RC 1: Vector Analysis

1 Vectors

- vector component form:
  An arbitrary can be expanded in terms of these basis vectors:

$$\vec{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

- dot product:

$$\vec{A} \cdot \vec{B} = ABcos\theta_{\vec{A}\vec{B}}$$

  - Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
  - Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
  - Not associative: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot \vec{C}$
    ~~e.g.~~ $\cancel{(\vec{a_x} \cdot \vec{a_y}) \cdot \vec{a_z} \neq \vec{a_x} \cdot (\vec{a_y} \cdot \vec{a_z})}$
  - For the three edges A, B, C in a triangle, $C^2 = A^2 + B^2 - 2ABcos(\theta_{A,B})$
  - In component form, the dot product can be written as

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

- cross product:

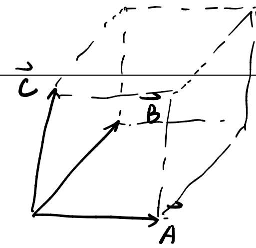
$$\vec{A} \times \vec{B} = \vec{a_n}|ABsin\theta_{\vec{A}\vec{B}}|$$

  - The cross product is always perpendicular to both $\vec{A}, \vec{B}$, the direction follows right hand rule.
  - Not Commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (have opposite directions).
  - Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
  - Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
    e.g. $\vec{a_x} \times (\vec{a_x} \times \vec{a_y}) = \vec{a_x} \times \vec{a_z} = -\vec{a_y}$,
    $(\vec{a_x} \times \vec{a_x}) \times \vec{a_y} = 0 \neq -\vec{a_y}$
  - In component form, the cross product can be written as

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\,\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{z}}\end{aligned}$$

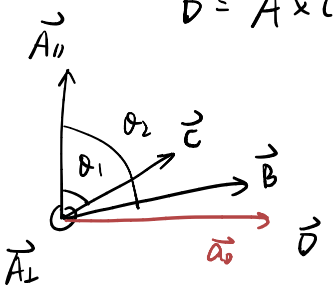
  or in a more neat way,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



- $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = Volume$
- BAC-CAB: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

  Prove:



$$\vec{D} = \vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}_{\parallel}\times(\vec{B}\times\vec{C}) = |\vec{A}_{\parallel}||\vec{B}||\vec{C}|\sin(\theta_2-\theta_1)\,\vec{a}_D$$

$$= |\vec{A}_{\parallel}||\vec{B}||\vec{C}|(\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)\,\vec{a}_D$$

$$= \Big[\underbrace{(|\vec{A}_{\parallel}||\vec{C}|\cos\theta_1)\,\vec{B}\sin\theta_2}_{=|\vec{B}|} - (|\vec{A}_{\parallel}||\vec{B}|\cos\theta_2)\,|\vec{C}|\sin\theta_1\Big]\vec{a}_D$$

$$|\vec{D}| = \vec{D}\cdot\vec{a}_D \qquad (\vec{A}\cdot\vec{B})$$

$$|\vec{D}| = (\vec{A}\cdot\vec{C})\,\vec{B}\cdot\vec{a}_D - (\vec{A}\cdot\vec{B})\,\vec{C}\cdot\vec{a}_D$$

~~1.1 Exercise~~

$$\Rightarrow \vec{D} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

- ~~P.2.4 Show that, if $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$~~ and $\vec{A}\times\vec{B} = \vec{A}\times\vec{C}$, where $\vec{A}$ is not a null vector, then $\vec{B} = \vec{C}$.

