# Mid2 RC Part3

## 1 Lorentz's Force Equation

- **Electric Force:**
$$\boldsymbol{F_e} = q\boldsymbol{E}$$
- **Magnetic Force:**
$$\boldsymbol{F_m} = q\boldsymbol{u} \times \boldsymbol{B}$$
**Notice:** The magnetic force is derived in experiments. Defined as $\boldsymbol{B}$: magnetic flux density ($Wb/m^2$=Tesla)
- Lorentz's Force Equation:
$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

**Ex.1**

A positive point charge $q$ of mass $m$ is injected with a velocity $\boldsymbol{u} = u_{x0}\boldsymbol{a_x} + u_{y0}\boldsymbol{a_y} + u_{z0}\boldsymbol{a_z}$ into the $y > 0$ region where a uniform magnetic field $\boldsymbol{B} = B_0\boldsymbol{a_x}$ and a uniform electric field $\boldsymbol{E} = E_0\boldsymbol{a_z}$ exist. Obtain the equation of motion of the charge.

# 2 Fundamental Postulates

| differential form | integral form | Comment |
|---|---|---|
| $\nabla \cdot \boldsymbol{B} = 0$ | $\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$ | $\boldsymbol{B}$ is solenoidal |
| $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ | $\oint_C \boldsymbol{B} \cdot dl = \mu_0 I$ | Ampere's circuital law |

where $\mu_0$ is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \quad H/m$.
**Conversion of magnetic flux:** no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves.
Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \boldsymbol{J} = \frac{\nabla \cdot (\nabla \times \boldsymbol{B})}{\mu_0} = 0$$

which is consistent with the formula

$$\nabla \cdot \boldsymbol{J} = \frac{\partial \rho}{\partial t} = 0$$

for steady current.

# 3 Vector Magnetic Potential & Biot-Savart Law

As $\nabla \cdot \boldsymbol{B} = 0$, $\boldsymbol{B}$ is solenoidal, thus could be expressed as:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (T) \tag{1}$$

where $\boldsymbol{A}$ is called the **vector magnetic potential**.
Magnetic flux $\Phi$:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot dl$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \boldsymbol{A} = 0$, we have

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) \\ &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\ &= -\nabla^2 \boldsymbol{A} \end{aligned}$$

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

The solution is then

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv' \tag{2}$$

For a thin wire with cross-sectional area $S$, $dv' = Sdl'$, current flow is entirely along the wire, we then have

$$\boldsymbol{J}dv' = JSdl' = Idl'$$

Thus, Eq 2 becomes:

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{dl'}{R}$$

Based on this form and properties of differentiation, we can get **Biot-Savart law**:

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2}$$

The formula for Biot-Savart law could also be written as:

$$\boldsymbol{B} = \oint_{C'} d\boldsymbol{B}$$

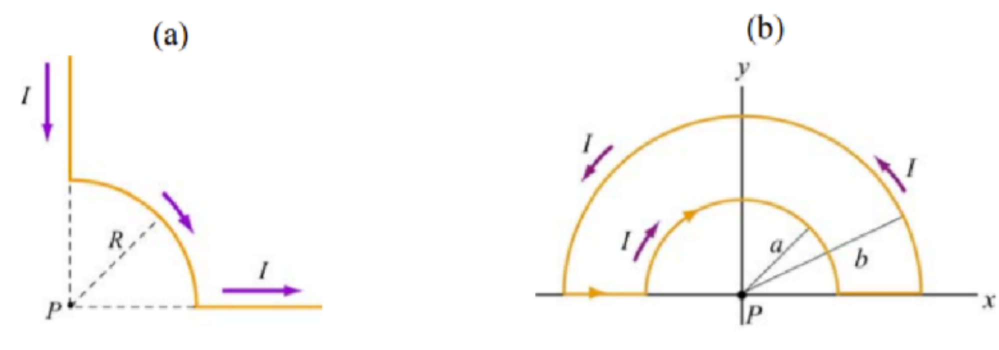
and

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'} \times \boldsymbol{R}}{R^3}\right)$$

**Comment**: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine $\boldsymbol{B}$ from $I$ in a circuit if a closed path cannot be found where $\boldsymbol{B}$ has a constant magnitude.

**Ex.2**

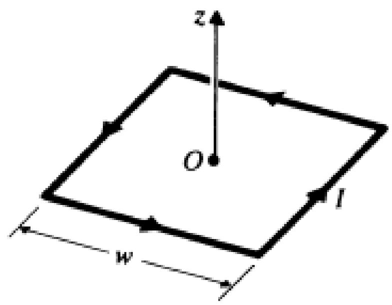
Find the magnetic field at $P$ due to the following current distribution by using Biot-Savart Law.



