RC 4: Capacitors, Energy in Electrostatic Fields, Electrostatic Solutions and Method of Images

1 Capacitance and Capacitors

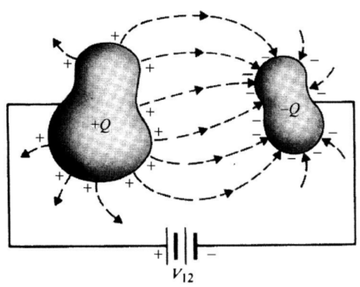
1.1 Capacitance

Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.

$$C = \frac{Q}{V}$$

1.2 Capacitor

Components: Two conductors with arbitrary shapes are separated by free space or dielectric medium.



Notice: The capacitance is independent of Q and V, which means a capacitor has a capacitance even no voltage is applied to it and no free charges exist on its conductors.

Steps to calculate capacitance:

1 Choose a proper coordinate system.

2 Assume $+Q, -Q$ on the conductors.

3 Find E from Q (like, Gauss's law, $D_n = \epsilon E_n = \rho_s$)

4 Find $V_{12} = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$

5 $C = Q/V_{12}$

Series Connections of Capacitors:

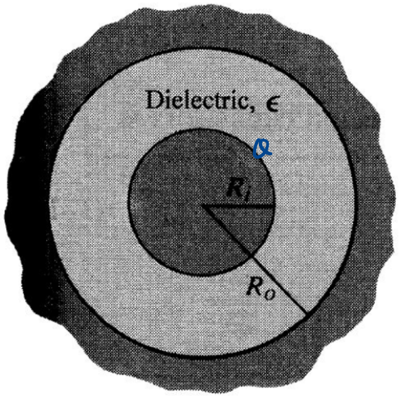
$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

Parallel Connections of Capacitors:

$$C_{\parallel} = C_1 + C_2 + \ldots + C_n$$

Ex4.1

A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o. The space in between is filled with a dielectric of permitivity ϵ. Determine the capacitance.



Suppose the charge in Capacitance is Q (positive side)

$$D \cdot 4\pi R^2 = Q \Rightarrow D = \frac{Q}{4\pi R^2}$$

$$E = \frac{D}{\varepsilon} \quad (R_i \le R < R_o)$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon R^2}, \quad R_i \le R < R_o$$

$$\Delta V = V_{R_i} - V_{R_o} = \int_{R_i}^{R_o} E \cdot dR = \frac{Q}{4\pi\varepsilon}\int_{R_i}^{R_o}\frac{1}{R^2}dR = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon}{\frac{1}{R_i} - \frac{1}{R_o}}$$

2 Energy in Electrostatic Fields

The potential energy of N discrete charges at rest:

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

For a continuous distribution of charge, the energy is

$$W_e = \frac{1}{2}\int \rho V dv$$

$$W_e = \frac{1}{2}\int (\nabla \cdot \mathbf{D}) V dv$$

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V$$

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D}) dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V dv$$

$$= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E} dv$$

$$W_e = \frac{1}{2}\int_{\text{allspace}} \mathbf{D} \cdot \mathbf{E} dv$$

$$W_e = \frac{1}{2}\int_{\text{allspace}} \epsilon E^2 dv$$

3 Uniqueness of Electrostatic Solutions

Uniqueness Theorem: A solution of Poisson's equation $\nabla^2 V = -\frac{\rho}{\epsilon}$ that satisfies the given boundary conditions is a unique solution.

Poisson's equation:

$$\nabla^2 V = -\frac{\rho_f}{\epsilon} = -\frac{\rho}{\epsilon_0}$$

where ρ_f is the free charge density, ϵ is the absolute permitivity, and ρ is the total charge density (free charge density + induced charge density).

Laplace's equation:

$$\nabla^2 V = 0$$

which is a special case of Poisson's equation ($\rho = 0$ everywhere)

Steps to solve boundary condition problem:

1 Write the expression of $V, \mathbf{D}$, and $\mathbf{E}$ according to the configuration, like symmetry or properties of some configuration.

