Chapter 6:
Static Magnetic Fields

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Many slides courtesy of Sung-Liang Chen

Everything you need to know about static magnetic fields in this course...

Main parts in static magnetic fields:

(I) Maxwell’s magnetostatic equations

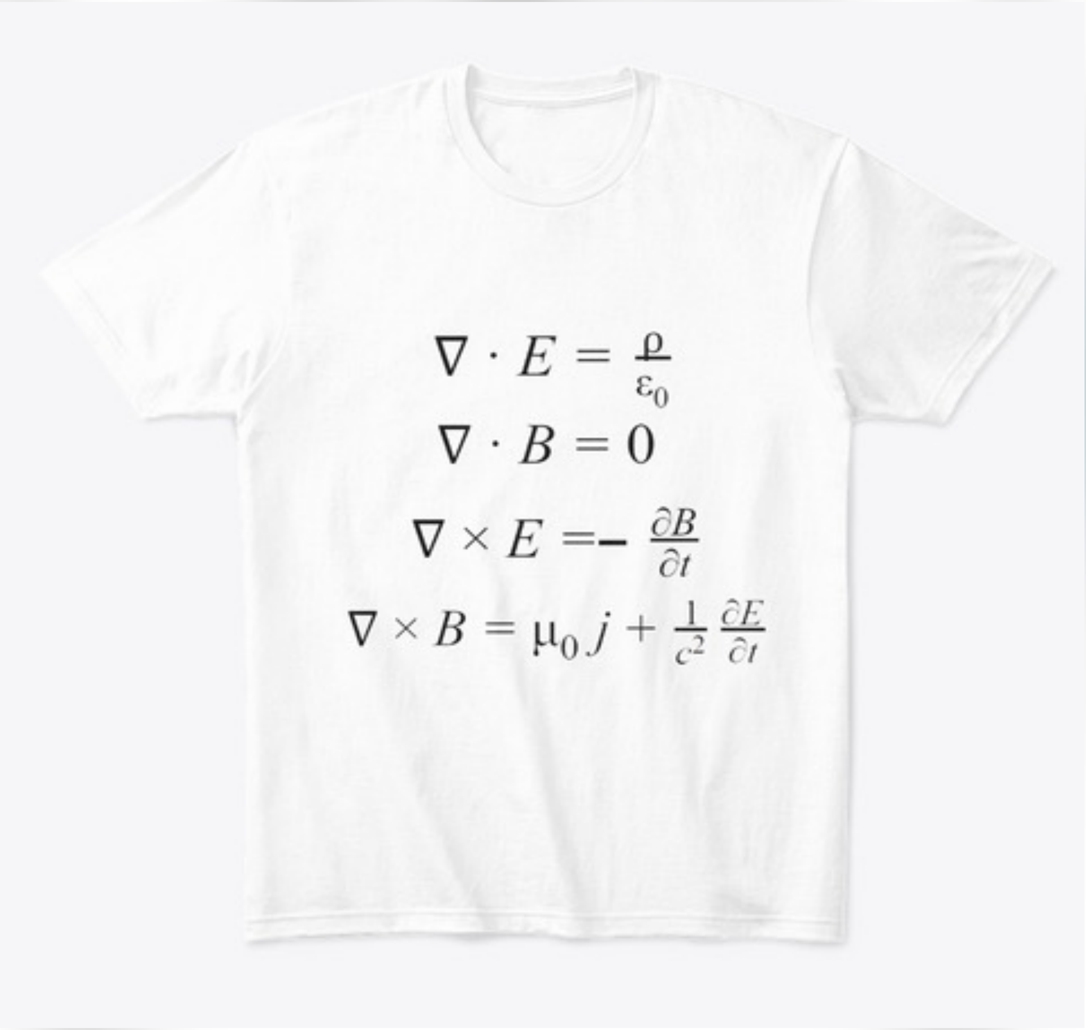
(II) Magnetisation

(III) Vector potential for magnetic field

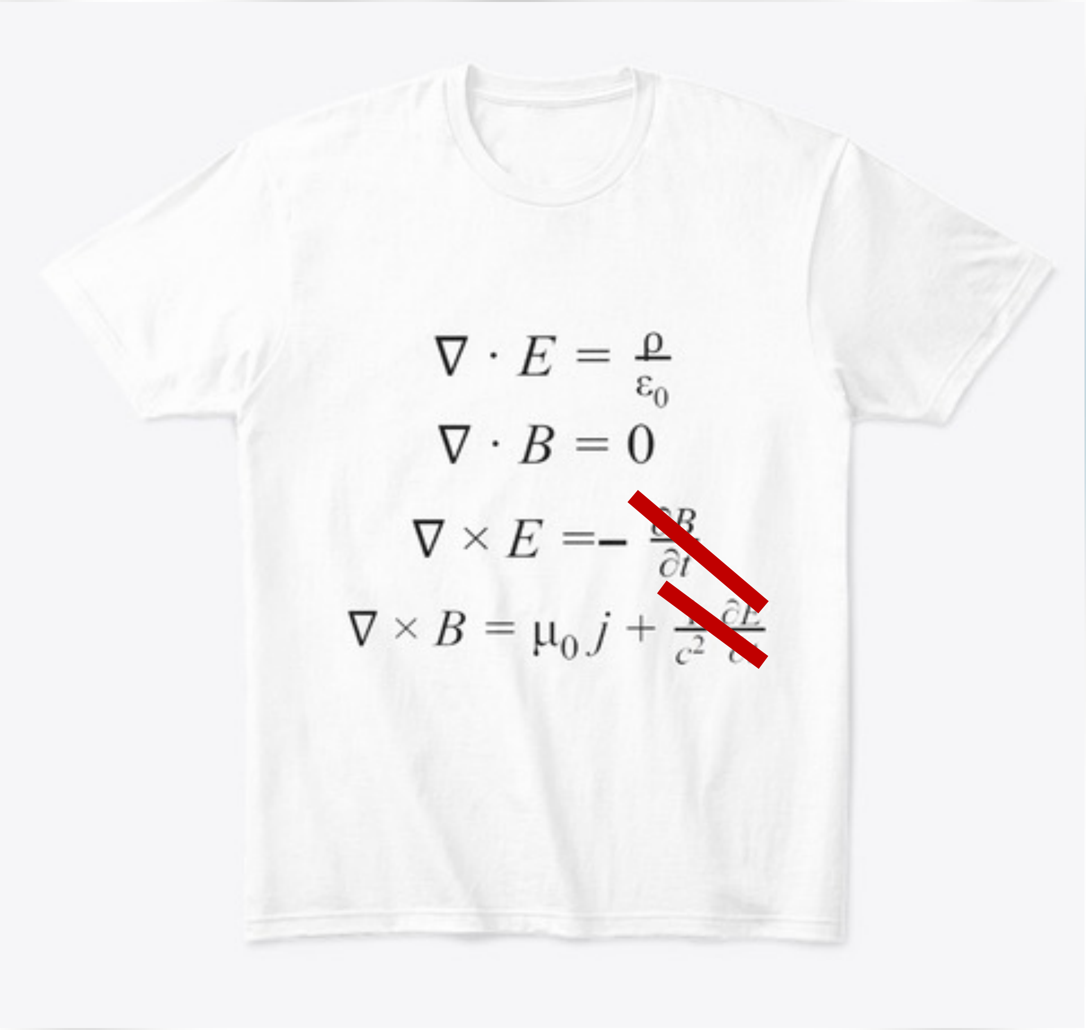
(IV) Boundary conditions

(V) Stored magnetic energy

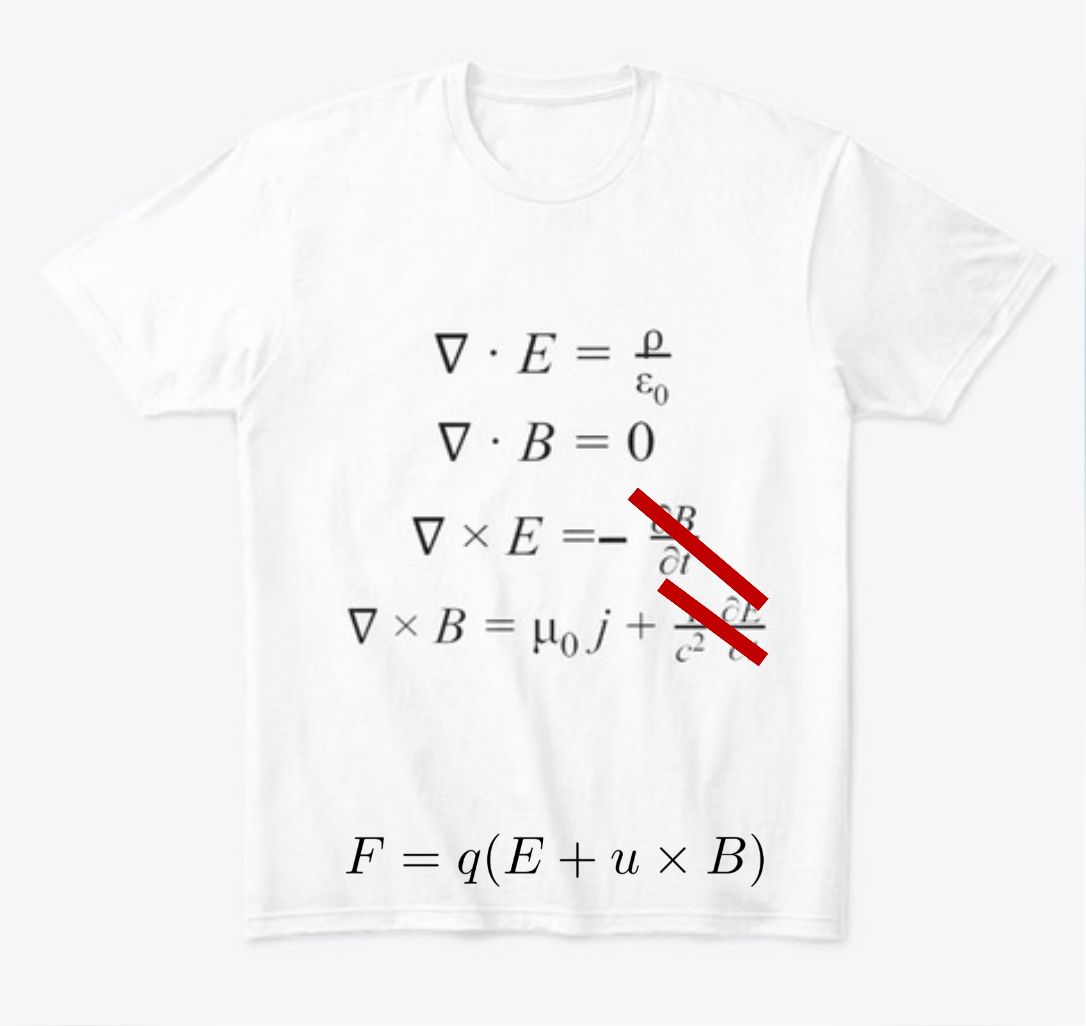
Everything we need in electrostatics and magnetostatics



Everything we need in electrostatics and magnetostatics



Everything we need in electrostatics and magnetostatics



Everything we need to know in magnetostatics

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

$$F = qu \times B$$

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

Analogy between electrostatics and magnetostatics

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

Electrostatics	Magnetostatics
$\boxed{\mathbf{E}}$	$\boxed{\mathbf{B}}$
$\mathbf{D}$	$\mathbf{H}$
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Postulates of Magnetostatics in Free Space	
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$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

Electrostatic and magnetostatic equations

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\dfrac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
$\boxed{\rho}$	$\boxed{\mathbf{J}}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$
$$\nabla \times B = \mu_0 J$$

No magnetic monopoles/charge: instead there is a current

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

Permittivity versus permeability

$$\nabla \times B = \mu_0 J$$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\boxed{\mathbf{P}}$	$\boxed{-\mathbf{M}}$
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Polarisation versus magnetisation

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
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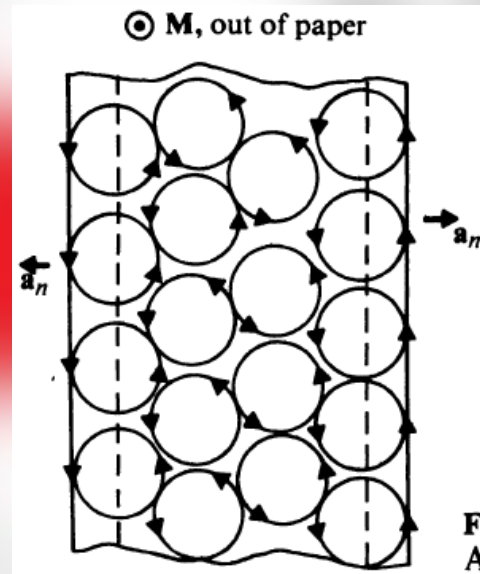

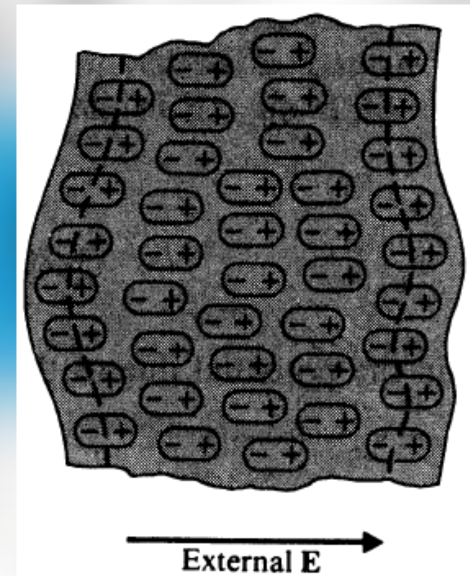

Polarisation versus magnetisation

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
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ϵ	$\dfrac{1}{\mu}$
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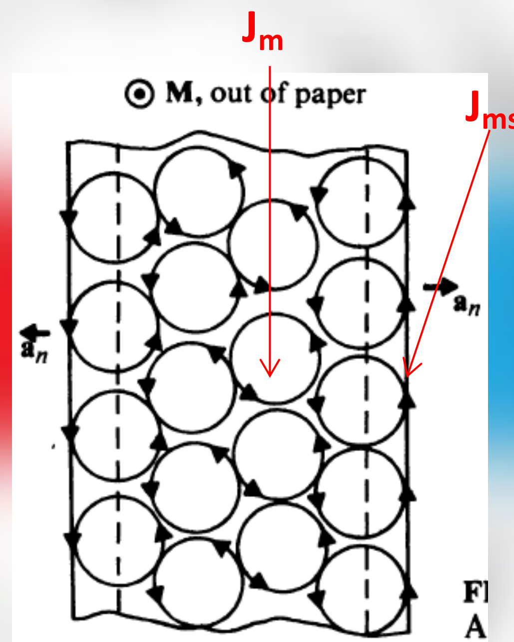
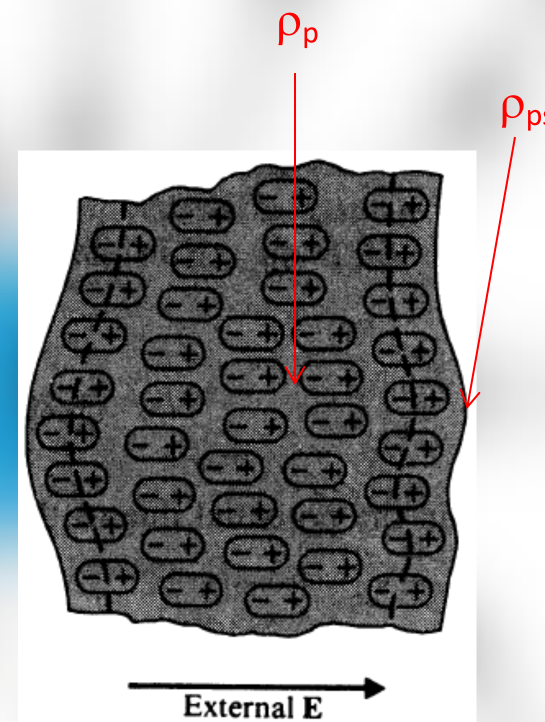


$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad \text{(A/m).}$$

$$\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\text{A/m}^2)$$



$$\rho_p = -\nabla \cdot \mathbf{P}.$$

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$

Polarisation versus magnetisation

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
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Displacement field and magnetic field intensity

Analogy between electrostatics and magnetostatics

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$\mathbf{E}$	$\mathbf{B}$
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$\cdot$	$\times$
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$$D = \epsilon_0 E + P$$

$$H = \frac{B}{\mu_0} - M$$

Displacement field and magnetic field intensity

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
E	**B**
D	**H**
ϵ	$\frac{1}{\mu}$
P	$-$**M**
ρ	**J**
V	**A**
$\cdot$	$\times$
$\times$	$\cdot$

For linear materials

$$P = \epsilon_0 \chi_e E$$

$$M = \chi_m H$$

Displacement field and magnetic field intensity

Types of magnetic materials

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

χ_m: magnetic susceptibility
μ_r: relative permeability

Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number).
Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number).
Ferromagnetic, if $\mu_r \gg 1$ (χ_m is a large positive number).

$$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$$

$$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$$

Types of magnetic materials

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

χ_m: magnetic susceptibility
μ_r: relative permeability

Compare to electric analogies: insulators, semiconductors and conductors

Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number).
Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number).
Ferromagnetic, if $\mu_r \gg 1$ (χ_m is a large positive number).

$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$

$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$

Boundary conditions for magnetostatic field

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),$$

Boundary conditions for magnetostatic field

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T)}.}$$

For linear materials

$$\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}.}$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),}$$

Boundary conditions for magnetostatic field

Electrostatics

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},}$$

Boundary conditions for magnetostatic field

Magnetostatics

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.}$$

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I.$$

$$H_{1t} - H_{2t} = J_{sn} \qquad \text{(A/m)},$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \text{(A/m)},}$$

Electrostatics

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},}$$

Boundary conditions for magnetostatic field

Magnetostatics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

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$$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \text{(A/m)},$$

Electrostatics

$$\nabla \times \mathbf{E} = 0$$

$$E_{1t} = E_{2t} \qquad \text{(V/m)},$$

$$J_{1n} = J_{2n} \qquad (\text{A/m}^2).$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.$$

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
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ρ	$\mathbf{J}$
$\boxed{V}$	$\boxed{\mathbf{A}}$
$\cdot$	$\times$
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Electric potential and vector potential

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

$$\nabla \times E = 0 \implies E = -\nabla V$$

$$\nabla \cdot B = 0 \implies B = \nabla \times A$$

Electric potential and vector potential

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
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$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 A = -\mu_0 J$$

Scalar and vector Poisson's equations

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
E	**B**
D	**H**
ϵ	$\frac{1}{\mu}$
P	$-$**M**
ρ	**J**
V	**A**
$\cdot$	$\times$
$\times$	$\cdot$

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{R} dV$$

$$A = \frac{\mu_0}{4\pi}\int_V \frac{J}{R} dV$$

Solutions to the scalar and vector Poisson's equations

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
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V	$\mathbf{A}$
$\cdot$	$\times$
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Also a scalar potential for the magnetic field in current-free region

$$\nabla \times B = 0 \implies$$

$$B = -\mu \nabla V_m$$

$$\nabla^2 V_m = 0$$

Scalar Poisson's Equation: Can also use method of images and boundary-value methods

Stored magnetic energy

Electrostatics

Work required to assemble a group of charges

$$W_e = \frac{1}{2}\int_{V'} \rho V \, dv \qquad \text{(J).}$$

Stored magnetic energy

Magnetostatics

Work required to send currents into conducting loops

Electrostatics

Work required to assemble a group of charges

$$W_e = \frac{1}{2}\int_{V'} \rho V \, dv \qquad \text{(J)}.$$

Stored magnetic energy

Magnetostatics

Work required to send currents into conducting loops

$$W_m = \int_{V'} w_m \, dv',$$

Linear medium

$$\mathbf{H} = \mathbf{B}/\mu,$$

$$w_m = \tfrac{1}{2}\,\mathbf{H} \cdot \mathbf{B} \qquad (\mathrm{J/m^3})$$

Electrostatics

Work required to assemble a group of charges

$$\boxed{W_e = \tfrac{1}{2} \int_{V'} \rho V \, dv \qquad (\mathrm{J}).}$$

Main parts in static magnetic fields:

(I) Maxwell’s magnetostatic equations

(II) Magnetisation

(III) Vector potential for magnetic field

(IV) Boundary conditions

(V) Stored magnetic energy

6-1 Introduction

- Electric charges at rest:

$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \times \mathbf{E} = 0.$$

  - Constitutive relation: $\mathbf{D} = \epsilon \mathbf{E},$
  - Electric force $\mathbf{F_e}$: $\boxed{\mathbf{F}_e = q\mathbf{E} \qquad \text{(N)}.}$
- Electric charges in motion:
  - Magnetic force $\mathbf{F_m}$: $\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)},}$

The force was found in experiments!
Defined as **B**: magnetic flux density (Wb/m^2=Tesla)

- Lorentz's force equation: $\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad \text{(N)},}$

6-2 Fundamental Postulates of Magnetostatics in Free Space

- Two postulates for **B** in free space:

Chap. 3

$$\boxed{\nabla \cdot \mathbf{B} = 0,}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.}$$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

$\mu_0 = 4\times10^{7}$ (H/m), permeability of free space

J: current density

Permeability is the measure of the ability of a material to support the formation of a magnetic field within itself. Hence, it is the degree of **magnetization** that a material obtains in response to an applied magnetic field.

$\nabla\cdot(\nabla\times\mathbf{B})=0$

$$\nabla \cdot \mathbf{J} = 0,$$

No magnetic charge

- Comparison of divergence **E** and **B**:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad\qquad \nabla \cdot \mathbf{B} = 0,$$

➔ No magnetic charge/monopole

The Law of Conservation of Magnetic Flux

- The integral form of **E**: $$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},}$$

  Q is the source of the total outward electric flux through any closed surface.

- The integral form of **B**: $$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0,}$$

  The total outward magnetic flux through any closed surface is zero.

  – **No magnetic flow sources**

  – The magnetic flux lines always close upon themselves

The Law of Conservation of Magnetic Flux

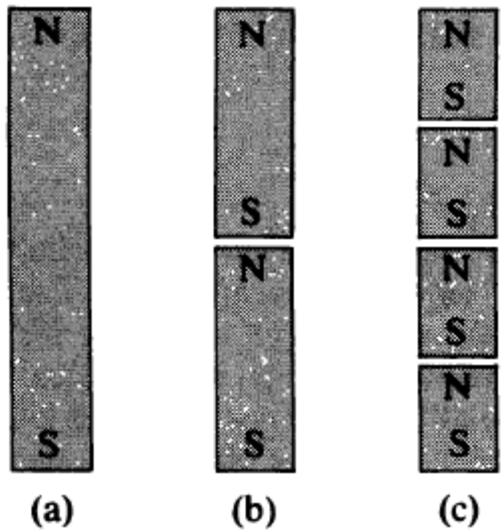
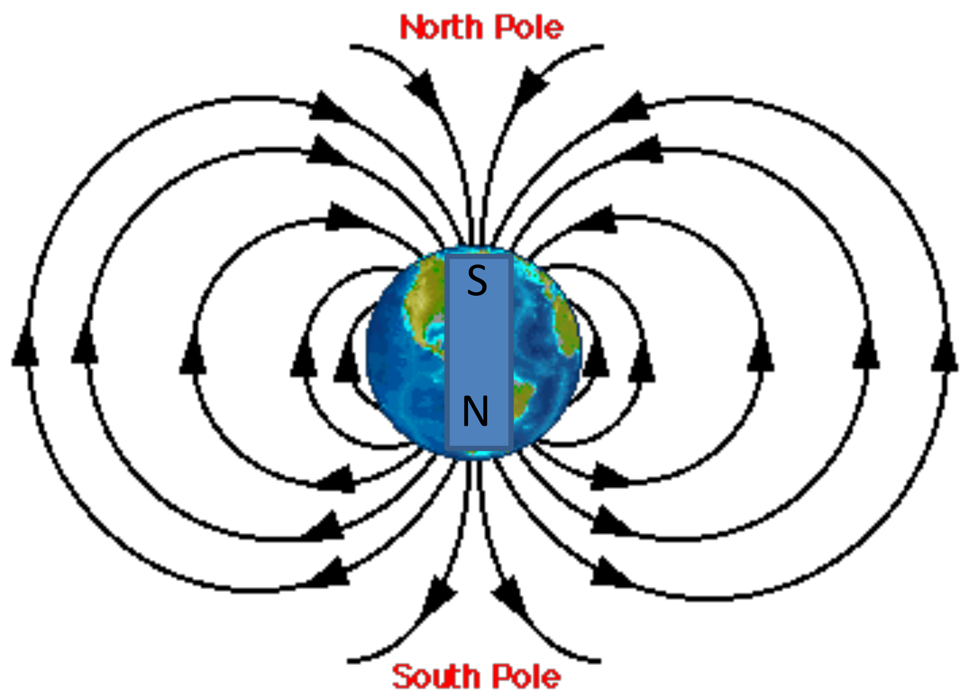


FIGURE 6–1
Successive division of a bar magnet.

The magnetic flux lines follow closed paths from one end of a magnet to the other end outside the magnet and then continue inside the magnet back to the first end.

The Earth's Magnetic Field



Ampere's Circuital Law

Chap.3

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

$$\boxed{\oint_C \mathbf{B} \cdot d\ell = \mu_0 I,}$$

C: the contour bounding the surface S

I: the total current through S

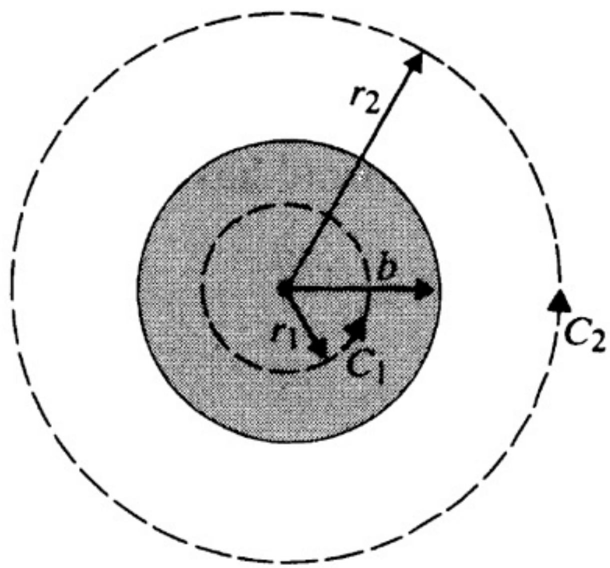
$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Gauss's law

Ampere's circuital law: the circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path.

