# Steady Magnetic Fields

## 1 Fundamental Postulates

| differential form | integral form | Comment |
|---|---|---|
| $\nabla \cdot \boldsymbol{B} = 0$ | $\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$ | $\boldsymbol{B}$ is solenoidal |
| $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ | $\oint_C \boldsymbol{B} \cdot dl = \mu_0 I$ | Ampere's circuital law ✓ |

where $\mu_0$ is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \quad H/m.$

**Conversion of magnetic flux:** no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves.

Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \boldsymbol{J} = \frac{\nabla \cdot (\nabla \times \boldsymbol{B})}{\mu_0} = 0$$

$$J = \frac{\nabla \times \vec{B}}{\mu_0}$$

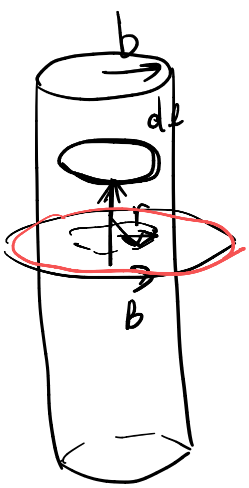
which is consistent with the formula

$$\nabla \cdot \boldsymbol{J} = \overset{=0.}{\frac{\partial \rho}{\partial t}} = 0$$

for steady current.

**Ex6.1**

An infinitely long, straight conductor with a circular cross section of radius $b$ carries a steady current $I$. Determine the magnetic flux density both inside and outside the conductor.

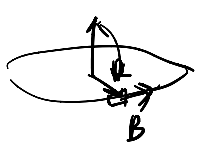



$$J = \frac{I}{2\pi b^2}$$

① $r<b$

$$\int \vec{B} \cdot d\vec{l} = |\vec{B}| \cdot 2\pi r = \mu_0 \cdot J \cdot \pi r^2$$

$$\Rightarrow \vec{B} = \frac{\mu_0 J r}{2} \hat{a}_\theta$$




② $r \geq b$

$$\int \vec{B} \cdot dl = |\vec{B}| \cdot 2\pi r = \mu_0 I$$

$$\Rightarrow |\vec{B}| = \frac{\mu_0 I}{2\pi r} \qquad \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{a}_\theta$$

## 2 Vector Magnetic Potential & Biot-Savart Law

As $\nabla \cdot \boldsymbol{B} = 0$, $\boldsymbol{B}$ is solenoidal, thus could be expressed as:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (T) \tag{1}$$

where $\boldsymbol{A}$ is called the **vector magnetic potential**.
Magnetic flux $\Phi$:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot dl$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \boldsymbol{A} = 0$, we have

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) \\ &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\ &= -\nabla^2 \boldsymbol{A} \end{aligned}$$

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

The solution is then

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv' \tag{2}$$

For a thin wire with cross-sectional area $S$, $dv' = Sdl'$, current flow is entirely along the wire, we then have

$$\boldsymbol{J} dv' = JSdl' = Idl'$$

Thus, Eq 2 becomes:

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{dl'}{R}$$

Based on this form and properties of differentiation, we can get **Biot-Savart law**:

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2}$$

The formula for Biot-Savart law could also be written as:

$$\boldsymbol{B} = \oint_{C'} d\boldsymbol{B}$$

and

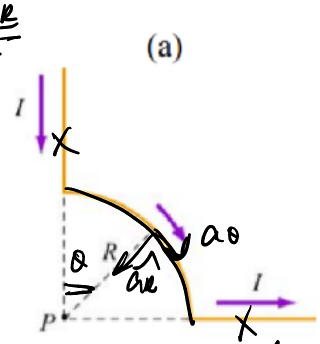
$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \left( \frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2} \right) = \frac{\mu_0 I}{4\pi} \left( \frac{d\boldsymbol{l'} \times \boldsymbol{R}}{R^3} \right)$$

**Comment**: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine $\boldsymbol{B}$ from $I$ in a circuit if a closed path cannot be found where $\boldsymbol{B}$ has a constant magnitude.