2 Simplify Poisson's equation or Laplace's equation based on the written expression.

3 Write out boundary conditions.

4 Solve the mathematical problem.

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z}\right)$$

- Cartesian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
- Cylindrical: $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$
- Spherical: $\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$

Ex4.2

Determine the E field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \le R \le b$ and $\rho = 0$ for $R > b$ by solving Poisson's and Laplace's equation for V.

① $0 < R \le b$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = -\frac{\rho}{\varepsilon_0} = \frac{\rho_0}{\varepsilon_0}$$

$$\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0 R^2}{\varepsilon_0} \Rightarrow \frac{dV_i}{dR} = \frac{1}{R^2}\left(\frac{\rho_0 R^3}{3\varepsilon_0} + C_1\right) = \frac{\rho_0 R}{3\varepsilon_0} + \frac{C_1}{R^2}^{=0}$$

Boundary Condition at $R=0$. E finite $\Rightarrow C_1 = 0$

$$\Rightarrow \boxed{\vec{E}_i = -\nabla V_i = -\frac{\rho_0 R}{3\varepsilon_0}\hat{a}_R \quad (0 \le R \le b)}$$

② $R > b$ $\quad \rho = 0$

$$\frac{dV_o}{dR} = \frac{C_2}{R^2} \qquad \text{at } R = b,\ E_o = E_i = -\frac{\rho_0 b}{3\varepsilon_0} \Rightarrow C_2 = \frac{\rho_0 b^3}{3\varepsilon_0}$$

$$\Rightarrow \boxed{E = -\hat{a}_R \frac{\rho_0 b^3}{3\varepsilon_0 R^2} \quad (R > b)}$$

4 Method of Images

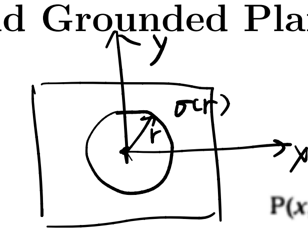
Note:

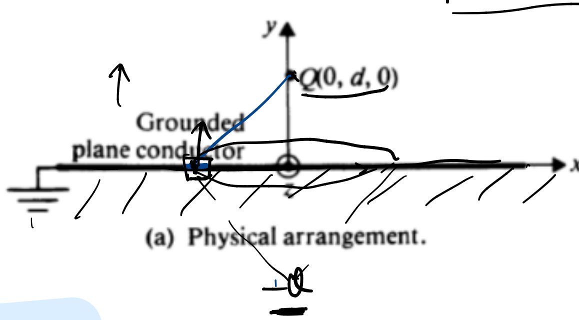
(1) Methods of images is a smart way to solve electrostatics to satisfy certain boundary conditions, utilizing equivalent image charge.(e.g. The voltage potential of a plate is 0 everywhere)

(2) The use of image charge is actually based on the uniqueness theorem of electrostatic solution.

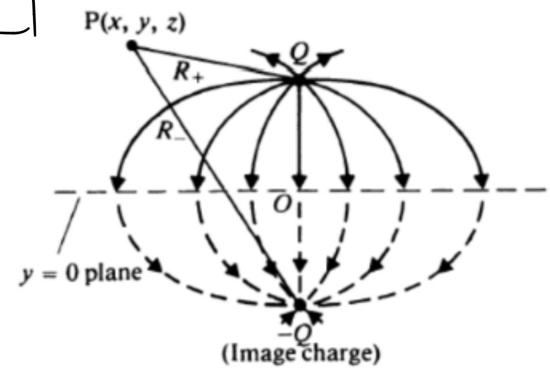
Case 1: Point Charge and Grounded Plane Conductor

$E_n \cdot ds = \frac{\sigma(r) \cdot ds}{\varepsilon_0}$





(a) Physical arrangement.



(b) Image charge and field lines.

Bondary condition:

$\nabla^2 V = 0$ is valid for all points above the plane. except $(0, d, 0)$

$V(x, 0, z) = 0$

Image charge $-Q$, at $(0, -d, 0)$

We find if there is an charge $-Q$ at $(0, -d, 0)$. All Boundary condition is satisfied.

