Chapter 2: Vector Analysis

Lecturer: Nana Liu

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JOINT INSTITUTE
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Many slides courtesy of Sung-Liang Chen

2-1 Introduction

- Vector algebra: addition, subtraction, and multiplication of vectors
- Orthogonal coordinate systems: Cartesian, cylindrical, and spherical coordinates
- Vector calculus: differentiation and integration of vectors; line, surface, and volume integrals; “del” operator; gradient, divergence, and curl operations.

Everything you need to know about vector analysis for EM in a few slides…

All the questions you can ask…

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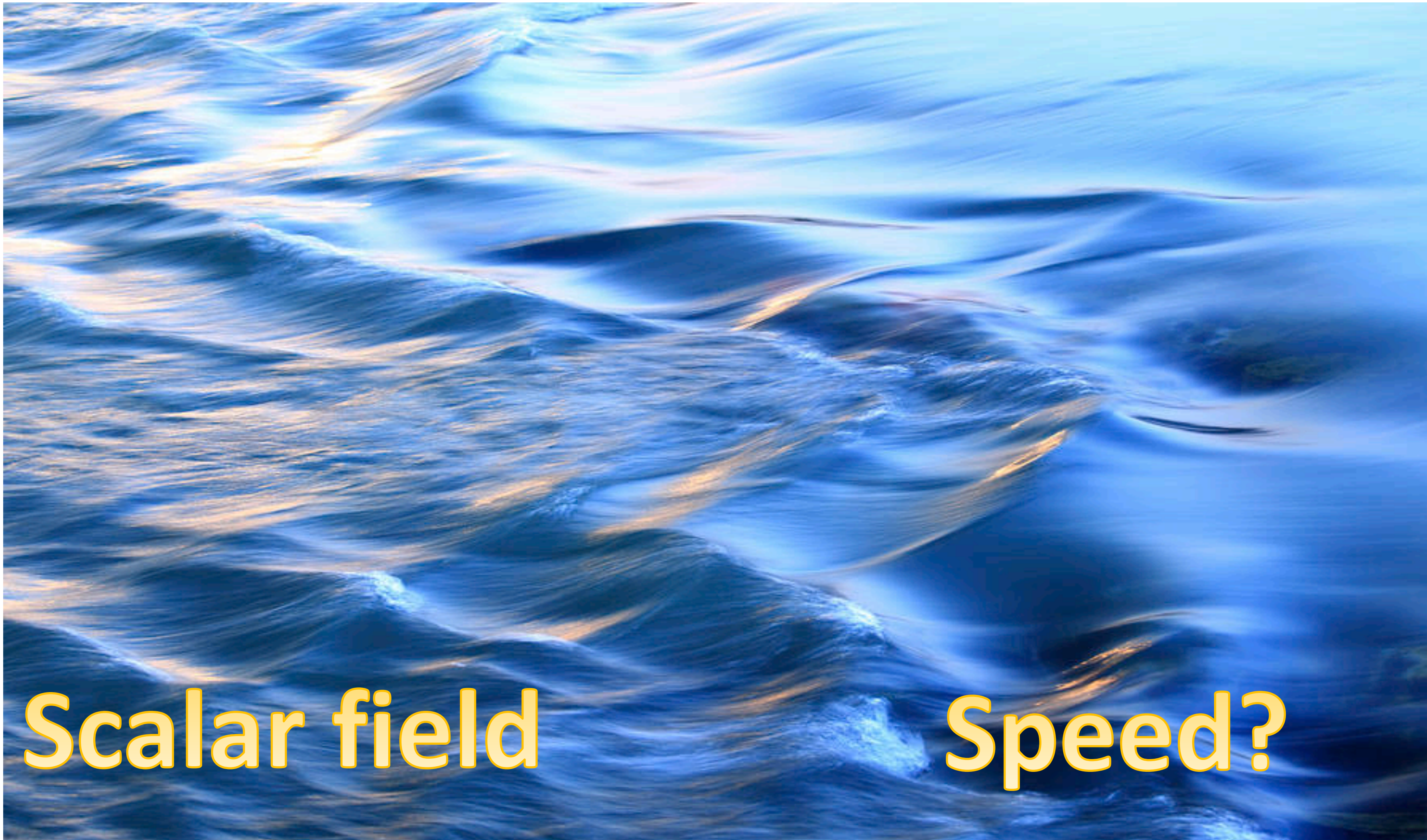

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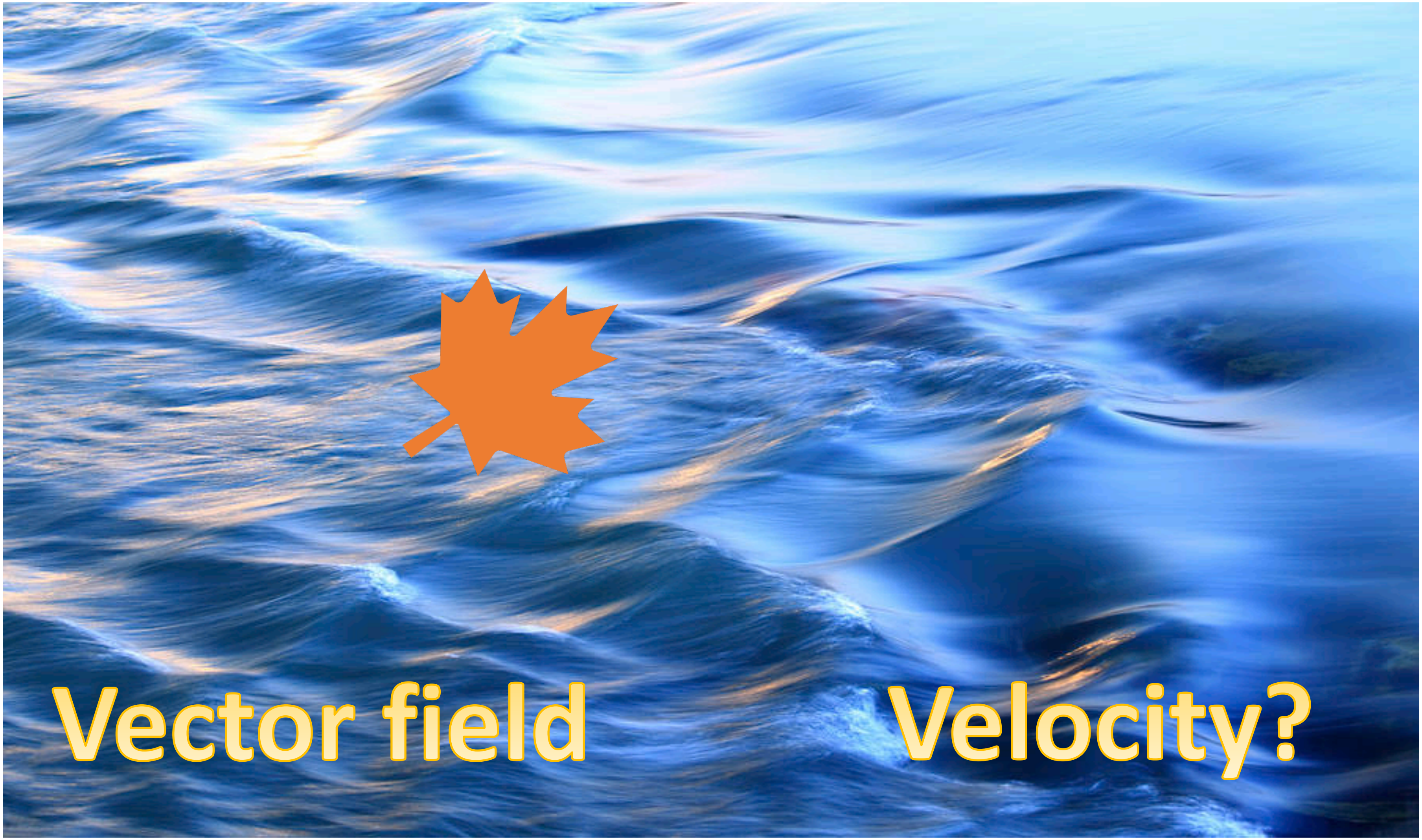

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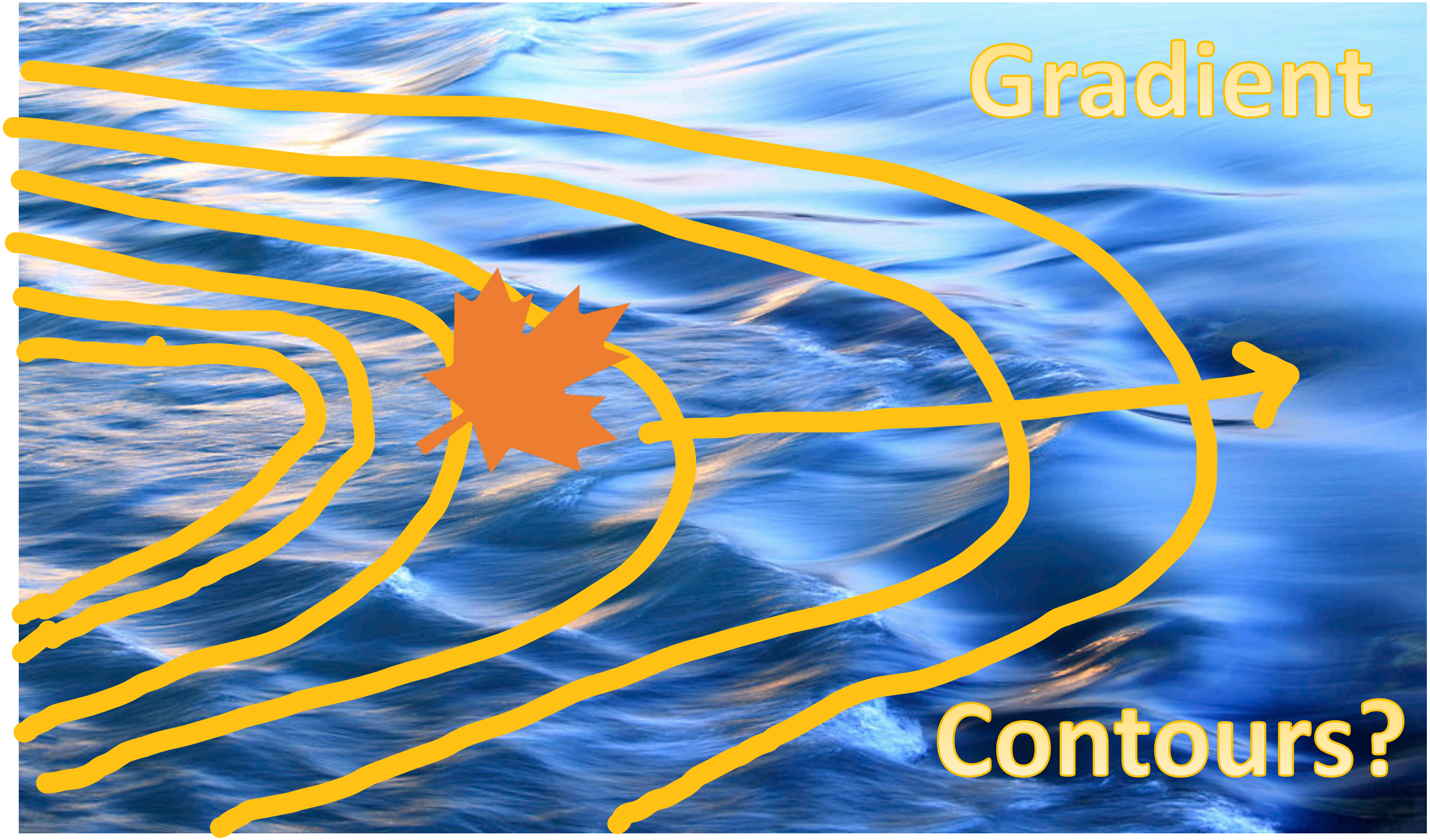

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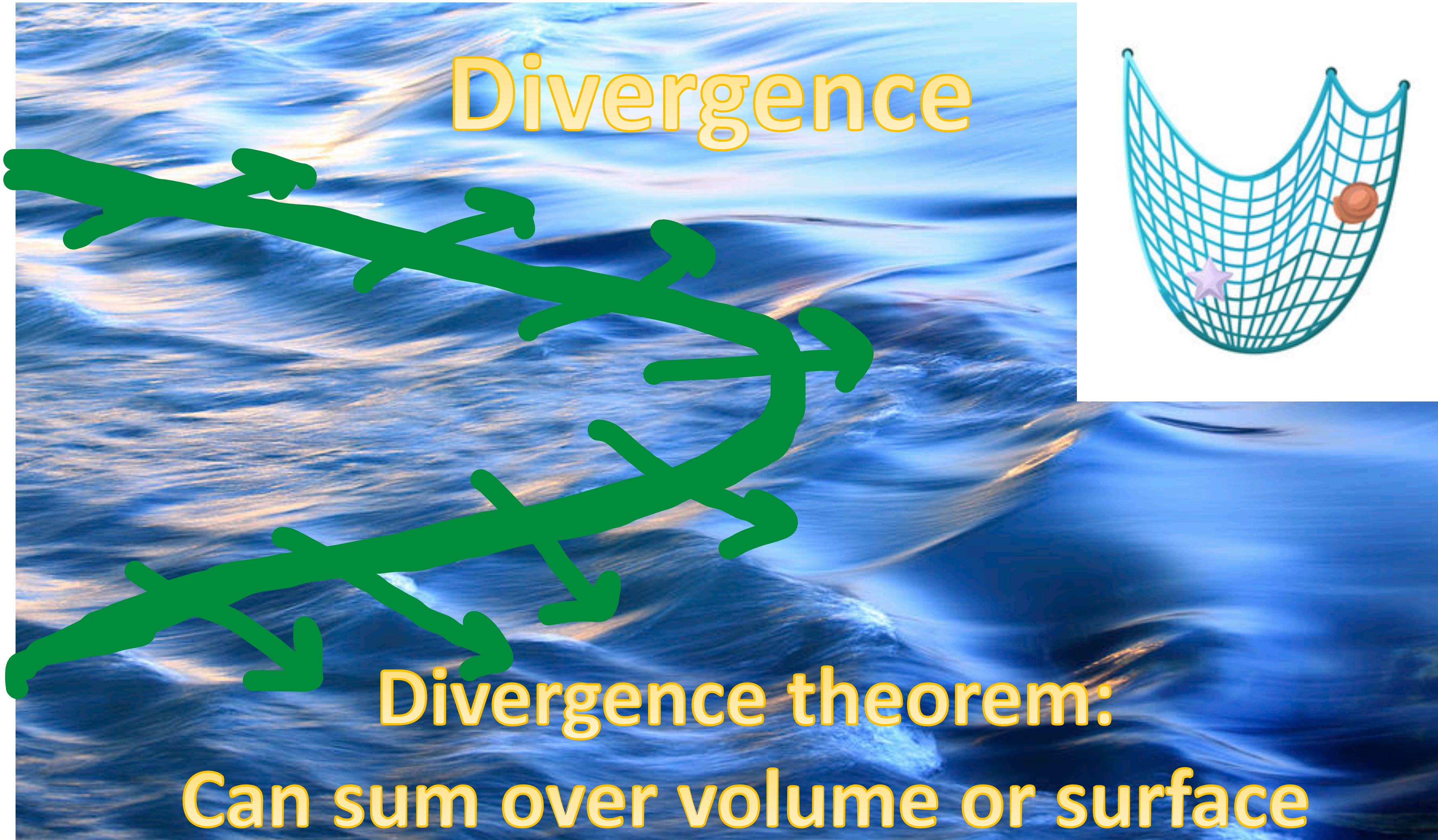

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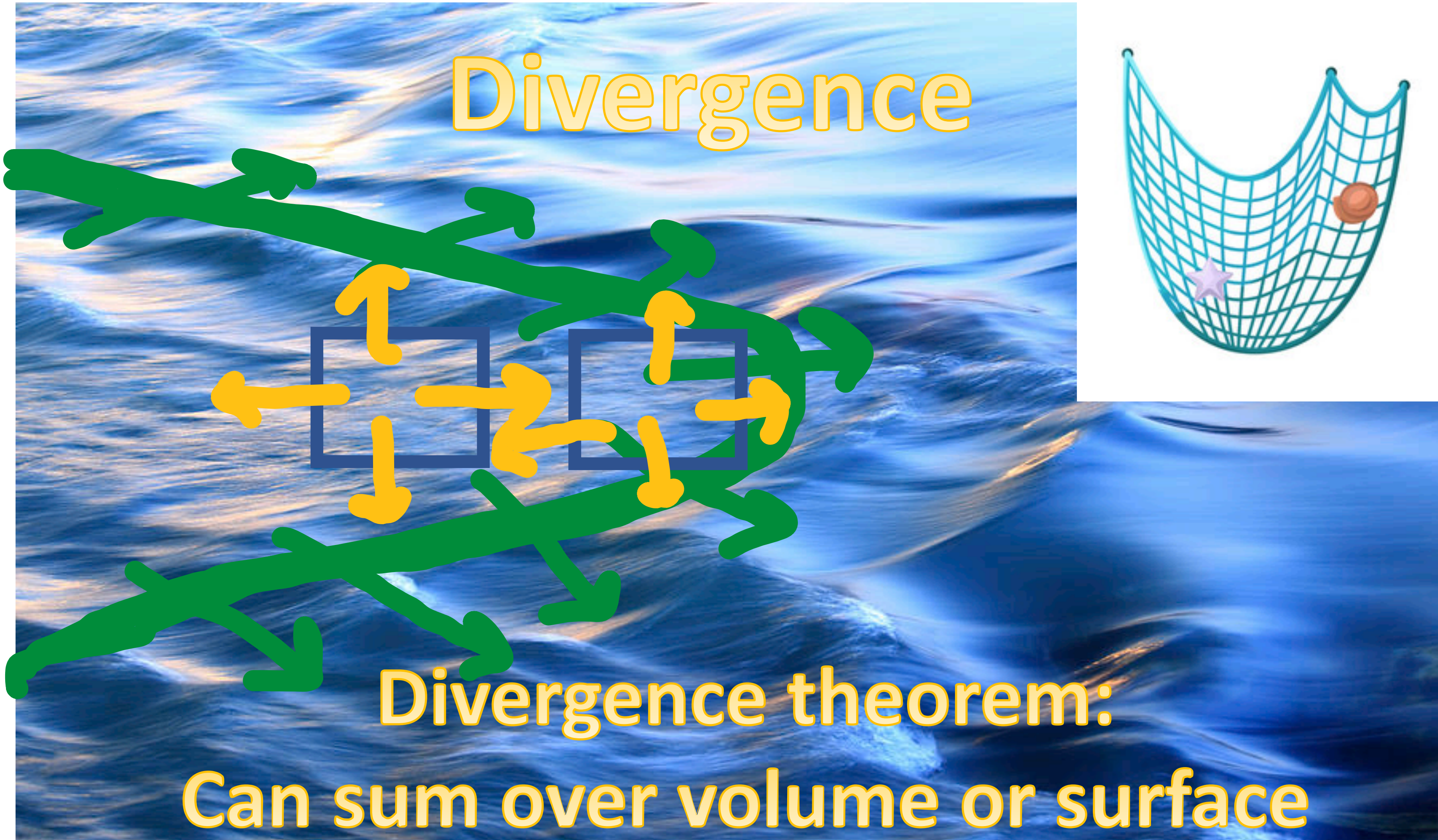

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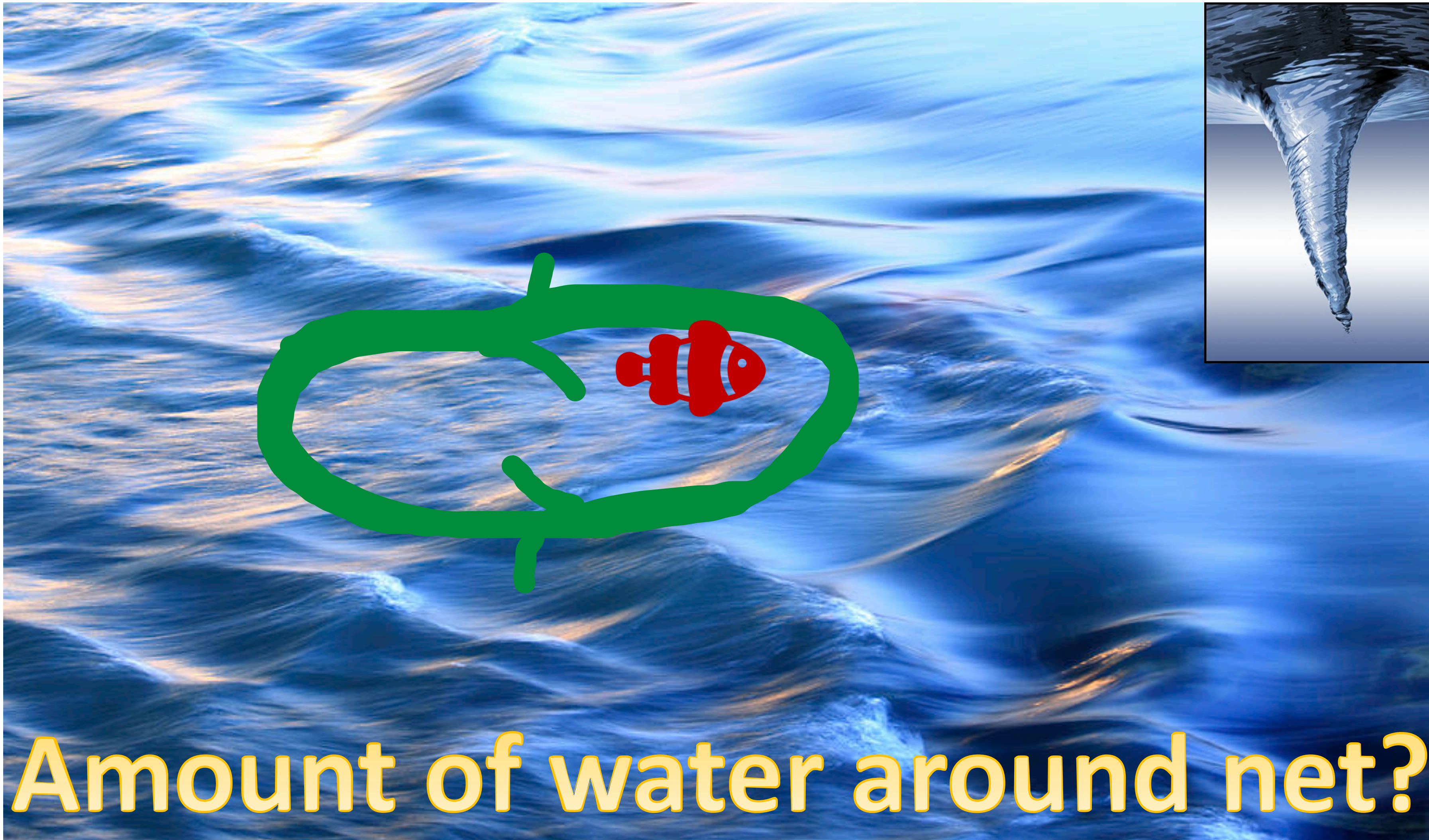

