ECE 2300
Recitation Class 5

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Pre-class

- Make-up Lecture on Sunday!
  - We are not meeting in person on Thursday (June 22nd)
  - Make-up Lecture arranged on Sunday original time

- Quiz this week!
  - After Sunday lecture (8:00 pm – 8:40 pm)
  - Same format as last quiz. Online student need to turn on at least one camara.
  - If you want to take online quiz, notify us beforehand!

5.1 Boundary Value Problem

- Laplace's Equation: ∇^2: laplace operator $= \nabla \cdot \nabla$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon} \xrightarrow[\text{simple medium}]{\text{no free charge}} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$\Downarrow$

in cartesian coordinate

$\Downarrow$ assume $V(x) = X(x)\,Y(y)\,Z(z)$

$$Y(y)Z(z)\frac{d^2X}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0$$

$\Downarrow$ $\frac{1}{V(x)} = \frac{1}{X(x)Y(y)Z(z)}$

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{f(y)} + \underbrace{\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}}_{f(z)} = 0$$

$f(x) + f(y) + f(z) = 0$

true for x, y, z.

$\Downarrow$

$f(x), f(y), f(z) \Rightarrow$ constant.

5.1 Boundary Value Problem

Cont.

$$\Rightarrow \frac{df(x)}{dx} = \frac{df(y)}{dy} = \frac{df(z)}{dz} = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right] = 0$$

$\Downarrow$ integrate both sides.

$f(x)$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = C \Rightarrow \text{denote } C = -k_x^2$$

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0$$

$f(x) + f(y) + f(z) = 0.$

$-k_x^2 - k_y^2 - k_z^2 = 0$

$k_x^2 + k_y^2 + k_z^2 = 0$

5.1 Boundary Value Problem

Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

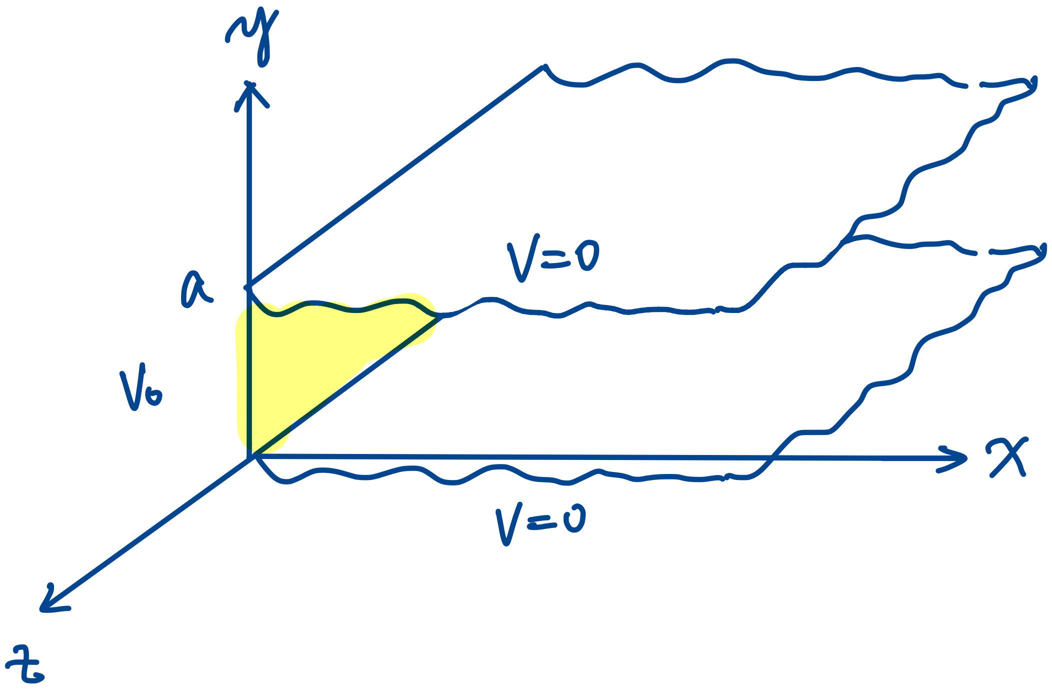
k_x^2	k_x	$X(x)$	Exponential forms[†] of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

k is real number

This is why taking square form.