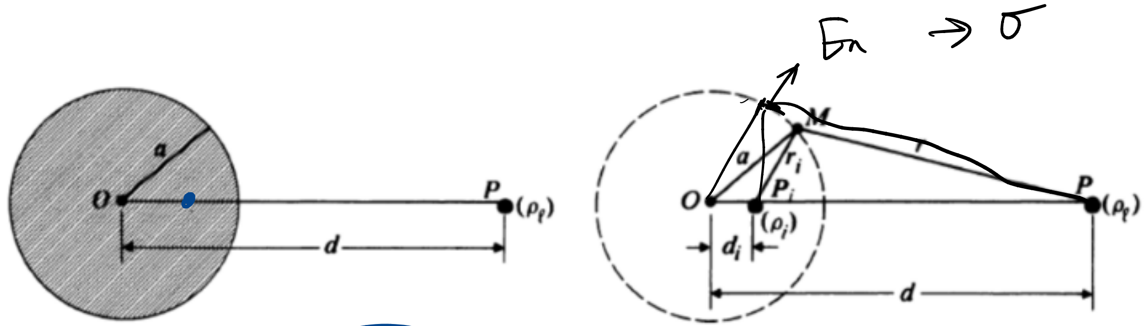
According to Uniqueness Theorem, this is the only solution for this electrostatic

Notice: the image charge is just an equivalent form of σ induced.

All charge are on the surface of the plane.

How does these charge comes from? To satisfy BC. the ground will give charge to the plane.

Case 2: Line Charge and Parallel Conducting Cylinder



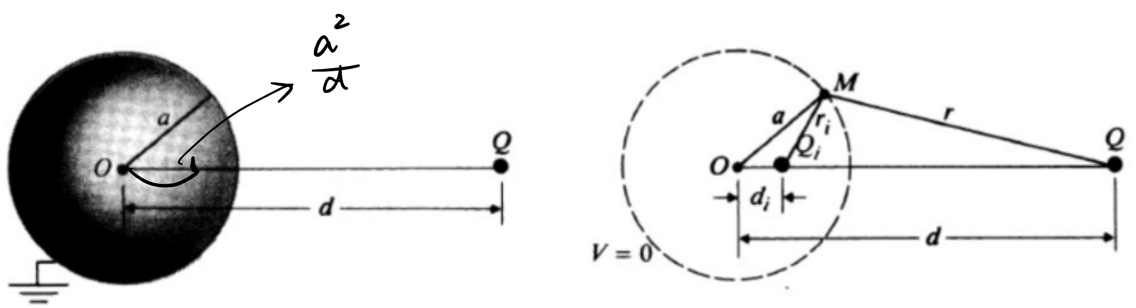
(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.

Bondary condition. $V(a) =$ Constant.

Image charge: $-\rho_\ell$, at $d_i = \frac{a^2}{d}$ away from point O.

Also, we can verify this satisfy bondary condition.

Case 3: Point Charge and Conducting Sphere



(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

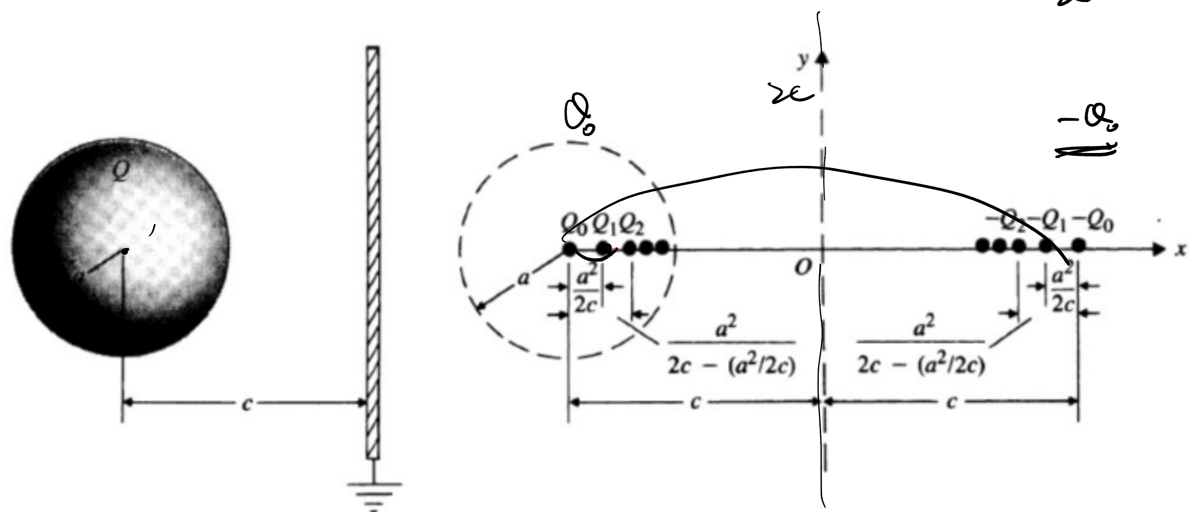
Bondary condition. $V(a) =$ Constant.

