

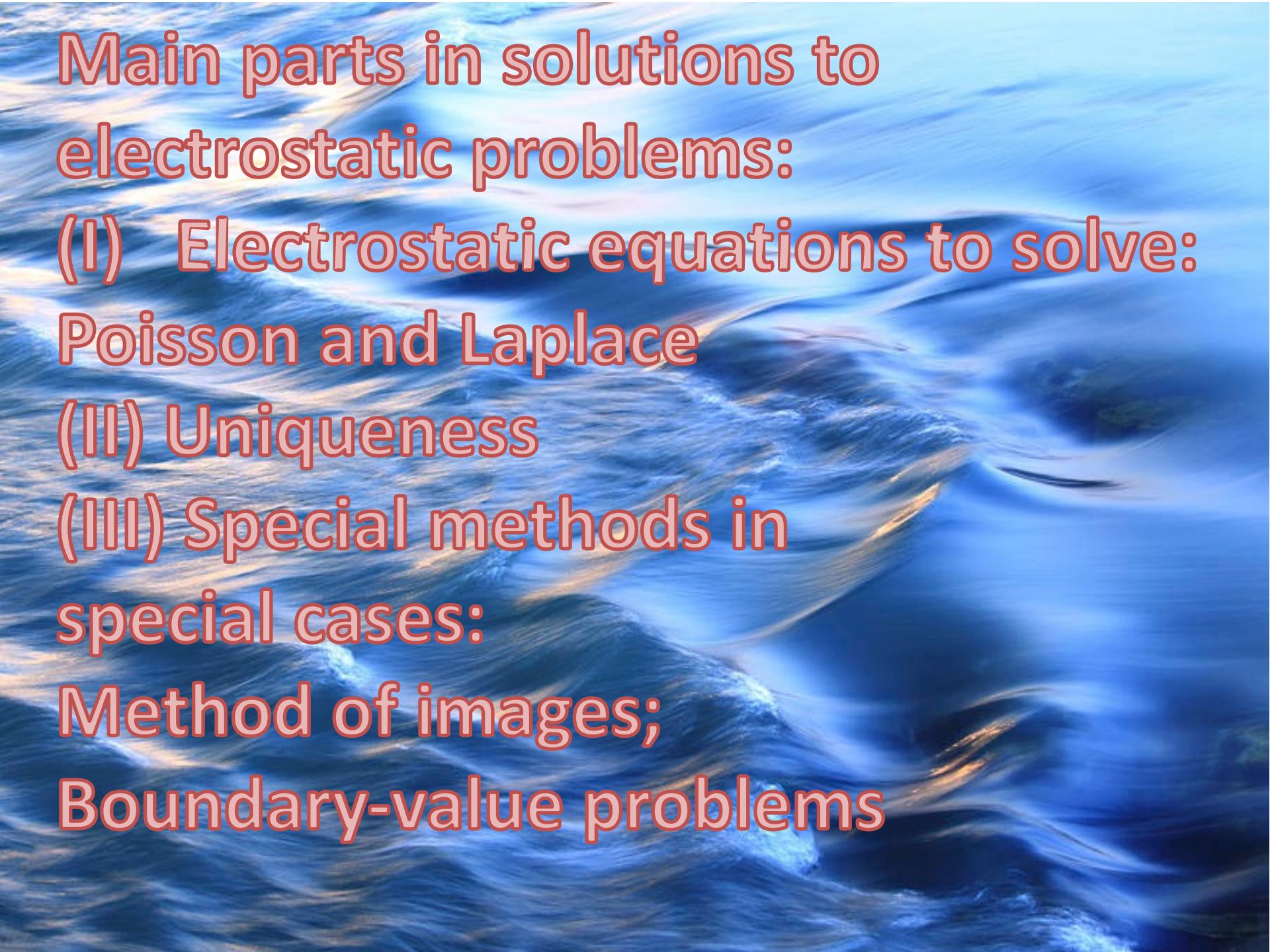
# Chapter 4: Electrostatic Problems

Lecturer: Nana Liu  
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JOINT INSTITUTE  
交大密西根学院

**Everything you need to know  
about solutions to  
electrostatic problems  
for this course...**



Main parts in solutions to electrostatic problems:

- (I) Electrostatic equations to solve:  
Poisson and Laplace
- (II) Uniqueness
- (III) Special methods in special cases:  
Method of images;  
Boundary-value problems

# What to solve for?

Charge density  $\rho$

$E$  field

Electric potential  $E = -\nabla V$

Force  $F = qE$

Work  $W = - \int F \cdot dl$

# Everything in electrostatics is here...

$$\nabla \cdot D = \rho$$

$$\nabla \times E = 0$$



# Poisson and Laplace for linear materials

Poisson:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Laplace:

$$\nabla^2 V = 0$$



# Poisson and Laplace for linear materials

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$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Laplace:

$$\nabla^2 V = 0$$

Solutions to  
these  
are unique!

# What to solve for?

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# Problem...

When charge density  $\rho$   
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When charge density  $\rho$   
is imperfectly known...

Method of images

Intelligent guess

Boundary-value problems

## 4-1 Introduction

- Electrostatic problems:  $\mathbf{E}$ ,  $V$ ,  $\rho$
- $\rho(\mathbf{r})$  known exactly everywhere  $\rightarrow \mathbf{E}(\mathbf{r}), V(\mathbf{r})$
- In practical problems,  $\rho(\mathbf{r})$  is not known everywhere (e.g., only partial  $\rho(\mathbf{r})$  known). We need techniques:
  - Method of images
  - Boundary-value problems

## 4-2 Poisson's and Laplace's Equations

Maxwell's 1<sup>st</sup> and 2<sup>nd</sup> equations

$$\nabla \cdot \mathbf{D} = \rho.$$
$$\nabla \times \mathbf{E} = 0.$$

$$\nabla \times \mathbf{E} = 0. \quad \rightarrow \quad \mathbf{E} = -\nabla V.$$

$$\nabla \cdot \mathbf{D} = \rho.$$



In a linear medium  
 $\mathbf{D} = \epsilon \mathbf{E}$ ,

$$\nabla \cdot \epsilon \mathbf{E} = \rho.$$



$$\mathbf{E} = -\nabla V.$$

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

# Poisson's Equations

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$



In a homogeneous medium  
 $\epsilon$  is a constant over space

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}.$$

Laplacian operator:  $\nabla^2 = \nabla \bullet \nabla$

Poisson's equation:

$\rho$  may be a function of space coordinates  
 $\epsilon$  must be a constant over space

# Poisson's Equation in Cartesian Coordinate

$$\nabla^2 V = \nabla \cdot \nabla V = \left( \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \right) \cdot \left( \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z} \right);$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}} \quad \rightarrow \quad \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \quad (\text{V/m}^2).}$$

# $\nabla^2$ in Cylindrical and Spherical Coordinates

- Cylindrical:

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

- Spherical:

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}.$$

# Laplace's Equation

- In a simple medium where there is no free charge,  $\rho=0$

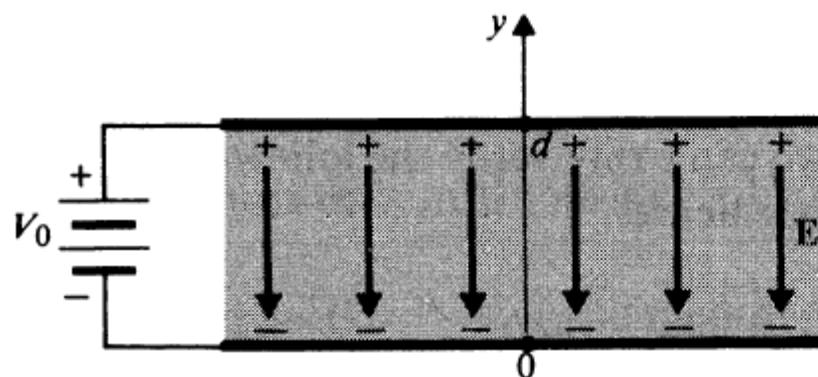
Laplace's Equation

$$\nabla^2 V = 0,$$

- Example to use Laplace's equation: a set of conductors at different potentials

Solve  $V$  by Laplace's equation  $\rightarrow \mathbf{E} = -\nabla V \rightarrow \rho_s = \epsilon E_n$   
(see example 4-1)

**EXAMPLE 4–1** The two plates of a parallel-plate capacitor are separated by a distance  $d$  and maintained at potentials 0 and  $V_0$ , as shown in Fig. 4–1. Assuming negligible fringing effect at the edges, determine (a) the potential at any point between the plates, and (b) the surface charge densities on the plates.



**FIGURE 4–1**  
A parallel-plate capacitor (Example 4–1).

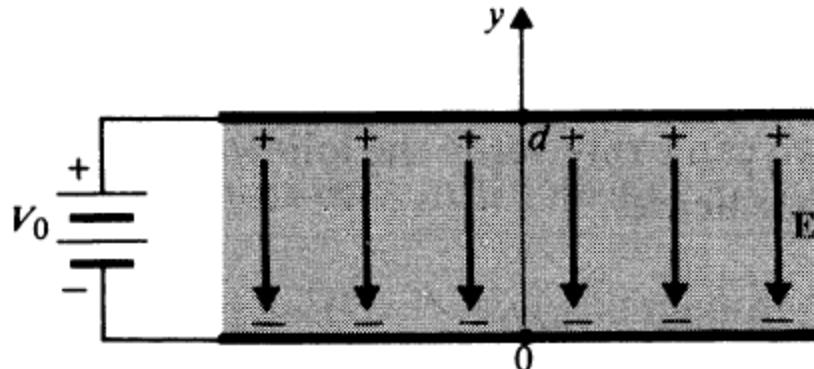
a) Laplace's equation is the governing equation for the potential between the plates, since  $\rho = 0$  there. Ignoring the fringing effect of the electric field is tantamount to assuming that the field distribution between the plates is the same as though the plates were infinitely large and that there is no variation of  $V$  in the  $x$  and  $z$  directions. Equation (4-7) then simplifies to

$$\frac{d^2V}{dy^2} = 0, \quad (4-11)$$

where  $d^2/dy^2$  is used instead of  $\partial^2/\partial y^2$ , since  $y$  is the only space variable here.

Integration of Eq. (4-11) with respect to  $y$  gives

$$\frac{dV}{dy} = C_1,$$



**FIGURE 4-1**  
A parallel-plate capacitor (Example 4-1).

where the constant of integration  $C_1$  is yet to be determined. Integrating again, we obtain

$$V = C_1 y + C_2. \quad (4-12)$$

Two boundary conditions are required for the determination of the two constants of integration,  $C_1$  and  $C_2$ :

$$\text{At } y = 0, \quad V = 0. \quad (4-13a)$$

$$\text{At } y = d, \quad V = V_0. \quad (4-13b)$$

Substitution of Eqs. (4-13a) and (4-13b) in Eq. (4-12) yields immediately  $C_1 = V_0/d$  and  $C_2 = 0$ . Hence the potential at any point  $y$  between the plates is, from Eq. (4-12),

$$V = \frac{V_0}{d} y. \quad (4-14)$$

The potential increases linearly from  $y = 0$  to  $y = d$ .

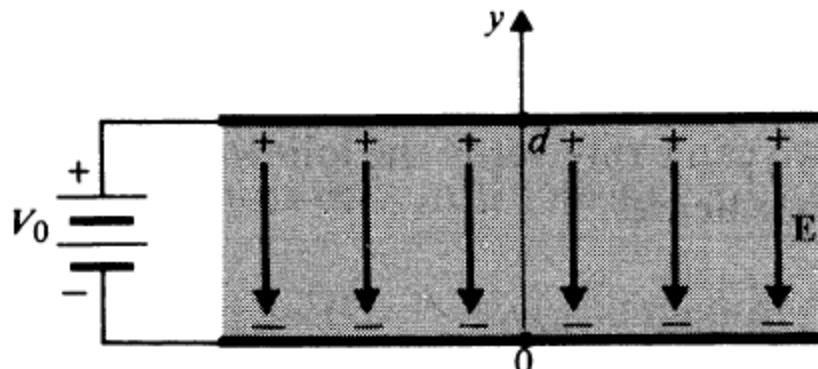


FIGURE 4-1  
A parallel-plate capacitor (Example 4-1).

- b) In order to find the surface charge densities, we must first find  $\mathbf{E}$  at the conducting plates at  $y = 0$  and  $y = d$ . From Eqs. (4–3) and (4–14) we have

$$\mathbf{E} = -\mathbf{a}_y \frac{dV}{dy} = -\mathbf{a}_y \frac{V_0}{d}, \quad (4-15)$$

which is a constant and is independent of  $y$ . Note that the direction of  $\mathbf{E}$  is opposite to the direction of increasing  $V$ . The surface charge densities at the conducting plates are obtained by using Eq. (3–72),

$$E_n = \mathbf{a}_n \cdot \mathbf{E} = \frac{\rho_s}{\epsilon}.$$

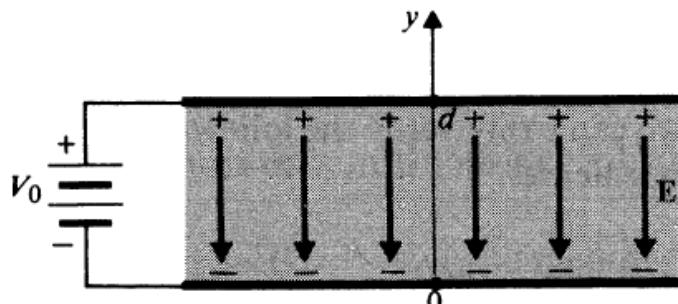
At the lower plate,

$$\mathbf{a}_n = \mathbf{a}_y, \quad E_{n\ell} = -\frac{V_0}{d}, \quad \rho_{s\ell} = -\frac{\epsilon V_0}{d}.$$

At the upper plate,

$$\mathbf{a}_n = -\mathbf{a}_y, \quad E_{nu} = \frac{V_0}{d}, \quad \rho_{su} = \frac{\epsilon V_0}{d}.$$

Electric field lines in an electrostatic field begin from positive charges and end in negative charges.

