Mid 2 RC Part 1: Static Electrics (Chap4)

1 Poisson's Equation and Laplace's Equation

Gauss's law.

1.1 Poisson's Equation:

$\nabla^2 = \nabla \cdot \nabla \Rightarrow \nabla^2 V = \nabla \cdot \underbrace{\nabla V}_{-E} = -\nabla \cdot E = -\frac{\rho}{\varepsilon}$

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

- In Cartesian System:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- In Cylindrical System:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial^2 \varphi} + \frac{\partial^2 V}{\partial z^2}$$

- In Spherical System:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$$

1.2 Laplace's Equation

For a simple medium with no free charge: $\rho = 0 \Rightarrow$ poisson's eq.

$$\nabla^2 V = 0$$

For problem involving conductors:

- use Laplace's Equation to obtain electric potential V.
- Use $E = -\nabla V$ to work out E.
- Use $\rho_s = \epsilon E$ to get charge density on the conductor surface.

Gauss's law.

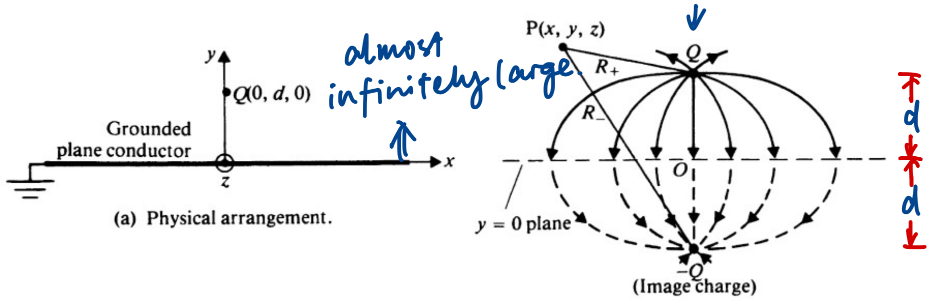
1.3 Uniqueness of Electrostatic Solutions

A solution of Poisson's Equation or Laplace's Equation that satisfies the given boundary conditions is a unique solution.

Fundamental reason why method of images is true.

2 Method of Images

- Point charge and conducting planes

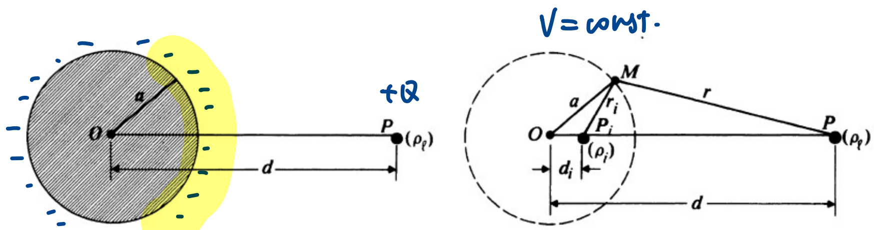


(a) Physical arrangement.

(Image charge)

(b) Image charge and field lines.

- Line Charge and Parallel Conducting Cylinder



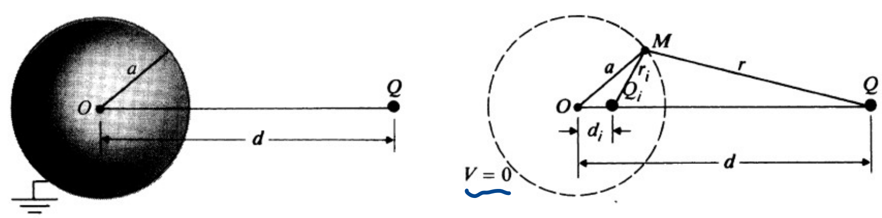
(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{const}$$

P_i : inverse point of P.

- Point Charge and a Conducting Sphere



(a) Point charge and grounded conducting sphere.

(b) Point charge and its image.

$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0$$

$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{const}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d}$$

$$\Rightarrow \quad Q_i = -\frac{a}{d} Q \qquad d_i = \frac{a^2}{d}.$$

- Charge Sphere and Grounded Plane

$$Q_1 = \left(\frac{a}{2c}\right) Q_0 = \alpha Q_0$$

$$Q_2 = \frac{a}{2c - \left(\frac{a^2}{2c}\right)} Q_1 = \frac{\alpha^2}{1-\alpha^2} Q_0$$

$$Q_3 = \frac{a}{2c - \frac{a^2}{\left(2c - \frac{a^2}{2c}\right)}} Q_2 = \frac{\alpha^3}{(1-\alpha^2)\left(1 - \frac{\alpha^3}{1-\alpha^3}\right)} Q_0$$

