RC1

1 Vectors

- dot product:

$$\vec{A} \cdot \vec{B} = ABcos\theta_{\vec{A}\vec{B}}$$

$\vec{A} \cdot \vec{B}$ scalar

  - Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
  - Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
  - Not associative: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot \vec{C}$
  - For the three edges A, B, C in a triangle, $C^2 = A^2 + B^2 - 2ABcos(\theta_{A,B})$
  - In component form, the dot product can be written as

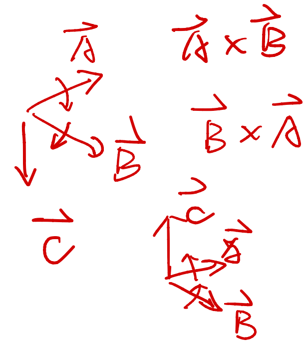
$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

- cross product:

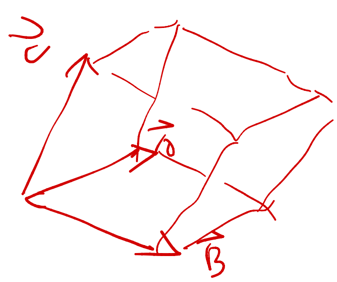
$\vec{A} \times \vec{B}$ vector

$$\vec{A} \times \vec{B} = \vec{a_n}|ABsin\theta_{\vec{A}\vec{B}}|$$

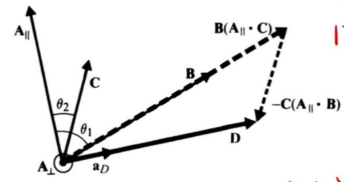
  - The cross product is always perpendicular to both $\vec{A}, \vec{B}$, the direction follows right hand rule.
  - Not commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$
  - Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
  - Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$



- $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = Volume$



- BAC-CAB: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$



$\vec{D} = \vec{A} \times (\vec{B} \times \vec{C}) = \vec{A}_\parallel \times (\vec{B} \times \vec{C})$

$|\vec{D}| = |\vec{A}_\parallel||\vec{B}||\vec{C}| \sin(\theta_1 - \theta_2)$

$= \underbrace{(|\vec{B}|\sin\theta_1)}_{\vec{B}\cdot\vec{a}_D} \underbrace{|\vec{A}_\parallel||\vec{C}|\cos\theta_2}_{\vec{A}_\parallel\cdot\vec{C}} - \underbrace{(|\vec{C}|\sin\theta_2)}_{\vec{C}\cdot\vec{a}_D} \underbrace{|\vec{A}_\parallel||\vec{B}|\cos\theta_1}_{\vec{A}_\parallel\cdot\vec{B}}$

$\Rightarrow |\vec{D}| = \left(\vec{B}\cdot(\vec{A}_\parallel\cdot\vec{C}) - \vec{C}\cdot(\vec{A}_\parallel\cdot\vec{B})\right)\vec{a}_D = \vec{D}\cdot\vec{a}_D$

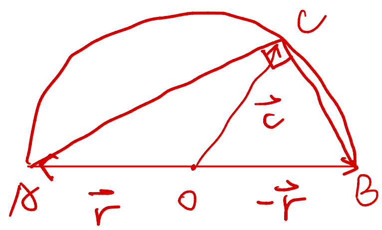
$\left(\vec{B}\cdot(\vec{A}_\parallel\cdot\vec{C}) - \vec{C}\cdot(\vec{A}_\parallel\cdot\vec{B})\right)\vec{A}_\parallel = 0 \Rightarrow$ no components on $\vec{a}_{D\perp}$ direction! ☆

$\vec{D} = \vec{B}\cdot(\vec{A}_\parallel\cdot\vec{C}) - \vec{C}\cdot(\vec{A}_\parallel\cdot\vec{B})$

credits to former TA Jiajun Sun & Pingchuan Ma.

1.1 Exercise

- (HW1-1) Prove that an angle inscribed in a semicircle is a right angle using vector analysis.



