RC4

1 Conductors and Dielectrics in Static Electric Field

- Conductors:
  - electrons migrate easily.
  - charges reach the surface and conductor redistribute themselves in a way that both the charge and the field vanish.
  - **static state conditions**:
    - inside the conductor:

      $$\boxed{\rho = 0,\ \mathbf{E} = 0}$$

      where $\rho = 0$ represents no charge in the interior
    - on the conductor surface (boundary conditions)

      $$\boxed{E_t = 0,\ E_n = \frac{\rho_s}{\epsilon_0}}$$

  - electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature
- Dielectrics (Insulators):
  - electrons are confined to their orbits.
  - **polarization charge densities/ bound-charge densities:**
    - **polarization mathbfor, P**:

      $$\boxed{\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p_k}}{\Delta v}}$$

      where the numerator represents the mathbfor sum of the induced dipole moment contained in a very small volume Δv.
    - charge distribution on surface density:

      $$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a_n}}$$

    - volume charge distribution density:

      $$\boxed{\rho_p = -\nabla \cdot \mathbf{P}}$$

      .

2 Electric Flux Density and Dielectric Constant

电位移矢量

- **electric flux density/electric displacement, $\mathbf{D}$**:

电通量密度

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (C/m^2)$$

- $$\nabla \cdot \mathbf{D} = \rho \quad (C/m^3)$$

free charge

where ρ is the volume density of free charges.

$\rho = \rho_f + \rho_p$; ρ_f: free, ρ_p: bound.

- Another form of **Gauss's law**:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad (\mathbf{C})$$

the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic**,

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_p}{\varepsilon_0}$

$\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_f + \rho_p$, $\rho_p = -\nabla \cdot \vec{P}$

$\nabla \cdot (\underbrace{\varepsilon_0 \vec{E} + \vec{P}}_{\vec{D}}) = \rho_f$

define $\varepsilon_r := 1 + \chi_e$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

$\varepsilon := \varepsilon_0 \varepsilon_r$

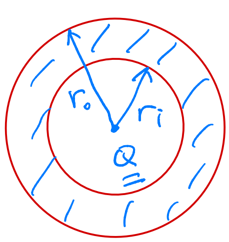
$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$$

where χ_e is a dimensionless quantity called electric susceptibility,

ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium,

ϵ is the absolute permittivity/permittivity of the medium (F/m). $\varepsilon_0 \varepsilon_r$

Example. A postive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine $\mathbf{E}, V, \mathbf{D}, \mathbf{P}$ as functions of the radial distance R.



① $\vec{D}$ (ρ_f)

$$\vec{D} = \frac{Q}{4\pi R^2} \vec{a}_R$$

② $\vec{D} = \varepsilon \vec{E}$

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 R^2} \vec{a}_R, & R > R_o \text{ or } R < R_i \\ \frac{Q}{4\pi\varepsilon_0 \varepsilon_r R^2} \vec{a}_R, & R_i \le R \le R_o \end{cases}$$

③ $$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \begin{cases} 0, & R > R_o \text{ or } R < R_i \\ \left(1 - \frac{1}{\varepsilon_r}\right) \frac{Q}{4\pi R^2} \vec{a}_R, & R_i < R < R_o \end{cases}$$

④ $R \to \infty$, $V = 0$

i) $R > R_o$, $V = \frac{Q}{4\pi\varepsilon_0 R}$, $V(R_o) = \frac{Q}{4\pi\varepsilon_0 R_o}$.

ii) $R_i \le R \le R_o$, $V_2 - V(R_o) = -\int_{R_o}^{R} \frac{Q}{4\pi\varepsilon_0 \varepsilon_r R^2} dR$

iii) $R < R_i$ $V_3 - V_2(R_i) = -\int_{R_i}^{R} \frac{Q}{4\pi\varepsilon_0 R^2} dR$.

$$V_3 = \frac{Q}{4\pi\varepsilon_0}\left[(1-\varepsilon_r)\frac{1}{R_o} - \left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$

- For anisotropic,

$\vec{D}.$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$D_x = \varepsilon_{11} E_x + \varepsilon_{12} E_y + \varepsilon_{13} E_z.$
$D_y = - -$
$D_z = - - - .$

