ECE 2300
Recitation Class 4

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Pre-class

- Quiz this week!
  - After Thursday lecture (8:00 pm – 8:40 pm)
  - Same format as last quiz. Online student need to turn on at least one camara.
  - If you want to take online quiz, notify us beforehand!

- Midterm 1 next week!
  - Location to be announced
  - Thursday June 15th 7-8:40 pm
  - Arrange your time well!

4.1.1 Recap - Conductors

- Definition:

An object or type of material allows the flow of charge in one or multiple directions

Equilibrium.

- Static state characteristics:
  - Inside:

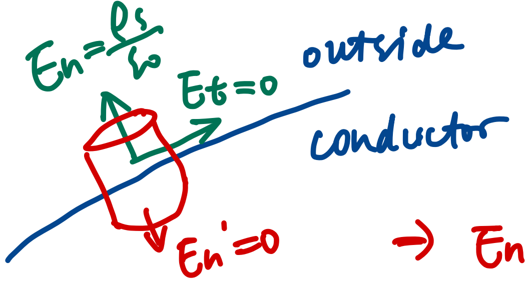
$\rho = 0$, $E = 0$

* equal potential within the conductor.

* system tend to have lowest potential energy.

  - Surface(Boundary):

$E_t = 0$ $E_n = \frac{\rho_s}{\varepsilon_0}$



$\rightarrow E_n \cdot S = \frac{\rho_s \cdot S}{\varepsilon_0} \Rightarrow E_n = \frac{\rho_s}{\varepsilon_0}.$

* if $E_t \neq 0$ ⇒ current & potential change

  - Outside:

Gauss's Law

4.1.2 Recap – Dielectrics

- Definition: slight change of + & - electric charge in opposite direction

⇑

Electrical insulator that can be polarized by external electric field.

⇓

a material in which current doesn't flow freely

- Polarization Vector:
  - Defined by dipole density:

$$P = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{n\Delta V} p_k}{\Delta V}$$

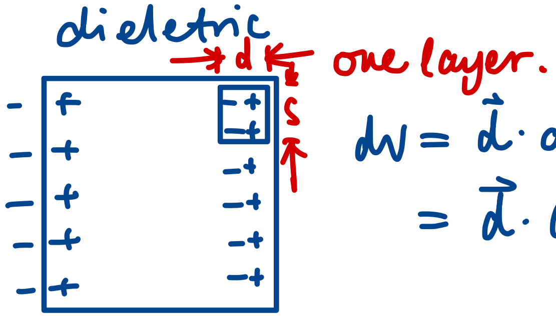
→ dipole moment

⇒ dipole moment density.

$$\vec{p} = q\vec{l}$$

4.1.2 Recap – Dielectrics

- Surface charge density: $\rho_{ps} = P \cdot a_n$



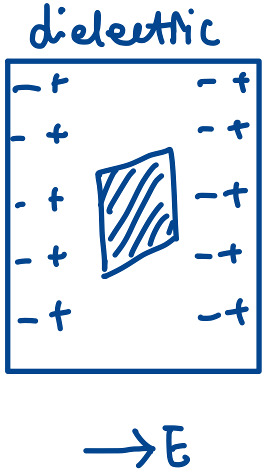
$$dV = \vec{d} \cdot ds$$
$$= \vec{d} \cdot \vec{a_n}\, ds$$

$$P = \frac{\Sigma p_k}{\Delta V} \Rightarrow \Sigma p_k = P \cdot \Delta V$$

$$\Sigma\, q \cdot d = P \cdot (d \cdot a_n) \cdot ds$$

$$\rho_{ps} = \frac{\Sigma q}{ds} = P \cdot a_n$$

- Volume charge density: $\rho_p = -\nabla \cdot P$



$$Q = -\oint \rho_s \cdot dS \quad \uparrow$$

reversed divergence the.

$$= \int_V (-\nabla \cdot P) \cdot dV \quad \downarrow$$

4.2.1 Electric Flux Density/Electric Displacement

- Definition: $\mathbb{D}$

it accounts for only the effects of free & bound charge within materials.