$\vec{CA}\cdot\vec{CB} = 0$

$\vec{CA} = \vec{r} - \vec{c}$, $\vec{CB} = -\vec{r} - \vec{c}$

$\vec{CA}\cdot\vec{CB} = (\vec{r} - \vec{c})\cdot(-\vec{r} - \vec{c})$

$= -|\vec{r}|^2 - \cancel{\vec{r}\cdot\vec{c}} + \cancel{\vec{r}\cdot\vec{c}} + |\vec{c}|^2$

$= -R^2 + R^2 = 0$

$\vec{CA} \perp \vec{CB}$

2 Coordinates

Three basis (u_1, u_2, u_3): number of linearly independent basis = dimension of the space. For the three types of coordinates we discuss, u_i is orthogonal to each other.

For arbitrary vector $\vec{A}$:

$$\vec{A} = \vec{a_{u1}} A_{u1} + \vec{a_{u2}} A_{u2} + \vec{a_{u3}} A_{u3}$$

,

Norm of $\vec{A}$:

$$|\vec{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

Cartesian

$|\vec{A}| = \sqrt{x^2 + y^2 + z^2}$

$A_x^2 + A_y^2 + A_z^2$

For a differential length dl,

$$dl = \vec{a_{u1}}(h_1 du_1) + \vec{a_{u2}}(h_2 du_2) + \vec{a_{u3}}(h_3 du_3)$$

, h_i is called metric coefficient.

metric coefficient

differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

differential area vector with a direction normal to the surface,

$$d\vec{s} = \vec{a_n} ds$$

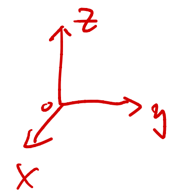
differential area ds_1 normal to the unit vector $\vec{a_{u1}}$.

$$ds_1 = h_2 h_3 du_2 du_3 \tag{1}$$

Note that for ds_i, the foot indices on the right hand side are the ones that don't show up on the left hand side.

2.1 Cartesian Coordinates

- $$(u_1, u_2, u_3) = (x, y, z)$$



- Right hand rule:

$$\vec{a_x} \times \vec{a_y} = \vec{a_z}$$

$\vec{a_x} \times \vec{a_y} = \vec{a_z}$

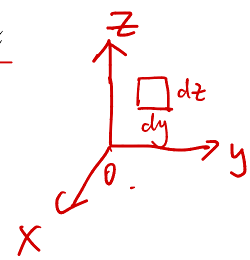
- $$\vec{A} = \vec{a_x} A_x + \vec{a_y} A_y + \vec{a_z} A_z$$

$\vec{a_x} \cdot \vec{a_y} = 0$

, where $\vec{a_i}$ is the basis for i-axis.

- differential length:

$$d\vec{l} = \vec{a_x} dx + \vec{a_y} dy + \vec{a_z} dz \tag{2}$$



- differential area:

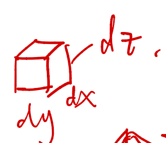
$$ds_x = dydz$$

, as $h_1 = h_2 = h_3 = 1$,
(ds_x is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).

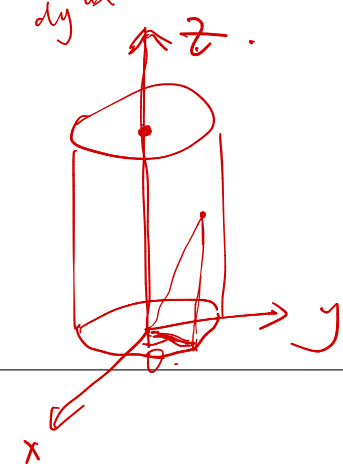
$ds_y = dxdz \quad ds_z = dxdy$

- differential volume:

$$dv = dxdydz$$



2.2 Cylindrical Coordinate



- $$(u_1, u_2, u_3) = (r, \phi, z)$$

- Right hand rule:

$$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$$

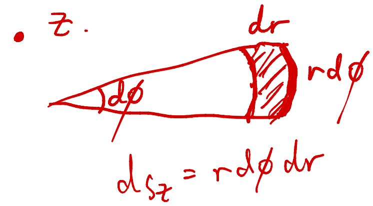
- $$\vec{A} = \vec{a_r}Ar + \vec{a_\phi}A_\phi + \vec{a_z}A_z$$