A B C D will be determined by the boundary conditions.

Ex.1 Boundary Condition



two semi-infinite plate

find the inner voltage distribution

For y:

(i) V=0 when y=0 . $V(x,0)=0$

(ii) V=0 when y=a, $V(x,a)=0$

For x:

(iii) $V=V_0$ when x=0 $V(0,y)=V_0$

(iv) $V\to 0$ when $x\to\infty$ $V(\infty,y)\to 0$

* Voltage distribution independent of z

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial^2 y}=0$$

Ex.1 Boundary Condition

$$\left.\begin{array}{l} Y\dfrac{d^2X}{dx^2}+X\dfrac{d^2Y}{dy^2}=0 \\ \Downarrow \\ \underbrace{\dfrac{1}{X}\dfrac{d^2X}{dx^2}}_{C_1}+\underbrace{\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}}_{C_2}=0 \end{array}\right\} \quad C_1+C_2=0 \quad \begin{array}{ll} \Downarrow & \Downarrow \\ K^2 & -K^2 \end{array}$$

$$\begin{cases} \dfrac{d^2X}{dx^2}-C_1X=0 \\ \dfrac{d^2Y}{dy^2}-C_2X=0 \end{cases} \Rightarrow \begin{cases} \dfrac{d^2X}{dx^2}=K^2X \\ \dfrac{d^2Y}{dy^2}=-K^2Y \end{cases}$$

$$\Rightarrow \begin{cases} X(x)=A\cdot e^{Kx}+B\cdot e^{-Kx} \\ Y(y)=C\cdot \sin ky+D\cdot \cos ky. \end{cases}$$

$$\Rightarrow V(x,y)=(Ae^{Kx}+Be^{-Kx})(C\cdot \sin ky+D\cdot \cos ky)$$

cond. (iv), $A=0$ cond.(i) $D=0$

$$\Rightarrow V(x,y)=B\cdot e^{-Kx}\cdot C\cdot \sin ky$$

$$=C\cdot e^{-Kx}\cdot \sin ky.$$

Ex.1 Boundary Condition

cond (ii)

$\Rightarrow V(x,a) = C_n \cdot e^{-kx} \sin ka = 0$

true for all x.

$\Rightarrow \sin kb = 0 \quad \Rightarrow k = \frac{n\pi}{a} \quad n = 1, 2, 3 \ldots$

$V_n(x,y) = C \cdot e^{-kx} \sin \frac{n\pi}{a} \cdot y$

$V(x,y) = \sum_{n=1}^{\infty} C_n \cdot e^{-kx} \sin \frac{n\pi}{a} \cdot y.$

$V(0,y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{a} y = V_0$

Using Fourier Transformation.

$\sum_{n=1}^{\infty} C_n \int_0^a \sin(\frac{n\pi}{a} y) \cdot \sin(n' \frac{\pi}{a} y) dy$

$= \int_0^a V_0 \sin(\frac{n'\pi}{a} y) dy$

$\Rightarrow C_n = \frac{2}{a} \int_0^a V_0 \sin(\frac{n\pi}{a} y) dy$

Ex.1 Boundary Condition

Steps for solving similar problems:

(i) boundary conditions.

(ii) write out the Laplace equation form

(iii) Use possible solution of $X''(x) + k_x X(x) = 0$

to find solution's general form.

(iv) Use the boundary conditions to find constants.

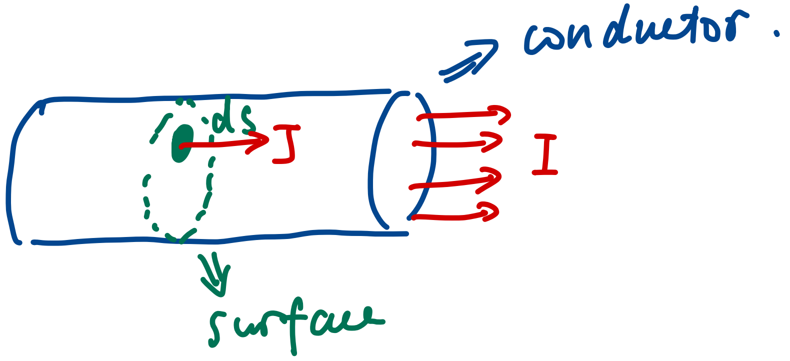
(v) Calculate accordingly.