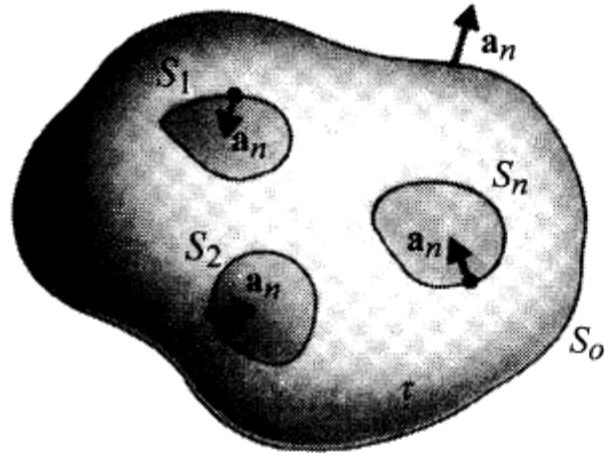


**FIGURE 4–1**  
A parallel-plate capacitor (Example 4–1).

## 4-3 Uniqueness of Electrostatic Solutions

- Uniqueness theorem: a solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

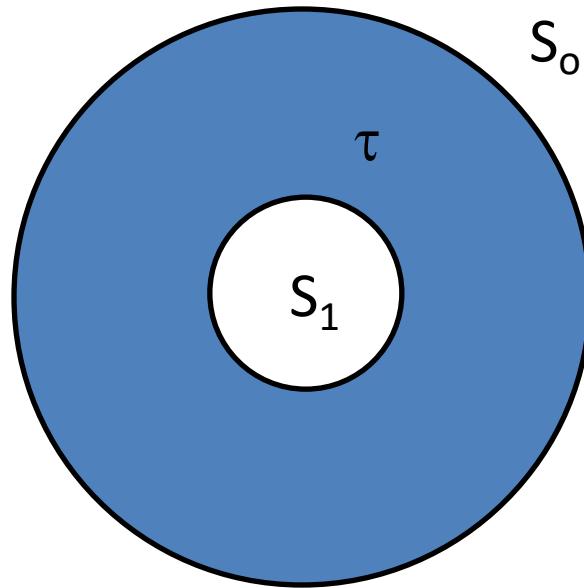
# Proof of Uniqueness Theorem



$S_o$ : outer surface enclosing the volume  $\tau$   
 $S_1, S_2, \dots, S_n$ : surfaces of **conducting bodies**

**FIGURE 4-2**  
Surface  $S_o$  enclosing volume  $\tau$  with conducting bodies.

- Volume  $\tau$  is bounded (enclosed) by a surface  $S_o$  and surfaces  $S_1, S_2, \dots, S_n$
- Inside  $S_o$ , many charged **conducting bodies** with surfaces  $S_1, S_2, \dots, S_n$  **at specified potentials**.



$S_1$ : Surface of a conducting body  
 $\tau$ : Volume bounded by  $S_0$  and  $S_1$

# Proof of Uniqueness Theorem

- Uniqueness theorem: there is only one solution of potential  $V$  in  $\tau$
- To prove uniqueness theorem, we assume two solutions  $V_1$  and  $V_2$  in  $\tau$ :

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon},$$

$$\nabla^2 V_2 = -\frac{\rho}{\epsilon}.$$

Also assume that  $V_1$  and  $V_2$  satisfy the same boundary conditions on  $S_1, S_2, \dots S_n$  and  $S_o$

# Proof of Uniqueness Theorem

- (i) Define a potential difference  $V_d$ :  $V_d = V_1 - V_2$ .

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon}, \quad - \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon}. \quad \rightarrow \quad \nabla^2 V_d = 0. \quad \text{in } \tau$$

- (ii) On conducting boundaries, **the potentials are specified**  $\rightarrow V_d=0$

Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions (= potentials are specified on surfaces of conducting bodies)** is a unique solution.

- (i) In  $\tau$ ,  $\nabla^2 V_d=0$
- (ii) On conducting boundaries,  $V_d=0$

# Proof of Uniqueness Theorem

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f;$$



letting  $f = V_d$  and  $\mathbf{A} = \nabla V_d$ ;

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2,$$

Integration over  $\tau$



$$\nabla^2 V_d = 0. \quad \text{in } \tau$$

$$\oint_{\underline{S}} (V_d \nabla V_d) \cdot \mathbf{a}_n ds = \int_{\tau} |\nabla V_d|^2 dv,$$

$S: S_1, S_2, \dots, S_n$ , and  $S_0$

# Proof of Uniqueness Theorem

$$\oint_S \underline{(V_d \nabla V_d) \cdot a_n} ds = \int_{\tau} |\nabla V_d|^2 dv,$$

1. For  $S_1, S_2, \dots S_n, V_d=0$
2. For  $S_o$

Consider the surface of a sphere with radius  $R \rightarrow \infty$

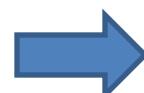
$$V_d \sim 1/R$$

$$\nabla V_d \sim 1/R^2$$

$$s \sim R^2$$

Thus, the integrand  $\sim 1/R$

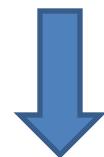
As  $R \rightarrow \infty$ , Left side  $\rightarrow 0$



$$\int_{\tau} |\nabla V_d|^2 dv = 0.$$

# Proof of Uniqueness Theorem

$$\int_{\tau} |\nabla V_d|^2 dv = 0.$$



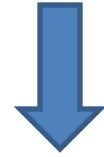
$|\nabla V_d|^2$  is nonnegative everywhere,

$$|\nabla V_d| = 0. \quad \text{everywhere in } \tau$$



$V_d$  is a constant everywhere in  $\tau$

(Thus, " $V_d$  in  $\tau$ " = " $V_d$  on surfaces")



We know  $V_d=0$  on surfaces  $S_1, S_2, \dots, S_n$

$V_d = 0$  everywhere in  $\tau$

That is,  $V_1 = V_2$  everywhere in  $\tau$ , and there is only one possible solution!

# Two Cases for the Uniqueness Theorem

- Known **potentials** of conducting bodies, which is just proved.
- Known **charge distributions** of conducting bodies:

$$(\rho_s = \epsilon E_n = -\epsilon \partial V / \partial n),$$



$\rho_s$  is known on conducting bodies

$$\nabla V_d = 0. \text{ on conducting body surfaces}$$

Substitute  $\nabla V_d = 0$ . on  $S_1, S_2, \dots, S_n$  into  $\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n ds = \int_{\tau} |\nabla V_d|^2 dv,$



$$\begin{aligned} & \oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n ds \\ &= 0 \text{ over } S_1, S_2, \dots, S_n. \end{aligned}$$

LHS  $\rightarrow 0$

Also,  $\oint_{S_0} (V_d \nabla V_d) \cdot \mathbf{a}_n ds = 0$  as  $R \rightarrow \infty$



$$\int_{\tau} |\nabla V_d|^2 dv = 0.$$

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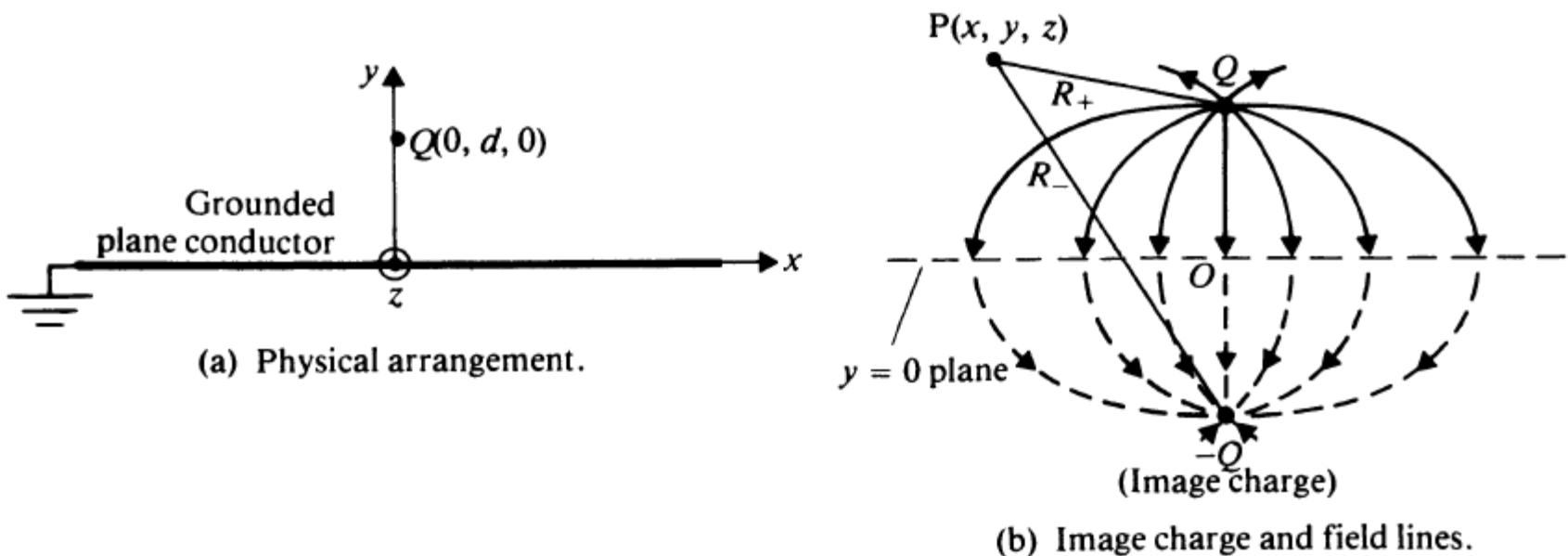


The same conclusion can be obtained!

## 4-4 Methods of Images

- Methods of images: replacing boundaries by appropriate **image charges** in lieu of a formal solution of Poisson's or Laplace's equation
  - Condition on boundaries unchanged
  - $V(R)$  can be determined easily

# Case: Point Charge and Grounded Plane Conductor



**FIGURE 4–3**  
Point charge and grounded plane conductor.

# Case: Point Charge and Grounded Plane Conductor

- Solved by Laplace eq.:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

Hold for  $y > 0$  except at the point charge

- 4 Conditions** should be satisfied:

- $V(x, 0, z) = 0$ .

- for points very close to Q

$$V \rightarrow \frac{Q}{4\pi\epsilon_0 R}, \text{ as } R \rightarrow 0,$$

- $V \rightarrow 0$  for points very far from Q

$$x \rightarrow \pm\infty, y \rightarrow +\infty, \text{ or } z \rightarrow \pm\infty$$

- Even functions w.r.t. x and z coordinates

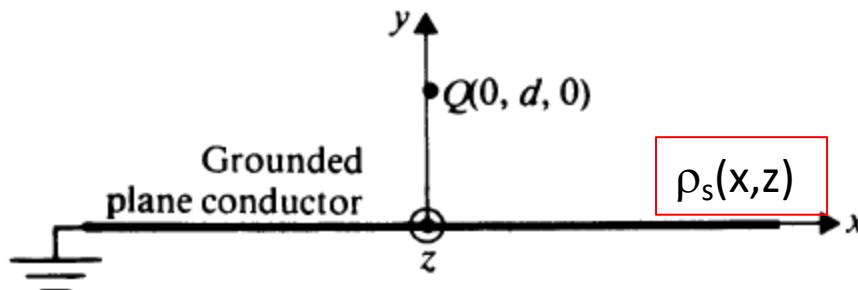
$$V(x, y, z) = V(-x, y, z) \quad V(x, y, z) = V(x, y, -z).$$

Difficult to solve...

# Case: Point Charge and Grounded Plane Conductor

- $+Q \rightarrow$  induce  $\rho_s$  on conducting plane

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + (y - d)^2 + z^2}} + \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s}{R_1} ds,$$

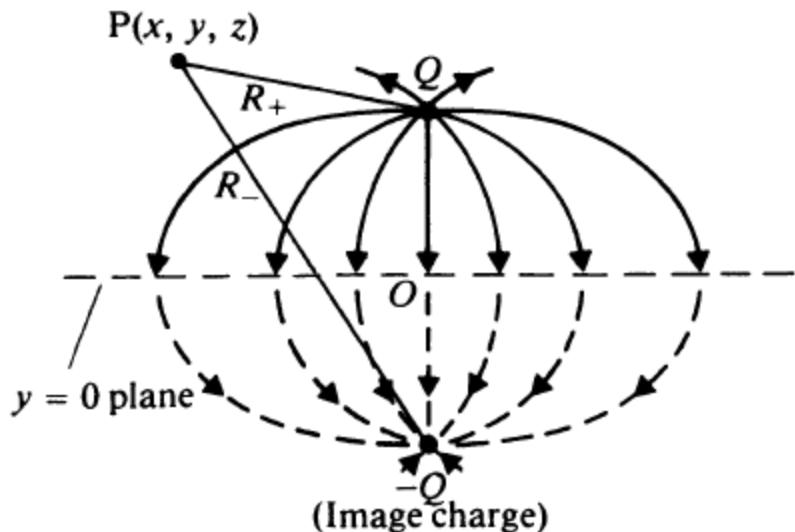


(a) Physical arrangement.

- $\rho_s(x, z)$  not easy to determine
- 2<sup>nd</sup> term is difficult to evaluate

## 4-4.1 Point Charge and Conducting Planes

- Image methods:
  - Remove the conductor
  - Replace with an image point charge  $-Q$  at  $y=-d$



(b) Image charge and field lines.

