Mid2 RC Part3

1 Lorentz's Force Equation

- **Electric Force:**

$$\boldsymbol{F_e} = q\boldsymbol{E}$$

- **Magnetic Force:**

$$\boldsymbol{F_m} = q\boldsymbol{u} \times \boldsymbol{B}$$

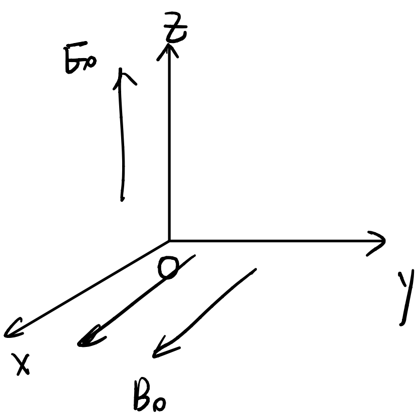
Notice: The magnetic force is derived in experiments. Defined as $\boldsymbol{B}$: magnetic flux density (Wb/m^2=Tesla)

- Lorentz's Force Equation:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

Ex.1

A positive point charge q of mass m is injected with a velocity $\boldsymbol{u} = u_{x0}\boldsymbol{a_x} + u_{y0}\boldsymbol{a_y} + u_{z0}\boldsymbol{a_z}$ into the $\boxed{y > 0}$ region where a uniform magnetic field $\boldsymbol{B} = B_0\boldsymbol{a_x}$ and a uniform electric field $\boldsymbol{E} = E_0\boldsymbol{a_z}$ exist. Obtain the equation of motion of the charge.



$$\vec{u} = (u_{x0}, u_{y0}, u_{z0}) = \underbrace{(u_{x0}, 0, 0)} + (0, u_{y0}, u_{z0})$$

$$\vec{E} = E_0 \hat{a_z}$$

$$\vec{B} = B_0 \hat{a_x} \quad (y>0)$$

$$u_x \times B = 0$$

$$\vec{u} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & u_{y0} & u_{z0} \\ B_0 & 0 & 0 \end{vmatrix} = B_0 \underset{u_z}{\dot{z}} \hat{y} - B_0 \underset{u_y}{\dot{y}} \hat{z}$$

$$\vec{F} = m \cdot \vec{a}$$

$$m(\ddot{y}\hat{y} + \ddot{z}\hat{z}) = q(E_0\hat{z} + B_0\dot{z}\hat{y} - B_0\dot{y}\hat{z})$$

$$\begin{cases} m\ddot{y} = qB_0\dot{z} \\ m\ddot{z} = q(E_0 - B_0\dot{y}) \end{cases} \Rightarrow \begin{matrix} \ddot{y} = \frac{qB_0}{m}\dot{z} \\ \ddot{z} = \frac{qB_0}{m}\left(\frac{E_0}{B_0} - \dot{y}\right) \end{matrix} \leftarrow \omega_0 = \frac{qB_0}{m} \Rightarrow \begin{cases} \ddot{y} = \omega_0\dot{z} \\ \ddot{z} = \omega_0\left(\frac{E_0}{B_0} - \dot{y}\right) \end{cases}$$

$$\underbrace{\frac{d\dot{y}}{dt} = \omega_0\dot{z}}_{\ddot{z}} \rightarrow \frac{d^2(\dot{y})}{dt^2} = \omega_0\ddot{z} = \omega_0^2\left(\frac{E_0}{B_0} - \dot{y}\right)$$

$$\dot{y} \rightarrow y.$$

$$\begin{cases} y = C_1 \cos\omega t + C_2 \sin\omega t + \dfrac{E_0}{B_0} t + C_3 \\ z = C_2 \cos\omega t + \underline{C_1 \sin\omega t} + C_4 \end{cases}$$

$C_1 \sim C_4$.

at $t=0$.

$$\begin{cases} y = ? & z = ? \\ \dot{y} = ? & \dot{z} = ? \end{cases}$$

$t=0$.

