Mid2 Part1

1 Steady Electric Currents

1.1 Current Density and Ohm's Law

J

$$I = \int_S J \cdot ds \quad (A)$$

where J is the volume current density or current density, defined by

$$J = Nqu \quad (A/m^2)$$

,

where N is the number of charge carriers per unit volume, each of charges q moves with a velocity u.

Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$J = \rho u \quad (A/m^2)$$

For conduction currents,

$$J = \sigma E \quad (A/m^2)$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $u = -\mu_e E \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.

Materials where $J = \sigma E \quad (A/m^2)$ holds are called ohmic media. The form can be referred as the point form of Ohm's law.

Thus, the resistance is defined as

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

$\rho \frac{l}{S}$ $\rho = \frac{1}{\sigma}$

resistivity conductivity.

where l is the length of the homogeneous conductor, S is the area of the uniform cross section. The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

1.2 Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma} J \cdot dl = 0$$

$\frac{J}{\sigma}$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

1.3 Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

⇒ conservation of charge.

,

where ρ is the volume charge density.

For steady currents, as $\partial \rho / \partial t = 0$, $\nabla \cdot J = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.

Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$

permittivity

conductivity.

1.4 Power Dissipation and Joule's Law

For a given volume V that the total electric power converted to heat is:

$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv$$

$$P = \int_L E d\ell \int_S J ds = VI = I^2 R$$

1.5 Boundary Conditions

1.5.1 Governing Equations for Steady Current Density

- Differential form:

$$\nabla \cdot \mathbf{J} = 0$$
$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

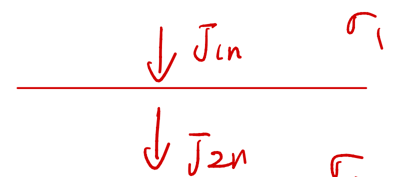
$\vec{J} = \sigma \vec{E}$.

$\nabla \times \vec{E} = 0$.

- Integral form:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$$
$$\oint_C \frac{1}{\sigma}\mathbf{J} \cdot d\ell = 0$$

1.5.2 Boundary Conditions



- Normal Component:

$$J_{1n} = J_{2n}$$

- Tangential Component:

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

$E_{1t} = E_{2t}$.

$E_{1t} = \frac{J_{1t}}{\sigma_1}$, $E_{2t} = \frac{J_{2t}}{\sigma_2}$.

Combining with boundary conditions of electric field:

$$J_{1n} = J_{2n} \quad \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$$
$$D_{1n} - D_{2n} = \rho_s \quad \rightarrow \cancel{\epsilon_1 E_{1n} = \epsilon_2 E_{2n}}$$

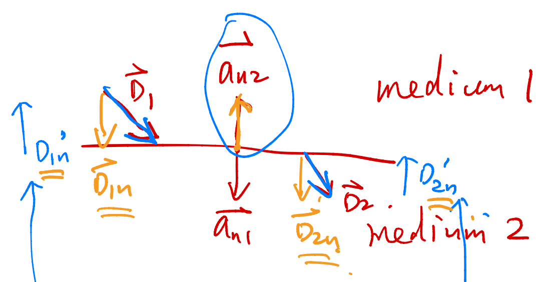
$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$.

Surface charge density on the interface:

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}$$

If medium 2 is a much better conductor than medium 1:

$$\rho_s = \epsilon_1 E_{1n} = D_{1n}$$



$\vec{a}_{n2} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$.

$D_{1n} - D_{2n} = \rho_s$.

$D_{2n} - D_{1n} = \rho_s$

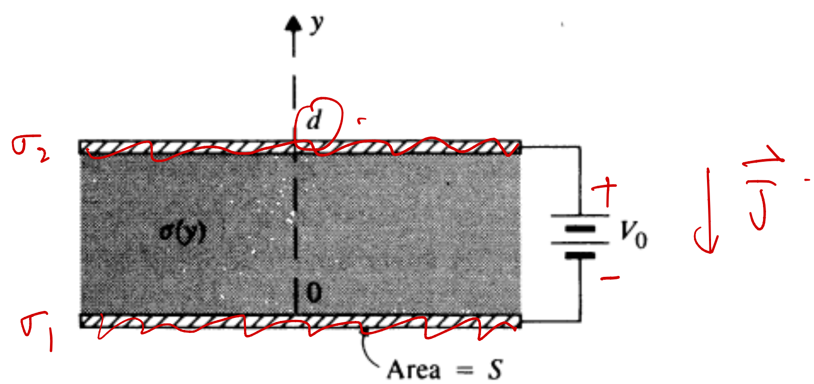
1.6 Example

- (HW5-2) The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as shown in the figure. Determine

  a) the total resistance between the plates.

  b) the surface charge densities on the pates.

  c) the volume charge density and the total amount of charge between the plates.



a) $\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$

$\vec{J} = -\vec{a}_y J_0 \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = -\vec{a}_y \frac{J_0}{\sigma(y)}$

$V_0 = -\int_0^d \vec{E}\cdot\vec{a}_y\, dy = \int_0^d \frac{J_0\, dy}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}} = \frac{J_0 d}{\sigma_2 - \sigma_1}\ln\frac{\sigma_2}{\sigma_1}$