$\vec{A}\cdot(\vec{B}-\vec{C}) = 0$ $\quad$ $\vec{A}\perp(\vec{B}-\vec{C})$

$\vec{A}\times(\vec{B}-\vec{C}) = 0$ $\quad$ $\vec{A}\parallel(\vec{B}-\vec{C})$ $\quad$ $\Rightarrow \vec{B}-\vec{C}=0$

$\vec{A}$ not null vector

2 Coordinates

For now, consider 3-D space. A point P in 3d space (or its position vector r) can be defined using Cartesian coordinates (x, y, z), by $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$, where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the standard basis vectors. In the Cartesian system, the standard basis vectors can be derived from the derivative of the location of point P with local coordinate

$$\mathbf{e}_x = \frac{\partial \mathbf{r}}{\partial x}; \mathbf{e}_y = \frac{\partial \mathbf{r}}{\partial y}; \mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z}$$

Applying the same derivatives to the curvilinear system locally at point P defines the natural basis vectors:

$$\mathbf{h}_u = \frac{\partial \mathbf{r}}{\partial u}; \mathbf{h}_v = \frac{\partial \mathbf{r}}{\partial v}; \mathbf{h}_w = \frac{\partial \mathbf{r}}{\partial w}.$$

Such a basis, whose vectors change their direction and/or magnitude from point to point is called a local basis. These may not have unit length, and may also not be orthogonal. **In**

the case that they are orthogonal at all points where the derivatives are well-defined, we define the Lamé coefficients(or metric coefficient) h_i (after Gabriel Lamé) by

$$h_u = |\mathbf{h}_u|\,; h_v = |\mathbf{h}_v|\,; h_w = |\mathbf{h}_w|$$

and the curvilinear orthonormal basis vectors $\boldsymbol{e_i}$ by

$$\mathbf{e}_u = \frac{\mathbf{h}_u}{h_u}; \mathbf{e}_v = \frac{\mathbf{h}_v}{h_v}; \mathbf{e}_w = \frac{\mathbf{h}_w}{h_w}$$

In orthogonal curvilinear coordinates, since the total differential change in $\mathbf{r}$ is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw = h_u du \mathbf{e}_u + h_v dv \mathbf{e}_v + h_w dw \mathbf{e}_w$$

To be more general:

Three basis (u_1, u_2, u_3): number of linearly independent basis = dimension of the space. For the three types of coordinates we discuss, u_i is orthogonal to each other.

For arbitrary vector $\vec{A}$:

$$\vec{A} = \vec{a_{u1}} A_{u1} + \vec{a_{u2}} A_{u2} + \vec{a_{u3}} A_{u3}$$

Norm of $\vec{A}$:

$$|\vec{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

For a differential length dl,

$$dl = \vec{a_{u1}}(h_1 du_1) + \vec{a_{u2}}(h_2 du_2) + \vec{a_{u3}}(h_3 du_3)$$

h_i is called metric coefficient.

differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

differential area vector with a direction normal to the surface,

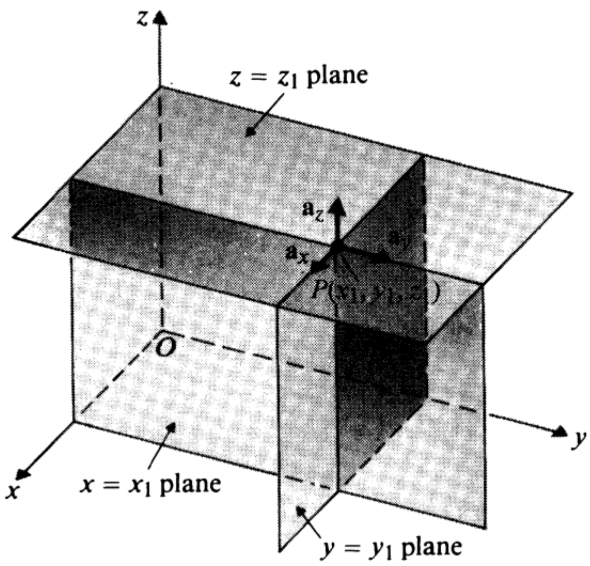
$$d\vec{s} = \vec{a_n} ds$$

differential area ds_1 normal to the unit vector $\vec{a_{u1}}$.

$$ds_1 = h_2 h_3 du_2 du_3 \tag{1}$$

Note that for ds_i, the foot indices on the right hand side are the ones that don't show up on the left hand side.