- differential length:

$$d\vec{l} = \vec{a_r}dr + \vec{a_\phi}rd\phi + \vec{a_z}dz \tag{3}$$

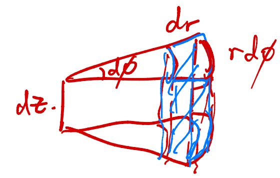
r

, as $h_1 = 1, h_2 = r, h_3 = 1$



- differential area:

$$ds_r = rd\phi dz \quad ds_z = rd\phi dr \quad ds_\phi = drdz$$



- differential volume:

$$dv = rdrd\phi dz$$

- From cylindrical coordinate to Cartesian coordinate:

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}}$$

- conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

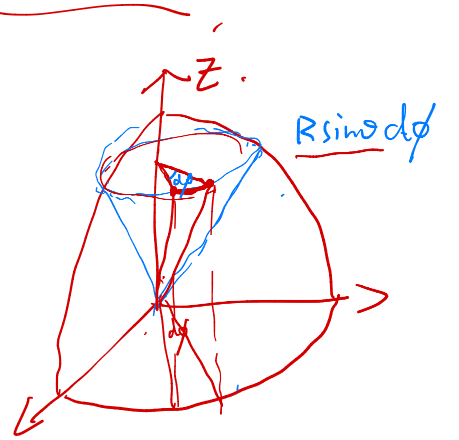
a)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

b)

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \arctan\frac{y}{x} \\ z = z \end{cases}$$

2.3 Spherical Coordinate



- $$(u_1, u_2, u_3) = (R, \theta, \phi)$$

- Right hand rule:

$$\vec{a_R} \times \vec{\theta} = \vec{\phi}$$

- $$\vec{A} = \vec{a_R}A_R + \vec{a_\theta}A_\theta + \vec{a_\phi}A_\phi$$

- differential length:

R θ φ

$$d\vec{l} = \vec{a_R}dR + \vec{a_\theta}Rd\theta + \vec{a_\phi}R\sin\theta d\phi \tag{4}$$

, as $h_1 = 1, h_2 = R, h_3 = R\sin\theta$.

- differential area:

$$ds_R = R^2\sin\theta d\theta d\phi \quad ds_\theta = R\sin\theta dRd\phi \quad ds_\phi = RdRd\theta$$

- differential volume:

$d R \, d\theta \, d\phi$

$$dv = R^2 \sin\theta dR d\theta d\phi$$

- conversion of quantities between Cartesian coordinate and Spherical coordinate:

a)
$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

b)
$$\begin{cases} R = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \\ \phi = \arctan\frac{y}{x} \end{cases}$$

- From Spherical coordinate to Cartesian coordinate:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

2.4 Exercise

- (HW1-2) A field is expressed in spherical coordinates by $\mathbf{E} = \mathbf{a_R}(25/R^2)$.

  a) Find $|\mathbf{E}|$ and E_x at point $P(-3, 4, -5)$.

  b) Find the angle that $\mathbf{E}$ makes with the vector $\mathbf{B} = \mathbf{a_x}2 - \mathbf{a_y}2 + \mathbf{a_z}$ at point P.

a) $\vec{E} = \vec{a}_R \frac{25}{(-3)^2+4^2+(-5)^2} = \frac{1}{2}\vec{a}_R$ $\quad |\vec{E}| = \frac{1}{2}$ $\quad \frac{1}{2} \cdot \frac{-3}{\sqrt{(-3)^2+4^2+(-5)^2}} = E_x$

b) $\vec{E}$ $\vec{B}$ unit vector

$\vec{a}_E = \frac{1}{\sqrt{50}}(-3, 4, -5)$ $\quad \vec{a}_B = \frac{1}{\sqrt{2^2+(-2)^2+1}} \cdot (2, -2, 1)$

$\cos\theta = \vec{a}_E \cdot \vec{a}_B$ — —

3 Vector Calculus

3.1 Integrals

- Line integral: $\int_C \mathbf{F} \cdot d\mathbf{l}$, where $\mathbf{F}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (e.g. $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$). If we are integrating over a closed loop, we write $\oint \mathbf{F} \cdot d\mathbf{l}$.
- Surface integral: $\int_S \mathbf{F} \cdot d\mathbf{S}$. If we are integrating over a closed surface, we write $\oint \mathbf{F} \cdot d\mathbf{S}$.
- Volume integral: $\int_{\mathcal{V}} \mathbf{F} dv$. In Cartesian coordinates, $dv = dxdydz$. Specially, $\int_{\mathcal{V}} \mathbf{F} dv = \int_{\mathcal{V}} (F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}) dv = \hat{\mathbf{x}} \int_{\mathcal{V}} F_x dv + \hat{\mathbf{y}} \int_{\mathcal{V}} F_y dv + \hat{\mathbf{z}} \int_{\mathcal{V}} F_z dv$

3.1.1 Exercise

- EXAMPLE 2-14: Given $\vec{F} = \vec{a_x}xy - \vec{a_y}2x$, evaluate the scalar line integral along the quarter-circle shown in Fig. 2-21.

