RC 3: Dielectrics and Boundary Condition

1 Conductors and dielectrics in static electric field

- conductors:
  - Orbiting electrons are loosely held by an atom and migrate easily from on atom to another.
  - **static state conditions**:
    - inside the conductor:

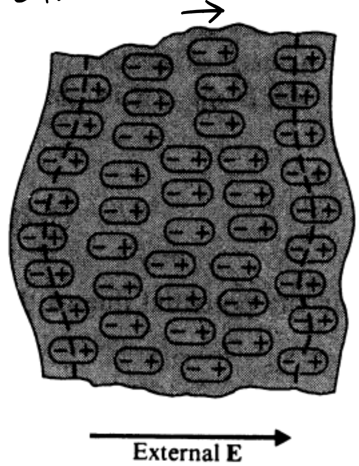
      $$\rho = 0, \boldsymbol{E} = 0$$

      , where $\rho = 0$ represents no charge in the interior
    - on the conductor surface (boundary conditions)

      $$E_t = 0, E_n = \frac{\rho_s}{\epsilon_0}$$

- semiconductos: ← doping
  - Relatively small number of freely movable charges.
- insulators(dielectrics):
  - Electrons are confined to their orbits.
  - External electric field $E_{external}$ polarizes a dielectric material and create electric dipoles. The induced electric dipoles (result in an induced electricfiled $E_{induced}$) will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

$E_{in} = \vec{E}_{external} - E_{induced}$



← $E_{induced}$

Figure 1: A cross section of a polarized dielectric medium

– **polarization charge densities/ bound-charge densities:**

∗ **Polarization vector, *P*** (Measures the density of electric dipoles):

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p_k}}{\Delta v}$$

where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv.

∗ Charge distribution on surface density (Polarization surface charge densities):

$$\rho_{ps} = \boldsymbol{P} \cdot \boldsymbol{a_n}$$

$\uparrow \vec{a}_n$ dielectric $\vec{a}_n$

∗ Volume charge distribution density (Polarization bound-charge densities):

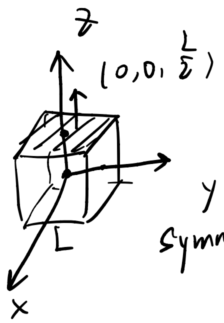
$$\rho_p = -\nabla \cdot \boldsymbol{P}$$

Ex.1

The polarization in a dielectric cube of side L, centered at the origin is given by $\boldsymbol{P} = P_0(\boldsymbol{a_x}x + \boldsymbol{a_y}y + \boldsymbol{a_z}z)$.

a) Determine the surface and volume bound-charge densities.

b) Show that the total bound charge is zero.



(a) $\rho_{ps} = \vec{P} \cdot \vec{a}_n = P_0 \frac{L}{2} \hat{a}_z \cdot \hat{a}_z = \frac{P_0 L}{2}$

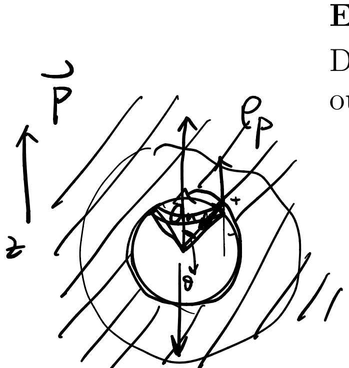
symmetric → All six surface $\rho_{ps} = \frac{P_0 L}{2}$

$\rho_p = -\nabla \cdot \vec{P} = -P_0\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z\right) = -3P_0$

(b) $6L^2 \cdot \frac{P_0 L}{2} + L^3(-3P_0) = 0$ verified.

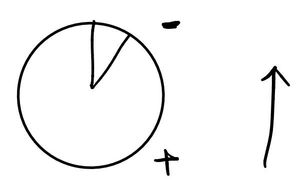
Ex.2

Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\boldsymbol{P}$ exists.

