Mid 1 RC Part 3: Static Electric Fields

1 Conductors and dielectrics in static electric field

- conductors:

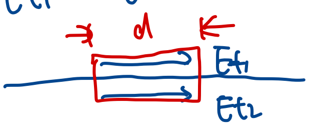
  * current could be in any direction.

  - electrons migrate easily.
  - charges reach the surface and conductor redistribute the charges in a way that the field vanishes.
  - **static state conditions**:

    * conservativeness

    $E_{t1} \cdot d + E_{t2} \cdot d + 0 = 0$

    $E_{t1} = -E_{t2}$

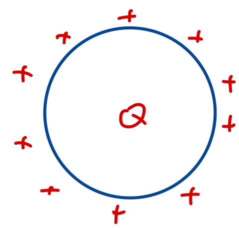
    

    * inside the conductor:

      charge volume density.

      $$\rho = 0, \boldsymbol{E} = 0$$

      where $\rho = 0$ represents no charge in the interior

    * on the conductor surface (boundary conditions)

      $$E_t = 0, E_n = \frac{\rho_s}{\epsilon_0}$$

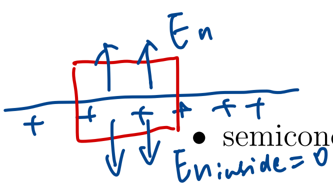
      if $E \neq 0 \Rightarrow V \neq 0$

      $\rightarrow$ charge move

      $\rightarrow V \downarrow \rightarrow V = 0$

      It is an equal-potential body.

      $E_n \cdot S = \frac{\rho_s \cdot S}{\varepsilon_0} \Rightarrow E_n = \frac{\rho_s}{\varepsilon_0}$

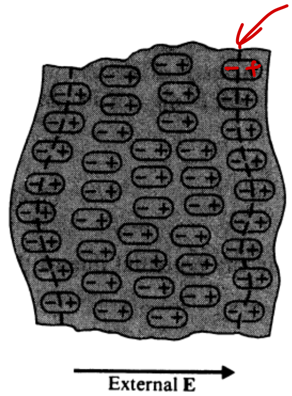
      

- semiconductors:
  - relatively small number of freely movable charges.

- insulators(dielectrics):
  - electrons are confined to their orbits.
  - external electric field polarizes a dielectric material and create electric dipoles. The induced electric dipoles will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

dipoles.

E_{induce}: electric field by induced charges.



$\rightarrow p, P$

Figure 1: A cross section of a polarized dielectric medium

$\leftarrow E_{induce}$.

- **polarization charge densities/ bound-charge densities:**
  - **polarization vector, $\boldsymbol{P}$**:

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p_k}}{\Delta v}$$

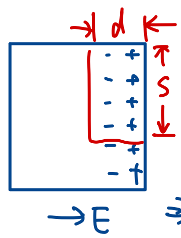
where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv.

  - charge distribution on surface density:

$$\rho_{ps} = \boldsymbol{P} \cdot \boldsymbol{a_n}$$

  - volume charge distribution density:

$$\rho_p = -\nabla \cdot \boldsymbol{P}$$



$dV = d \cdot S$

$= \vec{d} \cdot \vec{a}_n \cdot S$ not ⊥

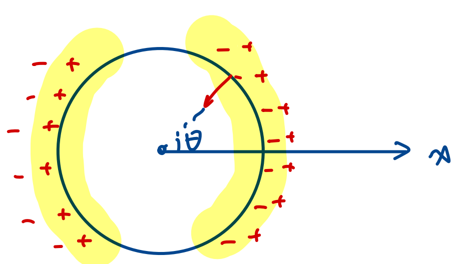
$P = \frac{\Sigma p_K}{\Delta V}$

$\Rightarrow P \cdot \Delta V = \Sigma p_K$

$P \cdot \cancel{\vec{d}} \cdot \vec{a}_n \cdot S = \Sigma q \cdot \cancel{\vec{d}}$

$P \cdot \vec{a}_n = \Sigma q / S$

Ex.1 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization **P** exists.