$$\begin{cases} y = 0. & z = 0 \\ \dot{y} = u_{y0} & \dot{z} = u_{z0} \end{cases}$$

$$\begin{cases} C_1 = -\dfrac{m u_{z0}}{qB_0} \\ C_2 = -\dfrac{m}{qB_0}\left(\dfrac{E_0}{B_0} - u_{y0}\right) \\ C_3 = -C_1 \\ C_4 = -C_2 \end{cases}$$

2 Fundamental Postulates

differential form	integral form	Comment
$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$	$\boldsymbol{B}$ is solenoidal
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$	$\oint_C \boldsymbol{B} \cdot dl = \mu_0 I$	Ampere's circuital law

where μ_0 is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \quad H/m$.
Conversion of magnetic flux: no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves.
Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \boldsymbol{J} = \frac{\nabla \cdot (\nabla \times \boldsymbol{B})}{\mu_0} = 0$$

which is consistent with the formula

$$\nabla \cdot \boldsymbol{J} = \frac{\partial \rho}{\partial t} = 0$$

$\to 0$

for steady current.

3 Vector Magnetic Potential & Biot-Savart Law

As $\nabla \cdot \boldsymbol{B} = 0$, $\boldsymbol{B}$ is solenoidal, thus could be expressed as:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (T) \tag{1}$$

where $\boldsymbol{A}$ is called the **vector magnetic potential**.
Magnetic flux Φ:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot dl$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \boldsymbol{A} = 0$, we have

$$\begin{aligned}\nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) \\ &= \nabla(\underbrace{\nabla \cdot \boldsymbol{A}}_{\to 0}) - \nabla^2 \boldsymbol{A} \\ &= -\nabla^2 \boldsymbol{A}\end{aligned}$$

$\approx \mu_0 J$

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

The solution is then

$$\boxed{\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv'} \tag{2}$$

For a thin wire with cross-sectional area S, $dv' = Sdl'$, current flow is entirely along the wire, we then have

$$\boldsymbol{J}dv' = JSdl' = Idl'$$

Thus, Eq 2 becomes:

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{dl'}{R}$$

Based on this form and properties of differentiation, we can get **Biot-Savart law**:

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2}$$

The formula for Biot-Savart law could also be written as:

$$\boldsymbol{B} = \oint_{C'} d\boldsymbol{B}$$

and

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'} \times \boldsymbol{a_R}}{R^2}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'} \times \boldsymbol{R}}{R^3}\right)$$

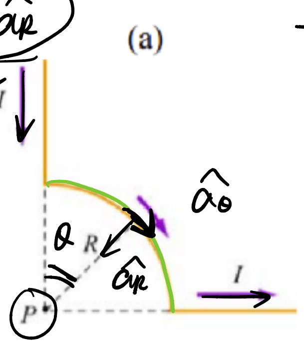
Comment: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine $\boldsymbol{B}$ from I in a circuit if a closed path cannot be found where $\boldsymbol{B}$ has a constant magnitude.

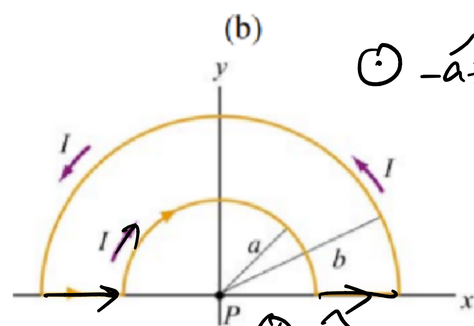
Ex.2

Find the magnetic field at P due to the following current distribution by using Biot-Savart Law.

$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{dl \times \hat{a}_R}{R^2}$

$\hat{a}_z$ ⊗





⊙ $-\hat{a}_z$

⊗ $\hat{a}_z$

⊗

(a)

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{R\,d\theta\,(\hat{a}_\theta \times \hat{a}_R)}{R^2}, \quad \hat{a}_\theta \times \hat{a}_R = \hat{a}_z$$

$$= \frac{\mu_0 I}{4\pi}\frac{d\theta}{R}\hat{a}_z \quad \propto \frac{1}{R}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi R}\int_0^{\pi/2} d\theta \cdot \hat{a}_z$$

$$= \frac{\mu_0 I}{8R}\hat{a}_z$$

(b) ① $R = a$

Inner: $\vec{B}_1 = \frac{\mu_0 I}{4\pi a}\int_0^{\pi} d\theta\ \hat{a}_z = \frac{\mu_0 I}{4a}\hat{a}_z$