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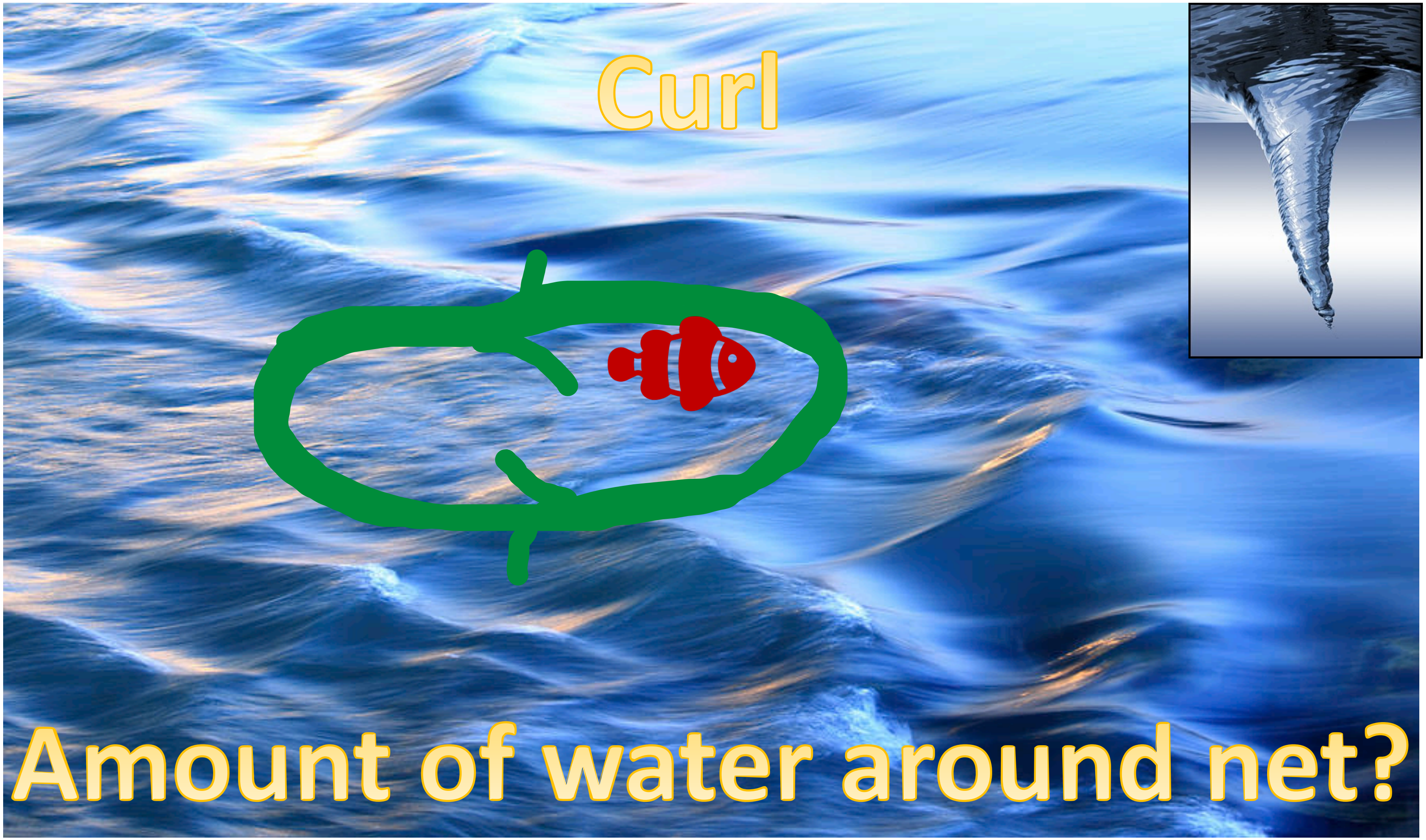

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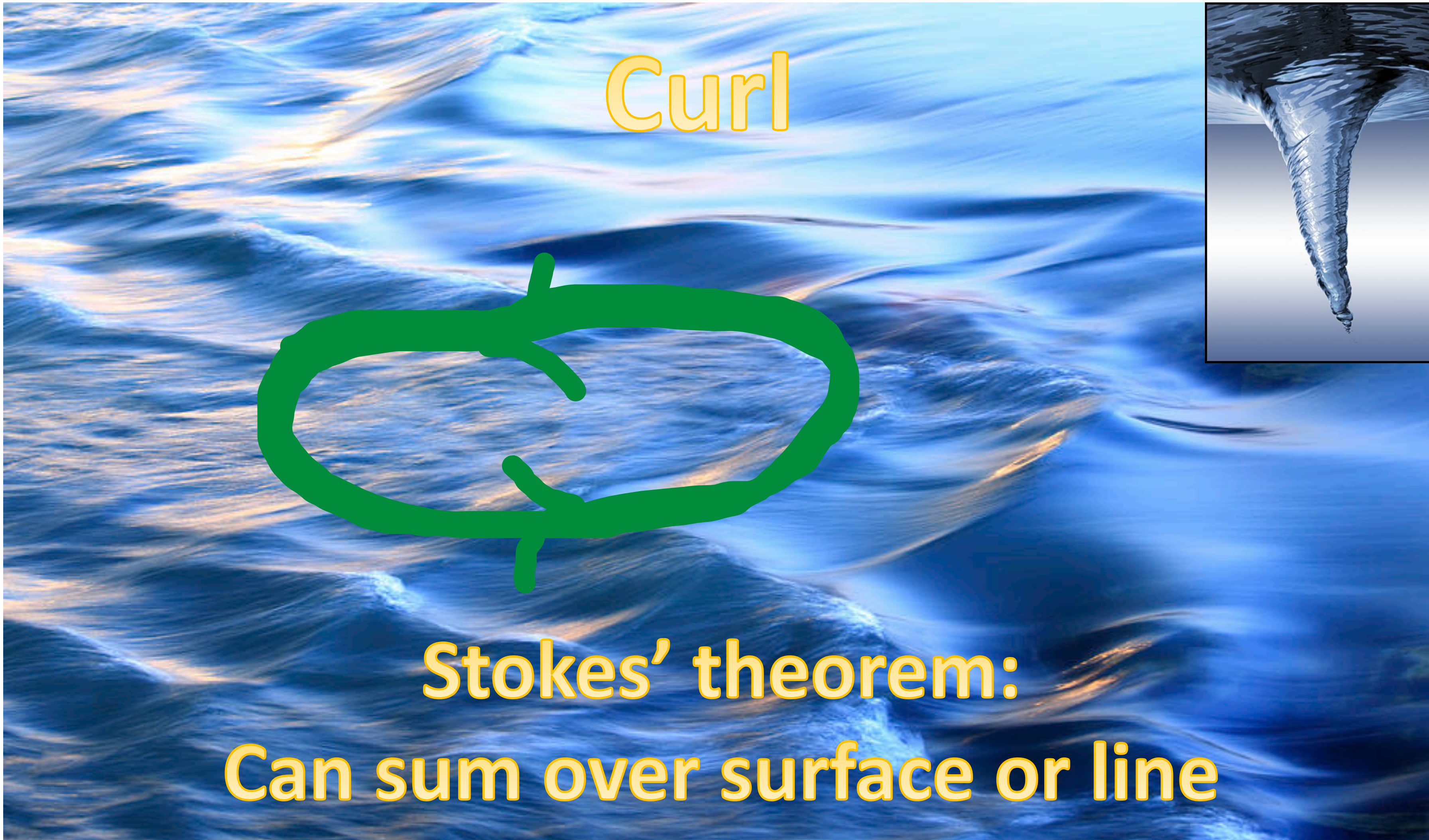

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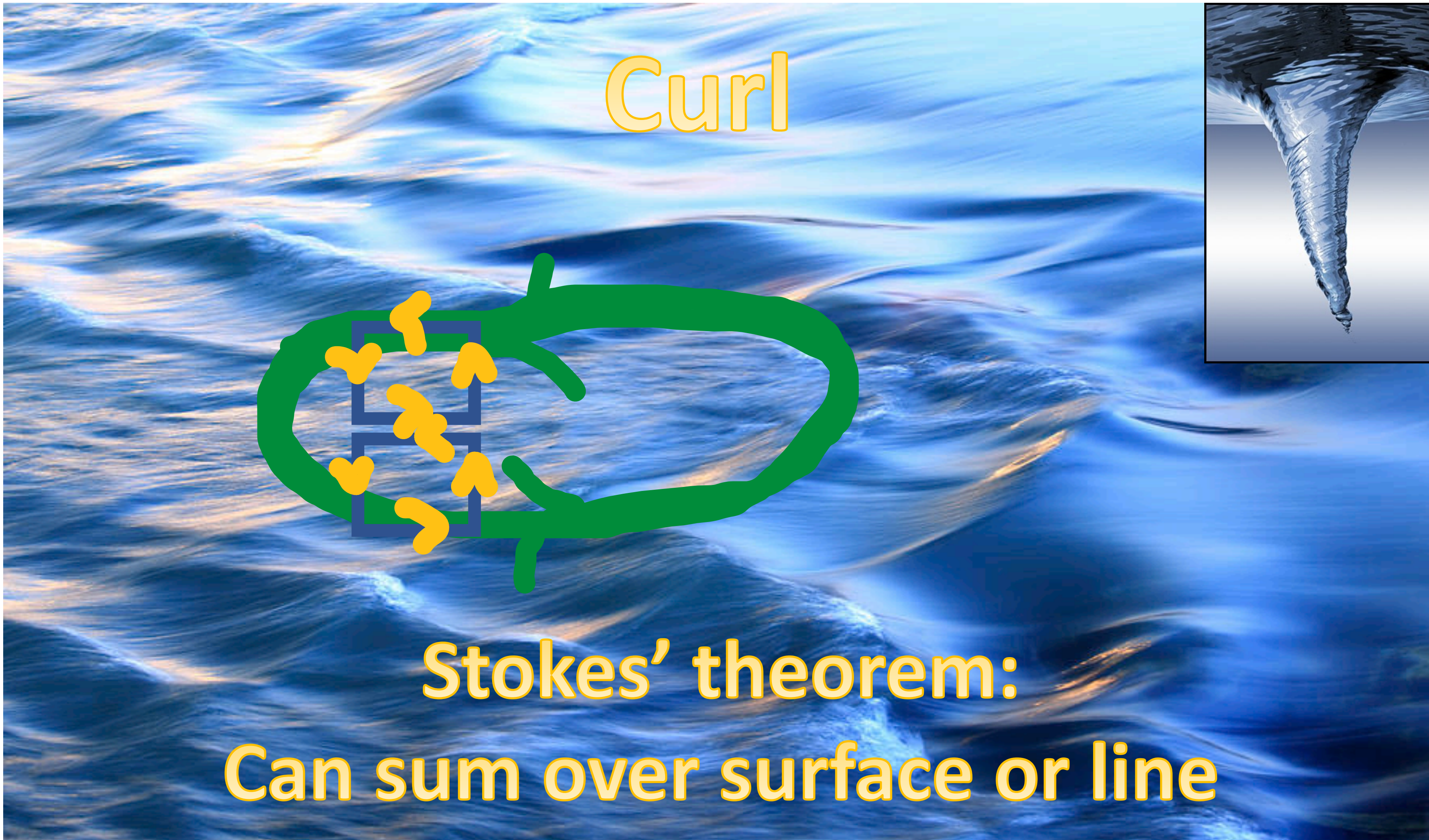

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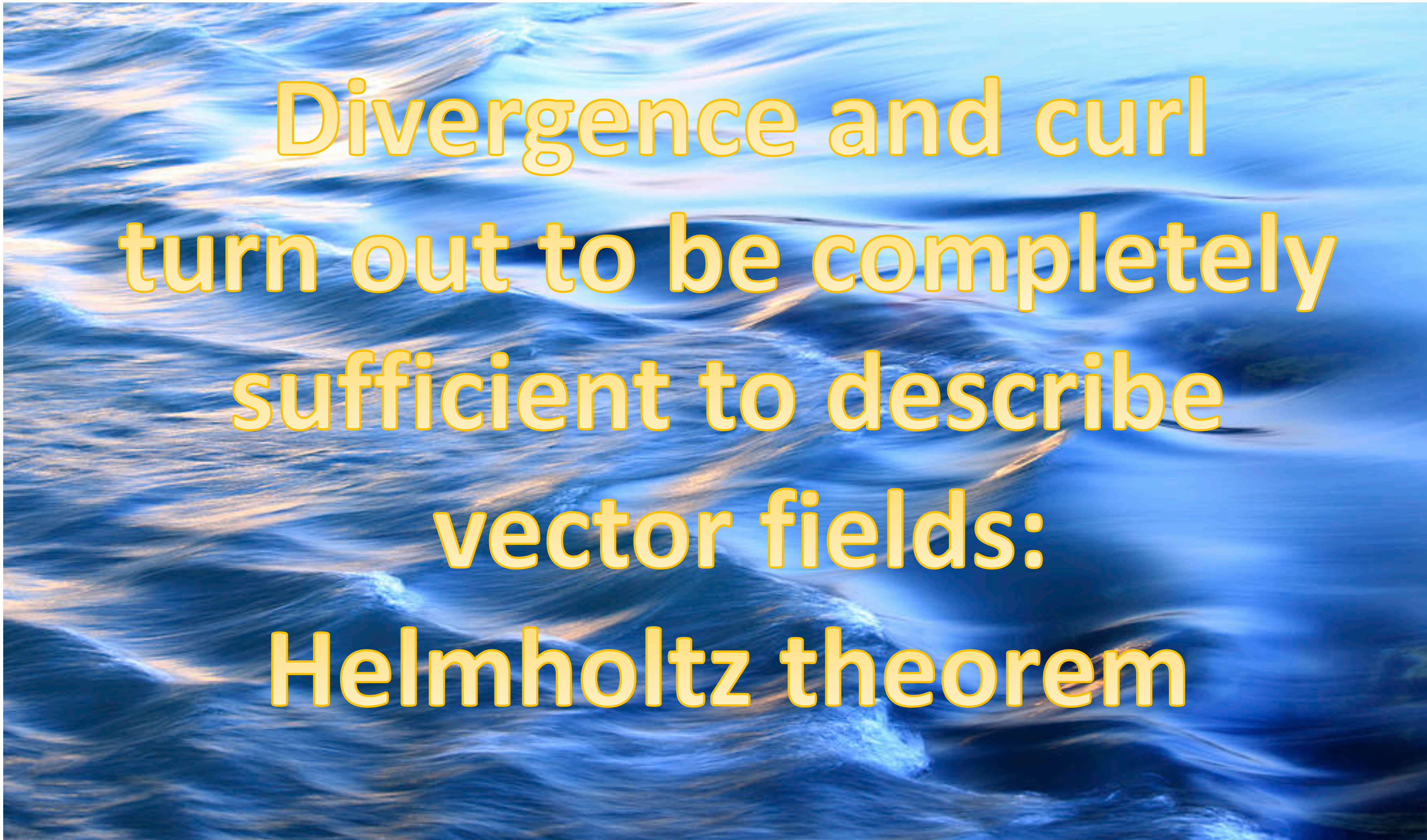

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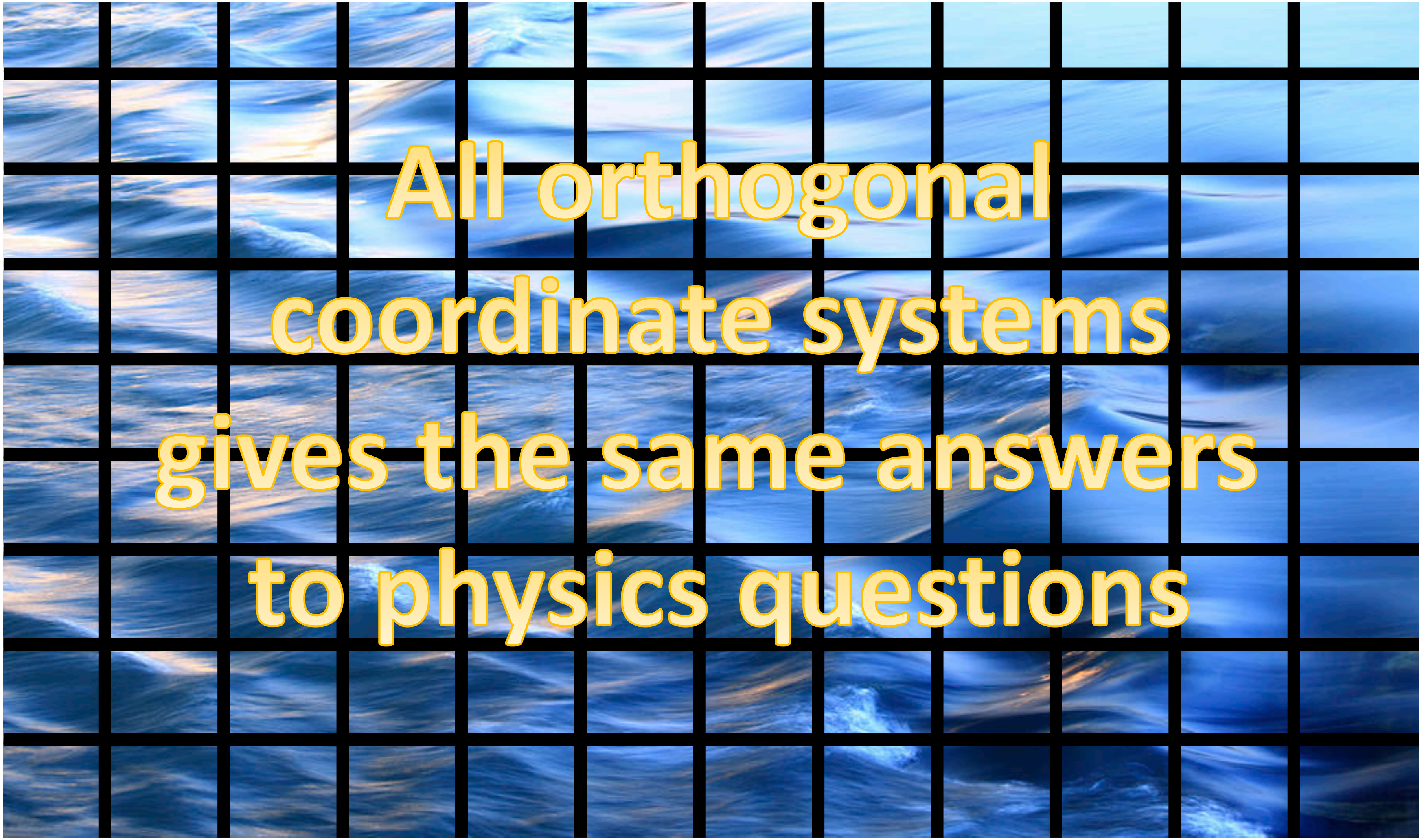


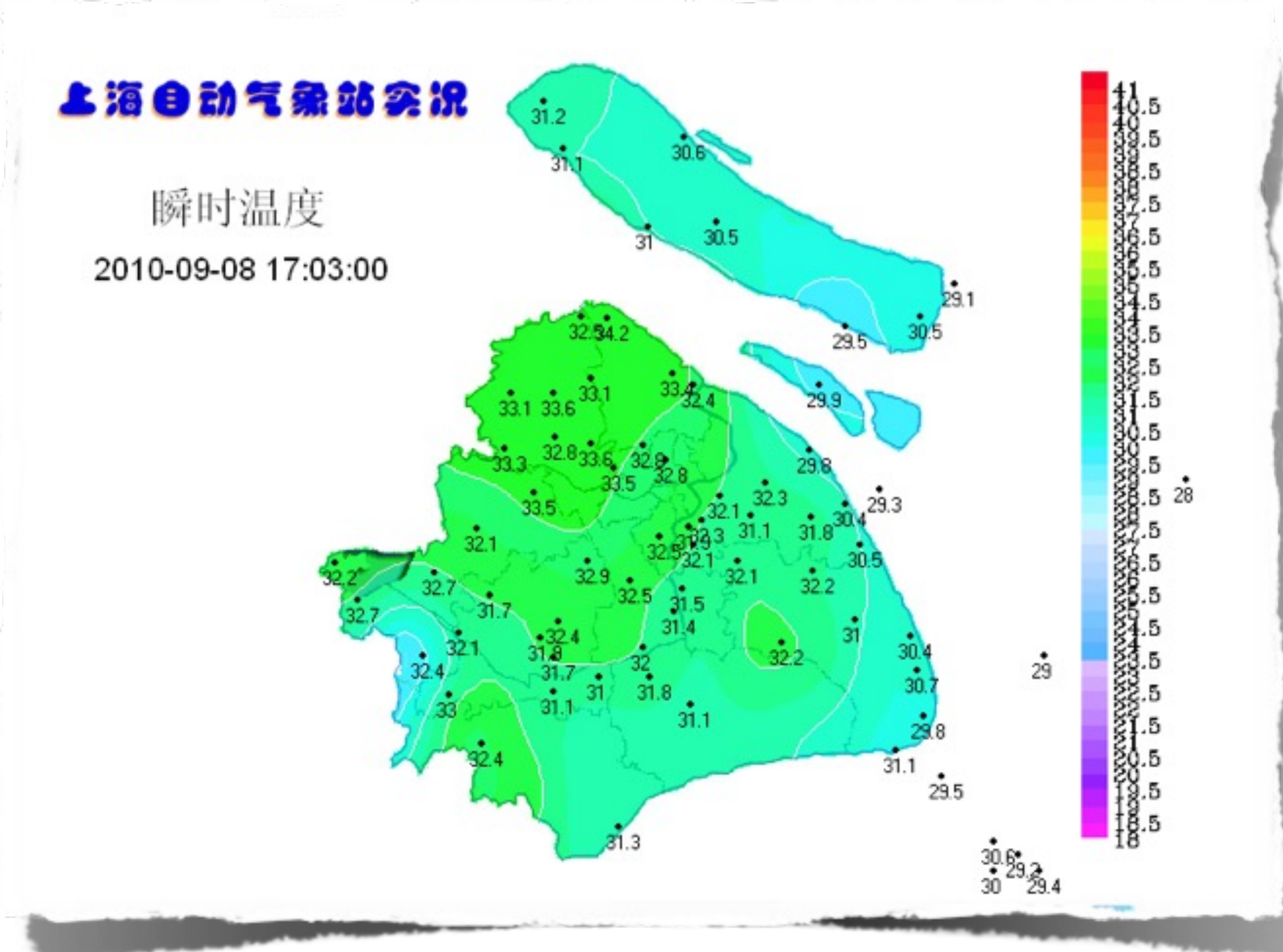
Vector algebra

Definition - Merriam Webster

- Scalar: a quantity that has a magnitude describable by a real number and no direction.
- Vector: a quantity that has magnitude and direction.

