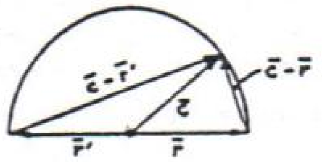
HW1-1

P.2-11



$$\bar{r}' = -\bar{r}, \qquad r' = r = c.$$

$$(\bar{c}-\bar{r}')\cdot(\bar{c}-\bar{r}) = (\bar{c}+\bar{r})\cdot(\bar{c}-\bar{r}) = 0.$$

$$\therefore \quad (\bar{c}-\bar{r}') \perp (\bar{c}-\bar{r}).$$

HW1-2

P.2-17 a) $\bar{E}_P = \bar{a}_R \dfrac{25}{(-3)^2+4^2+(-5)^2} = \bar{a}_R \dfrac{1}{2}.$

$$(E_P)_x = \frac{1}{2}\left(\frac{-3}{\sqrt{(-3)^2+4^2+(-5)^2}}\right) = -0.212.$$

b) $\bar{a}_R = \dfrac{1}{\sqrt{50}}(-\bar{a}_x 3 + \bar{a}_y 4 - \bar{a}_z 5)$, $\bar{a}_B = \dfrac{\bar{B}}{B} = \dfrac{1}{3}(\bar{a}_x 2 - \bar{a}_y 2 + \bar{a}_z).$

$$\theta = \cos^{-1}(\bar{a}_R \cdot \bar{a}_B) = \cos^{-1}\left(-\frac{19}{3\sqrt{50}}\right) = 154^\circ$$

HW1-3

P.2-21 $\displaystyle\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_{P_1}^{P_2} (y\,dx + x\,dy).$

a) $x = 2y^2$, $dx = 4y\,dy$; $\displaystyle\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_1^2 (4y^2dy + 2y^2dy) = 14$

b) $x = 6y-4$, $dx = 6\,dy$; $\displaystyle\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_1^2 [6y\,dy + (6y-4)]dy = 14.$

Equal line integrals along two specific paths do not necessarily imply a conservative field. $\bar{E}$ is a conservative field in this case because $\bar{E} = \bar{\nabla}(xy+c)$.

HW1-4

P.2-26 In spherical coordinates, $\bar{\nabla}\cdot\bar{A} = \dfrac{1}{R^2}\dfrac{\partial}{\partial R}(R^2 A_R)$, if $\bar{A} = \bar{a}_R A_R$.

a) $\bar{A} = f_1(\bar{R}) = \bar{a}_R R^n$, $\quad A_R = R^n$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}.$$

b) $\bar{A} = f_2(\bar{R}) = \bar{a}_R \dfrac{k}{R^2}$, $\quad A_R = kR^{-2}$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0.$$

HW1-5

P. 2-29 $\oint_S \bar{A}\cdot d\bar{s} = \left(\int_{\substack{top\\face}} + \int_{\substack{bottom\\face}} + \int_{walls}\right) \bar{A}\cdot d\bar{s}.$

Top face $(z=4)$: $\bar{A} = \bar{a}_r r^2 + \bar{a}_z 8, \quad d\bar{s} = \bar{a}_z ds.$

$$\int_{\substack{top\\face}} \bar{A}\cdot d\bar{s} = \int_{\substack{top\\face}} 8\, ds = 8(\pi 5^2) = 200\pi.$$

Bottom face $(z=0)$: $\bar{A} = \bar{a}_r r^2; \quad d\bar{s} = -\bar{a}_z ds.$

$$\int_{\substack{bottom\\face}} \bar{A}\cdot d\bar{s} = 0.$$

Walls $(r=5)$: $\bar{A} = \bar{a}_r 25 + \bar{a}_z 2z, \quad d\bar{s} = \bar{a}_r ds.$

$$\int_{walls} \bar{A}\cdot d\bar{s} = 25\int_{walls} ds = 25(2\pi 5\times 4) = 1000\pi.$$

$$\therefore \oint_S \bar{A}\cdot d\bar{s} = 200\pi + 0 + 1000\pi = 1{,}200\pi.$$