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{R_+} - \frac{1}{R_-} \right),$$

$$R_+ = [x^2 + (y - d)^2 + z^2]^{1/2},$$
$$R_- = [x^2 + (y + d)^2 + z^2]^{1/2}.$$

**Solution**     $V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{R_+} - \frac{1}{R_-} \right),$

- By direct substitution, we can verify:
  - Laplace's Eq. is satisfied
  - **4 conditions** are satisfied
- In view of uniqueness theorem, the solution is the only solution.

Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions (  $V(y=0)=0$  in this case )** is a unique solution.

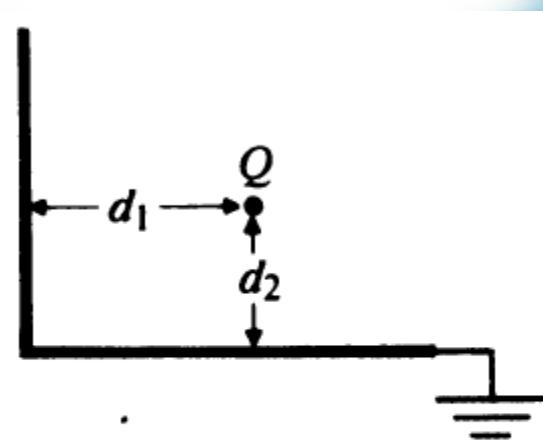
- $\mathbf{E} = -\nabla V$

Extremely simple!

# Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

**EXAMPLE 4-3** A positive point charge  $Q$  is located at distances  $d_1$  and  $d_2$ , respectively, from two grounded perpendicular conducting half-planes, as shown in Fig. 4-4(a). Determine the force on  $Q$  caused by the charges induced on the planes.

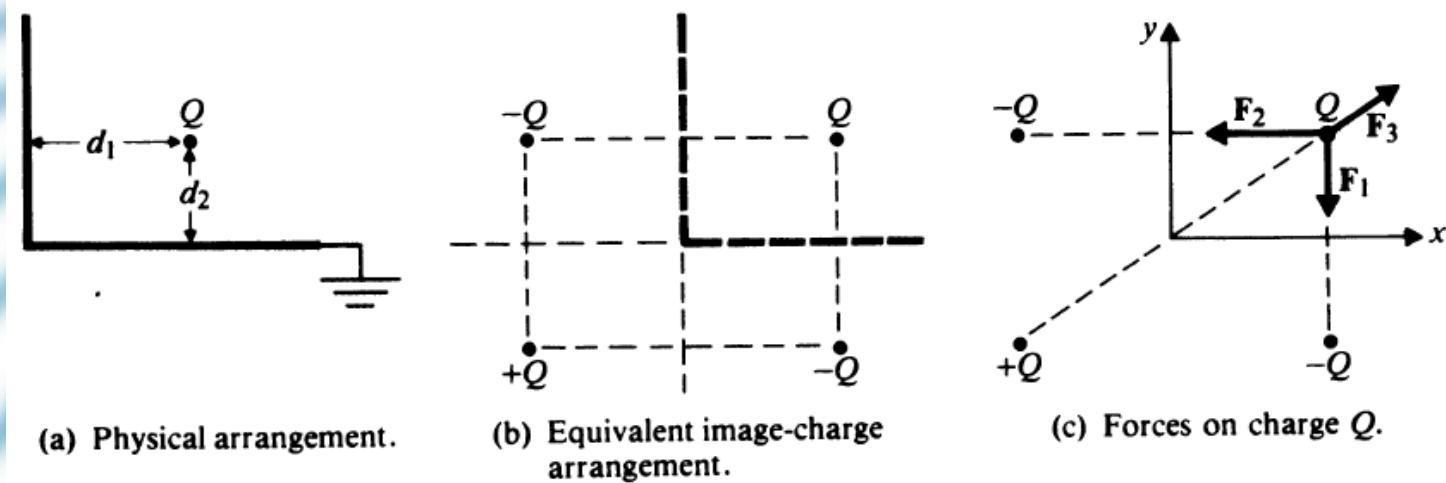


(a) Physical arrangement.

# Quick classroom exercise

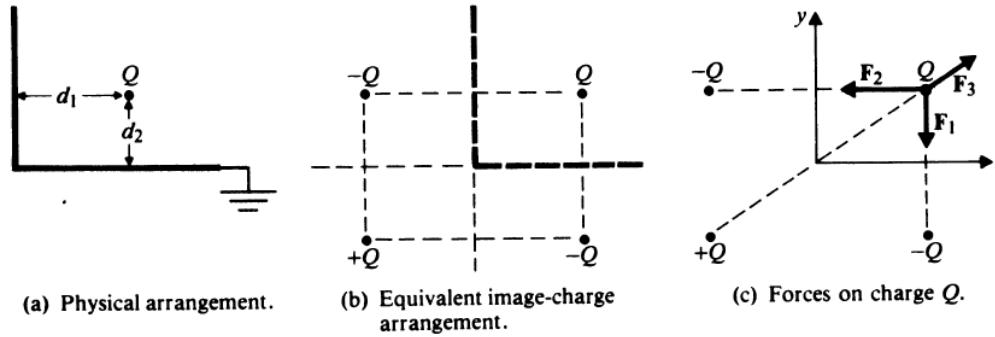
**Solution** A formal solution of Poisson's equation, subject to the zero-potential boundary condition at the conducting half-planes, would be quite difficult. Now an image charge  $-Q$  in the fourth quadrant would make the potential of the horizontal half-plane (but not that of the vertical half-plane) zero. Similarly, an image charge  $-Q$  in the second quadrant would make the potential of the vertical half-plane (but not that of the horizontal plane) zero. But if a third image charge  $+Q$  is added in the third quadrant, we see from symmetry that the image-charge arrangement in Fig. 4–4(b) satisfies the zero-potential boundary condition on both half-planes and is electrically equivalent to the physical arrangement in Fig. 4–4(a).

Negative surface charges will be induced on the half-planes, but their effect on  $Q$  can be determined from that of the three image charges. Referring to Fig. 4–4(c),



**FIGURE 4–4**  
Point charge and perpendicular conducting planes.

# Quick classroom exercise



**FIGURE 4-4**  
Point charge and perpendicular conducting planes.

we have, for the net force on  $Q$ ,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3,$$

where

$$\mathbf{F}_1 = -\mathbf{a}_y \frac{Q^2}{4\pi\epsilon_0(2d_2)^2},$$

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{Q^2}{4\pi\epsilon_0(2d_1)^2},$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\epsilon_0[(2d_1)^2 + (2d_2)^2]^{3/2}} (\mathbf{a}_x 2d_1 + \mathbf{a}_y 2d_2).$$

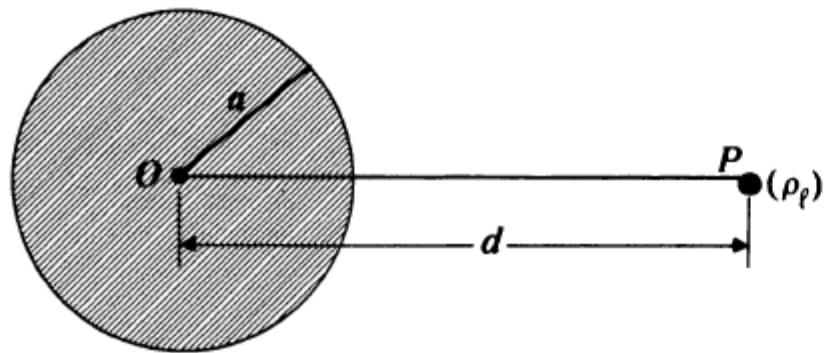
Therefore,

$$\mathbf{F} = \frac{Q^2}{16\pi\epsilon_0} \left\{ \mathbf{a}_x \left[ \frac{d_1}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_1^2} \right] + \mathbf{a}_y \left[ \frac{d_2}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_2^2} \right] \right\}.$$

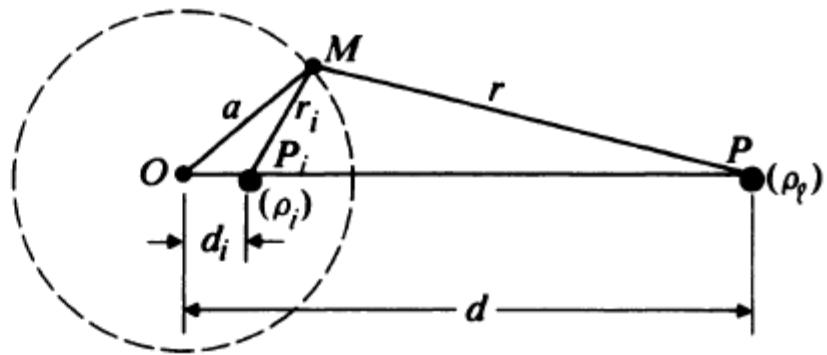
# Notes of Image Methods

- The solution **cannot** be used to calculate  $V$  or  $E$  **in the  $y<0$  region.**
- For  $y<0$  region,  $E=0$ ,  $V=0$ .

## 4-4.2 Line Charge and Parallel Conducting Cylinder



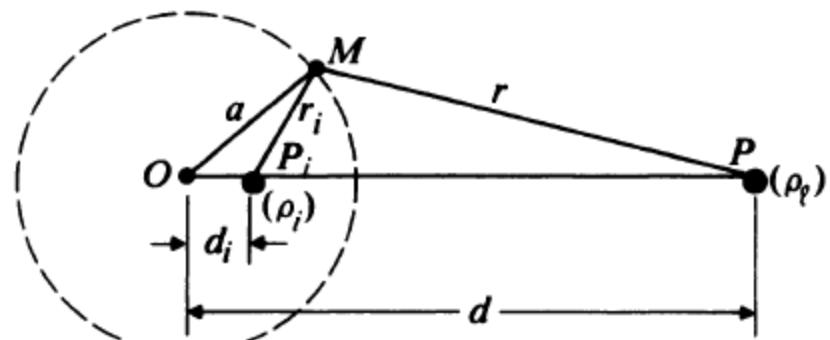
(a) Line charge and parallel conducting cylinder.



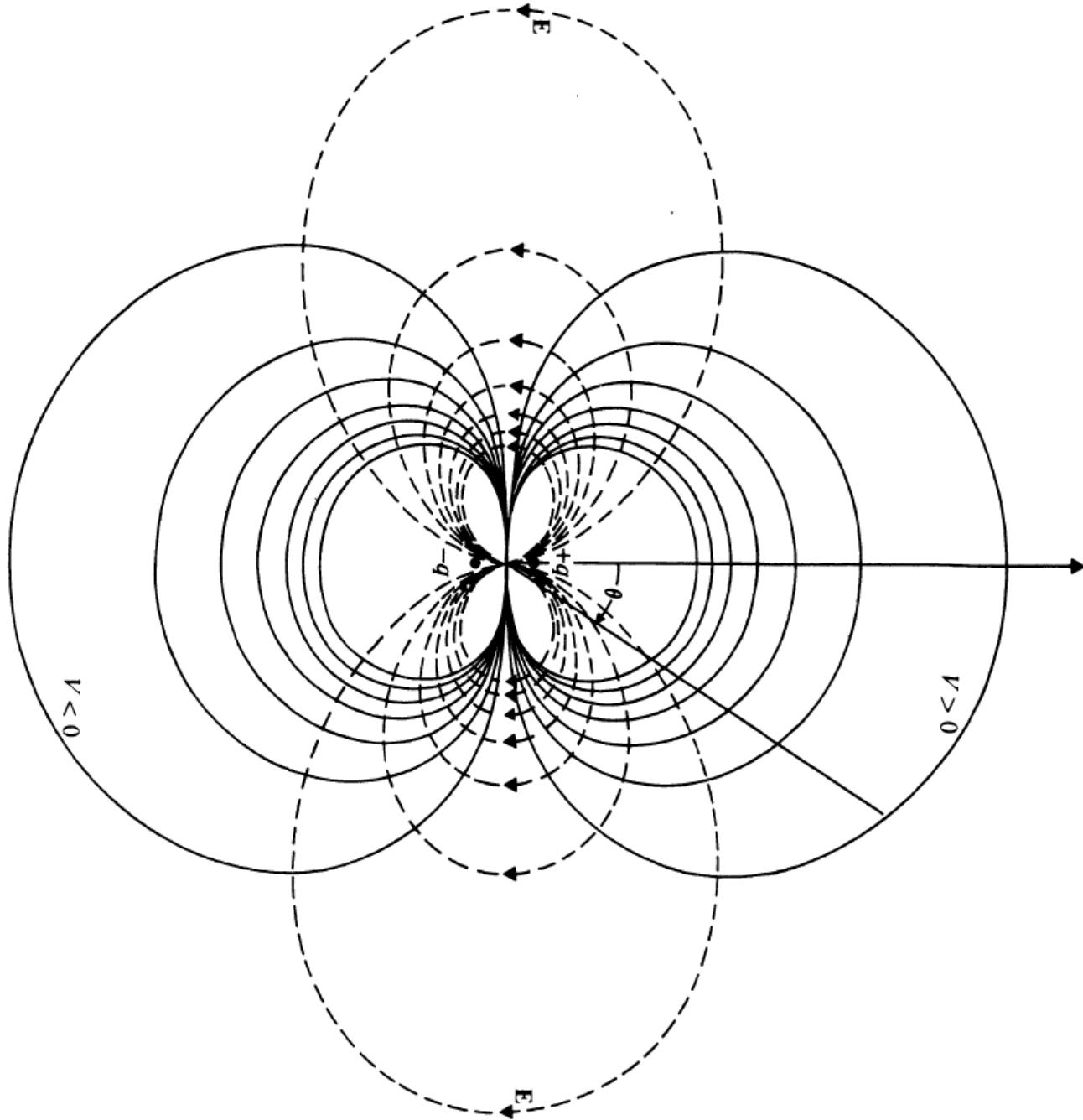
(b) Line charge and its image.