* True electric intensity. (only generated by free charge)

- Expression:
  - Relation with E and P:

$$\underset{\substack{\uparrow\\ \text{free charge}}}{\mathbb{D}} = \underset{\substack{\nwarrow\\ \text{total charge}}}{\varepsilon_0 E} + \mathbb{P} \leftarrow \text{induced charge}$$

$$\boxed{\rho_{total} = \rho_{free} + \rho_{induced}.}$$

4.2.1 Electric Flux Density/Electric Displacement

- Integration Form: (Gauss's Law for free charge)

$$\oint_s D\cdot ds = Q_{free} \quad (C)$$

$$\oint_s E\cdot ds = \frac{\rho_{enclosed.}}{\varepsilon_0} \Leftarrow \rho_{total} = \underline{\rho_{free}} + \rho_{induce.}$$

- Differential Form:

$$\nabla\cdot D = \rho_{free} \quad (C/m^3)$$

4.2.2 Electric Displacement in Isotropic Medium

- Relation between Polarization Vector and Field:

$$P = \varepsilon_0 \cdot \chi_\varepsilon E \quad \propto \quad E$$

$$D = \varepsilon_0 E + \varepsilon_0 \chi_\varepsilon E = \varepsilon_0 \underbrace{(1+\chi_e)} E = \overset{\varepsilon}{\overset{\Uparrow}{\varepsilon_0 \varepsilon_r}} E$$

$\varepsilon_0 \cdot \chi_\varepsilon$: polarizebility

χ_ε : dimensionless quantity called electric susceptibility.

ε_r : relative permittivity.

ε : absolute permittivity.

4.2.3 Electric Dis. for anisotropic Medium

- General anisotropic medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$D_x = \varepsilon_{11} E_x + \varepsilon_{12} E_y + \varepsilon_{13} E_z$

$D_y = \cdots$

$D_z = \cdots$

- Biaxial:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$D_x = \varepsilon_1 E_x$

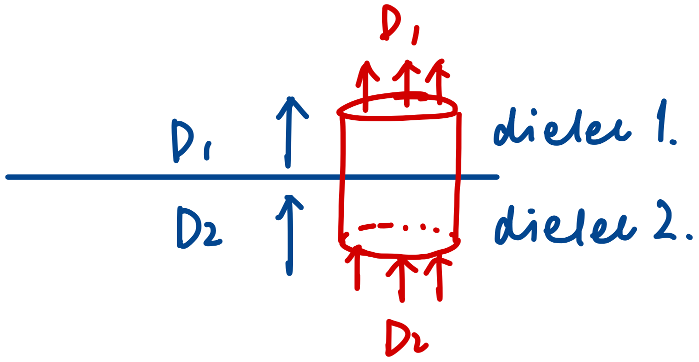
* uniaxial $\varepsilon_1 = \varepsilon_2$

* isotropic $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$

4.3 Boundary Conditions

- Normal:

$(D_1 - D_2) \cdot a_n = \rho_s$



* Gauss's Law

$D_1 \cdot S - D_2 \cdot S = \rho_s \cdot S \Rightarrow (D_1 - D_2) \cdot a_n = \rho_s$

↑

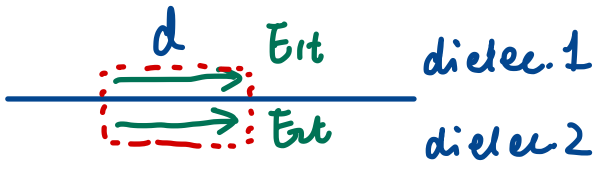
dot product

4.3 Boundary Conditions

- Tangential:

$E_{1t} = E_{2t}$

$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$, ε_1 & ε_2 are actual permittivity.



* For the red loop integral:

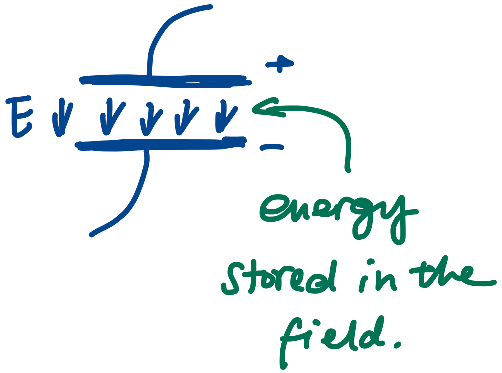
$E_{1t}\cdot d - E_{2t}\cdot d = 0$ * conservative

$E_{1t} - E_{2t} = 0$

4.4.1 Capacitors

- Definition:

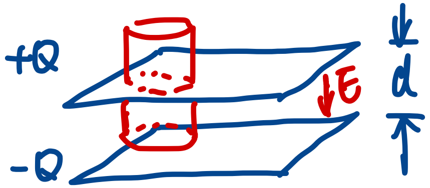
Device that store electric energy in an electric field by virtue of cumulating electric charges on two close surface insulated from each other.