2.1 Cartesian Coordinates



$$(u_1, u_2, u_3) = (x, y, z)$$

Right hand rule:

$$\vec{a_x} \times \vec{a_y} = \vec{a_z}$$

$$\vec{A} = \vec{a_x}A_x + \vec{a_y}A_y + \vec{a_z}A_z$$

where $\vec{a_i}$ is the basis for i-axis.
Differential length:

$$d\vec{l} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz \quad (2)$$

Differential area:

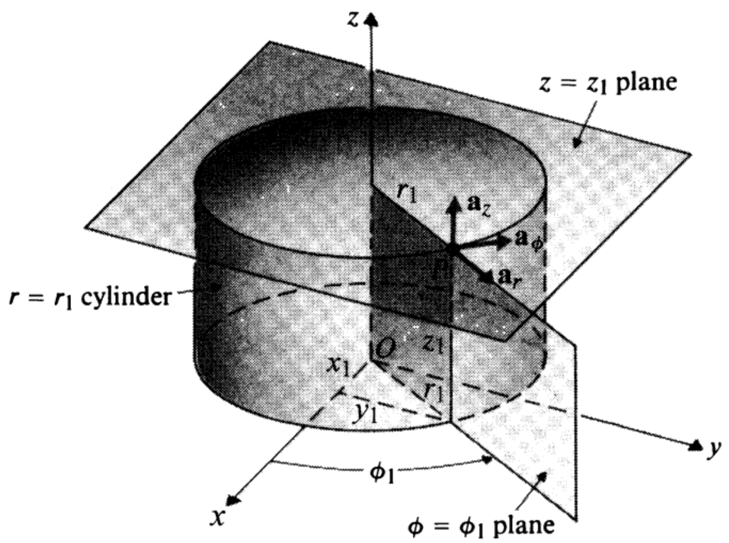
$$ds_x = dydz$$

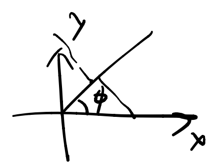
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

as $h_1 = h_2 = h_3 = 1$,
(ds_x is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).
Differential volume:

$$dv = dxdydz$$

2.2 Cylindrical Coordinate





$$(u_1, u_2, u_3) = (r, \phi, z)$$ $\rightarrow \vec{r} = R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y} + z\hat{z}$

Right hand rule:

$$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$$

$\frac{\partial r}{\partial R} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} = \hat{a_r}$

$$\vec{A} = \vec{a_r}A_r + \vec{a_\phi}A_\phi + \vec{a_z}A_z$$

$\frac{\partial r}{\partial \phi} = -R\sin\phi\,\hat{x} + R\cos\phi\,\hat{y} = R\hat{a_\phi}$

Differential length:

$$d\vec{l} = \vec{a_r}dr + \vec{a_\phi}rd\phi + \vec{a_z}dz \tag{3}$$

$\frac{\partial r}{\partial z} = \hat{z} = \hat{a_z}$

as $h_1 = 1, h_2 = r, h_3 = 1$

Differential area:

$$ds_r = rd\phi dz$$

Differential volume:

$$dv = rdrd\phi dz$$

Conversion of Cartesian coordinate and Cylindrical coordinate:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

$$\begin{bmatrix} r \\ 0 \\ z \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

a)

$$\begin{cases} x = rcos\phi \\ y = rsin\phi \\ z = z \end{cases}$$

$\underline{\underline{A_\phi \neq \phi}}$

b)

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \phi = arctan\frac{y}{x} \\ z = z \end{cases}$$

$x\cos\phi + y\sin\phi = r$

$x(-\sin\phi) + y\cos\phi = 0$

$z = z$

Question 1 What is the difference between these conversion equations and the conversion equations related to A_r, A_ϕ, A_z?