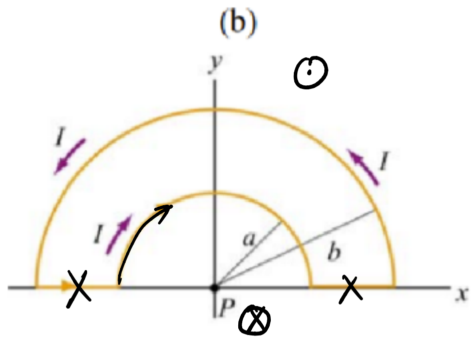
**Ex6.2**

Find the magnetic field at $P$ due to the following current distribution by using Biot-Savart Law.

$$\vec{dB} = \frac{\mu_0 I}{4\pi}\frac{dl \times \hat{a}_R}{R^2}$$




$\hat{a}_z$ ⊗




⊗ $\hat{a}_z$

(a) $$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{R\,d\theta\,\hat{a}_\theta \times \hat{a}_R}{R^2}$$

$$= \frac{\mu_0 I}{4\pi}\frac{d\theta}{R}\hat{a}_z$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\frac{\pi}{2}}{R} = \frac{\mu_0 I}{8R}\hat{a}_z$$

(b) $R = a$: $$d\vec{B}_1 = \frac{\mu_0 I}{4\pi}\frac{d\theta}{a}\hat{a}_z$$

$R = b$, $$d\vec{B}_2 = \frac{\mu_0 I}{4\pi}\frac{d\theta}{b}(-\hat{a}_z)$$

$$\vec{B} = \int_0^\pi d\vec{B}_1 + \int_0^\pi dB_2 = \frac{\mu_0 I}{4\pi}\left(\frac{\pi}{a} - \frac{\pi}{b}\right)$$

$$= \frac{\mu_0 I}{4}\left(\frac{b-a}{ab}\right)\hat{a}_z$$

## 3 Magnetic Field of Some Common Construction

1. Similarly for magnetic flux density at **the center of a square loop**, with side $w$ carrying a direct current $I$, is:

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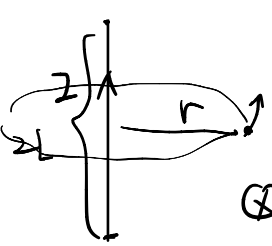
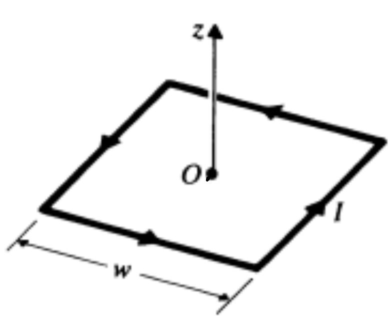


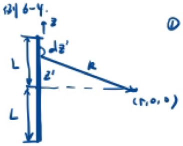

⊗ $\hat{a}_z$

$$\vec{B} = \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}\hat{a}_z$$

$$\boldsymbol{B} = \boldsymbol{a}_z\frac{2\sqrt{2}\mu_0 I}{\pi w}$$



例 6-4.




① 先求矢量磁位 $\vec{A}$，再由 $\vec{B}=\nabla\times\vec{A}$ 求 B

$$\vec{A}=\hat{a}_z\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz'}{\sqrt{z'^2+r^2}}$$

$$=\hat{a}_z\frac{\mu_0 I}{4\pi}\left[\ln\left(z'+\sqrt{z'^2+r^2}\right)\right]\Big|_{-L}^{L}$$

$$=\hat{a}_z\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$$

$$\Rightarrow \vec{B}=\nabla\times A=\nabla\times(a_z A_z)=\hat{a}_r\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\hat{a}_\phi\frac{\partial A_z}{\partial r}=-\hat{a}_\phi\frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right]=\boxed{\hat{a}_\phi\frac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}}$$