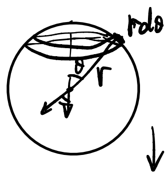


$\vec{P} = P_0 \hat{a}_z$

$\rho_{ps} = \vec{P} \cdot \vec{a}_n = -P_0 \cos\theta$



$dE = \frac{1}{4\pi\varepsilon_0} \frac{-P_0\cos\theta \cdot 2\pi r\sin\theta \cdot r d\theta}{r^2} \cos\theta = \frac{-P_0 \cos^2\theta \sin\theta d\theta}{2\varepsilon_0}$



$E = \int_0^\pi dE = \frac{-P_0}{2\varepsilon_0}\int_0^\pi -\cos^2\theta \, d\cos\theta = \frac{+P_0}{2\varepsilon_0} \frac{1}{3}\cos^3\theta \Big|_1^{-1} = -\frac{P_0}{3\varepsilon_0}$

E direction $|\vec{E}| = \left|\frac{P_0}{3\varepsilon_0}\right|$ same as $P_0 \hat{a}_z$

2 Electric Flux Density and Dielectric Constant

- **electric flux density/electric displacement, $\boldsymbol{D}$**:

$$① \quad \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \quad (C/m^2)$$

- $$\nabla \cdot \boldsymbol{D} = \rho \quad (C/m^3)$$

  where ρ is the volume density of free charges.

$\rho = \rho_{free} + \rho_{induced}$

$\nabla \cdot E = \frac{\rho'}{\epsilon_0} \rightarrow \rho_{free} + \rho_{induced}$

- Another form of **Gauss's law**:

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q \quad (\boldsymbol{C})$$

  the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic** (χ_e is independent of the space coordinate),

$$② \quad \boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$$

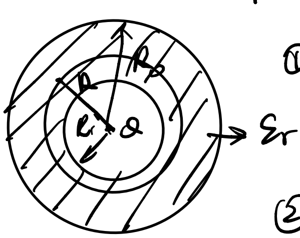
$$③ \quad \boldsymbol{D} = \epsilon_0(1 + \chi_e)\boldsymbol{E} = \epsilon_0 \epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E}$$

大于1

where χ_e is a dimensionless quantity called electric susceptibility, and ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium and ϵ is the absolute permittivity/permittivity of the medium (F/m).

Ex.3 $\epsilon_0 \cdot \epsilon$ (F/m)

A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine $\boldsymbol{E}, V, \boldsymbol{D}, \boldsymbol{P}$ as functions of the radial distance R.



① $D_{(R)} \cdot 4\pi R^2 = Q \Rightarrow D(R) = \frac{Q}{4\pi R^2}$

R arbitrary

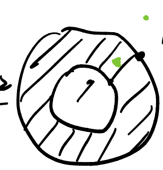
② $\vec{D} = \epsilon \vec{E}$

$$\vec{E}_{(R)} = \begin{cases} \frac{Q}{4\pi\epsilon_0 R^2} & R<R_i / R>R_o \\ \frac{Q}{4\pi\epsilon_0\epsilon_r R^2} & R_i<R<R_o \end{cases}$$

③ $$\vec{P} = (\vec{D} - \epsilon_0\vec{E}) = \begin{cases} 0 & R<R_i / R>R_o \\ \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\epsilon_r}\right) & R_i<R<R_o \end{cases}$$

④ $V = -\int \vec{E} \cdot d\vec{l}$

$V_\infty = 0$. V_∞



1. $V(R_o) = ?$

$V(R_o) - 0 = \int_{R_o}^{\infty} E \cdot dR$

$= \int_{R_o}^{\infty} \frac{Q}{4\pi\epsilon R^2} dR = \frac{Q}{4\pi\epsilon}\left(0 + \frac{1}{R_o}\right)$

$= \frac{Q}{4\pi\epsilon R_o}$

2. $V(R_i) - V(R_o) = \int_{R_i}^{R_o} E \cdot dR$

$= \int_{R_i}^{R_o} \frac{Q}{4\pi\epsilon_0\epsilon_r R^2} dR = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$