Ampere's Circuital Law: Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

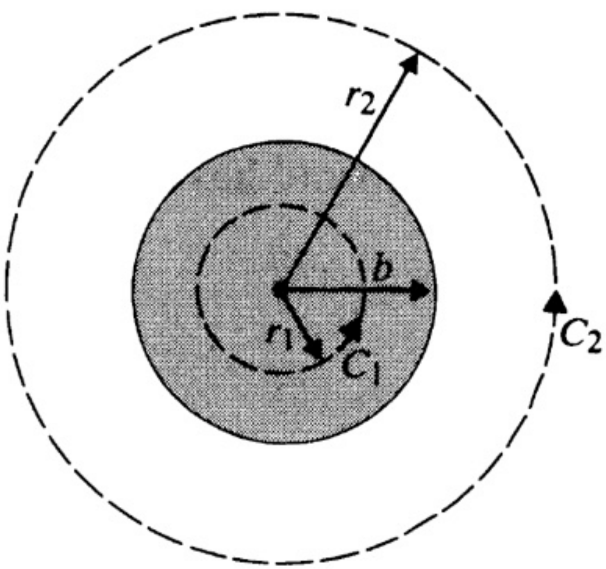


$$\boxed{\oint_C \mathbf{B} \cdot d\ell = \mu_0 I,}$$

Ampere's Circuital Law: Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

If we align the conductor along the z-axis, the magnetic flux density $\mathbf{B}$ will be ϕ-directed and will be constant along any circular path around the z-axis.



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Ampere’s Circuital Law: Example

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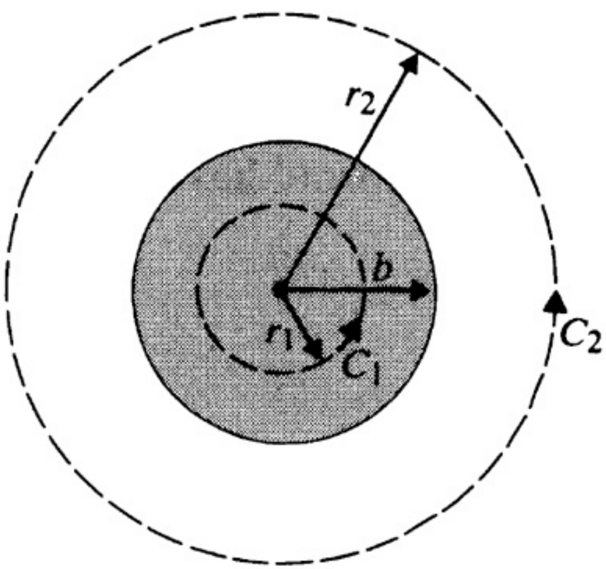


$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,$$

Inside the conductor:

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1}, \qquad d\boldsymbol{\ell} = \mathbf{a}_\phi r_1 \, d\phi$$

$$\oint_{C_1} \mathbf{B}_1 \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B_{\phi 1} r_1 \, d\phi = 2\pi r_1 B_{\phi 1}.$$

The current through the area enclosed by C_1 is

$$I_1 = \frac{\pi r_1^2}{\pi b^2} I = \left(\frac{r_1}{b}\right)^2 I.$$

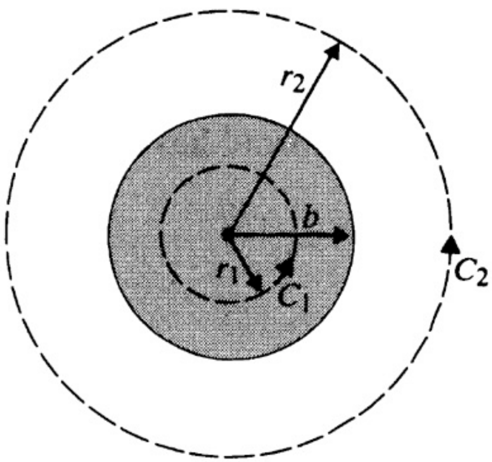
Therefore, from Ampère’s circuital law,

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1} = \mathbf{a}_\phi \frac{\mu_0 r_1 I}{2\pi b^2}, \qquad r_1 \le b.$$

Ampere's Circuital Law: Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

If we align the conductor along the z-axis, the magnetic flux density $\mathbf{B}$ will be ϕ-directed and will be constant along any circular path around the z-axis.



Outside the conductor:

$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2}, \qquad d\boldsymbol{\ell} = \mathbf{a}_\phi r_2\, d\phi$$

$$\oint_{C_2} \mathbf{B}_2 \cdot d\boldsymbol{\ell} = 2\pi r_2 B_{\phi 2}.$$

Path C_2 outside the conductor encloses the total current I. Hence

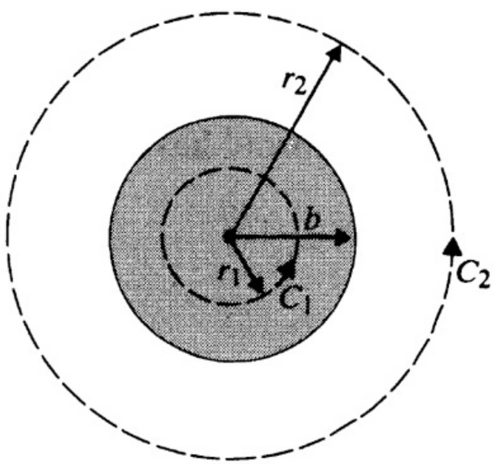
$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r_2}, \qquad r_2 \geq b.$$

$$\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,}$$

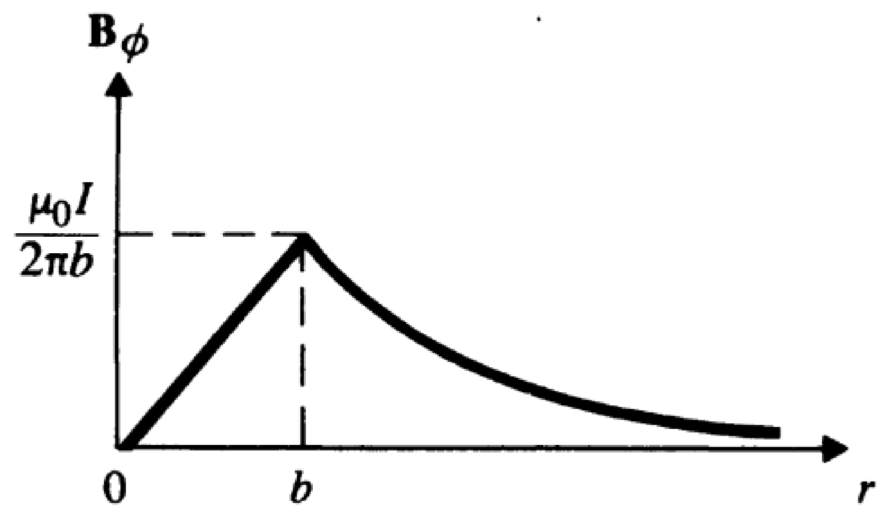
Ampere's Circuital Law: Example

An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

If we align the conductor along the z-axis, the magnetic flux density $\mathbf{B}$ will be ϕ-directed and will be constant along any circular path around the z-axis.



$$\boxed{\oint_C \mathbf{B} \cdot d\ell = \mu_0 I,}$$



Magnetic flux density of an infinitely long circular conductor carrying a current I out of paper

A Summary

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

6-3 Vector Magnetic Potential

Chap.3

$$\nabla \cdot \mathbf{B} = 0$$

B is solenoidal
By null identity

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad (\mathrm{T}).}$$

where **A**: **vector** magnetic potential (Wb/m)

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{\mathbf{E} = -\nabla V}$$

In magnetostatics: **J** ➔ **A** ➔ **B**
In electrostatics: ρ ➔ V ➔ **E**

$\nabla\cdot\mathbf{A}$=?

- To specify a vector, we should specify its curl and divergence. We have $\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T).}}$, how to choose $\nabla\cdot\mathbf{A}$?

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}.$$

Vector identity:

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\underline{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}.}$$

Definition of Laplacian of **A**

- In Cartesian: $\nabla^2 \mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z.$

(Similar to Laplacian of V)

- In other coordinates: should use the definition

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

To simplify the equation, we choose

$$\boxed{\nabla \cdot \mathbf{A} = 0,}$$

Vector's Poisson's equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

In Cartesian coordinate:

$$\nabla^2 A_x = -\mu_0 J_x,$$
$$\nabla^2 A_y = -\mu_0 J_y,$$
$$\nabla^2 A_z = -\mu_0 J_z.$$

By comparison

Solution:

$$A_x = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_x}{R} dv'.$$

Combine 3 components

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv' \qquad \text{(Wb/m)}.}$$

Analogy in electrostatics:

Scalar's Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Solution:

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv'.$$

Review The Analogy

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv' \qquad \text{(Wb/m).}$$

In magnetostatics: **J** ➔ **A** ➔ **B**

In electrostatics: ρ ➔ V ➔ **E**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv'.$$

Relation of Magnetic Flux Φ and Magnetic Vector Potential **A**

$$\underline{\Phi} = \int_S \underline{\mathbf{B}} \cdot d\mathbf{s}.$$

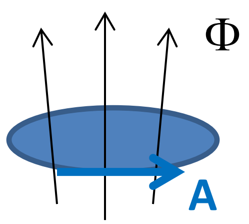
Magnetic flux

Magnetic flux density

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell} \qquad \text{(Wb)}.$$

Physical significance of **A**: line integral of **A** around any closed path = the total Φ passing through the area closed by the path

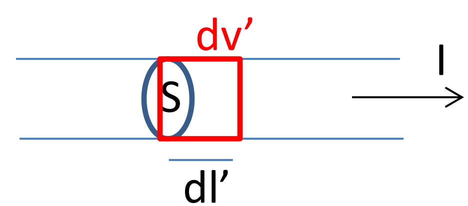


6-4 The Biot-Savart Law and Applications

- The magnetic field due to a current-carrying circuit.
- For a thin wire with cross-sectional area S (dv'=Sdl'), we have

J: (A/m^2)
JS=I

$$\mathbf{J}\,dv' = JS\,d\ell' = I\,d\ell',$$



$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\,dv' \qquad \text{(Wb/m).}}$$

$$\mathbf{J}\,dv' = JS\,d\ell' = I\,d\ell',$$

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \qquad \text{(Wb/m),}}$$

C' is closed because current must flow in a closed path.

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \qquad \text{(Wb/m)},}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[\frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \right]$$

$$= \frac{\mu_0 I}{4\pi} \oint_{C'} \underline{\nabla \times \left(\frac{d\boldsymbol{\ell}'}{R} \right)}.$$

unprimed curl: curl to field points
primed integration: to source coordinates

$$\nabla \times (f\mathbf{G}) = f\nabla \times \mathbf{G} + (\nabla f) \times \mathbf{G}.$$

with $f = 1/R$ and $\mathbf{G} = d\boldsymbol{\ell}'$,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \underline{\left[\frac{1}{R} \nabla \times d\boldsymbol{\ell}' + \left(\nabla \frac{1}{R} \right) \times d\boldsymbol{\ell}' \right]}.$$

(1) (2)

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \underline{\left[\frac{1}{R} \nabla \times d\boldsymbol{\ell}' + \left(\nabla \frac{1}{R} \right) \times d\boldsymbol{\ell}' \right]}.$$

(1) (2)

For term (1): primed and unprimed coordinates are independent ➔ 0

For term (2):

$$\frac{1}{R} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2};$$

R is from source to field

$$\underline{\nabla\left(\frac{1}{R}\right)} = \mathbf{a}_x \frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \mathbf{a}_y \frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \mathbf{a}_z \frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

Quotient rule and chain rule

$$= -\frac{\mathbf{a}_x(x - x') + \mathbf{a}_y(y - y') + \mathbf{a}_z(z - z')}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}$$

$$= -\frac{\mathbf{R}}{R^3} = \underline{-\mathbf{a}_R \frac{1}{R^2}},$$

$\mathbf{a}_R$: unit vector from source to field

$\boxed{\text{Useful!}}$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \left[\frac{1}{R} \cancel{\nabla \times d\boldsymbol{\ell}'} + \underline{\left(\nabla \frac{1}{R} \right)} \times d\boldsymbol{\ell}' \right].$$

$$\nabla\left(\frac{1}{R}\right) = -\mathbf{a}_R \frac{1}{R^2},$$

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \quad \text{(T).}}$$

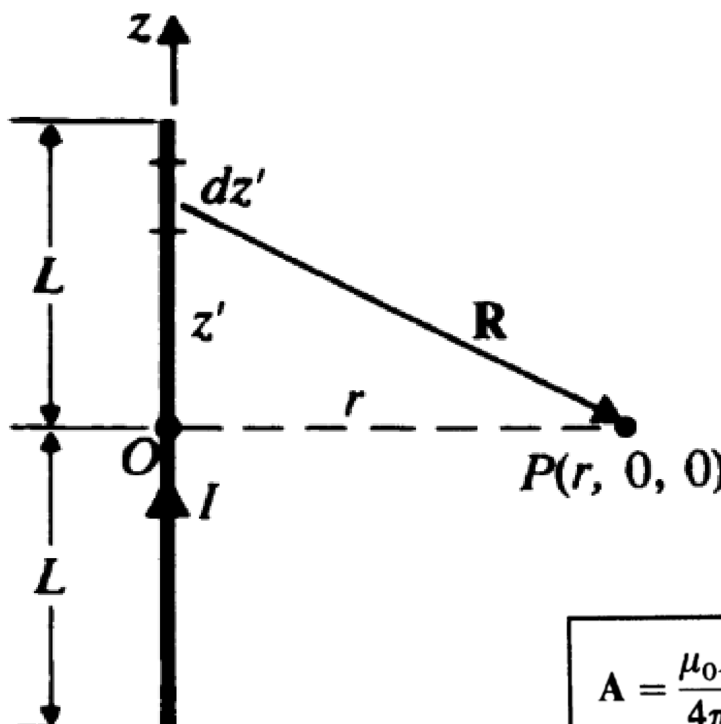
Biot-Savart law: **B** due to a current element Idl'

Comparison with Ampere's circuital law: $\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,}$

Because **B** does not have to be constant along C, **Biot-Savart law is more general** to determine B than Ampere's circuital law (although the former is more difficult in calculation).

6-4 The Biot-Savart Law and Applications

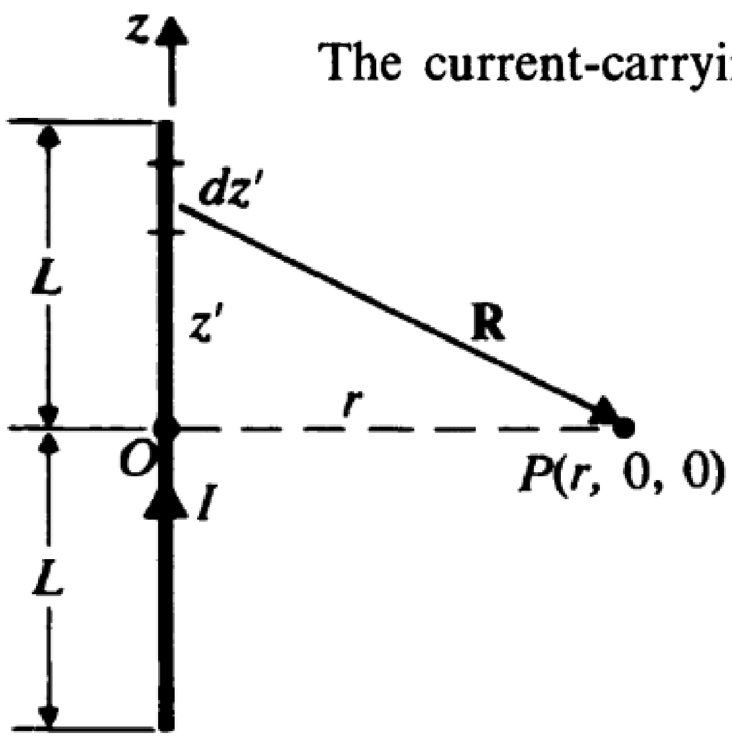
A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density **B** at a point located at a distance r from the wire in the bisecting plane: (a) by determining the vector magnetic potential **A** first, and (b) by applying Biot-Savart law.



$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\ell'}{R} \qquad \text{(Wb/m)},$$

6-4 The Biot-Savart Law and Applications

Currents exist only in closed circuits. Hence the wire in the present problem must be a part of a current-carrying loop with several straight sides. Since we do not know the rest of the circuit, Ampère's circuital law cannot be used to advantage.



The current-carrying line segment is aligned with the z-axis.

$$d\boldsymbol{\ell}' = \mathbf{a}_z \, dz'$$

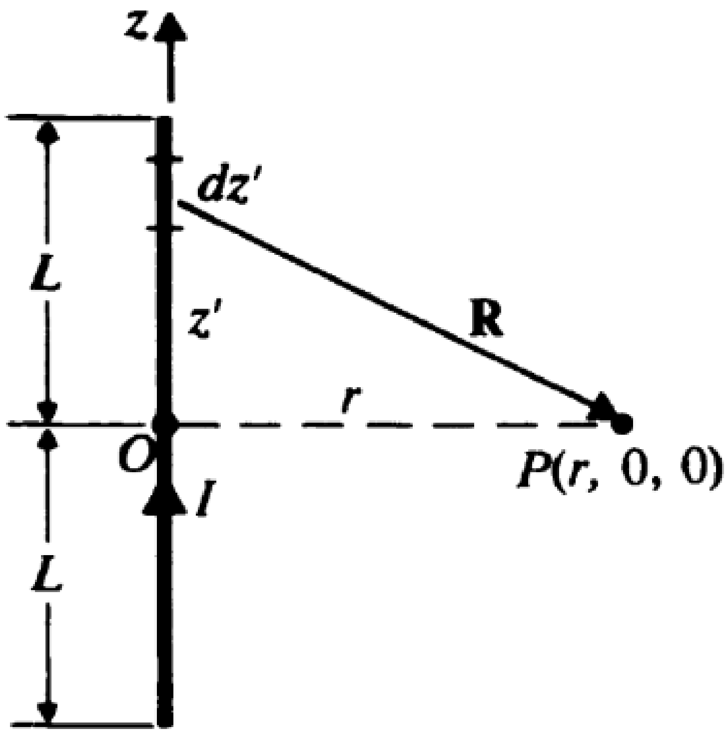
Substituting $R = \sqrt{z'^2 + r^2}$

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \qquad \text{(Wb/m)},}$$

6-4 The Biot-Savart Law and Applications

Substituting $R = \sqrt{z'^2 + r^2}$

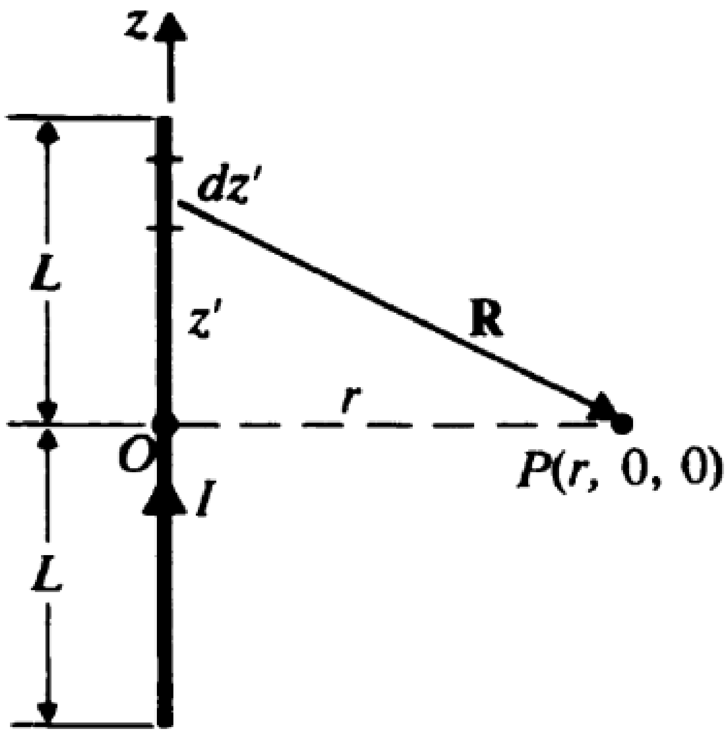
$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\ell'}{R} \qquad \text{(Wb/m)},}$$



$$\begin{aligned}
\mathbf{A} &= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz'}{\sqrt{z'^2 + r^2}} \\
&= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \left[\ln\left(z' + \sqrt{z'^2 + r^2}\right)\right]\Big|_{-L}^{L} \\
&= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L}.
\end{aligned}$$

6-4 The Biot-Savart Law and Applications

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{a}_z A_z) = \mathbf{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \mathbf{a}_\phi \frac{\partial A_z}{\partial r}$$



Cylindrical symmetry around the wire assures that $\partial A_z/\partial \phi = 0$. Thus,

$$\mathbf{B} = -\mathbf{a}_\phi \frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L}\right]$$

$$= \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2 + r^2}}.$$

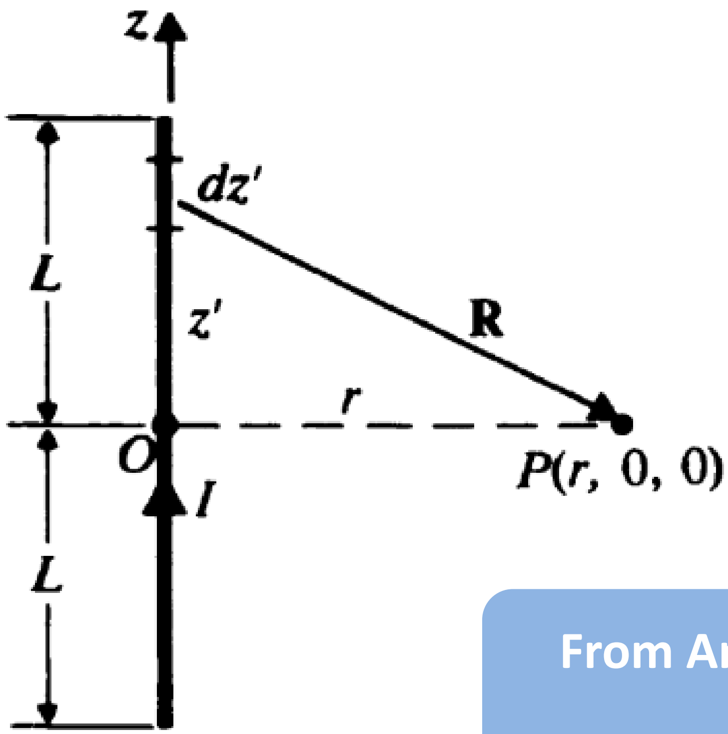
6-4 The Biot-Savart Law and Applications

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{a}_z A_z) = \mathbf{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \mathbf{a}_\phi \frac{\partial A_z}{\partial r}$$

Cylindrical symmetry around the wire assures that $\partial A_z/\partial\phi = 0$. Thus,

$$\mathbf{B} = -\mathbf{a}_\phi \frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right]$$

$$= \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}.$$



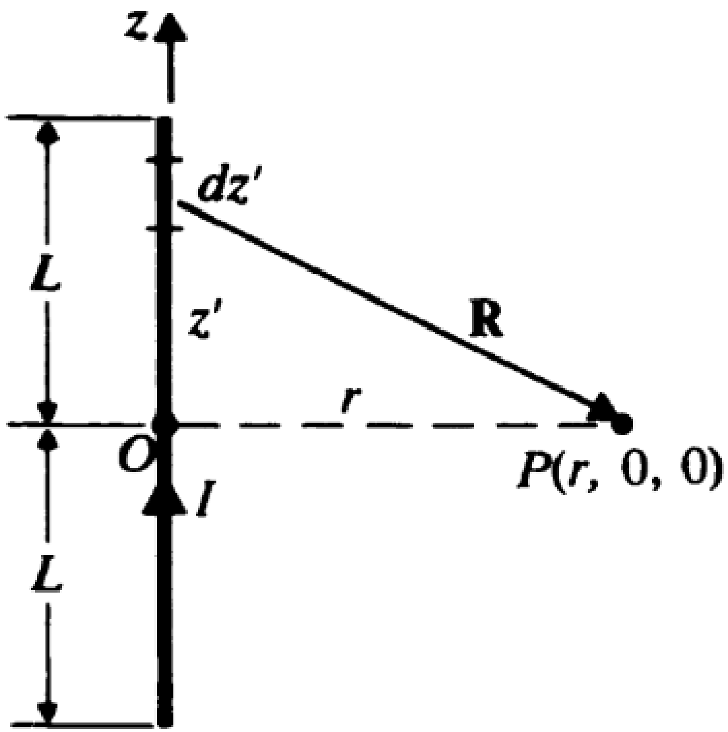
When $r \ll L$

$$\mathbf{B}_\phi = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}$$

From Ampere's Law for the infinitely long wire

6-4 The Biot-Savart Law and Applications

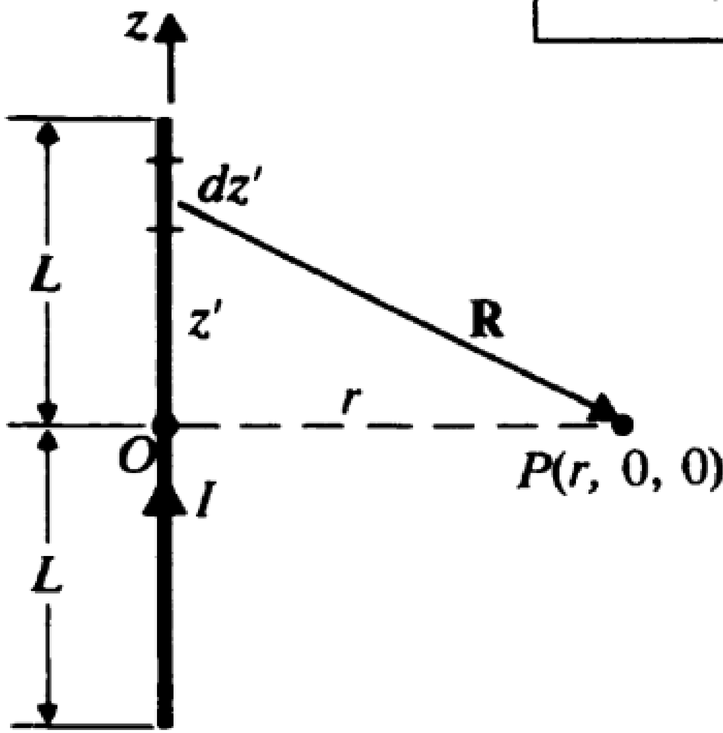
A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density $\mathbf{B}$ at a point located at a distance r from the wire in the bisecting plane: (a) by determining the vector magnetic potential $\mathbf{A}$ first, and (b) by applying Biot-Savart law.