**FIGURE 4-5**  
Cross section of line charge and its image in a parallel, conducting, circular cylinder.

1. Cylinder surface is an equi-potential surface  
 → image must be **a parallel line charge** ( $\rho_i$ ) inside the cylinder
2. By symmetry of line OP,  $\rho_i$  should be on OP



(b) Line charge and its image.



Solid line: equi-potential surface

**FIGURE 3–15**  
Equipotential and electric field lines of an electric dipole (Example 3–8).

Assume

$$\rho_i = -\rho_\ell.$$

(intelligent guess)

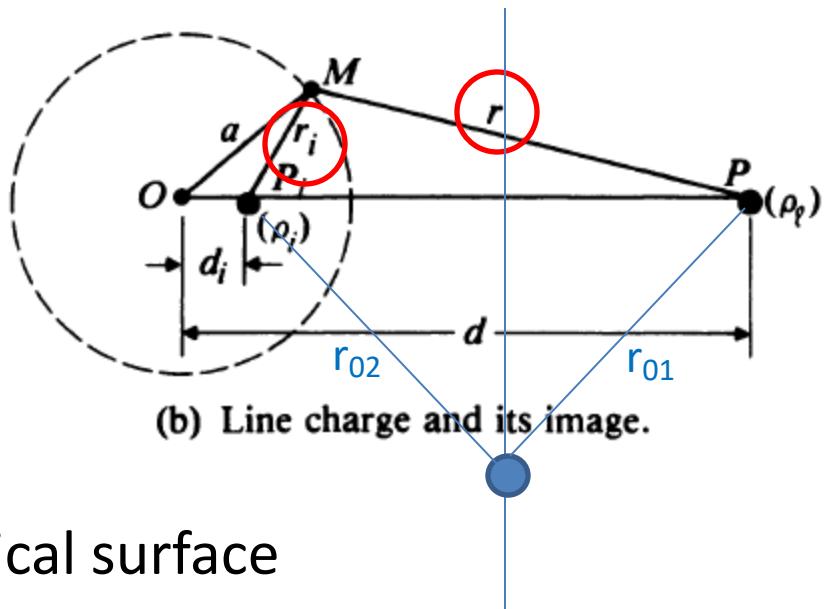
- Voltage due to  $\rho_l$

$$V = - \int_{r_0}^r E_r dr = -\frac{\rho_\ell}{2\pi\epsilon_0} \int_{r_0}^r \frac{1}{r} dr$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_0}{r}.$$

Reference point,  $V=0$

Point of interest



- Voltage due to  $\rho_l$  and  $\rho_i$  on cylindrical surface

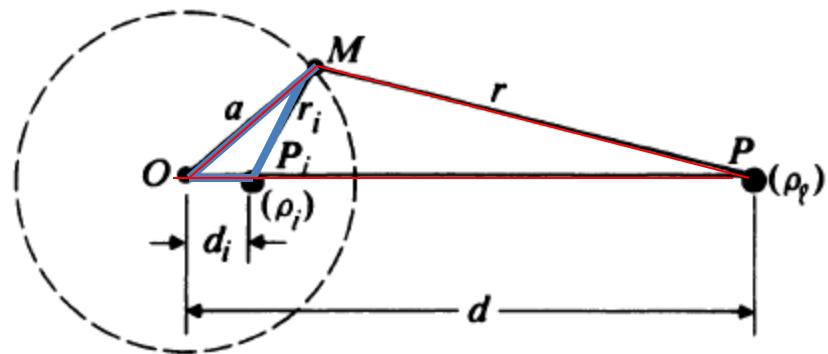
$$V_M = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_{01}}{r} - \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_{02}}{r_i}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_i}{r}.$$

Choosing the same reference point with equidistance from  $\rho_l$  and  $\rho_i$  so that  $\ln r_0$  terms cancel.

- To make  $V_M = \text{constant}$

$$\frac{r_i}{r} = \text{Constant.}$$



(b) Line charge and its image.

- To make M coincide with the cylindrical surface ( $OM=a$ ),  $P_i$  should be chosen to make the two triangles  $OMP_i$  and  $OPM$  similar. (Otherwise,  $r_i/r=\text{constant over the cylindrical surface cannot be satisfied.}$ )



$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$



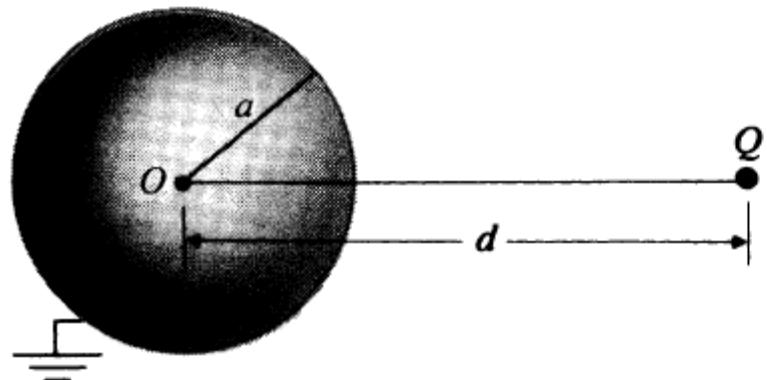
$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$



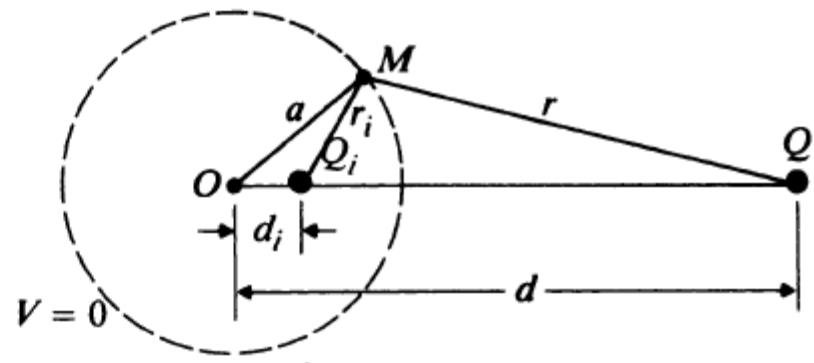
$$d_i = \frac{a^2}{d}$$

$P_i$  is called the **inverse point** of  $P$

## 4-4.3 Point Charge and Conducting Sphere



(a) Point charge and grounded conducting sphere.

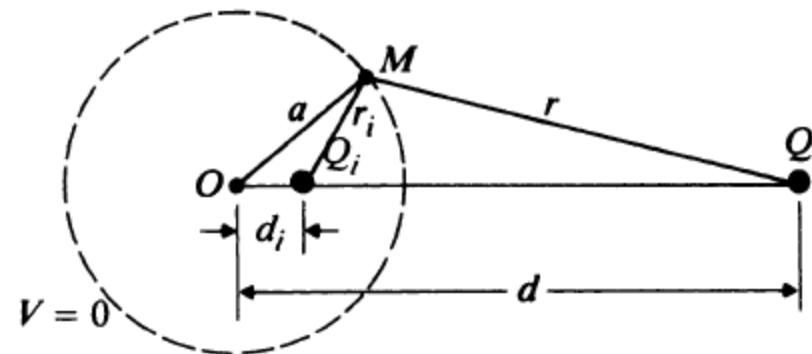


(b) Point charge and its image.

**FIGURE 4-11**  
Point charge and its image in a grounded sphere.

# Intelligent Guess

- By symmetry,  $Q_i$ 
  - Negative
  - Inside the sphere
  - On line OQ
- $Q_i \neq -Q$ , otherwise the equi-potential surface (**V=0**) is a plane
- Thus, both  $Q_i$  and  $d_i$  should be solved

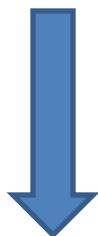


(b) Point charge and its image.

$$V_M = \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{Q_i}{r_i} \right) = 0,$$



$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{Constant.}$$



$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

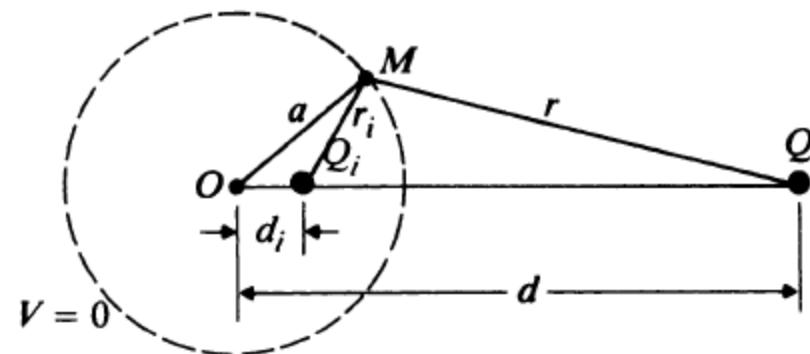
$$-\frac{Q_i}{Q} = \frac{a}{d}$$



$$Q_i = -\frac{a}{d} Q$$

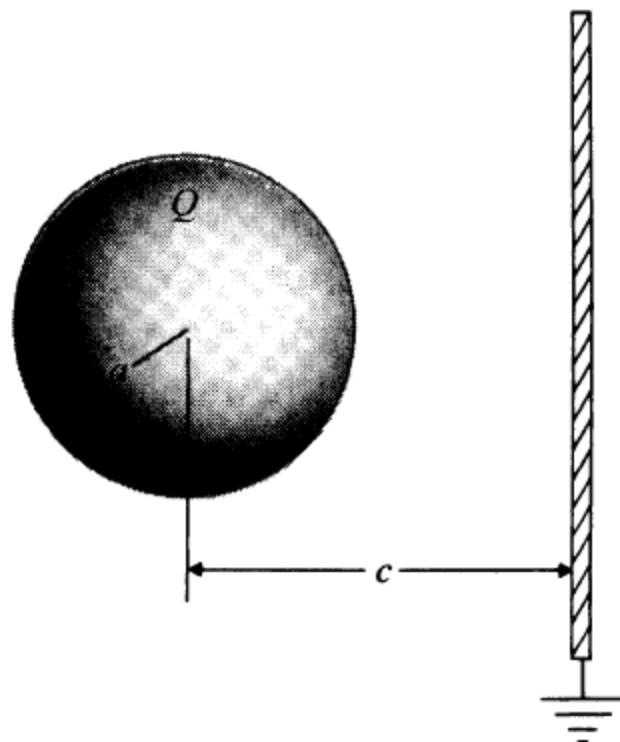
$$d_i = \frac{a^2}{d}.$$

$Q_i$  is called the **inverse point** of  $Q$

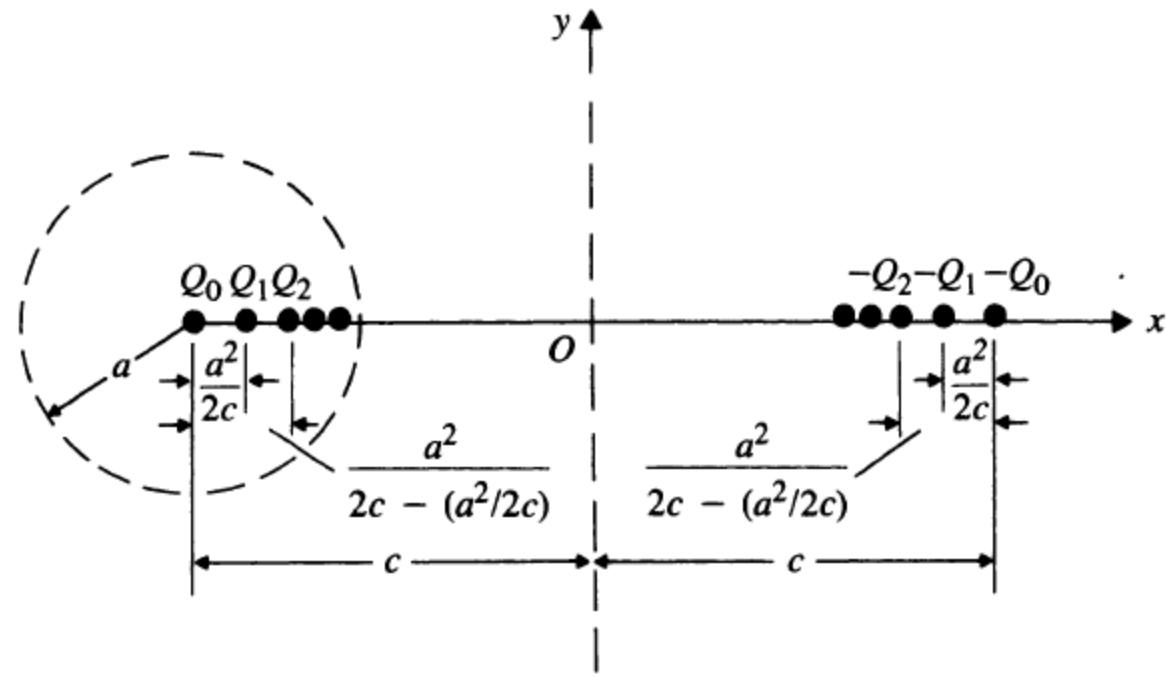


(b) Point charge and its image.

## 4-4.4 Charged Sphere and Grounded Plane



(a) Physical arrangement.