$\vdots$

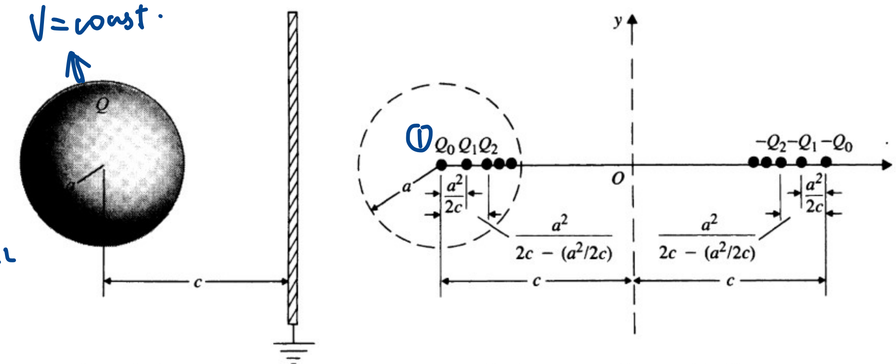
① at the centre Q_0 $\Rightarrow$ generate $-Q_0$

② $-Q_0$ on the other side $\Rightarrow$ generate Q_1

③ $Q_1 \Rightarrow -Q_1$

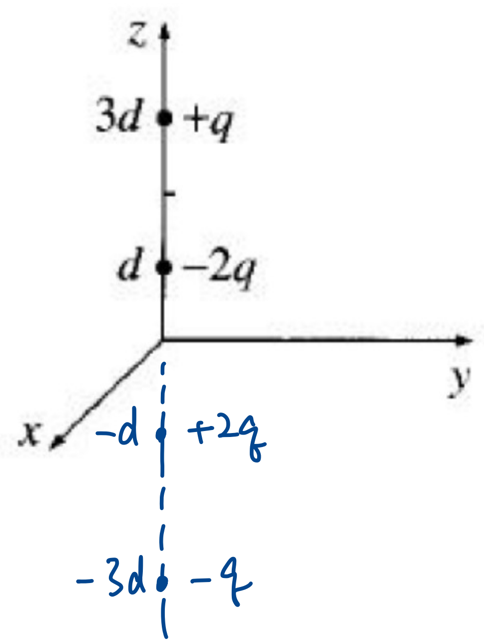
④ $-Q_1 \Rightarrow Q_2$

$\vdots$



(a) Physical arrangement. (b) Two groups of image point charges.

Example Find the force on the charge +q in the figure below. Note that the xy plane is grounded conductor plane.



$$F = F_{(-2q)} + F_{(2q)} + F_{(-q)}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right)$$

$\vdots$

3 Boundary Value problem in Cartesian Coordinates

3.1 Boundary Condition Problem

- In order to find specific voltage on conductor systems without isolated free charge.
- **General idea:** Use boundary condition to find coefficients for general solution form from Laplace equation.
- **Types of boundary condition:** (1) Dirichlet: V is specific; (2) Neumann: $\frac{dV}{dn}$ is specified on boundaries (3) Mixed: V is specific on some boundaries; $\frac{dV}{dn}$ is specified on some boundaries.
- **Solution Form:** Separation of variables, which means $V(x, y, z) = X(x)Y(y)Z(z)$. When the potential or normal derivative is specified, and it coincide with coordinate surfaces of an orthogonal, curvilinear coordinate system.

3.2 Boundary condition value in Cartesian Coordinate

(1) Laplace's Equation for V in Cartesian coordinates is

$V(x,y,z) = X(x)Y(y)Z(z)$

$$\nabla^2 V = 0 \Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad Y(y)Z(z)\frac{d^2X(x)}{dx^2} + \cdots$$

(2) To use the Separation of variables and take it into Laplace's Equation.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

$$f(x) + f(y) + f(z) = 0$$

make the constant $-k^2$

(3) In order to satisfied all x,y,z values, these three parts should be constant. Then we can get

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2, \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2, \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -k_z^2$$

$$k_x^2 + k_y^2 + k_z^2 = 0$$

(4) List the boundary conditions we got.

$f(x) = -kx^2$

(5) The general solution formats for above differential equation $\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0$ are:

k_x^2	k_x	$X(x)$	Exponential forms † of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

We need to choose the proper form of solution given boundary condition.
If V is independent of x, We can see X(x)=0;
If V goes to infinity or 0 as x goes to infinity, we choose k_x^2 is negative.

(6) Find the coefficients through boundary condition.