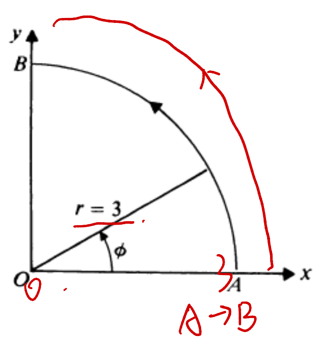


FIGURE 2–21
Path for line integral (Example 2–14).

$x^2 + y^2 = 9$

$y = \sqrt{9 - x^2}$

$(\sqrt{9-x^2})'$

$\frac{-2x}{2\sqrt{9-x^2}} dx$

$\vec{a}_x \cdot \vec{a}_y = 0$

  - By Cartesian coordinate:

$\vec{F} \cdot d\vec{l} = (\vec{a}_x xy - \vec{a}_y 2x)(\vec{a}_x dx + \vec{a}_y dy)$

$= x\sqrt{9-x^2}\, dx - 2x\, d\sqrt{9-x^2}$

$\int_3^0 x\sqrt{9-x^2}\, dx - 2x\, d\sqrt{9-x^2} = \cdots = -9(1+\frac{\pi}{2})$

  - By Cylindrical coordinate:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$

express A_r, A_ϕ, A_z with A_x, A_y, A_z
xy, $-2x$, 0

3.2 Gradient: How Fast a Scalar Field Changes

First, consider what is the difference between a scalar quantity and a scalar field? Between a vector quantity and a vector field?

mountain.

$$\nabla V = \vec{a_n}\frac{dV}{dn}$$

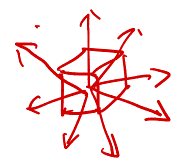
∇V at certain point is a vector.

$$\nabla V = \vec{a_{u1}}\frac{\partial V}{h_1 \partial u_1} + \vec{a_{u2}}\frac{\partial V}{h_2 \partial u_2} + \vec{a_{u3}}\frac{\partial V}{h_3 \partial u_3}$$

$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$

3.3 Divergence of a vector field

- Divergence of a vector field $\vec{A}$ at a point $div\vec{A}$ as the net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero:



$$div\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\,ds}{\Delta v} \quad (5)$$

Δv

source: net positive divergence; sink: net negative divergence. zero divergence: no source/sink.

- $div\vec{A}$ at certain point is a scalar.
- For Cartesian coordinate,

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- $\nabla \cdot \vec{A} \equiv div\vec{A}$ $\quad \nabla \cdot \vec{A}$
- $$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

3.3.1 Exercise

- (HW1-4) Find the divergence of the following radial vector fields:

  a) $f_1(\mathbf{R}) = \mathbf{a_R} R^n$,

  b) $f_2(\mathbf{R}) = \mathbf{a_R} \frac{k}{R^2}$.

In spherical coordinates, $\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R)$ if $\vec{A} = \vec{a_R} A_R$

a) $\vec{A} = f_1(\vec{R}) = \vec{a_R} R^n$, $A_R = R^n$

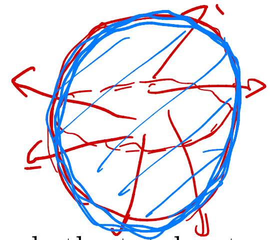
$\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}$

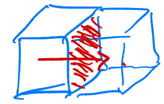
b) $\vec{A} = f_2(\vec{R}) = \vec{a_R}\frac{k}{R^2}$, $A_R = \frac{k}{R^2}$

$\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0$

3.4 Divergence Theorem

3D → 2D.