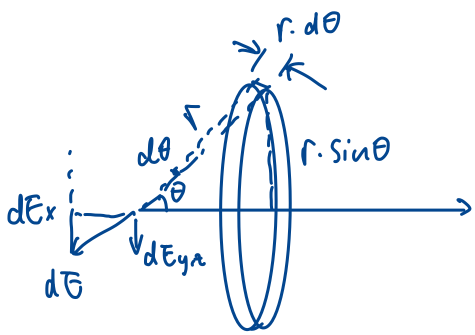


Assume $\vec{P} = \vec{a}_x \cdot P$

$\rho_{ps}(\theta) = \vec{P} \cdot \vec{a}_n$ from dielectrics to vacuum.

$= P \cdot \vec{a}_x \cdot \vec{a}_n$

$= -P \cdot \cos\theta$



$ds = 2\pi \cdot r \cdot \sin\theta \cdot r \cdot d\theta$

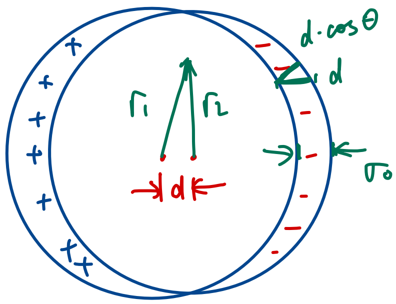
$dq = ds \cdot \rho_{ps}(\theta) = -P\cos\theta \cdot 2\pi r \cdot \sin\theta \cdot r \cdot d\theta$

$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2}$

$dE_x = dE \cdot \cos\theta = \frac{P\cos^2\theta\sin\theta}{2\varepsilon_0} d\theta$

$E = \int_0^\pi dE_x = \frac{P}{3\varepsilon_0}$.

*



ρ

one is positively charged.

one is negatively charged.

$-\rho$

$$\rho \cdot d(\theta) \Rightarrow \sigma(\theta) = \underbrace{\rho \cdot d}_{\sigma_0} \cdot \cos\theta = \sigma_0 \cdot \cos\theta.$$

$$\vec{E}_- = -\frac{\rho}{3\varepsilon_0} \vec{r_2}$$

$$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\vec{r_1}|^3} \vec{r_1}$$

$$= \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{\cancel{\frac{4}{3}\pi}|\cancel{\vec{r_1}}|^3 \cdot \rho}{|\cancel{\vec{r_1}}|^3} \cdot \vec{r_1}$$

$$= \frac{\rho}{3\varepsilon_0} \vec{r_1}$$

$$E_+ + E_- = \frac{\rho}{3\varepsilon_0}(\vec{r_1} - \vec{r_2}) = \frac{\rho\vec{d}}{3\varepsilon_0} = \frac{\sigma_0}{3\varepsilon_0} \cdot \frac{\vec{d}}{|d|}$$

2 Electric Flux Density and Dielectric Constant

- **electric flux density/electric displacement, $\boldsymbol{D}$**:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \quad (C/m^2)$$

free charge total induced charges.

- $$\nabla \cdot \boldsymbol{D} = \rho \quad (C/m^3)$$

  where ρ is the volume density of free charges.

- Another form of **Gauss's law**:

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q_{free} \quad (\boldsymbol{C})$$

$\oint_s E \cdot ds = Q_{encl}$

  the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic**,

* polarizability.

$p = \alpha E \longrightarrow$ applied field.

dipole moment

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$$

$$\boldsymbol{D} = \epsilon_0 (1 + \chi_e) \boldsymbol{E} = \epsilon_0 \epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E}$$

  where χ_e is a dimensionless quantity called electric susceptibility,

  ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium,

  ϵ is the absolute permittivity/permittivity of the medium (F/m).

- For anisotropic,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For bi-axial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For uni-axial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

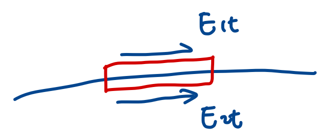
- dielectric breakdown: electric field is very strong, causes permanent dislocations and damage in the material.

  dielectric strength: the maximum electric field intensity that a dielectric material can withstand without breakdown.

3 Boundary Conditions for Electrostatic Fields

- the tangential component of an $\boldsymbol{E}$ field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$

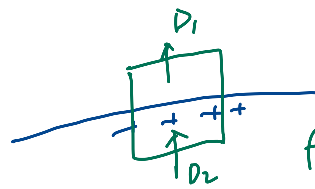


or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

- The normal component of $\boldsymbol{D}$ field is discontinuous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$$\boldsymbol{a_{n2}} \cdot (\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_s$$



or

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

free charge.

