Q1

(a)

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assume $\vec{A} = (x_a, y_a, z_a)$

$\vec{B} = (x_B, y_B, z_B)$

$\vec{x} = (x, y, z)$

$\Rightarrow$

$p = x_a x + y_a y + z_a z$

$$\vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ x_a & y_a & z_a \\ x & y & z \end{vmatrix} = (y_a z - z_a y,\ z_a x - x_a z,\ x_a y - y_a x)$$

$$= (x_B, y_B, z_B)$$

$$\Rightarrow \begin{cases} y_a z - z_a y = x_B \\ z_a x - x_a z = y_B \\ x_a y - y_a x = z_B \end{cases}$$

$\&\ p = x_a x + y_a y + z_a z$

$\longrightarrow$ obviously enough to determine (x, y, z)

$$\Rightarrow \begin{cases} x = \dfrac{-y_a z_B + y_B z_a + x_a p}{x_a^2 + y_a^2 + z_a^2} \\ y = \dfrac{z_B}{x_a} + \dfrac{y_a}{x_a} \cdot \dfrac{-y_a z_B + y_B z_a + x_a p}{x_a^2 + y_a^2 + z_a^2} \\ z = \dfrac{p}{z_a} - \dfrac{y_a z_B}{x_a z_a} + \dfrac{x_a^2 - y_a}{x_a z_a} \cdot \dfrac{-y_a z_B + y_B z_a + x_a p}{x_a^2 + y_a^2 + z_a^2} \end{cases}$$

Therefore, $\vec{X}$ is known if we have known $x_a, y_a, z_a, x_B, y_B, z_B$ and p.

(b) Q1

In spherical coordinate system:

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial \left[R^2 \frac{\partial V}{\partial R}\right]}{\partial R}$$

$$= \frac{1}{R^2} \frac{\partial}{\partial R} \left[R^2 \frac{\partial}{\partial R}\left(\frac{\rho R^2}{2\varepsilon_0}\right)\right]$$

$$= \frac{1}{R^2} \frac{\partial}{\partial R} \left[R^2 \cdot \frac{\rho R}{\varepsilon_0}\right]$$

$$= \frac{3\rho}{\varepsilon_0}$$

$\Rightarrow \nabla^2 V = -\frac{\rho'}{\varepsilon_0}$ where $\rho' = -3\rho$

$\Updownarrow$

this equation clearly shows that the scalar potential satisf~~ies~~ies Poisson's equation

Q1(c)

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad ①$$

since $V = xyz$ ② in 3D

then $\frac{\partial^2 V}{\partial x^2} = 0$. $\frac{\partial^2 V}{\partial y^2} = 0$. $\frac{\partial^2 V}{\partial z^2} = 0$

consider general form of matric

$$A = \frac{\partial^2 V}{\partial x^2} \; ③ \quad B = \begin{vmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial xy} \\ \frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial y^2} \end{vmatrix} \; ④$$

$$C = \begin{vmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial x \partial z} \\ \frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial z} \\ \frac{\partial^2 V}{\partial x \partial z} & \frac{\partial^2 V}{\partial y \partial z} & \frac{\partial^2 V}{\partial z^2} \end{vmatrix} \; ⑤$$

$$\Rightarrow B = \begin{vmatrix} 0 & z \\ z & 0 \end{vmatrix} = -z^2$$

$$C = \begin{vmatrix} 0 & z & y \\ y & 0 & x \\ z & x & 0 \end{vmatrix} = 2xy^2$$