- Equation for description:
  - General Form:

$$C = \frac{Q}{U} \quad \begin{matrix} \rightarrow \text{charge stored.} \\ \rightarrow \text{voltage diff.} \end{matrix}$$

  - *Related to only surface area & distance:



$$E \cdot S = \frac{\rho_s \cdot S}{\varepsilon_0} \Rightarrow E = \frac{\rho_s}{\varepsilon_0}$$

$$\nearrow Q = \rho_s \cdot S$$

$$V = E \cdot d \Rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{\rho_s}{\varepsilon_0} d} = \frac{\varepsilon_0 S}{d}$$

4.4.2 Find Capacitance

- Step1: Assume charges Q & $-Q$ on conductors.
- Step2: Find E with Q using Coulomb's Law. Gauss's Law ...
- Step3: Find V by evaluating $V_{12} = -\int_2^1 E\,dl$.

  From plate carrying $-Q$ to plate carrying $+Q$
- Step4: Find C using Q/V

4.4.3 Connected Capacitors

- Series Connection:

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

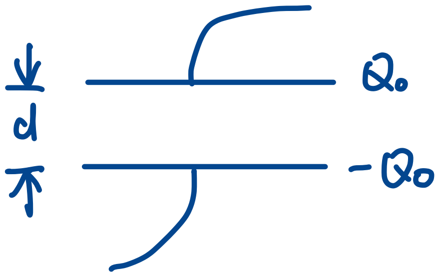
- Parallel Connection:

$$C_{\parallel} = C_1 + C_2 + \cdots + C_n$$

Ex.1 Electric Displacement

Suppose we have a capacitor with capacitance of C, what is the capacitance C′ after we inserted a dielectric with relative permittivity of εr?

① Before insertion:



generated by free charge only.

$$E_{int} \cdot S = \frac{\rho \cdot S}{\varepsilon_0}$$

$$E_{int} = \frac{\rho}{\varepsilon_0} = \frac{Q_0}{\varepsilon_0 \cdot S}$$

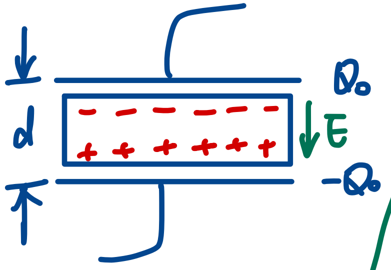
$$V_{int} = \frac{\rho \cdot d}{\varepsilon_0} = \frac{Q_0 d}{\varepsilon_0 \cdot S}$$

$$C = \frac{Q}{V_{int}} = \frac{Q_0}{\frac{Q_0 \cdot d}{\varepsilon_0 \cdot S}} = \frac{\varepsilon_0 S}{d}$$

For vacuum: $\varepsilon_r = 1$

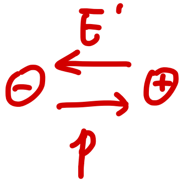
$\Rightarrow$ $D = \varepsilon_0 \cdot \varepsilon_r \cdot E = \varepsilon_0 E$

② After insertion.



$$E_{now} = E_{int} + E'$$

by induced charge.



$$E' = \frac{\rho_{induce}}{\varepsilon_0} = -\frac{P}{\varepsilon_0}$$

$\rho_{induce} \Rightarrow P \cdot a_n$

$$E_{now} = \frac{Q_0}{\varepsilon_0 \varepsilon_r S}$$

$$V_{now} = \frac{Q_0 d}{\varepsilon_0 \cdot S}$$

$$C = \frac{\varepsilon_0 \cdot \varepsilon_r \cdot S}{d}$$

multiply by ε_0

$$\varepsilon_0 E_{now} = \varepsilon_0 E_{int} + \varepsilon_0 \cdot E'$$

$$\varepsilon_0 \cdot E = D - P \Rightarrow \varepsilon_0 E + P = D$$

Ex.1 Electric Displacement

Suppose we have a capacitor with capacitance of C, what is the capacitance C’ after we inserted a dielectric with relative permittivity of εr? $\varepsilon_0\varepsilon_r$