$D_1 \cdot S - D_2 \cdot S = \rho_s \cdot S$

4 Capacitance and Capacitors

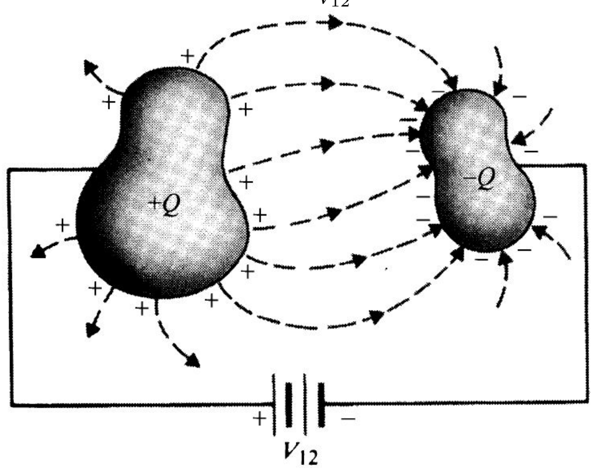
4.1 Capacitance

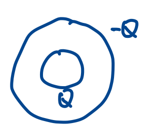
- Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.

- $C = \frac{Q}{V}$ $(F = C/V)$

4.2 Capacitor

- **Components:** two conductors with arbitrary shapes are separated by free space or dielectric medium. $C = \frac{Q}{V_{12}}$





$\frac{Q}{V_{12}} \rightarrow V_{12} = \int_1^2 \vec{E} \cdot d\vec{l}$

- **Capacitance:**
  Its Capacitance is independent of V and Q, which means a capacitor has a capacitance even no voltage is applied to it and no free charges exist on its conductors.

- **How to calculate its capacitance**:

  1. Choose a proper coordinate system
  2. Assume $+Q, -Q$ on the conductors
  3. Find $\mathbf{E}$ from Q (like, Gauss's law, $D_n = \epsilon E_n = \rho_s$)
  4. Find $V_{12} = -\int_2^1 \mathbf{E}\cdot d\mathbf{l}$
  5. $C = Q/V_{12}$

- **Series Connections of Capacitors:**

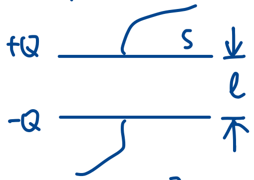
$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_n}$$

- **Parallel Connections of Capacitors:**

$$C_{||} = C_1 + C_2 + ... + C_n$$

Ex.2 Suppose we have a parallel conductor plane capacitor with capacitance of C, what is the new capacitance C' if we insert a dielectric with relative permittivity of ϵ_r?

Before insertion:



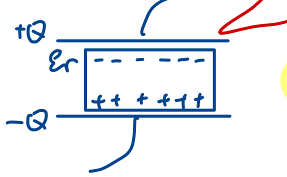
$E \cdot ds = \dfrac{\frac{Q}{S}\cdot ds}{\varepsilon_0}$

by free charges $E = \dfrac{Q}{\varepsilon_0 S}$

$V = \int_l E\cdot dl = \dfrac{Ql}{\varepsilon_0 S}$

$C = \dfrac{Q}{V} = \dfrac{\varepsilon_0 S}{l}$

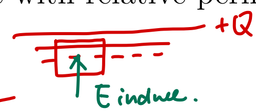
After insertion:



$E' = \dfrac{E}{\varepsilon_r} = \dfrac{Q}{\varepsilon_r \varepsilon_0 S}$

$V = \dfrac{Ql}{\varepsilon_r \varepsilon_0 S}$

$C = \dfrac{Q}{V} = \dfrac{\varepsilon_r \varepsilon_0 S}{l}$



$E' = E - E_{induce}$

$\varepsilon_0 E' = \underbrace{\varepsilon_0 E}_{D} - \underbrace{\varepsilon_0 \cdot E_{induce}}_{P}$