3. $V(R) - V(R_i) = \int_{R}^{R_i} E \cdot dR = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_i}\right)$

$R < R_i$ $V(R) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_0}\left(1-\frac{1}{\varepsilon_r}\right) + \frac{1}{R} + \frac{1}{R_i}(\frac{1}{\varepsilon_r}$

$R_i < R < R_0$ $V(R) - V(R_0) = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{R} - \frac{1}{R_0}\right)$

$V(R) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\varepsilon_r R} + \left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_0}\right)$

$R > R_0$ $V(R) = \frac{Q}{4\pi\varepsilon_0 R}$

- For anisotropic,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For biaxial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

  For uniaxial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

- dielectric breakdown: electric field is very strong, causes permanent dislocations and damage in the material.

  dielectric strength: the maximum electric field intensity that a dielectric material can withstand without breakdown.

3 Boundary Conditions for Electrostatic Fields

- the tangential component of an $\boldsymbol{E}$ field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$

  or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

$\vec{D}_2$ D_1 ε (大多数)

bounded charge

if no free charge in the surface $\vec{D}_{1n} = \vec{D}_{2n}$

- The normal component of $\boldsymbol{D}$ field is discontinous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$\vec{E}_{1n} \neq \vec{E}_{2n}$

$$\boldsymbol{a_{n2}} \cdot (\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_s$$

  or

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

$E_{1t} = E_{2t}$

$D = \varepsilon E$

$(\vec{D}_{2n} - \vec{D}_{1n}) = \rho_s$

general

- In terms of potential,

$$V_1 = V_2$$

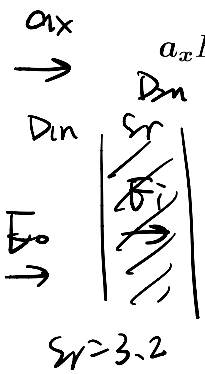
$$\epsilon_1 \frac{\partial V_1}{\partial n} - \epsilon_2 \frac{\partial V_2}{\partial n} = -\rho_{sf}$$

$E = -\frac{\partial V}{\partial x}$

$\rho_s = 0$ for most cases.

Ex.4

A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\boldsymbol{E_o} = \boldsymbol{a_x} E_o$ in free space. Determine $vect E_i, \boldsymbol{D_i}, \boldsymbol{P_i}$ inside the lucite.



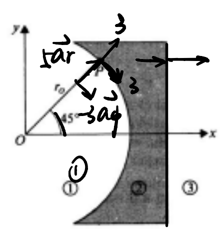
$$\vec{D}_{1n} = \vec{D}_{2n}$$

$$\vec{D}_{1n} = \varepsilon_0 \vec{E}_0 = \vec{D}_{2n} = \varepsilon_0 \varepsilon_r \vec{E}_i \Rightarrow \begin{cases} \vec{D}_i = \varepsilon_0 \vec{E}_0 \\ \vec{E}_i = \dfrac{\vec{E}_0}{\varepsilon_r} \end{cases}$$

$$\vec{D}_i = \varepsilon_0 \vec{E}_i + \vec{P}_i \Rightarrow \vec{P}_i = \vec{D}_i - \varepsilon_0 \vec{E}_i = \varepsilon_0 \vec{E}_0 \left(1 - \frac{1}{\varepsilon_r}\right)$$

Ex.5

Dielectric lenses can be used to collimate electromagnetics fields. In Fig **??**, the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\boldsymbol{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\boldsymbol{a_r}5 - \boldsymbol{a_\phi 3}$, what must be the dielectric constant of the lenses in order that $\boldsymbol{E_3}$ in region 3 is parallel to the x-axis?



ε_r ?