$$\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{\ell}' \times \mathbf{R}}{R^3}\right) \qquad \text{(T).}}$$

6-4 The Biot-Savart Law and Applications

$$\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{\ell}' \times \mathbf{R}}{R^3}\right) \qquad \text{(T)}.}$$

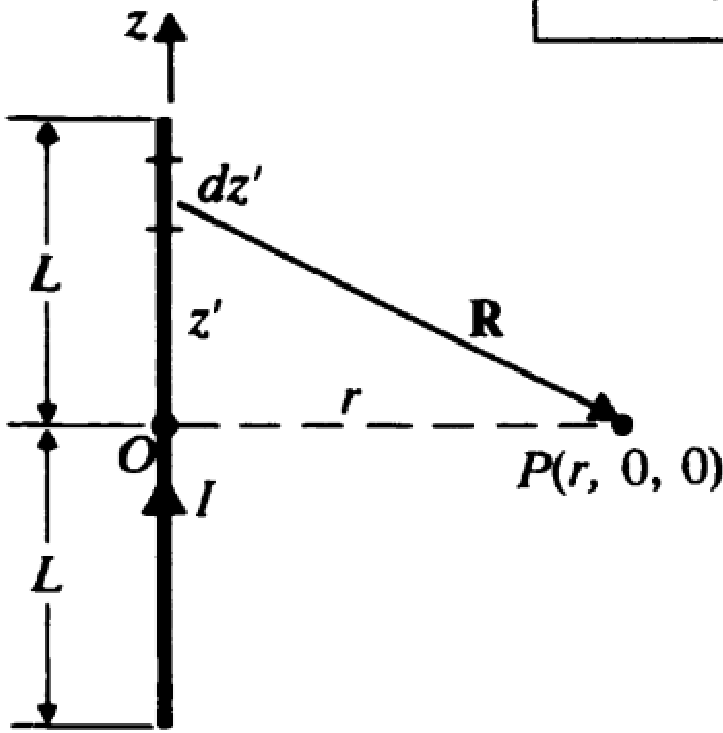


$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_z\, dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r\, dz'.$$

6-4 The Biot-Savart Law and Applications

$$\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{\ell}' \times \mathbf{R}}{R^3}\right) \qquad \text{(T).}}$$



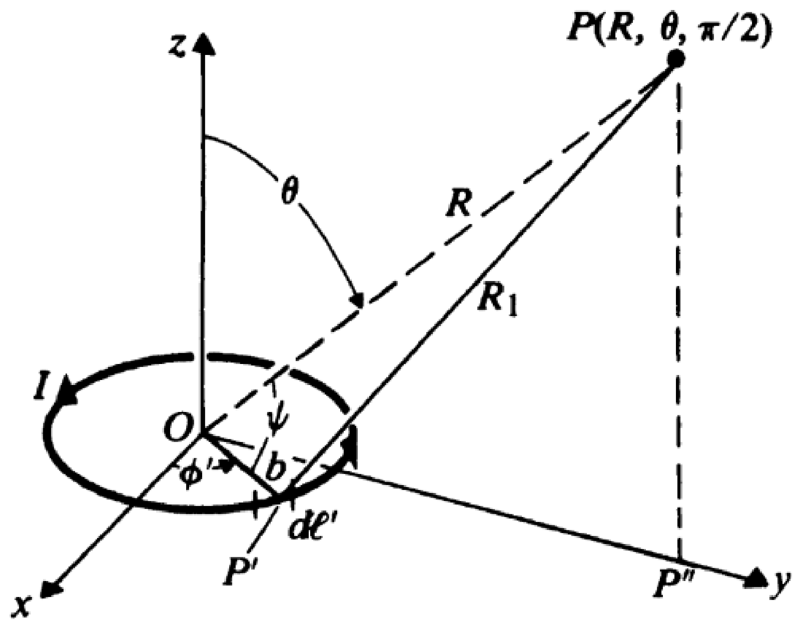
$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_z\, dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r\, dz'.$$

$$\mathbf{B} = \int d\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{r\, dz'}{(z'^2 + r^2)^{3/2}}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2 + r^2}},$$

6-5 The Magnetic Dipole: Example

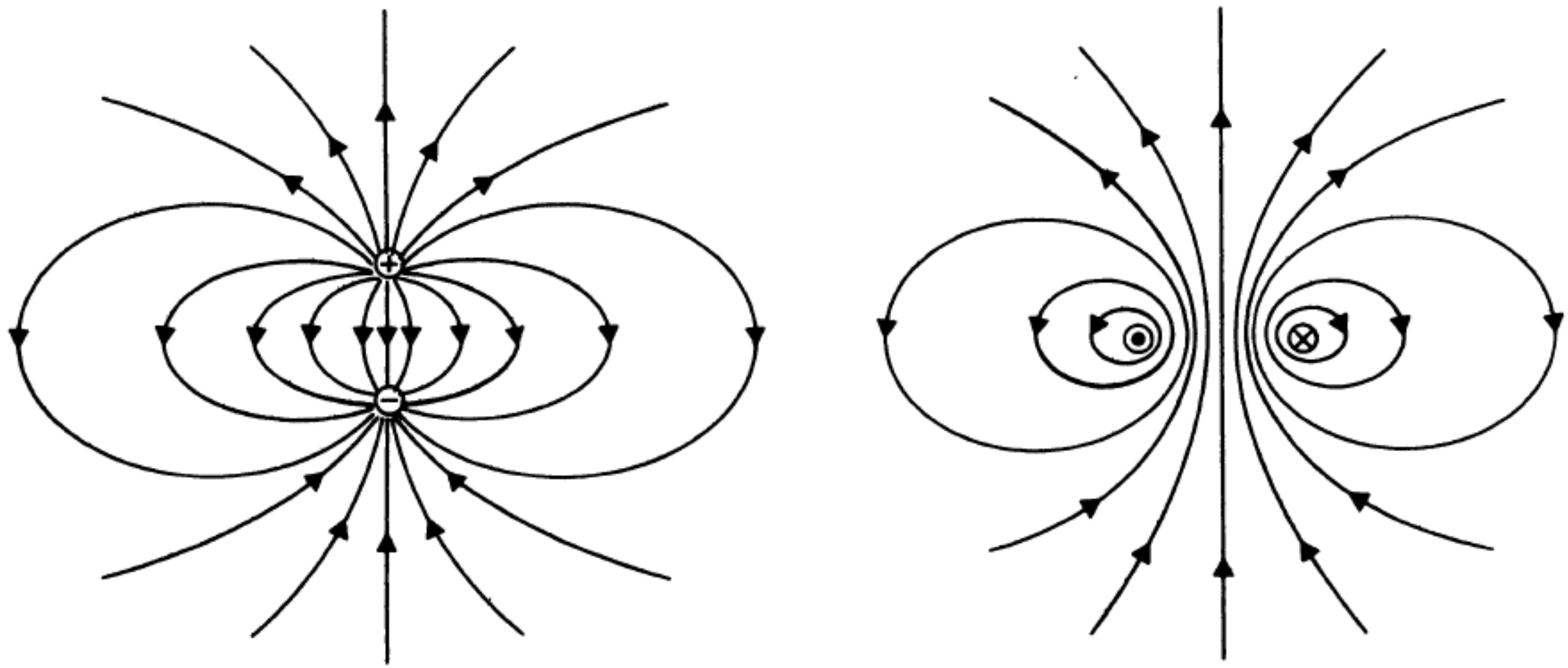


$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta.$$

$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R \, 2\cos\theta + \mathbf{a}_\theta \sin\theta),$$

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).$$

$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),$$



(a) Electric dipole.

(b) Magnetic dipole.

FIGURE 6–9
Electric field lines of an electric dipole and magnetic flux lines of a magnetic dipole.

- Difference: **E** terminated on the charges; **B** continuous
- Fields (**E** and **B**) are similar far from the dipole (e-dipole and m-dipole)

$$\boxed{\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta.}$$

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 (I\pi b^2)}{4\pi R^2} \sin\theta$$

$$\boxed{\mathbf{a}_z \times \mathbf{a}_R = \mathbf{a}_\phi \sin\theta}$$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},}$$

where $\mathbf{m} = \mathbf{a}_z I\pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \qquad (\mathrm{A\cdot m^2})$

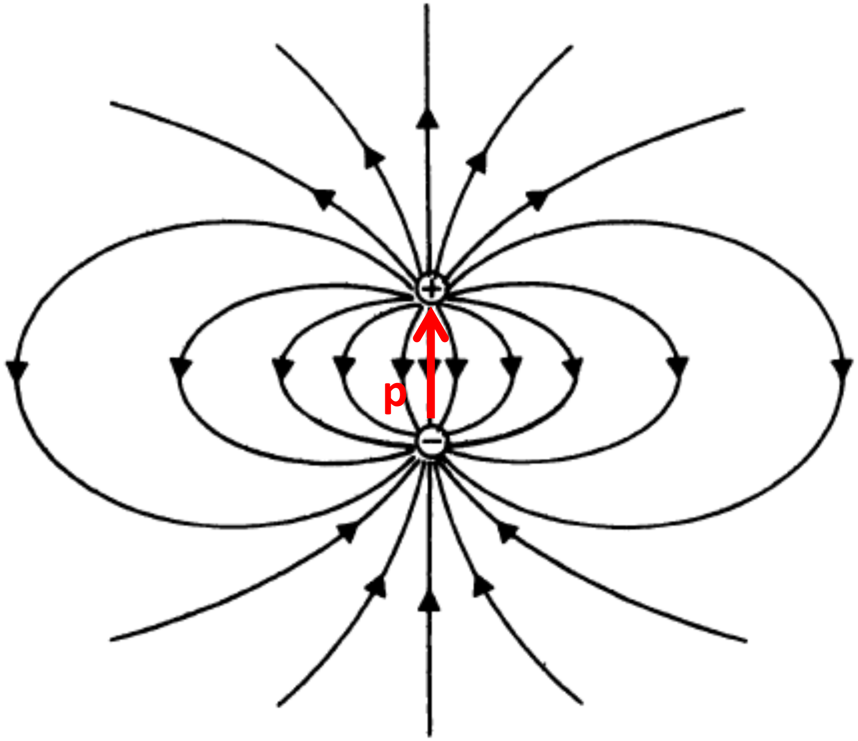
Defined as **magnetic dipole moment**

$$\boxed{\mathbf{B} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),}$$

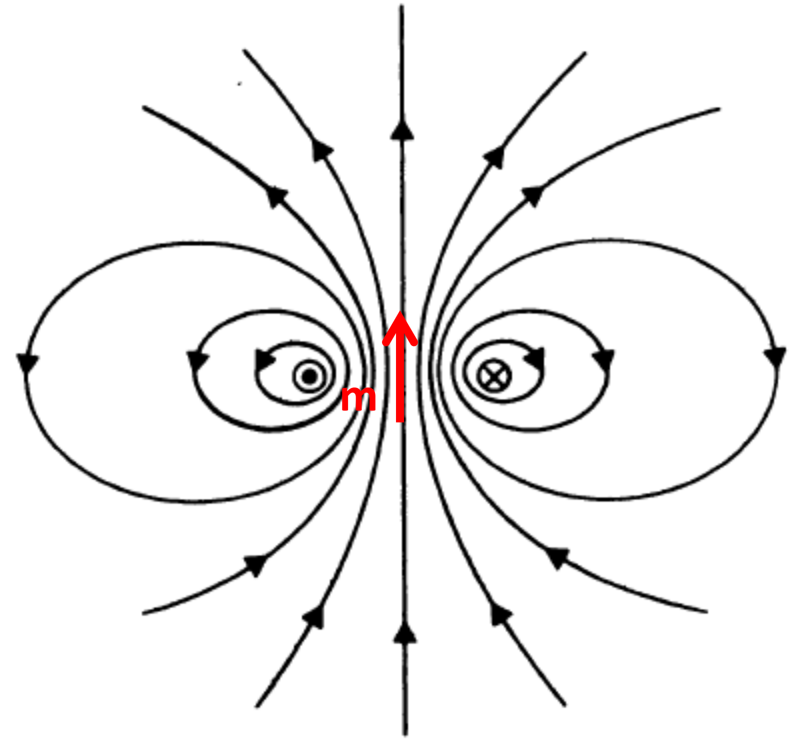
$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3} (\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T)}.}$$

$\boxed{\mathbf{p} = q\mathbf{d}}$

Electric dipole moment



(a) Electric dipole.



(b) Magnetic dipole.

Electric dipole moment **p**

Magnetic dipole moment **m**

p->**m**

$1/\varepsilon_0 \to \mu_0$

$\bullet \to \times$

Right-hand rule: **m** along thumb and **direction of current** along fingers

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)},$$

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)},$$

We call a small current-carrying loop a **magnetic dipole**

6-5.1 Scalar Magnetic Potential

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

For a region J=0

$$\nabla \times \mathbf{B} = 0.$$

By null identity

$$\mathbf{B} = -\mu_0 \nabla V_m,$$

V_m: scalar magnetic potential (A)

$$\boxed{\mathbf{E} = -\nabla V}$$

Analogy

V_m	V
$\mathbf{B}/\mu_0$	$\mathbf{E}$

Analogous to electric potential

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\mathbf{B} = -\mu_0 \nabla V_m,$$

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V)}.$$

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \mathbf{B} \cdot d\ell.$$

E $\downarrow$ $1/(4\pi\varepsilon_0)$

B$/\mu_0$ $\downarrow$ $(\mu_0/(4\pi))\,/\mu_0$

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv' \qquad \text{(V)}.$$

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R}\, dv' \qquad \text{(A)}.$$

No μ_0 in V_m

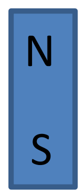
If there ***were*** magnetic charges

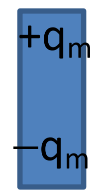
Fictitious magnetic charges:

1. A mathematical (not physical) model
2. Helpful in the discussion of magnetostatics from electrostatics knowledge

Magnetic Source

- Traditional view: magnetic pole (a magnetic dipole)





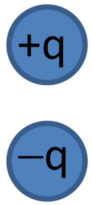
Fictitious magnetic charges

$$\mathbf{m} = q_m \mathbf{d} = \mathbf{a}_n I S.$$

- Microscopic view: circulating current

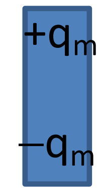
Potential due to a dipole

Electric dipole



$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},$$

Magnetic dipole



$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \qquad \text{(A)}.$$

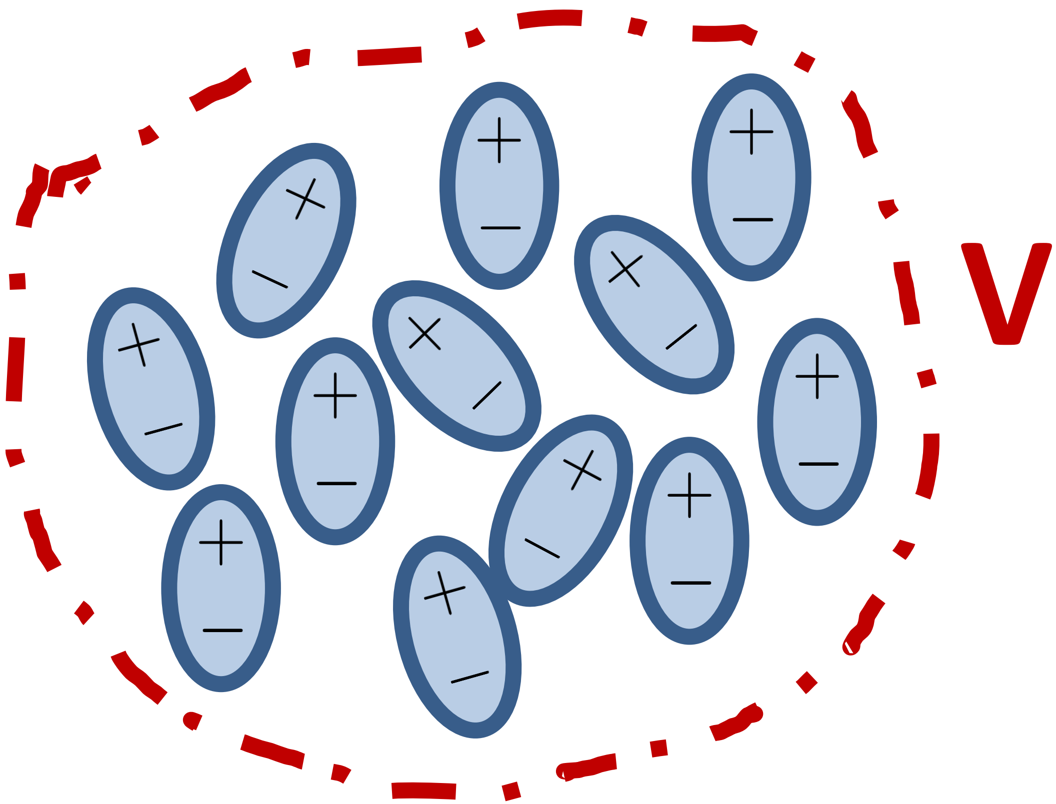
No μ_0 in V_m

$$\boxed{\mathbf{B} = -\mu_0 \nabla V_m,}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T)}.}$$

How to use fictitious magnetic charges…

- Recall from chapter 3

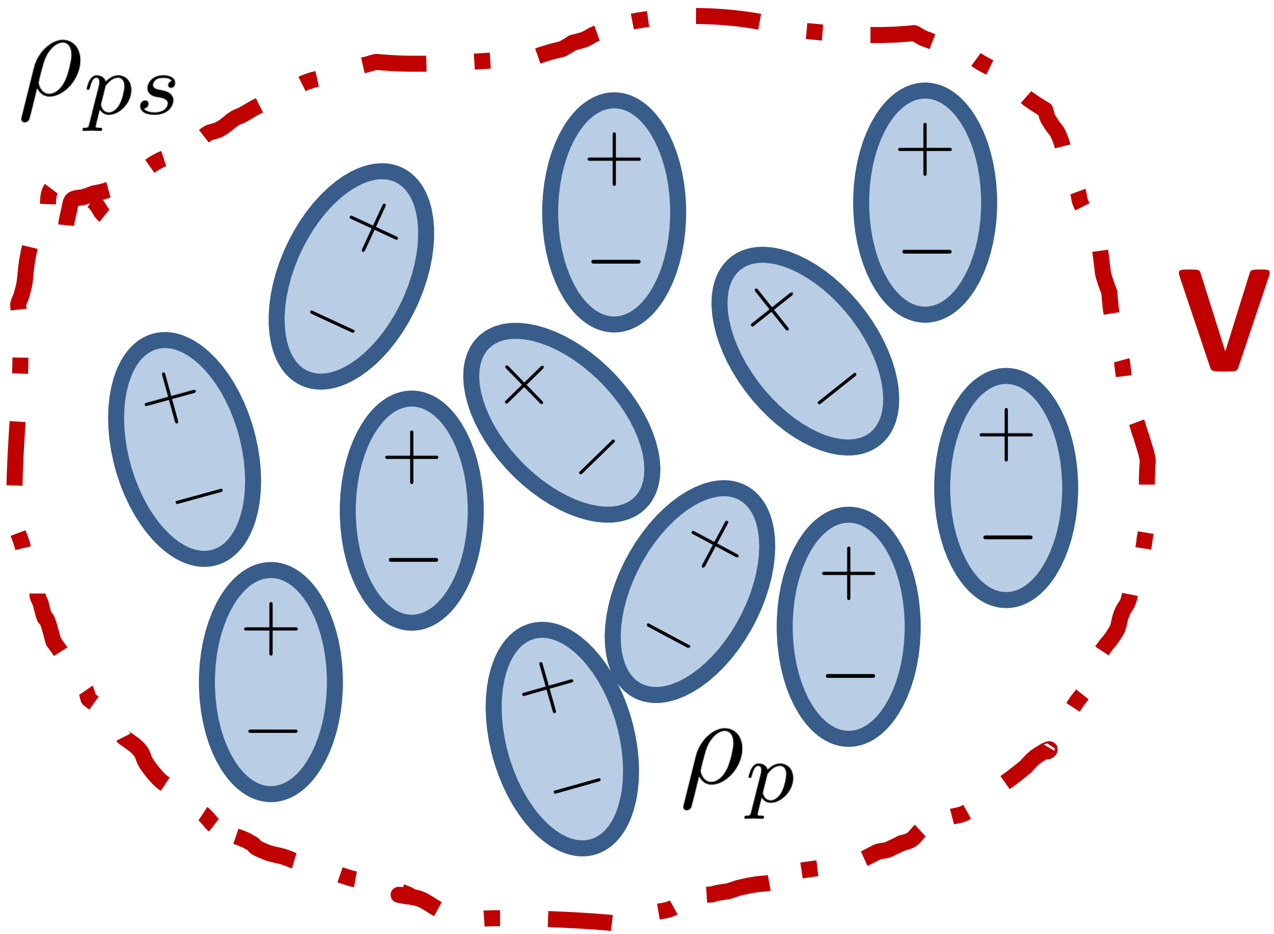


Charge Distributions of Polarized Dielectrics

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

Polarization surface charge densities,
and polarisation bound-charge densities



$$\boxed{V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dv'.$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R}\, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(-\nabla' \cdot \mathbf{P})}{R}\, dv',$$

V = contribution of surface charge distribution

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

+

contribution of volume charge distribution

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{ps}}{R}\, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_p}{R}\, dv'.$$

In magnetostatics without external current just replace V with scalar magnetic potential…

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}$$

$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2}$$

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\mathbf{M} \cdot \mathbf{a}_R}{R^2}\, dv'$$

$$V_m = \frac{1}{4\pi} \oint_{S'} \frac{\mathbf{M} \cdot \mathbf{a}_n'}{R}\, ds' + \frac{1}{4\pi} \int_{V'} \frac{-(\nabla' \cdot \mathbf{M})}{R}\, dv'$$

In magnetostatics without external current just replace V with scalar magnetic potential…

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \Rightarrow V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2}$$

$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\text{A/m})}$$

Fictitious magnetization surface charge density

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\text{A/m}^2).}$$

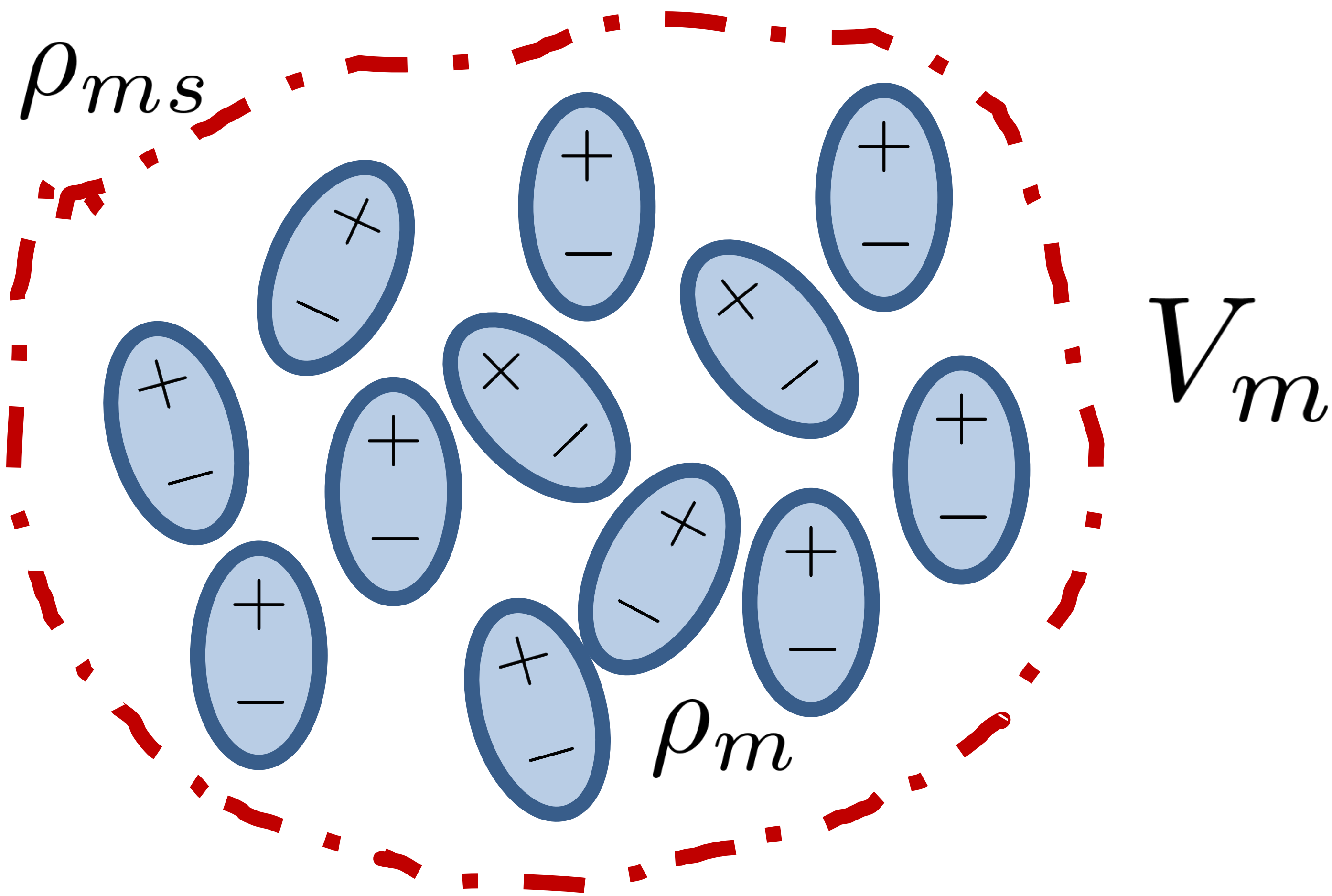
Fictitious magnetization volume charge density

$$V_m = \frac{1}{4\pi} \oint_{S'} \frac{\mathbf{M} \cdot \mathbf{a}_n'}{R} \, ds' + \frac{1}{4\pi} \int_{V'} \frac{-(\nabla' \cdot \mathbf{M})}{R} \, dv'$$

$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\text{A/m})}$$

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\text{A/m}^2).}$$

Fictitious surface charge densities, and bound-charge densities



6-6.1 Equivalent Magnetization Charge Densities

- In a **current-free** region, we may define V_m
- **B** can be found by $\mathbf{B} = -\mu_0 \nabla V_m,$

$$\boxed{V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \qquad \text{(A).}}$$

$$\boxed{d\mathbf{m} = \mathbf{M}\, dv'}$$

$$dV_m = \frac{\mathbf{M} \cdot \mathbf{a}_R}{4\pi R^2}\ dv'$$

integration

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\mathbf{M}\cdot\mathbf{a}_R}{R^2}\,dv'.$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2}\,dv',$$