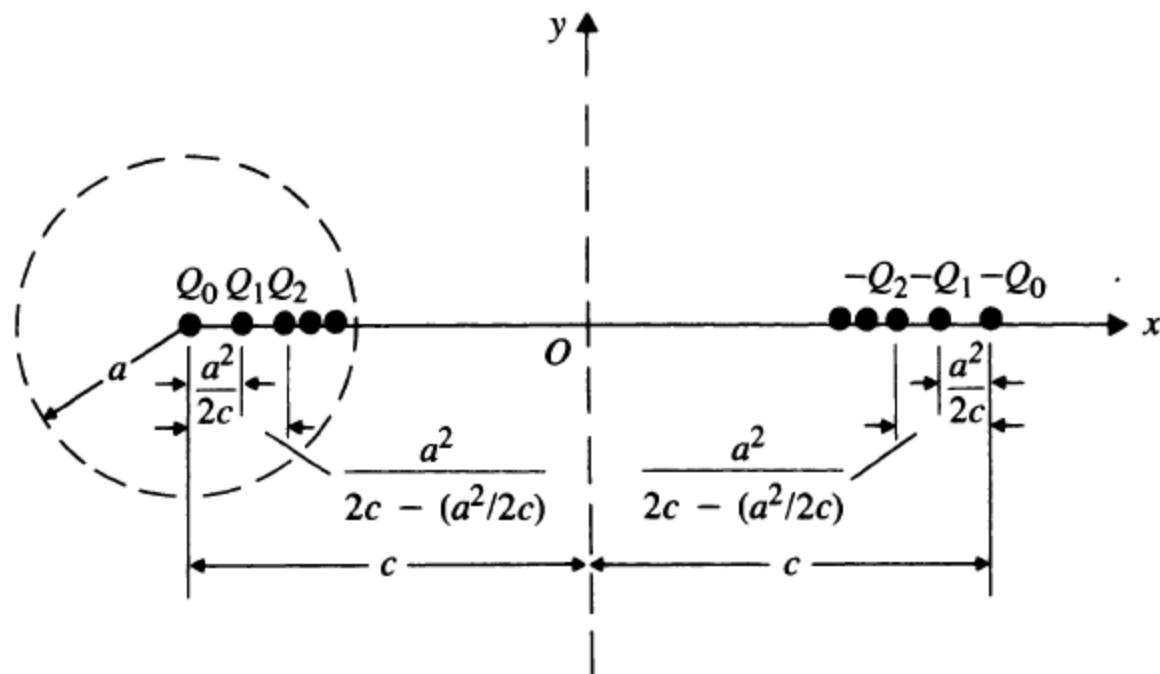


(b) Two groups of image point charges.

**FIGURE 4-13**

Charged sphere and grounded conducting plane.

- Assume that a charge  $Q_0$  is at the center.
- The sphere and the plane must be equi-potential surfaces.
- Method of images: The charged sphere and grounded plane can be replaced by charges



(b) Two groups of image point charges.

$Q_0$  at  $(-c, 0)$   $\rightarrow$  the sphere is an equi-potential plane



To make  $yz$  plane equi-potential ( $V=0$ )

$-Q_0$  at  $(c, 0)$



Destroy equi-potential of the sphere

To make the sphere surface equi-potential

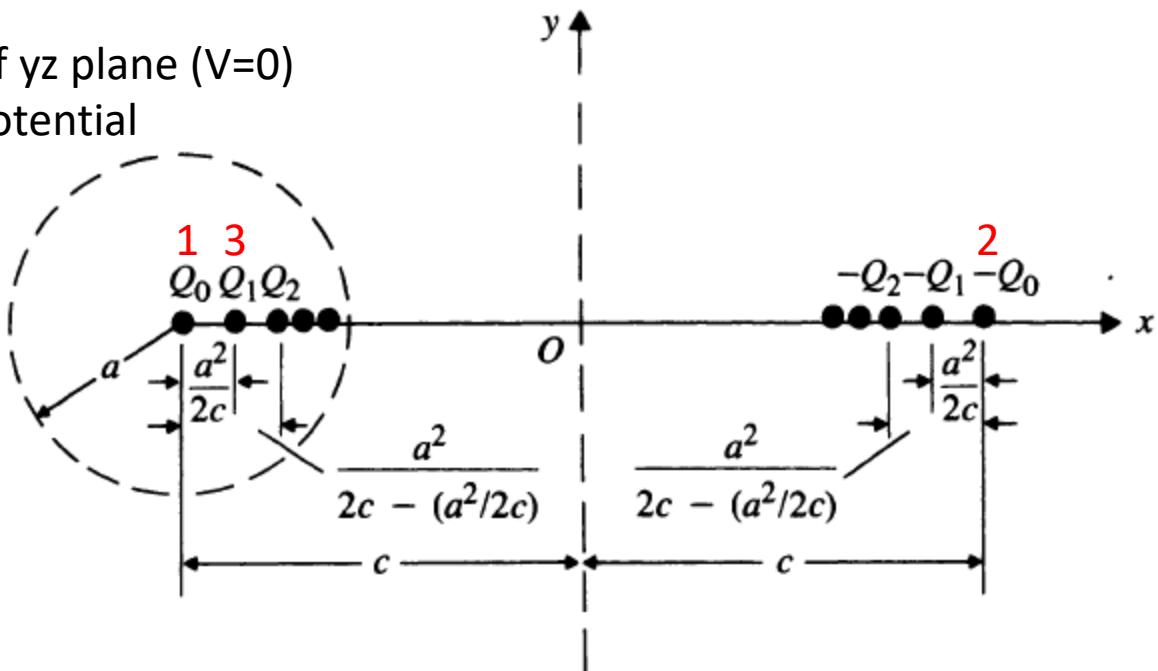
$Q_1$  inside the sphere



Destroy equi-potential of  $yz$  plane ( $V=0$ )

To make  $yz$  plane equi-potential

$-Q_1$



(b) Two groups of image point charges.

$$Q_1 = \left(\frac{a}{2c}\right) Q_0 = \alpha Q_0,$$

$$Q_2 = \frac{a}{\left(2c - \frac{a^2}{2c}\right)} Q_1 = \frac{\alpha^2}{1 - \alpha^2} Q_0, \quad \alpha = \frac{a}{2c}.$$

$$Q_3 = \frac{a}{2c - \frac{a^2}{\left(2c - \frac{a^2}{2c}\right)}} Q_2 = \frac{\alpha^3}{(1 - \alpha^2)\left(1 - \frac{\alpha^3}{1 - \alpha^3}\right)} Q_0,$$

⋮

$Q_0$  to  $Q_1 \rightarrow Q_1 = (a/d_1)Q_0 ; d_{i1} = a^2/d_1$

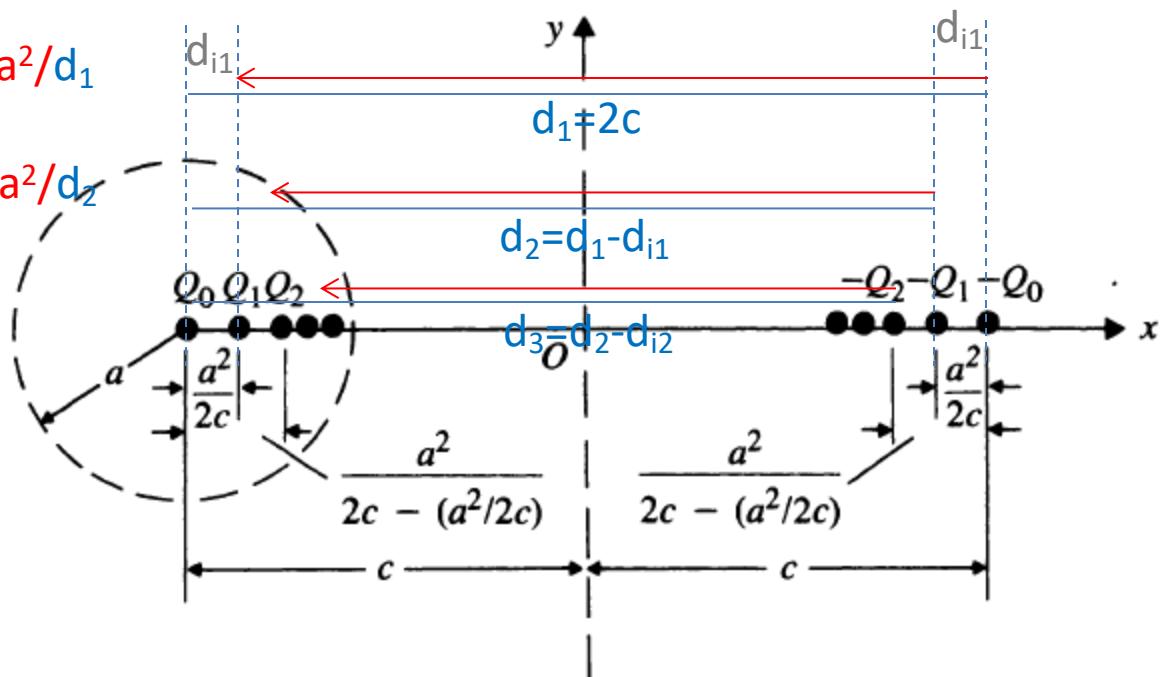
$Q_1$  to  $Q_2 \rightarrow Q_2 = (a/d_2)Q_1 ; d_{i2} = a^2/d_2$

$Q_2$  to  $Q_3 \dots$

$$Q_i = -\frac{a}{d} Q$$

$$d_i = \frac{a^2}{d}.$$

d: from the charge to the sphere center



(b) Two groups of image point charges.

- Total charge on the sphere

$$Q = Q_0 + Q_1 + Q_2 + \dots$$

$$= Q_0 \left( 1 + \alpha + \frac{\alpha^2}{1 - \alpha^2} + \dots \right).$$

- The V on the sphere

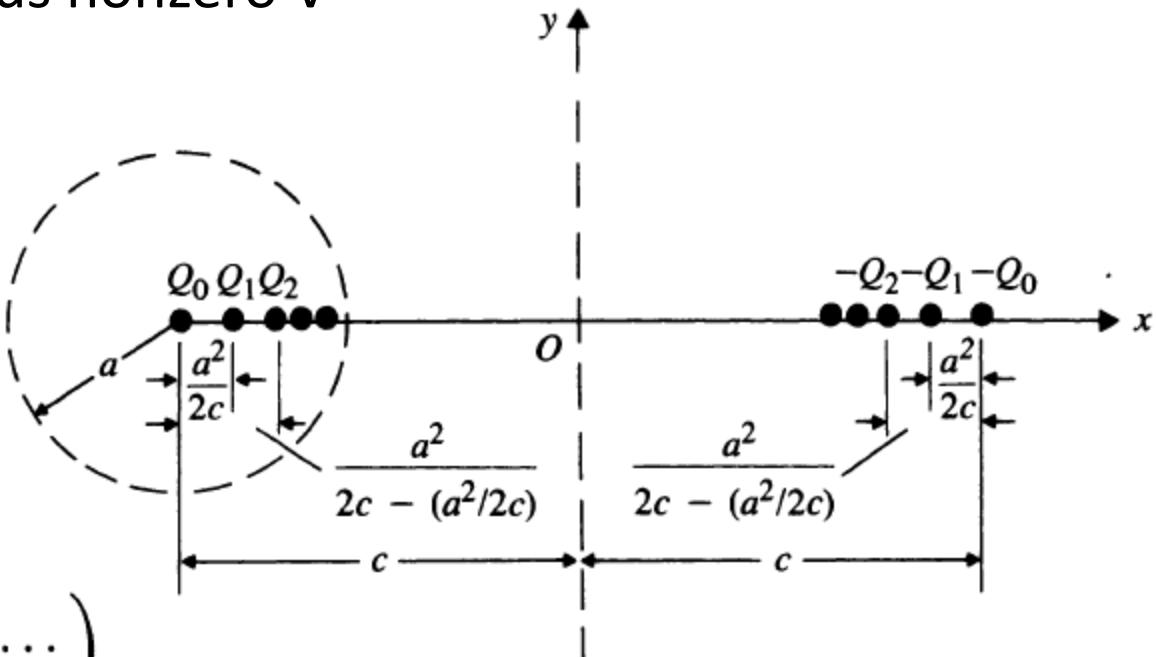
Pairs  $(-Q_0, Q_1), (-Q_1, Q_2), \dots$  yield zero potential on the sphere

Only  $Q_0$  at the center yields nonzero V

$$V_0 = \frac{Q_0}{4\pi\epsilon_0 a}.$$

- The C between the sphere and the conducting plane

$$C = \frac{Q}{V_0} = 4\pi\epsilon_0 a \left( 1 + \alpha + \frac{\alpha^2}{1 - \alpha^2} + \dots \right),$$



(b) Two groups of image point charges.

## 4-5 Boundary-Value Problems in Cartesian Coordinates

- Method of images: useful for the case with **isolated free charges**
- Laplace's equation: can be used to solve the case **w/o isolated free charges** (**Example 4-1: charges on conductors**)
  - Known boundary values (potential or its normal derivative specified), so called boundary-value problems.

# Three Types of Boundary Conditions

- Dirichlet:  $V$  is specified on boundaries
- Neumann:  $dV/dn$  is specified on boundaries
- Mixed:  $V$  specified on some boundaries;  $dV/dn$  specified over the remaining boundaries.