$\Uparrow$

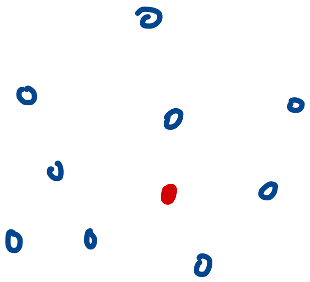
① we should replace ε_0 with ε in equations where ε_0 is needed.

② verify the relationship of $D=\varepsilon_0 E+P$

4.5 Energy of Electric Field

- For discrete charges:

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$



V_k : generated by others.

Q_k: red one.

*why ½? Energy between a pair of electrons are being double counted.

- For continuous charges:

$$W_e = \frac{1}{2}\int \rho v \cdot dv$$

4.5 Energy of Electric Field (Some Eq.s)

- $W_e = \frac{1}{2} \int_{allspace} D \cdot E \, dv$

$\downarrow$

$D = \varepsilon E$

- $W_e = \frac{1}{2} \int_{allspace} \varepsilon E^2 \, dv$

4.6 Method of Image Charge

- Key Point:

replacement of certain elements in the original layout with imaginary charges, which replicates the boundary condition of the problem.

4.6 Method of Image Charge

- Why legal?
  - Uniqueness Theorem!

* Poisson Equation: $\nabla^2 V = -\frac{\rho}{\varepsilon}$

* gradient $V \to E$

divergence $E \to$ Gauss's Law.

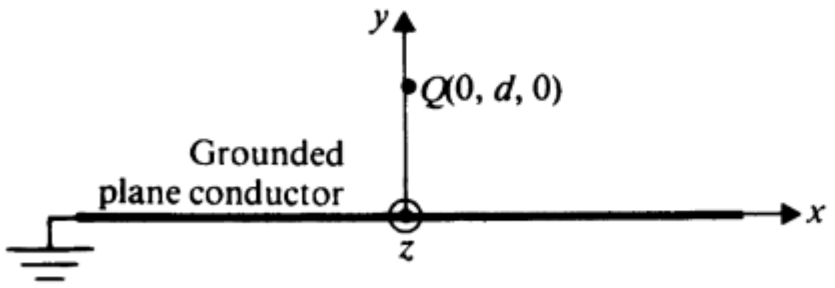
* if there is no free charge:

Laplace Equation: $\nabla^2 V = 0$

$\Rightarrow$ A solution of either Poisson's or Laplace's equation that satisfies the given boundary is unique solution.

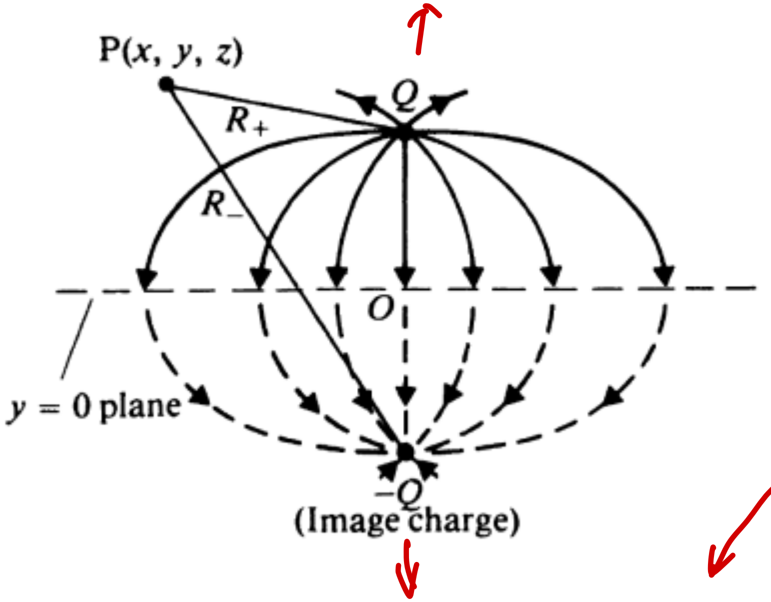
4.6 Method of Image Charge

- Examples:
  - Point charge and a conducting plane



(a) Physical arrangement.