Following similar steps as in section 3-7.1

$$V_m = \frac{1}{4\pi}\oint_{S'} \frac{\mathbf{M}\cdot\mathbf{a}'_n}{R}\,ds' + \frac{1}{4\pi}\int_{V'} \frac{-(\nabla'\cdot\mathbf{M})}{R}\,dv',$$

$$\boxed{\rho_{ms} = \mathbf{M}\cdot\mathbf{a}_n \qquad (\text{A/m})}$$

$$\boxed{\rho_m = -\nabla\cdot\mathbf{M} \qquad (\text{A/m}^2).}$$

$$V = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\mathbf{P}\cdot\mathbf{a}'_n}{R}\,ds' + \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{(-\nabla'\cdot\mathbf{P})}{R}\,dv',$$

$$\boxed{\rho_{ps} = \mathbf{P}\cdot\mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla\cdot\mathbf{P}.}$$

- A **polarized dielectric** may be replaced by an equivalent ρ_p and ρ_{ps}

Polarization charge density

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

- A **magnetized body** may be replaced by an equivalent ρ_m and ρ_{ms}

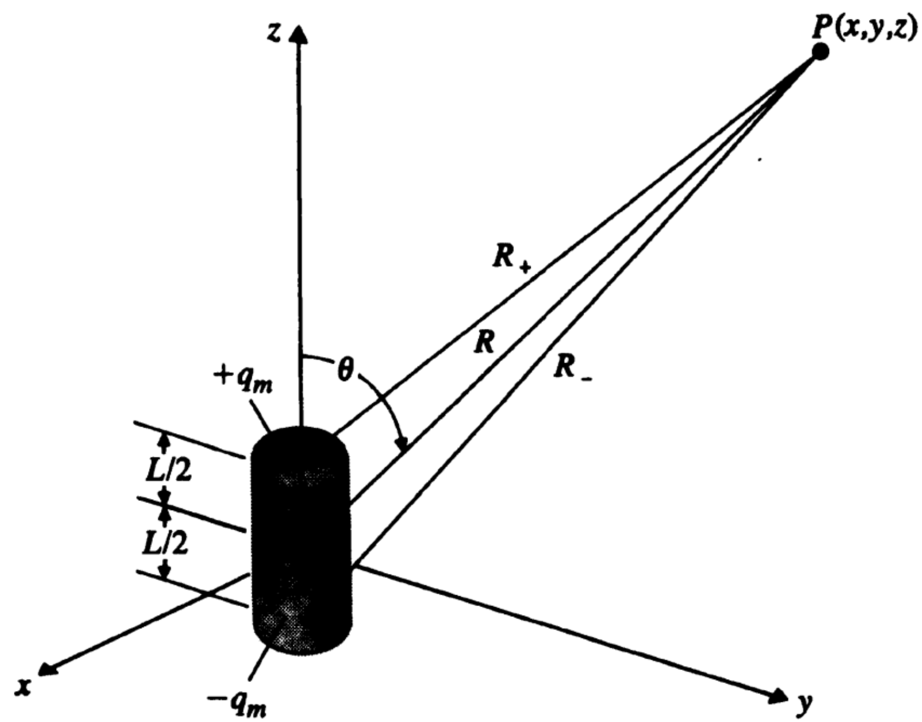
Magnetization charge density

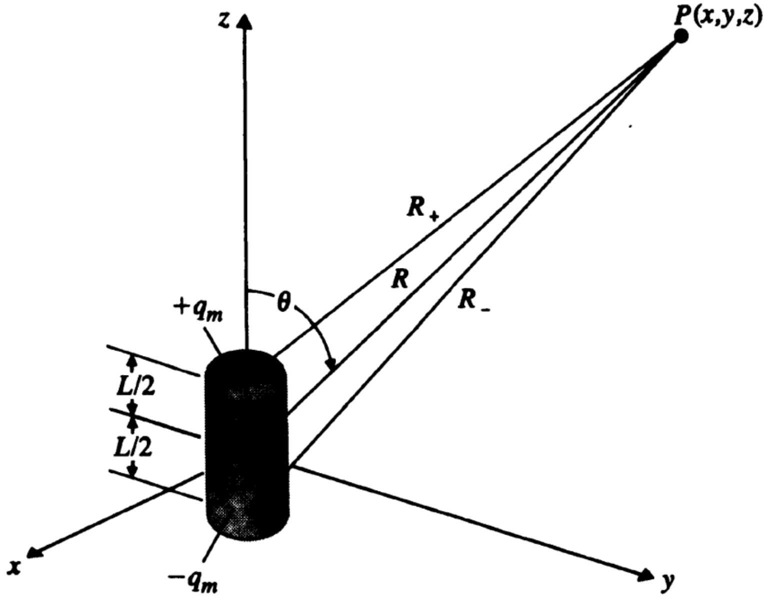
$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\mathrm{A/m})}$$

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\mathrm{A/m^2}).}$$

Example

A cylindrical bar magnet of radius b and length L has a uniform magnetization $\mathbf{M} = \mathbf{a}_z M_0$ along its axis. Use the equivalent magnetization charge density concept to determine the magnetic flux density at an arbitrary distant point.





$$\mathbf{M} = \mathbf{a}_z M_0$$

$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\mathrm{A/m})}$$

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\mathrm{A/m^2}).}$$

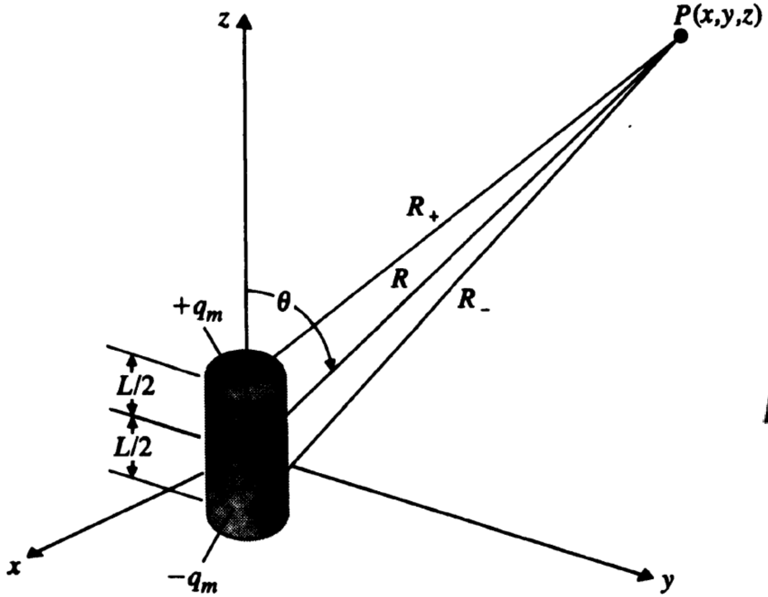
$$\rho_{ms} = \begin{cases} M_0 & \text{on top face,} \\ -M_0 & \text{on bottom face,} \\ 0 & \text{on side wall;} \end{cases}$$

$$\rho_m = 0 \qquad \text{in the interior.}$$

At a distant point the total equivalent magnetic charges on the top and bottom faces appear as point charges: $q_m = \pi b^2 \rho_{ms} = \pi b^2 M_0$. We have at $P(x, y, z)$

$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\text{A/m})}$$

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\text{A/m}^2).}$$



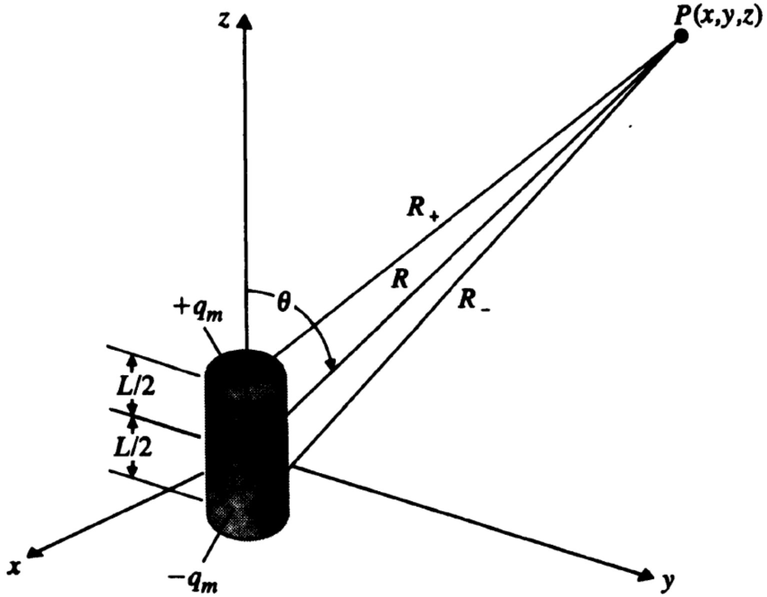
$$\rho_{ms} = \begin{cases} M_0 & \text{on top face,} \\ -M_0 & \text{on bottom face,} \\ 0 & \text{on side wall;} \end{cases}$$

$$\rho_m = 0 \qquad \text{in the interior.}$$

At a distant point the total equivalent magnetic charges on the top and bottom faces appear as point charges: $q_m = \pi b^2 \rho_{ms} = \pi b^2 M_0$. We have at $P(x, y, z)$

$$V_m = \frac{q_m}{4\pi}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

Think about analogy to electric dipole



$$V_m = \frac{q_m}{4\pi}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

If $R \gg b$

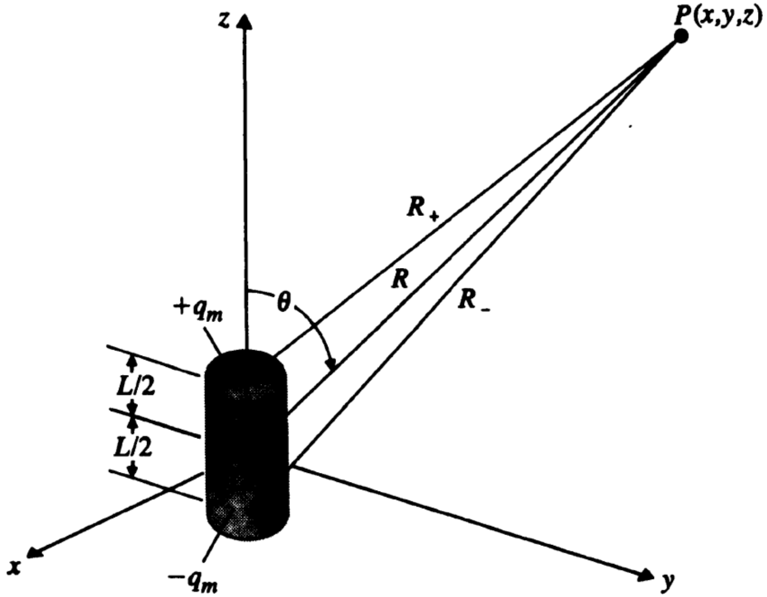
$$V_m = \frac{q_m L \cos\theta}{4\pi R^2} = \frac{(\pi b^2 M_0) L \cos\theta}{4\pi R^2}$$

$$= \frac{M_T \cos\theta}{4\pi R^2},$$

where $M_T = \pi b^2 L M_0$ is the total dipole moment of the cylindrical magnet.

$$\mathbf{B} = -\mu_0 \nabla V_m$$

$$\mathbf{B} = -\mu_0 \nabla V_m = \frac{\mu_0 M_T}{4\pi R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta)$$



$$V_m = \frac{q_m}{4\pi}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

If $R \gg b$

$$V_m = \frac{q_m L \cos\theta}{4\pi R^2} = \frac{(\pi b^2 M_0) L \cos\theta}{4\pi R^2}$$

$$= \frac{M_T \cos\theta}{4\pi R^2},$$

where $M_T = \pi b^2 L M_0$ is the total dipole moment of the cylindrical magnet.

$$\mathbf{B} = -\mu_0 \nabla V_m$$

$$\mathbf{B} = -\mu_0 \nabla V_m = \frac{\mu_0 M_T}{4\pi R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta)$$

No extra mathematical effort necessary once we already know electrostatics!

Little summary

Electric potential

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)},$$

$$\nabla \times \mathbf{E} = 0.$$

J = 0

Fictitious magnetic charge and Magnetic scalar potential

$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \quad \text{(A)}.$$

$$\nabla \times \mathbf{B} = 0.$$

V_m holds at any points **with no currents**

B is conservative

J ≠ 0

Circulating current and Magnetic vector potential

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)},$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

B is nonconservative

We looked at when the external current J=0…What happens now when J is not 0? How to exploit analogies between electric fields and magnetic fields?

$J \neq 0$

Electric potential

Circulating current and
Magnetic vector potential

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},$$

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},$$

$$\nabla \times \mathbf{E} = 0.$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

6-6 Magnetization and Equivalent Current Densities

- Microscopic viewpoint: orbiting electrons ➔ atomic currents ➔ magnetic dipoles **m**

- Without external **B**: random orientation of magnetic dipoles ➔ no net magnetic moment, $\sum \mathbf{m}=0$
- With external **B**: alignment of magnetic dipoles ➔ induced magnetic moment **m**

Let $\mathbf{m}_k$: magnetic dipole moment of an atom

Define magnetization vector **M**

n: number density
$n\Delta v=N$: total #

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\,\Delta v} \mathbf{m}_k}{\Delta v} \qquad \text{(A/m)},$$

M: density of total magnetic dipoles

$$d\mathbf{m} = \mathbf{M}\, dv'$$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},}$$

$$\boxed{d\mathbf{m} = \mathbf{M}\, dv'}$$

$$d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2}\, dv'.$$

Recall polarization vector **P**

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}.$$

R: source to field

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv'.$$

See section 3-7.1

$$\mathbf{A} = \int_{V'} d\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv',$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'.$$

By using the vector identity

$$\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) = \frac{1}{R}\nabla' \times \mathbf{M} - \nabla' \times \left(\frac{\mathbf{M}}{R}\right)$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' \underline{- \frac{\mu_0}{4\pi}\int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv'}.$$

By using the vector identity (Prob. 6-20)

$$\int_{V'} \nabla' \times \mathbf{F}\, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{s}',$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' \underline{+ \frac{\mu_0}{4\pi}\oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds'},$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R}\, dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R}\, ds',$$

Comparisons with

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\, dv'}$$

$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2})}$$

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\mathrm{A/m}).}$$

Prime is omitted for simplicity

For a given **M**

➔Find equivalent **magnetization current densities** $\mathbf{J}_m$ and $\mathbf{J}_{ms}$

➔**A**

➔**B**=∇×**A**

See section 3-7.1

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

Magnetization **M**

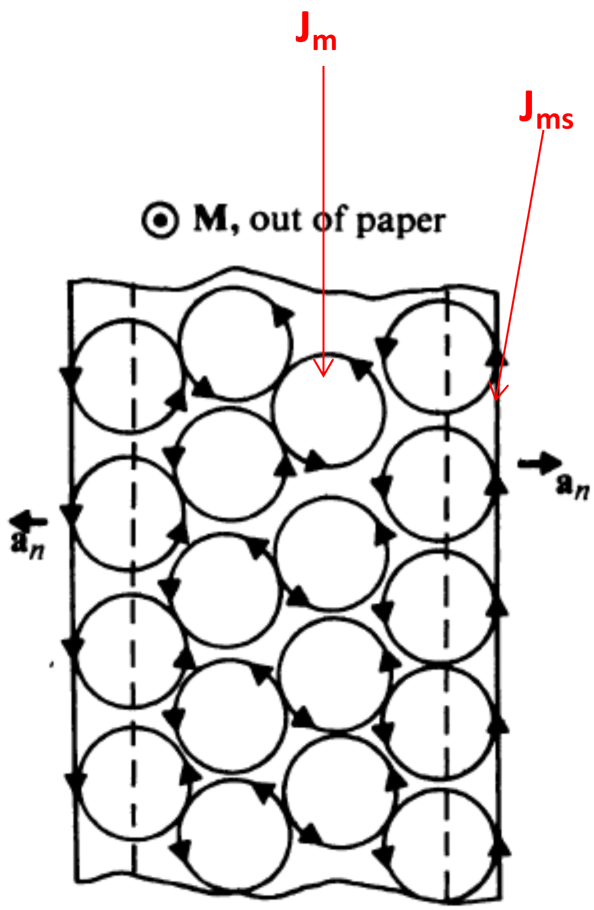


FIGURE 6–10
A cross section of a magnetized material.

Polarization **P**

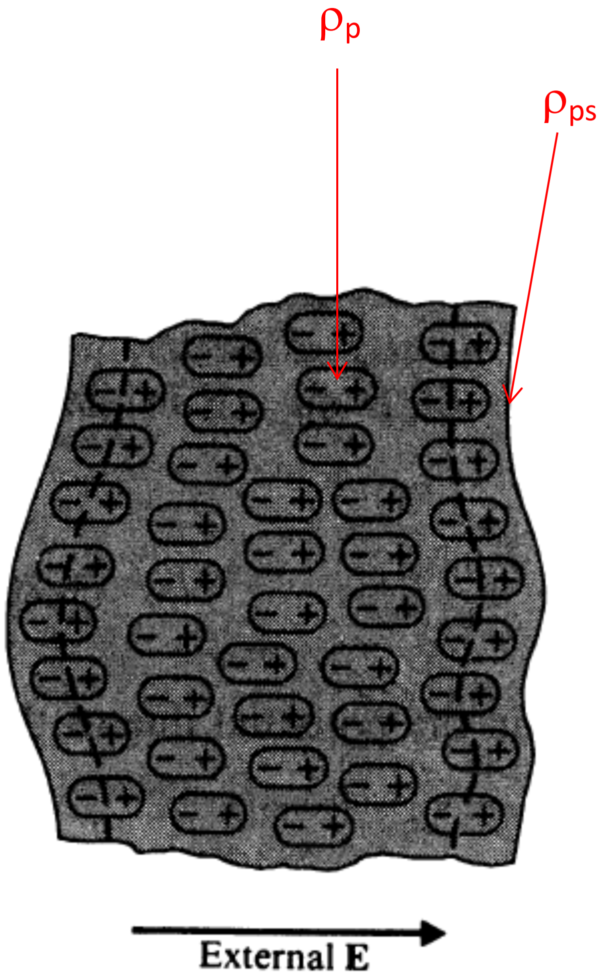
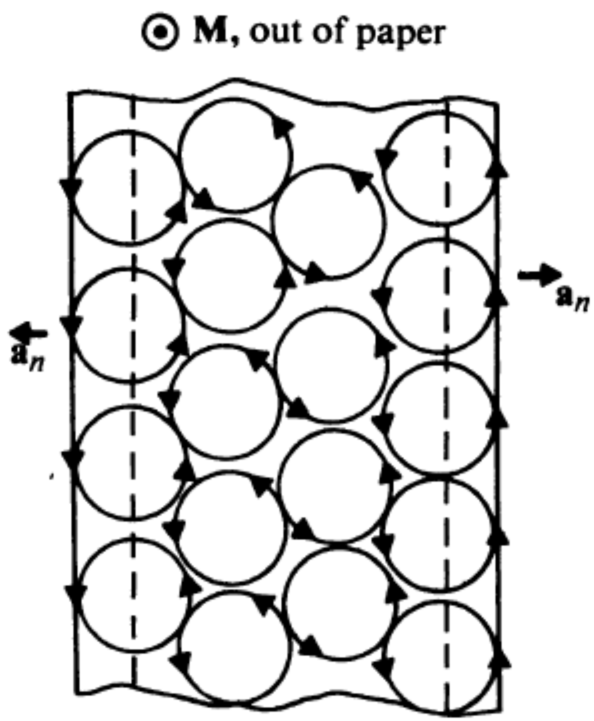


FIGURE 3–20
A cross section of a polarized dielectric medium.



$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2})}$$

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\mathrm{A/m}).}$$

- Check direction of $\mathbf{J}_{ms}$
- If **M** is uniform inside ➔ currents inside cancel ➔ $\mathbf{J}_m$=0 (or ∇×(constant)=0)

6-7 Magnetic Field Intensity and Relative Permeability

- Review: External $\mathbf{E}_{\mathbf{ext}}$ applied to a dielectric material ➔ induced dipole moments inside the dielectric material (and thus, $\mathbf{E}_{\mathbf{induced}}$)

➡ **E** inside the dielectric $\mathbf{E}_{\mathbf{total}} = \mathbf{E}_{\mathbf{ext}} + \mathbf{E}_{\mathbf{induced}}$

➡ $$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p).}$$

Magnetic Side

- External **B_{ext}** applied to a **magnetic** material ➔ induced dipole moments inside the **magnetic** material (and thus, **$B_{induced}$**)

➔ **B** inside the magnetic **$B_{total} = B_{ext} + B_{induced}$**

➔ $$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J}_m = \mathbf{J} + \nabla \times \mathbf{M}$$

$$\nabla \times \underline{\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right)} = \mathbf{J}.$$

Defined as **H**: magnetic field intensity

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\mathrm{A/m}).}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}.$$

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\mathrm{A/m}).}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\mathrm{A/m^2}),}$$

J: volume density of **free** current

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

ρ: volume density of **free** charges

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\text{A/m}^2),}$$

integration

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Stokes' theorem

$$\boxed{\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I \qquad (\text{A}),}$$

C: the contour bounding the surface S

Another form of Ampere's circuital law: the circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bound by the path.

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\text{C/m}^3),}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \int_V \rho\, dv$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\text{C}).}$$

Another form of Gauss's law: …

If the magnetic properties of the medium is *linear* and *isotropic*, then **M ~ H**

$$\mathbf{M} = \chi_m \mathbf{H},$$

χ_m: magnetic susceptibility

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\mathrm{A/m}).}$$

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\boxed{\begin{aligned} \mathbf{B} &= \mu_0(1 + \chi_m)\mathbf{H} \\ &= \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \qquad (\mathrm{Wb/m^2}) \end{aligned}}$$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$$

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}), \end{aligned}}$$

$$\boxed{\begin{aligned}\mathbf{B} &= \mu_0(1+\chi_m)\mathbf{H} \\ &= \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \qquad (\mathrm{Wb/m^2})\end{aligned}}$$

Or $$\boxed{\mathbf{H} = \frac{1}{\mu}\mathbf{B} \qquad (\mathrm{A/m}),}$$

where $$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}}$$

μ: permeability

μ_r: relative permeability

For a simple medium (linear, isotropic, homogeneous), χ_m and μ_r are constants.

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0(1+\chi_e)\mathbf{E} \\ &= \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E} \qquad (\mathrm{C/m^2}),\end{aligned}}$$

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

Relative permeability μ_r

- Ferromagnetic materials: iron, nickel, and cobalt. μ_r is very large (i.e., easy to be magnetized)

Analogy between electrostatics and magnetostatics

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

6-9 Behavior of Magnetic Materials

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

χ_m: magnetic susceptibility
μ_r: relative permeability

Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number).
Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number).
Ferromagnetic, if $\mu_r \gg 1$ (χ_m is a large positive number).

$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$ $\qquad$ $\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$

1. Diamagnetic

- Without external magnetic field ➔ no net magnetic dipole moments, $\mathbf{m}=0$
- With external magnetic field ➔ a net magnetic dipole moment (induced magnetization **M**) , $\mathbf{m}\neq 0$
- According to Lenz's law, **M** opposes $\mathbf{B}_{\mathbf{ext}}$, thus reducing the **B**. Thus, $\chi_m<0$ (~ -10^{-5})

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mathbf{B}/\mu_0=\mathbf{H}+\mathbf{M} \qquad\qquad \varepsilon_0\mathbf{E}=\mathbf{D}-\mathbf{P}$$

- Like **P** opposes **E**, reducing the E in dielectrics. B is reduced in diamagnetics.

$$\mathbf{B}/\mu_0=\mathbf{H}+\mathbf{M} \qquad\qquad \varepsilon_0\mathbf{E}=\mathbf{D}-\mathbf{P}$$

- No permanent magnetism: the induced magnetic moments disappears when applied field is withdrawn.
- Diamagnetic materials: bismuth, copper, lead, mercury, germanium, silver, gold, diamond
- Due to mainly orbiting electrons

2. Paramagnetic

- Without external magnetic field ➔ there is net magnetic dipole moments, **m**≠0
- With external magnetic field ➔ a very weak induced magnetization **M** (similar to diamagnetic effect)
- **M** is in the direction of $\mathbf{B}_{ext}$, thus increasing the **B**. Thus, $\chi_m>0$ (~ 10^{-5})

$$\mathbf{M} = \chi_m \mathbf{H},$$

$\mathbf{B}/\mu_0=\mathbf{H}+\mathbf{M}$ $\qquad$ $\varepsilon_0\mathbf{E}=\mathbf{D}-\mathbf{P}$

- Paramagnetic materials: aluminum, magnesium, titanium, and tungsten
- Due to mainly spinning electrons

3. Ferromagnetic

- Magnetization can be many orders of magnitude larger than that of paramagnetic substances.
- Magnetized domain:

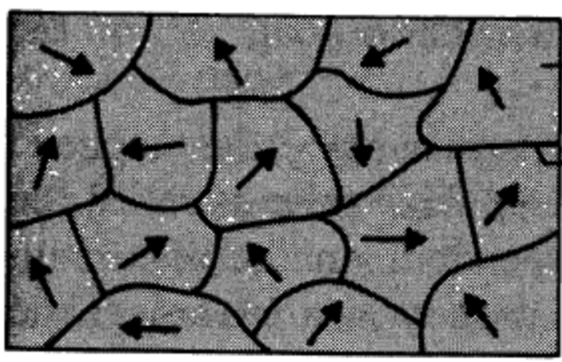


FIGURE 6–16
Domain structure of a polycrystalline ferromagnetic specimen.

  – These domains are fully magnetized even in the absence of an applied magnetic field.