Examples of Scalars and Vectors

- Today's highest temperature 26°C, lowest temperature 18°C
- Wind level 3 to level 4 from south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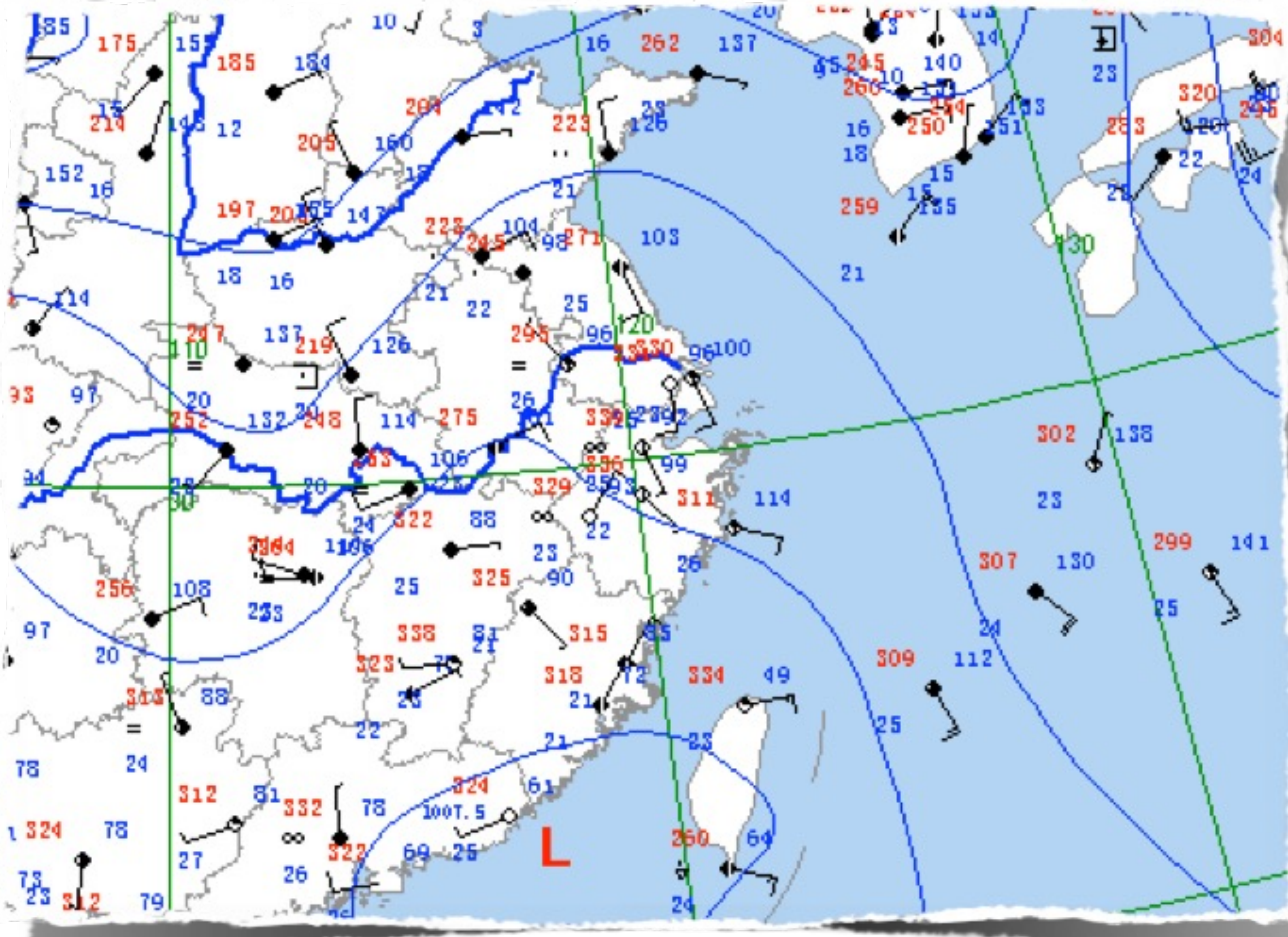
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2-2 Vector Addition and Subtraction

- Vector representation:

$$\mathbf{A} = \mathbf{a}_A A,$$

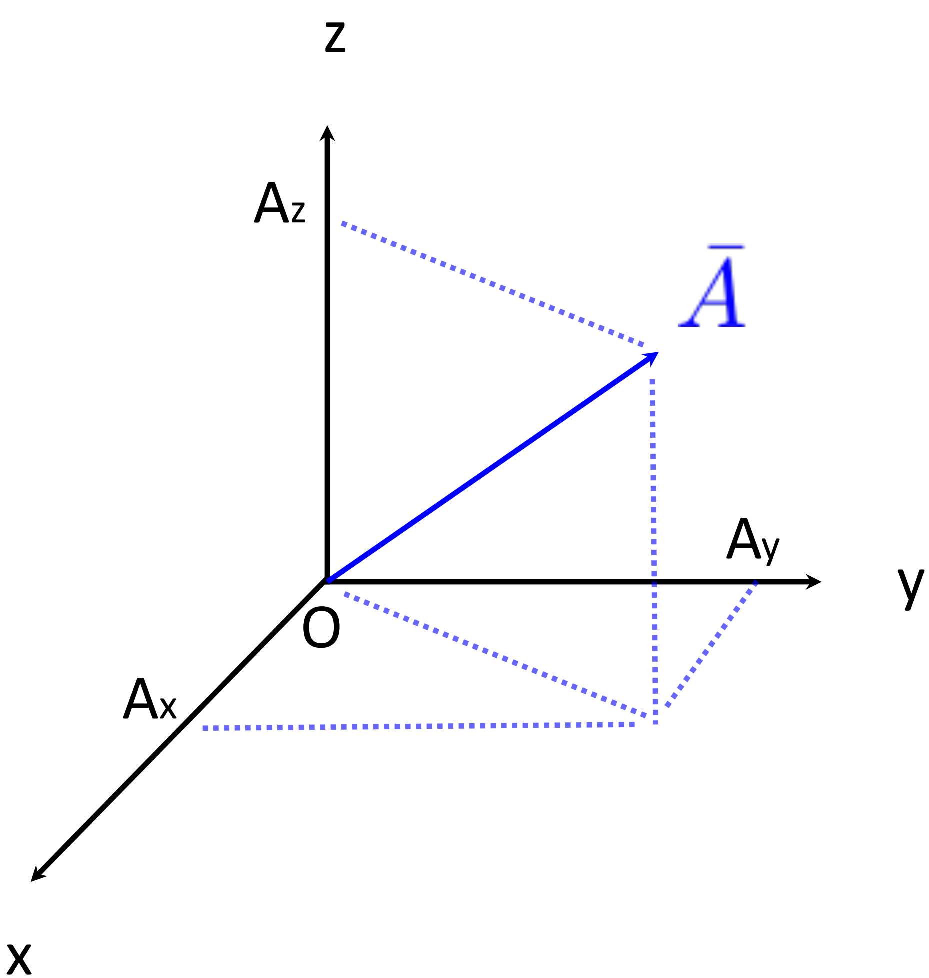
$$A = |\mathbf{A}|$$

Vector magnitude (same as scalar)

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$

Unit vector, has magnitude of 1, only denotes direction

Vector Representation - Cartesian Coordinate System

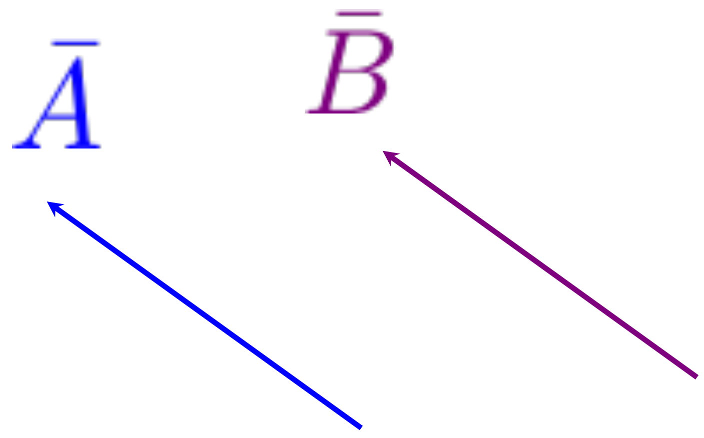


Hat to represent unit vector

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

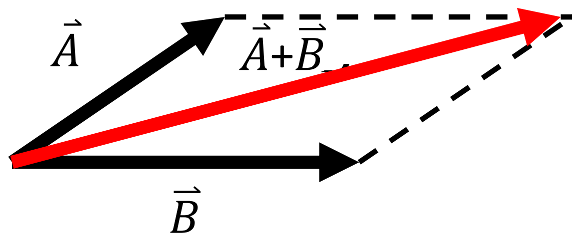
Equality of Two Vectors

- Two vectors **A** and **B** are said to be equal if they have equal magnitudes and unit vectors.

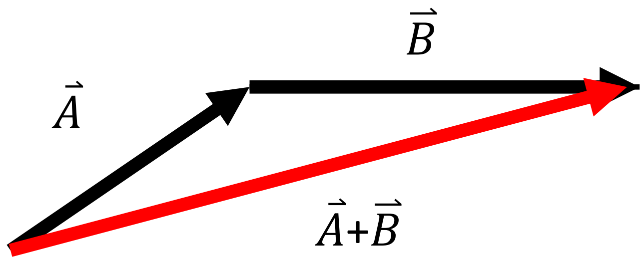


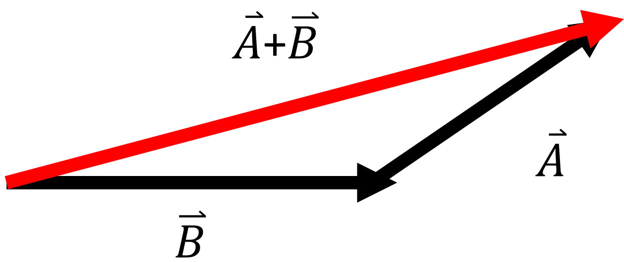
Vector Addition

parallelogram rule



head-to-tail rule

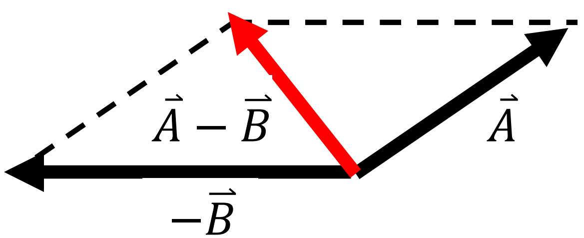




Commutative law $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

Vector Subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



Vector subtraction is mathematically the same as vector addition, except the direction of the subtracted vector is reversed.

Vector Operations

Commutative Law of Vector Addition: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

The order of vector addition does not matter.

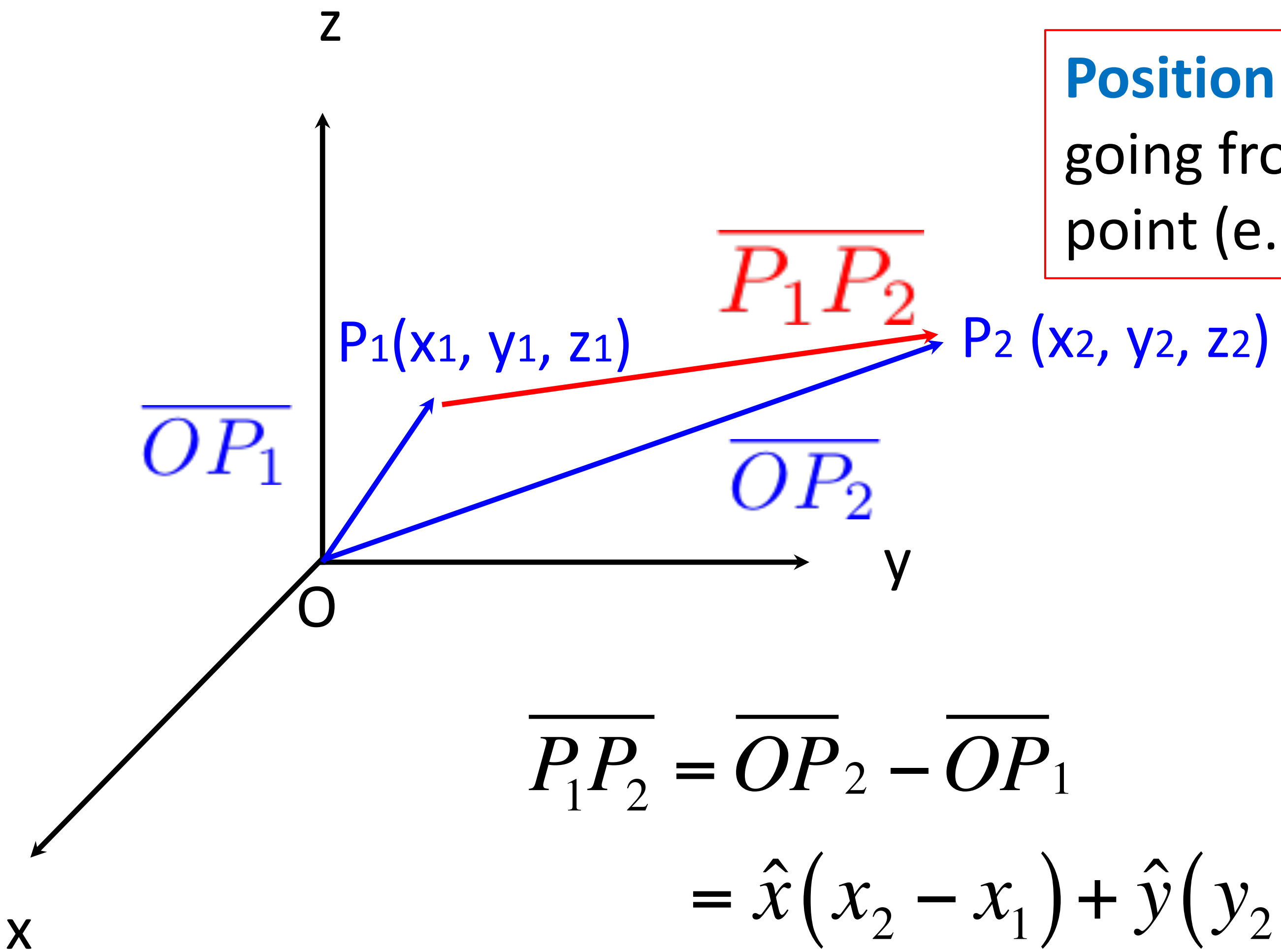
Associative Law of Vector Addition: $\vec{A}+\left(\vec{B} + \vec{C}\right) = \left(\vec{A} + \vec{B}\right) + \vec{C}$

Again, the order of vector addition does not matter.

Distributive Law of Vector Addition: $n(\vec{A}+\vec{B})= n\vec{A} + n\vec{B}$

The order of vector magnification does not matter.

Position and Distance



Position vector: The vector going from the origin to the point (e.g., $\overline{OP_1}$)

$$\overline{P_1P_2} = \overline{OP}_2 - \overline{OP}_1$$

$$= \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)$$

$$\left|\overline{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2-3 Products of Vectors

- Vector Multiplication - Simple Product

$$\vec{B} = k\vec{A}$$

$$\vec{B} = B\hat{B} = k\vec{A} = kA\hat{A}$$

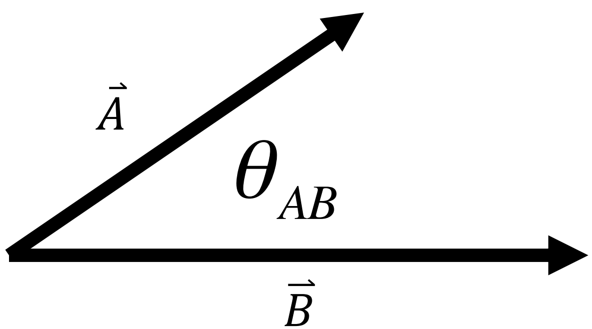
$$\therefore B = kA$$

$$\therefore \hat{B} = \hat{A}$$

does not change the direction but magnitude

Scalar Multiplication-Scalar or Dot Product

Dot Product: $\vec{A} \cdot \vec{B} = |A||B|cos\theta_{AB}$



The dot product (inner product) of two vectors produces a scalar quantity.

Commutative Law of Dot Product: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

The order of dot product multiplication does not matter.

Distributive Law of Dot Product: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Scalar Product in XYZ coordinate

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

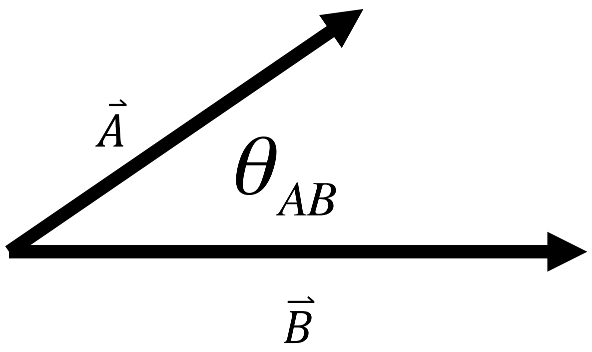
If the DOT product of two non-zero vectors is zero, then they are perpendicular to each other!

$$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$$

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$$

Vector Multiplication - Vector or Cross Product

Cross Product: $\vec{A} \times \vec{B} = |A||B| sin\theta_{AB} \hat{C}$ $\hat{C} = \hat{A} \times \hat{B}$



Direction: The cross product (outer product) of two vectors produces a vector that is perpendicular to both of the original vectors and in a direction that obeys the permutation principle. (Right-Hand Rule)

Non-Commutative Law of Cross Product:

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Non-Associative Law of Cross Product:

The order of cross product multiplication **does matter**.

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$
$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) \neq -(\vec{A} \times \vec{B}) \times \vec{C}$$

Associative Case of Triple Cross Product

Associative when one of the vectors is zero, or when A and C are parallel

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

To prove this, note that $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ and likewise $(A \times B) \times C = -(C \cdot B)A + (C \cdot A)B$

Assume the two are equal, and note that the dot product commutes, and we find

$$(A \cdot B)C = (C \cdot B)A$$

Then,

$$\frac{A}{A \cdot B} = \frac{C}{C \cdot B} \implies C = kA.$$

So C is some scalar multiple of A.

Vector Product in XYZ coordinate

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$$

$$\hat{x} \times \hat{y} = \hat{z}$$

$$\hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \hat{y}$$

Vector Product in Matrix Format

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{x}\left(A_y B_z - A_z B_y\right) + \hat{y}\left(A_z B_x - A_x B_z\right) + \hat{z}\left(A_x B_y - A_y B_x\right)$$

(Proof: see Example 2-4)

Product of Three Vectors

- Note: not all the products of three vectors are meaningful!

$$\bar{A}\left(\bar{B}\cdot\bar{C}\right) \qquad \bar{A}\times\left(\bar{B}\cdot\bar{C}\right)$$

$$\bar{A}\cdot\left(\bar{B}\times\bar{C}\right) \qquad \bar{A}\times\left(\bar{B}\times\bar{C}\right)$$

Product of Three Vectors

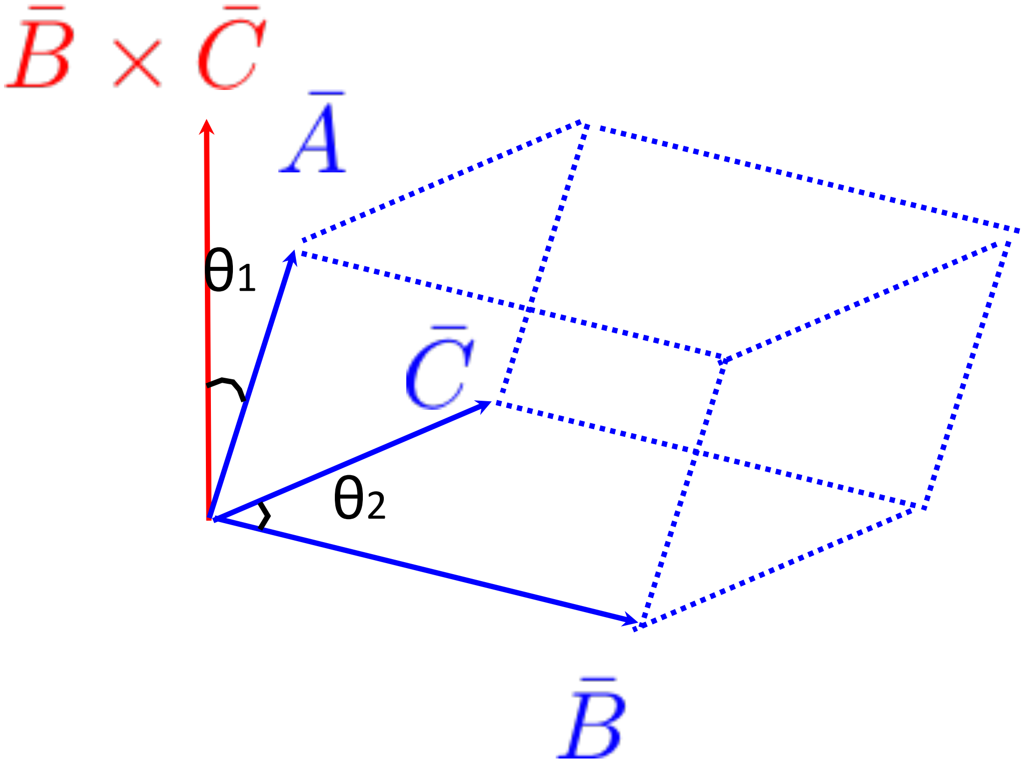
- Note: not all the products of three vectors are meaningful!

$$\bar{A}\left(\bar{B}\cdot\bar{C}\right) \qquad \boxed{\bar{A}\times\left(\bar{B}\cdot\bar{C}\right)}$$

$$\bar{A}\cdot\left(\bar{B}\times\bar{C}\right) \qquad \bar{A}\times\left(\bar{B}\times\bar{C}\right)$$

Scalar Triple Product

- The product is a scalar, equal to the volume of the parallelepiped.



$$\bar{A} \cdot \left(\bar{B} \times \bar{C} \right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

- the product is a vector
- "Back-cab" (BAC-CAB) rule

$$\bar{A} \times \left(\bar{B} \times \bar{C}\right) = \bar{B}\left(\bar{A} \cdot \bar{C}\right) - \bar{C}\left(\bar{A} \cdot \bar{B}\right)$$

(See Example 2-3)

Orthogonal coordinate systems

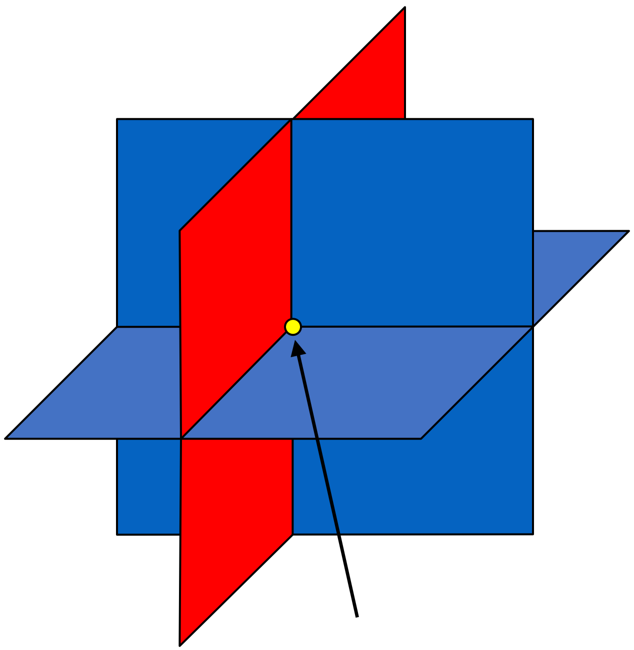
2-4 Orthogonal Coordinate Systems

- Different coordinate systems can be more suitable for different problems
- The physics, the EM quantities, are **not changed** if using different orthogonal coordinate systems.
- A point in 3D space can be determined by the intersection of three surfaces.
- If these three surfaces are mutually orthogonal to each other, this sets up an orthogonal coordinate system.

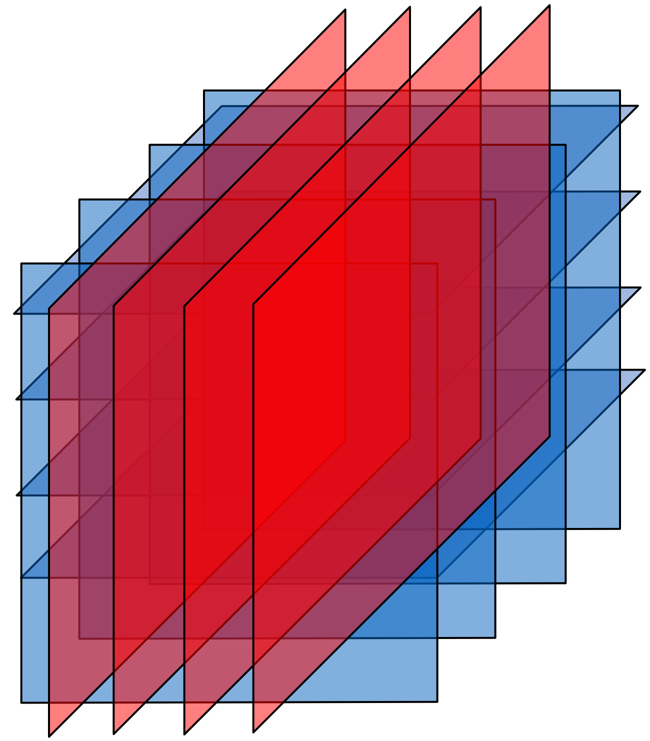
Orthogonal Coordinate Systems

- 3 surfaces determine 1 point
- u_1=constant; u_2=constant; u_3=constant
  ➔ 3 surfaces
  (In Cartesian, u_1=x, u_2=y, u_3=z)
- Should u be length?
  Should the surface be planes?
- If 3 surfaces are perpendicular to each other, we have an orthogonal coordinate system.

Coordinate systems

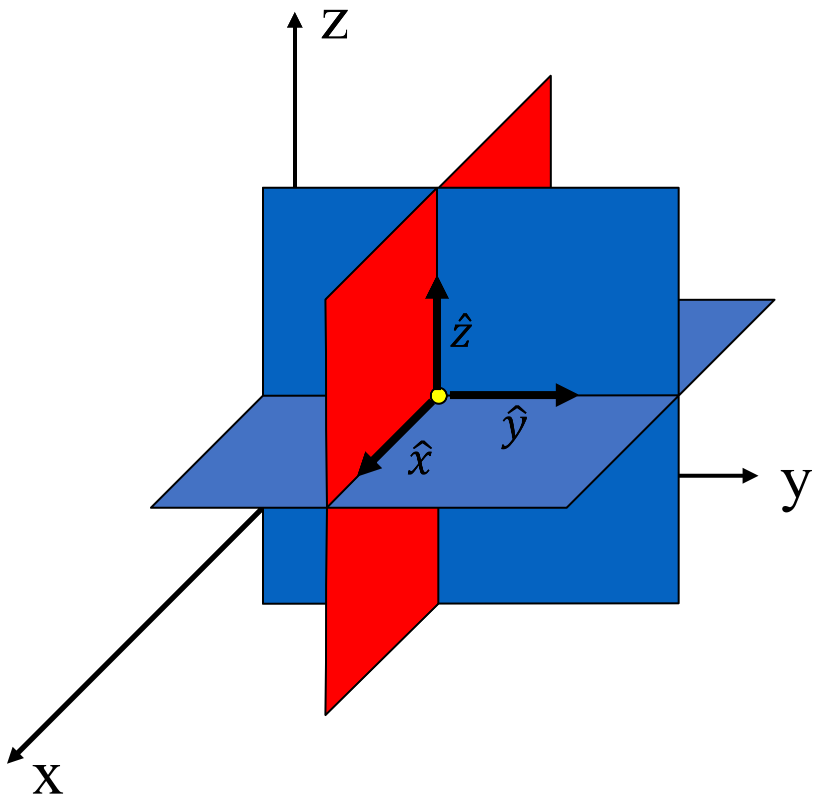


Point in space is an intersection of coordinate surfaces.



To describe location of each point in space we need 3 sets of surfaces.

