上 海 交 通 大 学 试 卷

（ 2022~ 2023~3 Academic Year/Summer Semester ）

Class No.____________________ Name in English or Pinyin: ____________________

Student ID No.______________ Name in Hanzi(if applicable): ____________________

ECE2300J Electromagnetics

Midterm Exam 1

Thursday June 15th 2023.

7:00 pm -8:40 pm

The exam paper has 8 pages in total.

You are to abide by the University of Michigan-Shanghai Jiao Tong University Joint Institute (UM-SJTU JI) honor code. Please sign below to signify that you have kept the honor code pledge.

THE UM-SJTU JI HONOR CODE

I accept the letter and spirit of the honor code:

I have neither given nor received unauthorized aid on this examination, nor have I concealed any violations of the Honor Code by myself or others.

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Exercises No. 题号	Points 得分	Grader's Signature 流水批阅人签名
1		
2		
3		
4		
5		
6		
Total 总分		

Question 1 [15]

Compute the line integral of:

$$v = 6\hat{x} + yz^2\hat{y} + (3y + z)\hat{z}$$

along the triangular path shown in the figure below. Check your answer using Stokes' theorem.

① $x = z = 0$

$v \cdot dl = yz^2 dy = 0 \qquad \int v \cdot dl = 0$ 2'

② $x = 0$; $z = 2 - 2y$;

$v \cdot dl = (yz^2) dy + (3y + z) dz$

$= y(2-2y)^2 dy - (3y + 2 - 2y) 2dy$

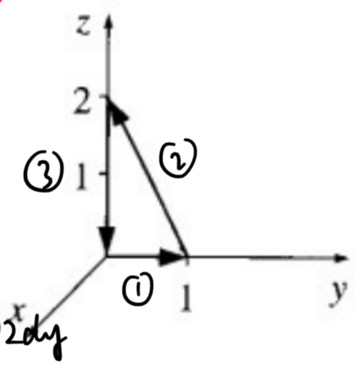


Figure 1: Question 1

$\int v \cdot dl = 2\int_1^0 (2y^3 - 4y^2 + y - 2)\, dy$

$= 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Big|_1^0 = \frac{14}{3}$ 3'

② $x = y = 0$; $z : 2 \to 0$

$v \cdot dl = (3y + z) dz = z\, dz$

$\int v \cdot dl = \int_2^0 z\, dz = \frac{z^2}{2}\Big|_2^0 = -2.$ 3'

Total: $0 + \frac{14}{3} - 2 = \frac{8}{3}$ 2'

Stoke's Theoreom:

$(\nabla \times v)_x = \frac{\partial}{\partial y}(3y + z) - \frac{\partial}{\partial z}(yz^2) = 3 - 2yz.$ 2'

$\int (\nabla \times v) \cdot da = \iint (3 - 2yz)\, dy dz = \int_0^1 \left[\int_0^{2-2y} (3 - 2yz)\, dz\right] dy$ 2'

$= -y^4 + \frac{8}{3}y^3 - 5y^2 + 6y \Big|_0^1 = -1 + \frac{8}{3} - 5 + 6 = \frac{8}{3}$ 1'

Question 2 [15]

(a) Find the electric field (both magnitude and direction) at a distance z above the midpoint between two equal charges q, which are apart from each other by a distance d. A figure indication is shown below. Check if your answer is consistent with what you would expect when $z \gg d$.

(b) Repeat part (a), only this time make the right-hand charge $-q$ instead of $+q$.

a) No horizontal components.

$$\vec{E}_{total} = \vec{E}_{1y} + \vec{E}_{2y}$$
$$= 2 * \vec{E}_{1y}$$
$$= 2E\cos\theta$$

$$\cos\theta = \frac{z}{\sqrt{z^2+(\frac{d}{2})^2}}$$ 2'

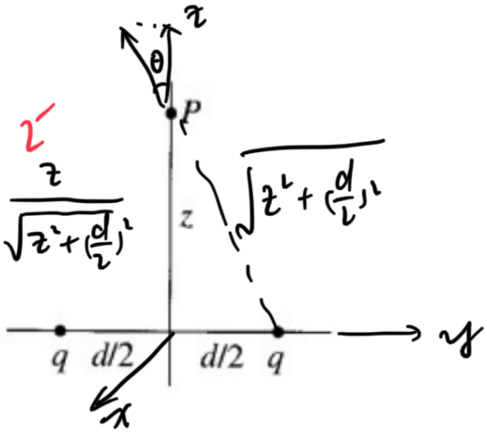


Figure 2: Question 2

$$\Rightarrow \vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{zq}{(z^2+(\frac{d}{2})^2)^{\frac{3}{2}}} \hat{z}$$ 4'

$z \gg d$ $(\frac{d}{2})^2 \rightarrow$ ignore.

$$\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{q}{z^2} \hat{z}$$ 1.5'

like 2 charges.

b). No vertical components.