# 4 Magnetic Field of Some Common Construction

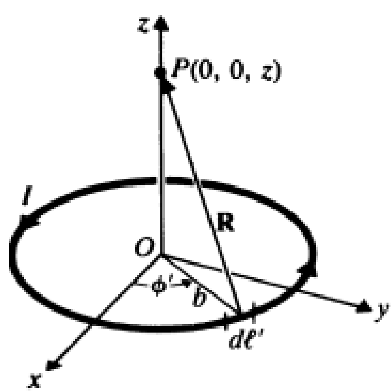
1. Similarly for magnetic flux density at **the center of a square loop**, with side $w$ carrying a direct current $I$, is:

$$\boldsymbol{B} = \boldsymbol{a_z}\frac{2\sqrt{2}\mu_0 I}{\pi w}$$

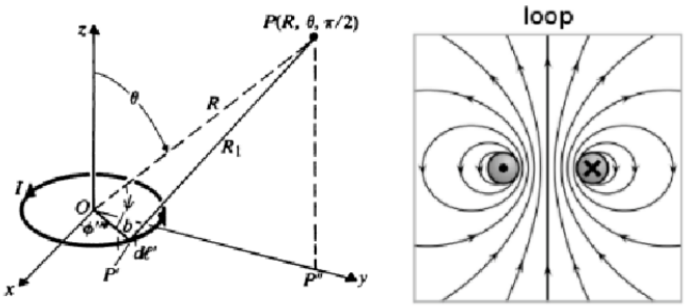



2. For magnetic flux density ata point on the axis of a circular loop of radiu $b$ that carries a direct current $I$,

$$\boldsymbol{B} = \boldsymbol{a_z}\frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$$



# 5 Magnetic Dipole




(a) Visualization (b) Cross-section

**Definition of the magnetic dipole:** We call a small current-carrying loop a magnetic dipole

$$\boldsymbol{m} = I \int d\boldsymbol{S}$$

The direction is determined by the right-hand rule. (along with the current direction)

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{a_R}}{4\pi R^2}$$

In spherical coordinates, the vector potential of a magnetic dipole can be written as

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 m \sin\theta}{4\pi R^2} \boldsymbol{a_\phi}$$

Hence, we can compute the magnetic field of a magnetic dipole

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \nabla \times \boldsymbol{A} = \frac{\mu_0 m}{4\pi R^3}(2\cos\theta \boldsymbol{a_r} + \sin\theta \boldsymbol{a_\theta})$$

Written in a coordinate-free form,

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \frac{\mu_0}{4\pi R^2}[3(\boldsymbol{m} \cdot \boldsymbol{a_r})\boldsymbol{a_r} - \boldsymbol{m}]$$

Compared with the electric field density of an electric dipole, we can find that we just replace $\frac{1}{\epsilon_0}$ with $\mu_0$, and replace $\boldsymbol{p}$ with $\boldsymbol{m}$.

# 6 Scalar magnetic potential

If a region is current free, i.e. $\boldsymbol{J} = 0$,

$$\nabla \times \boldsymbol{B} = 0$$

thus $\boldsymbol{B}$ can be expressed as the gradient of a scalar field.

Assume

$$\boldsymbol{B} = -\mu_0 \nabla V_m$$

where $V_m$ is called the **scalar magnetic potential**, the negative sign is conventional, $\mu_0$ is the permeability of free space.

Thus, between two points $P_1, P_2$,

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \boldsymbol{B} \cdot dl$$

If there were magnetic charges with a volume density $\rho_m$ in a volume $V'$, we could find $V_m$ from:

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R} dv'$$

**Notice:** This is only a mathematical model, isolated magnetic charges have never been found. Then we could obtain $\boldsymbol{B}$ by Eq 6.

For a bar magnet the fictitous magnetic charges $+q_m, -q_m$ assumed to be separated by $d$ (magnetic dipole), the scalar magnetic potential $V_m$ is given by:

$$V_m = \frac{\boldsymbol{m} \cdot \boldsymbol{a_R}}{4\pi R^2}$$

and it holds at any points with no currents.

# 7 Magnetization and Equivalent Current Densities

## 7.1 Basics

Define **magnetization vector**, $\boldsymbol{M}$, as

$$\boldsymbol{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{m_k}}{\Delta v}$$

which is the volume density of magnetic dipole moment,

1. The effect of magnetization is vector is equivalent to both
   (a) a volume current density:
   $$\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$$
   (b) a surface current density:
   $$\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$$
2. Then we can determine $A$ by:
   $$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a'_n}}{R} ds'$$
3. Then we could obtain $\boldsymbol{B}$ from $\boldsymbol{A}$.

## 7.2 Equivalent Magnetization Charge Densities

In current-free region, a manetized body may be replaced by

1. an equivalent/fictitous magnetization surface charge density

$$\rho_{ms} = \boldsymbol{M} \cdot \boldsymbol{a_n}$$

2. an equivalent/fictitous magnetization volume charge density

$$\rho_m = -\nabla \cdot \boldsymbol{M}$$

**Approaches to get $\boldsymbol{B}$**

(1) Given $\boldsymbol{I}$ or $\boldsymbol{J}$, applying Biot-Savart Law $\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{\boldsymbol{dl'} \times \boldsymbol{a_R}}{R^2}$ or utilizing scalar potential $\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv'$ and $B = \nabla \times \boldsymbol{A}$ to get $\boldsymbol{B}$

(2) Given magnetism $\boldsymbol{M}$, we can either get $\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$ and $\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$ together with $\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a'_n}}{R} ds'$ to get $A$ and then find $B$,
**or** utilizing $\rho_{ms} = \boldsymbol{M} \cdot \boldsymbol{a_n}$ and $\rho_m = -\nabla \cdot \boldsymbol{M}$ to get scalar magnetic potential $V_m$, and then apply $\boldsymbol{B} = -\mu_0 \nabla V_m$ to get $\boldsymbol{B}$.