

Mid1 Part1

1 Vectors

- Dot product:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{\mathbf{AB}}$$

- Commutative: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
- Distributive: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
- Not associative: $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \neq (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$
- For the three edges A, B, C in a triangle, $C^2 = A^2 + B^2 - 2AB \cos(\theta_{A,B})$
- In component form, the dot product can be written as

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned}$$

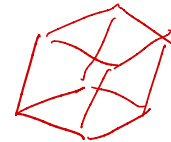
- Cross product:

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB \sin \theta_{\mathbf{AB}}|$$

- The cross product is always perpendicular to both \mathbf{A}, \mathbf{B} , the direction follows right hand rule.
- Not commutative: $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$ $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- Distributive: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
- Not associative: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

- $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \text{Volume}$

- BAC-CAB: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$



2 Coordinates

Three basis (u_1, u_2, u_3) : number of linearly independent basis = dimension of the space.
For the three types of coordinates we discuss, u_i is orthogonal to each other.

For arbitrary vector \mathbf{A} :

$$\mathbf{A} = \mathbf{a}_{u1} A_{u1} + \mathbf{a}_{u2} A_{u2} + \mathbf{a}_{u3} A_{u3}$$

Norm of \mathbf{A} :

$$|\mathbf{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

For a differential length $d\ell$,

$$d\ell = \mathbf{a}_{u1}(h_1 du_1) + \mathbf{a}_{u2}(h_2 du_2) + \mathbf{a}_{u3}(h_3 du_3)$$

where h_i is called metric coefficient.

$h_i du_i$

Differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

Differential area vector with a direction normal to the surface,

$$d\mathbf{s} = \mathbf{a}_n ds$$

right hand rule.

Differential area $d\mathbf{s}_1$ normal to the unit vector \mathbf{a}_{u1} .

$$\underline{d\mathbf{s}_1} = \underline{h_2 h_3} du_2 du_3$$

$$d\vec{s}_2 = \dots$$

$$d\vec{s}_3 = \dots$$

Note that for $d\mathbf{s}_i$, the foot indices on the right hand side are the ones that don't show up on the left hand side.

2.1 Cartesian Coordinates

•

$$(u_1, u_2, u_3) = (x, y, z)$$

• Right hand rule:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

$$\vec{a}_y \times \vec{a}_z$$

$$\vec{a}_z \times \vec{a}_x$$

•

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

where \mathbf{a}_i is the basis for i-axis.

• Differential length:

$$d\ell = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$$

• Differential area:

$$ds_x = dydz$$

direction!

as $h_1 = h_2 = h_3 = 1$,

($d\mathbf{s}_x$ is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).

• Differential volume:

$$\underline{dv = dx dy dz}$$

2.2 Cylindrical Coordinate

•

$$(u_1, u_2, u_3) = (r, \phi, z)$$

- Right hand rule:

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z$$

•

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

- Differential length:

$$d\ell = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_z dz \quad (1)$$

as $h_1 = 1, h_2 = r, h_3 = 1$

- Differential area:

$$d\mathbf{s}_r = r d\phi dz \quad d\mathbf{s}_\phi = r dr dz \quad d\mathbf{s}_z = dr d\phi$$

- Differential volume:

$$dv = r dr d\phi dz$$

- From cylindrical coordinate to Cartesian coordinate:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

a)

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \\ z = z \end{cases}$$

b)

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \arctan \frac{y}{x} \\ z = z \end{cases}$$

2.3 Spherical Coordinate

•

$$(u_1, u_2, u_3) = (R, \theta, \phi)$$

- Right hand rule:

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

•

$$\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$

- Differential length:

$$d\ell = \mathbf{a}_R dR + \mathbf{a}_\theta R d\theta + \mathbf{a}_\phi R \sin \theta d\phi \quad (2)$$

as $h_1 = 1, h_2 = R, h_3 = R \sin \theta$.

