RC 3: Dielectrics and Boundary Condition

1 Conductors and dielectrics in static electric field

- conductors:
  - Orbiting electrons are loosely held by an atom and migrate easily from on atom to another.
  - **static state conditions**:
    - inside the conductor:
      $$\rho = 0, \boldsymbol{E} = 0$$
      , where $\rho = 0$ represents no charge in the interior
    - on the conductor surface (boundary conditions)
      $$E_t = 0, E_n = \frac{\rho_s}{\epsilon_0}$$
- semiconductos:
  - Relatively small number of freely movable charges.
- insulators(dielectrics):
  - Electrons are confined to their orbits.
  - External electric field $E_{external}$ polarizes a dielectric material and create electric dipoles. The induced electric dipoles (result in an induced electricfiled $E_{induced}$) will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

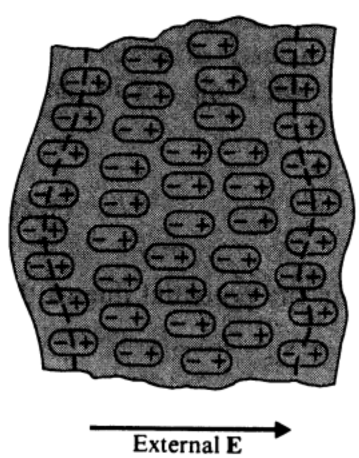


Figure 1: A cross section of a polarized dielectric medium

- **polarization charge densities/ bound-charge densities:**
  * **Polarization vector, $\boldsymbol{P}$** (Measures the density of electric dipoles):

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p_k}}{\Delta v}$$

    where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv.
  * Charge distribution on surface density (Polarization surface charge densities):

$$\rho_{ps} = \boldsymbol{P} \cdot \boldsymbol{a_n}$$

  * Volume charge distribution density (Polarization bound-charge densities):

$$\rho_p = -\nabla \cdot \boldsymbol{P}$$

Ex.1

The polarization in a dielectric cube of side L, centered at the origin is given by $\boldsymbol{P} = P_0(\boldsymbol{a_x}x + \boldsymbol{a_y}y + \boldsymbol{a_z}z)$.

a) Determine the surface and volume bound-charge densities.

b) Show that the total bound charge is zero.

Ex.2

Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\boldsymbol{P}$ exists.

2 Electric Flux Density and Dielectric Constant

- **electric flux density/electric displacement, $\boldsymbol{D}$**:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \quad (C/m^2)$$

- $$\nabla \cdot \boldsymbol{D} = \rho \quad (C/m^3)$$

  where ρ is the volume density of free charges.

- Another form of **Gauss's law**:

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q \quad (\boldsymbol{C})$$

  the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic** (χ_e is independent of the space coordinate),

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$$

$$\boldsymbol{D} = \epsilon_0 (1 + \chi_e) \boldsymbol{E} = \epsilon_0 \epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E}$$

  where χ_e is a dimensionless quantity called electric susceptibility, and ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium and ϵ is the absolute permittivity/permittivity of the medium (F/m).

  Ex.3

  A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine $\boldsymbol{E}, V, \boldsymbol{D}, \boldsymbol{P}$ as functions of the radial distance R.

- For anisotropic,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For biaxial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For uniaxial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

- dielectric breakdown: electric field is very strong, causes permanent dislocations and damage in the material.

  dielectric strength: the maximum electric field intensity that a dielectric material can withstand without breakdown.

3 Boundary Conditions for Electrostatic Fields

- the tangential component of an $\boldsymbol{E}$ field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$

  or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

- The normal component of $\boldsymbol{D}$ field is discontinous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$$\boldsymbol{a_{n2}} \cdot (\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_s$$

  or

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

- In terms of potential,

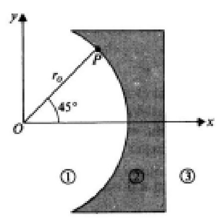
$$V_1 = V_2$$
$$\epsilon_1 \frac{\partial V_1}{\partial n} - \epsilon_2 \frac{\partial V_2}{\partial n} = -\rho_{sf}$$

Ex.4

A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\boldsymbol{E_o} = \boldsymbol{a_x} E_o$ in free space. Determine $vectE_i, \boldsymbol{D_i}, \boldsymbol{P_i}$ inside the lucite.

Ex.5

Dielectric lenses can be used to collimate electromagnetics fields. In Fig **??**, the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\boldsymbol{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\boldsymbol{a_r}5 - \boldsymbol{a_\phi 3}$, what must be the dielectric constant of the lenses in order that $\boldsymbol{E_3}$ in region 3 is parallel to the x-axis?



Ex.6

The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

Ex.7

Two dielectric media with permittivities ϵ_1 and ϵ_2 are separated by a charge-free boundary as shown in Fig 2. The electric field intensity in medium 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the elctric field intensity at point P_2 in medium 2.

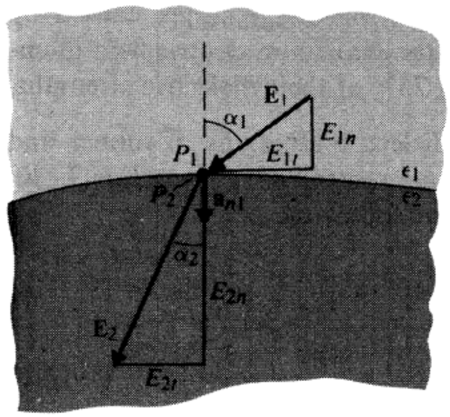


Figure 2: Boundary conditions at the interface between two dielectric media