Midterm 1 RC Part 2: Static Electric Field

1 Electrostatics in Free Space

1.1 Basic Concepts

Electrostatics:

i. electric charges are **at rest(not moving)**;

ii electric field **do not change with time**.

Static electric charges (source) in free space → electric field

1.2 Electric field intensity

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

1.3 Fundamental Postulates of Electrostatics

- Differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (divergence)$$

$$\nabla \times \mathbf{E} = 0 \quad (curl)$$

  where ρ is the volume charge density of free charges (C/m^3), ϵ_0 is the permittivity of free space, a universal constant.

- Integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

$$\oint_C \mathbf{E} \cdot d\ell = 0$$

  where Q is the total charge contained in volume V bounded by surface S. Also, the scalar line integral of the static electric field intensity around any closed path vanishes.

$\mathbf{E}$ is **not solenoidal** (unless $\rho = 0$), but **irrotational (conservative)**

2 Coulomb's Law

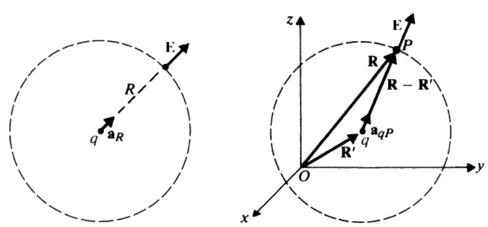
2.1 Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V}/\mathrm{m})$$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\,(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R}'\right|^3} \quad (\mathbf{V}/\mathrm{m})$$



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGURE due to a point charge.

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force F_{12} is experienced by q_2 due to the electric field intensity $\boldsymbol{E_{12}}$ of q_1 at q_2. Then we have:

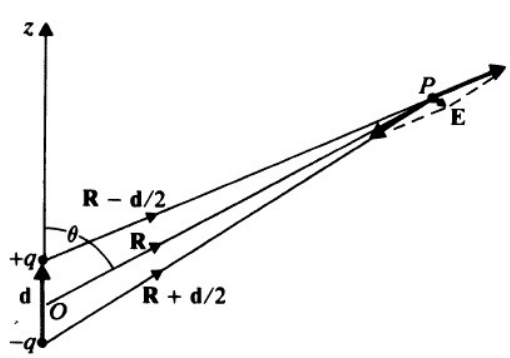
$$\boldsymbol{F_{12}} = q_2 \boldsymbol{E_{12}} = \boldsymbol{a_R} \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

- **several point charges:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k\,(\mathbf{R} - \mathbf{R}'_k)}{\left|\mathbf{R} - \mathbf{R}'_k\right|^3}$$

2.2 Electric Dipole

- **Electric Field**



general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]$$

- **Electric Dipole Moment**
  Definition:

$$\mathbf{p} = q\mathbf{d}$$

  where q is the charge, vector $\mathbf{d}$ goes from $-q$ to $+q$.

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

- **Electric Field:** (spherical coordinate)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta\right) \quad (\mathrm{V/m})$$

An interesting approach to obtain the electric field of electric dipole

- First, divide the electric dipole into two dipoles (one dipole moment is along with $\vec{R}$, while another dipole moment is perpendicular to $\vec{R}$).
- Then calculate the electric filed caused by each dipole. Owing to the great symmetry, the electric filed of one dipole is in the direction of $\mathbf{a}_R$, while another one is in the direction of $\mathbf{a}_\theta$.

For detailed derivative process, you can refer to my annotated RC2.

2.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R\frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

  , where dv' is the differential volume element.

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'}\mathbf{a}_R\frac{\rho_\ell}{R^2}d\ell' \quad (\mathbf{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'}\mathbf{a}_R\frac{\rho_s}{R^2}ds' \quad (\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'}\mathbf{a}_R\frac{\rho}{R^2}dv' \quad (\mathbf{V/m})$$

2.4 Possible Problems

Given distribution of charges (discrete point charges or charge density), calculate the electric field density at arbitrary point.

Tip 1: Usually, the knowledge of coordinates will be used to calculate the electric field density.

Tip 2: Steps: Find the electric field density of symmetric infinitesimals $d\vec{E}$ (or a point charge $\vec{E}_i$) in a certain coordinate. Then integrate $d\vec{E}$ (or sum up all $\vec{E}_i$) to obtain E.

Tip 3: Since $\vec{E}$ is a vector, don't forget to include both value and direction of $\vec{E}$.

Sample Problems
HW2-1, HW2-3, HW2-5, RC2-1, RC2-2, RC2-3, RC2-4.

3 Gauss's Law and Application

3.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

3.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

3.3 Possible Problems

Given **symmetric** distribution of charges (discrete point charges or charge density), calculate the electric field density at arbitrary point **with Gausses's law**.

Step 1: Take a closed **Gaussian surface** (Arbitrary position on the surface has the same value of electric field in the normal direction).

Step 2: Calculate the total charge enclosed by the Gaussian surface Q.

Step 3: Apply Gauss's Law to calculate the electric field density.

Sample Problems
RC2-5, RC2-6, RC2-7, RC2-8.

3.4 Several Useful Models (Paste on your CTPP!)

Note: The charge distribution should be **uniform**.

different models	E(magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r\epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3}(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0}(r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0}(r > R) \end{cases}$

4 Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

the reason for the negative sign: consistent with the convention that in going against the **E** field, the electric potential V increases.

- **Electric Potential Difference:**

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$$

- **Electric Potential due to a Charge Distribution**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

i. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

ii. **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'} \frac{\rho_s}{R} ds' \quad (V)$$

iii. **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R} dv' \quad (V)$$

Sample Problems

RC2-9