Unit Vectors in Orthonormal Systems



Unit vectors are defined as the normal direction of each coordinate surface. **The choice of direction for each coordinate surface is not unique. Here, the Cartesian coordinate system is shown.**

General 3D, Right-handed, Curvilinear, and Orthogonal Coordinate System

Three curved surfaces: u_1, u_2, u_3

Three base vectors: $\hat{a}_{u_1}$ $\hat{a}_{u_2}$ $\hat{a}_{u_3}$

$$\bar{A} = \hat{a}_{u_1} A_{u_1} + \hat{a}_{u_2} A_{u_2} + \hat{a}_{u_3} A_{u_3}$$

***General expression*, convenient to write down formulas in different coordinate systems.**

Base Vectors, Expression, and Length

$$\mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3},$$
$$\mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1},$$
$$\mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2}.$$

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0$$

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1.$$

$$\boxed{\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}.}$$

$$A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}.$$

Differential Elements

- Note: u_i may not be a length!

Differential length: $dl_i = \underline{h_i} du_i$

Metric coefficient

- Example:

In 2D polar coordinate: $(u_1, u_2) = (r, \phi)$

A differential change $d\phi\ (=du_2)$ in $\phi\ (=u_2)$ ➡ A differential length change?

$$d\ell_2 = r\, d\phi$$

The metric coefficient for $\phi\ (=u_2)$: $h_2 = r$

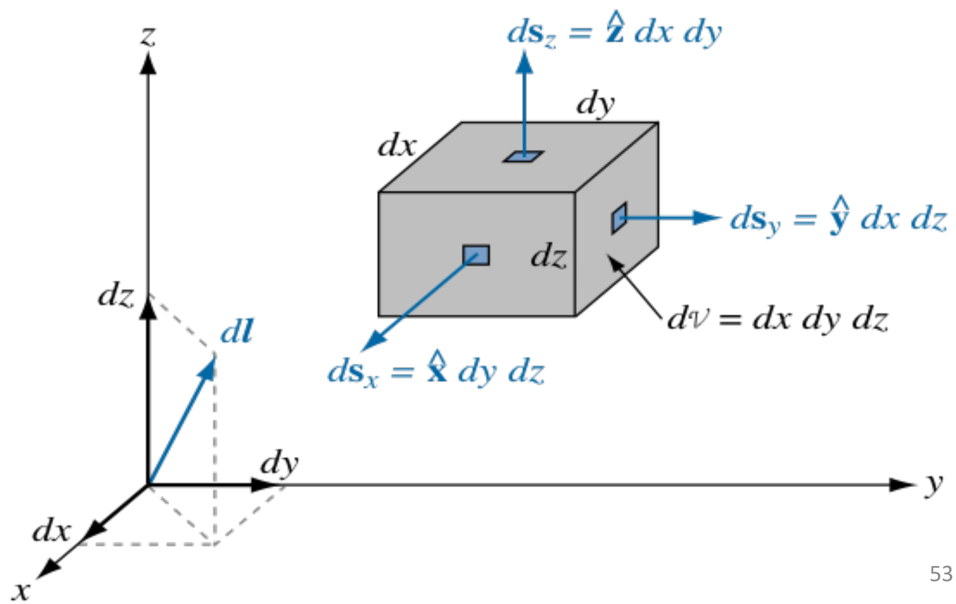
h_2 in this case is a function of u_1, **not a constant**.

Differential Elements

Differential length: $d\bar{\ell} = \mathbf{a}_{u_1} \underline{d\ell_1} + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3$

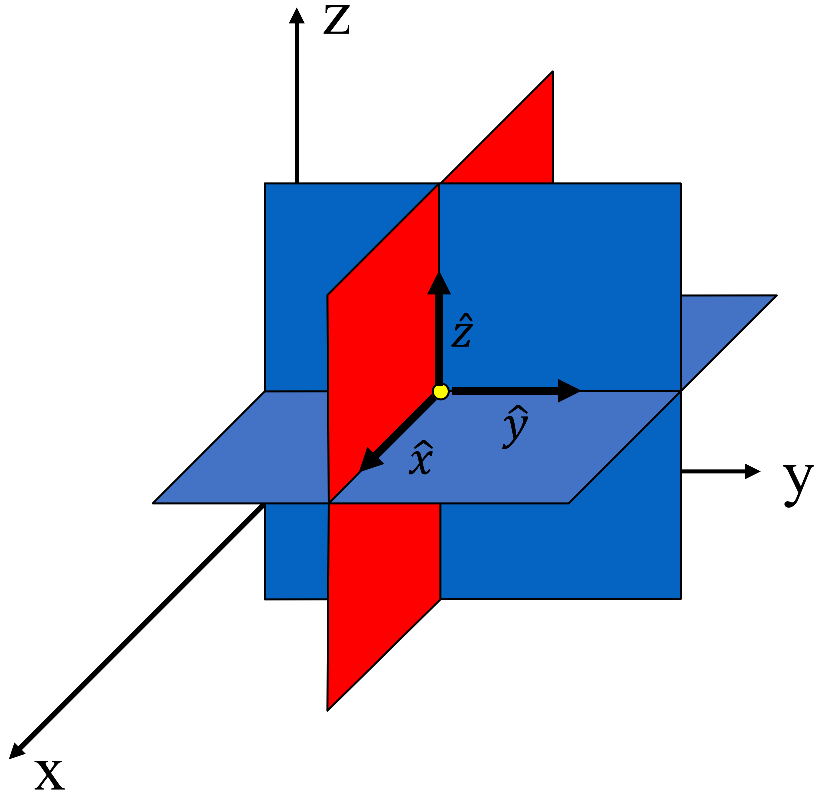
$$d\bar{l} = \hat{a}_{u_1}\,\underline{(h_1 du_1)} + \hat{a}_{u_2}\,(h_2 du_2) + \hat{a}_{u_3}\,(h_3 du_3)$$

Magnitude:
$$\begin{aligned} d\ell &= [(d\ell_1)^2 + (d\ell_2)^2 + (d\ell_3)^2]^{1/2} \\ &= [(h_1\,du_1)^2 + (h_2\,du_2)^2 + (h_3\,du_3)^2]^{1/2}. \end{aligned}$$



Cartesian Coordinate System

$$(u_1, u_2, u_3) = (x, y, z)$$



Coordinate system where all coordinate surfaces are planar is called Cartesian. Note that in this system, **the direction** of coordinate vectors is **the same at every point** in space

Differential Elements

Differential surfaces: $d\bar{s} = \hat{a}_n ds$

$$ds_1 = dl_2 dl_3 = h_2 h_3 du_2 du_3$$

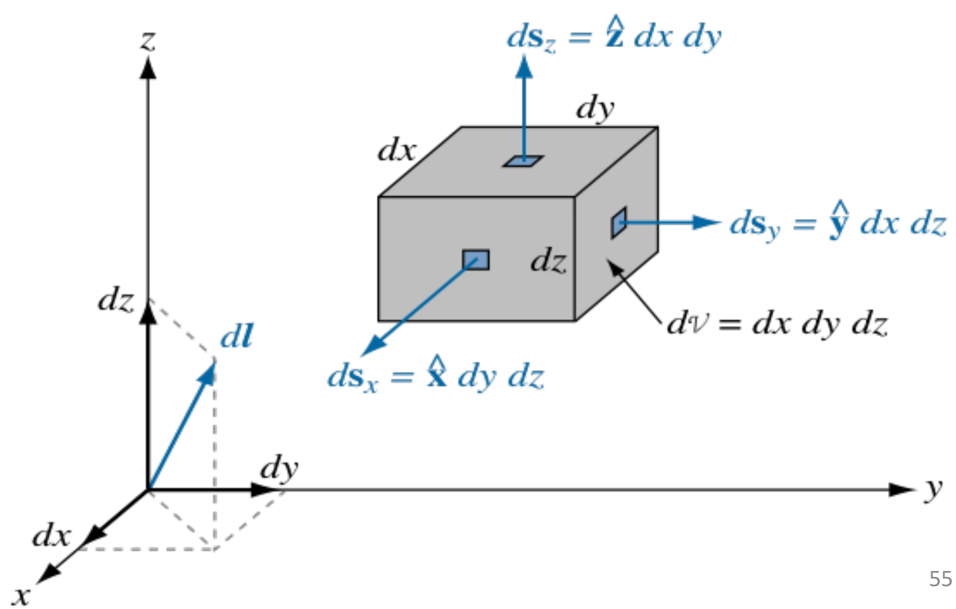
$$\boxed{ds_1 = h_2 h_3 \, du_2 \, du_3.}$$

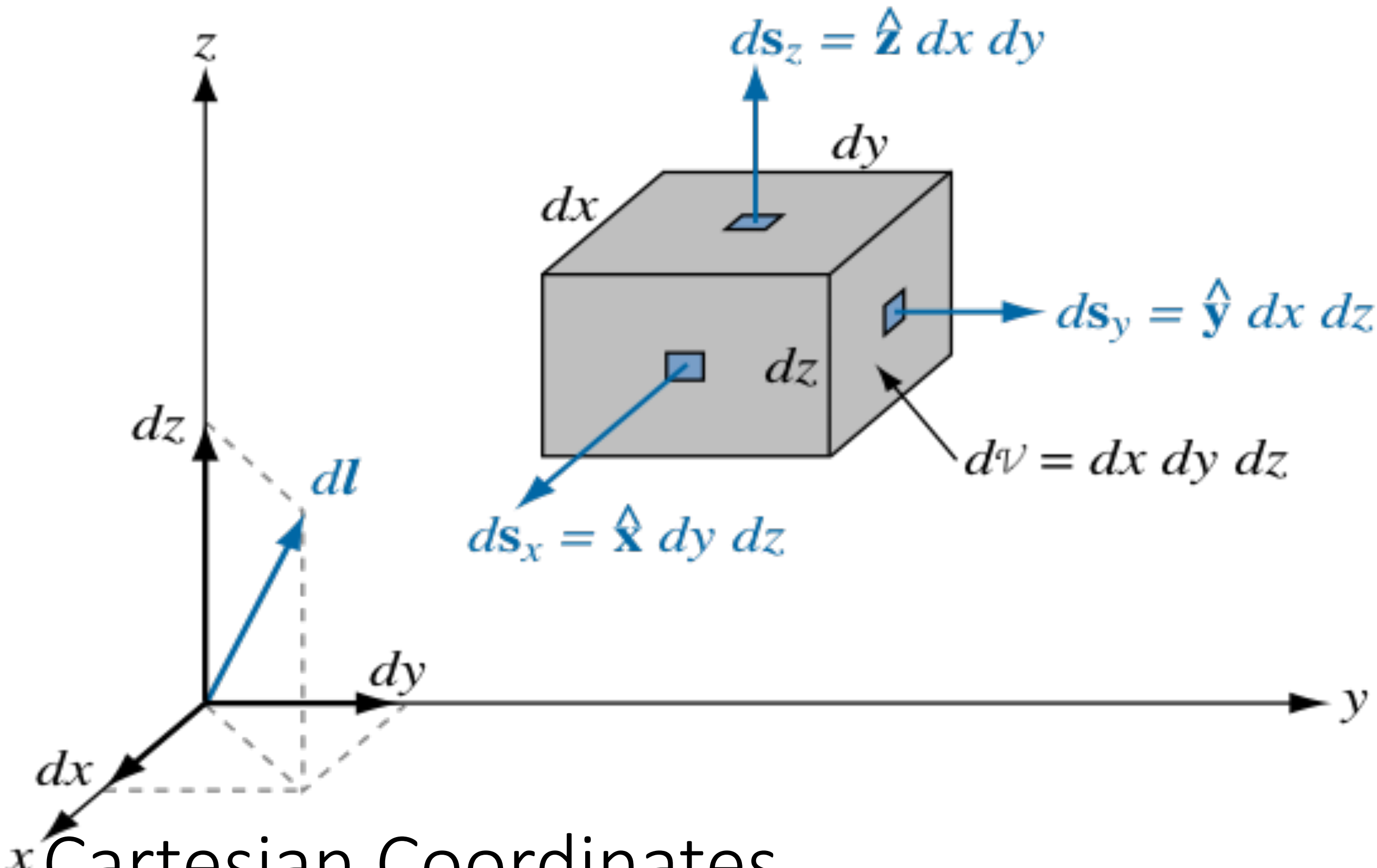
$$\boxed{ds_2 = h_1 h_3 \, du_1 \, du_3}$$

$$\boxed{ds_3 = h_1 h_2 \, du_1 \, du_2.}$$

Differential volume:

$$\boxed{dv = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3.}$$





Cartesian Coordinates

$$(x, y, z)$$

Cartesian Coordinate System

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$
$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$
$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y.$$

$$\overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1.$$

$$\boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.}$$

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z,}$$

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x) \\ &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.\end{aligned}$$

See Example 2-4

Cartesian Coordinate System

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\,du_1) + \mathbf{a}_{u_2}(h_2\,du_2) + \mathbf{a}_{u_3}(h_3\,du_3).}$$

$$\boxed{ds_1 = h_2 h_3\,du_2\,du_3.}$$

$$\boxed{ds_2 = h_1 h_3\,du_1\,du_3}$$

$$\boxed{ds_3 = h_1 h_2\,du_1\,du_2.}$$

$$\boxed{dv = h_1 h_2 h_3\,du_1\,du_2\,du_3.}$$

$$\boxed{h_1 = h_2 = h_3 = 1}$$

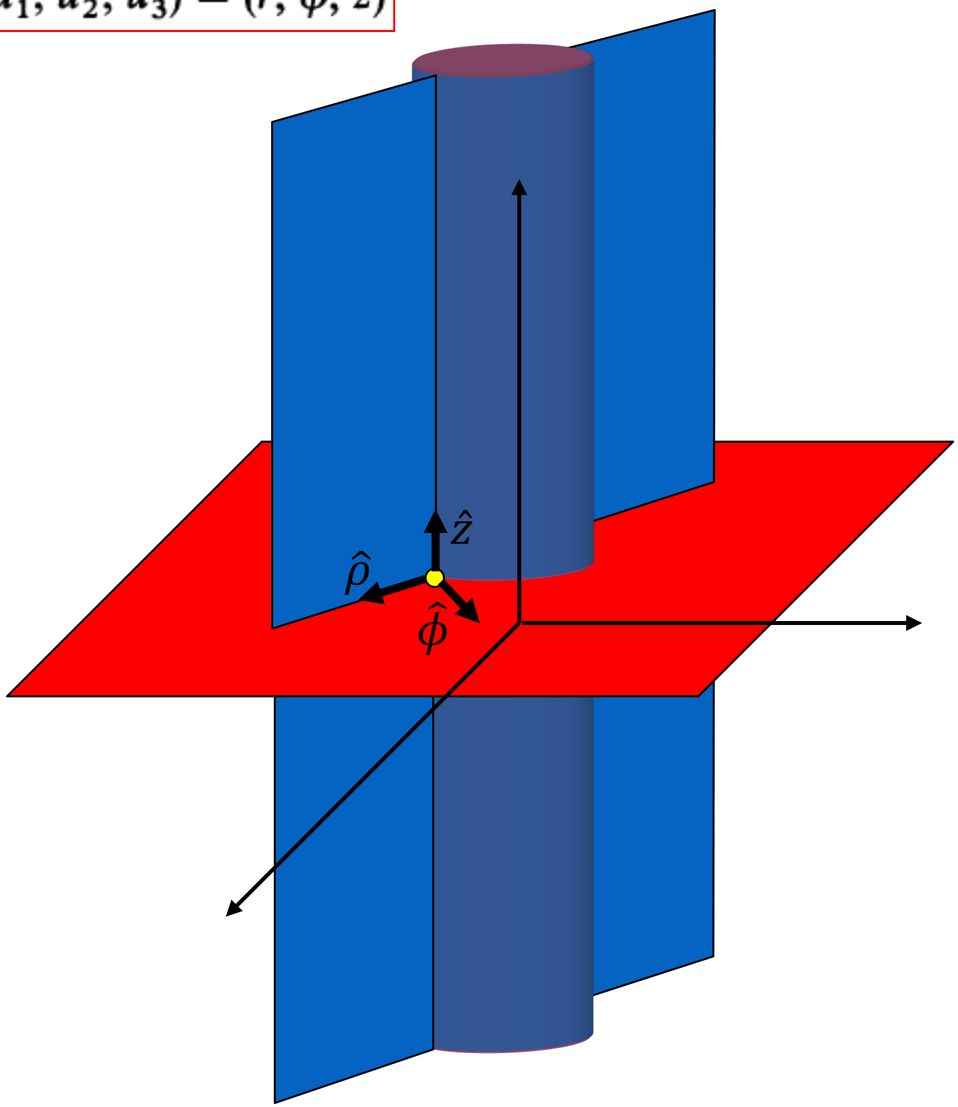
$$\boxed{d\ell = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz;}$$

$$\boxed{\begin{aligned} ds_x &= dy\,dz,\\ ds_y &= dx\,dz,\\ ds_z &= dx\,dy; \end{aligned}}$$

$$\boxed{dv = dx\,dy\,dz.}$$

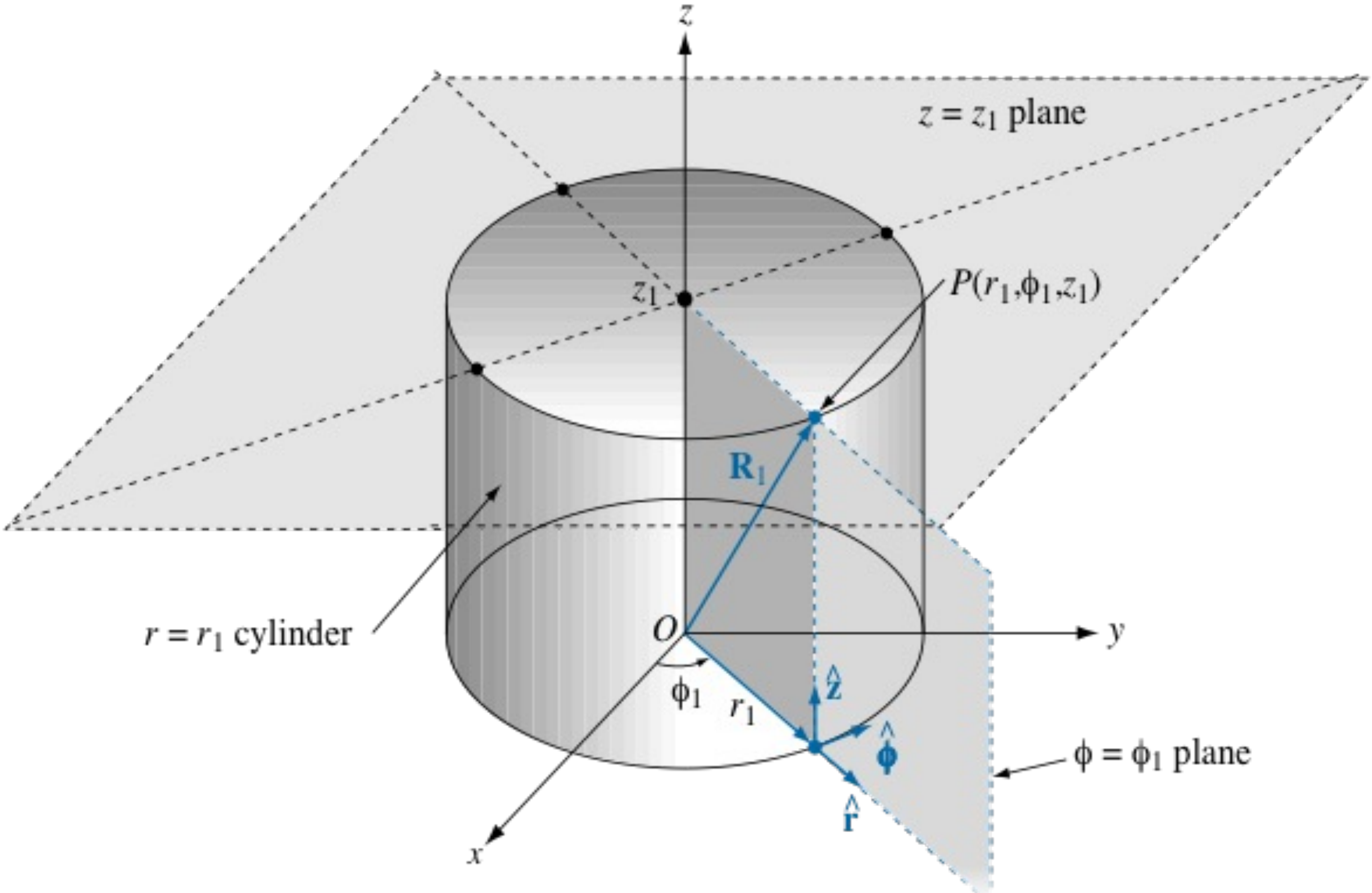
Cylindrical Coordinate System

$(u_1, u_2, u_3) = (r, \phi, z)$



In a cylindrical coordinate system, **a cylindrical surface** and **two planar surfaces** all orthogonal to each other define location of a point in space. The coordinate vectors are $\hat{\rho}$ $\hat{\phi}$ and $\hat{z}$.

Note, in this system the direction of coordinate vectors **changes** from point to point



Cylindrical Coordinates

$$(r, \phi, z)$$

Cylindrical Coordinate System

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z,$$
$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r,$$
$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi.$$

$$\boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z.}$$

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3).}$$

$$\boxed{ds_1 = h_2 h_3\, du_2\, du_3.}$$

$$\boxed{ds_2 = h_1 h_3\, du_1\, du_3}$$

$$\boxed{ds_3 = h_1 h_2\, du_1\, du_2.}$$

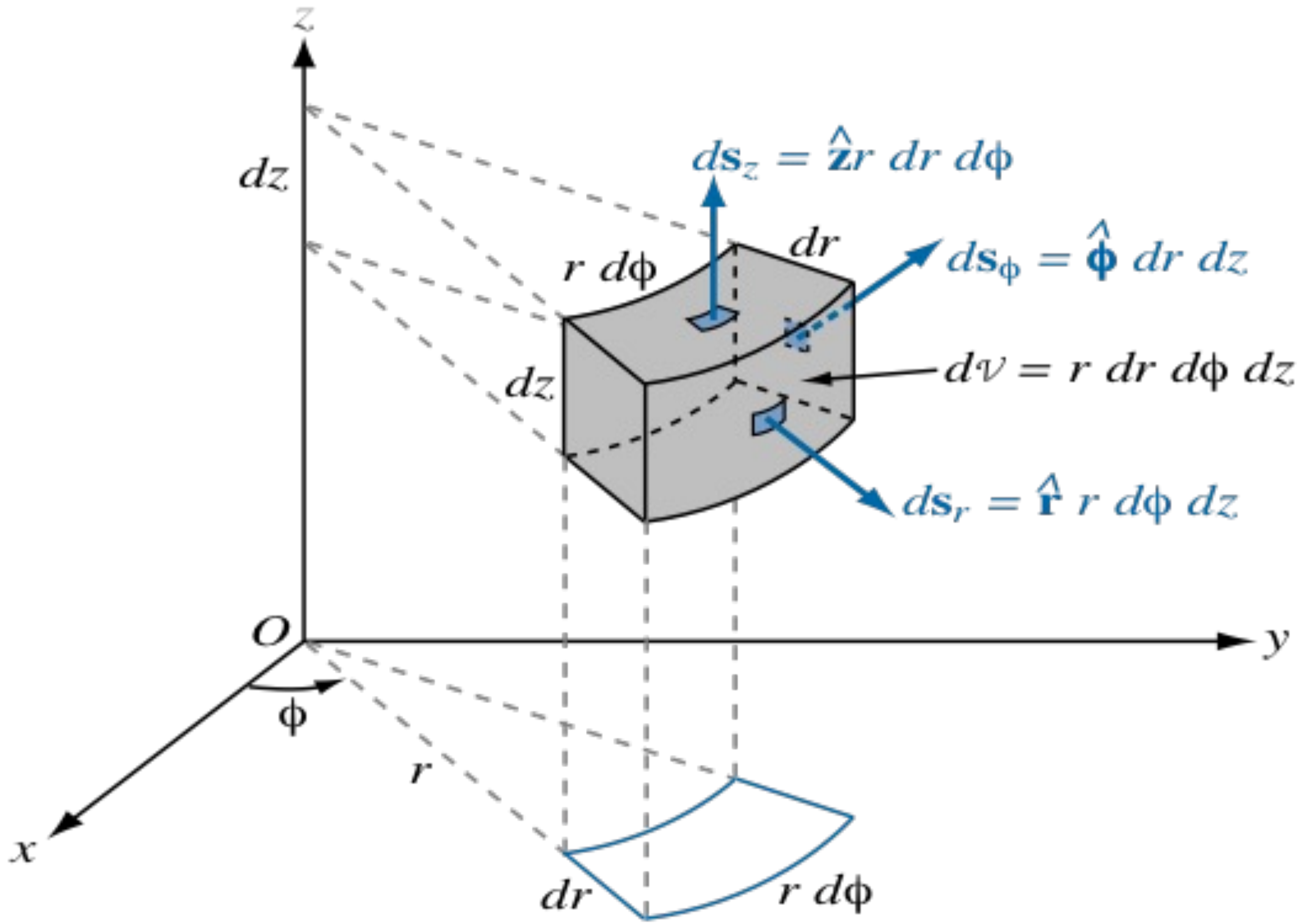
$$\boxed{dv = h_1 h_2 h_3\, du_1\, du_2\, du_3.}$$

$$h_1 = h_3 = 1$$
$$h_2 = r$$

$$\boxed{d\ell = \mathbf{a}_r\, dr + \mathbf{a}_\phi \underline{r\, d\phi} + \mathbf{a}_z\, dz.}$$