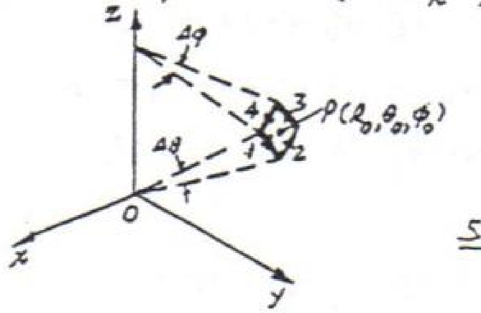
$$\bar{\nabla}\cdot\bar{A} = 3r+2, \quad \int_V \bar{\nabla}\cdot\bar{A}\, dv = \int_0^4\int_0^{2\pi}\int_0^5 \bar{\nabla}\cdot\bar{A}\; r\,dr\,d\phi\,dz = 1{,}200\pi = \oint_S \bar{A}\cdot d\bar{s}.$$

HW1-6

$$
\begin{aligned}
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \nabla\cdot\left((A_yB_z - A_zB_y)\,\mathbf{i} + (A_zB_x - A_xB_z)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x) \\
&= \left(\frac{\partial A_yB_z}{\partial x} - \frac{\partial A_zB_y}{\partial x}\right) + \left(\frac{\partial A_zB_x}{\partial y} - \frac{\partial A_xB_z}{\partial y}\right) + \left(\frac{\partial A_xB_y}{\partial z} - \frac{\partial A_yB_x}{\partial z}\right) \\
&= \left(A_y\frac{\partial B_z}{\partial x} + \frac{\partial A_y}{\partial x}B_z - A_z\frac{\partial B_y}{\partial x} - \frac{\partial A_z}{\partial x}B_y\right) + \left(A_z\frac{\partial B_x}{\partial y} + \frac{\partial A_z}{\partial y}B_x - A_x\frac{\partial B_z}{\partial y} - \frac{\partial A_x}{\partial y}B_z\right) + \left(A_x\frac{\partial B_y}{\partial z} + \frac{\partial A_x}{\partial z}B_y - A_y\frac{\partial B_x}{\partial z} - \frac{\partial A_y}{\partial z}B_x\right) \\
&= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&\quad + A_x\left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) + A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + A_z\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) \\
&= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&\quad - A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \\
&= (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})\cdot\left(\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}\right) \\
&\quad - (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot\left(\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{k}\right) \\
&= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})
\end{aligned}
$$

HW1-7

P.2-35 Eq. (2-126): $(\bar{\nabla}\times\bar{A})_R = \lim_{\Delta S_R\to 0}\frac{1}{\Delta S_R}\left(\oint_{C_R}\bar{A}\cdot d\bar{\ell}\right)$, ①



where $\Delta S_R = R_0^2\sin\theta_0\,\Delta\theta\,\Delta\phi$ ② and the contour consists of the four sides 1,2,3,&4.

Side 1: $d\bar{\ell} = \bar{a}_\theta\,(R_0\Delta\theta)$,
$\bar{A}\cdot d\bar{\ell} = A_\theta(R_0,\theta_0,\phi_0-\frac{\Delta\phi}{2})R_0\Delta\theta$,
where $A_\theta(R_0,\theta_0,\phi_0-\frac{\Delta\phi}{2}) = A_\theta(R_0,\theta_0,\phi_0) - \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\Big|_{(R_0,\theta_0,\phi_0)} + H.O.T.$

$$\int_{side\,1}\bar{A}\cdot d\bar{\ell} = \left\{\left[A_\theta - \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\Delta\theta. \quad ③$$

Side 3: $d\bar{\ell} = -\bar{a}_\theta(R_0\Delta\theta)$, $\bar{A}\cdot d\bar{\ell} = A_\theta(R_0,\theta_0,\phi_0+\frac{\Delta\phi}{2})R_0\Delta\theta$,

$$\int_{side\,3}\bar{A}\cdot d\bar{\ell} = -\left\{\left[A_\theta + \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\Delta\theta. \quad ④$$

Combining ③ and ④:

$$\int_{\substack{sides\\1\&3}}\bar{A}\cdot d\bar{\ell} = \left(-\frac{\partial A_\theta}{\partial\phi} + H.O.T.\right)\Big|_{(R_0,\theta_0,\phi_0)} R_0\Delta\theta\Delta\phi. \quad ⑤$$

