RC5

1 Separation of Variables

1.1 Boundary Value problem in Cartesian Coordinates

We have Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If we assume $V(x, y, z) = X(x)Y(y)Z(z)$, then we have

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

Then we know that

$$\frac{d^2X(x)}{dx^2} + k_x^2X(x) = 0, \quad \frac{d^2Y(y)}{dy^2} + k_y^2Y(y) = 0, \quad \frac{d^2Z(z)}{dz^2} + k_z^2Z(z) = 0$$

where k_x^2, k_y^2 and k_z^2 is constant and

$$k_x^2 + k_y^2 + k_z^2 = 0$$

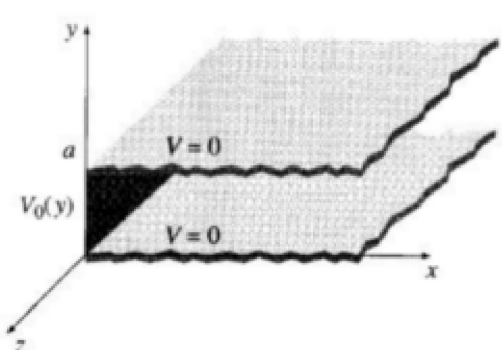
Possible Solutions of $X''(x) + k_x^2X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1e^{jkx} + D_1e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2e^{kx} + D_2e^{-kx}$

Where k is real. And constant A and B should be determined by boundary conditions.

1.1.1 Example

Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated form the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."



Since the configuration is independent of z, the Laplace's equation can be simplified as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The boundary conditions are

(i) $V = 0$ when $y = 0$

(ii) $V = 0$ when $y = a$

(iii) $V = V_0(y)$ when $x = 0$

(iv) $V \to 0$ as $x \to \infty$

$$V(x, y) = X(x)Y(y)$$

Substitute it into the 2-dimensional Laplace's equation,

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Then,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Hence the first term only depends on x and the second term only depends on y. The only way the above equation can be true is by requiring

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0$$

One of these constants is positive, the other negative (or perhaps both are zero). In general, one must investigate all the possibilities; however, in our particular problem we need C_1 positive and C_2 negative, for reasons that will appear in a moment. Thus

$$\frac{d^2X}{dx^2} = k^2X, \quad \frac{d^2Y}{dy^2} = -k^2Y$$

We then have

$$X(x) = Ae^{kx} + Be^{-kx}, \quad Y(y) = C\sin ky + D\cos ky$$

Therefore

$$V(x, y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky)$$

This is the appropriate separable solution to Laplace's equation; it remains to impose the boundary conditions, and see what they tell us about the constants. To begin at the end, condition (iv) requires that A equal zero. Absorbing B into C and D, we are left with

$$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$

Condition (i) demands that $D = 0$. Therefore,

$$V(x,y) = Ce^{-kx}\sin ky$$

Meanwhile (ii) yields $sin(ka) = 0$, from which it follows that

$$k = \frac{n\pi}{a}, \quad (n = 1, 2, 3, \ldots)$$

(At this point you can see why I chose C_1 positive and C_2 negative: If X were sinusoidal, we could never arrange for it to go to zero at infinity, and if Y were exponential we could not make it vanish at both 0 and a. Incidentally, $n = 0$ is no good, for in that case the potential vanishes everywhere. And we have already excluded negative n's.) We cannot fit boundary condition (iii) for arbitrary $V_0(y)$. What to do next? Since Laplace's equation is linear,

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a)$$

Then we can fit (iii) by requiring

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

Recall that those eigen-functions are orthogonal to each other. We use the Fourier's trick.

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \int_0^a V_0(y)\sin(n'\pi y/a)\,dy$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2}, & \text{if } n' = n. \end{cases}$$

$$\text{Then, } C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

1.2 Boundary Value problem in Spherical Coordinates

We have Laplace's equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

We assume that the solution is independent of ϕ, then we have

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Assume $V(r,\theta) = R(r)\Theta(\theta)$

Putting this into Eq. (40), and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1)$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a partial differential equation into ordinary differential equations.

The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)R$$

has the general solution

$$R(r)=Ar^l+\frac{B}{r^{l+1}}$$

A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation by boundary conditions.

The angular equation is:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1)\sin\theta\Theta$$

Its solutions are **Legendre polynomials** in the variable $cos\theta$:

$$\Theta(\theta)=P_l(\cos\theta)$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula**:

$$P_l(x)\equiv\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l$$

The first few Legendre polynomials are listed below:

$$\begin{aligned}
&P_0(x)=1\\
&P_1(x)=x\\
&P_2(x)=\left(3x^2-1\right)/2\\
&P_3(x)=\left(5x^3-3x\right)/2\\
&P_4(x)=\left(35x^4-30x^2+3\right)/8\\
&P_5(x)=\left(63x^5-70x^3+15x\right)/8
\end{aligned}$$

In the case of azimuthal symmetry, then, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r,\theta)=\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

As before, separation of variables yields an infinite set of solutions, one for each l. The general solution is the linear combination of separable solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

1.2.1 Example

An uncharged grounded conducting sphere of radius b is placed in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_z E_0$. Determine the potential distribution $V(R,\theta)$ outside the sphere.
To determine the potential distribution $V(R,\theta)$ for $R \geq b$, we note the following boundary conditions:

$$\begin{aligned} &V(b,\theta) = 0 \\ &V(R,\theta) = -E_0 z = -E_0 R\cos\theta, \quad \text{for } R \gg b. \end{aligned}$$

And the interpretation of the second boundary condition is that the original E_0 is not disturbed at points very far away from the sphere. And we assume the general form of $V(R,\theta)$ is

$$V(R,\theta) = \sum_{n=0}^{\infty} \left[A_n R^n + B_n R^{-(n+1)} \right] P_n(\cos\theta), \quad R \geq b.$$