$|E_{\phi i}| = |E_{ri}|$ → enable parallel

$|E_{\phi i}| = |E_{\phi o}| = 3$

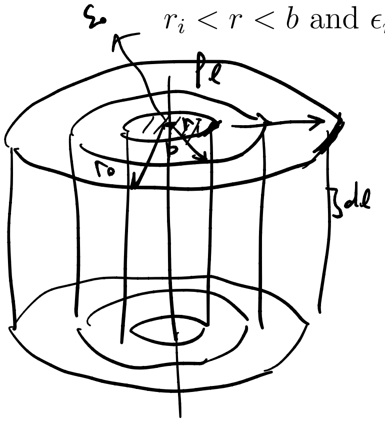
$|E_{ri}| = 3$

$$E_{ri} \cdot \varepsilon_0 \varepsilon_r = E_{ro} \cdot \varepsilon_0 \Rightarrow \varepsilon_r = \frac{E_{ro}}{E_{ri}} = \frac{5}{3}$$

Ex.6

The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

ρ_ℓ.



$$D \cdot 2\pi r \cancel{d\ell} = \rho_\ell \cancel{d\ell}$$

$$C_\ell = \frac{Q_\ell}{V_{ri} - V_{ro}} \quad = \rho_\ell$$

$$\vec{D} = \frac{\rho_\ell}{2\pi r} \hat{a}_r$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{E} = \begin{cases} \hat{a}_r \dfrac{\rho_\ell}{2\pi \varepsilon_{r1} r} = E_1 & r_i < r < b \\ \hat{a}_r \dfrac{\rho_\ell}{2\pi \varepsilon_{r2} r} = E_2 & b < r < r_o \end{cases}$$

$$V_{ri} - V_{ro} = \int_{r_i}^{b} |\vec{E}_1| dr + \int_{b}^{r_o} |\vec{E}_2| dr = \frac{\rho_\ell}{2\pi} \left(\frac{1}{\varepsilon_{r1}} \ln \frac{b}{r_i} + \frac{1}{\varepsilon_{r2}} \ln \frac{r_o}{b} \right)$$

$$C_\ell = \frac{\rho_\ell}{\Delta V} = 2\pi \Big/ \left(\frac{1}{\varepsilon_{r1}} \ln \frac{b}{r_i} + \frac{1}{\varepsilon_{r2}} \ln \frac{r_o}{b} \right)$$

Ex.7

Two dielectric media with permittivities ϵ_1 and ϵ_2 are separated by a charge-free boundary as shown in Fig 2. The electric field intensity in medium 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the elctric field intensity at point P_2 in medium 2.

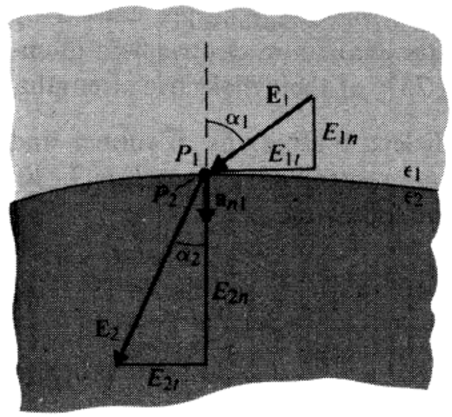
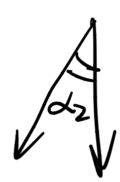


Figure 2: Boundary conditions at the interface between two dielectric media

$$\underline{D_{1n}} = D_{2n} \Leftrightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \Leftrightarrow \epsilon_1 E_1 \cos\alpha_1 = \epsilon_2 E_{2n} \Rightarrow E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_1 \cos\alpha_1$$

$$\underline{E_{1t}} = E_{2t} \Leftrightarrow E_{2t} = E_1 \sin\alpha_1$$

$$\underline{\tan\alpha_2} = \frac{E_{2t}}{E_{2n}} = \frac{\epsilon_1}{\epsilon_2} \tan\alpha_1$$



$$|E_2| = \sqrt{(E_{2t})^2 + (E_{2n})^2}$$