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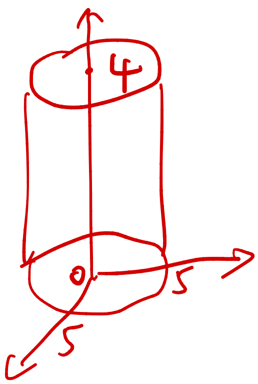
$$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

3.4.1 Exercise

- (HW1-5) For vector function $\mathbf{A} = \mathbf{a_r} r^2 + \mathbf{a_z} 2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r = 5$, $z = 0$, and $z = 4$.



$\nabla \cdot \vec{A} = \frac{1}{r}\left(\frac{\partial}{\partial r} r^3 + \frac{\partial}{\partial z} 2rz\right) = 3r + 2 \Rightarrow \int_V \nabla\cdot\vec{A}\, dv = \int_0^4 \int_0^{2\pi} \int_0^5 (3r+2)\, r\, dr\, d\phi\, dz = 1200\pi.$

top face $(z=4)$: $\vec{A} = \vec{a}_r r^2 + \vec{a}_z 8$, $d\vec{s} = \vec{a}_z ds$

$\int \vec{A}\cdot d\vec{s} = \int 8 ds = 8\cdot(\pi 5^2) = 200\pi$

bottom face $(z=0)$: $\vec{A} = \vec{a}_r r^2$, $d\vec{s} = -\vec{a}_z ds \Rightarrow \int \vec{A}\cdot d\vec{s} = 0$

wall $(r=5)$: $\vec{A} = \vec{a}_r 25 + \vec{a}_z 2z$, $d\vec{s} = \vec{a}_r ds$

$\int \vec{A}\cdot d\vec{s} = 25\int ds = 25(2\pi\times 5\times 4) = 1000\pi$

$\oint_S \vec{A}\cdot d\vec{s} = 1000\pi + 0 + 200\pi = 1200\pi = \int_V \nabla\cdot\vec{A}\, dv$

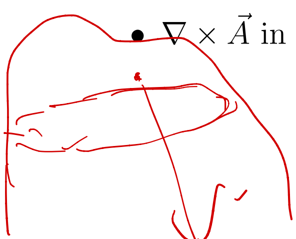
3.5 Curl of a vector field

$\nabla \times \vec{A}.$

- $$curl\vec{A} \equiv \nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} [\vec{a_n} \oint_C \vec{A} \cdot d\vec{l}]_{max}$$

  $\Delta s \to 0$.

  The curl of a vector field $\vec{A}$, denoted by $curl\vec{A}$ or $\nabla \times \vec{A}$, is a vector whose magnitude is the maximum net circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).



- $\nabla \times \vec{A}$ in a general coordinate:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a_{u1}} h_1 & \vec{a_{u2}} h_2 & \vec{a_{u3}} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- curl-free vector field ($\nabla \times \vec{A} = 0$): **irrotational** or **conservative field**

3.5.1 Exercise

- (HW1-3) Given a vector function $\mathbf{E} = \mathbf{a_x} y + \mathbf{a_y} x$, evaluate the scalar line integral $\int \mathbf{E} \cdot dl$ from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$

  a) along the parabola $x = 2y^2$,

$P_2(8,2,-1)$.

$P_1(2,1,-1)$

$x = 6y - 4$

$\vec{a}_x dx + \vec{a}_y dy$.

b) along the straight line joining the two points.

Is this **E** a conservative field?

(a) $x = 2y^2$ $\quad dx = 4y\,dy$

$\int \vec{E}\cdot d\vec{l} = \int (\vec{a}_x y + \vec{a}_y x)(\vec{a}_x dx + \vec{a}_y dy)$

$= \int y\,dx + x\,dy$

$= \int_1^2 (4y^2 dy + 2y^2 dy) = 14$

(b) $x = 6y - 4$, $dx = 6dy$

$\int \vec{E}\cdot d\vec{l} = \int y\,dx + x\,dy$

$= \int_1^2 (6y\,dy + (6y-4)\,dy)$

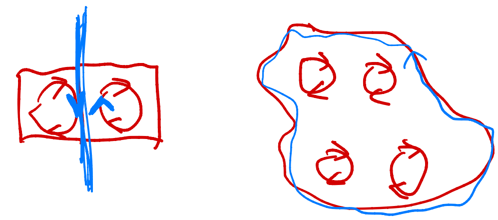
$= 14$

conservative!