For biaxial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

For uniaxial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

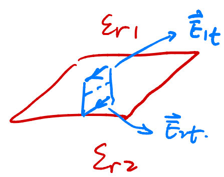
3 Boundary Conditions for Electrostatic Fields

$\oint \vec{E} \cdot d\vec{l} = 0$

$(\vec{E}_{1t} - \vec{E}_{2t}) \cdot d = 0$

- The tangential component of an **E** field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$



$\vec{E}_{1t}$ $\vec{E}_{2t}$

$\vec{D} = \varepsilon \vec{E}.$

or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

$\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$

- The normal component of **D** field is discontinous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$$\mathbf{a_{n2}} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$$

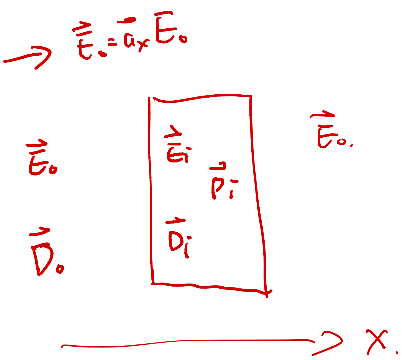
or

no free charge: $D_{1n} = D_{2n}$

free charge

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

Example. A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\mathbf{E_o} = \mathbf{a_x} E_o$ in free space. Determine $\mathbf{E_i}, \mathbf{D_i}, \mathbf{P_i}$ inside the lucite.



no free charge

$\vec{D}_i = \vec{D}_0 .$

$\vec{D}_0 = \varepsilon_0 \vec{E}_0 \Rightarrow \vec{D}_i = \varepsilon_0 \vec{E}_0 = \vec{D}_i = \varepsilon_0 \varepsilon_r \vec{E}_i.$

$\Rightarrow \vec{E}_i = \frac{1}{\varepsilon_r} \vec{E}_0$

$\vec{P}_i = \vec{D}_i - \varepsilon_0 \vec{E}_i = (1 - \frac{1}{\varepsilon_r}) \varepsilon_0 \vec{E}_0 .$

3.1 Exercise

- (HW3-4) Dielectric lenses can be used to collimate electromagnetic fields. As shown in the figure below, the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\mathbf{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\mathbf{a_r}5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that $\mathbf{E_3}$ in region ~~3 is parallel~~ to the x-axis?

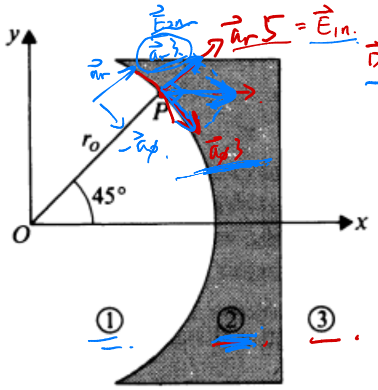
$\vec{a}_r 5 = \vec{E}_{1n}$

$D_{1n} = D_{2n}$

$\varepsilon_0 \varepsilon_r \underset{3}{\vec{E}_{2n}} = \varepsilon_0 \underset{5}{\vec{E}_{1n}}$

$\vec{E}_{1t} = -\vec{a}_\phi 3$

$\vec{E}_{1n} = \vec{a}_r 5$



$\varepsilon_r = \frac{5}{3}$

$\vec{E}_{2t} = -\vec{a}_\phi 3.$

$\vec{E}_{2n} = \vec{a}_r (3).$

4 Capacitor

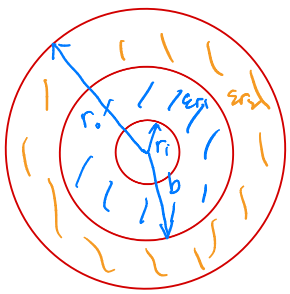
$$C = \frac{Q}{V_{12}}$$

How to find capacitance?

1. Choose an appropriate coordinate system for the given geometry.
2. Assume charges $+Q$ and $-Q$ on the conductors. $+Q$ $-Q$
3. Find $\mathbf{E}$ from Q by Coulomb's law, Gauss's law, or other relations. $\vec{E} = \frac{\vec{D}}{\varepsilon}$
4. Find V_{12} by evaluating
$$V_{12} = -\int_2^1 \mathbf{E} d\ell$$
from the conductor carrying $-Q$ to the other carrying $+Q$.
5. Find C by taking the ratio Q/V_{12}.