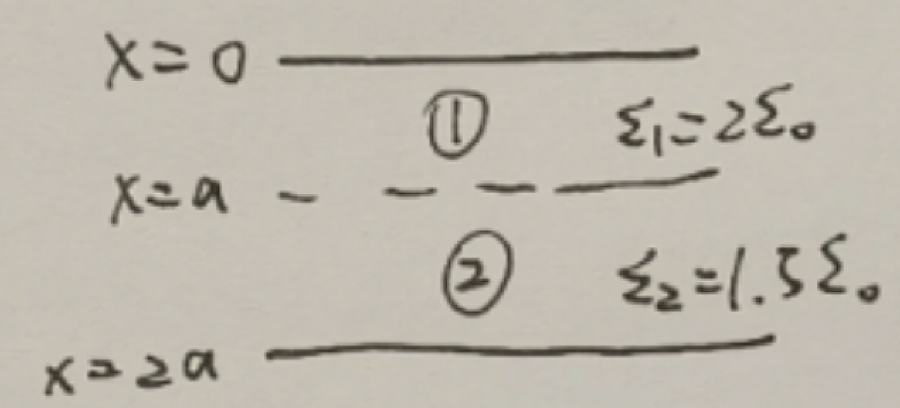
$A = 0$

since $A = 0 \Rightarrow$ saddle point

$\Rightarrow$ no maximum or minimum within the region

Q2

a) $\int \vec{D} \cdot d\vec{s} = \sigma \int ds$



$\Rightarrow \vec{D} = \sigma$

b) $E = \dfrac{D}{\varepsilon}$

$\Rightarrow$ In slab 1, $E_1 = \dfrac{\sigma}{2\varepsilon_0}$

In slab 2, $E_2 = \dfrac{\sigma}{1.5\varepsilon_0} = \dfrac{2\sigma}{3\varepsilon_0}$

c) we know $\vec{D} = \varepsilon_0 \vec{E} + \vec{D}$

In slab 1, we have $\vec{D_1} = \varepsilon_0 \vec{E_1} + \vec{P_1}$

$\Rightarrow \vec{P_1} = \vec{D_1} - \varepsilon_0 \vec{E_1} = (\sigma - \varepsilon_0 \cdot \dfrac{\sigma}{2\varepsilon_0})\hat{n}$

Therefore, we can get $\vec{P_1} = \dfrac{\sigma}{2}\hat{n}$

Similarly, $\vec{P_2} = \vec{D_2} - \varepsilon_0 \vec{E_2} = \dfrac{1}{3}\sigma\hat{n}$

d) $V = -\int_{2a}^{a} E_2\, dx - \int_{a}^{0} E_1\, dx = \dfrac{\sigma a}{\varepsilon_0}\left(\dfrac{1}{2} + \dfrac{1}{1.5}\right)$

$= \dfrac{7}{6}\dfrac{\sigma a}{\varepsilon_0}$

(e)

$\sigma_{bound1} = \vec{P_1} \cdot \hat{n} = -\dfrac{\sigma}{2}$

$\sigma_{bound2} = -\dfrac{\sigma}{3}$

(f)

In slab 1: $E_1 = \dfrac{1}{\varepsilon_0}(\sigma + \sigma_{bound1}) = \dfrac{\sigma}{2\varepsilon_0}$ same as (b)

In slab 2: $E_2 = \dfrac{1}{\varepsilon_0}(\sigma + \sigma_{bound2}) = \dfrac{2\sigma}{3\varepsilon_0}$ same as (b)

Hence Proved

Q3

a) $K_x = 0$, $K_y = 0$

$X(x) = A_0 x + B_0$

$Y(y) = C_0 y + D_0$

Let $B_0 = 0$

$\Rightarrow X(x) = A_0 x$

Let $C_0 = 0$

$\Rightarrow Y(y) = D_0$

For Catesian System:

general solutions: $V(x) = X(x)\,Y(x)$

$\Rightarrow V(x) = A_0 D_0 x \underset{(x=a)}{=} A_0 D_0 a$

Now let $V = V_0$

$\Rightarrow A_0 D_0 = \frac{V_0}{a}$

$\Rightarrow V(x) = \frac{V_0}{a} x$

(b)

