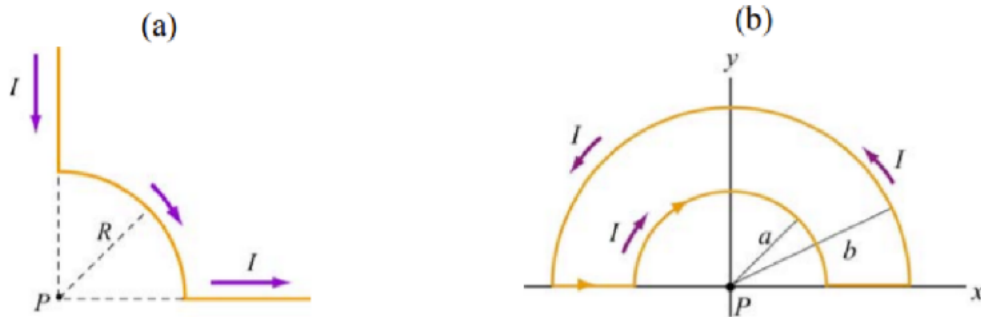
and

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2} \right) = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)$$

Comment: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine \mathbf{B} from I in a circuit if a closed path cannot be found where \mathbf{B} has a constant magnitude.

Ex6.2

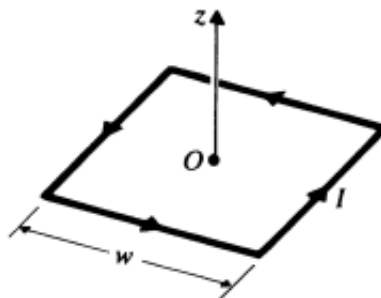
Find the magnetic field at P due to the following current distribution by using Biot-Savart Law.



3 Magnetic Field of Some Common Construction

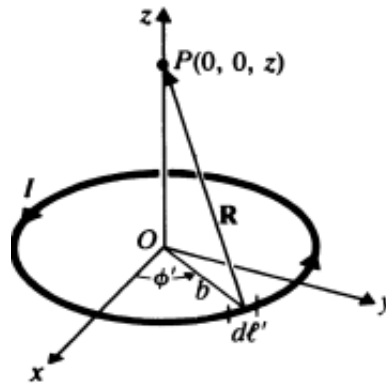
1. Similarly for magnetic flux density at the center of a square loop, with side w carrying a direct current I , is:

$$B = a_z \frac{2\sqrt{2}\mu_0 I}{\pi w}$$

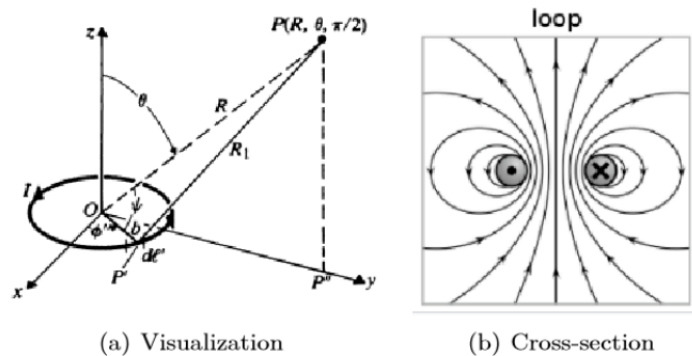


2. For magnetic flux density at a point on the axis of a circular loop of radius b that carries a direct current I ,

$$\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$$



4 Magnetic Dipole



Definition of the magnetic dipole: We call a small current-carrying loop a magnetic dipole

$$\mathbf{m} = I \int d\mathbf{S}$$

The direction is determined by the right-hand rule. (along with the current direction)

$$\mathbf{A}_{dip}(\mathbf{R}) = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2}$$

In spherical coordinates, the vector potential of a magnetic dipole can be written as

$$\mathbf{A}_{dip}(\mathbf{R}) = \frac{\mu_0 m \sin \theta}{4\pi R^2} \mathbf{a}_\phi$$

Hence, we can compute the magnetic field of a magnetic dipole

$$\mathbf{B}_{dip}(\mathbf{R}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi R^3} (2 \cos \theta \mathbf{a}_r + \sin \theta \mathbf{a}_\theta)$$

Written in a coordinate-free form,

$$\mathbf{B}_{dip}(\mathbf{R}) = \frac{\mu_0}{4\pi R^2} [3(\mathbf{m} \cdot \mathbf{a}_r) \mathbf{a}_r - \mathbf{m}]$$

Compared with the electric field density of an electric dipole, we can find that we just replace $\frac{1}{\epsilon_0}$ with μ_0 , and replace \mathbf{p} with \mathbf{m} .

5 Scalar magnetic potential

If a region is current free, i.e. $\mathbf{J} = 0$,

$$\nabla \times \mathbf{B} = 0$$

thus \mathbf{B} can be expressed as the gradient of a scalar field.

Assume

$$\mathbf{B} = -\mu_0 \nabla V_m$$

where V_m is called the **scalar magnetic potential**, the negative sign is conventional, μ_0 is the permeability of free space.

Thus, between two points P_1, P_2 ,

$$V_{m2} - V_{m1} = - \int_{P_1}^{P_2} \frac{1}{\mu_0} \mathbf{B} \cdot d\mathbf{l}$$

If there were magnetic charges with a volume density ρ_m in a volume V' , we could find V_m from:

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R} dv'$$

Notice: This is only a mathematical model, isolated magnetic charges have never been found. Then we could obtain \mathbf{B} by Eq 5.

For a bar magnet the fictitious magnetic charges $+q_m, -q_m$ assumed to be separated by d (magnetic dipole), the scalar magnetic potential V_m is given by:

$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2}$$

and it holds at any points with no currents.

6 Magnetization and Equivalent Current Densities

6.1 Basics

Define **magnetization vector**, \mathbf{M} , as

$$\mathbf{M} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v}$$

which is the volume density of magnetic dipole moment,

1. The effect of magnetization is vector is equivalent to both

(a) a volume current density:

$$\mathbf{J}_m = \nabla \times \mathbf{M}$$

(b) a surface current density:

$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n$$

2. Then we can determine \mathbf{A} by:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds'$$

3. Then we could obtain \mathbf{B} from \mathbf{A} .

6.2 Equivalent Magnetization Charge Densities

In current-free region, a magnetized body may be replaced by

1. an equivalent/fictitious magnetization surface charge density

$$\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n$$

2. an equivalent/fictitious magnetization volume charge density

$$\rho_m = -\nabla \cdot \mathbf{M}$$

Ex6.3

Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius b , length L , and axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$.