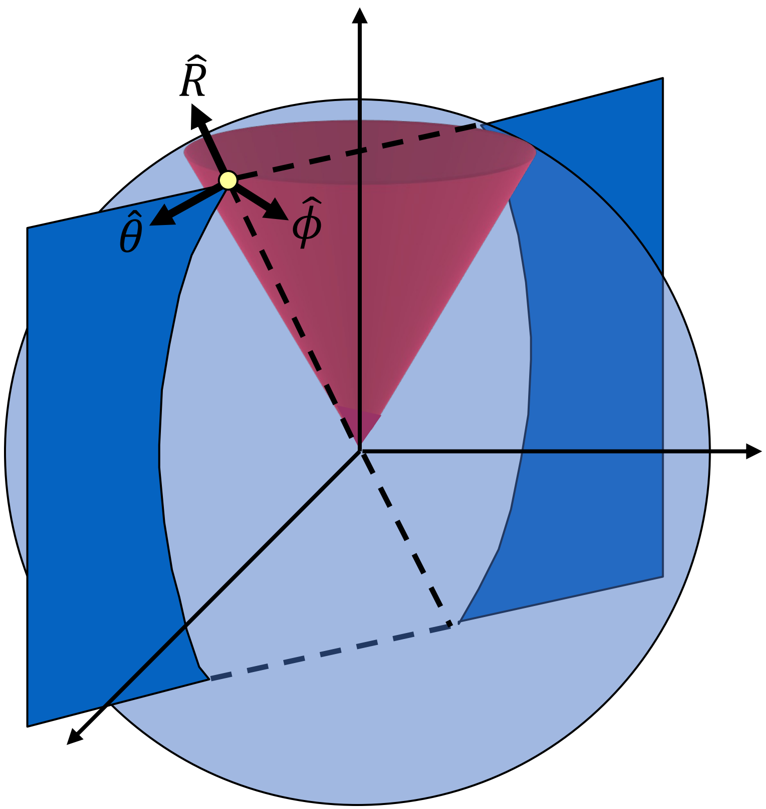
$$ds_r = r\, d\phi\, dz,$$
$$ds_\phi = dr\, dz,$$
$$ds_z = r\, dr\, d\phi,$$

$$\boxed{dv = r\, dr\, d\phi\, dz.}$$

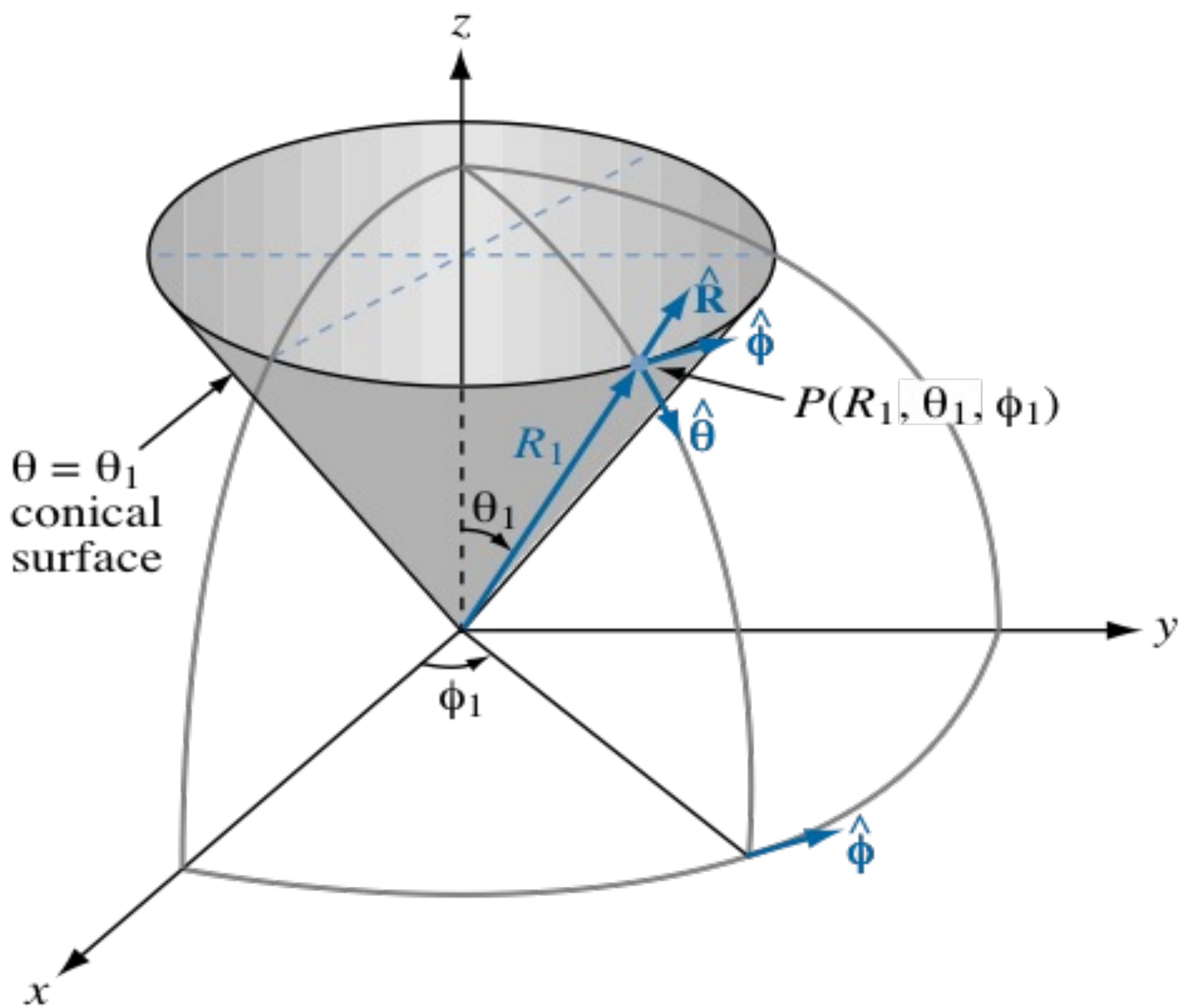

z
dz
$d\mathbf{s}_z = \hat{\mathbf{z}} r\, dr\, d\phi$
dr
r dφ
$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz$
dz
$d\mathcal{V} = r\, dr\, d\phi\, dz$
$d\mathbf{s}_r = \hat{\mathbf{r}}\, r\, d\phi\, dz$
O
y
φ
r
x
dr
r dφ

Spherical Coordinate System

$$(u_1, u_2, u_3) = (R, \theta, \phi)$$



In a spherical coordinate system the surfaces of **spheres, cones and planes** are coordinate surfaces. The corresponding coordinate vectors also **change** their direction from point to point



Spherical Coordinates

$$(R, \theta, \phi)$$

Spherical Coordinate System

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi,$$
$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R,$$
$$\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta.$$

$$\boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi.}$$

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3).}$$

$$\boxed{ds_1 = h_2 h_3\, du_2\, du_3.}$$

$$\boxed{ds_2 = h_1 h_3\, du_1\, du_3}$$

$$\boxed{ds_3 = h_1 h_2\, du_1\, du_2.}$$

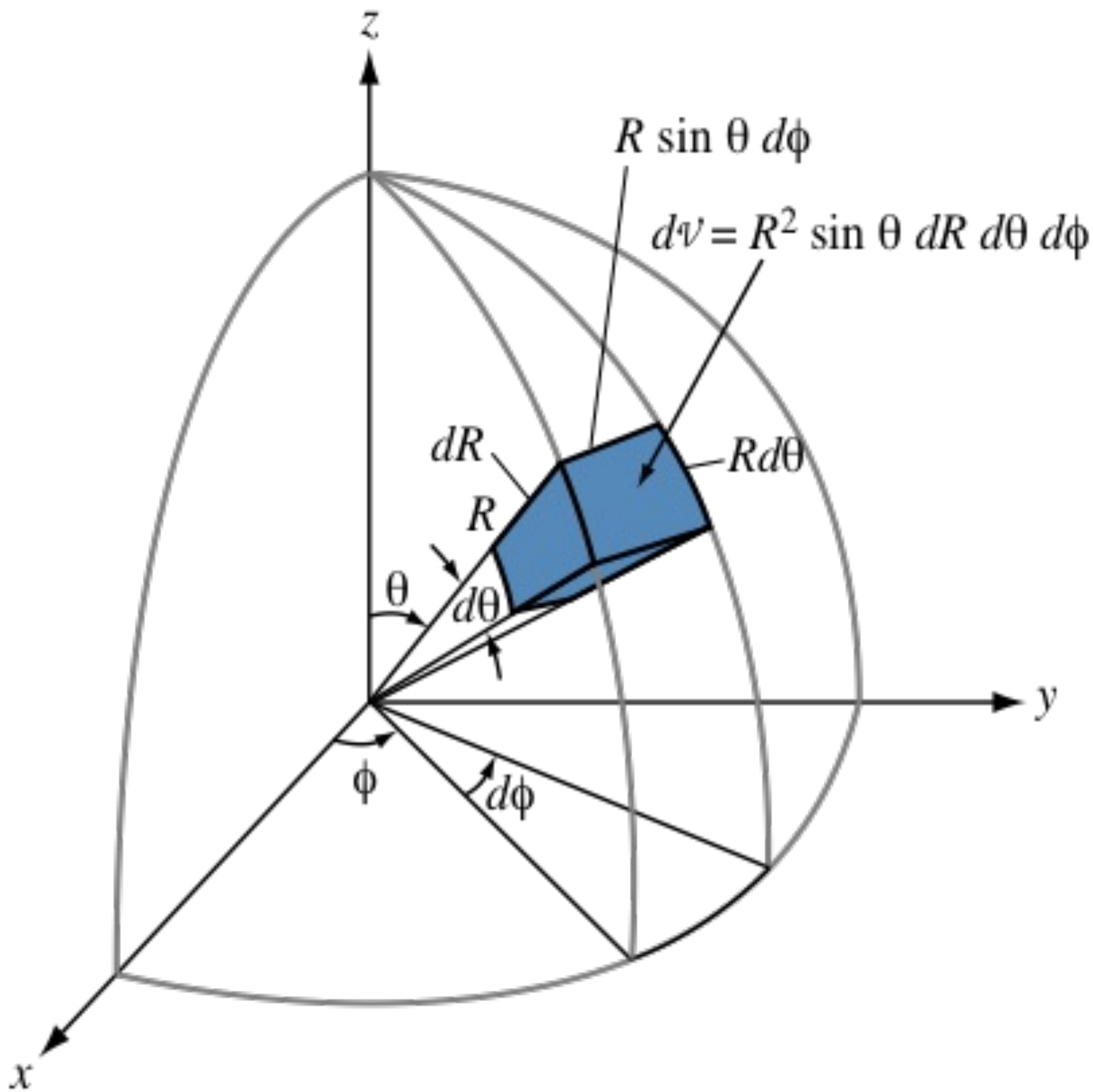
$$\boxed{dv = h_1 h_2 h_3\, du_1\, du_2\, du_3.}$$

$$\boxed{d\ell = \mathbf{a}_R\, dR + \mathbf{a}_\theta R\, d\theta + \mathbf{a}_\phi R \sin\theta\, d\phi.}$$

$$h_2 = R$$
$$h_3 = R\sin\theta$$

$$ds_R = R^2 \sin\theta\, d\theta\, d\phi,$$
$$ds_\theta = R\sin\theta\, dR\, d\phi,$$
$$ds_\phi = R\, dR\, d\theta,$$

$$\boxed{dv = R^2 \sin\theta\, dR\, d\theta\, d\phi.}$$


z
$R \sin\theta \, d\phi$
$d\mathcal{V} = R^2 \sin\theta \, dR \, d\theta \, d\phi$
dR
$R d\theta$
R
θ
$d\theta$
y
ϕ
$d\phi$
x

Metric Coefficients

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin\theta\,dR\,d\theta\,d\phi$

Cartesian Coordinates: Vector Operation Example

$$\vec{A} = 4\hat{x} - 2\hat{y} + 3\hat{z} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = (4,-2,3) \qquad \vec{B} = 1\hat{x} - 1\hat{y} + 7\hat{z} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = (1,-1,7)$$

Vector Addition:

$$\vec{A} + \vec{B} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 10 \end{bmatrix}$$

Vector Dot Product:

$$\vec{A} \cdot \vec{B} = (4\hat{x} - 2\hat{y} + 3\hat{z}) \cdot (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \cdot \vec{B} = 4\hat{x} \cdot (1\hat{x} - 1\hat{y} + 7\hat{z}) - 2\hat{y} \cdot (1\hat{x} - 1\hat{y} + 7\hat{z}) + 3\hat{z} \cdot (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \cdot \vec{B} = 4 + 2 + 21 = 27$$

Here we applied the **distributive law** and the dot product property that unit vectors only in the same direction are non-zero

$$\hat{\delta}_i \cdot \hat{\delta}_j = \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}$$

Vector Operation Example (Continued)

$$\vec{A} = 4\hat{x} - 2\hat{y} + 3\hat{z} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = (4,-2,3) \qquad \vec{B} = 1\hat{x} - 1\hat{y} + 7\hat{z} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = (1,-1,7)$$

Cross Product:

$$\vec{A} \times \vec{B} = (4\hat{x} - 2\hat{y} + 3\hat{z}) \times (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \times \vec{B} = 4\hat{x} \times (1\hat{x} - 1\hat{y} + 7\hat{z}) - 2\hat{y} \times (1\hat{x} - 1\hat{y} + 7\hat{z}) + 3\hat{z} \times (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \times \vec{B} = (-28\hat{y} - 4\hat{z}) + (-14\hat{x} + 2\hat{z}) + (3\hat{x} + 3\hat{y})$$

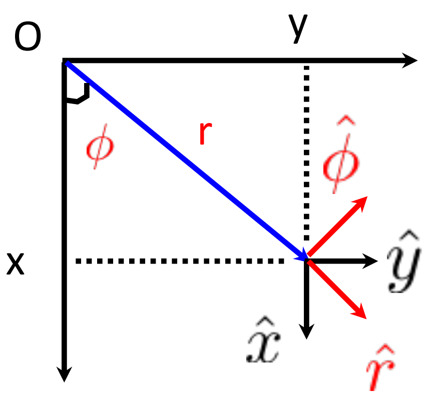
$$\vec{A} \times \vec{B} = (-11\hat{x} - 25\hat{y} - 2\hat{z})$$

The cross product is thus the determinant of the matrix:

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Coordinate Transformation - Cartesian to Cylindrical

$$\bar{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$$

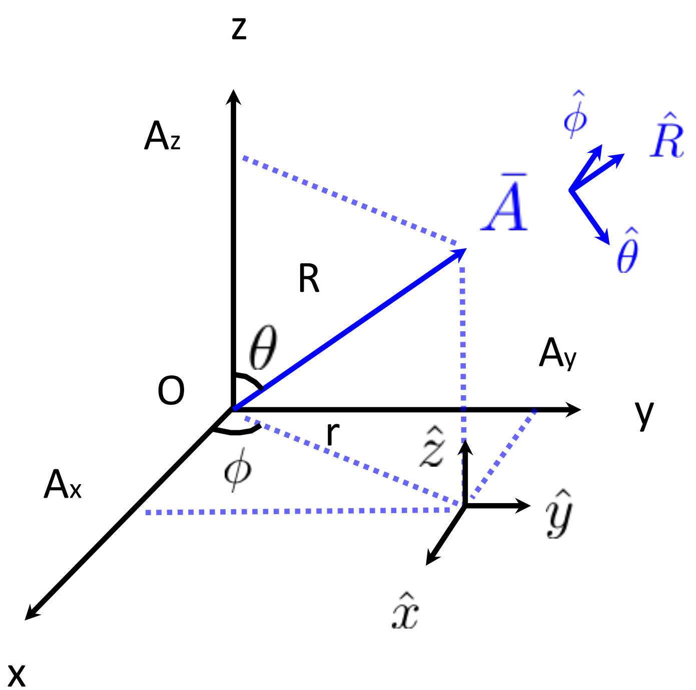


$$\boxed{\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \\ z &= z. \end{aligned}}$$

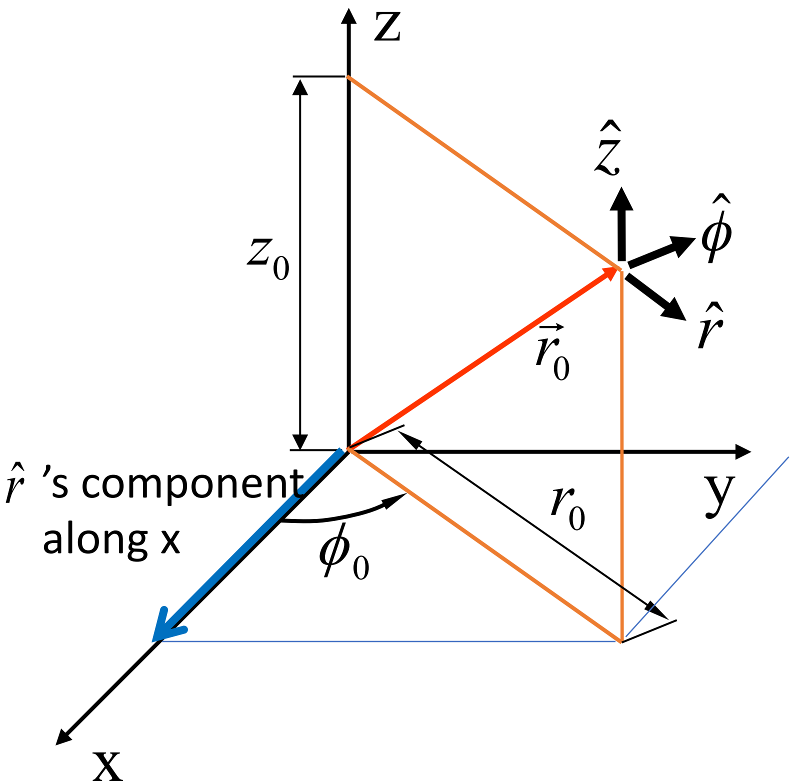
$$\boxed{\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \phi &= \tan^{-1}\frac{y}{x}, \\ z &= z. \end{aligned}}$$

Coordinate Transformation - Cartesian to Spherical

$$\bar{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$



Coordinate Transformations: Cylindrical & Cartesian



The **projection** of the Cylindrical coordinates **unit vectors** along the Cartesian coordinates unit vector directions are:

$$\hat{r}\cdot\hat{x} = \cos\varphi_0 \qquad \hat{\phi}\cdot\hat{x} = -\sin\phi_0 \qquad \hat{z}\cdot\hat{x} = 0$$

$$\hat{r}\cdot\hat{y} = \sin\varphi_0 \qquad \hat{\phi}\cdot\hat{y} = \cos\phi_0 \qquad \hat{z}\cdot\hat{y} = 0$$

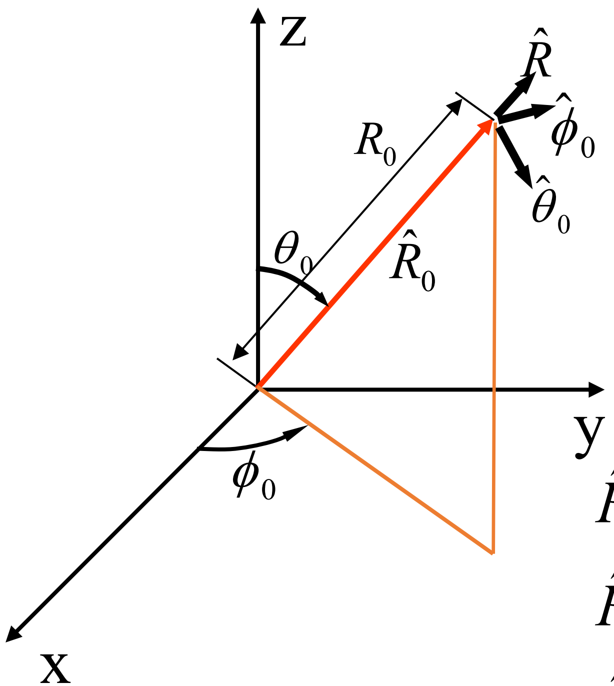
$$\hat{r}\cdot\hat{z} = 0 \qquad \hat{\phi}\cdot\hat{z} = 0 \qquad \hat{z}\cdot\hat{z} = 1$$

Thus, the unit coordinate transformation matrix is given by:

$$\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\varphi_0 & \sin\varphi_0 & 0\\ -\sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \qquad \begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\varphi_0 & -\sin\varphi_0 & 0\\ \sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix}$$

$\hat{r}$ decompsed into $\hat{x},\hat{y},\hat{z}$components

Coordinate Transformations: Spherical & Cartesian



The **projection** of the Spherical coordinates unit vectors along the Cartesian coordinates unit vector directions are:

$$\hat{R}\cdot\hat{x} = \cos\varphi_0 \sin\theta_0 \quad \hat{\theta}\cdot\hat{x} = \cos\phi_0 \cos\theta_0 \quad \hat{\phi}\cdot\hat{x} = -\sin\phi_0$$

$$\hat{R}\cdot\hat{y} = \sin\varphi_0 \sin\theta_0 \quad \hat{\theta}\cdot\hat{y} = \sin\phi_0 \cos\theta_0 \quad \hat{\phi}\cdot\hat{y} = \cos\phi_0$$

$$\hat{R}\cdot\hat{z} = \cos\theta_0 \quad \hat{\theta}\cdot\hat{z} = -\sin\theta_0 \quad \hat{\phi}\cdot\hat{z} = 0$$

Thus, the unit coordinate transformation matrix is given by:

$$\begin{bmatrix} \hat{R} \\ \hat{\theta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 \sin\theta_0 & \sin\varphi_0 \sin\theta_0 & \cos\theta_0 \\ \cos\varphi_0 \cos\theta_0 & \sin\varphi_0 \cos\theta_0 & -\sin\theta_0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

A• ~ on both sides

A•x=A_x

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}=\begin{bmatrix}\cos\varphi_0 & -\sin\varphi_0 & 0\\ \sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix}$$

$$\Rightarrow$$

$$\boxed{\begin{bmatrix}A_x\\ A_y\\ A_z\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_r\\ A_\phi\\ A_z\end{bmatrix}.}$$

Coordinate Transformations: Cylindrical & Cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}.$$

A_r, A_ϕ, and A_z may themselves be functions of r, ϕ, z. In that case, they should be converted into functions of x, y, and z (using the below relations) in the final answer.

See Example 2-9

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z.$$

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x}, \quad z = z.$$

Comparisons of Two Expressions

Vector by variables

(x, y, z)

(r, ϕ, z)

(R, θ, ϕ)

- Advantages:
  - ✓ Convenient to specify a point
- Disadvantages:
  - × Need to convert to Cartesian coordinates before vector analysis (addition, subtraction, multiplication, etc.)

Vector by base vector

$$\boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.}$$

$$\boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z.}$$

$$\boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi.}$$

- Advantages:
  - ✓ Convenient to do vector analysis
  - ✓ Convenient in formula derivations
- Disadvantages:
  - × To get a specific point, use the coordinate transformations to Cartesian

    Also see Example 2-11

- **Unified basis with its base vector at the same direction (e.g., a_θ for two different vectors A_1 and A_2 are in the same direction (see next slide))**

Proof of Vector Addition in Spherical Coordinates

$$\boldsymbol{A_1} = \boldsymbol{a_R} A_{R1} + \boldsymbol{a_\theta} A_{\theta 1} + \boldsymbol{a_\phi} A_{\phi 1}$$

$$\boldsymbol{A_2} = \boldsymbol{a_R} A_{R2} + \boldsymbol{a_\theta} A_{\theta 2} + \boldsymbol{a_\phi} A_{\phi 2}$$

$$\boldsymbol{A_1} + \boldsymbol{A_2} = ?$$

$$\boldsymbol{A_1} + \boldsymbol{A_2} = \boldsymbol{a_R}(A_{R1} + A_{R2}) + \boldsymbol{a_\theta}(A_{\theta 1} + A_{\theta 2}) + \boldsymbol{a_\phi}(A_{\phi 1} + A_{\phi 2})$$

The formula in Cartesian Coordinates applies to Spherical Coordinates.

we are OK!