$$\vec{E} = 2\vec{E}_{1z}$$
$$= 2\vec{E} \cdot \sin\theta$$

$$\sin\theta = \frac{d}{\sqrt{z^2+(\frac{d}{2})^2}}$$ 2'

$$\Rightarrow \vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{dq}{(z^2+(\frac{d}{2})^2)^{\frac{3}{2}}} \hat{y}$$ 4'

$z \gg d$ $\frac{d}{(z^2+(\frac{d}{2})^2)^{\frac{3}{2}}} \rightarrow 0$

$$\Rightarrow \vec{E} \rightarrow 0.$$ 1.5'

Question 3 [15]

Suppose the electric field in some region is found to be $E = kr^3\hat{r}$ (where k is some constant), in spherical coordinates.

(a) Please find the charge density ρ.

(b) Please find the total charge contained in a sphere of radius R, centered at the origin. (Do this in two different ways)

a) $\rho = \varepsilon_0 \nabla \cdot E$ 1'

$= \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 * kr^3)$

$= \varepsilon_0 \frac{1}{r^2} k(5r^4)$

$= 5\varepsilon_0 k r^2.$ 3'

b) ① Gauss's Law:

$Q_{enc} = \varepsilon_0 \cdot \oint E \cdot da$ 2'

$= \varepsilon_0 \cdot kr^3 \cdot 4\pi r^2 \Big|_{r=R}$

$= 4\pi \varepsilon_0 k R^5$ 3'

② Integration:

$Q = \int_V \rho \cdot dv$ 3'

$= \int_0^R 5\varepsilon_0 k r^2 \cdot 4\pi r^2 \cdot dr$

$= 20\pi\varepsilon_0 k \int_0^R r^4 dr$

$= 4\pi \varepsilon_0 k R^5.$ 3'

Question 4 [15]

A hollow spherical shell carries charge density of:

$$\rho = \frac{k}{r^2}$$

in the region $a \leq r \leq b$. Find the electric field in the following three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$

Plot $|E|$ as a function of r after you obtain the result.

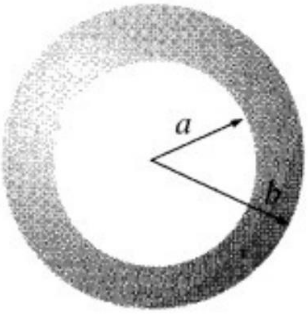


Figure 3: Question 4

Region I: $0 < r < a$.

$Q_{enc} = 0$

Apply Gauss's law.

$\boxed{\vec{E} = 0}$ [3']

Region II: $a < r < b$

Apply Gauss's law

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0} \quad [1']$$

$$\Leftrightarrow |\vec{E}| 4\pi r^2 = \frac{1}{\varepsilon_0} \int_a^r 4\pi r^2 \cdot \rho(r) \cdot dr \quad [1']$$

$$= \frac{1}{\varepsilon_0} \int_a^r 4\pi r^2 \cdot \frac{k}{r^2} dr = \frac{1}{\varepsilon_0} 4\pi k (r-a) \Rightarrow |\vec{E}| = \frac{k}{\varepsilon_0 r^2}(r-a) \quad [1']$$

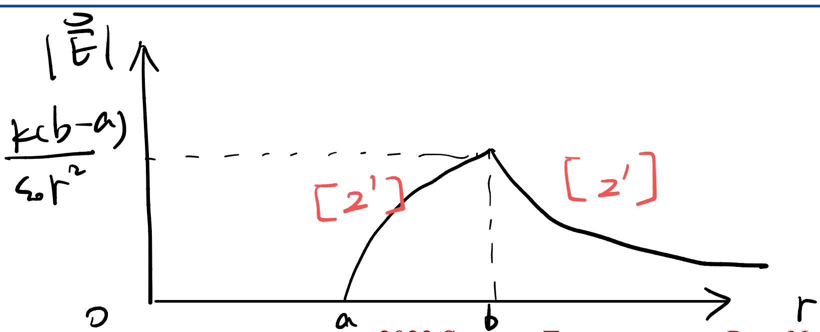
Hence, $\boxed{\vec{E} = \frac{k}{\varepsilon_0 r^2}(r-a)\hat{r}}$ [1']

Region III $r \geq b$

Apply Gauss's law, $\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0}$ [1']

$$\Leftrightarrow |\vec{E}| \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \int_a^b 4\pi r^2 \cdot \rho(r) \cdot dr = \frac{1}{\varepsilon_0} \int_a^b 4\pi r^2 \cdot \frac{k}{r^2} dr \Rightarrow |\vec{E}| = \frac{k}{\varepsilon_0 r^2}(b-a) \quad [1'] \quad [1']$$

Hence $\boxed{\vec{E} = \frac{k}{\varepsilon_0 r^2}(b-a)\hat{r}}$ [1']



Question 5 [20']

Three charges are situated at the corners of a square with side length a, as shown in the figure below:

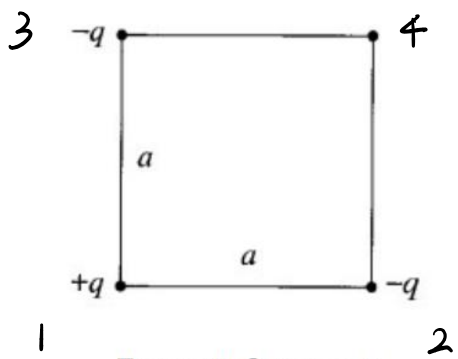


Figure 4: Question 5

(a) How much work does it take to bring in another charge, $+q$ from infinitely far away to the fourth corner of this square?