- Differential area:

$$d\mathbf{s}_R = R^2 \sin \theta d\theta d\phi \quad d\mathbf{s}_\theta = R \sin \theta dR d\phi \quad d\mathbf{s}_\phi = R dR d\theta$$

- Differential volume:

$$dv = R^2 \sin \theta dR d\theta d\phi$$

- conversion of quantities between Cartesian coordinate and Spherical coordinate:

a)

$$\begin{cases} x = R \sin \theta \cos \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \theta \end{cases}$$

b)

$$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan \frac{\sqrt{x^2 + y^2}}{z} \\ \phi = \arctan \frac{y}{x} \end{cases}$$

- From Spherical coordinate to Cartesian coordinate:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

3 Vector Calculus

3.1 Integrals

- Line integral: $\int_C \mathbf{F} \cdot d\mathbf{l}$, where \mathbf{F} is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (e.g. $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$). If we are integrating over a closed loop, we write $\oint \mathbf{F} \cdot d\mathbf{l}$.
 $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$
- Surface integral: $\int_S \mathbf{F} \cdot d\mathbf{S}$. If we are integrating over a closed surface, we write $\oint \mathbf{F} \cdot d\mathbf{S}$.
- Volume integral: $\int_V \mathbf{F} dv$. In Cartesian coordinates, $dv = dxdydz$. Specially, $\int_V \mathbf{F} dv = \int_V (F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}} + F_z \hat{\mathbf{z}}) dv = \hat{\mathbf{x}} \int_V F_x dv + \hat{\mathbf{y}} \int_V F_y dv + \hat{\mathbf{z}} \int_V F_z dv$

3.2 Gradient: How Fast a Scalar Field Changes

First, consider what is the difference between a scalar quantity and a scalar field? Between a vector quantity and a vector field?

$$\nabla V = \mathbf{a}_n \frac{dV}{dn}$$

∇V V : scalar

∇V at certain point is a vector.

∇V : vector

$$\nabla V = \mathbf{a}_{u1} \frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u2} \frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u3} \frac{\partial V}{h_3 \partial u_3}$$

↓

$$\text{Cartesian: } \nabla V = \hat{\mathbf{a}}_x \frac{\partial V}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial V}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial V}{\partial z}$$

3.3 Divergence of a vector field

- Divergence of a vector field \mathbf{A} at a point $\text{div}\mathbf{A}$ as the net outward flux of \mathbf{A} per unit volume as the volume about the point tends to zero:

$$\nabla \cdot \vec{A} \quad \text{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v} \quad (3)$$

- Source: Net positive divergence
- Sink: Net negative divergence.

- $\nabla \cdot \mathbf{A}$ at certain point is a scalar.
- In Cartesian coordinate,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- Generally,

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

- Divergenceless vector field ($\nabla \cdot \mathbf{A} = 0$): **Solenoidal field**.

3.4 Divergence Theorem

$$\int_V \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

3.5 Curl of a vector field

-

$$\text{curl}\mathbf{A} = \nabla \times \mathbf{A} = \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} [\mathbf{a}_n \oint_C \mathbf{A} \cdot d\mathbf{l}]_{\max}$$

The curl of a vector field \mathbf{A} , denoted by $\text{curl}\mathbf{A}$ or $\nabla \times \mathbf{A}$, is a vector whose magnitude is the maximum net circulation of \mathbf{A} per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).

- $\nabla \times \mathbf{A}$ in a general coordinate:

$$\nabla \times \vec{A} \quad \vec{A}: \text{vector} \quad \nabla \times \vec{A}: \text{vector} \quad \nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u1} h_1 & \mathbf{a}_{u2} h_2 & \mathbf{a}_{u3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Cartesian:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- Curl-free vector field ($\nabla \times \mathbf{A} = 0$): **Irrotational or conservative field**

3.6 Stokes's Theorem

$\{D \rightarrow 2D,$

$$\int_S (\nabla \times \mathbf{A}) d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$



The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

3.7 Null Identities

$\nabla \times (\nabla V) \equiv 0$ ↗ scalar.