⊚ Proof of vector addition

$$\vec{A}_3 = \vec{A}_1 + \vec{A}_2$$

$$\hat{a}_x A_{x3} + \hat{a}_y A_{y3} + \hat{a}_z A_{z3} = \hat{a}_x(A_{x1}+A_{x2}) + \hat{a}_y(A_{y1}+A_{y2}) + \hat{a}_z(A_{z1}+A_{z2})$$

$$\vec{A}_1 = \hat{a}_R A_{R1} + \hat{a}_\theta A_{\theta 1} + \hat{a}_\phi A_{\phi 1}$$

$$\vec{A}_2 = \hat{a}_R A_{R2} + \hat{a}_\theta A_{\theta 2} + \hat{a}_\phi A_{\phi 2}$$

$$\Rightarrow \vec{A}_3 = \vec{A}_1 + \vec{A}_2 = \hat{a}_R(A_{R1}+A_{R2}) + \hat{a}_\theta(A_{\theta 1}+A_{\theta 2}) + \hat{a}_\phi(A_{\phi 1}+A_{\phi 2})$$

$$\hat{a}_R A_{R3} + \hat{a}_\theta A_{\theta 3} + \hat{a}_\phi A_{\phi 3}$$

?

Car $\vec{A}_1 = \vec{A}_1$ Sph

$\vec{A}_2 = \vec{A}_2$

$\vec{A}_3 = \vec{A}_3$

using Eqs. 75-77

using Eqs. (2-75)-(2-77), we have

① By A_{R1}, A_{R2}

$$A_{x1} + A_{x2} = \left(\frac{A_{R1}x}{\sqrt{x^2+y^2+z^2}} + \frac{A_{\theta 1}xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{A_{\phi 1}y}{\sqrt{x^2+y^2}}\right) + \left(\frac{A_{R2}x}{\sqrt{x^2+y^2+z^2}} + \frac{A_{\theta 2}xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{A_{\phi 2}y}{\sqrt{x^2+y^2}}\right)$$

$$= \left[\frac{(A_{R1}+A_{R2})x}{\sqrt{x^2+y^2+z^2}} + \frac{(A_{\theta 1}+A_{\theta 2})xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{(A_{\phi 1}+A_{\phi 2})y}{\sqrt{x^2+y^2}}\right]$$

similarly, $A_{y3} = \frac{(A_{R1}+A_{R2})\cdots}{\cdots} + \frac{(A_{\theta 1}+A_{\theta 2})\cdots}{\cdots} - \frac{(A_{\phi 1}+A_{\phi 2})\cdots}{\cdots}$

$A_{z3} = \frac{(A_{R1}+A_{R2})\cdots}{\cdots} - \frac{(A_{\theta 1}+A_{\theta 2})\cdots}{\cdots}$

② By A_{R3}

$$A_{x3} = \left[\frac{A_{R3}x}{\sqrt{x^2+y^2+z^2}} + \frac{A_{\theta 3}xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{A_{\phi 3}y}{\sqrt{x^2+y^2}}\right]$$

similarly, for A_{y3}, A_{z3}

We are ok for $A_{x1}+A_{x2}=A_{x3}$. By comparing ① and ②

we have $A_{R1}+A_{R2}=A_{R3}$, ...

Vector calculus

2-5 Integrals Containing Vector Functions

$$\int_V \mathbf{F}\, dv,$$

$$\int_C V\, d\boldsymbol{\ell},$$

$$\oint_C V\, d\boldsymbol{\ell}.$$

$$\int_C \mathbf{F} \cdot d\boldsymbol{\ell},$$

$$\int_S \mathbf{A} \cdot d\mathbf{s}.$$

$$\int_C V\, d\boldsymbol{\ell} = \int_C V(x, y, z)[\mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz],$$

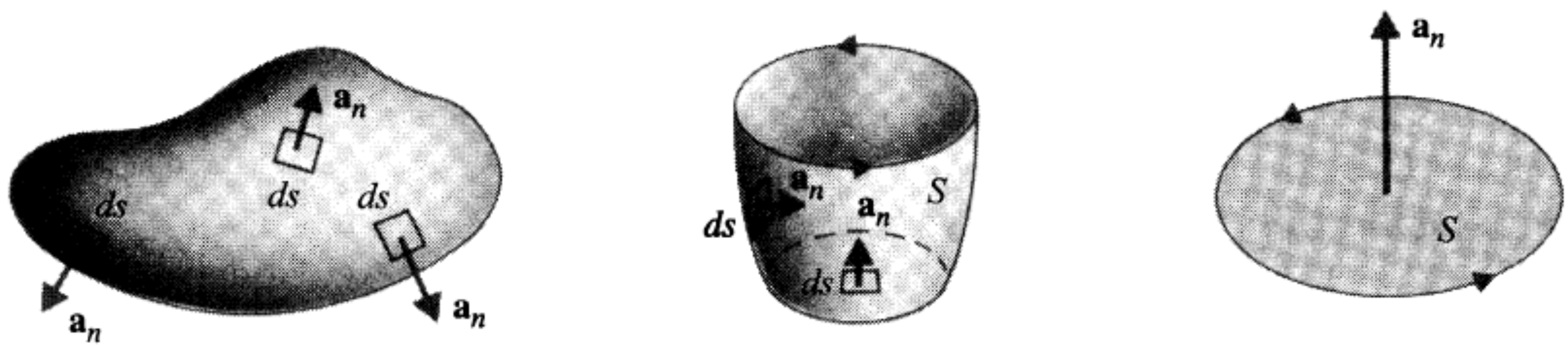
$$\int_C V\, d\boldsymbol{\ell} = \mathbf{a}_x \int_C V(x, y, z)\, dx + \mathbf{a}_y \int_C V(x, y, z)\, dy + \mathbf{a}_z \int_C V(x, y, z)\, dz.$$

$$\int_S \mathbf{A} \cdot d\mathbf{s},$$

Flux of vector field **A** flowing through the area S

$$d\mathbf{s} = \mathbf{a}_n \, ds$$

1. If S is a closed surface → $\mathbf{a}_n$ is in the outward direction
2. If S is an open surface → $\mathbf{a}_n$ is decided by right-hand rule (thumb)



(a) A closed surface. (b) An open surface. (c) A disk.

FIGURE 2–22
Illustrating the positive direction of $\mathbf{a}_n$ in scalar surface integral.

Vector Integration: Line, Surface, Volume

Electromagnetics frequently requires the integral of the potential, current, field, etc., over elements of different spatial extent.

Cartesian

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

$$d\vec{S} = \hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy$$

$$dV = dxdydz$$

Cylindrical

$$d\vec{l} = \hat{\rho}d\rho + \hat{\phi}\rho d\phi + \hat{z}dz$$

$$d\vec{S} = \hat{\rho}\rho d\phi dz + \hat{\phi}d\rho dz + \hat{z}\rho d\rho d\phi$$

$$dV = \rho d\rho d\phi dz$$

Spherical

$$d\vec{l} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

$$d\vec{S} = \hat{r}r^2\sin\theta d\theta d\phi + \hat{\theta}r\sin\theta drd\phi + \hat{\phi}rdrd\theta$$

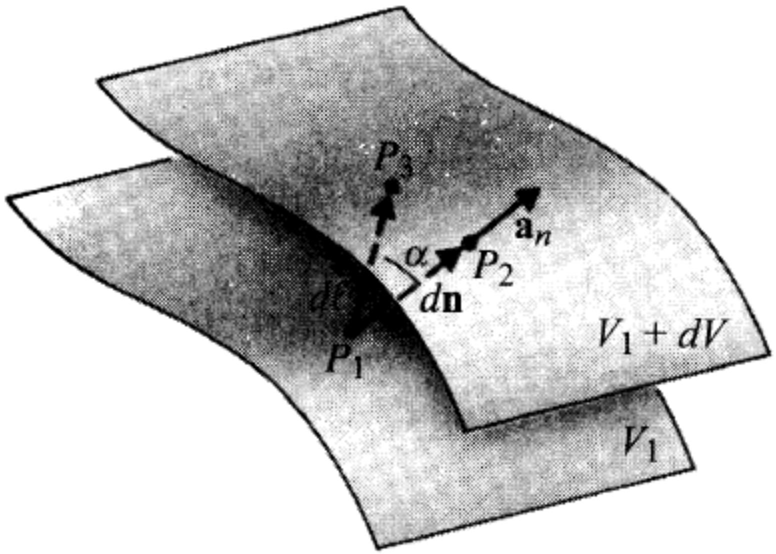
$$dV = r^2\sin\theta drd\phi d\phi$$

2-6 Gradient of a Scalar Field

- In general, fields(t,u_1,u_2,u_3)
- At a given time, fields(u_1,u_2,u_3)
  - Considering a scalar field, we have

$$V(u_1, u_2, u_3),$$

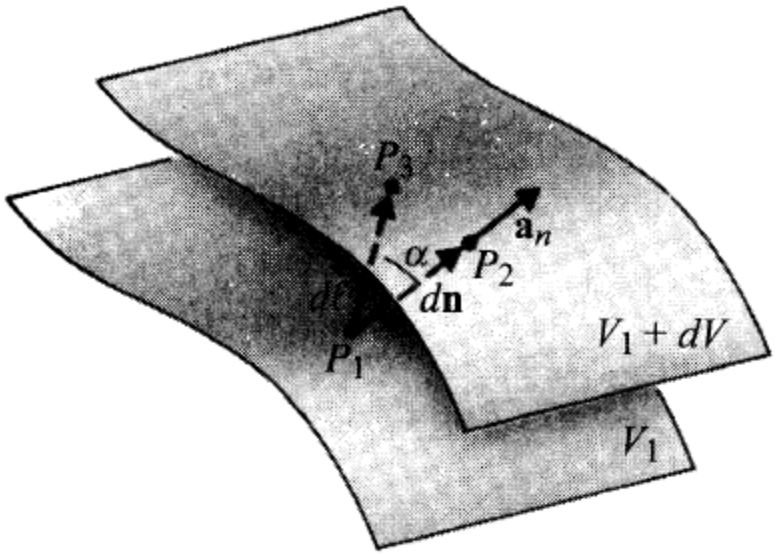
V is a scalar field, not volume



$V(u_1, u_2, u_3)$,

FIGURE 2–24
Concerning gradient of a scalar.

- 2 constant-*v* surfaces
- P_1 on surface V_1; P_2 and P_3 on surface V_1+dV
- Space rate of change: dV/dl
- *d*n is the shortest distance between the two surfaces



$V(u_1, u_2, u_3),$

FIGURE 2–24
Concerning gradient of a scalar.

- Q1: Maximum space rate of change?

- Q2: What's the ratio between dV/dn (P_1 to P_2) and dV/dl (P_1 to P_3)?

dV/dn

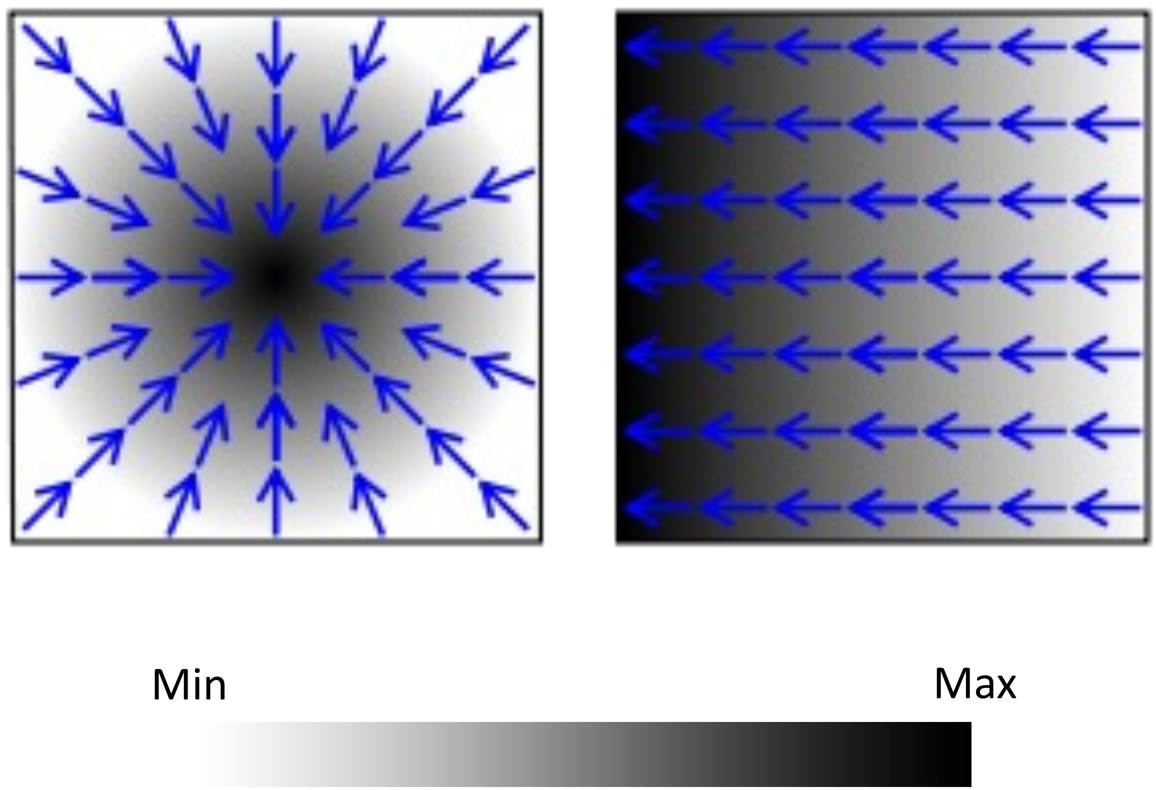
$\cos\alpha$

Gradient of a Scalar Field

- Describes the maximum space rate of change of a scalar field at a given time, which is a vector along that direction.

$$\boxed{\mathbf{grad}\ V \triangleq \mathbf{a}_n \frac{dV}{dn}.}$$

$$\boxed{\nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}.}$$


Min
Max

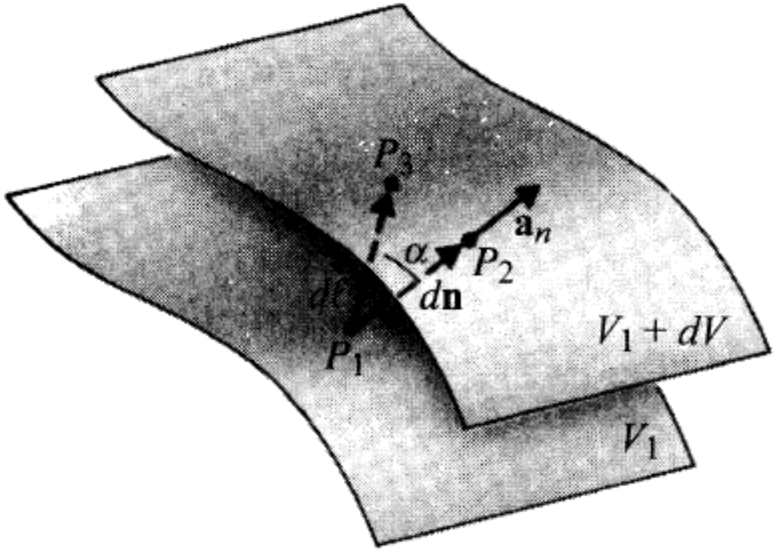


FIGURE 2–24
Concerning gradient of a scalar.

$$\frac{dV}{d\ell} = \frac{dV}{dn}\frac{dn}{d\ell} = \frac{dV}{dn}\cos\alpha$$

$$= \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_\ell = (\nabla V) \cdot \mathbf{a}_\ell.$$

dV/dl: the component of ∇V along $\mathbf{a}_l$ direction

$\mathbf{A} = A_x\mathbf{x} + A_y\mathbf{y} + A_z\mathbf{z}$; $A_x = \mathbf{A} \bullet \mathbf{x}$

$$\frac{dV}{d\ell} = (\nabla V) \cdot \mathbf{a}_\ell.$$

$$d\bar{\ell} = \mathbf{a}_\ell d\ell.$$

$$\boxed{dV = (\nabla V) \cdot d\bar{\ell},}$$

$$dV = \frac{\partial V}{\partial \ell_1} d\ell_1 + \frac{\partial V}{\partial \ell_2} d\ell_2 + \frac{\partial V}{\partial \ell_3} d\ell_3,$$

Total differential change

partial change at one coordinate direction

$$dV = \frac{\partial V}{\partial \ell_1} d\ell_1 + \frac{\partial V}{\partial \ell_2} d\ell_2 + \frac{\partial V}{\partial \ell_3} d\ell_3,$$

$$\boxed{dV = (\nabla V) \cdot d\overline{\boldsymbol{\ell}},}$$

$$\begin{aligned} d\overline{\boldsymbol{\ell}} &= \underline{\mathbf{a}_{u_1} d\ell_1 + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3} \\ &= \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3). \end{aligned}$$

$$\nabla V = \underline{\mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial \ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{d\ell_3}}$$

$$\boxed{\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3\, \partial u_3}.}$$

$$\boxed{\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3\, \partial u_3}.}$$

take V out

Gradient operator
(general expression)

$$\nabla \equiv \left(\mathbf{a}_{u_1} \frac{\partial}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3\, \partial u_3} \right)$$

E.g., in Cartesian coordinate

$$\boxed{\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}.}$$

Grad in cylindrical and spherical coordinates

Similarly, in cylindrical coordinate

$$\boxed{\begin{aligned} h_1 &= h_3 = 1 \\ h_2 &= r \end{aligned}}$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla \equiv \left(\mathbf{a}_{u_1}\frac{\partial}{h_1\,\partial u_1} + \mathbf{a}_{u_2}\frac{\partial}{h_2\,\partial u_2} + \mathbf{a}_{u_3}\frac{\partial}{h_3\,\partial u_3}\right)$$

In spherical coordinate

$$\boxed{\begin{aligned} h_2 &= R \\ h_3 &= R\sin\theta \end{aligned}}$$

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

Integrating Gradients Along an Open Path

For the special case of the integral of a gradient function, the integral is simply the difference of the value of the function at the endpoints:

$$W = \int_L \underline{\nabla V \cdot d\vec{l}}$$

$$W = \int_L \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \int_L \underline{dV} = V(r_f) - V(r_i)$$

This result is known as the fundamental theorem of calculus, which can be re-written in many forms, which we will show later. This also implies that the integral of **a gradient function over any closed integral vanishes**.

$$W = \oint \nabla V \cdot d\vec{l} = \oint dV = V(r_f) - V(r_i) = 0$$

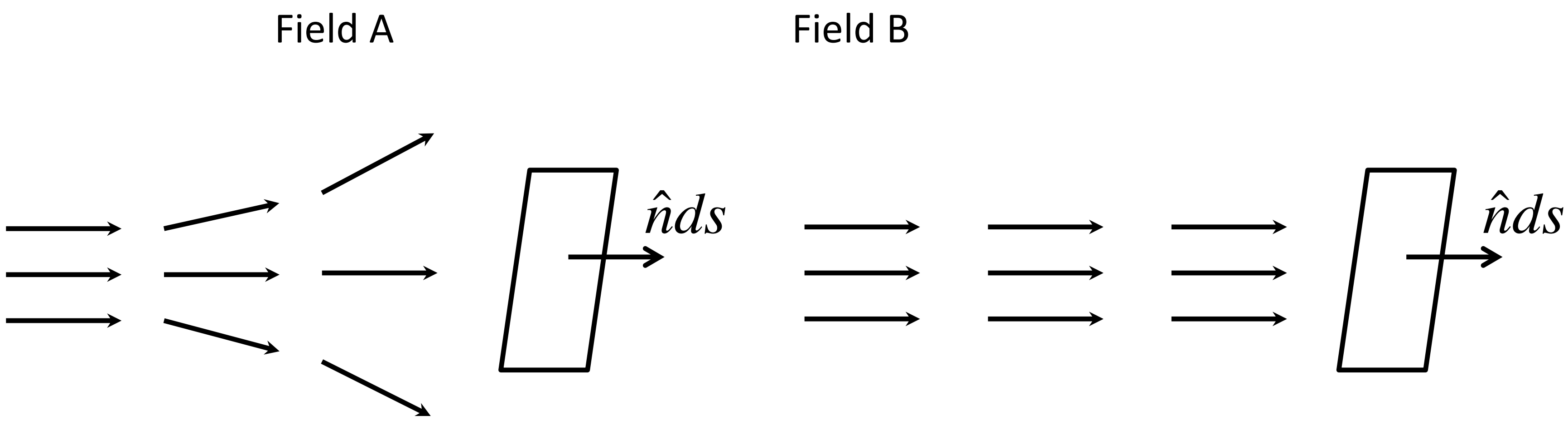
2-7 Divergence of a Vector Field

- Spatial derivative of a **vector** field

    → Divergence, Curl

- Divergence: A concept on the (flow) source or sink

Flux Lines for A Vector Field



The vector field **strength** is measured by the number of flux lines passing through a unit surface normal to the vector (i.e., **density**).

High density ⇔ Strong field

Flux and Flux Density

- **Flux**:
  - A flow (e.g., an incompressible fluid such as water)
  - In general, a continuous moving on or passing by as of a stream

v • dS , water amount/time (where ***v*** is the flow speed), m^3/s

$$\bar{A} \cdot d\bar{S}$$

- **Flux density**:

- The total flux crossing an enclosed surface S which bounds a volume V is:

$$\frac{\bar{A} \cdot d\bar{S}}{\left|d\bar{S}\right|} = \frac{\bar{A} \cdot \hat{n} dS}{dS}$$

$$\oint_S \bar{A} \cdot d\bar{S}$$

Water amount/time *through (outward or inward)* V

Divergence

- The net outward flux of **A** **per unit volume** as the volume about the point tends to zero.

$$div\bar{A} = \lim \frac{\oint \bar{A} \cdot d\bar{S}}{\Delta V}$$

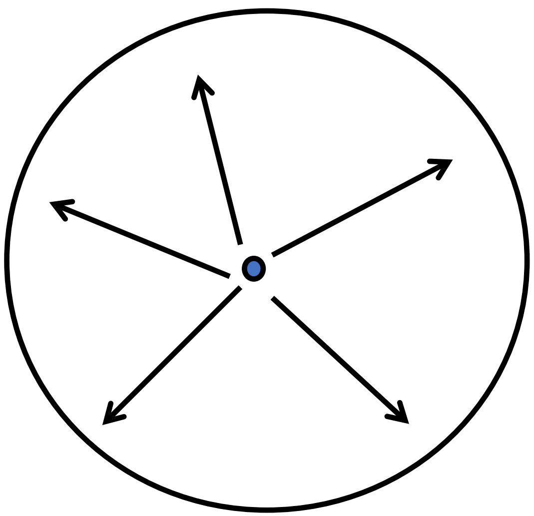
Outward water **density**/time

- diverge – to move or extend in different directions from a common point; move away from.