Outer: $\vec{B}_2 = \frac{\mu_0 I}{4\pi b}\int_0^{\pi} d\theta\,(-\hat{a}_z) = -\frac{\mu_0 I}{4b}\hat{a}_z$

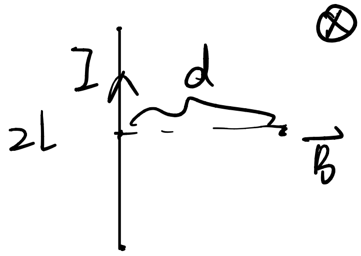
$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{a}_z$$

4 Magnetic Field of Some Common Construction

1. Similarly for magnetic flux density at **the center of a square loop**, with side w carrying a direct current I, is:

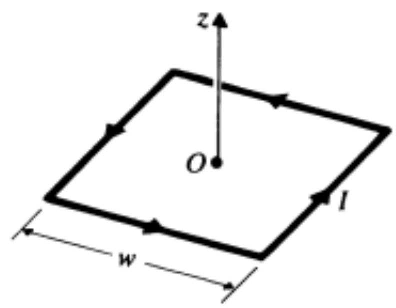
③

$\otimes \ \hat{a_z}$



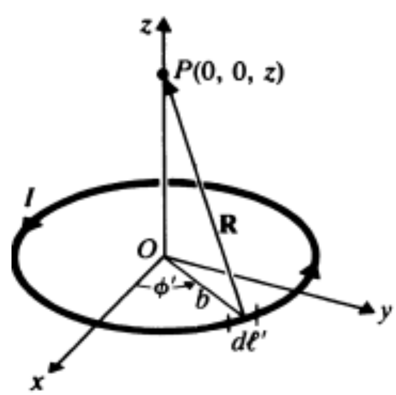
$$\vec{B} = \frac{\mu_0 I L}{2\pi d\sqrt{L^2+r^2}}\hat{a_z}$$

$$\boldsymbol{B} = \boldsymbol{a_z}\frac{2\sqrt{2}\mu_0 I}{\pi w}$$

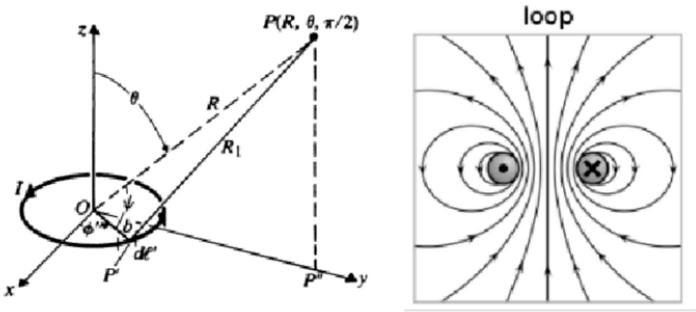


2. For magnetic flux density ata point on the axis of a circular loop of radiu b that carries a direct current I,

$$\boldsymbol{B} = \boldsymbol{a_z}\frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$$



5 Magnetic Dipole



(a) Visualization (b) Cross-section

Definition of the magnetic dipole: We call a small current-carrying loop a magnetic dipole

$$\boldsymbol{m} = I \int d\boldsymbol{S}$$

The direction is determined by the right-hand rule. (along with the current direction)

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{a_R}}{4\pi R^2}$$

In spherical coordinates, the vector potential of a magnetic dipole can be written as

$$\boldsymbol{A}_{dip}(\boldsymbol{R}) = \frac{\mu_0 m \sin\theta}{4\pi R^2} \boldsymbol{a_\phi}$$

Hence, we can compute the magnetic field of a magnetic dipole

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \nabla \times \boldsymbol{A} = \frac{\mu_0 m}{4\pi R^3}(2\cos\theta \boldsymbol{a_r} + \sin\theta \boldsymbol{a_\theta})$$

Written in a coordinate-free form,

$$\boldsymbol{B_{dip}}(\boldsymbol{R}) = \frac{\mu_0}{4\pi R^2}[3(\boldsymbol{m} \cdot \boldsymbol{a_r})\boldsymbol{a_r} - \boldsymbol{m}]$$

Compared with the electric field density of an electric dipole, we can find that we just replace $\frac{1}{\epsilon_0}$ with μ_0, and replace $\boldsymbol{p}$ with $\boldsymbol{m}$.