$\nabla \times \vec{A} = 0 \Leftrightarrow \vec{A} = \nabla V.$

- The curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is **curl-free**, then it can be expressed as the **gradient** of a **scalar field**.

$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$ ↗ vector.

$\nabla \cdot \vec{A} = 0 \Leftrightarrow \vec{A} = \nabla \times \vec{B}.$

- The divergence of the curl of any vector field is identically zero.
- Another interpretation: If a vector field is **divergenceless**, then it can be expressed as the **curl** of another **vector field**.

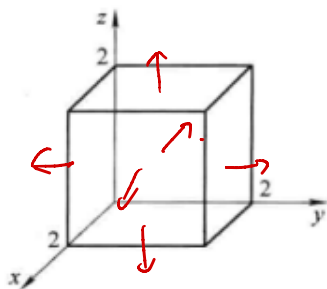
4 Helmholtz's Theorem

A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

5 Quiz 1 Recap

Question 1

- (a) Use the cube of side length 2 in the following picture and function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3xz)\hat{\mathbf{z}}$ to verify the divergence theorem.



$$\textcircled{1} \nabla \cdot \vec{v} = y + 2z + 3x.$$

$$\textcircled{2} \int_V (\nabla \cdot \vec{v}) dV = \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) dx dy dz = \dots$$

$$\textcircled{3} \text{ front: } x=2, d\vec{S} = \vec{a}_x dy dz.$$

$$\text{back: } x=0, d\vec{S} = -\vec{a}_x dy dz.$$

$$\text{top: } z=2, d\vec{S} = \vec{a}_z dx dy$$

$$\text{bottom: } z=0, d\vec{S} = -\vec{a}_z dx dy$$

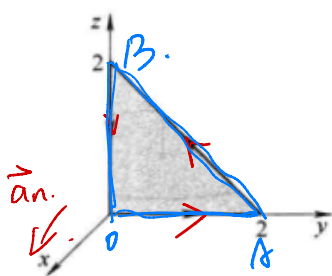
$$\text{left: } y=0, d\vec{S} = -\vec{a}_y dx dz$$

$$\text{right: } y=2, d\vec{S} = \vec{a}_y dx dz.$$

$$\int_S = \sum \dots$$

verified.

- (b) Use the triangle in the following picture and function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3xz)\hat{\mathbf{z}}$ to verify Stokes' theorem.



$$\textcircled{1} \nabla \times \vec{v} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \dots$$

$$\textcircled{2} d\vec{S} = \vec{a}_x dy dz$$

$$\int_0^2 \int_0^{2-y} (\nabla \times \vec{v}) \cdot \vec{a}_x dy dz.$$

$$\textcircled{3} \text{ i) } \int_0^A \dots \text{ ii) } \int_A^B \dots \text{ iii) } \int_B^0 \dots$$

$$\int_C = \sum \int_0^A + \int_A^B + \int_B^0 (= \int_S (\nabla \times \vec{v}) d\vec{S}),$$

verified.

6 Exercise

- Show that $\mathbf{F} = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

$$\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \underline{V}$$

$$\nabla \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \times \underline{\vec{A}}.$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \vec{a}_x(z-x) + \vec{a}_y(y-y) + \vec{a}_z(z-z) = 0.$$

$$\vec{F} = \nabla V \Rightarrow V = xyz.$$

$$\vec{F} = \nabla \times \underline{\vec{A}} \quad \begin{cases} \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} = yz \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy \end{cases}$$

$$\underline{\vec{A}} = \frac{1}{4} \left[x(z^2 - y^2) \vec{a}_x + y(x^2 - z^2) \vec{a}_y + z(y^2 - x^2) \vec{a}_z \right].$$