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0$$

3.6 Stokes's Theorem

2D → 1D.



$$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

3.6.1 Exercise

- Given $\mathbf{F} = \mathbf{a_x}xy - \mathbf{a_y}2x$, verify Stokes's theorem over a quarter-circular disk with a radius 3 in the first quadrant, as was shown in Fig. 2-21.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -\vec{a}_z(2+x)$$

$$\int_S (\nabla \times \vec{F}) d\vec{S} = \oint_C \vec{F}\cdot d\vec{l},$$

$$\int_S (\nabla\times\vec{F})\, d\vec{s} = \int_0^3 \int_0^{\sqrt{9-x^2}} -\vec{a}_z(2+x)\cdot \vec{a}_z\, dx\,dy = \cdots = \boxed{-9(1+\tfrac{\pi}{2})}$$

From B to O: $x=0$, $\vec{F}\cdot d\vec{l} = \vec{F}\cdot(-\vec{a}_y dy) = 2x\,dy = 0$

From O to A: $y=0$, $\vec{F}\cdot d\vec{l} = \vec{F}\cdot(\vec{a}_x dx) = xy\,dx = 0$

3.7 Null Identities

$\Rightarrow \oint_{ABOA} \vec{F}\cdot d\vec{l} = \int_A^B \vec{F}\cdot d\vec{l} = \boxed{-9(1+\tfrac{\pi}{2})}$

(example 2-14)

$$\nabla \times (\nabla V) \equiv 0$$

- The curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.

$$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

- The divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field.

$$\begin{cases} \nabla \cdot \vec{A} = 0. \\ \vec{A} = \nabla \times \vec{B} \end{cases}$$

3.8 Other useful vector properties

$$\begin{cases} \nabla \times \vec{A} = 0 \\ \vec{A} = \nabla B. \end{cases}$$

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$
$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$$
$$\nabla \times (\psi \mathbf{A}) = \psi(\nabla \times \mathbf{A}) + \nabla\psi \times \mathbf{A}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times (\nabla\psi) = \mathbf{0}$$
$$\nabla \cdot (\nabla\psi) = \nabla^2\psi \text{ (scalar Laplacian)}$$
$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A} \text{ (vector Laplacian)}$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

4 Helmholtz's Theorem

Helmholtz's Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

4.1 Exercise

- (HW1-8) Given a vector function $\mathbf{F} = \mathbf{a_x}(x + c_1 z) + \mathbf{a_y}(c_2 x - 3z) + \mathbf{a_z}(x + c_3 y + c_4 z)$.
  a) Determine the constants c_1, c_2, and c_3 if $\mathbf{F}$ is irrotational.
  b) Determine the constant c_4 if $\mathbf{F}$ is also solenoidal.
  c) Determine the scalar potential function V whose negative gradient equals $\mathbf{F}$.

a) $$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + c_1 z & c_2 x - 3z & x + c_3 y + c_4 z \end{vmatrix} = \vec{a}_x(c_3 + 3) + \vec{a}_y(c_1 - 1) + \vec{a}_z c_2 = 0$$

$$\begin{cases} c_3 + 3 = 0 \\ c_1 - 1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = -3 \end{cases}$$

$$\vec{F} = \vec{a}_x(x+z) - \vec{a}_y 3z + \vec{a}_z(x-3y+C_4 z)$$

$$\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(3z) + \frac{\partial}{\partial z}(x-3y+C_4 z).$$

(b). $\nabla \cdot \vec{F} = 1 + 0 + C_4 = 0 \Rightarrow C_4 = -1$

(c). $\vec{F} = \vec{a}_x(x+z) - \vec{a}_y 3z + \vec{a}_z(x-3y-z)$.

$$-V = \frac{1}{2}x^2 + xz + f(y,z).$$

$$\frac{\partial}{\partial y} f(y,z) = -3z.$$

$$f(y,z) = -3yz + g(z).$$

$$-V = \frac{1}{2}x^2 + xz - 3yz + g(z).$$

$$\frac{\partial}{\partial z} = x - 3y + g'(z)$$

$$g(z) = -\frac{1}{2}z^2 + C$$

$$-V = \frac{1}{2}x^2 + xz - 3yz - \frac{1}{2}z^2$$

$$V = -\frac{1}{2}x^2 - xz + 3yz + \frac{1}{2}z^2 + C.$$