- Without external magnetic field ➔ no net magnetization (due to random orientation in the various domains)
- With external magnetic field ➔ the domains aligned with applied magnetic field grow ➔ **B** is increased ($\chi_m > 0$)

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$$

- Due to mainly spinning electrons
- Ferromagnetic materials: cobalt, nickel, and iron

Hysteresis for ferromagnetic materials

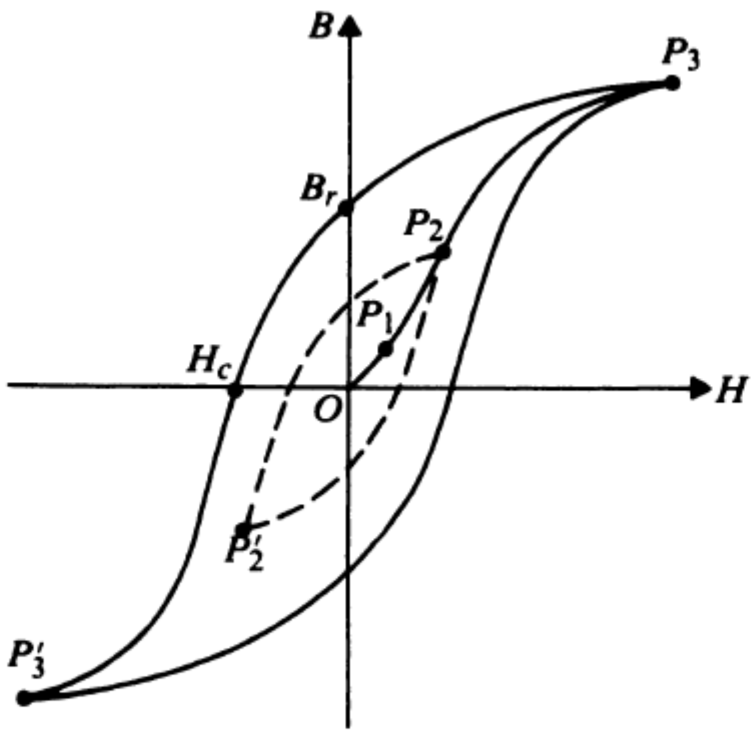


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

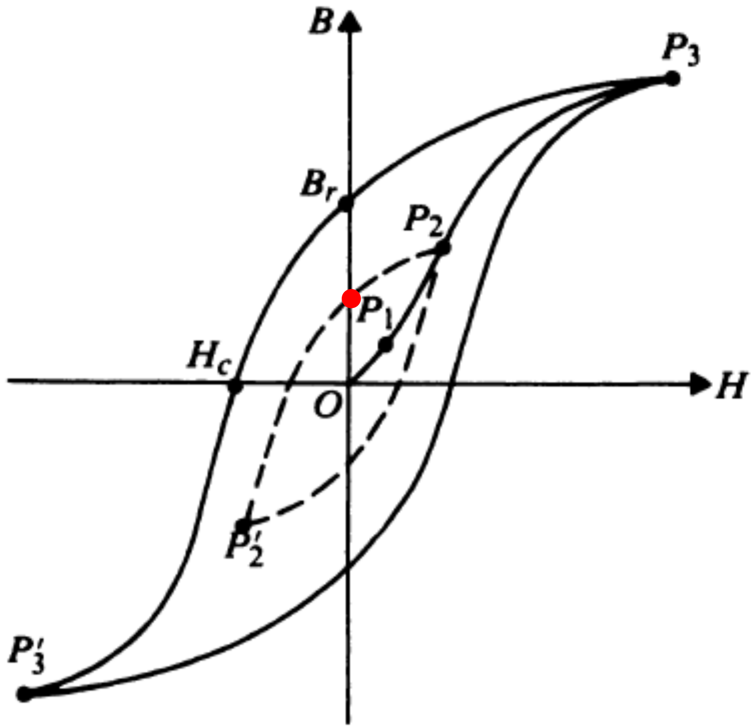


FIGURE 6–17
Hysteresis loops in the *B–H* plane for ferromagnetic material.

1. For weak applied fields (e.g., to P_1), magnetization are reversible.
2. For stronger applied fields (e.g., to P_2), magnetization are no longer reversible.
   - Forward: OP_1P_2
   - Backward: P_2P_2'

   Magnetization (*B*) lags the field (*H*), called **hysteresis** (“to lag” in Greek word)

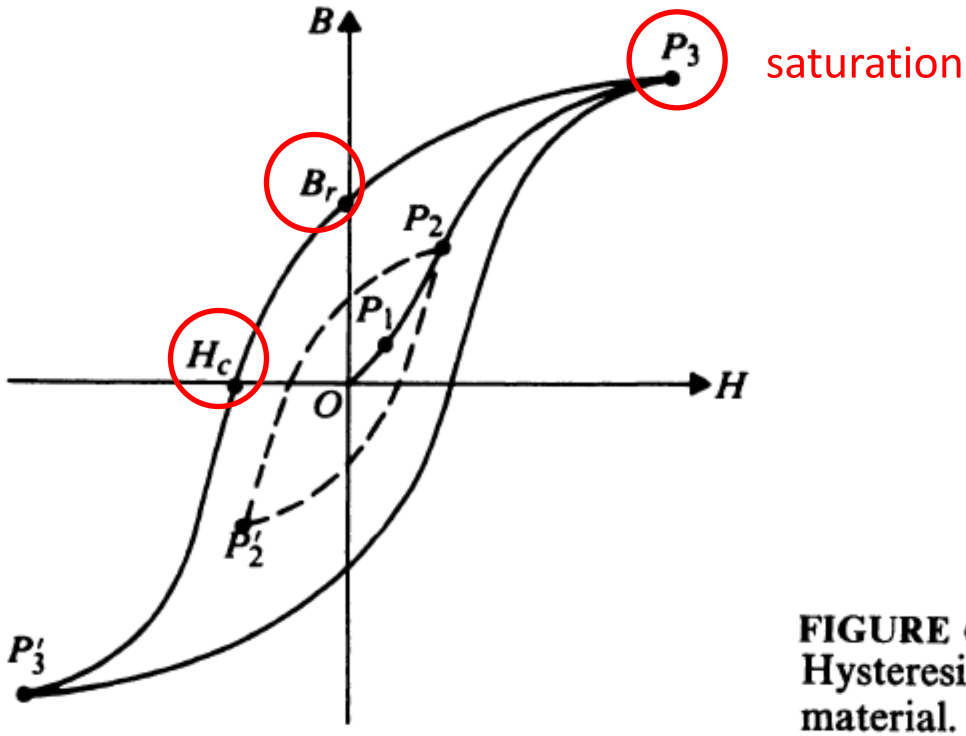


FIGURE 6–17
Hysteresis loops in the *B–H* plane for ferromagnetic material.

3. For much stronger fields (e.g., to P_3) ➔ total alignment of microscopic magnetic dipole moments with the applied field ➔ reached the saturation
   - **Permanent magnets**: If H is reduced to 0, B does not go to 0 but the value B_r (called residual or remnant **B**).
   - Coercive field intensity: to make B back to 0, a H_c in the opposite direction is necessary.

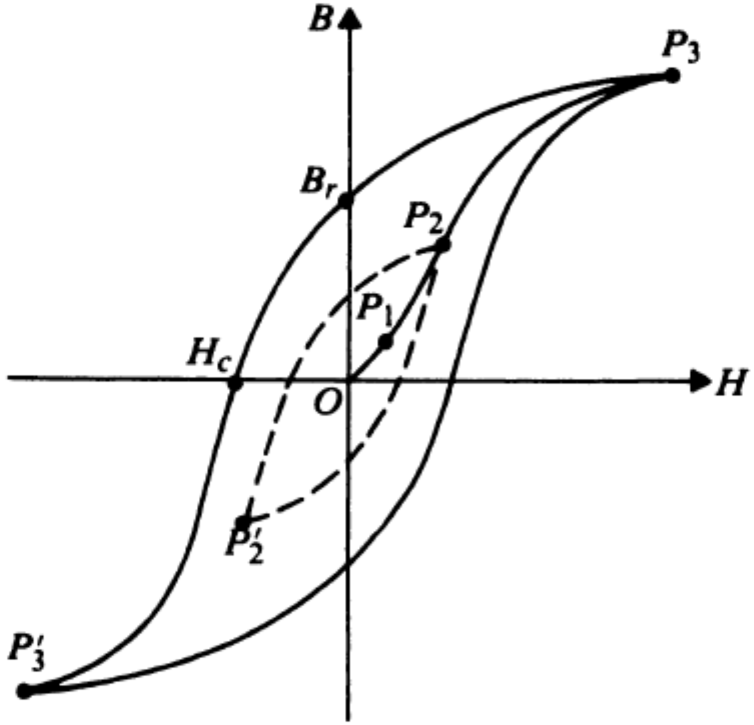


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

$$\mathbf{B} = \mu \mathbf{H}$$

- Linear: μ = **B/H** ➔ $\mu = \Delta B/\Delta H$ (slope passing through the origin)
- Linear & Hysteresis: μ = **B/H** ➔ μ (history of H)
  Hysteresis: Incremental permeability = dB/dH (local slope)

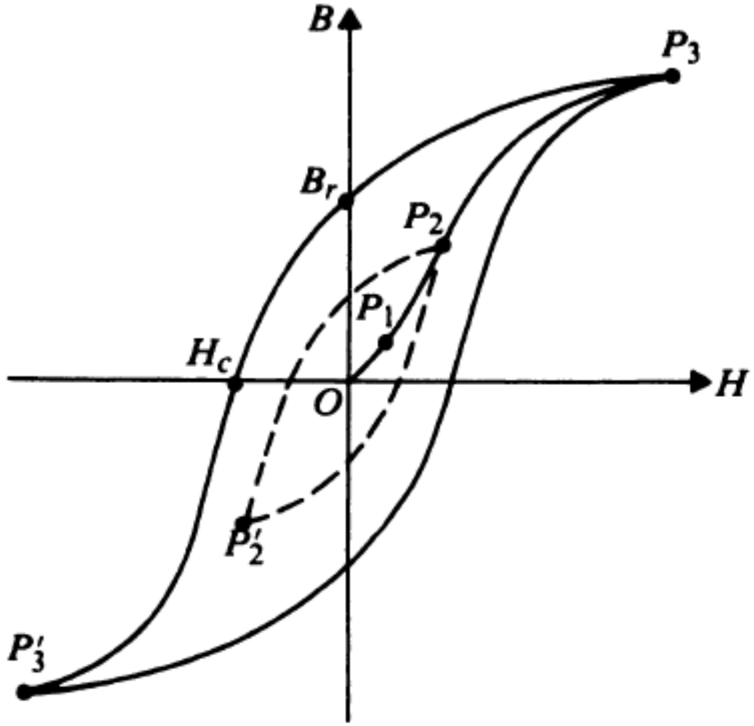


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

Soft materials

Q: How to have a large magnetization for a very small applied field?

A: **Tall narrow** hysteresis loops

The area of hysteresis loop = energy loss per unit volume per cycle (when the hysteresis loop is traced once per cycle)

hysteresis energy loss is the energy lost in the form of heat in overcoming the friction encountered during domain-wall motion and domain rotation.

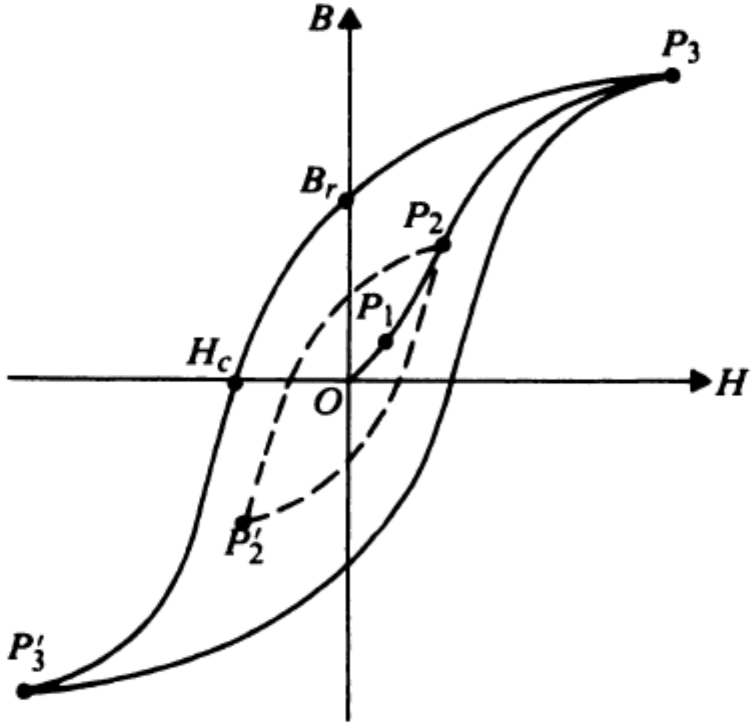


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

Hard materials

Q: How to have good permanent magnets?

A: **Fat** hysteresis loops (i.e., large H_c)

Temperature Effects on Ferromagnetic Materials

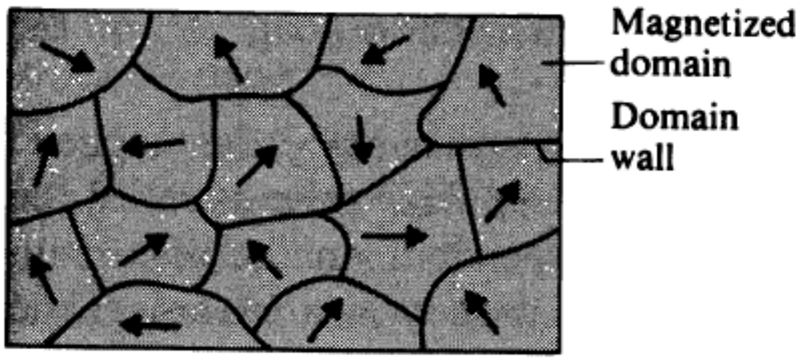
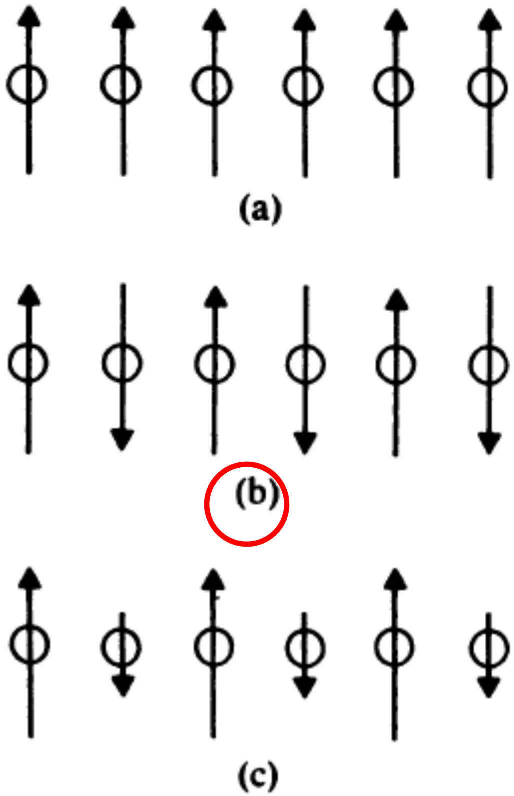


FIGURE 6–16
Domain structure of a polycrystalline ferromagnetic specimen.

- Under curie temperature T_c → well-defined magnetized domain (T_c=770°C for iron)
- Above curie temperature → loses magnetization, reducing to paramagnetic substances

Anti-ferromagnetic



When T>T_c → paramagnetic

FIGURE 6–18
Schematic atomic spin structures for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials.

- (a) Ferromagnetic: parallel alignments of electron spins (in a magnetized domain)
- (b) Anti-ferromagnetic: antiparallel alignments of electron spins ➔ no net magnetic moment

Ferri-magnetic

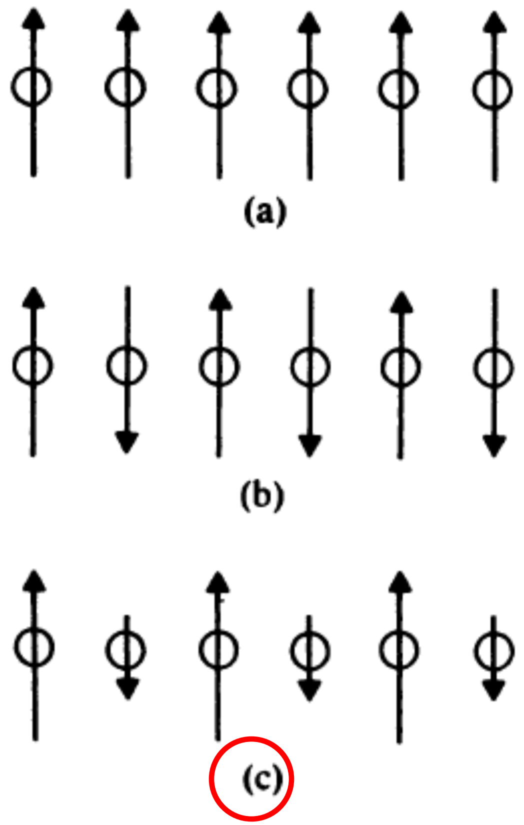


FIGURE 6–18
Schematic atomic spin structures for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials.

- (c) Ferri-magnetic: alternating alignments of electron spins with unequal magnitudes ➔ nonzero net magnetic moment
  - Due to partial cancelation, B_{ferri} ~ 1/10 B_{ferro}

	Diamagnetic	**Paramagnetic**	**Ferromagnetic**
μ_r	≤ 1	≥ 1	$>>1$
χ_m	Small negative	Small positive	Larger positive
M ($\mathbf{M}=\chi_m\mathbf{H}$)	$//-\mathbf{H}$	$//\mathbf{H}$	$//\mathbf{H}$
B in the material ($\mathbf{B}/\mu_0=\mathbf{H}+\mathbf{M}$)	Reduced	Increased	Increased
Mainly due to	Orbiting electrons	Spinning electrons	
When $\mathbf{B}_{ext}=0$	Net **m** = 0	Net m $\neq$ 0 (very weak)	Net m $\neq$ 0 (hysteresis)

6-10 Boundary Conditions for Magnetostatic Field

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T).}}$$

The normal component of **B** is continuous across an interface

For linear media,

$$\mathbf{B}_1 = \mu_1 \mathbf{H}_1 \text{ and } \mathbf{B}_2 = \mu_2 \mathbf{H}_2,$$

$$\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}.}$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),}$$

J_{sn}: the surface current density on the interface **normal** to the contour C (J_{sn} is along the thumb when the fingers of right hand follow the path C)

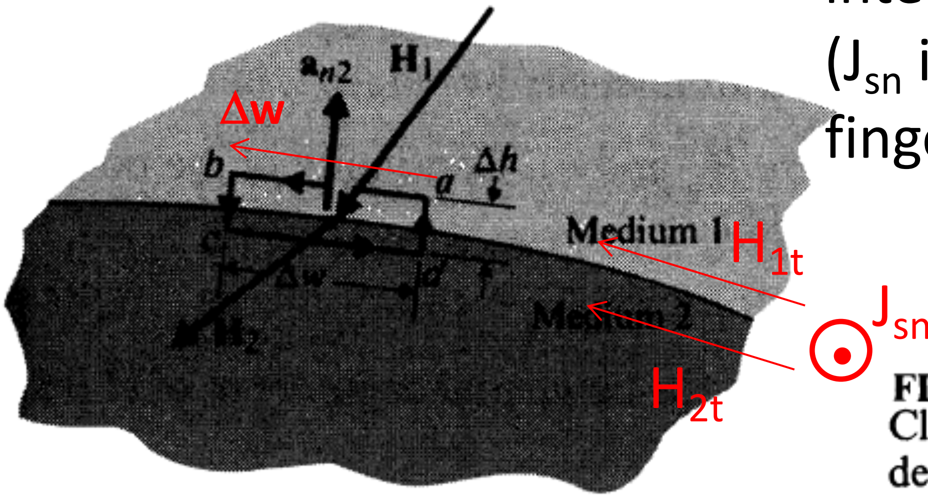


FIGURE 6–19
Closed path about the interface of two media for determining the boundary condition of H_t.

Magnetostatics

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I.$$

Let bc=da=Δh → 0

$$\oint_{abcda} \mathbf{H} \cdot d\boldsymbol{\ell} = \mathbf{H}_1 \cdot \Delta\mathbf{w} + \mathbf{H}_2 \cdot (-\Delta\mathbf{w}) = J_{sn}\Delta w$$

$$H_{1t} - H_{2t} = J_{sn} \qquad \text{(A/m)},$$

along the finger – opposite to the finger

Electrostatics

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},}$$

Or $$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad (\text{A/m}),}$$

The tangential component of the **H** field is discontinuous across an interface where a free surface current exists

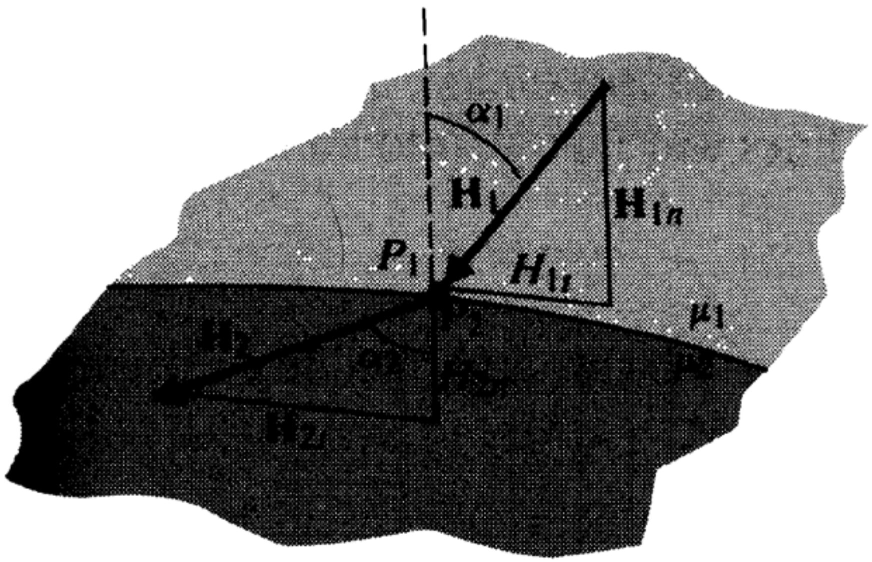
- For finite σ of two media ➔ only J, no J_s ➔ H_t continuous
- If infinite σ for one medium ➔ J_s exists ➔ H_t discontinuous

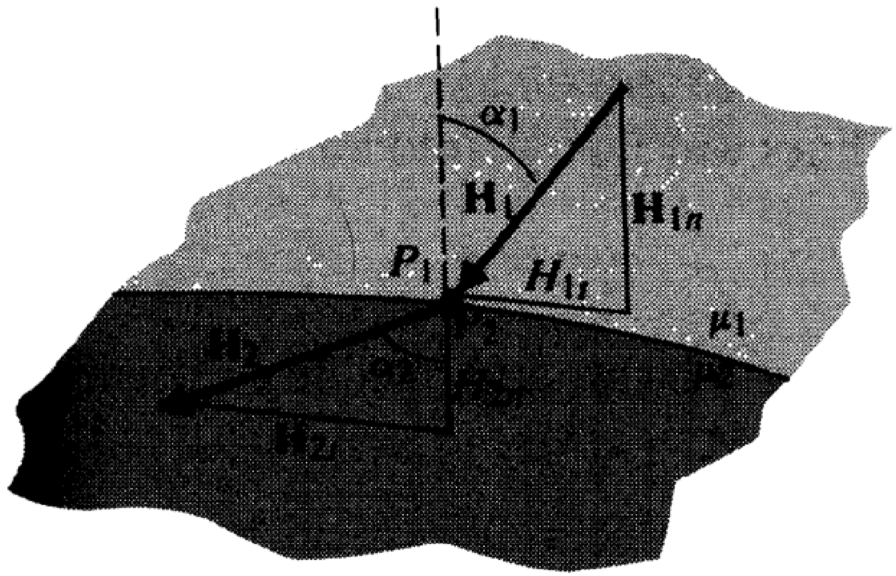
Recall in Chap.5: $\boxed{J_{1n} = J_{2n} \qquad (\text{A/m}^2).}$ $\boxed{\dfrac{J_{1t}}{J_{2t}} = \dfrac{\sigma_1}{\sigma_2}.}$ (see Prob. 6-30)

Example

Two magnetic media with permeabilities μ_1 and μ_2 have a common boundary

The magnetic field intensity in medium 1 at the point P_1 has a magnitude H_1 and makes an angle α_1 with the normal. Determine the magnitude and the direction of the magnetic field intensity at point P_2 in medium 2.