We know that $J = \sigma E$

Then if $E = \Delta V$

$\Rightarrow J = -\sigma \Delta V$

$J = -\sigma\, a_x\, \Delta V / \Delta x$

since $V = \frac{V_0}{a} x$

$\Rightarrow J = -\sigma\, a_x \frac{\Delta (\frac{V_0}{a}) x}{\Delta x}$

$\Rightarrow J = -a_x\, \sigma \frac{V_0}{a}$

Q4 (a)

according to Lorentz force :

$$\vec{F} = q \cdot (\vec{v} \times \vec{B})$$

now $\vec{v} = \hat{a}_y u_0$. $\vec{B} = \vec{a}_x B_0$

$$\Rightarrow \vec{F} = q(\hat{a}_y u_0 \times \vec{a}_x B_0)$$

$$= q\, \hat{a}_z\, u_0 B_0$$

since $\vec{F}$ is perpendicular to the velocity of the particle, It will cause the particle to move in a circular path in the 'z' direction

radius : $r = \dfrac{m u_0}{q B_0}$ (since $\dfrac{mv^2}{r} = Bvq$)

Q4

(b)

(i) $E = a_z \Sigma_0$ and $B = a_x B_0$

Magnetic force $\Rightarrow M = Bev = m\frac{v^2}{r}$

$\Rightarrow v = \frac{Ber}{m}$ < electrons will rotate magnetic field at right angle to the field

electronic force $F = ev = ma$

$\Rightarrow F = \frac{m^2 s}{t^2}$. $x = vt$

$\frac{eV}{d} = 2my\, v^2/x^2$

$\Rightarrow$ electron path : $y = x^2 \cdot [\frac{eV}{2dmV}]$

$\Leftrightarrow$ electron moves in parabola in electric field at right angle

(ii) same electrons will move in ~~same~~ same velocity but opposite direction

in $E = -a_z \Sigma_0$ and $B = -a_z B_0$

Q5

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{J}{R}\, dv'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{V'} \frac{J \times \hat{a}_R}{R^2}\, dv'$$

need to prove

Assume $\vec{R} = \vec{r} - \vec{r}'$

$$A(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\, dv' \quad ①$$

since $B(\vec{r}) = \nabla \times A(\vec{r})$ ②

$$\Rightarrow B(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \nabla \times \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\, dv' \quad ③$$

we know that $\nabla \times (f(\vec{r})\, G(\vec{r})) = f(\vec{r})\, \nabla \times G(\vec{r}) - G(\vec{r})\, \nabla \times f(\vec{r})$

Here $f(\vec{r}) = \frac{1}{\vec{r}-\vec{r}'}$, $G(\vec{r}) = J(\vec{r}')$

$$\Rightarrow \nabla \times \left(\frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = \frac{1}{|\vec{r}-\vec{r}'|} \cdot \nabla \times J(\vec{r}') - J(\vec{r}') \times \nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \quad ④$$

We can get that $\nabla \times J(\vec{r}') = 0$ ⑤

$$\Rightarrow \nabla \times \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2} \quad ⑥$$

From ④ and ⑥:

$$\nabla \times \left(\frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = \frac{J(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad ⑦ \quad = \frac{J(\vec{r}') \times \hat{a}_R}{|\vec{r}-\vec{r}'|^2}$$

Putting ⑦ back to ③

$$\Rightarrow B(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|^2}$$

$$\|$$

$$\frac{\mu_0}{4\pi}\int_{V'} \frac{J \times \hat{a}_R}{R^2}\, dv'$$

Hence Proved.

Q6

(a) $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

ensure the conservation of fluid mass along a stream line. stating that the mass entering a ~~se~~ certain region is equal to the mass exiting that region.

(b) $\sum_i I_i = 0$

The sum of currents entering a junction equals to the sum of currents leaving it, ensuring charge conservation and establishing the principle of current balance in electrical circuits.

(c)

We equate the mass flow rate in fluid dynamic to electric current flow in an electrical circuits. This equation of continuity becomes $A_1 I_1 = A_2 I_2$, where A_1 and A_2 are areas, and I_1 and I_2 are currents. Kirchhoff's current law states that the sum of currents entering a junction equals to the sum of currents leaving it. No extra assumptions are ~~need~~ needed.