$R = \frac{V_0}{\underbrace{I}_{J_0 S}} = \frac{d}{(\sigma_2-\sigma_1)S}\ln\frac{\sigma_2}{\sigma_1}$

c) $\rho = \nabla\cdot\vec{D} = \frac{d}{dy}(\varepsilon_0 \vec{E})$

$= -\varepsilon_0 J_0 \frac{d}{dy}\left(\frac{1}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}}\right)$

$= \varepsilon_0 J_0 \frac{(\sigma_2-\sigma_1)/d}{(\sigma_1+(\sigma_2-\sigma_1)\frac{y}{d})^2}$

b) upper plate: $\rho_s = \varepsilon_0 E_y(d) = \frac{\varepsilon_0 J_0}{\sigma_2} = \frac{\varepsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_2 d \ln(\sigma_2/\sigma_1)}$.

Lower plate: $\rho_s = -\varepsilon_0 E_y(0) = -\frac{\varepsilon_0 J_0}{\sigma_1} = -\frac{\varepsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_1 d \ln(\sigma_2/\sigma_1)}$

$Q = -$ ___

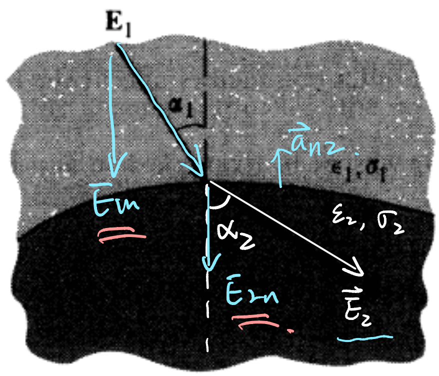
1.7 Quiz 3 Recap

- Two lossy dielectric media with permittivities and conductivities (ϵ_1, σ_1) and (ϵ_2, σ_2) are in contact. An electric field with a magnitude E_1 is incident from medium 1 upon the interface at an angle α_1 measured from the common normal, as shown in the figure.

  a) Find the magnitude and direction of $\mathbf{E}_2$ in medium 2.

  b) Find the surface charge density at the interface.

  c) Compare the results in parts (a) and (b) with the case in which both media are perfect dielectrics.



b.c.

a) $E_{1t} = E_{2t} \Rightarrow E_1 \sin\alpha_1 = E_2 \sin\alpha_2$. ①

$J_{1n} = J_{2n} \Rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \Rightarrow \sigma_1 E_1 \cos\alpha_1 = \sigma_2 E_2 \cos\alpha_2$ ②.

$|E_2| = |E_1| \sqrt{\sin^2\alpha_1 + \left(\frac{\sigma_1}{\sigma_2}\cos\alpha_1\right)^2}$. $\tan\alpha_2 = \frac{\sigma_2}{\sigma_1}\tan\alpha_1 \Rightarrow \alpha_2 = \arctan\left(\frac{\sigma_2}{\sigma_1}\tan\alpha_1\right)$.

b). $D_{2n} - D_{1n} = \rho_s \Rightarrow \epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s$.

$\rho_s = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right) E_1 \cos\alpha_1$

c) $\boxed{\sigma_1 = \sigma_2 = 0}$ from (b) $\rho_s = 0$

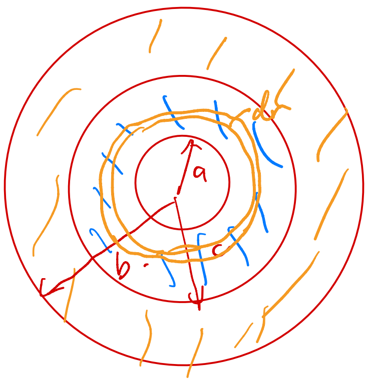
from (a). $\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\epsilon_2}{\epsilon_1}$, $E_2 = E_1 \sqrt{\sin^2\alpha_1 + \left(\frac{\epsilon_1}{\epsilon_2}\cos\alpha_1\right)^2}$.

- A d-c voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ϵ_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ϵ_2 and conductivity σ_2 in the region $c < r < b$. Determine

  a) the current density in each region,

  b) the surface charge densities on the inner and outer conductors and at the interface between the two dielectrics.

$$R = \frac{l}{\sigma S}.$$



a) $R_1 = \int_a^c \frac{1}{\sigma} \frac{dr}{2\pi r L} = \frac{1}{2\pi\sigma_1 L} \ln\frac{c}{a}.$

$$R_2 = \frac{1}{2\pi\sigma_2 L} \ln\frac{b}{c}.$$

$$I = \frac{V_0}{R_1 + R_2} = \frac{2\pi\sigma_1\sigma_2 L V_0}{\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)}.$$

$$J_1 = J_2 = \frac{I}{2\pi r l} = \frac{\sigma_1\sigma_2 V_0}{r(\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a))}.$$

b). $\rho_{sa} = \varepsilon_1 E_1 \Big|_{r=a} = \text{-- — ——}$

$\rho_{sb} = -\varepsilon_2 E_2 \Big|_{r=b} = \text{— ——}$

$\rho_{sc} = -(\varepsilon_1 E_1 - \varepsilon_2 E_2) \Big|_{r=c} = \text{— — ——— —}$