$$\text{if } r\ll L \Rightarrow \vec{B}_\phi=a_\phi\frac{\mu_0 I}{2\pi r}$$

② 用毕奥一萨伐尔定律

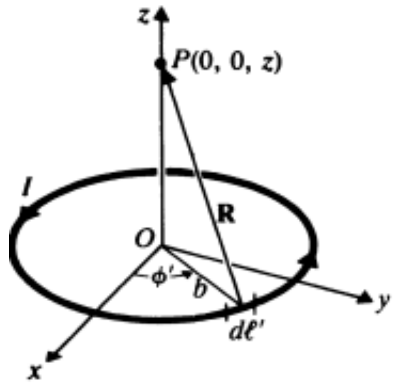
$$\vec{R}=\hat{a}_r r-\hat{a}_z z'$$

$$d\vec{l}'\times\vec{R}=\hat{a}_z dz'\times(a_r r-a_z z')=\hat{a}_\phi r\,dz'$$

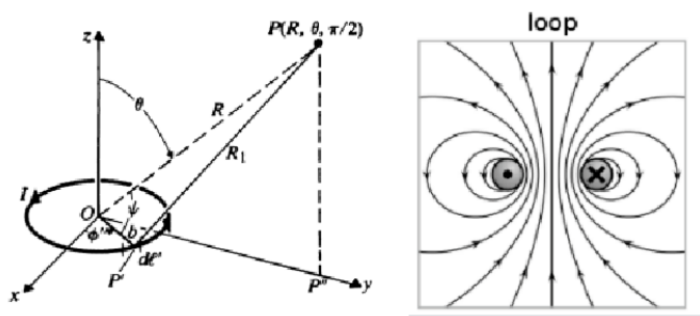
$$\vec{B}=\int d\vec{B}=\hat{a}_\phi\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{r\,dz'}{(z'^2+r^2)^{\frac{3}{2}}}=\boxed{\hat{a}_\phi\frac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}}$$

2. For magnetic flux density ata point on the axis of a circular loop of radiu $b$ that carries a direct current $I$,

$$\boldsymbol{B} = \boldsymbol{a}_z \frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$$


# 4 Magnetic Dipole



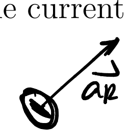

(a) Visualization (b) Cross-section

**Definition of the magnetic dipole:** We call a small current-carrying loop a magnetic dipole

$$\boldsymbol{m} = I \int d\boldsymbol{S}$$

The direction is determined by the right-hand rule. (along with the current direction)

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{a_R}}{4\pi R^2}$$




In spherical coordinates, the vector potential of a magnetic dipole can be written as

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 m \sin\theta}{4\pi R^2} \boldsymbol{a_\phi}$$

Hence, we can compute the magnetic field of a magnetic dipole

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \nabla \times \boldsymbol{A} = \frac{\mu_0 m}{4\pi R^3}(2\cos\theta \boldsymbol{a_r} + \sin\theta \boldsymbol{a_\theta})$$

Written in a coordinate-free form,

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \frac{\mu_0}{4\pi R^2}[3(\boldsymbol{m}\cdot\boldsymbol{a_r})\boldsymbol{a_r} - \boldsymbol{m}]$$

$E_{dip}(\vec{R}) = \frac{1}{4\pi\epsilon R^2}[3(\vec{p}\cdot\hat{a}_r)\hat{a}_r - \vec{p}]$

Compared with the electric field density of an electric dipole, we can find that we just replace $\frac{1}{\epsilon_0}$ with $\mu_0$, and replace $\boldsymbol{p}$ with $\boldsymbol{m}$.

# 5 Scalar magnetic potential

If a region is current free, i.e. $\boldsymbol{J} = 0$,

$$\nabla \times \boldsymbol{B} = 0$$

thus $\boldsymbol{B}$ can be expressed as the gradient of a scalar field.

Assume

$$\boldsymbol{B} = -\mu_0 \nabla V_m$$

$E = -\nabla V$

where $V_m$ is called the **scalar magnetic potential**, the negative sign is conventional, $\mu_0$ is the permeability of free space.