5.2.1 Steady Electric Currents

- Current Density: J A/m^2

The amount of current flows through a unit surface.

$$I = \int_S J \cdot ds$$



$$J = N \cdot q \cdot u = \rho \cdot u$$

charges

free charge density. $\rho = N \cdot q$.

velocity.

Number of charge carrier

5.2.2 Steady Electric Currents

- Ohm's Law:

$J = \sigma \cdot E \quad (A/m^2)$

↓

conductivity

$\sigma = \rho_e \mu_e$

charge density of electron

$\rho_e = -Ne$

$\mu_e = -\frac{u}{E}$

electron mobility.

⇒ Materials have $J = \sigma E$ characteristic

Ohmic Material.

$$\begin{cases} R = \dfrac{l}{\sigma S} \\ G = \dfrac{1}{R} = \dfrac{\sigma S}{l} \end{cases}$$

→ length

→ cross section

↓ conductance

5.2.3 Kirchhoff's Laws

- Voltage Law:

Along the circuit, emf change (voltage rises) equals voltage drop across resistance.

$$\sum_j V_j = \sum_k R_k \cdot I_k$$

$\Downarrow$ algebraic sum of emf.

$\Downarrow$ algebraic sum on resistor pot.

Reason behind it: conservativeness of electric field.

$$\oint_C E \cdot dl = 0 \Rightarrow \oint_C \frac{J}{\sigma} \cdot dl = 0$$

5.2.3 Kirchhoff's Laws

- Current Law:

Algebraic Sum of all currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

Reasons behind it: No charge/electrons were generated within each junction

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \xrightarrow[\text{current}]{\text{steady}} \nabla \cdot J = 0$$

5.2.4 Joule's Law

- This is a Law about power and heat generated on (a) resistor(s)

$$p = \int_V E \cdot J \, dv$$

$$p = \int_L E \, dl \int_S J \cdot ds = V \cdot I = I^2 R .$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$V \qquad\qquad I$$

5.2.5 Boundary conditions

- Steady current density:
  - Differential Form:

$$\nabla \cdot J = 0 \longrightarrow \text{Kirchoff's current law.}$$

$$\nabla \times \left(\frac{J}{\sigma}\right) = 0$$

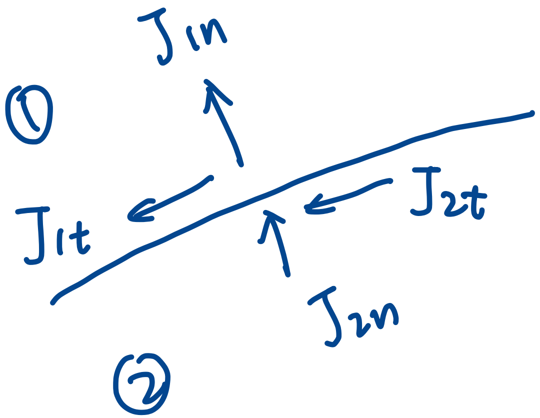
  - Integral Form:

$$\oint_S J \cdot ds = 0$$

$$\oint_S \frac{1}{\sigma} J \cdot dl = 0 \longrightarrow \text{Kirchoff's voltage law.}$$

5.2.5 Boundary conditions

- At the connecting surface of different conductors:
  - Normal: $J_{1n} = J_{2n}$
  - Tangential: $\dfrac{J_{1t}}{J_{2t}} = \dfrac{\sigma_1}{\sigma_2}$



would be verified by checking voltage.

$$\begin{cases} J_{1n} = J_{2n} \Rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \\ D_{1n} - D_{2n} = \rho_s \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \end{cases}$$

Surface Charge:

$$\rho_s = (\varepsilon_1 \frac{\sigma_2}{\sigma_1} - \varepsilon_2) E_{2n}$$
$$= (\varepsilon_1 - \varepsilon_2 \frac{\sigma_1}{\sigma_2}) E_{1n}$$

$$\rho_s = \varepsilon_1 E_{1n} = D_{1n}.$$

Thank You

Credit to Deng Naihao for this slides & information