V_m

6 Scalar magnetic potential

$$E_{dip}(\vec{R}) = \frac{1}{4\pi\varepsilon_0 R^2}(3(\vec{p}\cdot\vec{a_r})\vec{a_r} - \vec{p})$$

If a region is current free, i.e. $\boldsymbol{J} = 0$,

$$\nabla \times \boldsymbol{B} = 0$$

thus $\boldsymbol{B}$ can be expressed as the gradient of a scalar field.

Assume

$$\boldsymbol{B} = -\mu_0 \nabla V_m$$

$\vec{E} = -\nabla V_E$

where V_m is called the **scalar magnetic potential**, the negative sign is conventional, μ_0 is the permeability of free space.

Thus, between two points P_1, P_2,

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \boldsymbol{B} \cdot dl$$

If there were magnetic charges with a volume density ρ_m in a volume V', we could find V_m from:

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R} dv'$$

Notice: This is only a mathematical model, isolated magnetic charges have never been found. Then we could obtain $\boldsymbol{B}$ by Eq 6.

For a bar magnet the fictitous magnetic charges $+q_m, -q_m$ assumed to be separated by d (magnetic dipole), the scalar magnetic potential V_m is given by:

$$V_m = \frac{\boldsymbol{m} \cdot \boldsymbol{a_R}}{4\pi R^2}$$

and it holds at any points with no currents. $(J=0)$

7 Magnetization and Equivalent Current Densities

7.1 Basics

Define **magnetization vector**, $\boldsymbol{M}$, as

$$\boldsymbol{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{m_k}}{\Delta v}$$

which is the volume density of magnetic dipole moment,

1. The effect of magnetization is vector is equivalent to both
   (a) a volume current density:
   $$\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$$
   (b) a surface current density:
   $$\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$$
2. Then we can determine A by:
   $$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a'_n}}{R} ds'$$
3. Then we could obtain $\boldsymbol{B}$ from $\boldsymbol{A}$. $\vec{B} = \nabla \times A \Leftarrow$

7.2 Equivalent Magnetization Charge Densities

In current-free region, a manetized body may be replaced by

1. an equivalent/fictitous magnetization surface charge density

$$\rho_{ms} = \boldsymbol{M} \cdot \boldsymbol{a_n}$$

2. an equivalent/fictitous magnetization volume charge density

$$\rho_m = -\nabla \cdot \boldsymbol{M}$$

Approaches to get $\boldsymbol{B}$

(1) Given $\boldsymbol{I}$ or $\boldsymbol{J}$, applying Biot-Savart Law $\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{\boldsymbol{dl'} \times \boldsymbol{a_R}}{R^2}$ or utilizing scalar potential $\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv'$ and $B = \nabla \times \boldsymbol{A}$ to get $\boldsymbol{B}$

(2) Given magnetism $\boldsymbol{M}$, we can either get $\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$ and $\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$ together with $\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a'_n}}{R} ds'$ to get A and then find B,
① **or** utilizing ① $\rho_{ms} = \boldsymbol{M} \cdot \boldsymbol{a_n}$ and $\rho_m = -\nabla \cdot \boldsymbol{M}$ to get scalar magnetic potential V_m, and then apply $\boldsymbol{B} = -\mu_0 \nabla V_m$ to get $\boldsymbol{B}$.

→ ② V_m → ③ $\vec{B} = -\mu_0 \nabla V_m$

② ① $\vec{M}$ → ② J_m . J_{ms} → ③ $\vec{A}$, $\vec{B}$

③