Q1

1. Step 1: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} = \frac{d}{dx} f(t - R\sqrt{\mu\varepsilon}) \frac{d}{dt}(t - R\sqrt{\mu\varepsilon}) = f'(t - R\sqrt{\mu\varepsilon})$

Step 2: $\frac{\partial u}{\partial R} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial R} = \frac{d}{dx} f(t - R\sqrt{\mu\varepsilon}) \frac{d}{dR}(t - R\sqrt{\mu\varepsilon}) = f'(t - R\sqrt{\mu\varepsilon})(-\sqrt{\mu\varepsilon})$

Step 3: $\frac{\partial^2 u}{\partial t^2} = \frac{d}{dt}[f'(t - R\sqrt{\mu\varepsilon})] = \frac{d}{d(t - R\sqrt{\mu\varepsilon})} f'(t - R\sqrt{\mu\varepsilon}) \frac{d}{dt}(t - R\sqrt{\mu\varepsilon}) = f''(t - R\sqrt{\mu\varepsilon})$

Step 4: $\frac{\partial^2 u}{\partial R^2} = \frac{d}{dR}[f'(t - R\sqrt{\mu\varepsilon})(-\sqrt{\mu\varepsilon})] = (-\sqrt{\mu\varepsilon})[f''(t - R\sqrt{\mu\varepsilon})(-\sqrt{\mu\varepsilon})]$

$= \mu\varepsilon f''(t - R\sqrt{\mu\varepsilon})$

$\Rightarrow \frac{\partial^2 u}{\partial R^2} - \mu\varepsilon \frac{\partial^2 u}{\partial t^2} = 0$

Hence Proved.

Q 1

2.

Maxwell's equation:

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad ①$$

$$\nabla \times H = J_f + \frac{\partial D}{\partial t} \quad ②$$

$$\nabla \cdot B = 0 \quad ③$$

$$\nabla \cdot D = \rho_f \quad ④$$

Also from vector identities:

$$\nabla \times (\nabla V) = 0 \quad ⑤$$

$$\nabla \cdot (\nabla \times A) = 0 \quad ⑥$$

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A \quad ⑦$$

From ③ and ⑥:

$$B = \nabla \times A \quad ⑧$$

Then from ①:

$$\nabla \times \left(E + \frac{\partial B}{\partial t}\right) = 0 \quad ⑨$$

From ⑤ and ⑨:

$$E + \frac{\partial B}{\partial t} = -\nabla V \quad ⑩$$

Since $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

$$\Rightarrow \nabla \times (\nabla \times A) = \mu J_f + \frac{1}{c^2}\left[-\nabla\left(\frac{\partial V}{\partial t}\right) - \frac{\partial^2 A}{\partial t^2}\right] \quad ⑪$$

where $c^2 = \frac{1}{\mu\varepsilon}$

From ⑦ and ⑪

$$-\mu J_f = \nabla^2 A - \nabla\left[\nabla \cdot A + \frac{1}{c^2}\frac{\partial V}{\partial t}\right] - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \quad ⑫$$

If we let $\nabla \cdot A = -\frac{1}{c^2}\frac{\partial V}{\partial t}$ ⑬

$$\Rightarrow \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu J_f \quad ⑭$$

Using ⑩ and ④

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho_f}{\varepsilon} \quad ⑮$$

From ⑬ and ⑭:

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho_f}{\varepsilon} \quad ⑯$$

$$\Leftrightarrow \nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Q2

Maxwell's equation:

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\nabla D = \rho$$

From the question

$$B = \nabla \times A \quad ③$$

$$E = -\nabla V - \frac{\partial A}{\partial t} \quad ④$$

Substitute $\frac{B}{\mu}$ for H in ①:

$$\nabla \times \frac{B}{\mu} = J + \varepsilon \frac{\partial E}{\partial t} \quad ⑤$$

From ③ ④ and ⑤:

$$\nabla \times \left(\frac{\nabla \times A}{\mu}\right) = J - \nabla \varepsilon \frac{\partial V}{\partial t} - \varepsilon \frac{\partial^2 A}{\partial t^2} \quad ⑥$$

consider the gauge condition for potentials in an homogenous medium

$$\nabla . (\varepsilon A) + \mu \varepsilon^2 \frac{\partial V}{\partial t} = 0 \quad ⑦$$

$$\Rightarrow \frac{\partial V}{\partial t} = -\frac{\nabla (\varepsilon A)}{\mu \varepsilon^2}$$

From ⑥:

$$\nabla \times \left(\frac{\nabla \times A}{\mu}\right) = J \cdot \varepsilon \nabla \left(\frac{-\nabla \varepsilon A}{\mu \varepsilon^2}\right) - \varepsilon \frac{\partial^2 A}{\partial t^2}$$

$$-\nabla \times \left(\frac{\nabla \times A}{\mu}\right) + \varepsilon \nabla \left(\frac{-\nabla (\varepsilon A)}{\mu \varepsilon^2}\right) - \varepsilon \frac{\partial^2 A}{\partial t^2} = -J \quad ⑧$$

equ ⑧ represents the wave equation for vector potential.

From ②:

$$\nabla . (\varepsilon E) = \rho$$

Then from ④:

$$\nabla . \left(\varepsilon \left(-\nabla V - \frac{\partial A}{\partial t}\right)\right) = \rho$$

$$\Rightarrow \nabla . \varepsilon \nabla V + \frac{\partial \nabla \varepsilon A}{\partial t} = -\rho \quad ⑨$$

From ⑦: $\nabla . (\varepsilon A) = -\mu \varepsilon^2 \frac{\partial V}{\partial t}$

since $\nabla . (\varepsilon A) = -\mu \varepsilon^2 \frac{\partial V}{\partial t}$

$$\Rightarrow \nabla . (\varepsilon \nabla V) + \frac{\partial \left(-\mu \varepsilon^2 \frac{\partial V}{\partial t}\right)}{\partial t} = -\rho$$

$$\Leftrightarrow \varepsilon \nabla^2 V - \mu \varepsilon^2 \frac{\partial^2 V}{\partial t^2} = -\rho$$

$$\Rightarrow \nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} \quad ⑩$$

equ ⑩ represents the wave equation scalar potential.

From ⑩ and ⑧,

we know the wave equation for vector potential is

$$-J = -\nabla \times \frac{(\nabla \times A)}{\mu} + \varepsilon \nabla \left(\frac{-\nabla (\varepsilon A)}{\mu \varepsilon^2}\right) - \varepsilon \frac{\partial^2 A}{\partial t^2}$$

scalar potential is

$$\nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$