# Separation of Variables

Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$



By separation of variables

$$V(x, y, z) = X(x)Y(y)Z(z),$$

$$Y(y)Z(z) \frac{d^2 X(x)}{dx^2} + X(x)Z(z) \frac{d^2 Y(y)}{dy^2} + X(x)Y(y) \frac{d^2 Z(z)}{dz^2} = 0,$$



$$\times 1/X(x)Y(y)Z(z)$$

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0.$$

$$f(x) + f(y) + f(z) = 0$$

$f(x) + f(y) + f(z) = 0$  to be satisfied for all values of  $x, y, z$

$f(x)$ : function of  $x$  only

$f(y)$ : function of  $y$  only

$f(z)$ : function of  $z$  only



$f(x), f(y), f(z)$  must be a constant



$df(x)/dx=0, df(y)/dy=0, df(z)/dz=0$



$$df(x)/dx=0$$

$$\frac{d}{dx} \left[ \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} \right] = 0,$$



Integration on both sides

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2,$$

Integration constant



$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0.$$

### Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

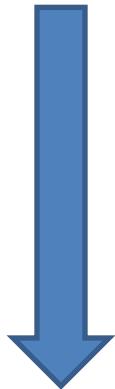
$k_x^2$	$k_x$	$X(x)$	Exponential forms <sup>†</sup> of $X(x)$
0	0	$A_0 x + B_0$	
+	$k$	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
-	$jk$	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{jkx} + D_2 e^{-jkx}$

k is real

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0.$$

$$\frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$\frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0,$$



$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0.$$

$$\begin{array}{ccccc} f(x) & + & f(y) & + & f(z) \\ -k_x^2 & & -k_y^2 & & -k_z^2 \end{array} = 0$$

$k_x^2 + k_y^2 + k_z^2 = 0$ , which should be satisfied.

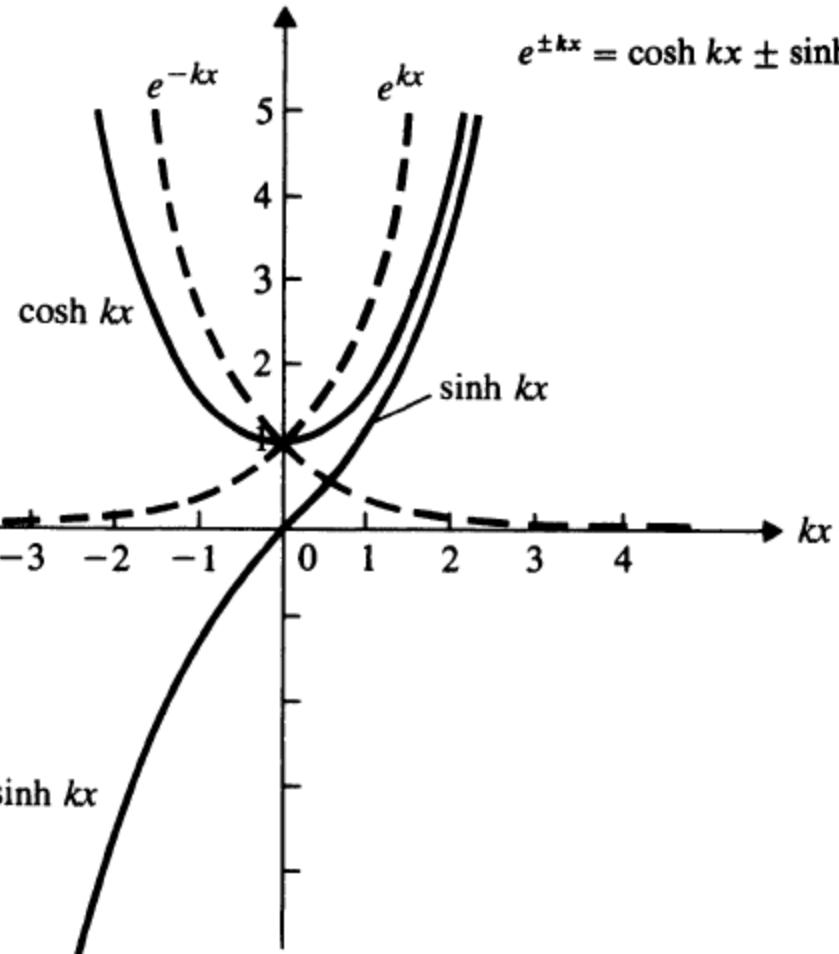
## Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

$k_x^2$	$k_x$	$X(x)$	Exponential forms <sup>†</sup> of $X(x)$
0	0	$A_0 x + B_0$	
+	$k$	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
-	$jk$	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

$k$  is real

$$e^{\pm jkx} = \cos kx \pm j \sin kx, \quad \cos kx = \frac{1}{2}(e^{jkx} + e^{-jkx}), \quad \sin kx = \frac{1}{2j}(e^{jkx} - e^{-jkx});$$

$$e^{\pm kx} = \cosh kx \pm \sinh kx, \quad \cosh kx = \frac{1}{2}(e^{kx} + e^{-kx}), \quad \sinh kx = \frac{1}{2}(e^{kx} - e^{-kx}).$$



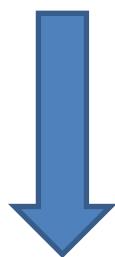
A, B or C, D will be determined  
by boundary conditions

**FIGURE 4-14**  
Hyperbolic and exponential functions.

## 4-6 Boundary-Value Problems in Cylindrical Coordinates

Laplace's equation     $\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$

General solution: Bessel functions



Assuming  $z$  independent

$$\frac{\partial^2 V}{\partial z^2} = 0$$

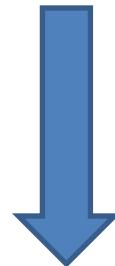
$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} = 0.$$



By separation of variables

$$V(r, \phi) = R(r)\Phi(\phi),$$

$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] + \frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$



To hold for all values of r and  $\phi$

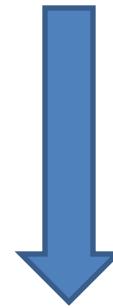
$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] + \frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$

$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = k^2$$

$$\frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2} = -k^2, \quad \xrightarrow{\text{rewrite}} \quad \frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$

$$V(r, \phi) = R(r)\Phi(\phi),$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$



For circular cylindrical configurations, if  $\phi$  is unrestricted,  $\Phi(\phi)$  is periodic (same values at a certain  $\phi$ )

- $k$  must be an integer  $\rightarrow n$
- $\sinh, \cosh$  are not periodic!

$$e^{jk\phi} \neq e^{jk(\phi+2\pi)}$$
$$e^{jn\phi} = e^{jn(\phi+2\pi)}$$

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$

$$V(r, \phi) = \underline{R(r)}\Phi(\phi),$$

$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = k^2$$

  $k \rightarrow n$   
Product rule

$$r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} - n^2 R(r) = 0,$$

Solution:  $R(r) = A_r r^n + B_r r^{-n}$ .

(Verified by direct substitution)

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$

$$R(r) = A_r r^n + B_r r^{-n}.$$



Combine the two solutions

$$V(r, \phi) = R(r)\Phi(\phi),$$

$$V_n(r, \phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A'_n \sin n\phi + B'_n \cos n\phi), \quad n \neq 0.$$

$$\text{where } A_n = A_r A_\phi; \quad B_n = A_r B_\phi;$$

$$A'_n = B_r A_\phi; \quad B'_n = B_r B_\phi;$$

If region of interest (ROI) includes  $r=0$ , 2<sup>nd</sup> term cannot exist.

If region of interest (ROI) includes  $r=\infty$ , 1<sup>st</sup> term cannot exist

# A Special Case: k=0

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$

$$\frac{r}{R(r)} \frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = k^2$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$

$$\frac{d}{dr} \left[ r \frac{dR(r)}{dr} \right] = 0,$$



for k=0

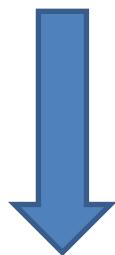
$$\Phi(\phi) = A_0\phi + B_0$$

$$R(r) = C_0 \ln r + D_0,$$

## 4-7 Boundary-Value Problems in Spherical Coordinates

Laplace's equation

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$



Assuming  $\phi$  independent

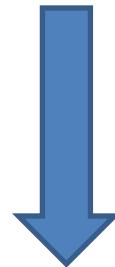
$$\frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0.$$



By separation of variables

$$V(R, \theta) = \Gamma(R)\Theta(\theta).$$

$$\frac{1}{\Gamma(R)} \frac{d}{dR} \left[ R^2 \frac{d\Gamma(R)}{dR} \right] + \frac{1}{\Theta(\theta) \sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] = 0.$$



To hold for all values of R and θ

$$\frac{1}{\Gamma(R)} \frac{d}{dR} \left[ R^2 \frac{d\Gamma(R)}{dR} \right] = k^2$$

$$\frac{1}{\Theta(\theta) \sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] = -k^2,$$

$$V(R, \theta) = \underline{\Gamma(R)} \Theta(\theta).$$

$$\frac{1}{\Gamma(R)} \frac{d}{dR} \left[ R^2 \frac{d\Gamma(R)}{dR} \right] = k^2$$



$$R^2 \frac{d^2\Gamma(R)}{dR^2} + 2R \frac{d\Gamma(R)}{dR} - k^2\Gamma(R) = 0,$$

Solution:  $\Gamma_n(R) = A_n R^n + B_n R^{-(n+1)}$ .

where  $n(n+1) = k^2$ ,  
 $n=0,1,2\dots$  is a positive integer

(Verified by direct substitution)

$$V(R, \theta) = \Gamma(R)\Theta(\theta).$$

$$\frac{1}{\Theta(\theta) \sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] = -k^2,$$



$$n(n + 1) = k^2,$$

$$\frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] + n(n + 1)\Theta(\theta) \sin \theta = 0, \quad \text{Legendre's equation}$$

Solution:  $\Theta_n(\theta) = P_n(\cos \theta)$ . Legendre's functions  
if involving full range of  $\theta=[0,\pi]$

For integer values of  $n$ ,

### Several Legendre Polynomials

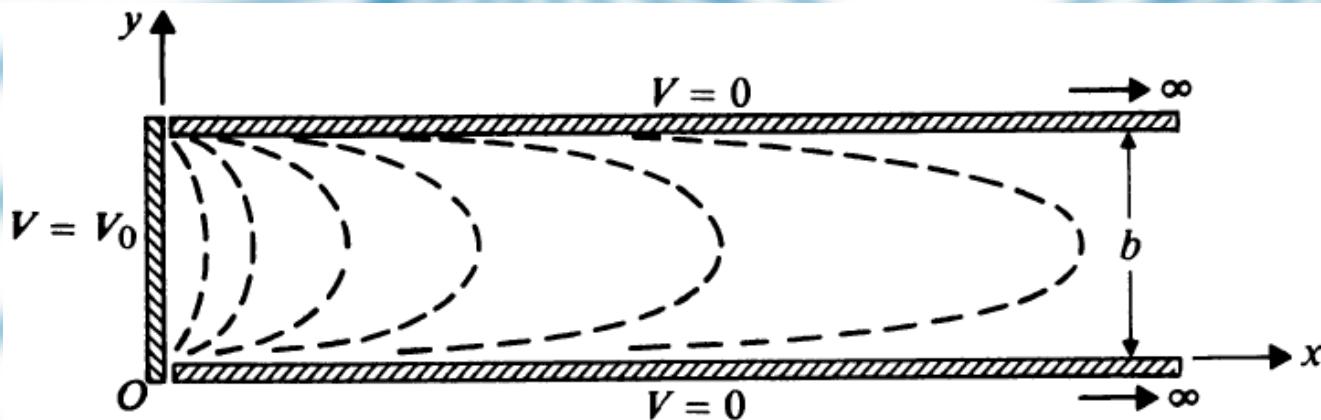
$n$	$P_n(\cos \theta)$
0	1
1	$\cos \theta$
2	$\frac{1}{2}(3 \cos^2 \theta - 1)$
3	$\frac{1}{2}(5 \cos^3 \theta - 3 \cos \theta)$

The Legendre polynomials are **orthogonal**

# Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

**EXAMPLE 4–6** Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance  $b$ . A third electrode perpendicular to and insulated from both is maintained at a constant potential  $V_0$  (see Fig. 4–15). Determine the potential distribution in the region enclosed by the electrodes.



**FIGURE 4–15**  
Cross-sectional figure for Example 4–6. The plane electrodes are infinite in  $z$ -direction.

# Quick classroom exercise

**TABLE 4-1**  
**Possible Solutions of  $X''(x) + k_x^2 X(x) = 0$**

$k_x^2$	$k_x$	$X(x)$	Exponential forms <sup>†</sup> of $X(x)$
0	0	$A_0 x + B_0$	
+	$k$	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
-	$jk$	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

# Quick classroom exercise

**Solution** Referring to the coordinates in Fig. 4–15, we write down the boundary conditions for the potential function  $V(x, y, z)$  as follows.

With  $V$  independent of  $z$ :

$$V(x, y, z) = V(x, y). \quad (4-90a)$$

In the  $x$ -direction:

$$V(0, y) = V_0 \quad (4-90b)$$

$$V(\infty, y) = 0. \quad (4-90c)$$

In the  $y$ -direction:

$$V(x, 0) = 0 \quad (4-90d)$$

$$V(x, b) = 0. \quad (4-90e)$$

Condition (4–90a) implies  $k_z = 0$ , and from Table 4–1,

$$Z(z) = B_0. \quad (4-91)$$

The constant  $A_0$  vanishes because  $Z$  is independent of  $z$ . From Eq. (4–89) we have

$$k_y^2 = -k_x^2 = k^2, \quad (4-92)$$

where  $k$  is a real number. This choice of  $k$  implies that  $k_x$  is imaginary and that  $k_y$  is real. The use of  $k_x = jk$ , together with the condition of Eq. (4–90c), requires us to choose the exponentially decreasing form for  $X(x)$ , which is

$$X(x) = D_2 e^{-kx}. \quad (4-93)$$

In the  $y$ -direction,  $k_y = k$ . Condition (4–90d) indicates that the proper choice for  $Y(y)$  from Table 4–1 is

$$Y(y) = A_1 \sin ky. \quad (4-94)$$

Combining the solutions given by Eqs. (4–91), (4–93), and (4–94) in Eq. (4–82), we obtain an appropriate solution of the following form:

$$\begin{aligned} V_n(x, y) &= (B_0 D_2 A_1) e^{-kx} \sin ky \\ &= C_n e^{-kx} \sin ky, \end{aligned} \quad (4-95)$$

where the arbitrary constant  $C_n$  has been written for the product  $B_0 D_2 A_1$ .