4.1 Exercise

- (HW3-5) The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.



ρ_ℓ => line charge density (free charge)

① $\vec{D} = \vec{a}_r \dfrac{\rho_\ell}{2\pi r}$

② $\vec{E} = \begin{cases} \vec{a}_r \cdot \dfrac{\rho_\ell}{2\pi\varepsilon_0\varepsilon_{r1} r}, & r_i < r < b \\ \vec{a}_r \cdot \dfrac{\rho_\ell}{2\pi\varepsilon_0\varepsilon_{r2} r}, & b < r < r_o \end{cases}$

③ $V = -\int_{r_o}^{r_i} \vec{E}\cdot d\vec{r}$

$= \dfrac{\rho_\ell}{2\pi\varepsilon_0}\left(\dfrac{1}{\varepsilon_{r1}}\ln\dfrac{b}{r_i} + \dfrac{1}{\varepsilon_{r2}}\ln\dfrac{r_o}{b}\right).$

④ $C = \dfrac{\rho_\ell}{V} = \dfrac{2\pi\varepsilon_0}{\frac{1}{\varepsilon_{r1}}\ln\frac{b}{r_i} + \frac{1}{\varepsilon_{r2}}\ln\frac{r_o}{b}}$

5 Energy in Electrostatic Fields

The potential energy of N discrete charges at rest:

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

For a continuous distribution of charge, the energy is

$$W_e = \frac{1}{2}\int \rho V dv$$

$\nabla \cdot \vec{D} = \rho$

$$W_e = \frac{1}{2}\int \underbrace{(\nabla \cdot \mathbf{D})}_{} V dv$$

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V$$

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V dv$$

$$= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E} dv$$

$$W_e = \frac{1}{2}\int_{allspace} \mathbf{D} \cdot \mathbf{E} dv$$

$$W_e = \frac{1}{2}\int_{allspace} \epsilon E^2 dv$$

6 Uniqueness Theorem of Electrostatic Solution

- Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

ρ: free charge

- If ρ is 0 everywhere, then we have Laplace's equation:

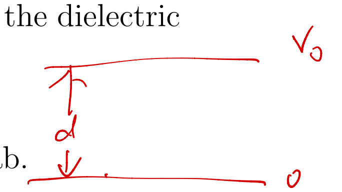
$$\nabla^2 V = 0$$

A solution of Poisson's equation that satisfies the given boundary condition is a unique solution. Steps to solve boundary condition problem:

- Write the expression of V, **D**, and **E** according to the configuration, like symmetry or properties of some configuration.
- Simplify Poisson's equation or Laplace's equation based on the written expression.
- Write out boundary conditions.
- Solve the mathematical problem.

6.1 Exercise

- (HW4-1) The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance d and maintained at potentials V_0 and 0, respectively. A dielectric slab of dielectric constant 6.0 and uniform thickness of $0.8d$ is placed over the lower plate. Assuming negligible fringing effect, determine
  a) the potential and electric field distribution in the dielectric slab,
  b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate,
  c) the surface charge densities on the upper and lower plates.
  d) Compare the results in part b) with those without the dielectric slab.

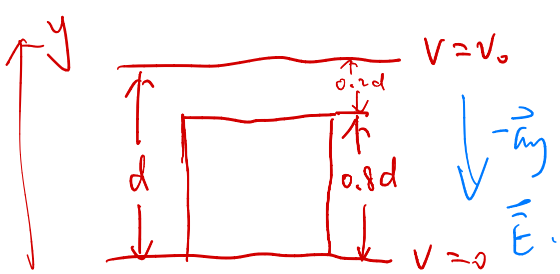


$\nabla^2 V = 0 \quad V = C_1 y + C_2$

$V(d) = V_0$

$V(0) = 0$

d - dielectric, a - air



$\nabla^2 V = 0$.

$V_d = C_1 y + C_2$, $\vec{E}_d = -\vec{a}_y C_1$, $\vec{D}_d = -\vec{a}_y \varepsilon_0 \varepsilon_r C_1$

$V_a = C_3 y + C_4$, $\vec{E}_a = -\vec{a}_y C_3$, $\vec{D}_a = -\vec{a}_y \varepsilon_0 C_3$.

b.c, $y = 0$, $V_d = 0$, $y = d$, $V_a = V_0$.