$y = x\tan\phi \Rightarrow \phi = \arctan\frac{y}{x}$

$r = \sqrt{x^2+y^2}$

(r, ϕ, z)

将 x. y. z 用 r. φ. z 换掉.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r} = r\hat{r} + \underbrace{A_\phi}_{0}\hat{\phi} + z\hat{z}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \begin{matrix} r \\ 0 \\ z \end{matrix}$$

Question 2 When we calculate the norm of $\vec{A}$ in cylindrical coordinate, we can use

$$|\vec{A}| = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$

Is

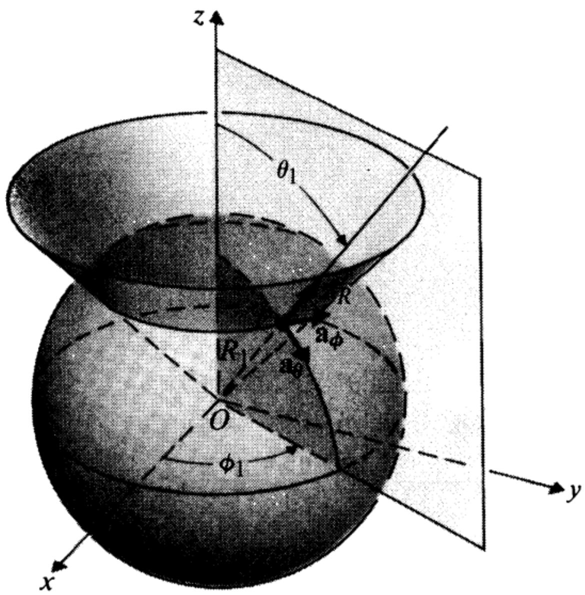
$$|\vec{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2} \tag{4}$$

also true? Can we prove Eq. 4 is the valid equation to calculate norm in cylindrical coordinate?

$|\vec{A}| = \sqrt{(A_r\cos\phi - A_\phi\sin\phi)^2 + (A_r\sin\phi + A_\phi\cos\phi)^2 + A_z^2}$

$= \sqrt{A_r^2 + A_\phi^2 + A_z^2}$

2.3 Spherical Coordinate



$$(u_1, u_2, u_3) = (R, \theta, \phi)$$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r} = R\sin\theta\cos\phi\,\hat{x} + R\sin\theta\sin\phi\,\hat{y} + R\cos\theta\,\hat{z}$

Right hand rule:

$$\vec{a_R} \times \vec{\theta} = \vec{\phi}$$

$$\vec{A} = \vec{a_R}A_R + \vec{a_\theta}A_\theta + \vec{a_\phi}A_\phi$$

$\frac{\partial \vec{r}}{\partial R} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} = \hat{a_R}$

Differential length:

$$d\vec{l} = \vec{a_R}dR + \vec{a_\theta}Rd\theta + \vec{a_\phi}Rsin\theta d\phi \tag{5}$$

as $h_1 = 1, h_2 = R, h_3 = Rsin\theta$.

$\frac{\partial \vec{r}}{\partial \theta} = R\cos\phi\cos\theta\,\hat{x} + R\cos\theta\sin\phi\,\hat{y} - R\sin\theta\,\hat{z} = R\hat{a_\theta}$

Differential area:

$$ds_R = R^2 sin\theta d\theta d\phi$$

based on Eq 1.

Differential volume:

$$dv = R^2 sin\theta dR d\theta d\phi$$

$\frac{\partial \vec{r}}{\partial \phi} = -R\sin\theta\sin\phi + R\sin\theta\cos\phi = R\sin\theta\,\hat{a_\phi}$

Conversion of Cartesian coordinate and Spherical coordinate

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Conversion of quantities between Cartesian coordinate and Spherical coordinate:

a)

$$\begin{cases} x = Rsin\theta cos\phi \\ y = Rsin\theta sin\phi \\ z = Rcos\theta \end{cases}$$

b)

$$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = arctan\frac{\sqrt{x^2+y^2}}{z} \\ \phi = arctan\frac{y}{x} \end{cases}$$