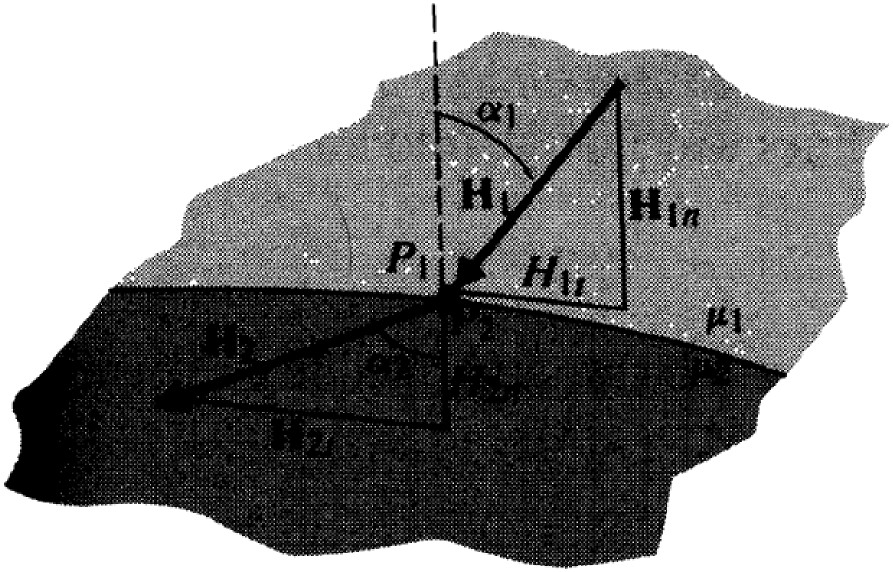


The desired unknown quantities are H_2 and α_2. Continuity of the normal component of **B** field requires

$$\mu_2 H_2 \cos \alpha_2 = \mu_1 H_1 \cos \alpha_1$$

Since neither of the media is a perfect conductor, the tangential component of **H** field is continuous. We have

$$H_2 \sin \alpha_2 = H_1 \sin \alpha_1.$$

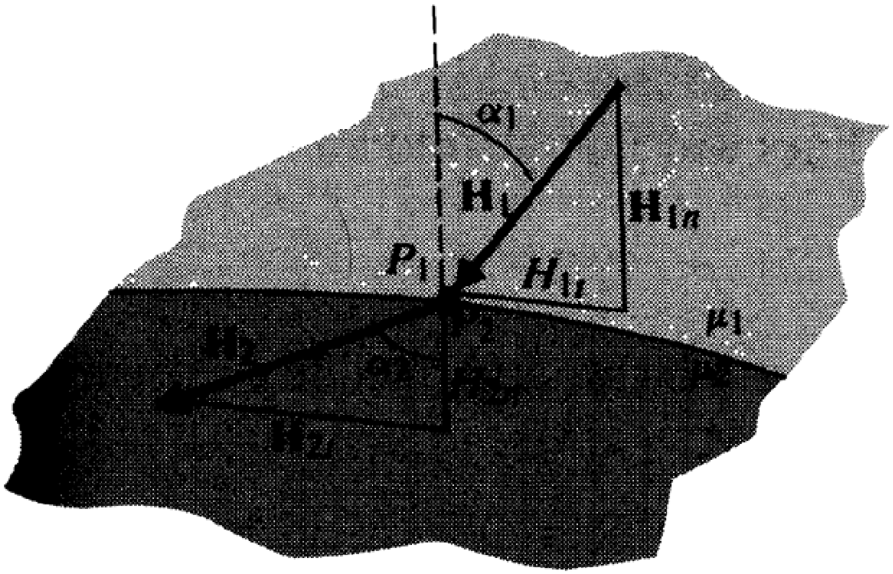


$$\mu_2 H_2 \cos\alpha_2 = \mu_1 H_1 \cos\alpha_1$$

$$H_2 \sin\alpha_2 = H_1 \sin\alpha_1$$

$$\boxed{\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\mu_2}{\mu_1}}$$

$$\alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right)$$

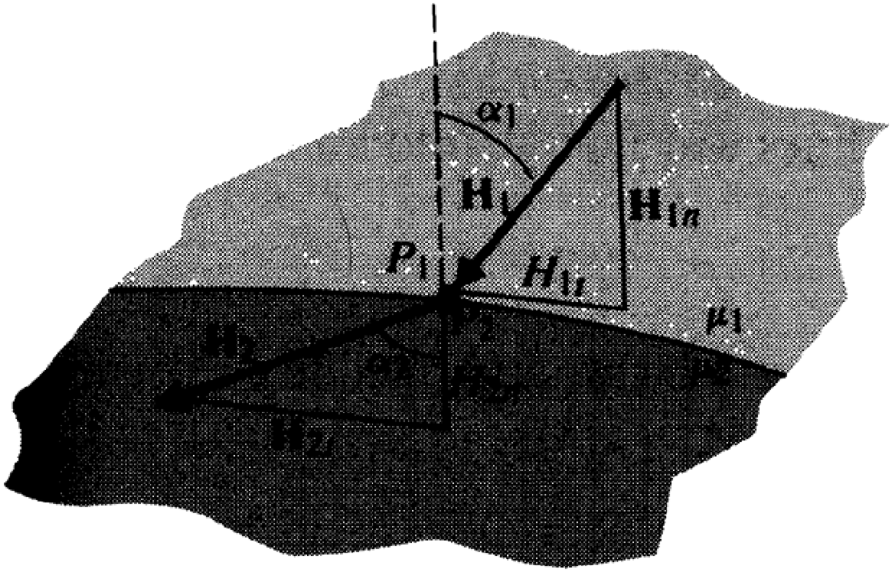


$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\mu_2}{\mu_1}}$$

$$\alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan \alpha_1\right)$$

$$H_2 = \sqrt{H_{2t}^2 + H_{2n}^2} = \sqrt{(H_2 \sin \alpha_2)^2 + (\mathrm{H}_2 \cos \alpha_2)^2}$$

$$\boxed{H_2 = H_1\left[\sin^2 \alpha_1 + \left(\frac{\mu_1}{\mu_2}\cos \alpha_1\right)^2\right]^{1/2}.}$$



medium 1 is nonmagnetic

medium 2 is ferromagnetic

$$\mu_2 \gg \mu_1$$

α_2 will be nearly 90°

$$\mu_1 \gg \mu_2$$

α_2 will be nearly zero

$$\alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right)$$

Analogous Boundary-value Problems

Magnetostatics

In current-free regions, $\nabla \times (\mathbf{B}/\mu) = 0$

$$\mathbf{B} = -\mu \nabla V_m.$$

$\nabla \cdot \mathbf{B} = 0$

And assume a constant μ

$$\nabla^2 V_m = 0.$$

Thus, the techniques (method of images and method of separation of variables) discussed in Chap. 4 for solving boundary-value problems (BVPs) can be adapted to solving analogous magnetostatic BVPs.

Electrostatics

In charge-free regions, $\nabla \times \mathbf{E} = 0$

Laplace's equation

$$\nabla^2 V = 0,$$

6-11 Inductances and Inductors

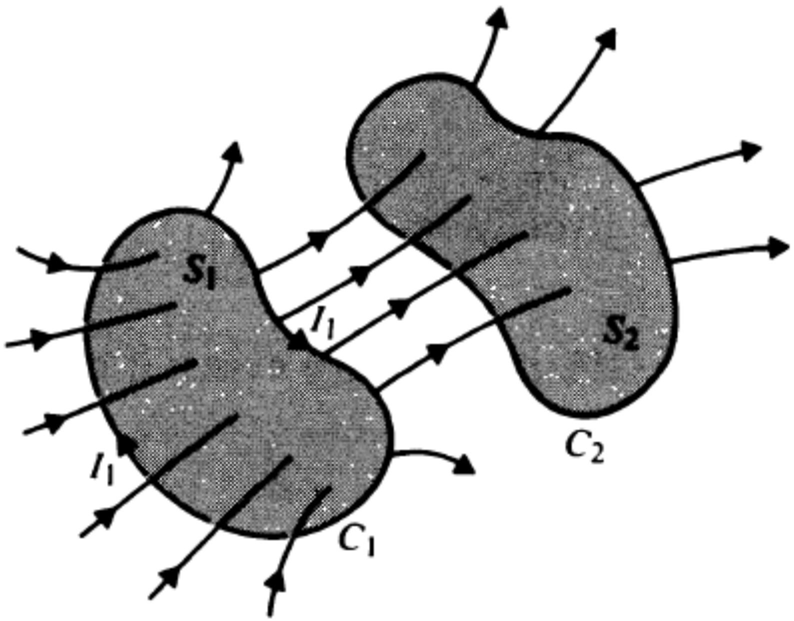


FIGURE 6–22
Two magnetically coupled loops.

I_1 ➔ Φ_1 ➔ part of Φ_1 (Φ_{12}) passes through S_2

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \qquad \text{(Wb)}.$$

From Biot-Savart law: B ~ I

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \qquad \text{(T)}.}$$

$\rightarrow$ $\Phi_{12} \sim I_1$

1 turn for C_2, mutual flux $\Phi_{12} = \underline{L_{12}} I_1,$

The proportionality constant L_{12} (called mutual inductance)

If N turns for C_2, flux linkage $\Lambda_{12} = N_2 \Phi_{12} \qquad \text{(Wb)},$

$\boxed{\Phi \cong BS,}$

$\rightarrow$ $\Phi \sim S$

Thus, the general expression: $\Lambda_{12} = L_{12} I_1 \qquad \text{(Wb)}$

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \qquad \text{(H)}.}$$

The mutual inductance between two circuits is then the magnetic flux linkage with one circuit (Λ_{12}) per unit current in the other (I_1)

For linear media, μ is a constant

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \quad \text{(T)}.$$

B ~ I

$$\Phi_{12} = L_{12} I_1, \quad \Rightarrow \quad \boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \quad \text{(H)}.}$$

For linear media only

For nonlinear media, μ is a function of I

B(I)

$$\Phi_{12}(I) \quad \Rightarrow \quad \boxed{L_{12} = \frac{d\Lambda_{12}}{dI_1} \quad \text{(H)}.}$$

In general

Self-inductance

$I_1 \rightarrow \Phi_1 \rightarrow$ part of Φ_1 (Φ_{11}) passes through S_1

If N_1 turns for C_1, flux linkage $\Lambda_{11} = N_1\Phi_{11} > N_1\Phi_{12}$.

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Self-inductance

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1} \qquad \text{(H)},}$$

For linear media only

$$\boxed{L_{11} = \frac{d\Lambda_{11}}{dI_1} \qquad \text{(H)}.}$$

In general

Inductor

- A conductor arranged in an appropriate shape (such as a conducting wire wound as a coil) to supply a certain amount of self-inductance is called an **inductor**.
- The procedure to determine self-inductance of an inductor: **From I to Λ**
  - 1. Choose an appropriate coordinate system
  - 2. Assume I in the conducting wire
  - 3. Find B from I by Ampere's circuital law (for symmetric case) or Biot-Savart law (otherwise)

– 4. Find the flux linkage with each turn, Φ, from **B**

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s},$$

– 5. Find the total flux linkage Λ

$$\Lambda = N\Phi$$

– 6. Find L by $L = \Lambda / I$

- The procedure to determine mutual-inductance L_{12}: slight modification

  I_1 ➔ B_1 ➔ **Φ_{12} by integrating B_1 over S_2** ➔ $\Lambda_{12} = N_2 \Phi_{12}$ ➔ $L_{12} = \Lambda_{12} / I_1$

$$L_{12} = L_{21}?$$

<Proof>

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \qquad \text{(Wb).}$$

$$\Lambda_{12} = N_2 \Phi_{12} \qquad \text{(Wb),}$$

$$L_{12} = \frac{\Lambda_{12}}{I_1} \qquad \text{(H).}$$

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 .$$

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1$$

$\mathbf{A}_1$: vector magnetic potential

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{s}_2$$

$$= \frac{N_2}{I_1} \oint_{C_2} \mathbf{A}_1 \cdot d\boldsymbol{\ell}_2 .$$

$$\mathbf{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\boldsymbol{\ell}_1}{R}.$$

$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R},$$

1 turn for C_1
1 turn for C_2

Neumann formula for mutual inductance

$$\boxed{L_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R} \qquad \text{(H)},}$$

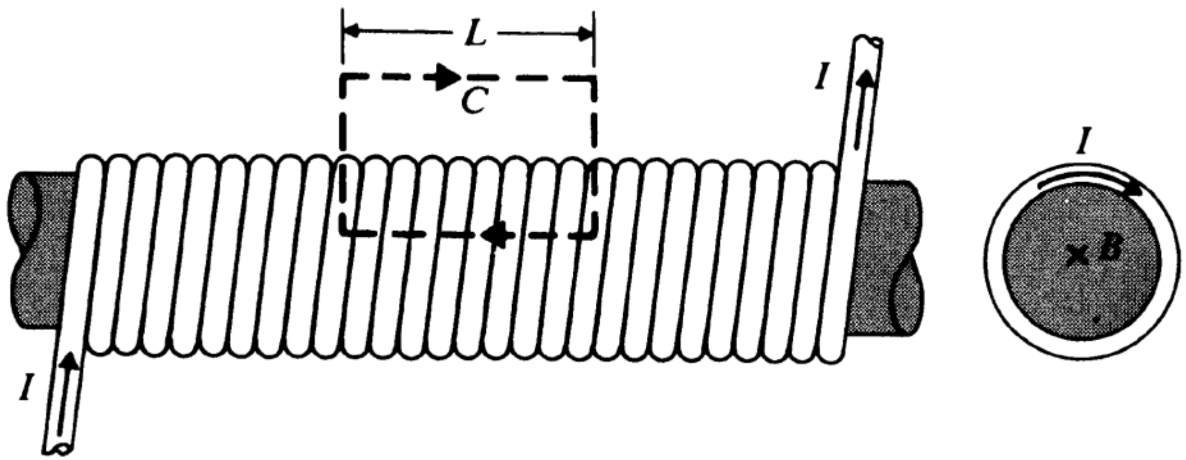
N_1 turns for C_1
N_2 turn for C_2

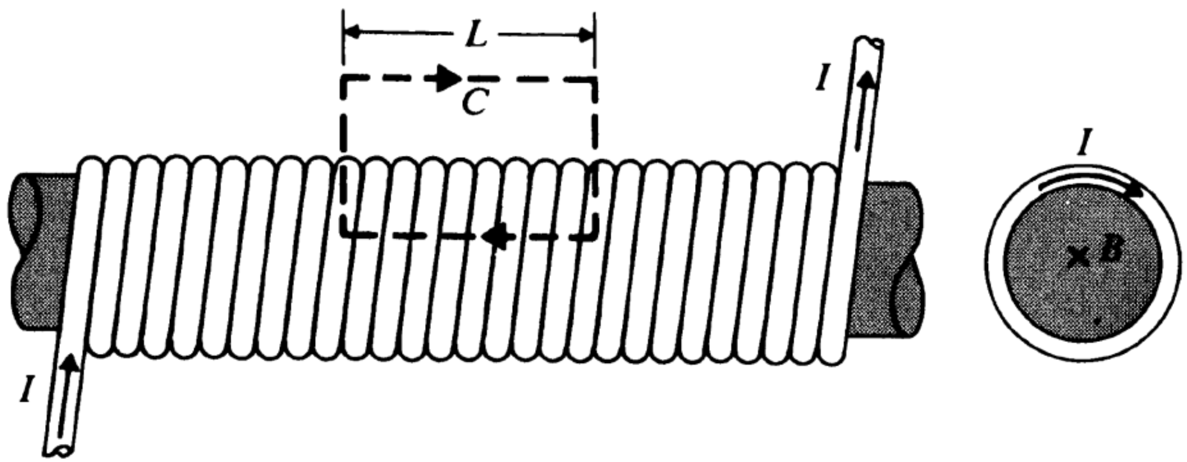
Mutual inductance:

- dependent on the geometrical shape and the physical arrangement of coupled circuits
- independent of currents (for linear media where μ is a constant)
- interchanging subscript 1 and 2 does not change the value ➔ $L_{12}=L_{21}$

Example

Determine the magnetic flux density inside an infinitely long solenoid with air core having n closely wound turns per unit length and carrying a current I as shown





As a direct application of Ampère's circuital law. It is clear that there is no magnetic field outside of the solenoid. To determine the **B**-field inside, we construct a rectangular contour C of length L that is partially inside and partially outside the solenoid. By reason of symmetry the **B**-field inside must be parallel to the axis. Applying Ampère's circuital law, we have

$$BL = \mu_0 nLI$$

$$\boxed{B = \mu_0 nI}$$

Example

Assume that N turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown in Fig. 6–23. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil.

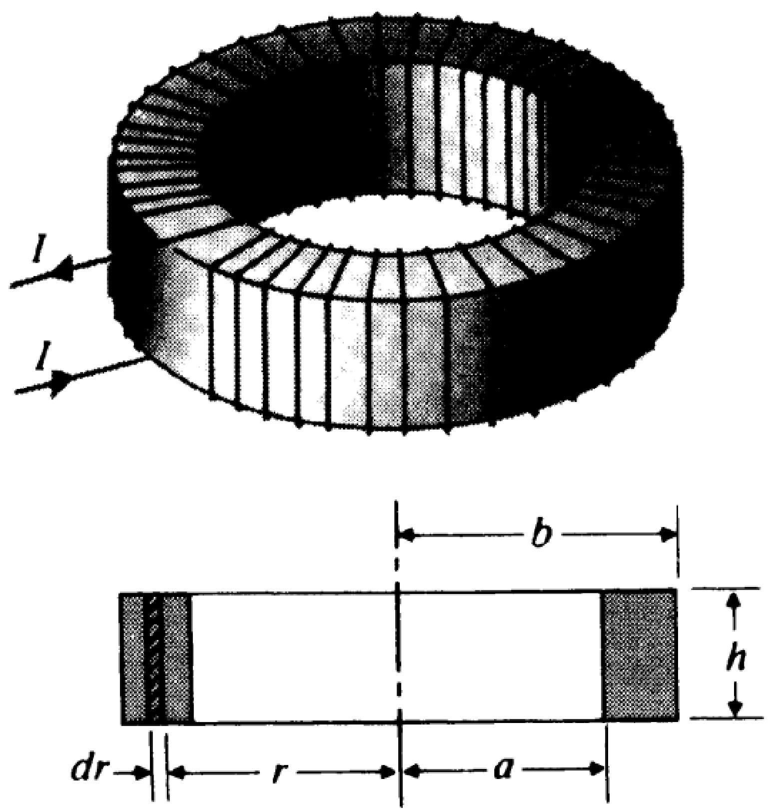
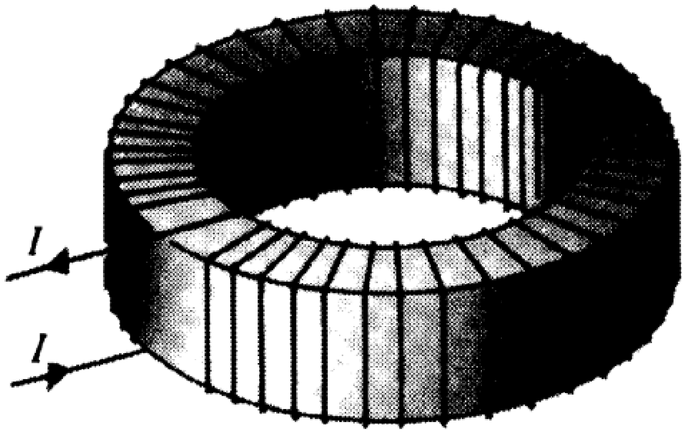


FIGURE 6–23
A closely wound toroidal coil

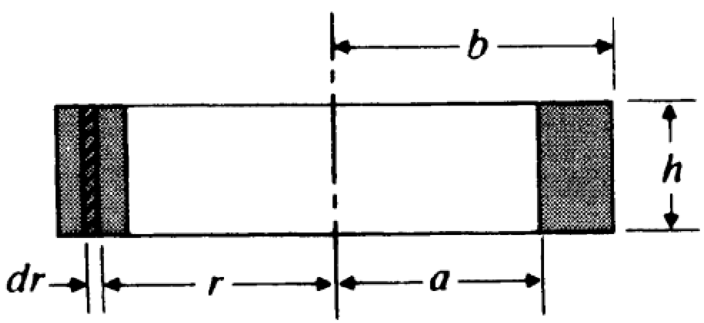
It is clear that the cylindrical coordinate system is appropriate



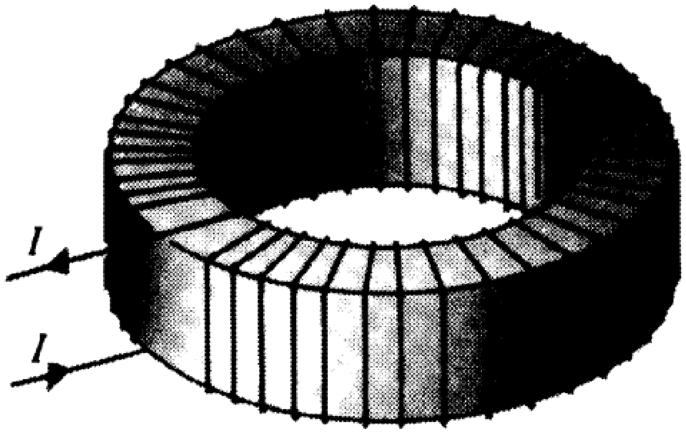
$$\mathbf{B} = \mathbf{a}_\phi B_\phi,$$

$$d\boldsymbol{\ell} = \mathbf{a}_\phi r d\phi,$$

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B_\phi r\, d\phi = 2\pi r B_\phi.$$



It is clear that the cylindrical coordinate system is appropriate



$$\mathbf{B} = \mathbf{a}_\phi B_\phi,$$

$$d\boldsymbol{\ell} = \mathbf{a}_\phi r d\phi,$$

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B_\phi r\, d\phi = 2\pi r B_\phi.$$

This result is obtained because both B_ϕ and r are constant around the circular path C. Since the path encircles a total current NI, we have

$$2\pi r B_\phi = \mu_0 NI$$

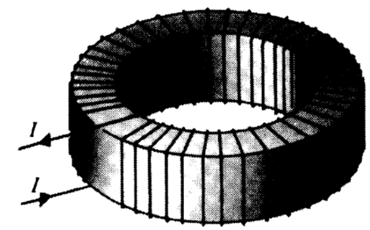
and

$$B_\phi = \frac{\mu_0 NI}{2\pi r}.$$

Next we find

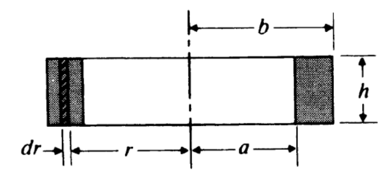
$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \left(\mathbf{a}_\phi \frac{\mu_0 N I}{2\pi r} \right) \cdot (\mathbf{a}_\phi h\, dr)$$

$$= \frac{\mu_0 N I h}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 N I h}{2\pi} \ln \frac{b}{a}.$$



The flux linkage Λ is $N\Phi$ or

$$\Lambda = \frac{\mu_0 N^2 I h}{2\pi} \ln \frac{b}{a}.$$



Finally, we obtain

$$L = \frac{\Lambda}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} \qquad \text{(H)}.$$

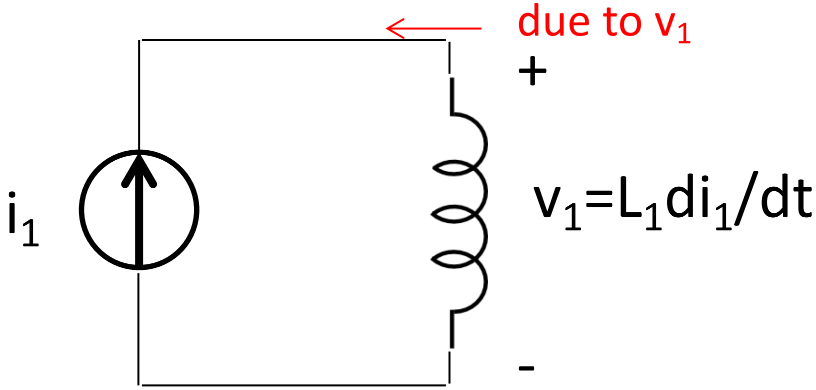
6-12 Magnetic Energy

- For DC, the inductor behaves like short circuit.
- Exact AC case: retardation and radiation effects should be considered (Chaps. 7 and 8)
- Here, we consider **quasi-static conditions**: the current vary very slowly in time (low frequency, or long wavelength)

- In section 3-11, work is required to assemble a group of charges (stored electric energy)
- Here, work is required to send currents into conducting loops (stored magnetic energy)

Stored Magnetic Energy

- A current generator increases the current i_1 from 0 to I_1:
- Work must be done to **overcome** this induced v_1



$$W_1 = \int v_1 i_1 \, dt = L_1 \int_0^{I_1} i_1 \, di_1 = \tfrac{1}{2} L_1 I_1^2.$$

For linear media

$$L_1 = \Phi_1 / I_1$$

$$W_1 = \tfrac{1}{2} I_1 \Phi_1,$$

Chap.3

$$W_e = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},$$

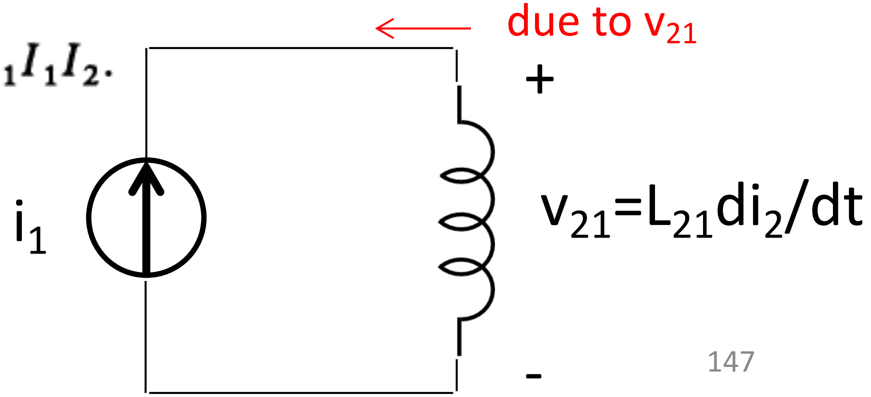
Two loops C_1 and C_2 to I_1 and I_2

- Initially, i_1=0, i_2=0
- Step 1: increase i_1 from 0 to I_1

$$W_1 = \tfrac{1}{2}L_1 I_1^2.$$

- Step 2: increase i_2 from 0 to I_2
  - Work must be done to **overcome** the induced v_{21} $=L_{21}di_2/dt$ (to keep i_1 constant at I_1)

$$W_{21} = \int v_{21} I_1 \, dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2.$$



  - Work in C_2

$$W_{22} = \tfrac{1}{2}L_2 I_2^2.$$

Total work required: $W_2 = \frac{1}{2}L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2$

$$= \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} L_{jk} I_j I_k.$$

Generalization to a system of N loops carrying currents $I_1, I_2, \ldots, I_n$:

$$\boxed{W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \qquad \text{(J)},}$$

The stored magnetic energy for a current I through a single inductor with inductance L:

$$\boxed{W_m = \frac{1}{2} L I^2 \qquad \text{(J).}}$$

Alternative Derivation

Consider a kth loop of N magnetically coupled loops

The work done in the kth loop in time dt

$$dW_k = \underline{v_k i_k} \, dt = i_k \, d\phi_k,$$

power

$\boxed{v_k = d\phi_k/dt.}$

$d\phi_k$: change in flux ϕ_k linking with the kth loop due to the change of currents in all the coupled loops
(di ➔ $d\phi$ ➔ v_k)

The differential work done to the system $$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k \, d\phi_k.$$

$$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k \, d\phi_k.$$

$$i_k = \alpha I_k, \qquad \phi_k = \alpha \Phi_k$$

I_k and Φ_k are final values,
α increases from 0 to 1

The total magnetic energy $$W_m = \int dW_m = \sum_{k=1}^{N} I_k \Phi_k \int_0^1 \alpha \, d\alpha$$

$$\boxed{W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k \qquad \text{(J)},}$$

For linear media

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j,$$

$$\boxed{W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \qquad \text{(J)},}$$

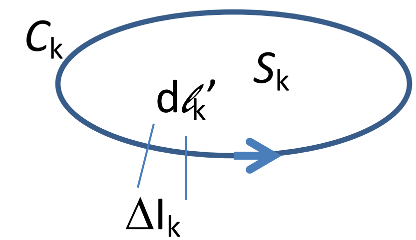
6-12.1 Magnetic Energy in Terms of Field Quantities

$$\boxed{W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k \qquad \text{(J)},}$$

$$\Phi_k = \int_{S_k} \mathbf{B} \cdot \mathbf{a}_n \, ds'_k = \oint_{C_k} \mathbf{A} \cdot d\ell'_k,$$

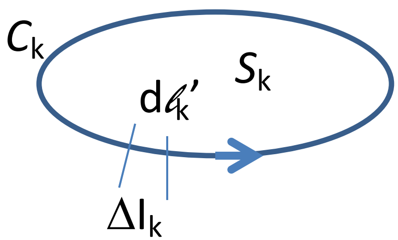
$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