Side 2: $d\bar{\ell} = \bar{a}_\phi R_0\sin(\theta_0+\frac{\Delta\theta}{2})\Delta\phi$, $\bar{A}\cdot d\bar{\ell} = A_\phi(R_0,\theta_0+\frac{\Delta\theta}{2},\phi_0)R_0\sin(\theta_0+\frac{\Delta\theta}{2})\Delta\phi$,

$$\int_{side\,2}\bar{A}\cdot d\bar{\ell} \cong \left\{\left[A_\phi + \frac{\Delta\theta}{2}\frac{\partial A_\phi}{\partial\theta}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\left(\sin\theta_0 + \frac{\Delta\theta}{2}\cos\theta_0\right)\Delta\phi. \quad ⑥$$

Side 4:

$$\int_{side\,4}\bar{A}\cdot d\bar{\ell} \cong -\left\{\left[A_\phi - \frac{\Delta\theta}{2}\frac{\partial A_\phi}{\partial\theta}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\left(\sin\theta_0 - \frac{\Delta\theta}{2}\cos\theta_0\right)\Delta\phi. \quad ⑦$$

Combining ⑥ and ⑦:

$$\int_{\substack{sides\\2\&4}}\bar{A}\cdot d\bar{\ell} = \frac{\partial A_\phi}{\partial\theta}\Big|_{(R_0,\theta_0,\phi_0)} R_0\sin\theta_0\Delta\theta\Delta\phi + A_\phi R_0\cos\theta_0\Delta\theta\Delta\phi + H.O.T.$$

$$= \frac{\partial}{\partial\theta}(A_\phi\sin\theta)\Big|_{(R_0,\theta_0,\phi_0)} R_0\Delta\theta\Delta\phi + H.O.T. \quad ⑧$$

Substituting ②, ⑤, and ⑧ in ①, we obtain

$$(\bar{\nabla}\times\bar{A})_R = \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right],$$

where the subscript 0 has been dropped for simplicity.

P. 2-39. $\bar{F} = \bar{a}_x(x + c_1 z) + \bar{a}_y(c_2 x - 3z) + \bar{a}_z(x + c_4 y + c_5 z)$.

a) $\bar{F}$ irrotational $\longrightarrow \bar{\nabla} \times \bar{F} = 0$,

or $$\bar{a}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \bar{a}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \bar{a}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial F_y}\right) = 0,$$

which gives three equations:

$$\frac{\partial}{\partial y}(x + c_3 y + c_4 z) - \frac{\partial}{\partial z}(c_2 x - 3z) = 0 \longrightarrow c_3 + 3 = 0 \longrightarrow c_3 = -3$$

$$\frac{\partial}{\partial z}(x + c_1 z) - \frac{\partial}{\partial x}(x + c_3 y + c_4 z) = 0 \longrightarrow c_1 - 1 = 0 \longrightarrow c_1 = 1.$$

$$\frac{\partial}{\partial x}(c_2 x - 3z) - \frac{\partial}{\partial y}(x + c_1 z) = 0 \longrightarrow c_2 = 0.$$

b) $\bar{F}$ also solenoidal $\longrightarrow \bar{\nabla} \cdot \bar{F} = 0$,

or $$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0.$$

or $$\frac{\partial}{\partial x}(x + c_1 z) + \frac{\partial}{\partial y}(c_2 x - 3z) + \frac{\partial}{\partial z}(x + c_3 y + c_4 z) = 0,$$

or $$1 + c_4 = 0 \longrightarrow c_4 = -1.$$

c) $$\bar{F} = -\bar{\nabla} V \longrightarrow \bar{a}_x(x + z) - \bar{a}_y 3z + \bar{a}_z(x - 3y - z)$$
$$= -a_x \frac{\partial V}{\partial x} - a_y \frac{\partial V}{\partial y} - a_z \frac{\partial V}{\partial z}.$$

$$\frac{\partial V}{\partial x} = -(x + z) \longrightarrow V = -\frac{x^2}{2} - xz + f_1(y, z)$$

$$\frac{\partial V}{\partial y} = 3z \longrightarrow V = 3yz + f_2(x, z).$$

$$\frac{\partial V}{\partial z} = -x + 3y + z \longrightarrow V = -xz + 3yz + \frac{z^2}{2} + f_3(x, y).$$

$$\therefore \; V = -\frac{x^2}{2} - xz + 3yz + \frac{z^2}{2}.$$