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin\theta\,dR\,d\theta\,d\phi$

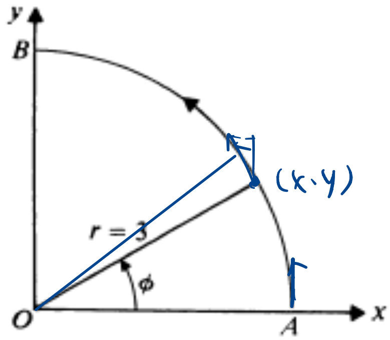
3 Vector Calculus

3.1 Integrals

- Line integral: $\int_C \mathbf{F} \cdot d\mathbf{l}$, where F is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (e.g. $d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$). And if we are integrating over a closed loop, we write $\oint \mathbf{F} \cdot d\mathbf{l}$.
- Surface Integrals: $\int_S \mathbf{F} \cdot d\mathbf{S}$. And if we are integrating over a closed surface, we write $\oint \mathbf{F} \cdot d\mathbf{S}$.
- Volume Integrals: $\int_V \mathbf{F}dv$ or $\int_V Tdv$. In Cartesian coordinates, $dv = dxdydz$. Specially, $\int_V \mathbf{F}dv = \int_V (F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}})\,dv = \hat{\mathbf{x}}\int_V F_xdv + \hat{\mathbf{y}}\int_V F_ydv + \hat{\mathbf{z}}\int_V F_zdv$.

3.1.1 Exercise

EXAMPLE 2-14: Given $\vec{F} = \vec{a_x}xy - \vec{a_y}\overset{x^2}{\cancel{2x}}$, evaluate the scalar line integral along the quarter-circle shown in Fig. 2-21.



$y = \sqrt{9-x^2}$ $\quad x > 0$

FIGURE 2–21
Path for line integral (Example 2–14).

(a) By Cartesian coordinate:

$\vec{F} = xy\hat{a_x} - x^2\hat{a_y}$

$\vec{dl} = dx\hat{a_x} + dy\hat{a_y} = dx\,a_x + \frac{-2x}{2\sqrt{9-x^2}}dx\,\hat{a_y}$

$\vec{F}\cdot\vec{dl} = \int_3^0 x\sqrt{9-x^2}\,dx + \frac{x^3}{\sqrt{9-x^2}}dx = -27$

(b) By Cylindrical coordinate:

$\vec{F} = xy\hat{a_x} - x^2\hat{a_y} = \overset{=0}{(xy\cos\phi - x^2\sin\phi)}\hat{a_r} + (-xy\sin\phi - x^2\cos\phi)a_\phi + 0\,\hat{a_z}$

$\hat{a_x} = \cos\phi\,\hat{a_r} - \sin\phi\,\hat{a_\phi}$ $\quad x = 3\cos\phi \quad y = 3\sin\phi$

$\hat{a_y} = \sin\phi\,a_r + \cos\phi\,a_\phi$ $\quad \vec{F}\cdot dl = \int_0^{\frac{\pi}{2}} -9\cos\theta\cdot 3\,d\theta = -27$

$\vec{dl} = dR\hat{a_R} + Rd\phi\,\hat{a_\phi} + dz\,\hat{a_z}$

3.2 Gradient of a Scalar Field

Gradient of a scalar field describes the maximum space rate of change of a scalar field at a given time, which is a vector along that direction. In other words, it measures how fast a scalar field changes.

$$\text{grad}\, f \triangleq \mathbf{a}_n \frac{df}{dn}$$

$df = \nabla f \cdot d\vec{r}$

Given a function $f(u, v, w)$ in a curvilinear coordinate system, we would like to find a form for the gradient operator.

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w$$

How to derive?