However, in view of second boundary condition at $R \gg b$, all A_n except A_1 must vanish, and $A_1 = -E_0$. We have

$$\begin{aligned} V(R,\theta) &= -E_0 R P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta) \\ &= B_0 R^{-1} + \left(B_1 R^{-2} - E_0 R \right)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \quad R \geq b. \end{aligned}$$

Now applying the first boundary condition at $R = b$, we require for arbitrary θ

$$0 = \frac{B_0}{b} + \left(\frac{B_1}{b^2} - E_0 b \right)\cos\theta + \sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos\theta),$$

from which we obtain

$$B_1 = E_0 b^3, \quad B_n = 0 \text{ for } n \geq 2 \text{ or } n = 0$$

We have, finally

$$V(R,\theta) = -E_0 \left[1 - \left(\frac{b}{R} \right)^3 \right] R\cos\theta, \quad R \geq b$$

1.3 Boundary Value problem in Cylindrical Coordinates

Laplace's equation in Cylindrical Coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Assuming V has no z dependence,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Assume

$$V(r,\phi) = R(r)\Phi(\phi)$$

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr} \right] + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

Therefore,

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = k^2$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0$$

has solution

$$R(r) = A_r r^n + B_r r^{-n}$$

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi$$

Therefore,

$$V_n(r,\phi) = r^n\left(A_n \sin n\phi + B_n \cos n\phi\right) + r^{-n}\left(A_n' \sin n\phi + B_n' \cos n\phi\right), \quad n \neq 0$$

In the special case where $k = 0$,

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

$$\Phi(\phi) = A_0\phi + B_0, \quad k = 0$$

and $A_0 = 0$ if there is no circumferential variations. Meanwhile,

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = 0$$

$$R(r) = C_0 \ln r + D_0, \quad k = 0$$

Therefore,

$$V(r) = C_1 \ln r + C_2$$

Thus, the general solution is

$$V(r,\phi) = a_0 + b_0 \ln r + \sum_{k=1}^{\infty}\left[r^k\left(a_k \cos k\phi + b_k \sin k\phi\right) + r^{-k}\left(c_k \cos k\phi + d_k \sin k\phi\right)\right]$$

2 Steady Electric Currents

2.1 Current Density and Ohm's Law

$$I = \int_S J \cdot ds \quad (A)$$

where J is the volume current density or current density, defined by

$$J = Nqu \quad (A/m^2)$$

,

where N is the number of charge carriers per unit volume, each of charges q moves with a velocity u.

Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$J = \rho u \quad (A/m^2)$$

For conduction currents,

$$J = \sigma E \quad (A/m^2)$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $u = -\mu_e E \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.

Materials where $J = \sigma E \quad (A/m^2)$ holds are called ohmic media. The form can be referred as the point form of Ohm's law.

Thus, the resistance is defined as

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

where l is the length of the homogeneous conductor, S is the area of the uniform cross section. The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

2.2 Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma} J \cdot dl = 0$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

2.3 Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

,

where ρ is the volume charge density.

For steady currents, as $\partial\rho/\partial t = 0$, $\nabla \cdot J = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.

Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$

2.4 Power Dissipation and Joule's Law

For a given volume V that the total electric power converted to heat is:

$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv$$

$$P = \int_L E d\ell \int_S J ds = VI = I^2 R$$

2.5 Boundary Conditions

2.5.1 Governing Equations for Steady Current Density

- Differential form:

$$\nabla \cdot \mathbf{J} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

- Integral form:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$$

$$\oint_C \frac{1}{\sigma}\mathbf{J} \cdot d\ell = 0$$

2.5.2 Boundary Conditions

- Normal Component:

$$J_{1n} = J_{2n}$$

- Tangential Component:

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Combining with boundary conditions of electric field:

$$\begin{aligned} J_{1n} = J_{2n} \quad &\rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \\ D_{1n} - D_{2n} = \rho_s \quad &\rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \end{aligned}$$

Surface charge density on the interface:

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}$$

If medium 2 is a much better conductor than medium 1:

$$\rho_s = \epsilon_1 E_{1n} = D_{1n}$$

2.6 Exercise

- (HW5-1) Lightning strikes a lossy dielectric sphere—$\epsilon = 1.2\epsilon_0$, $\sigma = 10$ (S/m)—of radius 0.1 (m) at time $t = 0$, depositing uniformly in the sphere a total charge 1 (mC).

  a) Calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value.

  b) Calculate the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its value. What happens to this energy?

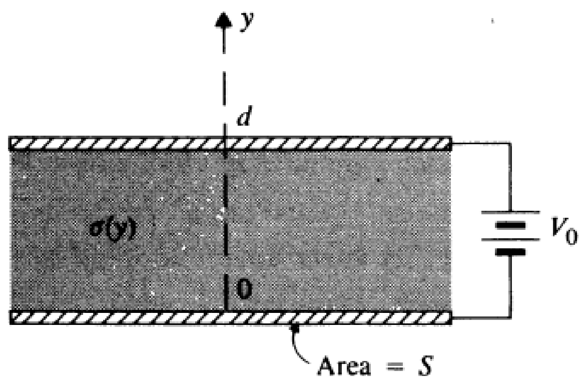
  c) Determine the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

- (HW5-2) The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as shown in the figure. Determine

  a) the total resistance between the plates.

  b) the surface charge densities on the pates.

  c) the volume charge density and the total amount of charge between the plates.



3 References

1. Naihao Deng, SU2020 VE230 RC5

2. Pingchuan Ma, FA2022 ECE2300J RC4