(b) How much work does it take to assemble the whole configuration of four charges?

(a) $V = \frac{1}{4\pi\varepsilon_0} \sum \frac{q_i}{r_{ij}}$ [2'] $= \frac{1}{4\pi\varepsilon_0}\left(\frac{-q}{a} + \frac{q}{\sqrt{2}a} + \frac{-q}{a}\right)$ [4'] $= \frac{q}{4\pi\varepsilon_0}\left(\frac{\sqrt{2}-4}{2}\right)$ [2']

$$\boxed{W_4 = qV = \frac{q^2}{4\pi\varepsilon_0 a}\left(\frac{\sqrt{2}-4}{2}\right)}$$ [2']

(b) $W_1 = 0$ [2']

$W_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{-q^2}{a}\right)$ [2']

$W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right)$ [2']

$W_4 = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{\sqrt{2}-4}{2}\right)$

Hence $W_{total} = \sum_{i=1}^{4} W_i$ [1']

$$= \boxed{\frac{(\sqrt{2}-4)}{4\pi\varepsilon_0}\frac{q^2}{a}}$$ [8']

Question 6 [20']

According to findings in quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density of:

$$\rho(r) = \frac{q}{\pi a^3} e^{-\frac{2r}{a}}$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom.

[Hint: Polarizability is the constant proportionality of the induced dipole moment to the field, namely, it is the α in the equation $p = \alpha E$. For solving this question, first calculate the electric field of the electron cloud $E_e(r)$, then expand the exponential, assuming $r \gg a$]

First find the field at radius r. applying Gauss's law:

$$\oint \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0} Q_{enc} \quad \Leftrightarrow \quad E\cdot 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0} \qquad [2']$$

$$Q_{enc} = \int_0^r \frac{q}{\pi a^3} e^{-\frac{2r'}{a}} \cdot 4\pi r'^2 dr' = \int_0^r \frac{4q}{a^3} r'^2 e^{-\frac{2r'}{a}} dr' \qquad [4']$$

$$= \frac{4q}{a^3}\left[-\frac{a}{2} e^{-\frac{2r'}{a}} \left(r'^2 + ar' + \frac{a^2}{2}\right)\right]\Big|_0^r \qquad [2']$$

$$= \frac{4q}{a^3}\cdot\left(-\frac{a}{2}\right)\cdot\left(e^{-\frac{2r}{a}}\left(r^2+ar+\frac{a^2}{2}\right) - \frac{a^2}{2}\right)$$

$$= q\left[1 - e^{-\frac{2r}{a}}\left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)\right] \qquad [4']$$

Hence $$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[1 - e^{-\frac{2r}{a}}\left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)\right]\hat{r}} \qquad [2']$$

Apply Taylor's expansion.

$$e^{-\frac{2r}{a}} = 1 - \left(\frac{2r}{a}\right) + \frac{1}{2!}\left(\frac{2r}{a}\right)^2 - \frac{1}{3!}\left(\frac{2r}{a}\right)^3 + \cdots$$

$$= 1 - \frac{2r}{a} + 2\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{r}{a}\right)^3 + \cdots$$

$$1 - e^{-\frac{2r}{a}}\left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right) = 1 - \left(1 - \frac{2r}{a} + 2\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{r}{a}\right)^3\right)\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)$$

since $r \ll a$

Only chose terms with order ≤ 3

$$= 1 - \left[1 + \frac{2r}{a} + \frac{2r^2}{a^2} - \frac{2r}{a} - \frac{4r^2}{a^2} - \frac{4r^3}{a^3} + 2\frac{r^2}{a^2} + 4\frac{r^3}{a^3} - \frac{4}{3}\left(\frac{r}{a}\right)^3\right]$$

$$= \boxed{\frac{4}{3}\left(\frac{r}{a}\right)^3 + \text{higher order terms}} \qquad [2']$$

Question 6

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Hence $\quad E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\left[\frac{4}{3}\left(\frac{r}{a}\right)^3\right] = \frac{qr}{3\pi\varepsilon a^3} = \frac{p}{3\pi\varepsilon a^3}$ [2']

$p = \alpha E$

Hence $\boxed{\alpha = 3\pi\varepsilon a^3}$ [2']