$h_u du = e_u\cdot d\vec{r} \Rightarrow du = \frac{e_u\cdot dr}{h_u}$

$df = \nabla f\cdot d\vec{r}$

$= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw$

$= \frac{1}{h_u}\frac{\partial f}{\partial u}e_u\cdot d\vec{r} + \frac{1}{h_v}\frac{\partial f}{\partial v}e_v\cdot d\vec{r} + \frac{1}{h_w}\frac{\partial f}{\partial w}e_w\cdot dw$

3.3 Divergence of a vector field

Divergence of a vector field $\vec{A}$ at a point $\operatorname{div}\vec{A}$ as the net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero:

$$\operatorname{div}\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\,\mathrm{d}s}{\Delta v}$$

source: net positive divergence; sink: net negative divergence. zero divergence: no source/sink.

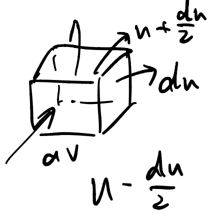
(1) $\operatorname{div}\vec{A}$ at certain point is a scalar.

(2) $\nabla \cdot \vec{A} \equiv \operatorname{div}\vec{A}$

(3)

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

How to derive?



$$\text{Flux}_u = \frac{\partial A_u h_v h_w\, dv\, dw}{\partial u}\, du$$

$$\int A \cdot n\, ds = \frac{\partial A_u h_v h_w}{\partial u}\, dv\, dw\, du + \frac{\partial A_v h_u h_w}{\partial v}\, dv\, du\, dw + \frac{\partial A_w du dv}{\partial v}\, du\, dv\, dw$$

$$dv = h_u h_v h_w\, du\, dv\, dw$$

$$\Rightarrow \frac{\int \vec{A} \cdot \vec{n}\, ds}{dv} = \frac{1}{h_u h_v h_w}\left(\frac{\partial A_u h_v h_w}{\partial u} + \frac{\partial A_v h_u h_w}{\partial v} + \frac{\partial A_w h_u h_v}{\partial w}\right)$$

flux density

3.4 Divergence Theorem

$$\int_V (\nabla \cdot \vec{A})\, dv = \oint_S \vec{A} \cdot d\vec{s}$$

The volume integral of the divergence of a vector field equals to the total outward flux of the vector through the surface that bounds the volume.

3.5 Curl of a vector field

$$curl\vec{A} \equiv \nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}[\vec{a_n} \oint_C \vec{A} \cdot d\vec{l}]_{max}$$

The curl of a vector field $\vec{A}$, denoted by $curl\vec{A}$ or $\nabla \times \vec{A}$, is a vector whose magnitude is the maximum net circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).

$\nabla \times \vec{A}$ in a general coordinate:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a_{u1}} h_1 & \vec{a_{u2}} h_2 & \vec{a_{u3}} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

curl-free vector field ($\nabla \times \vec{A} = 0$): **irrotational** or **conservative field**

3.5.1 Exercise

Show that $\nabla \times \vec{A} = 0$ if (HW 1-4)

(a) $\mathbf{A} = \mathbf{a}_\phi (k/r)$ in cylindrical coordinates, where k is a constant

(b) $\mathbf{A} = \mathbf{a}_R f(R)$ in spherical coordinates, where $f(R)$ is any function of the radial distance R.

$$\text{(a)} \quad \nabla \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r\frac{k}{r} & 0 \end{vmatrix} = 0$$

$$\text{(b)} \quad \nabla \times \vec{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{a}_R & \hat{a}_\theta R & \hat{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f(R) & 0 & 0 \end{vmatrix} = 0$$

3.6 Stokes's Theorem

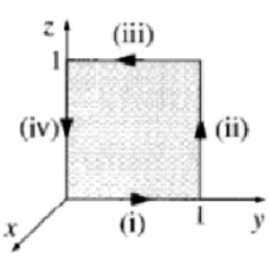
$$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

circulation density

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

3.6.1 Exercise

Suppose $\mathbf{v} = (2xz + 3y^2)\,\hat{\mathbf{y}} + (4yz^2)\,\hat{\mathbf{z}}$. Check Stokes's theorem for the square surface shown below.