$D = \varepsilon_0 E' + P$

$E_{induce}\cdot ds = \dfrac{\rho_{induce}\cdot ds}{\varepsilon_0}$

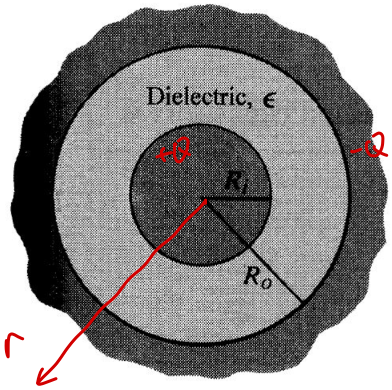
$\rho_{induce} = P\cdot a_n = P$

$\varepsilon_0 \cdot E_{induce} = P$

$D = \varepsilon E = \varepsilon_r \varepsilon_0 E$ $\varepsilon_r = 1$

$D = \varepsilon_0 E$

Ex.3 A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o. The space in between is filled with a dielectric of permittivity ϵ. Determine the capacitance.



$$\oint \vec{D} \cdot ds = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

$$\vec{E} = \frac{D}{\varepsilon} = \frac{Q}{4\pi \varepsilon r^2} \cdot \hat{r}$$

$$V = \int_{R_i}^{R_o} E \cdot dl = \frac{Q}{4\pi \varepsilon} \left[\frac{1}{R_i} - \frac{1}{R_o} \right]$$

$$C = \frac{Q}{V} \ldots$$

4.3 Electrostatic Energy and Forces

- Potential difference between P_1 to P_2

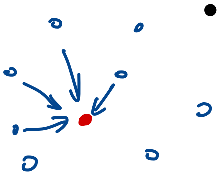
work done by moving from P_1 to P_2.

$$\frac{W_{12}}{q} = V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

- **Self Energy:** Work done to bring a charge Q_2 from infinitely far away to distance R_{12} with Q_1 (initially, Q_1 is in space)

$$W = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

- **Mutual Energy:** Potential energy of a group of N discrete point charges at rest



$$W_e = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k$$

why $\frac{1}{2}$: mutual energy is shared by 1 & 2

where $V_k = \frac{1}{4\pi\epsilon_0} \sum_{j=1\&j\neq k}^{N} \frac{Q_j}{R_{jk}}$ Note the W_e can be negative, for example, there are 2-point charge systems, and one charge is positive,the other is negative.

- **Electrostatic Energy (Volume) density w_e:** $W_e = \int_{v'} w_e dv$

4.3.1 Electrostatic Energy in terms of Field Quantities

- v' can be all space.

- **A continuous Charge Distribution of Density ρ**

$\nabla \cdot D = \rho$

$$W_e = \frac{1}{2}\int_{v} \rho V dv = \frac{1}{2}\int_{v'} (\nabla \cdot \mathbf{D}) V dv$$

Another expression:

$$W_e = \frac{1}{2}\int_{v'} \mathbf{D} \cdot \mathbf{E} dv$$

$D = \epsilon \cdot E$

- If it is a simple dielectric, it should be

$$W_e = \frac{1}{2}\int_{v'} \epsilon E^2 dv = \frac{1}{2}\int_{v'} \frac{D^2}{\epsilon} dv$$

4.3.2 Electrostatic Forces

Here we use **Principle of virtual displacement** to calculate Force in two situations.

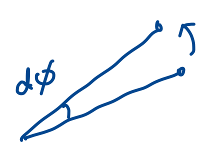
- **System of bodies with fixed charges**

  1. Mechanical work is from the reduced stored electrostatic energy

$$F_Q = -\nabla W_e (N)$$

  2. Electric torque rotates one of the bodies by $d\phi$(a virtual rotation) about an axis

$$T_Q = -\frac{\partial W_e}{\partial \phi} (N \cdot m)$$



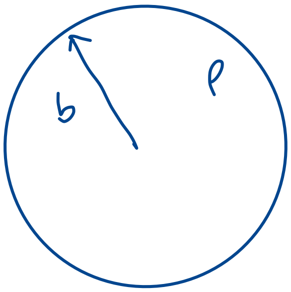
- **System of conducting bodies with Fixed Potentials**

  1. The fixed potential can be retained by connecting with an external source.

  2. $F_v = \nabla W_e$

  3. $T_v = \frac{\partial W_e}{\partial \phi}$

Example 3-22 Find the energy required to assemble a uniform sphere of charges of radius b and volume charge ρ.



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_r$$

$$W = \int dW$$

$$= \int V(Q)\cdot dQ$$

$$\vec{V} = \int_R \vec{E} = \frac{Q}{4\pi\varepsilon_0 R}\hat{a}_r$$

$$Q = \frac{4}{3}\pi R^3 \cdot \rho$$

$$dQ = 4\pi\rho R^2 \cdot dR$$