original charge



induce charges on the plane

equal on the effect.

image charge

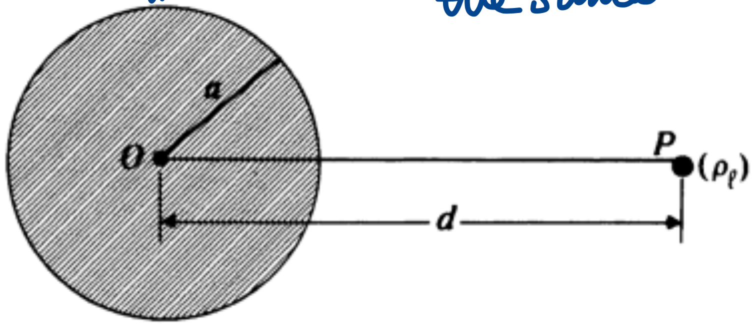
4.6 Method of Image Charge

- Examples:
  - Line charge and a parallel conducting cylinder

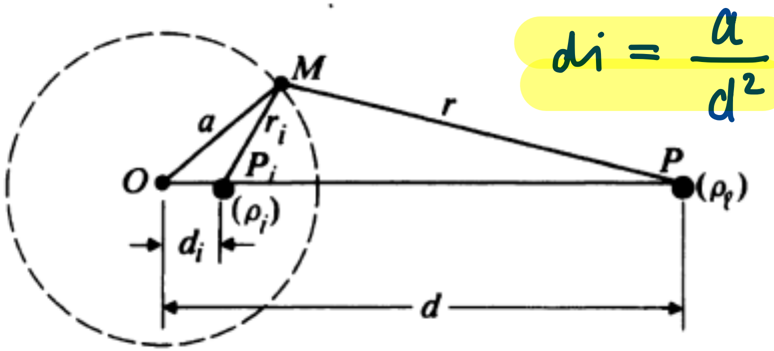
V of conducting cylinder should be the same
⇑

$\rho_i = -\rho_\ell$

$d_i = \frac{a}{d^2}$



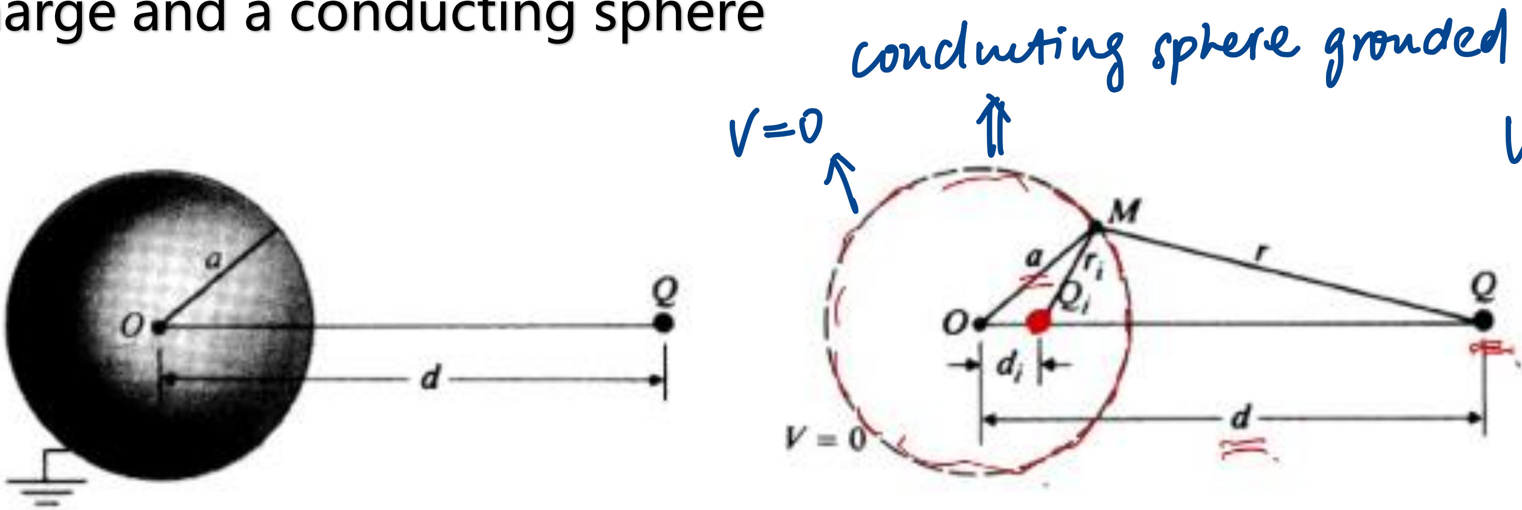
(a) Line charge and parallel conducting cylinder.