Image charge: $-\frac{a}{d}Q$, at $d_i = \frac{a^2}{d}$ away from point O.

Also, we can verify this satisfy bondary condition.

Case 4: Charged Sphere and Grounded Plane

$\frac{a^2}{2c}$ $\quad -\frac{a}{d}(-Q_0) = \frac{aQ_0}{2c}$, $d = 2c$



(a) Physical arrangement. (b) Two groups of image point charges.

Bondary condition. $V(a) = \text{Constant}$, $V_{plane} = 0$. initial Q_0. $(-c, 0)$

Image charge: Infinite many induced Charges

① to make $V_{plane} = 0$ $\quad -Q_0$. at $(c, 0)$

② to make $V(a) = \text{Constant}$ $\quad Q_1 = \frac{a}{2c} Q_0$ at $(-c + \frac{a^2}{2c}, 0)$

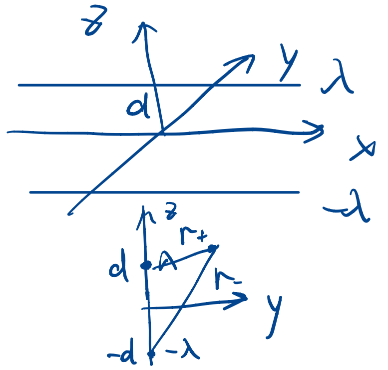
③ to make $V_{plane} = 0$. $\quad -Q_1$ at $(c_1 - \frac{a^2}{2c}, 0)$

Ex4.3

An infinitely long wire is uniformly charged with linear charge density of λ. The distance between the wire and the ground conductor plate is d. (The wire can be arranged parallel to the x axis and located above the x axis, and the conductor plate is the xy plane)

(a) Calculate the potential above the conductor plate.

(b) Calculate the surface density of the induced charge above the conductor plate is obtained.



(a) $V(x, y, z) = V(y, z) = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{r_\infty}{r_+} - \ln\frac{r_\infty}{r_-}\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_-}{r_+}\right)$

$= \frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}}$

(b) $\left.\begin{array}{l} E = \frac{\sigma}{\varepsilon_0} \\ E = -\frac{\partial V}{\partial z} \end{array}\right\} \Rightarrow \sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{\lambda d}{\pi(y^2+d^2)}$

$\boxed{\text{Check.}}$ $\lambda_{induced} = -\frac{\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{1}{y^2+d^2}\,dy = \frac{-\lambda d}{\pi}\frac{1}{(\frac{y}{d})^2+1}\frac{1}{d}\,d(\frac{y}{d})$, t, $t = \tan\theta$

$$-\frac{\lambda d}{\pi}\cdot\frac{1}{d}\cdot\int_{-\infty}^{\infty}\frac{1}{\tan^2\theta+1}\,d(\tan\theta)$$

$$= -\frac{\lambda}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\cdot\frac{1}{\cos^2\theta}\,d\theta = -\frac{\lambda}{\pi}\cdot\pi = -\lambda.$$

<u>verified</u>.