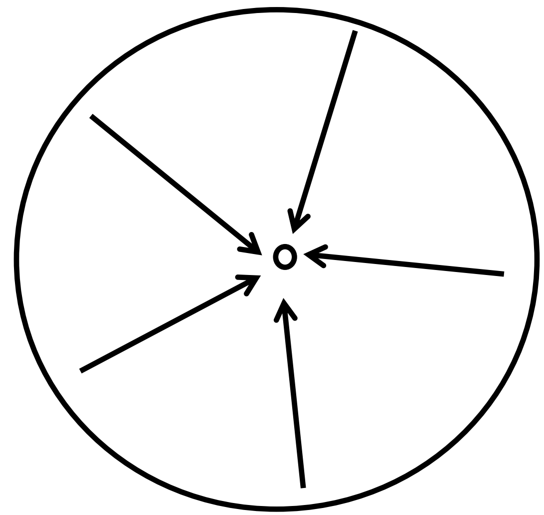
Divergence: flux density (over a tiny volume (closed surface))

Source or Sink of A Vector Field

Source



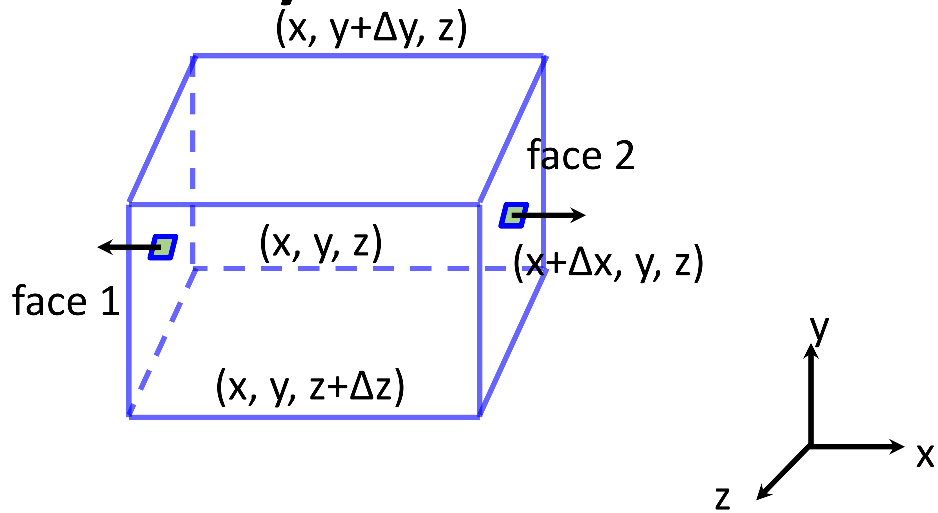
Sink



1. Divergence >0, source exists
2. Divergence <0, sink exists
3. Divergence = 0, divergenceless or solenoidal (pipe)

net outward flux > inward flux ➔ source ➔ divergence > 0 net outward flux < inward flux ➔ sink ➔ divergence < 0

Derivation of Divergence in the XYZ Coordinate System



Face 1: $$F_1 = \int_s \bar{A} \cdot \hat{n}_1 \, dS = \int_x \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right) \cdot \left(-\hat{x}dydz\right) = -A_x(1)\Delta y \Delta z$$

where $A_x(1)$ is the average value of A_x on face 1

Face 2: $$F_2 = \int_s \bar{A} \cdot \hat{n}_s \, dS = \int_x \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right) \cdot \left(\hat{x}dydz\right) = A_x(2)\Delta y \Delta z$$

$$A_x(2) = A_x(1) + \frac{\partial A_x}{\partial x}\Delta x$$ (H.O.T.: See Text, p. 48)

$$F_1 + F_2 = -A_x(1)\Delta y \Delta z + \left(A_x(1) + \frac{\partial A_x}{\partial x}\Delta x\right)\Delta y \Delta z = \frac{\partial A_x}{\partial x}\Delta V$$

Derivation of Divergence in the XYZ Coordinate System

Similarly,

$$F_3 + F_4 = \frac{\partial A_y}{\partial y}\Delta V$$

$$F_5 + F_6 = \frac{\partial A_z}{\partial z}\Delta V$$

So the total flux is

$$\sum_{i=1}^{6} F_i = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta V$$

$$div\bar{A} = \lim \frac{\oint \bar{A}\cdot d\bar{S}}{\Delta V}$$

And the divergence is defined as

$$div\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \bar{A}$$

General Expression

$$\boxed{\nabla \cdot \mathbf{A} \equiv \operatorname{div} \mathbf{A}.}$$

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right].}$$

In cylindrical coordinate system:

$$\boxed{\begin{aligned} h_1 &= h_3 = 1 \\ h_2 &= r \end{aligned}}$$

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

In spherical coordinate system:

$$\boxed{\begin{aligned} h_2 &= R \\ h_3 &= R \sin \theta \end{aligned}}$$

$$\nabla \cdot \bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

2-8 Divergence Theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_v \nabla \cdot \overline{A}\, dV = \oint_s \overline{A} \cdot d\overline{S}$$

- Conversion between volume integral and surface integral.

flux density × volume = flux
outward water density/time × volume = outward water amount/time (m^3/s)

Proof

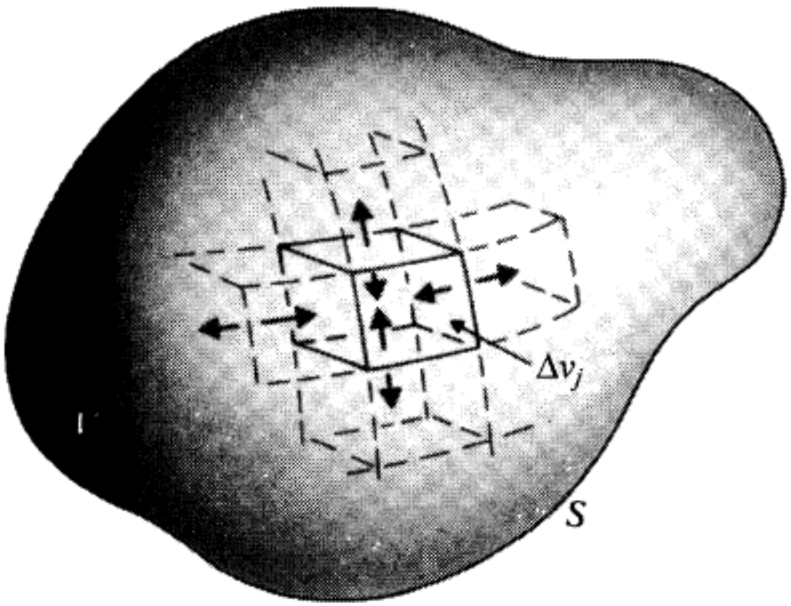


FIGURE 2–27
Subdivided volume for proof of divergence theorem.

$$(\nabla \cdot \mathbf{A})_j \, \Delta v_j = \oint_{s_j} \mathbf{A} \cdot d\mathbf{s}.$$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \, \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \mathbf{A} \cdot d\mathbf{s} \right].$$

Proof

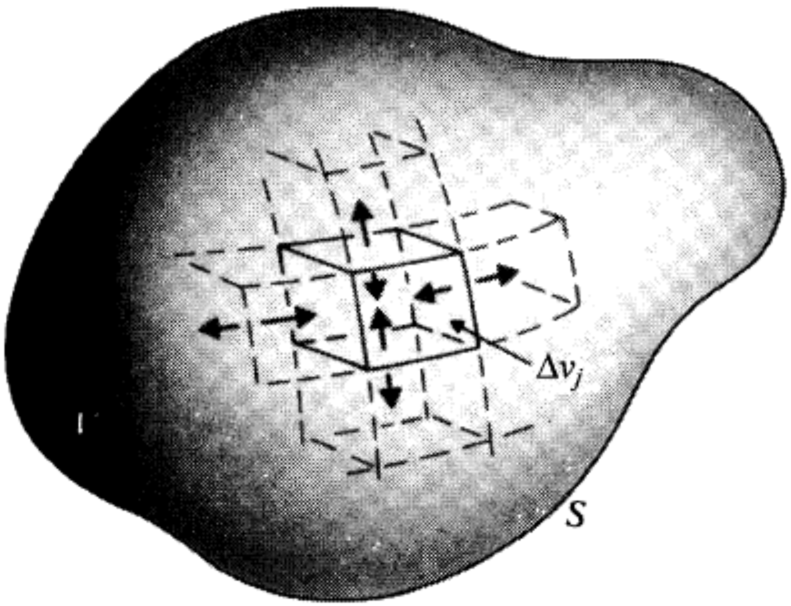


FIGURE 2–27
Subdivided volume for proof of divergence theorem.

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \, \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \mathbf{A} \cdot d\mathbf{s} \right].$$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \, \Delta v_j \right] = \int_V (\nabla \cdot \mathbf{A}) \, dv.$$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \int_{s_j} \mathbf{A} \cdot d\mathbf{s} \right] = \oint_S \mathbf{A} \cdot d\mathbf{s}.$$

Internal surfaces cancelled

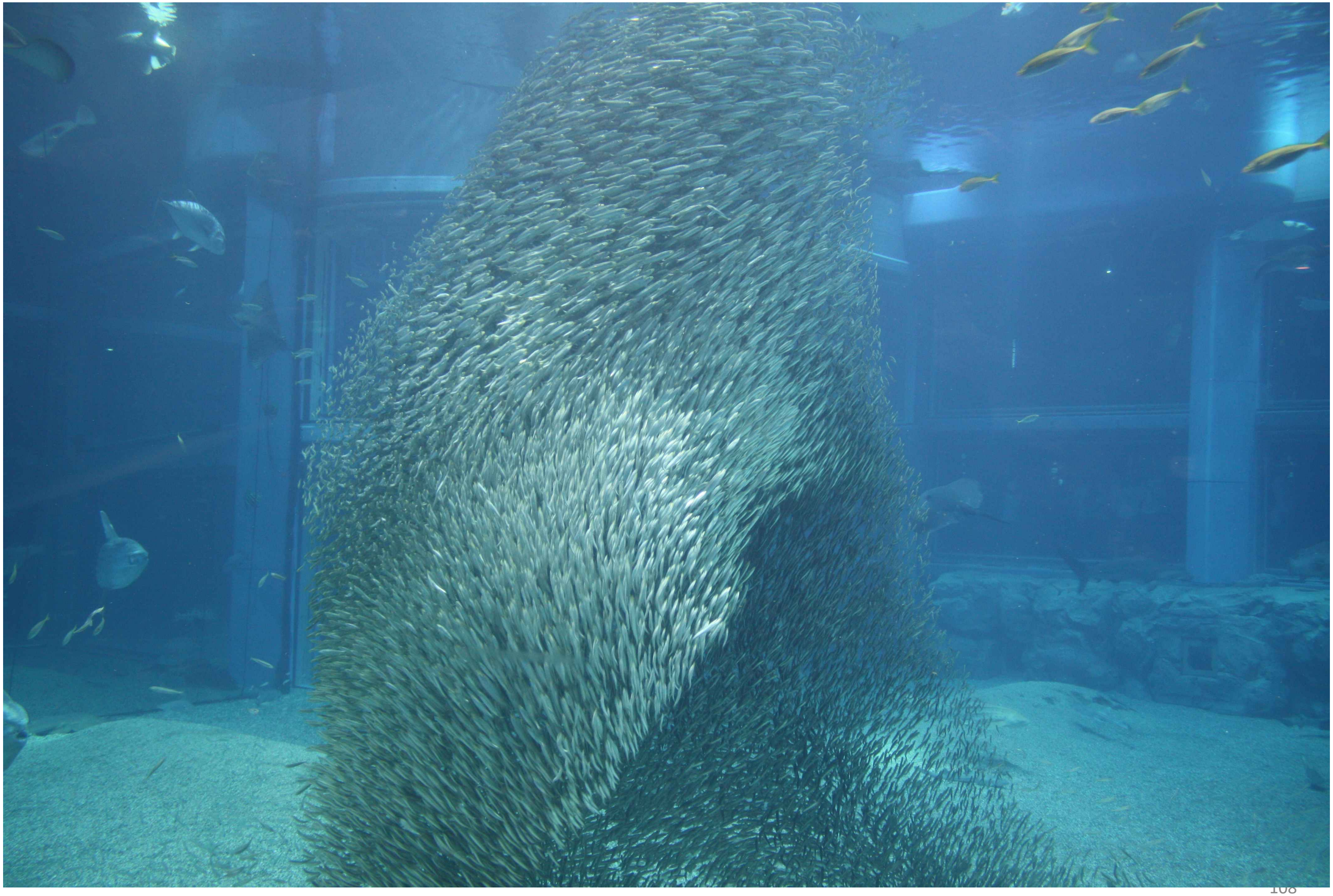
2-9 Curl of a Vector Field

- Divergence indicates whether there is a flow source or not.
- Curl indicates whether there is a vortex source or not.

Water whirling down a sink drain is an example of a vortex sink causing a circulation of fluid velocity

Fish Vortex!

Fish Vortex!



Circulation

- The net circulation of a vector field around a closed path is defined as:

$$\oint_c \overline{A} \cdot d\overline{l}$$

Example 1

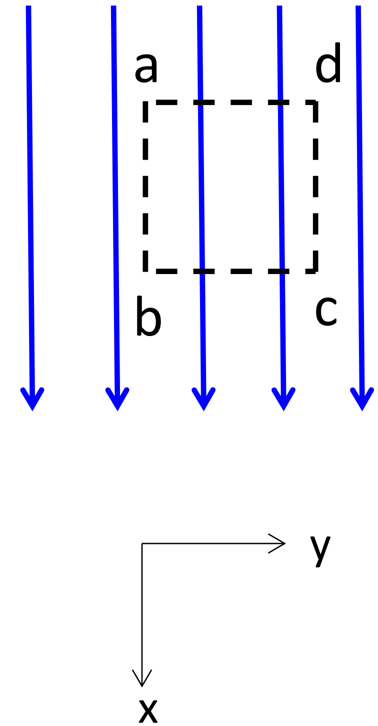
Given a uniform magnetic field $\overline{B} = \hat{x}B_0$

$$Circulation = \oint_s \overline{B} \cdot d\overline{l}$$

$$= \int_a^b \hat{x}B_0 \cdot \hat{x}\,dx + \int_b^c \hat{x}B_0 \cdot \hat{y}\,dy +$$

$$\int_c^d \hat{x}B_0 \cdot \hat{x}\,dx + \int_d^a \hat{x}B_0 \cdot \hat{y}\,dy$$

$$= B_0\Delta x \underline{- B_0\Delta x} = 0$$

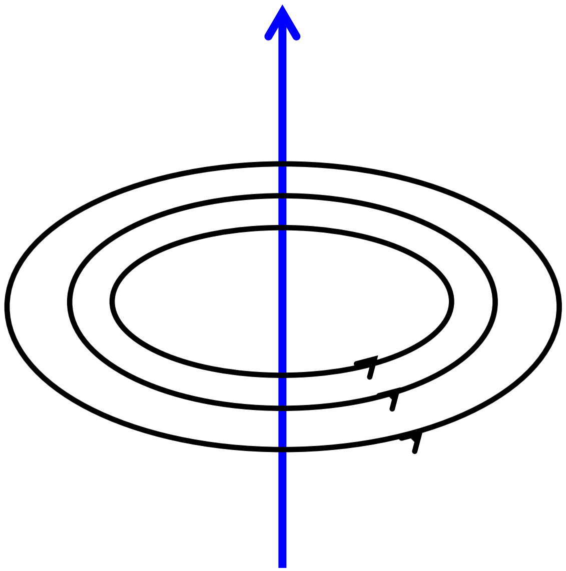


No net circulation exists!

Example 2

Magnetic field induced by an infinite wire carrying a DC current I in the positive z-axis direction, then

$$\overline{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r}$$



$$Circulation = \oint_c \overline{B} \cdot d\overline{l}$$

$$= \int_0^{2\pi} \hat{\phi}\frac{\mu_0 I}{2\pi r} \cdot \hat{\phi} r \, d\phi = \mu_0 I$$

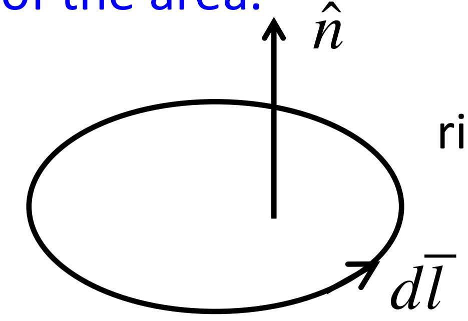
A net circulation exists!

Curl

- Curl is the circulation per unit area in short.

$$\nabla \times \bar{A} = \lim \frac{1}{\Delta s}\left(\hat{n} \oint_c \bar{A} \cdot d\bar{l} \right)$$

- Its magnitude is the **maximum** net circulation of A per unit area as the area tends to zero.
- Its direction is the normal direction of the area.



right-hand rule

Curl: **circulation** density (over a tiny **surface** (closed **path**))

Component of Curl

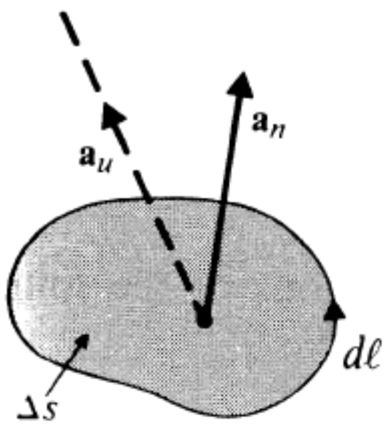


FIGURE 2–30
Relation between $\mathbf{a}_n$ and $d\ell$ in defining curl.

$$\boxed{(\nabla \times \mathbf{A})_u = \mathbf{a}_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u}\left(\oint_{C_u} \mathbf{A} \cdot d\ell\right),}$$

Normal of Δs_u is $\mathbf{a}_u$

The component of $\nabla \times \mathbf{A}$ along $\mathbf{a}_u$ direction

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Proof

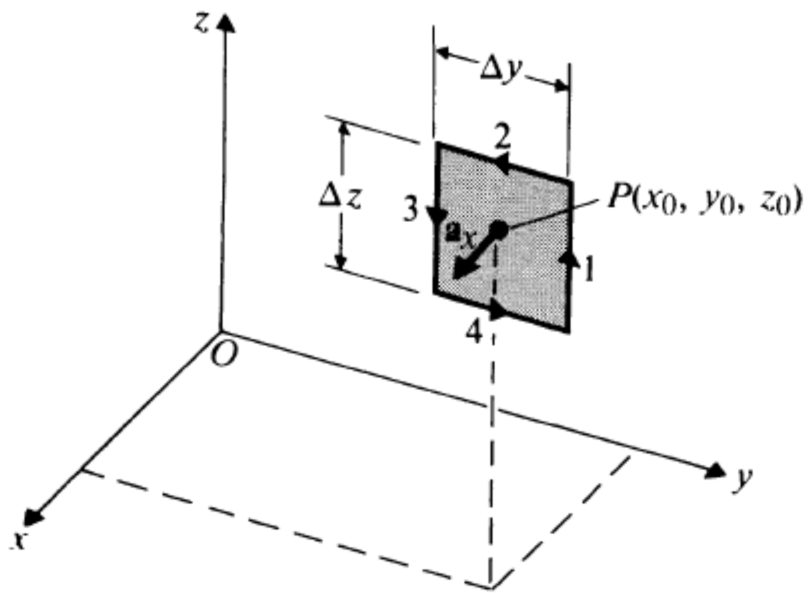


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$(\nabla \times \mathbf{A})_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \, \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,\,2,\,3,\,4}} \mathbf{A} \cdot d\boldsymbol{\ell} \right).$$

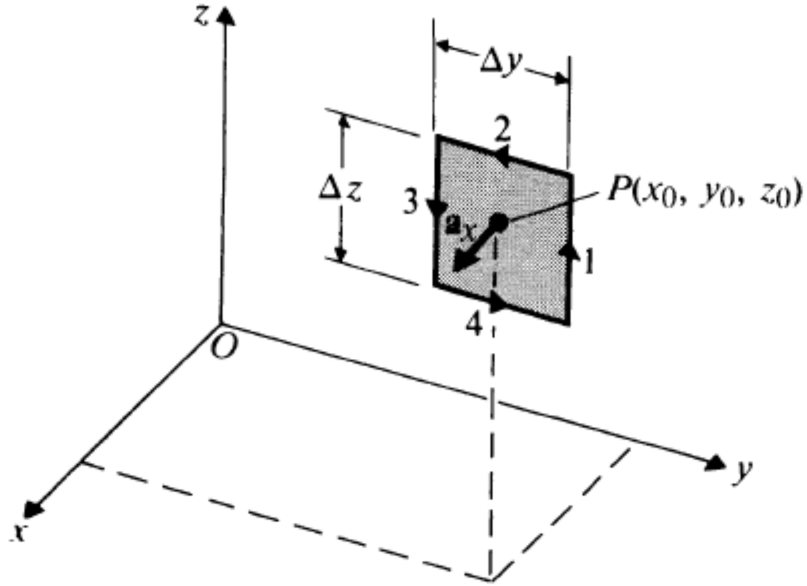


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.$$

Side 1: $d\boldsymbol{\ell} = \mathbf{a}_z \Delta z, \mathbf{A} \cdot d\boldsymbol{\ell} = A_z\left(x_0, y_0 + \dfrac{\Delta y}{2}, z_0\right) \Delta z,$

where $A_z\left(x_0, y_0 + \dfrac{\Delta y}{2}, z_0\right)$ can be expanded as a Taylor series:

$$A_z\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.},$$

$(\Delta y)^2, (\Delta y)^3$, etc.

$$\int_{\text{side 1}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\} \Delta z.$$

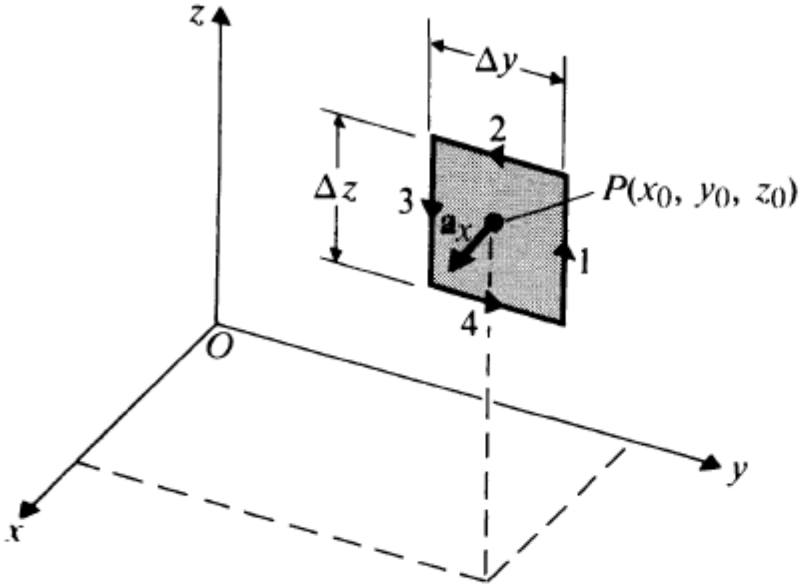


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.$$

Side 3: $\quad d\ell = -\mathbf{a}_z \Delta z, \ \mathbf{A} \cdot d\ell = A_z\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)-\Delta z,$

where

$$A_z\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.};$$

$$\int_{\text{side 3}} \mathbf{A} \cdot d\ell = \left\{A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\}(-\Delta z).$$

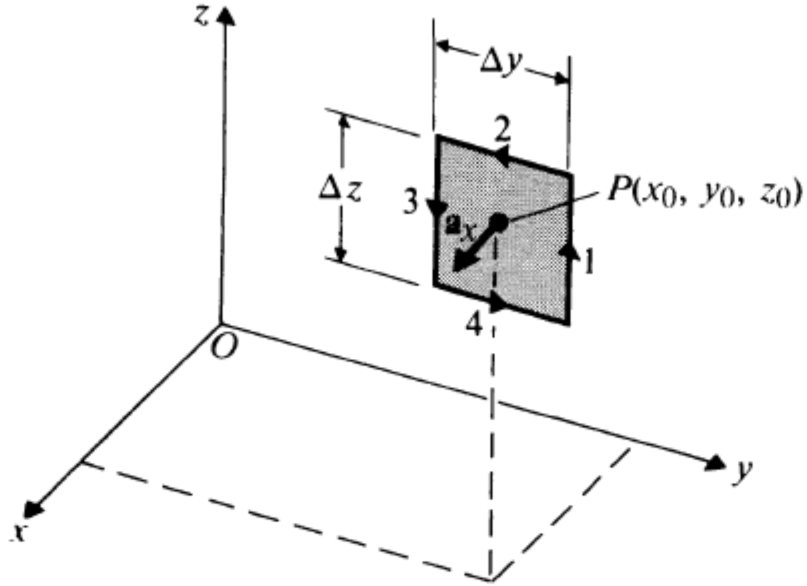


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$\int_{\text{side 1}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{ A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta z.$$

$$\int_{\text{side 3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{ A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta z).$$

$$\int_{\substack{\text{sides} \\ 1\,\&\,3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(\frac{\partial A_z}{\partial y} + \text{H.O.T.} \right)\bigg|_{(x_0, y_0, z_0)} \Delta y \, \Delta z.$$

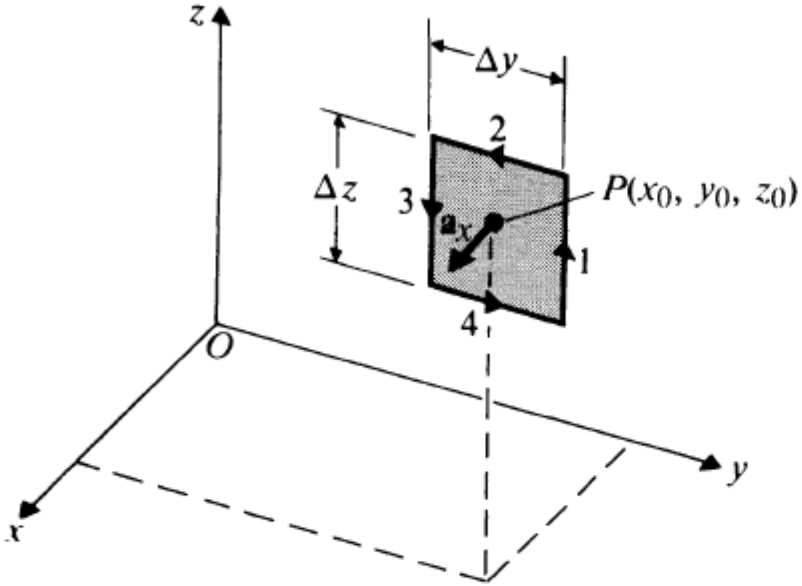


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$\int_{\substack{\text{sides}\\ 1\ \&\ 3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(\frac{\partial A_z}{\partial y} + \text{H.O.T.} \right) \bigg|_{(x_0,\, y_0,\, z_0)} \Delta y \, \Delta z.$$

$\boxed{\propto \Delta y, \Delta y^2, \text{etc.}}$

Similarly, $$\int_{\substack{\text{sides}\\ 2\ \&\ 4}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(-\frac{\partial A_y}{\partial z} + \text{H.O.T.} \right) \bigg|_{(x_0,\, y_0,\, z_0)} \Delta y \, \Delta z.$$

$\boxed{\propto \Delta z, \Delta z^2, \text{etc.}}$

As $\Delta y \rightarrow 0$ and $\Delta z \rightarrow 0$, HOT $\rightarrow 0$

$$(\nabla \times \mathbf{A})_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \, \Delta z} \left(\oint_{\substack{\text{sides}\\ 1,\,2,\,3,\,4}} \mathbf{A} \cdot d\boldsymbol{\ell} \right).$$

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

General Expression

$$\boxed{\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.}$$

2-10 Stokes' Theorem

- The surface integral of the curl of a vector field over **an open surface** is equal to the closed line integral of the vector along the contour bounding the surface.