Now, of the five boundary conditions listed in Eqs. (4–90a) through (4–90e) we have used conditions (4–90a), (4–90c), and (4–90d). To meet condition (4–90e), we require

$$V_n(x, b) = C_n e^{-kx} \sin kb = 0, \quad (4-96)$$

which can be satisfied, for all values of  $x$ , only if

$$\sin kb = 0$$

or

$$kb = n\pi$$

or

$$k = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots \quad (4-97)$$

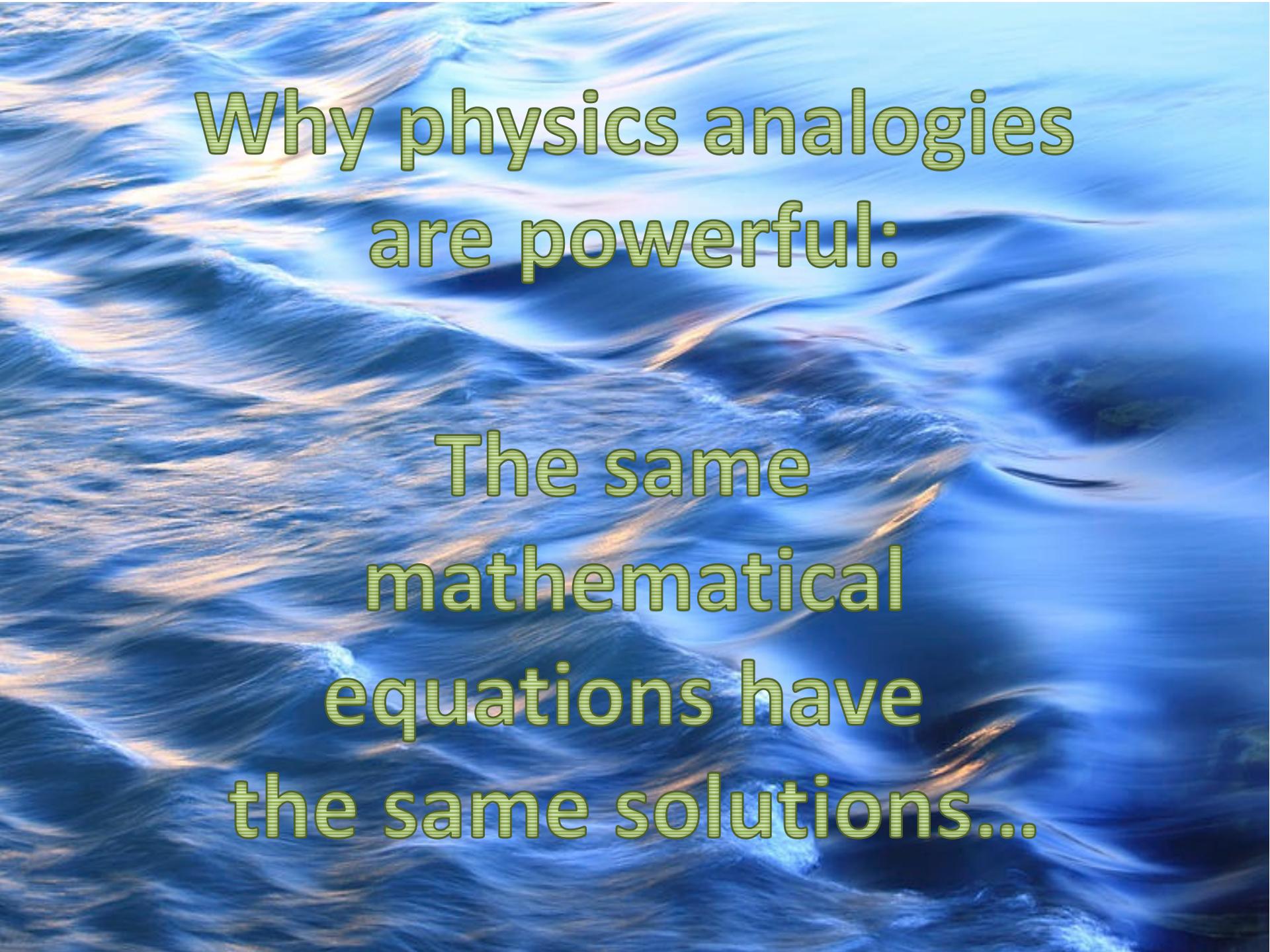
Therefore, Eq. (4–95) becomes

$$V_n(x, y) = C_n e^{-n\pi x/b} \sin \frac{n\pi}{b} y. \quad (4-98)$$



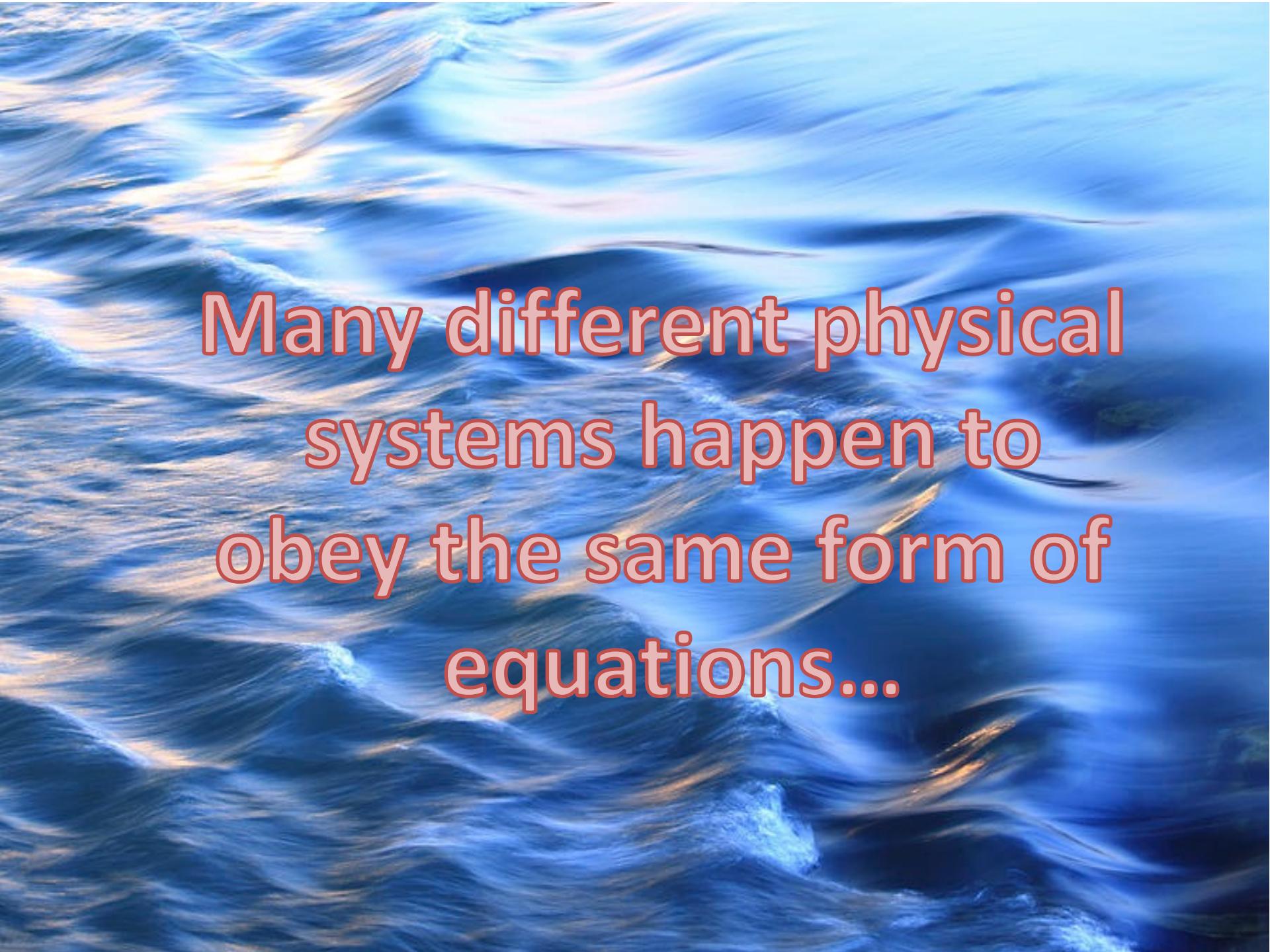
The background of the image is a vibrant, abstract representation of energy or fluid motion. It features deep blue and orange hues, with intricate, swirling patterns that resemble both ocean waves and celestial bodies like galaxies. The movement is depicted through a series of concentric, curved lines that radiate from the center, creating a sense of depth and motion.

# Electrostatic Analogies!!

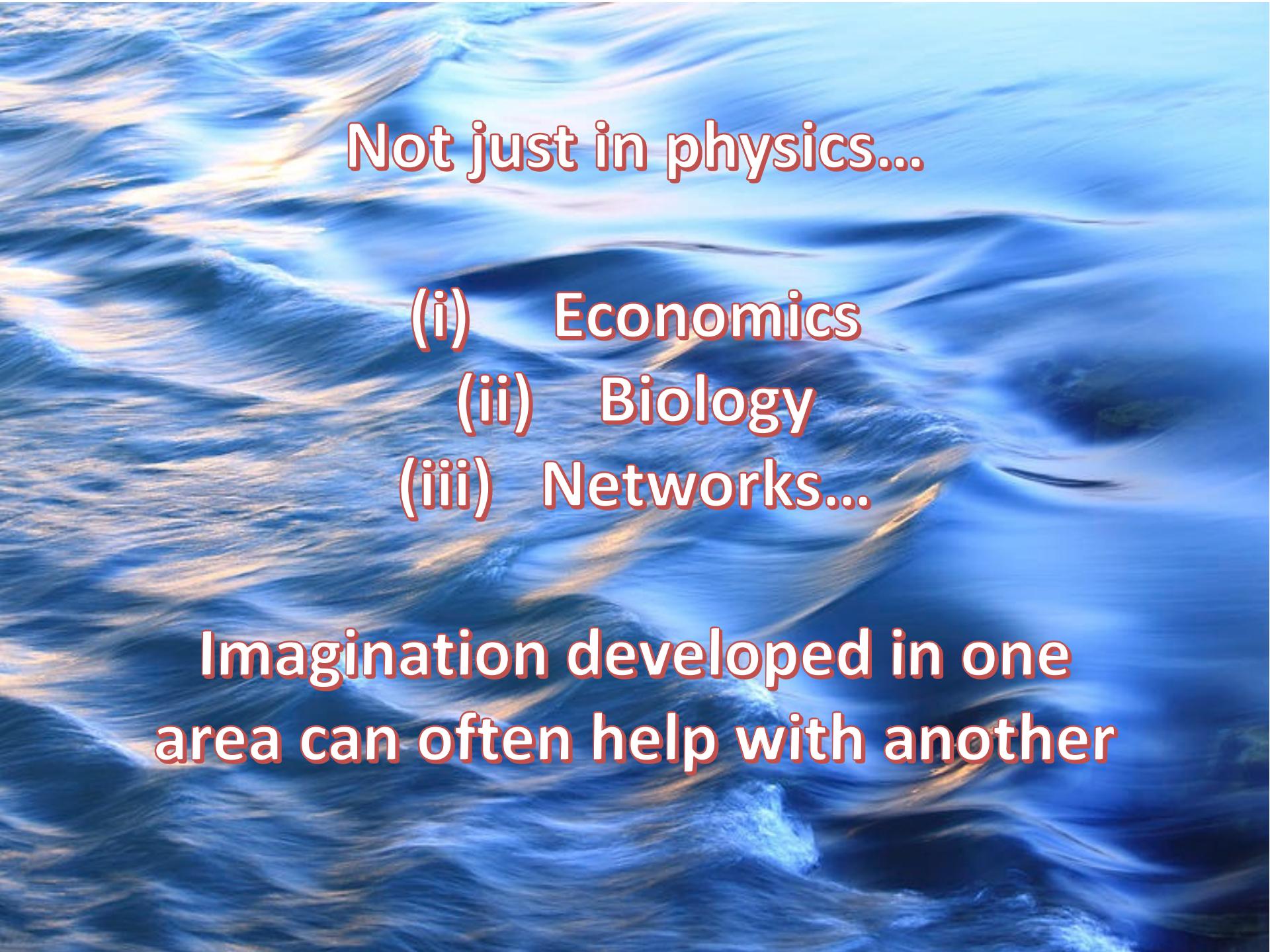


Why physics analogies  
are powerful:

The same  
mathematical  
equations have  
the same solutions...



Many different physical  
systems happen to  
obey the same form of  
equations...



Not just in physics...

- (i) Economics
- (ii) Biology
- (iii) Networks...