$y = 0.8d$ $V_a = V_d$, $\vec{D}_d = \vec{D}_a$.

$C_1 = \frac{V_0}{(0.8 + 0.2\varepsilon_r)d}$, $C_2 = 0$.

$C_3 = \frac{\varepsilon_r V_0}{(0.8 + 0.2\varepsilon_r)d}$, $C_4 = \frac{(1-\varepsilon_r)V_0}{1 + 0.25\varepsilon_r}$

(a) - -

(b) - -

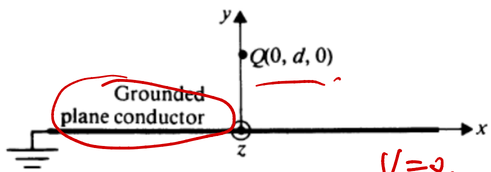
(c) $\vec{D}$

$\rho_s|_{y=d} = -(\vec{D}_d)|_{y=d} = \frac{5\varepsilon_0\varepsilon_r V_0}{(4+\varepsilon_r)d}$

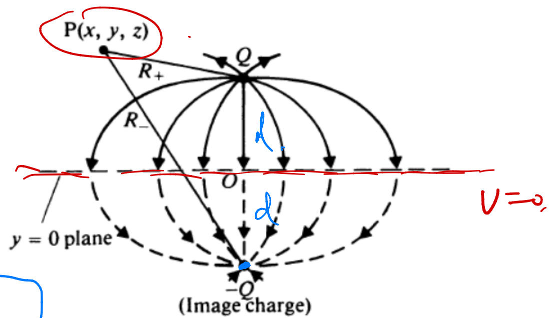
$\rho_s|_{y=0} = (\vec{D}_d)|_{y=0} = -\frac{5\varepsilon_0\varepsilon_r V_0}{(4+\varepsilon_r)d}$.

7 Method of Images

7.1 Point Charge and Conducting Planes



(a) Physical arrangement.



(b) Image charge and field lines.

$\nabla^2 V = 0$.

$V(x,0,z) = 0$.

$V \to \frac{Q}{4\pi\varepsilon_0 R}$, as $R \to 0$.

$x, y, z \to \infty$, $V = 0$.

$V(x, y, z) = V(-x, y, z)$

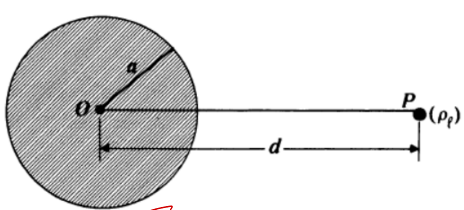
$V(x, y, z) = V(x, y, -z)$

$V(x, y, z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$.

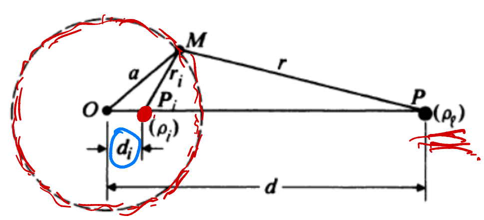
$R_+ = (x^2 + (y-d)^2 + z^2)^{\frac{1}{2}}$

$R_- = (x^2 + (y+d)^2 + z^2)^{\frac{1}{2}}$.

7.2 Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder.

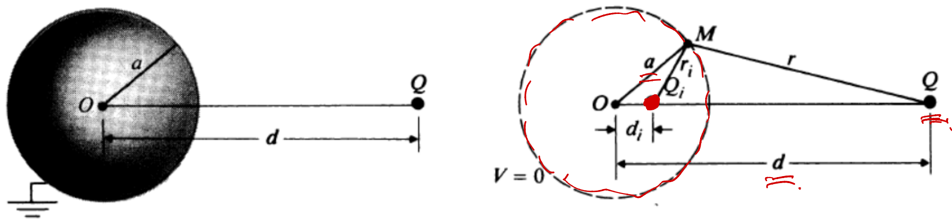


(b) Line charge and its image.

$\rho_i = -\rho_\ell$.