- Conversion between surface integral and line integral.

$$\int_s \left(\nabla \times \overline{A}\right) \cdot d\overline{S} = \oint_c \overline{A} \cdot d\overline{l}$$

$$\boxed{\text{circulation density} \times \text{area} = \text{circulation}}$$

Comparisons

Divergence theorem

$$\int_v \nabla \cdot \overline{A} dV = \oint_s \overline{A} \cdot d\overline{S}$$

flux density × volume = flux

Stokes' theorem

$$\int_s \left(\nabla \times \overline{A} \right) \cdot d\overline{S} = \oint_c \overline{A} \cdot d\overline{l}$$

circulation density × area = circulation

Proof

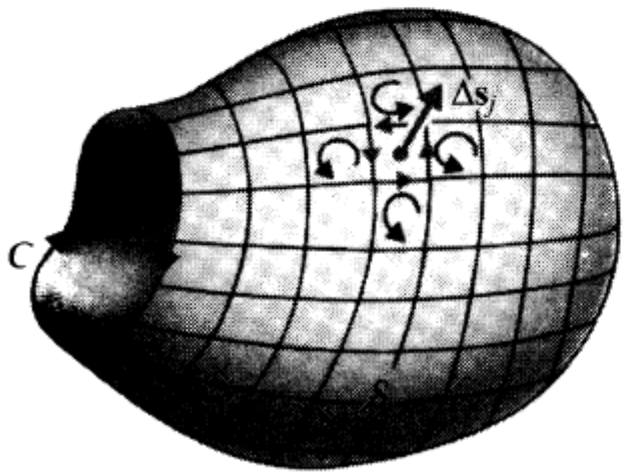


FIGURE 2–32
Subdivided area for proof of Stokes's theorem.

$$(\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \oint_{C_j} \mathbf{A} \cdot d\boldsymbol{\ell}.$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}.$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot d\boldsymbol{\ell} \right) = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}.$$

Interior line integrals cancel each other

2-11 Two Null Identities

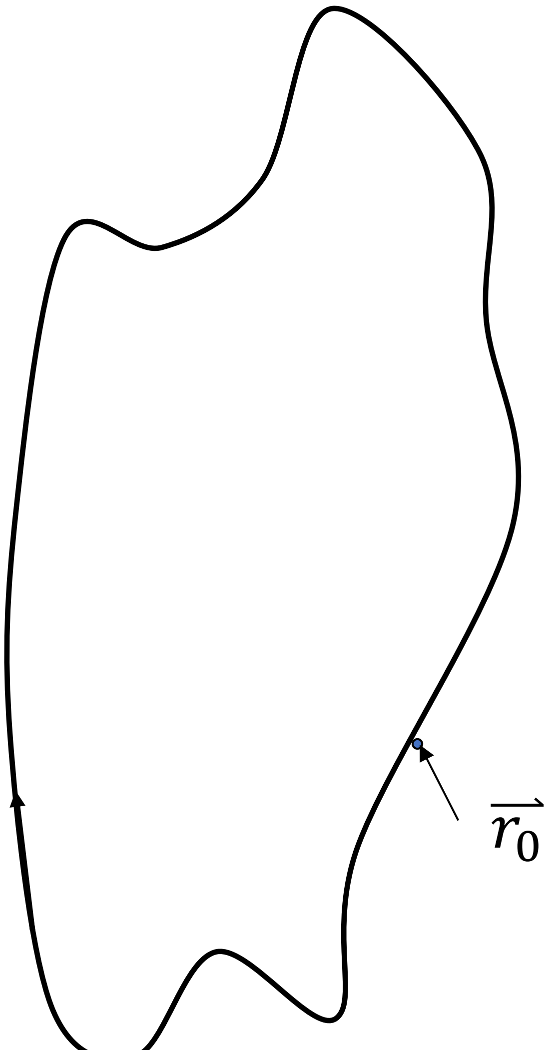
$$\nabla \times \left(\nabla V\right) = 0$$

Therefore, if a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \cdot \left(\nabla \times \boldsymbol{A}\right) = 0$$

Therefore, if a vector field is divergenceless, then it can be expressed as the curl of another vector field.

Proof: The curl of a gradient of a scalar is always zero



$$\oint_L \nabla V(\vec{r}) \cdot d\vec{l} = V(\vec{r_0}) - V(\vec{r_0}) = 0$$

$$\boxed{dV = \nabla V \cdot d\vec{l}}$$

Now consider $\nabla \times \nabla V(\vec{r})$

$$\hat{n} \cdot \nabla \times \nabla V(\vec{r}) = \lim_{S_n \to 0} \frac{\oint_{L_n} \nabla V(\vec{r}) \cdot d\vec{l}}{S_n} = \lim_{S_n \to 0} \frac{0}{S_n} = 0$$

Thus $\boxed{\nabla \times \nabla V(\vec{r}) = 0}$

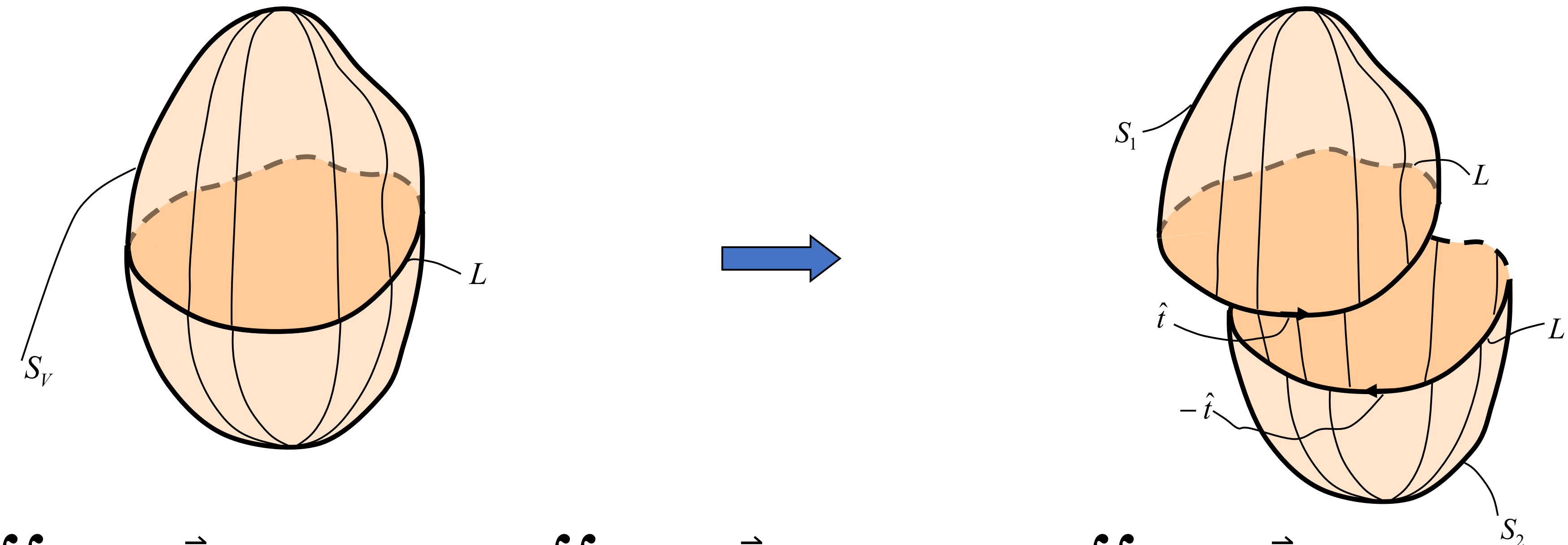
$$\underbrace{\nabla \times \left(\nabla V \right)}_{\mathbf{A}} = 0$$

A: Curl free = irrotational field = Conservative field

A conservative field can always be expressed:

$$\mathbf{A} = \nabla V$$

Proof: The divergence of a curl of a vector is always zero



$$\oiint_{S_V} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS = \iint_{S_1} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS + \iint_{S_2} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS =$$

$$\oint_{L_1} \vec{A}(\vec{r}) \cdot \hat{t} dl + \oint_{L_2} \vec{A}(\vec{r}) \cdot \left(-\hat{t}\right) dl = 0$$

n of S_1 is upward

n of S_2 is downward

$$\nabla \cdot \left(\nabla \times \vec{A}(\vec{r})\right) = \lim_{V \to 0} \frac{\oiint_{S_V} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS}{V} = \lim_{V \to 0} \frac{0}{V} = 0$$

$$\nabla \cdot \big(\underbrace{\nabla \times V}_{\mathbf{B}}\big) = 0$$

B: divergenceless

↓

A divergenceless field can always be expressed:

$$\mathbf{B} = \nabla \times V$$

Combination of Vector Operators: Laplacian

One of the important vector operators that we will commonly use is the divergence of the gradient of a scalar function:

$$\nabla \cdot \nabla V = \nabla^2 V$$

Let's first analyze its form in Cartesian coordinates. Knowing that the divergence of a vector is:

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

And assuming that ***A*** can be written as the gradient of a scalar:

$$\vec{A} = \nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

Replace A_x with $\partial V/\partial x$...

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

Laplacian in Cylindrical Coordinates

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\vec{A} = \nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial V}{\partial \varphi} + \hat{z}\frac{\partial V}{\partial z}$$

Replace A_r with $\partial u/\partial r$
Replace A_ϕ with $(1/r)\partial u/\partial \phi$
…

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial^2 \varphi} + \frac{\partial^2 V}{\partial z^2}}$$

Laplacian in Spherical Coordinates

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\vec{A} = \nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}}$$

Chain rule of vector differentiation

$$\nabla\left(f(\vec{r})g(\vec{r})\right) = f\nabla g + g\nabla f$$

$$\nabla\cdot\left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla\cdot\vec{G} + \vec{G}\cdot\nabla f$$

$$\nabla\times\left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla\times\vec{G} + \nabla f\times\vec{G}$$

$$\nabla\cdot\left(\vec{F}(\vec{r})\times\vec{G}(\vec{r})\right) = \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}$$

Example: Using chain show that: $\nabla\frac{e^{\beta r}}{r} = \hat{r}\left(\beta - \frac{1}{r}\right)\frac{e^{\beta r}}{r}$

Note that this relation is only valid when r is not equal to 0.

Other Useful Vector Identities

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) + \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$\nabla\cdot(\nabla\psi) = \nabla^2\psi$ (scalar Laplacian)

$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A}$ (vector Laplacian)

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

2-12 Helmholtz Theorem

- Any vector in 3D can be decomposable into a sum of the following vector fields

Divergence less ⇔ solenoidal ⇔ no (flow) source
Curl free ⇔ irrotational ⇔ no circulation

- 4 cases:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} = 0.$$
$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0.$$
$$\nabla \times \mathbf{F} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F} \neq 0.$$
$$\nabla \cdot \mathbf{F} \neq 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0.$$

Helmholtz Theorem

- A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.
- Intuitive illustration:
  - Divergence ⇔ flow source
    Curl ⇔ vortex source
  - If both the flow source and vortex source of a vector are specified
    ➔ the vector is determined

Helmholtz Theorem-Equations

$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s,$$

Irrotational = flow source

$$\begin{cases} \nabla \times \mathbf{F}_i = 0 \\ \nabla \cdot \mathbf{F}_i = g \end{cases}$$

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_i = g$$

$$\mathbf{F}_i = -\nabla V.$$

Solenoidal = Vortex source

$$\begin{cases} \nabla \cdot \mathbf{F}_s = 0 \\ \nabla \times \mathbf{F}_s = \mathbf{G}, \end{cases}$$

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_s = \mathbf{G}.$$

$$\mathbf{F}_s = \nabla \times \mathbf{A}.$$

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}.$$

Helmholtz Theorem-Proof

What we would like to prove is that any vector field can be described completely by **a scalar potential** and **a vector potential** if the source density and circulation density is everywhere known. Consider the following… Suppose you have a vector field ***V*** which satisfies the following:

$$\nabla \cdot \vec{V}(\vec{r}) = s(\vec{r}) \qquad \nabla \times \vec{V}(\vec{r}) = \vec{c}(\vec{r})$$

Where functions $s(\vec{r}), \vec{c}(\vec{r})$ are **known** quantities, located within finite regions of space,

$$\nabla \cdot \vec{c}(\vec{r}) = 0$$

s: flow source

c: vortex source

We will show that this problem has a unique solution in the form:

$$\vec{V}(\vec{r}) = -\nabla\varphi(\vec{r}) + \nabla\times\vec{A}(\vec{r})$$

$$\varphi(\vec{r}) = \frac{1}{4\pi}\iiint_{all\ space} \frac{s(\overrightarrow{r_s})}{|\vec{r}-\overrightarrow{r_s}|}dV_s \qquad \vec{A}(\vec{r}) = \frac{1}{4\pi}\iiint_{all\ space} \frac{\vec{c}(\overrightarrow{r_s})}{|\vec{r}-\overrightarrow{r_s}|}dV_s$$

- Where we denote φ as the scalar potential, whose gradient defines part of the vector field ***V***; and we denote ***A*** as the vector potential whose curl defines the other part of the vector field ***V***.
- What we are trying to show is that any vector field can be "decomposed" into a scalar (φ) and vector potential (***A***), which is curl-free and divergence-free, respectively.

$$\vec{V}(\vec{r}) = \underline{-\nabla\varphi(\vec{r})} + \nabla\times\vec{A}(\vec{r})$$

Part 1: flow source *s* (scalar potential φ)

First, let's consider a vector field which is **curl-free**:

$$\nabla\cdot\vec{V}(\vec{r}) = s(\vec{r}) \qquad\qquad \nabla\times\vec{V}(\vec{r}) = 0$$

Let's first show that **this vector field can be defined only by the gradient of a scalar potential φ** due to source distribution *s*.

$$\varphi(\vec{r}) = \frac{1}{4\pi}\iiint\limits_{all\ space}\frac{s(\vec{r_s})}{|\vec{r}-\vec{r_s}|}dV_s \qquad\qquad \vec{A}(\vec{r}) = \frac{1}{4\pi}\iiint\limits_{all\ space}\frac{\cancel{\vec{c}}(\vec{r_s})}{|\vec{r}-\vec{r_s}|}dV_s = 0$$

First we must show that when the vector field is defined by the gradient of a scalar potential, the vector field has no curl.

$$\vec{V}(\vec{r}) = -\nabla\varphi(\vec{r}) \qquad\qquad \nabla\times\vec{V}(\vec{r}) = -\nabla\times\nabla\varphi(\vec{r}) = 0$$

Next, we must show that the divergence of the vector field ***V*** leads to the source distribution *s*.

$$\nabla \cdot \vec{V}(\vec{r}) = -\nabla^2 \varphi(\vec{r}) = -\frac{1}{4\pi} \iiint_{all\ space} s(\overrightarrow{r_s}) \nabla^2 \frac{1}{|\vec{r} - \overrightarrow{r_s}|} dV_s = ?\ ?\ ?$$

$$\boxed{\varphi(\vec{r}) = \frac{1}{4\pi} \iiint_{all\ space} \frac{s(\overrightarrow{r_s})}{|\vec{r} - \overrightarrow{r_s}|} dV_s}$$

$$\boxed{-4\pi\delta(\boldsymbol{r} - \boldsymbol{r_s})}$$

$$= \iiint_{all\ space} s(\overrightarrow{r_s}) \delta(\vec{r} - \overrightarrow{r_s}) dV_s = s(\vec{r})$$

$$\nabla \cdot \vec{V}(\vec{r}) = s(\vec{r})$$

Thus, we have proven the point that the divergence of any vector field, which is defined by the gradient of a scalar potential, is equal to the source distribution *s*(***r***) which produces the scalar potential.

Assuming $$\vec{V}(\vec{r}) = -\nabla\varphi(\vec{r})$$

$$\Downarrow$$

Confirmed $$\nabla\times\vec{V}(\vec{r}) = 0 \qquad \nabla\cdot\vec{V}(\vec{r}) = s(\vec{r})$$

For a curl-free vector field, it can be defined only by the gradient of a scalar potential φ

$$\vec{V}(\vec{r}) = -\nabla\varphi(\vec{r}) + \underline{\nabla\times\vec{A}(\vec{r})}$$

Part 2: circulation source ***c*** (vector potential ***A***)

Now, let's consider a vector field which is source free:

$$\nabla\cdot\vec{V}(\vec{r}) = 0 \qquad\qquad \nabla\times\vec{V}(\vec{r}) = \vec{c}(\vec{r})$$

$$\varphi(\vec{r}) = \frac{1}{4\pi}\iiint_{all\ space}\frac{\cancel{\vec{s}}(\vec{r_s})}{|\vec{r}-\vec{r_s}|}dV_s = 0 \qquad \vec{A}(\vec{r}) = \frac{1}{4\pi}\iiint_{all\ space}\frac{\vec{c}(\vec{r_s})}{|\vec{r}-\vec{r_s}|}dV_s$$

First we must show that when the vector field is defined by the curl of a vector potential, the vector field has no divergence.

$$\vec{V}(\vec{r}) = \nabla\times\vec{A}(\vec{r}) \qquad\qquad \nabla\cdot\vec{V}(\vec{r}) = \nabla\cdot\nabla\times\vec{A}(\vec{r}) = 0$$

Next, we must show that the curl of the vector field ***V*** leads to the circulation distribution **c**.

$$\nabla\times\vec{V}(\vec{r}) = \nabla\times\nabla\times\vec{A}(\vec{r}) = \nabla\left(\nabla\cdot\vec{A}(\vec{r})\right) - \nabla^2\vec{A}(\vec{r})$$

$$\nabla^2 \vec{A}(\vec{r}) = \nabla^2 \frac{1}{4\pi} \iiint\limits_{all\ space} \vec{c}(\overrightarrow{r_s}) \frac{1}{|\vec{r} - \overrightarrow{r_s}|} dV_s = \ ?\ \ ?\ \ ?$$

$-4\pi\delta(\mathbf{r} - \mathbf{r}_s)$

$$= -\iiint\limits_{all\ space} \vec{c}(\overrightarrow{r_s}) \delta(\vec{r} - \overrightarrow{r_s}) dV_s = -\vec{c}(\vec{r})$$

As it turns out, the other term is zero, however it will not be derived here, i.e.,

$$\nabla\left(\nabla \cdot \vec{A}(\vec{r})\right) = 0$$

Hence: $\nabla \times \vec{V}(\vec{r}) = -\nabla^2 \vec{A}(\vec{r}) = \vec{c}(\vec{r})$

Assuming $\vec{V}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$

$\Downarrow$

Confirmed $\nabla \cdot \vec{V}(\vec{r}) = 0$ $\nabla \times \vec{V}(\vec{r}) = \vec{c}(\vec{r})$

For a source-free vector field, it can be defined only by the curl of a vector potential **A**

Thus, we have proven that the curl of any vector field, which is defined by the curl of a vector potential, is equal to the circulation distribution c(r). (see part 2)

$$\boxed{\nabla \times \vec{V}(\vec{r}) = \vec{c}(\vec{r})}$$

Likewise, the divergence of any vector field, which is defined by the gradient of a scalar potential, is equal to the scalar source distribution, s(r). (see part 1)

$$\boxed{\nabla \cdot \vec{V}(\vec{r}) = s(\vec{r})}$$

Effectively, this means that if we know the source distribution and circulation distribution everywhere in space then we can determine the vector field resulting from a scalar potential and vector potential everywhere.

$$\boxed{s \text{ and } \mathbf{c} \Rightarrow \varphi \text{ and } \mathbf{A} \Rightarrow \mathbf{V}}$$

Uniqueness

Uniqueness: How do we know if there is more than one solution to the vector field equations? Let's suppose that there are two possible solutions, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ which satisfy the same source and circulation distributions.

$$\nabla \cdot \overrightarrow{V_1}(\vec{r}) = s(\vec{r}) \qquad \nabla \cdot \overrightarrow{V_2}(\vec{r}) = s(\vec{r})$$

$$\nabla \times \overrightarrow{V_1}(\vec{r}) = \vec{c}(\vec{r}) \qquad \nabla \times \overrightarrow{V_2}(\vec{r}) = \vec{c}(\vec{r})$$

Let's define a new function ***W***, which is the difference between the two vector fields ***V*₁** and ***V*₂**:

$$\overrightarrow{W}(\vec{r}) = \overrightarrow{V_1}(\vec{r}) - \overrightarrow{V_2}(\vec{r})$$

$$\nabla \cdot \overrightarrow{W}(\vec{r}) = \nabla \cdot \overrightarrow{V_1}(\vec{r}) - \nabla \cdot \overrightarrow{V_2}(\vec{r}) = s(\vec{r}) - s(\vec{r}) = 0 \quad (1)$$

$$\nabla \times \overrightarrow{W}(\vec{r}) = \nabla \times \overrightarrow{V_1}(\vec{r}) - \nabla \times \overrightarrow{V_2}(\vec{r}) = \vec{c}(\vec{r}) - \vec{c}(\vec{r}) = 0 \quad (2)$$

The goal is to prove that vector field ***W*** is identically zero everywhere in space.

First, let's define the vector field ***W*** produced solely by sources as the gradient of a scalar potential ψ.

(2) ⇒ $\overrightarrow{W}(\vec{r}) = -\nabla\psi(\vec{r}) \longrightarrow \nabla \cdot \overrightarrow{W}(\vec{r}) = \boxed{-\nabla^2\psi(\vec{r}) = 0}$

By (1)

Let's now apply Gauss's divergence theorem to the vector:

$$\oiint \psi(\vec{r})\nabla\psi(\vec{r})\cdot\hat{n}dS = \iiint \nabla\cdot(\psi(\vec{r})\nabla\psi(\vec{r}))dV = \iiint \left(\psi(\vec{r})\nabla^2\psi(\vec{r}) + \nabla\psi(\vec{r})\cdot\nabla\psi(\vec{r})\right)dV$$

$$\boxed{\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi}$$

$$= \iiint (\nabla\psi(\vec{r}))^2 dV = \iiint (\overrightarrow{W})^2 dV$$

One very important assumption for vector fields is that all sources of potential are finite and **no sources exist at infinity**. This also implies that **the potential at infinity is also zero**. Therefore, the surface integral on the left hand side will vanish if we integrate over all space.

$$\underbrace{\oiint_{R\to\infty} \psi(\vec{r})}_{=0}\nabla\psi(\vec{r})\cdot\hat{n}dS = 0 = \iiint (\overrightarrow{W})^2 dV \qquad \therefore \overrightarrow{W} = 0 \Rightarrow \overrightarrow{V_1}(\vec{r}) - \overrightarrow{V_2}(\vec{r})$$

This condition is satisfied everywhere

Basic summary

1. Introduced scalar and vector fields
2. Gradient, divergence and curl defined
3. Fundamental theorem of calculus, Gauss' divergence theorem, Stokes' theorem
4. Helmholtz's theorem:

a) Every vector expressible by a sum of divergence-free and curl-free vectors. From the null theorems, these define a scalar and vector potential

b) These scalar and vector potentials are unique solutions

5. All of these results can be expressed in any orthogonal coordinate systems, and answers to physical problems are identical

Great video!

https://www.youtube.com/watch?v=rB83DpBJQsE