$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix} = (4z^2 - 2x)\,\hat{x} + 2z\,\hat{z}$$

$$d\vec{s} = dy\,dz\,\hat{x}$$

$$\int_S (\nabla \times \vec{v}) ds = \int_0^1 \int_0^1 4z^2 dz\,dy = \frac{4}{3}$$

$$\oint \vec{v} \cdot d\vec{l} = \int_0^1 3y^2 dy + \int_0^1 4z^2 dz + \int_1^0 3y^2 dy = \frac{4}{3}$$

3.7 Null identities

I

$$\nabla \times (\nabla V) \equiv 0$$

(1) The curl of the gradient of any scalar field is identically zero.

(2) Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.

(3) Since a curl-free vector field is irrotational or conservative, an irrotational/conservative vector field can always be expressed as the gradient of a scalar field.

II

$$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

(1) The divergence of the curl of any vector field is identically zero.

(2) Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.

(3) Divergenceless field is called solenoidal field, which will be further discussed in later classes.

3.8 Laplacian

The Laplacian of a scalar field is defined as the divergence of the gradient of that field:

$$\begin{aligned}\nabla^2 T = \nabla \cdot (\nabla T) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\end{aligned}$$

Occasionally, we shall speak of the Laplacian of a vector, $\nabla^2 v$. By this we mean a vector quantity whose x-component is the Laplacian of v_x, and so on:

$$\nabla^2 \mathbf{v} = \left(\nabla^2 v_x\right)\hat{\mathbf{x}} + \left(\nabla^2 v_y\right)\hat{\mathbf{y}} + \left(\nabla^2 v_z\right)\hat{\mathbf{z}}$$

4 Helmholtz's Theorem

Helmholtz's Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

(1) **Helmholtz's theorem is a basic theorem describing the uniqueness of electromagnetic field.**

(2) A vector field in space is uniquely determined by its divergence, curl and definite solution conditions.

(3) More specifically, a vector field can be expressed as the superposition of a divergenceless divergence field and a curl-free vector field.

4.1 Exercise

P.2 -39 Given a vector function $\mathbf{F} = \mathbf{a}_x (x + c_1 z) + \mathbf{a}_y (c_2 x - 3z) + \mathbf{a}_z (x + c_3 y + c_4 z)$. a) Determine the constants c_1, c_2, and c_3 if $\mathbf{F}$ is irrotational. b) Determine the constant c_4 if $\mathbf{F}$ is also solenoidal. c) Determine the scalar potential function V whose negative gradient equals F.

add space

$$\nabla \times \vec{F} = 0 \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + c_1 z & c_2 x - 3z & x + c_3 y + c_4 z \end{vmatrix}$$

$$= (c_3 + 3)\hat{x} + (c_1 - 1)\hat{y} + c_2 \hat{z} = 0$$

$c_2 = 0$

$c_1 = 1$

$c_3 = -3$

5 Other useful vector properties

$$\nabla(\psi \phi) = \phi \nabla \psi + \psi \nabla \phi$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$$

$$\nabla \times (\psi \mathbf{A}) = \psi (\nabla \times \mathbf{A}) + \nabla \psi \times \mathbf{A}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla \psi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \psi) = \nabla^2 \psi \text{ (scalar Laplacian)}$$

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A} \text{ (vector Laplacian)}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$$

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

(b) $\nabla \cdot \vec{F} = 0 \Rightarrow \nabla \cdot \vec{F}$

$$\vec{F} = (x + z)\hat{x} - 3z\hat{y} + (x - 3y + c_4 z)\hat{z}$$

$$1 + c_4 = 0 \Rightarrow c_4 = -1$$

(c) $\vec{F} = (x + z)\hat{x} - 3z\hat{y} + (x - 3y - z)\hat{z}$

$$\frac{1}{2}x^2 + xz + f(y, z)$$

$$V = \underline{\frac{1}{2}x^2 + xz - 3yz - \frac{1}{2}z^2 + C}$$