(b) Line charge and its image.

4.6 Method of Image Charge

- Examples:
  - Point charge and a conducting sphere



(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

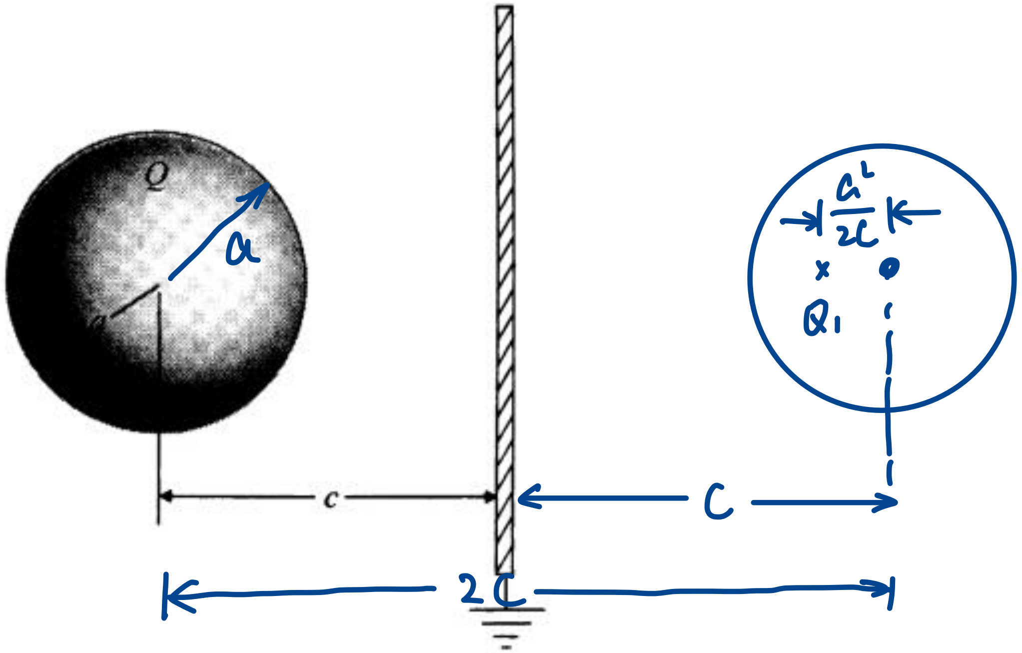
$V_{Qi} + V_Q = 0$

$Q_i = -\frac{a}{d} Q$

$* \; Q_i < Q$

$d_i = \frac{a^2}{d}$

Ex.2 Method of Image Charge



(a) Physical arrangement.

$$Q_1 = \frac{a}{2c} Q_0$$

$$Q_2 = \frac{a}{2c - \frac{a^2}{2c}} = \frac{a\cdot 2c}{4c^2 - a^2} = \frac{(\frac{a}{2c})^2}{1-(\frac{a}{2c})^2} Q_0$$

$$Q_3 = \frac{a}{2c - \frac{a^2}{2c} - \frac{a^2}{2c - \frac{a^2}{2c}}} = \frac{(\frac{a}{2c})^3}{(1-(\frac{a}{2c})^2)(1 - \frac{(\frac{a}{2c})^3}{1-(\frac{a}{2c})^3})} Q_0$$

$\vdots$

Thank You

Credit to Deng Naihao for this slides & information

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