Thus, between two points $P_1, P_2$,

analog

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0}\boldsymbol{B}\cdot dl$$

$V_2 - V_1 = -\int_{P_1}^{P_2} E\cdot dl$

If there were magnetic charges with a volume density $\rho_m$ in a volume $V'$, we could find $V_m$ from:

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\rho_m}{R} dv'$$

**Notice:** This is only a mathematical model, isolated magnetic charges have never been found. Then we could obtain $\boldsymbol{B}$ by Eq 5.

For a bar magnet the fictitous magnetic charges $+q_m, -q_m$ assumed to be separated by $d$ (magnetic dipole), the scalar magnetic potential $V_m$ is given by:

$\vec{m} = -q_m \cdot d$

$$V_m = \frac{\boldsymbol{m}\cdot\boldsymbol{a_R}}{4\pi R^2}$$

and it holds at any points with no currents.

# 6 Magnetization and Equivalent Current Densities

## 6.1 Basics

Define **magnetization vector**, $\boldsymbol{M}$, as

$$\boldsymbol{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{m_k}}{\Delta v}$$

which is the volume density of magnetic dipole moment,

1. The effect of magnetization is vector is equivalent to both
   (a) a volume current density:
   $$\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$$
   (b) a surface current density:
   $$\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$$
2. Then we can determine $A$ by:
$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\nabla' \times \boldsymbol{M}}{R}dv' + \frac{\mu_0}{4\pi}\oint_{S'}\frac{\boldsymbol{M}\times\boldsymbol{a}'_{\boldsymbol{n}}}{R}ds'$$
3. Then we could obtain $\boldsymbol{B}$ from $\boldsymbol{A}$. $\nabla \times \vec{A}$

## 6.2 Equivalent Magnetization Charge Densities

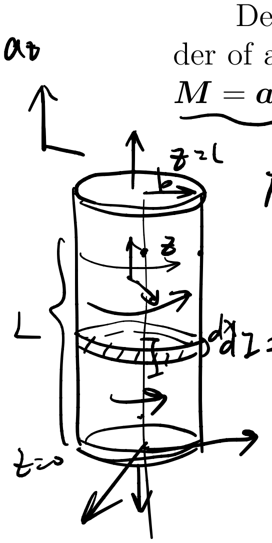
In current-free region, a manetized body may be replaced by

1. an equivalent/fictitous magnetization surface charge density
$$\rho_{ms} = \boldsymbol{M}\cdot\boldsymbol{a_n}$$
2. an equivalent/fictitous magnetization volume charge density
$$\rho_m = -\nabla\cdot\boldsymbol{M}$$

**Ex6.3**

$\vec{B}$

Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius $b$, length $L$, and axial magnetization $\boldsymbol{M} = \boldsymbol{a_z}M_0$.




$\vec{M} = \hat{a}_z M_0$

$dI = J_{ms}\cdot dz'$

① J.

$\vec{J}_m = \nabla\times\vec{M} = 0$

$\vec{J}_{ms} = \vec{M}\times\vec{a}_n = M_0\hat{a}_\phi$

$$d\vec{B} = \frac{\mu_0 M_0 dz' b^2}{2[(z-z')^2+b^2]^{3/2}}\hat{a}_z$$

$$\vec{B} = \hat{a}_z\frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2+b^2}} - \frac{z-L}{\sqrt{(z-L)^2+b^2}}\right]$$

②

top $\rho_{ms} = M\cdot a_n = M_0$

bottom $\rho_{ms} = M\cdot a_n' = -M_0$

bottom $V_m = \frac{M_0}{2}\left(\sqrt{b^2+z^2} - z\right)$

top $V_m = \frac{M_0}{2}\left[\sqrt{b^2+(z-L)^2} - (z-L)\right]$

$V_m = V_{m\,top} + V_{m\,bottom}$

$B = -\mu_0\nabla V_m$

=

P.