Imagination developed in one area can often help with another

# Electrostatic analogs:

- (i) Gravity
- (ii) Heat flow
- (iii) Stretched membrane
- (iv) Irrotational fluid

# Electrostatics

$$\nabla \cdot (\epsilon E) = \frac{\rho}{\epsilon_0}$$
$$\nabla \times E = 0$$

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$$\nabla \cdot (\epsilon E) = \frac{\rho}{\epsilon_0}$$

$$\nabla \times E = 0$$

$$E = -\nabla V$$

Poisson equation

$$\nabla \cdot (\epsilon \nabla V) = -\frac{\rho}{\epsilon_0}$$

# Newtonian Gravity

$$F = -\nabla U_g$$

$$F = \frac{Gm'}{|x' - x|^2}$$

$$U_g = -\frac{Gm'}{|x' - x|}$$

# Newtonian Gravity

$$F = -\nabla U_g$$

$$F = \frac{Gm'}{|x' - x|^2}$$

$$U_g = -\frac{Gm'}{|x' - x|}$$

$$N \text{ masses: } U_g = -G \sum_{i=1}^N \frac{m'_i}{|x'_i - x|}$$

$$\text{Smooth: } U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

# Newtonian Gravity

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# Newtonian Gravity

$$U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

$$\nabla^2 \left( \frac{1}{|x' - x|} \right) = -4\pi \delta(x' - x)$$

Poisson equation

$$\nabla^2 U_g = 4\pi G \rho(x)$$

The background of the image is a vibrant, abstract representation of energy or fluid motion. It features deep blue and teal hues, accented with bright orange and yellow streaks that suggest light passing through or reflecting off turbulent surfaces. The overall effect is one of intense, dynamic activity, possibly representing gravitational waves or complex astrophysical phenomena.

More complicated as we  
go beyond Newtonian  
gravity...

# Heat flow

$$\nabla \cdot h = \text{rate of heat out/unit time}$$

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Steady heat flow

$$\nabla \cdot h = s$$

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$\nabla \cdot h =$  rate of heat out/unit time

Steady heat flow

$$\nabla \cdot h = s$$

$$h = -K\nabla T$$

$K$  = thermal conductivity

$T$  = temperature

# Heat flow

$$\nabla \cdot h = s$$

$$h = -K\nabla T$$

Poisson equation

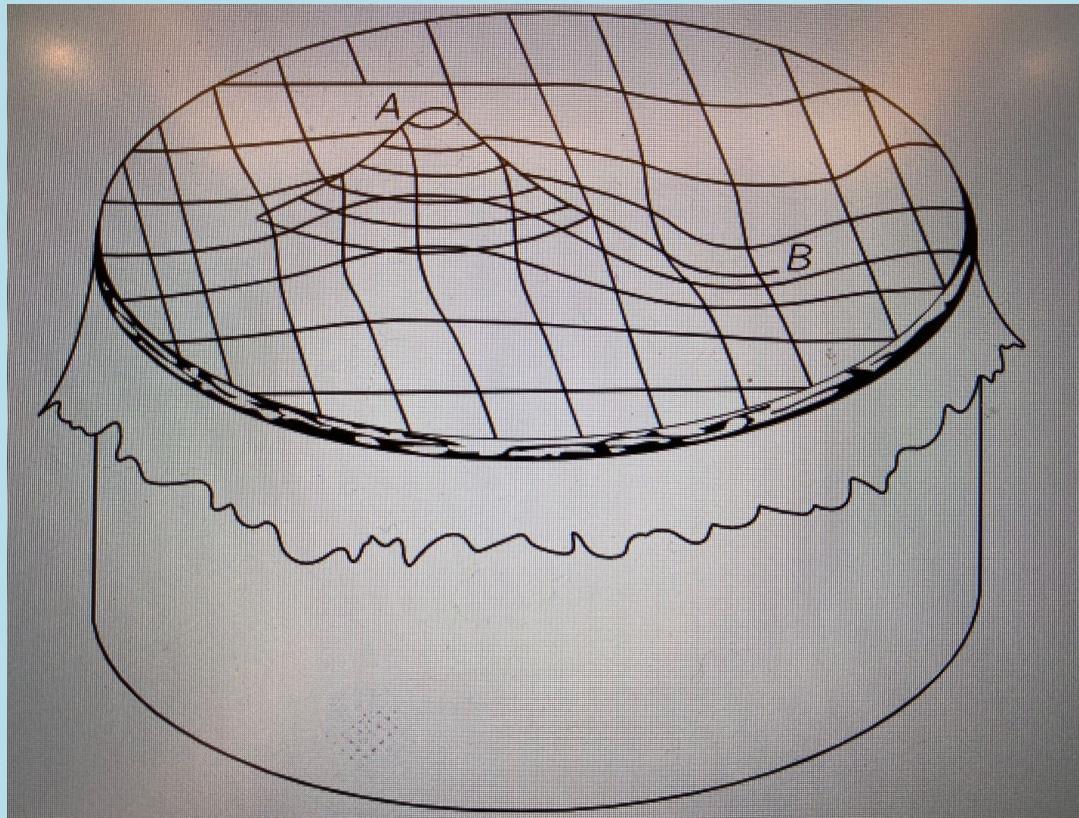
$$\nabla \cdot (K\nabla T) = -s$$

# Stretched membrane

Thin rubber sheet stretched over a membrane over cylindrical drum.

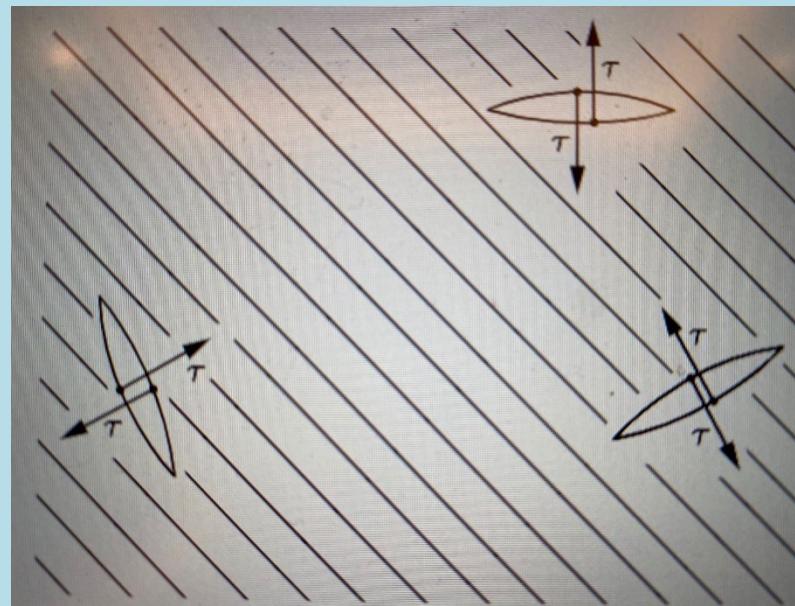
If sheet pushed up at A and down at B,  
what is shape of surface?

# Stretched membrane

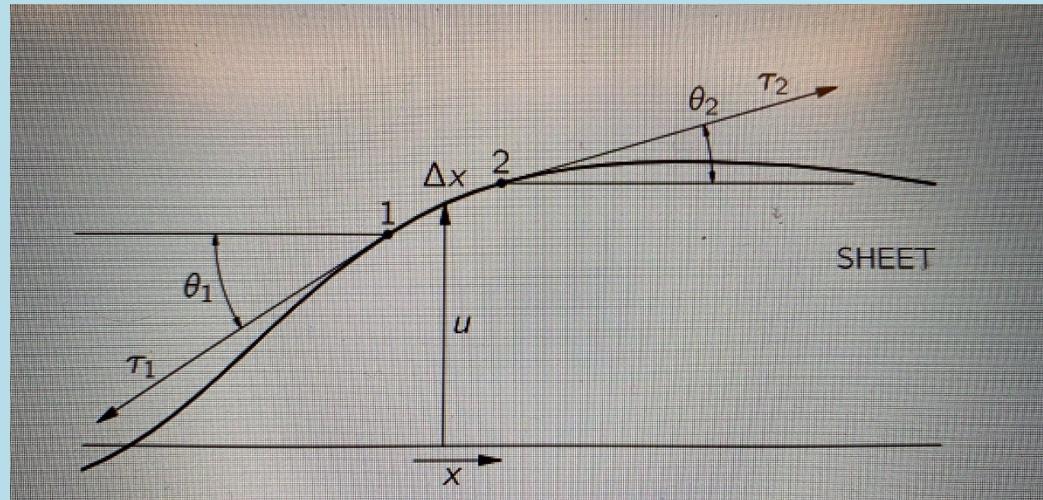


# Stretched membrane

Surface tension  $\tau$  = force/unit length across line



# Stretched membrane



$$\Delta F = \tau_2 \Delta y \sin \theta_2 - \tau_1 \Delta y \sin \theta_1$$

Small slopes...

# Stretched membrane

Small slopes...

$$\Delta F \sim \tau_2 \Delta y \sin \theta_2 - \tau_1 \Delta y \sin \theta_1$$

$$\sim \left( \tau_2 \left( \frac{\partial u}{\partial x} \right)_2 - \tau_1 \left( \frac{\partial u}{\partial x} \right)_2 \right) \Delta y$$

$$\sim \frac{\partial}{\partial x} \left( \tau \frac{\partial u}{\partial x} \right) \Delta x \Delta y$$

# Stretched membrane

Define  $f \equiv -\frac{\Delta F}{\Delta x \Delta y}$

$$f = -\nabla \cdot (\tau \nabla u)$$

Poisson equation  $\nabla^2 u = -\frac{f}{\tau}$

# Stretched membrane

Poisson equation

$$\nabla^2 u = -\frac{f}{\tau}$$

Since  $u$  is the height profile of membrane,  
does it make it easier to believe in the uniqueness  
solution of the electrostatic potential?

# Irrational fluid

Conservation of matter

$$\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t} = 0$$

$v$  = velocity

# Irrational fluid

Incompressible fluid

$$\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t} = 0 \implies$$

$$\nabla \cdot v = 0$$

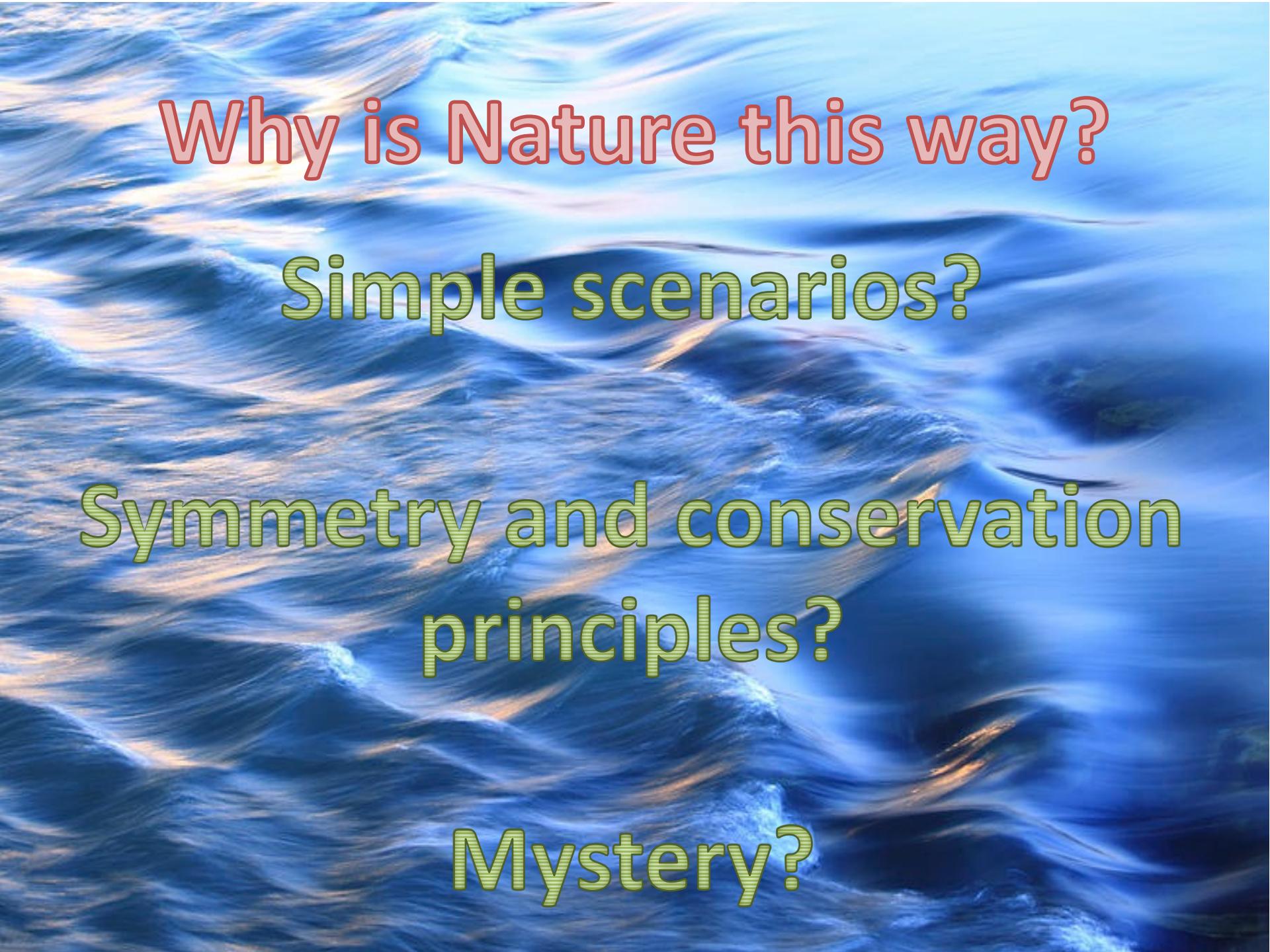
# Irrational fluid

Irrational fluid

$$\nabla \times v = 0 \implies v = -\nabla\phi$$

Incompressible fluid  $\nabla \cdot v = 0$

Laplace equation  $\nabla^2\phi = 0$



Why is Nature this way?

Simple scenarios?

Symmetry and conservation  
principles?

Mystery?