A single current-carrying loop



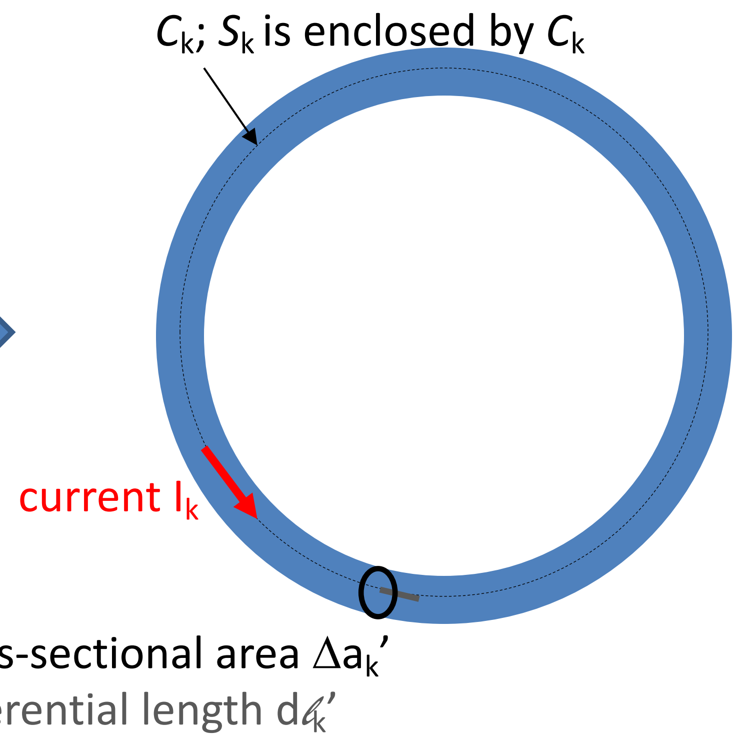
N contiguous filamentary current elements with a current ΔI_k (flowing in an infinitesimal cross-sectional area $\Delta a_k'$; $\Delta v_k' = \Delta a_k' d\ell_k'$)

A single current-carrying loop



N contiguous filamentary current elements with a current ΔI_k (flowing in an infinitesimal cross-sectional area $\Delta a_k'$; $\Delta v_k' = \Delta a_k' d\ell_k'$)

Top view



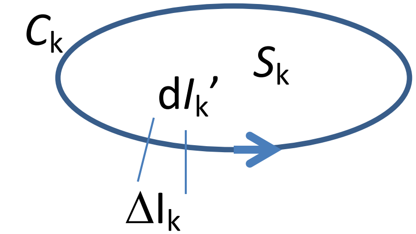
$$W_m = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \oint_{C_k} \mathbf{A} \cdot d\ell'_k.$$

$$\Delta I_k\, d\ell'_k = J(\Delta a'_k)\, d\ell'_k = \mathbf{J}\,\Delta v'_k.$$

As N → ∞, $\Delta v_k'$ becomes dv_k'

$$\boxed{W_m = \tfrac{1}{2}\int_{V'} \mathbf{A} \cdot \mathbf{J}\, dv' \qquad \text{(J)},}$$

A single current-carrying loop



N contiguous filamentary current elements with a current ΔI_k (flowing in an infinitesimal cross-sectional area $\Delta a_k'$; $\Delta v_k' = \Delta a_k' dl_k'$)

v': the volume of the loop or the linear medium in which J exists

$$W_e = \tfrac{1}{2}\int_{V'} \rho V\, dv \qquad \text{(J)}.$$

By vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}),$$

$$\mathbf{A} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{A} \times \mathbf{H})$$

$$\mathbf{A} \cdot \mathbf{J} = \mathbf{H} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{H}).$$

$$W_m = \tfrac{1}{2}\int_{V'} \mathbf{H} \cdot \mathbf{B}\, dv' - \tfrac{1}{2}\oint_{S'} (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{a}_n\, ds'.$$

$$W_m = \tfrac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}\,dv' - \tfrac{1}{2}\oint_{S'} (\mathbf{A}\times\mathbf{H})\cdot\mathbf{a}_n\,ds'.$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\,dv' \qquad \text{(Wb/m)}.$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\boldsymbol{\ell}'\times\mathbf{a}_R}{R^2} \qquad \text{(T)}.$$

$|\mathbf{A}| \sim 1/R$

$|\mathbf{H}| \sim 1/R^2$

$S' \sim R^2$

As $R \rightarrow \infty$, 2nd term $\rightarrow 0$

$$\boxed{W_m = \tfrac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}\,dv' \qquad \text{(J)}.}$$

For linear media

$$\mathbf{H} = \mathbf{B}/\mu,$$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu}\,dv' \qquad \text{(J)}}$$

$$\boxed{W_m = \tfrac{1}{2}\int_{V'} \mu H^2\,dv' \qquad \text{(J)}.}$$

The magnetic energy density

$$W_m = \int_{V'} w_m \, dv',$$

$$w_m = \tfrac{1}{2}\,\mathbf{H}\cdot\mathbf{B} \qquad (\mathrm{J/m^3})$$

$$w_m = \frac{B^2}{2\mu} \qquad (\mathrm{J/m^3})$$

$$w_m = \tfrac{1}{2}\mu H^2 \qquad (\mathrm{J/m^3}).$$

L can be calculated more easily by W_m formula here than using flux linkage (Λ):

$$W_m = \tfrac{1}{2} L I^2 \qquad (\mathrm{J}). \quad \Rightarrow \quad \boxed{L = \frac{2W_m}{I^2} \qquad (\mathrm{H}).}$$

6-13 Magnetic Forces and Torques

- A magnetic force $\mathbf{F_m}$ on a moving charge q

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}.$$

6-13.2 Forces and Torques on Current-Carrying Conductors

Magnetic force

dQ

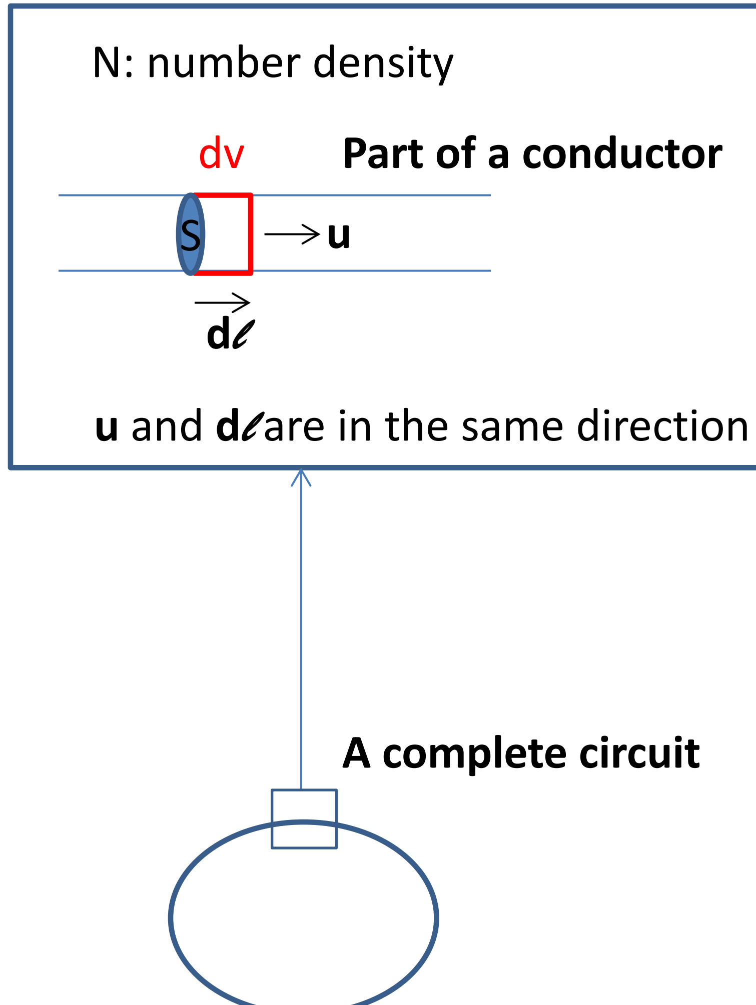
$$d\mathbf{F}_m = -NeS|d\ell|\mathbf{u} \times \mathbf{B}$$
$$= \underline{-NeS|\mathbf{u}|}\, d\ell \times \mathbf{B},$$

=JS=I (J=−*Ne*|**u**|)

$$\boxed{d\mathbf{F}_m = I\, d\ell \times \mathbf{B} \quad (\mathrm{N}).}$$

For a complete circuit

$$\boxed{\mathbf{F}_m = I \oint_C d\ell \times \mathbf{B} \quad (\mathrm{N}).}$$

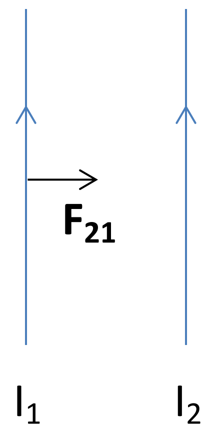


Two Circuits Carrying Currents

$$\mathbf{F}_{21} = I_1 \oint_{C_1} d\boldsymbol{\ell}_1 \times \mathbf{B}_{21},$$

Force $\mathbf{F}_{21}$ on circuit C_1

B from I_2 on circuit 1



Biot-Savart law (source: $_2$)

$$\mathbf{B}_{21} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}}}{R_{21}^2}.$$

$\mathbf{a}_{R21}$: from 2 (source) to 1 (field)

The Ampere's law of force

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},}$$

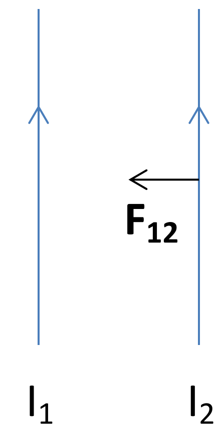
Comparison with Coulomb's law:

$$\mathbf{F}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},}$$

Interchanging subscript 1 and 2

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2}.$$



<Proof> Newton's third law in this case

First, $d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}}) \neq -d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})$,

So we need to check if $F_{12} = -F_{21}$!?

Expand the left side by back-cab rule:

$$\frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\boldsymbol{\ell}_2(d\boldsymbol{\ell}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}}(d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2)}{R_{21}^2}.$$

$$\frac{d\ell_1 \times (d\ell_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \underline{\frac{d\ell_2(d\ell_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2}} - \frac{\mathbf{a}_{R_{21}}(d\ell_1 \cdot d\ell_2)}{R_{21}^2}.$$

Double closed line integration

$$\oint_{C_1}\oint_{C_2} \frac{d\ell_2(d\ell_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} = \oint_{C_2} d\ell_2 \oint_{C_1} \frac{d\ell_1 \cdot \mathbf{a}_{R_{21}}}{R_{21}^2}$$

$$\boxed{\nabla_1(1/R_{21}) = -\mathbf{a}_{R_{21}}/R_{21}^2.}$$

$$= \oint_{C_2} d\ell_2 \oint_{C_1} d\ell_1 \cdot \left(-\nabla_1 \frac{1}{R_{21}}\right)$$

$$= -\oint_{C_2} d\ell_2 \underbrace{\oint_{C_1} d\left(\frac{1}{R_{21}}\right)}_{=0} = 0.$$

1st term of right side disappears

Thus,

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{d\ell_1 \times (d\ell_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2}$$

$$\mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{\mathbf{a}_{R_{21}}(d\ell_1 \cdot d\ell_2)}{R_{21}^2},$$

Interchanging subscript 1 and 2

$$\mathbf{a}_{R_{12}} = -\mathbf{a}_{R_{21}}$$

$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

A Circular Circuit Carrying Currents

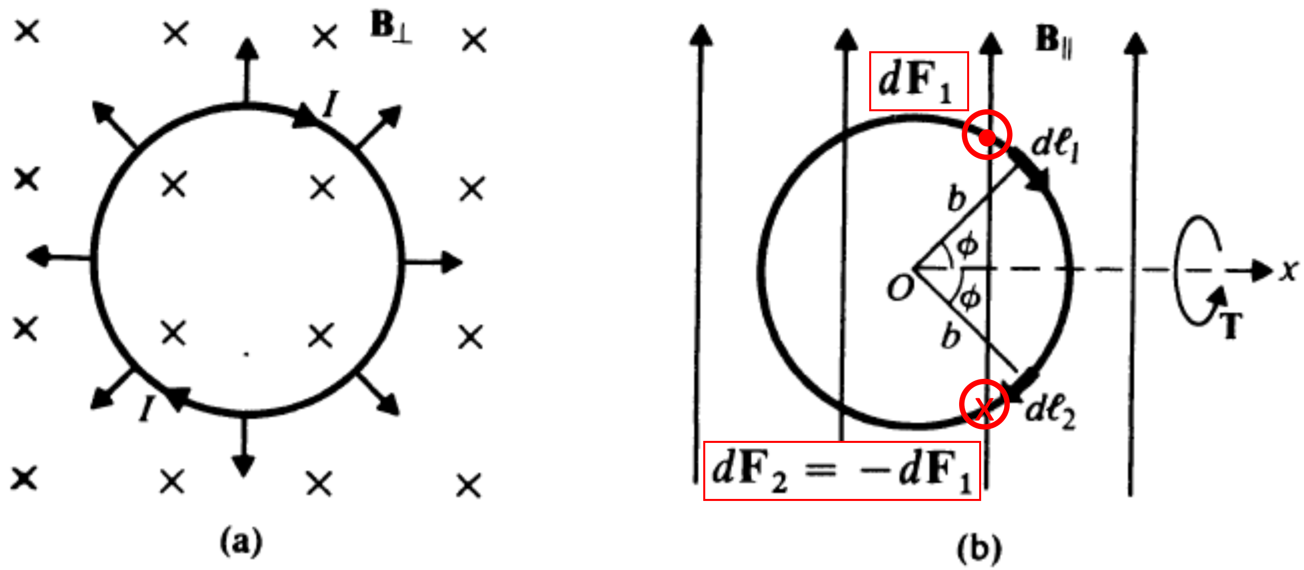
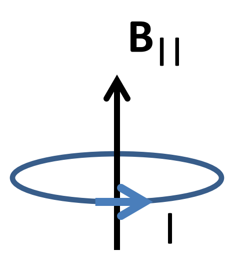
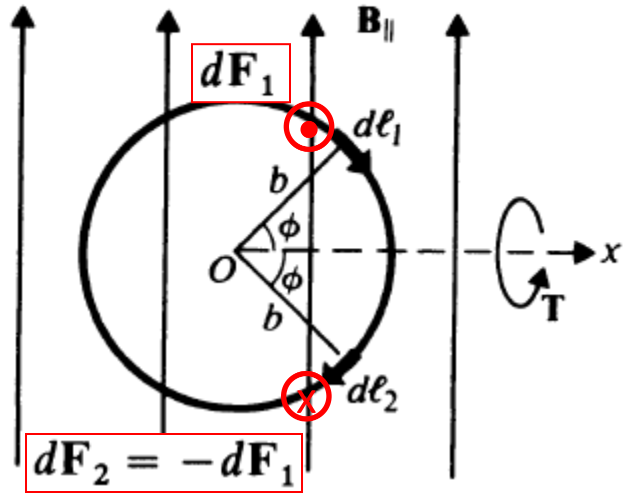


FIGURE 6–30
A circular loop in a uniform magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$.

(a) $\mathbf{B}_\perp$ ➔ $\mathbf{F}_m$ tends to expand the loop
(b) $\mathbf{B}_{//}$ ➔ $\mathbf{F}_m$ tends to rotates the loop about *x* axis
(or, tends to align the **B** (due to **I**) with $\mathbf{B}_{||}$)





The torque by d$\mathbf{F_1}$ and d$\mathbf{F_2}$

$$dF = |d\mathbf{F}_1| = |d\mathbf{F}_2|$$

$$d\ell = |d\ell_1| = |d\ell_2| = b\,d\phi.$$

Torque due to dl_1 and dl_2

$$d\mathbf{T} = \mathbf{a}_x(dF)\underbrace{2b\sin\phi}_{\text{arm}}$$

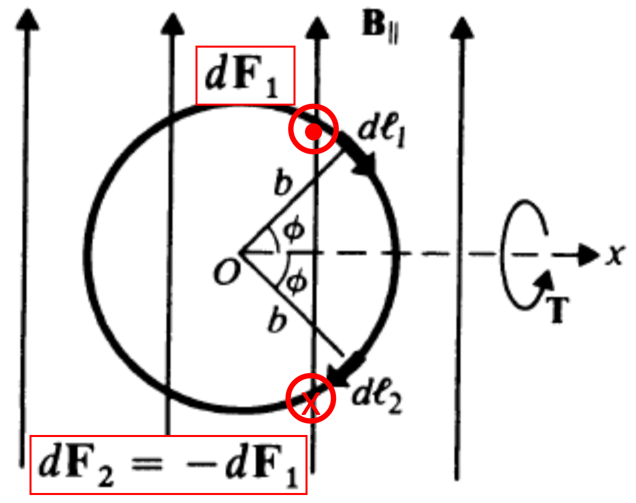
dF=I**dl**×**B**$_{||}$

$$= \mathbf{a}_x(I\,d\ell\,B_{||}\sin\phi)2b\sin\phi$$

$$= \mathbf{a}_x 2Ib^2B_{||}\sin^2\phi\,d\phi,$$

$$\mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2Ib^2B_{||}\int_0^{\pi}\sin^2\phi\,d\phi$$

$$= \mathbf{a}_x I(\pi b^2)B_{||}.$$



$$\mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2Ib^2B_{||} \int_0^{\pi} \sin^2 \phi \, d\phi$$
$$= \mathbf{a}_x I(\pi b^2)B_{||}.$$

By definition of magnetic dipole moment **m**

$$\boxed{\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n IS,}$$

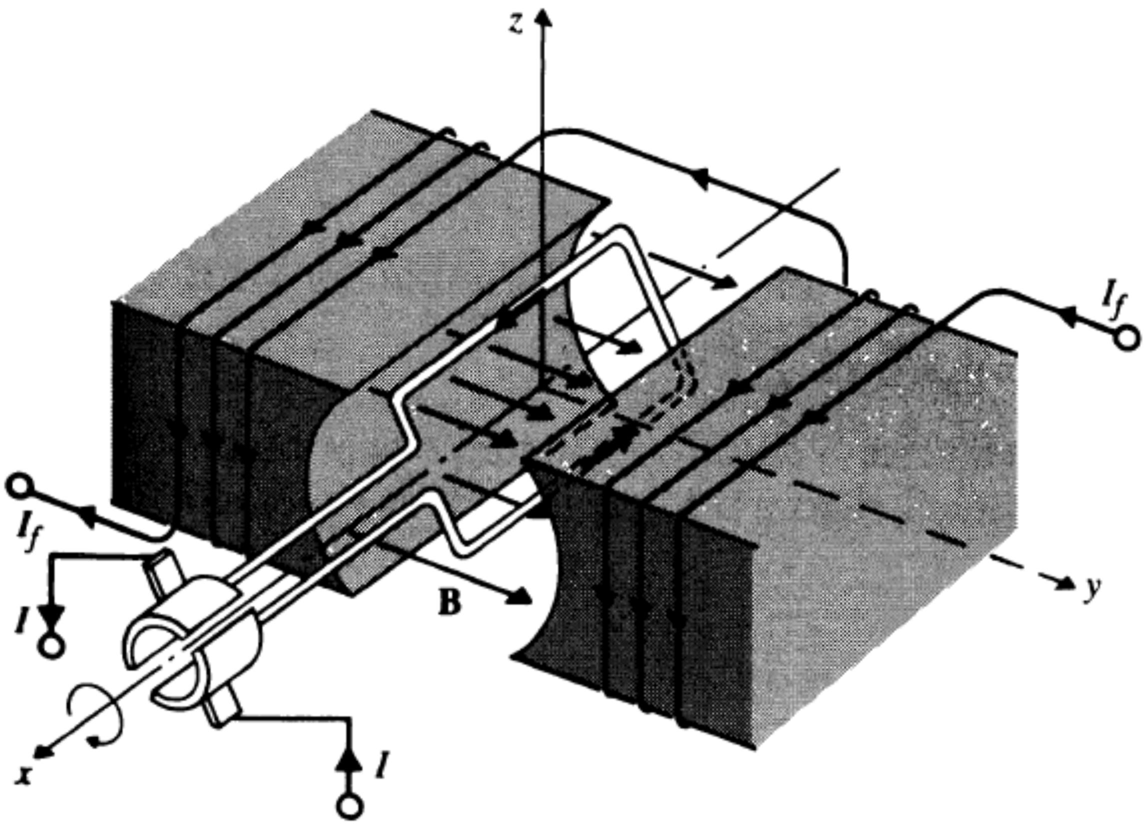
$$\mathbf{m} \times (\mathbf{B}_{\perp} + \mathbf{B}_{||}) = \mathbf{m} \times \mathbf{B}_{||}.$$

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\mathrm{N \cdot m}).}$$

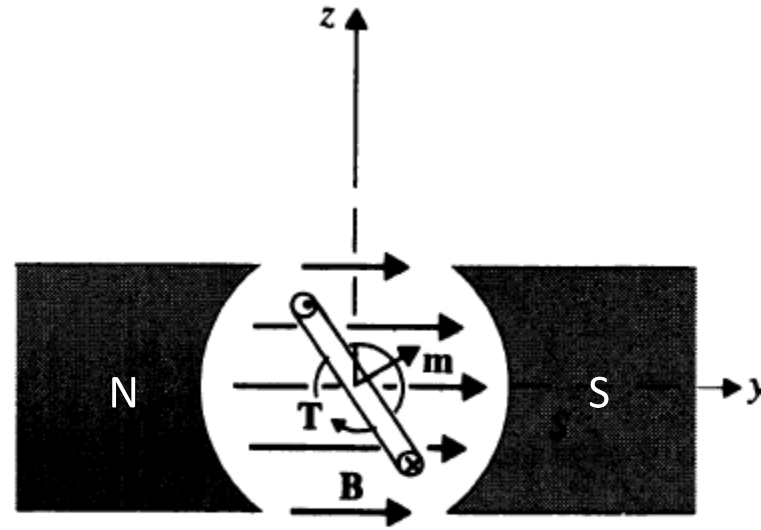
holds also for a planar loop of any shape under a uniform **B**

Microscopically, an applied **B** ➔ a torque to align magnetic dipoles along **B** in magnetic materials ➔ magnetization in the material

DC Motor



(a) Perspective view.

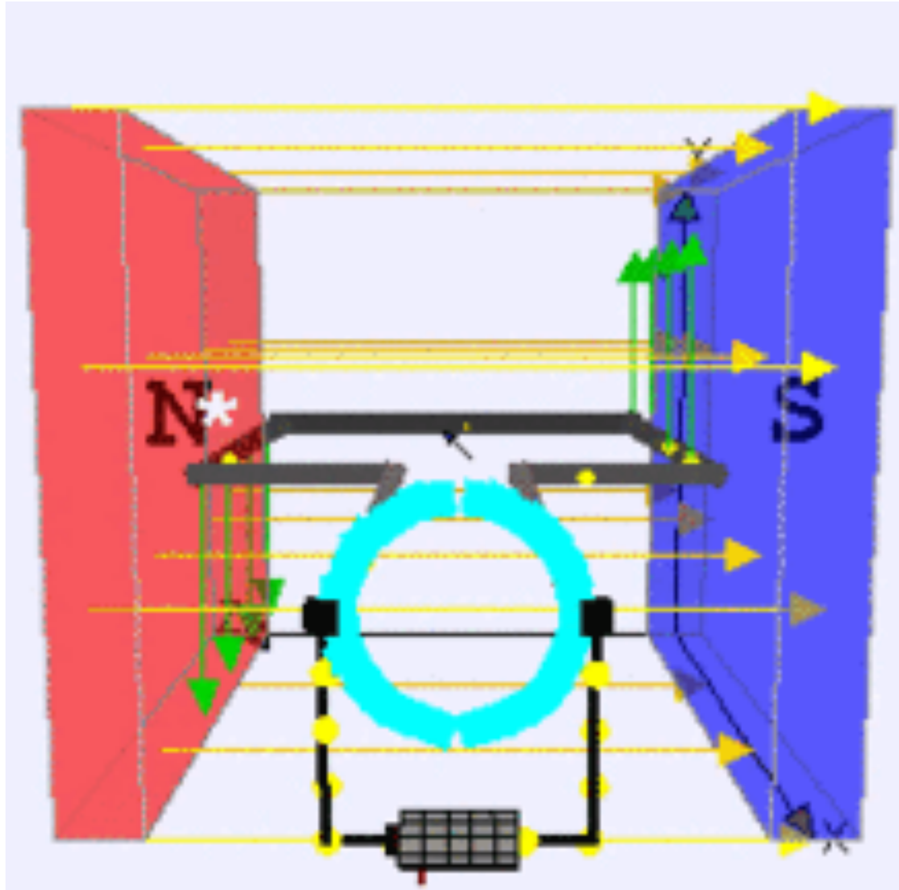


$$\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n IS,$$

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\mathrm{N \cdot m}).}$$

(b) Schematic view from $+x$ direction.

FIGURE 6–32
Illustrating the principle of operation of d-c motor.



Yellow: **B**
Green: force, **F**

When **m** // **B**, the direction of currents reverses

Extra slides

6-8 Magnetic Circuits

- Magnetic circuits: transformers, generators, motors, relays, magnetic recording devices, and so on.
- Electric circuits: to find V (and **E**) and I
  Magnetic circuits: to find A (and **H**) and Φ
- Analysis of magnetic circuits:

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Magnetomotive Force (mmf)

- Analogous to electromotive force (emf)
- Not a force in newtons, but in ampere (A)

$$\boxed{\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I \qquad \text{(A)},}$$

$$\Downarrow$$

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = NI = \mathscr{V}_m.$$

A toroidal core with N turns of wires wound

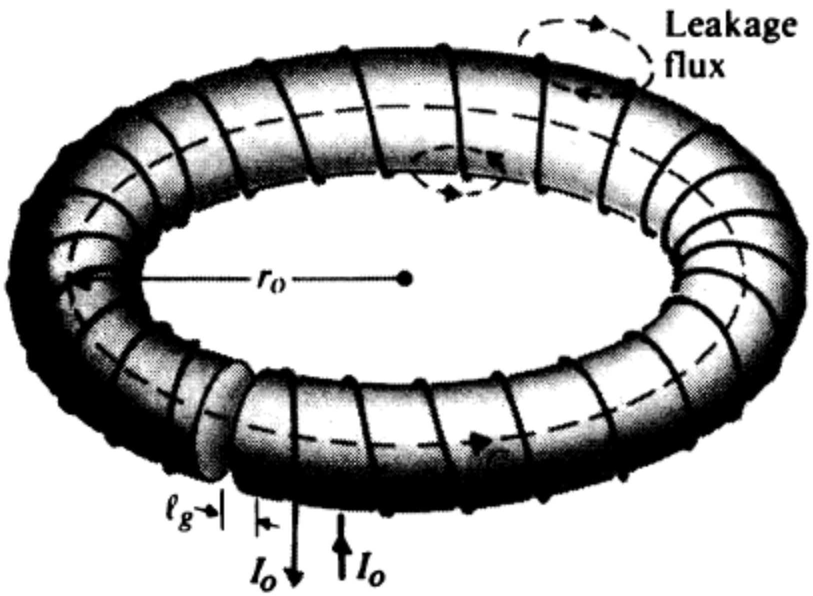


FIGURE 6–13
Coil on ferromagnetic toroid with air gap (Example 6–10).