$d_i = \frac{a^2}{d}$

7.3 Point Charge and a Conducting Sphere



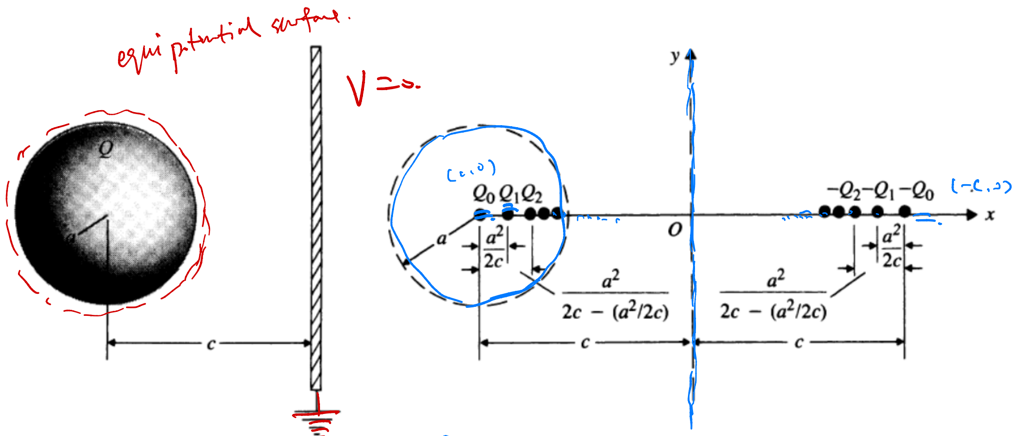
(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

$$Q_i = -\frac{a}{d} Q$$

$$d_i = \frac{a^2}{d}.$$

inverse point

7.4 Charge Sphere and Grounded Plane



(a) Physical arrangement. (b) Two groups of image point charges.

$$\alpha = \frac{a}{2c}$$

$$Q_1 = \frac{a}{2c} Q_0 = \alpha Q_0$$

$$Q_i = (\quad) Q_0.$$

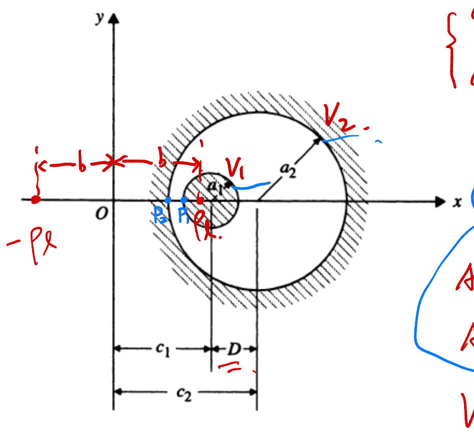
$$Q_2 = \frac{a}{2c - \frac{a^2}{2c}} Q_1 = \frac{\alpha^2}{1-\alpha^2} Q_0$$

$$Q_3 = \frac{\alpha^3}{(1-\alpha^2)(1-\frac{\alpha^3}{1-\alpha^3})} Q_0.$$

7.5 Example

- (HW4-4) A long wire of radius a_1 lies inside a conducting circular tunnel of radius a_2, as shown in Fig. . The distance between their axes is D.
  a) Find the capacitance per unit length.
  b) Determine the force per unit length on the wire if the wire and the tunnel carry equal and opposite line charges of magnitude ρ_ℓ.

a) put image charge ρ_ℓ and $-\rho_\ell$.



$$\begin{cases} c_1 - b = \dfrac{a_1^2}{c_1+b} \\ c_2 - b = \dfrac{a_2^2}{c_2+b} \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{1}{2D}(a_2^2 - a_1^2 - D^2) \\ c_2 = \dfrac{1}{2D}(a_2^2 - a_1^2 + D^2). \end{cases}$$

$$V = \frac{\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}$$

r_2 – distance to ρ_ℓ

r_1 – distance to $-\rho_\ell$.

At P_1: $r_2 = b + c_1 - a_1$, $r_1 = b - (c_1 - a_1)$

At P_2: $r_2 = b + c_2 - a_2$, $r_1 = b - (c_2 - a_2)$.

$V_1 - V_2 =$ — — — —

$$C = \frac{\rho_\ell}{V_1 - V_2}.$$

$Q \to \rho$

(b)

$$F = \frac{\rho_\ell^2}{2\pi\varepsilon_0 \cdot (2b)} = \text{— — — —}$$

$\downarrow$

$a_1, a_2, D.$