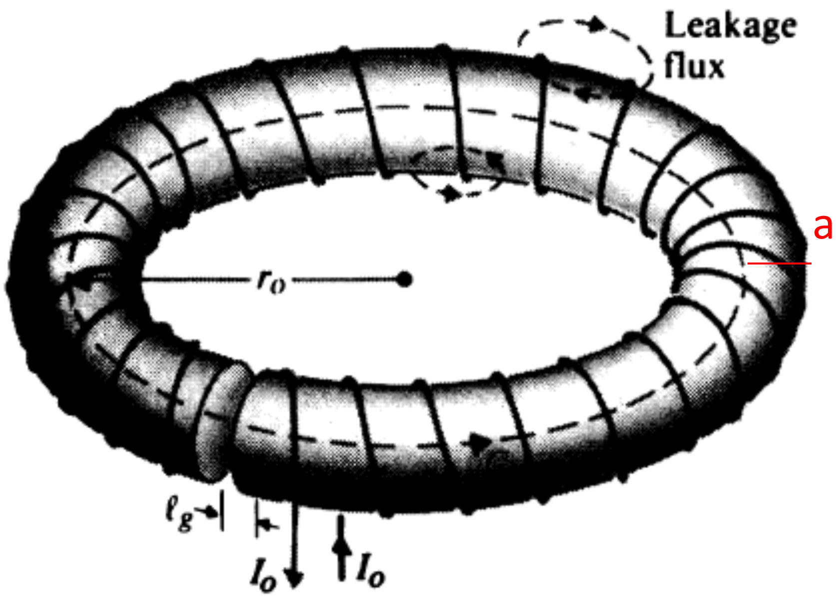
In the ferromagnetic core

$$\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{\mu_0 N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

In the air gap

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{\mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$



a: radius of the coil
S: cross section of the toroid core

FIGURE 6–13
Coil on ferromagnetic toroid with air gap (Example 6–10).

If a<<r_0, **B** ~ constant,

$$\Phi \cong BS,$$

$$\boxed{\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.}$$

$$\Phi = \frac{N I_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}.$$

$$\Phi = \frac{NI_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}.$$

$$\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g},$$

where $\mathscr{R}_f = \dfrac{2\pi r_o - \ell_g}{\mu S} = \dfrac{\ell_f}{\mu S},$ $\mathscr{R}_g = \dfrac{\ell_g}{\mu_0 S}.$

$$\ell_f = 2\pi r_o - \ell_g$$

ℓ_f: length of the ferromagnetic core

R_f: reluctance of the ferromagnetic core (H^{-1})
R_g: reluctance of the air gap

Ohm's law: $I=V/R$ Resistance $R=\ell/(\sigma S)$

Analogous to Electric Circuit

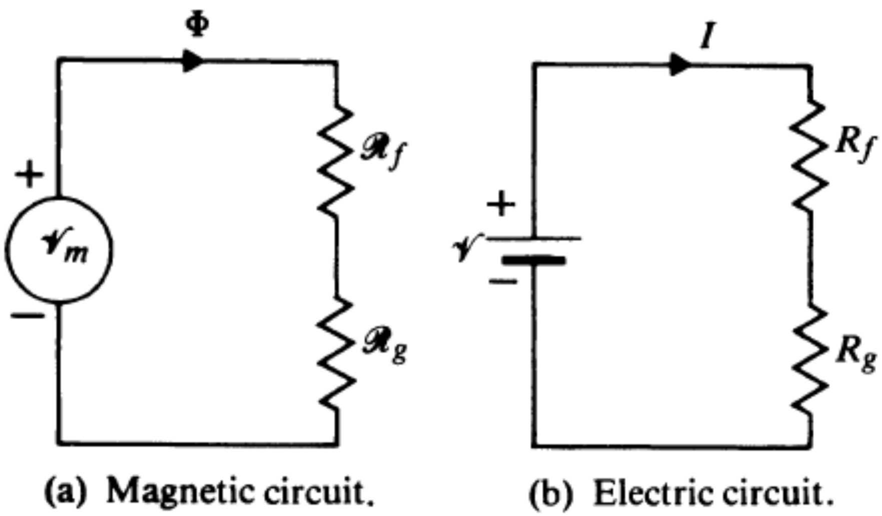


FIGURE 6–14
Equivalent magnetic circuit and analogous electric circuit for toroidal coil with air gap in Fig. 6–13.

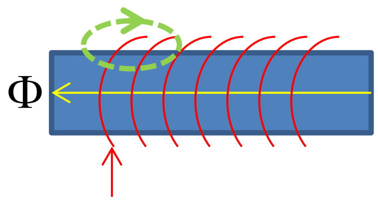
$$\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g}, \qquad I = \frac{\mathscr{V}}{R_f + R_g}.$$

Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m\ (= NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

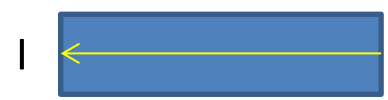
Difficulty in Analysis of Magnetic Circuits

- 1. Very difficult to account for leakage fluxes

Magnetic circuits



Electric circuits

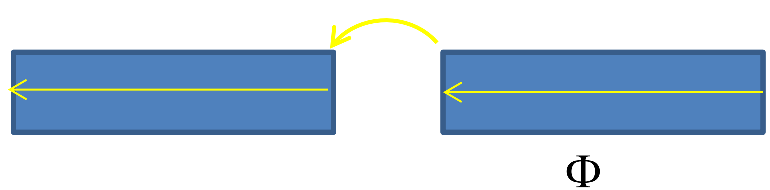


Leakage fluxes Φ outside ≠ 0
due to $\mu_0 \neq 0$ (in air)

Current I outside = 0
because of $\sigma=0$ (in air)

- 2. Difficult to account for fringing effect (at the air gap)

Magnetic circuits



- 3. **B** and **H** have a nonlinear relationship
  - **B**(**H**) or μ is not a constant

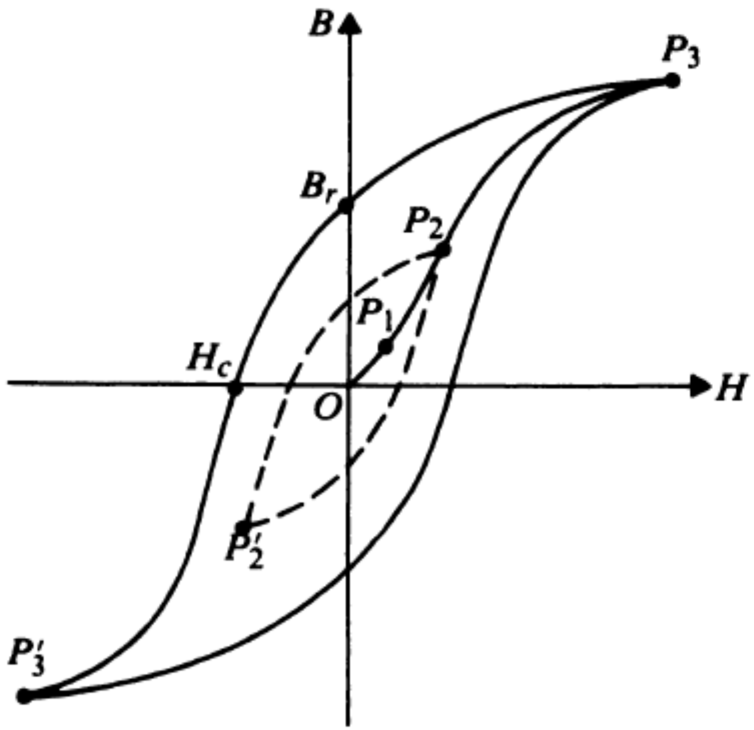


FIGURE 6–17
Hysteresis loops in the *B*–*H* plane for ferromagnetic material.

Analysis of Magnetic Circuits (w/ nonlinear B-H effect)

$$\boxed{\oint_C \mathbf{H} \cdot d\ell = NI_o.}$$

$$H_g\ell_g + H_f\ell_f = NI_o.$$

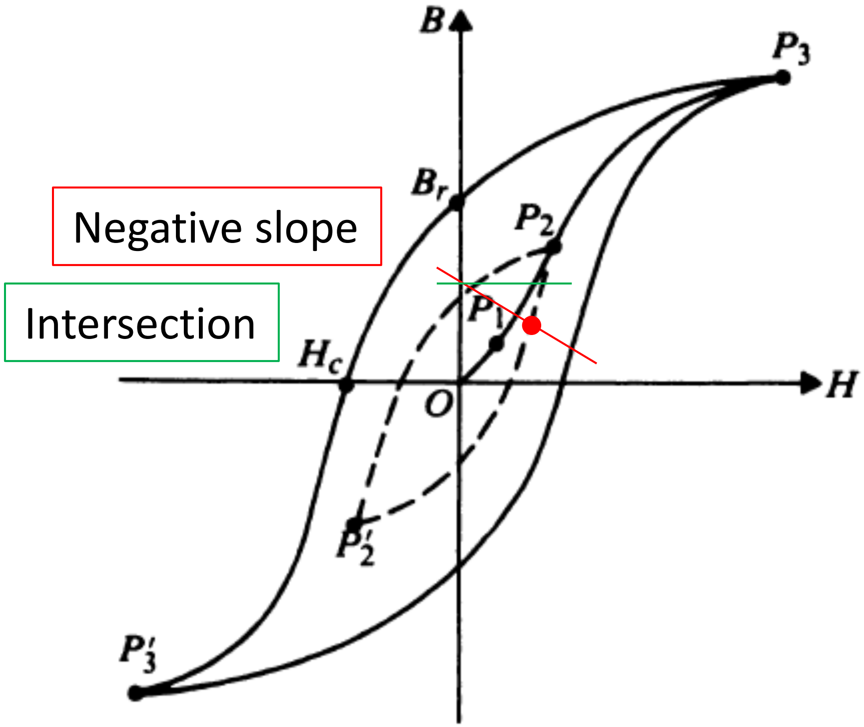
Neglect fringing effect ➔ $B_f = B_g$

$$B_f = \mu_0 H_g.$$

$$B_f + \underline{\mu_0 \frac{\ell_f}{\ell_g}} H_f = \underline{\frac{\mu_0}{\ell_g} NI_o}.$$

In B-H curve, Negative slope Intersection

Find the operating points in the B-H curve



Line equation can be determined.

With B_f ➔ H_f and μ can be obtained.

KVL and KCL in Magnetic Circuits

$$\boxed{\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g},}$$

$$\boxed{\sum_j N_j I_j = \sum_k \mathscr{R}_k \Phi_k.}$$

KVL: around a closed path in a magnetic circuit the algebraic sum of ampere-terns is equal to the algebraic sum of the products of the reluctances and fluxes

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{\sum_j \Phi_j = 0,}$$

KCL: the algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero

See Chap. 5

$$I = \frac{\mathscr{V}}{R_f + R_g}.$$

$$\nabla \cdot \mathbf{J} = 0.$$

6-13.3 Forces and Torques in Terms of Stored Magnetic Energy

- All current-carrying conductors and circuits experience $\mathbf{F_m}$ when situated in a magnetic field.
- In general, determination of $\mathbf{F_m}$ is tedious by Ampere's law of force (except for special symmetrical cases).

Ampere's law of force

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},}$$

- Alternative method: principle of virtual displacement

Case 1: System of Circuits with **Constant Flux Linkages**

Assume a virtual differential displacement dl

Assume **Constant Flux Linkages**

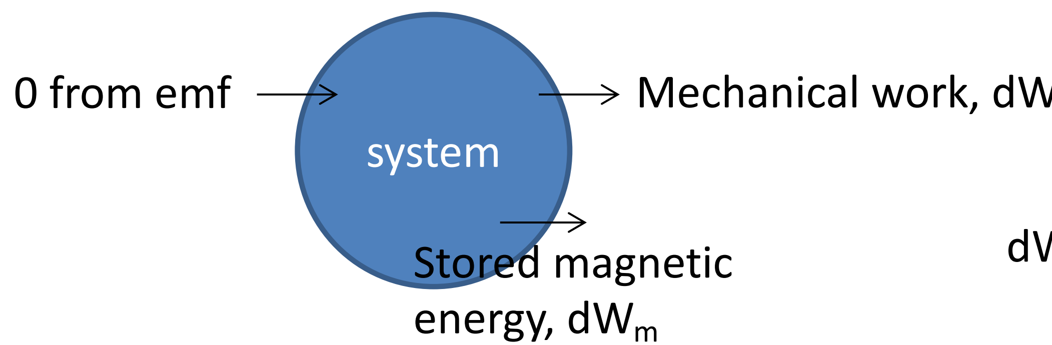
➔ $\Delta\Phi=0$

➔ emf = $d\Phi/dt = 0$

The source supplies no energy to the system

The mechanical work done by the system

$\mathbf{F}_\Phi \cdot d\ell,$



$dW+dW_m=0$

The mechanical work (increase) is provided by the stored magnetic energy (decrease, $dW_m<0$)

$$\mathbf{F}_\Phi \cdot d\ell = -dW_m = -(\nabla W_m)\cdot d\ell,$$

$$\mathbf{F}_\Phi \cdot d\boldsymbol{\ell} = -dW_m = -(\nabla W_m) \cdot d\boldsymbol{\ell},$$

$$\boxed{\mathbf{F}_\Phi = -\nabla W_m \qquad (\mathrm{N}).}$$

In Cartesian coordinates,

$$(F_\Phi)_x = -\frac{\partial W_m}{\partial x},$$

$$(F_\Phi)_y = -\frac{\partial W_m}{\partial y},$$

$$(F_\Phi)_z = -\frac{\partial W_m}{\partial z}.$$

For the case with a virtual rotation dϕ about an axis (say z axis)

$$\boxed{(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \qquad (\mathrm{N \cdot m}),}$$

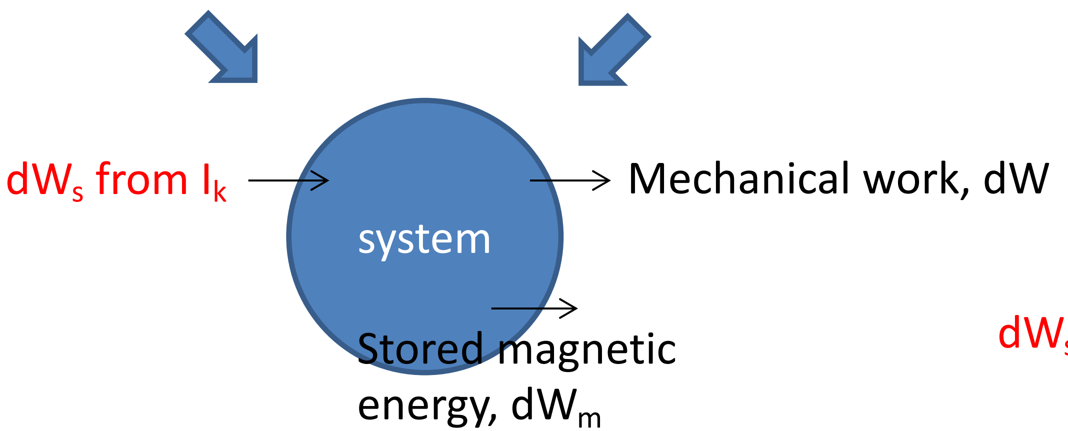
Case 2: System of Circuits with **Constant Currents**

Assume a virtual differential displacement dl

dl → dΦ

$$dW_s = \sum_k I_k \, d\Phi_k.$$

The source supplies energy to the system



$dW_s = dW + dW_m$

The mechanical work (increase) and the stored magnetic energy (increase, $dW_m>0$) are provided by dW_s

$$dW_s = dW + dW_m.$$

$$\boxed{W_m = \frac{1}{2}\sum_{k=1}^{N} I_k\Phi_k \qquad \text{(J)},}$$

$$\boxed{dW_s = \sum_k I_k\, d\Phi_k.}$$

$$dW_m = \frac{1}{2}\sum_k I_k\, d\Phi_k = \frac{1}{2}\, dW_s.$$

$$dW = \mathbf{F}_I \cdot d\boldsymbol{\ell} = dW_m$$
$$= (\nabla W_m) \cdot d\boldsymbol{\ell}$$

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},}$$

Similar to case 1 except for a sign change

For the case with a virtual rotation $d\phi$ about an axis (say z axis)

$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$

6-13.4 Forces and Torques in Terms of Mutual Inductance

- Method of virtual displacement ($d\ell$) for constant currents is powerful to determine the **F** and **T** between rigid-carrying circuits.
- The magnetic energy of two circuits with currents I_1 and I_2:

$$W_m = \tfrac{1}{2}L_1I_1^2 + L_{12}I_1I_2 + \tfrac{1}{2}L_2I_2^2.$$

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad (\text{N}),}$$

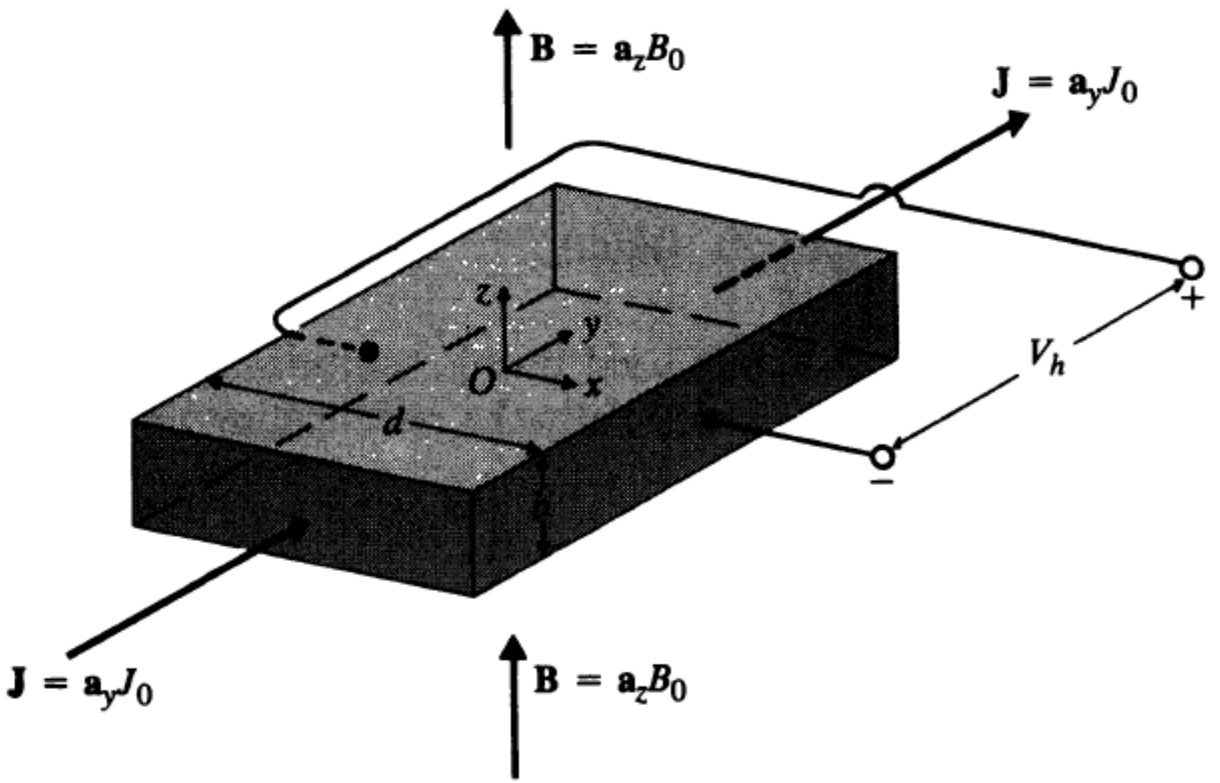
$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$

L_1 and L_2 (self inductance) remain constants given a virtual displacement $d\ell$

$$\boxed{\mathbf{F}_I = I_1 I_2 (\nabla L_{12}) \qquad (\text{N}).}$$

$$\boxed{(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$

6-13.1 Hall Effect

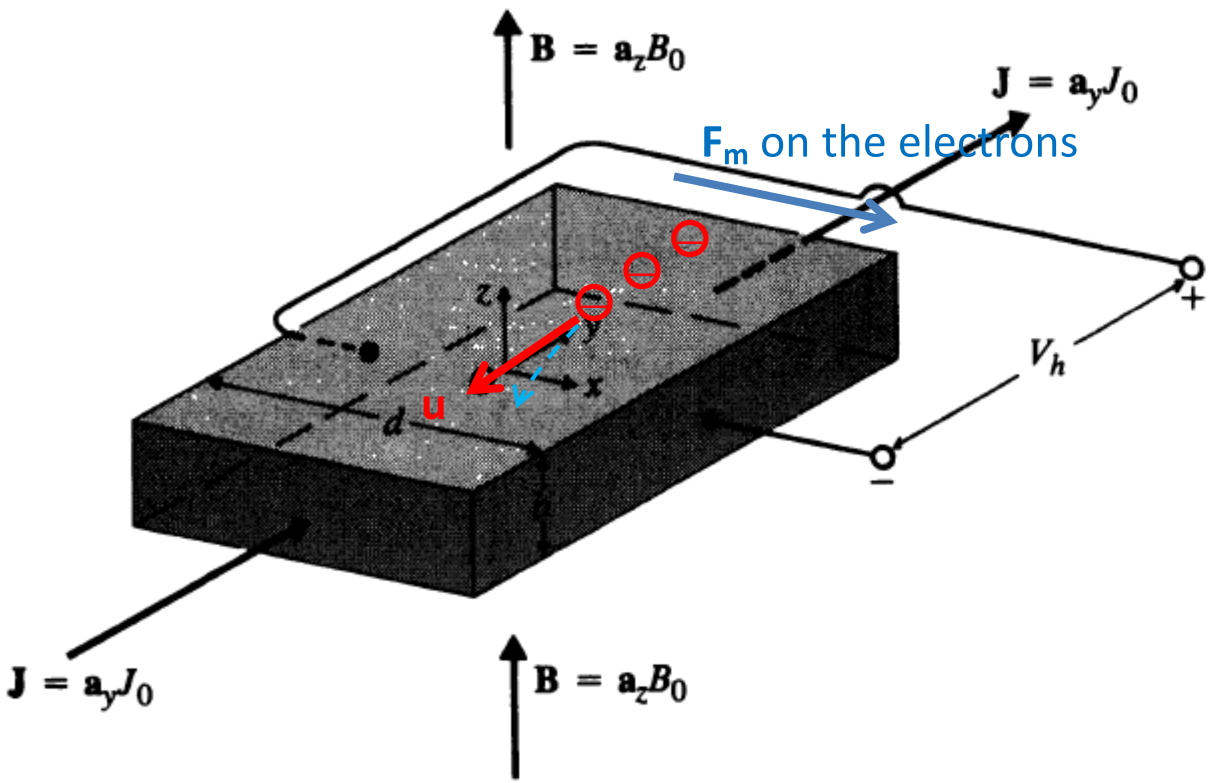


$$\boxed{\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u},}$$

FIGURE 6–28
Illustrating the Hall effect.

From $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ (N). ➡ There is a force $\perp$ **u**, $\perp$ **B**

Considering a n-type semiconductor (carriers: electrons): q is negative



$$\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}.}$$

$\mathbf{F_m} \,//\, -(-\mathbf{a_y})\times\mathbf{a_z} = \mathbf{a_x}$

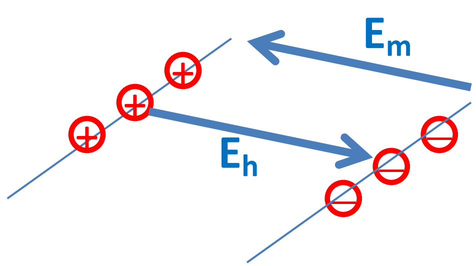
$\mathbf{E_m} = \mathbf{u}\times\mathbf{B}$

$\mathbf{E_m} \,//\, (-\mathbf{a_y})\times\mathbf{a_z} = -\mathbf{a_x}$

For electrons:

$$\mathbf{u} = -\mathbf{a}_y u_0$$

FIGURE 6–28
Illustrating the Hall effect.



$\mathbf{E_m}$: due to moving charge (magnetic force)
$\mathbf{E_h}$: due to charge accumulation

In the steady state, the net force on the charge carriers is zero.

F = q($\mathbf{E}_m$+$\mathbf{E}_h$) = 0

$\mathbf{E}_m$+$\mathbf{E}_h$ = 0

$$\mathbf{E}_h + \mathbf{u} \times \mathbf{B} = 0$$

or $\mathbf{E}_h = -\mathbf{u} \times \mathbf{B}$.

$\mathbf{E}_h$: Hall field

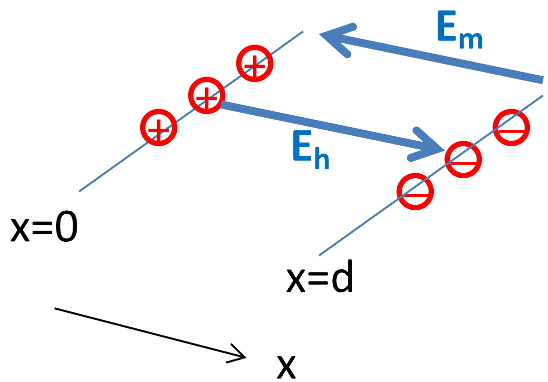
$\boxed{\mathbf{u} = -\mathbf{a}_y u_0}$

$$\mathbf{E}_h = -(-\mathbf{a}_y u_0) \times \mathbf{a}_z B_0 = \mathbf{a}_x u_0 B_0.$$

$$V_h = -\int_0^d E_h\, dx = u_0 B_0 d,$$

Hall effect can be used to measure the B field

V_h: Hall voltage



$$\boxed{\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u},}$$

$$\boxed{\begin{aligned}\mathbf{E}_h &= -(-\mathbf{a}_y u_0) \times \mathbf{a}_z B_0 \\ &= \mathbf{a}_x u_0 B_0.\end{aligned}}$$

$\mathbf{J}\times\mathbf{B}$ // $\mathbf{E}_h$

$$E_x/J_yB_z = \underline{1/Nq}$$

Hall coefficient, a characteristic of the material

- Considering a p-type semiconductor (carriers: + charges): $\mathbf{E_h}$ will be reversed, V_h will be in opposite polarity (see Fig. 6-28)
- Hall effect can be used to determine the sign of predominant charge carriers.