Example

Two infinite grounded metal plates lie parallel to the xz plane, one at y=0, the other at y=a(Fig.2). The left end, at x=0, is closed off with an infinite strip insulated ~~form~~ from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

1 For y:

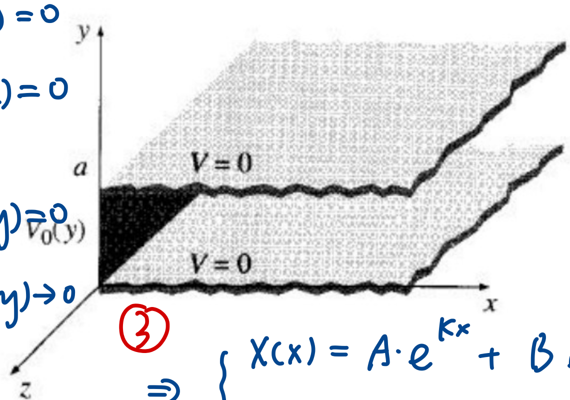
(i) $V=0$ when $y=0$, $V(x,0)=0$

(ii) $V=0$ when $y=a$, $V(x,a)=0$

For x:

(iii) $V=0$ when $x=0$, $V(0,y)=0$

(iv) $V\to 0$ when $x\to\infty$, $V(\infty,y)\to 0$



2 Voltage distribution independent of z.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial^2 y}=0$$

$$V(x,y)=X(x)\,Y(y)$$

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$$\Downarrow \qquad \Downarrow$$

$$k^2\ (C_1) \qquad -k^2\ (C_2)$$

$$\begin{cases}\dfrac{d^2X}{dx^2}-C_1X=0\\[2mm] \dfrac{d^2Y}{dy^2}-C_2X=0\end{cases}\Rightarrow\begin{cases}\dfrac{d^2X}{dx^2}=k^2X\\[2mm] \dfrac{d^2Y}{dy^2}=-k^2Y\end{cases}$$

3 $$\Rightarrow\begin{cases}X(x)=A\cdot e^{kx}+Be^{-kx}\\ Y(y)=C\cdot\sin ky+D\cdot\cos ky.\end{cases}$$

4 $$V(x,y)=(Ae^{kx}+\underline{B}e^{-kx})(\underline{C}\sin ky+D\cdot\cos ky)$$

cond. (iv) $\underline{A=0}$ cond. (i) $\underline{D=0}$

$$\left.\begin{aligned}&V(x,y)=C\cdot e^{-kx}\cdot\sin ky\\ &V(x,a)=0 \quad \text{true for all } x\end{aligned}\right\}\Rightarrow \begin{aligned}&\sin ka=0\\ &k=\frac{n\pi}{a}\quad n=1,2\cdots\end{aligned}$$

$$V_n(x,y)=C\cdot e^{-kx}\sin\frac{n\pi}{a}y.$$

$$V=\sum V_n(x,y)=\sum_{n=1}^{\infty}C_n\cdot e^{-kx}\cdot\sin\frac{n\pi}{a}y.$$

$$V(0,y)=\sum_{n=1}^{\infty}C_n\cdot\sin\frac{n\pi}{a}y=V_0.$$

5 $$C_n=\frac{2}{a}\int_0^a V_0\sin\frac{n\pi}{a}y\cdot dy.$$

4 Boundary-value Problems in Cylindrical Coordinates

(1) Laplace Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

(2) General solution: Assuming V is independent of Z. Symmetric.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0$$

(3) Separation of variables: $V(r,\phi)=R(r)\Phi(\phi)$

(4) Equations for $\Phi(\phi)$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0$$

Since the solution should be periodic among phi, we can get k=n and we should use $k^2>0$ be like

$$\Phi(\phi)=A_\phi\sin n\phi+B_\phi\cos n\phi$$

(5) Equations for R(r): After using separation of variables, we get

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=k^2$$

Which is a second order differential Equation

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}-n^2R(r)=0$$

And the general solution is

$$R(r)=A_r r^n+B_r r^{-n}$$

If we study the area including r=0, B_r=0, Otherwise, V goes to infinity at r=0
If we studay the area including r= ∞, S_r=0

(6) Equations for $V_n(r,\phi)$,

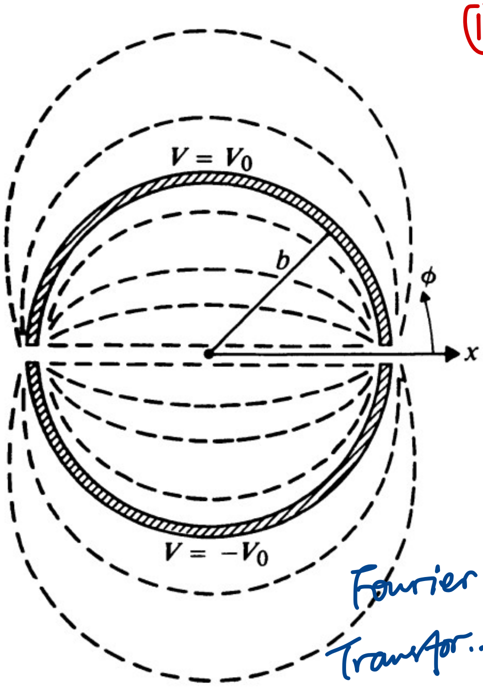
$$V_n(r,\phi)=r^n\left(A_n\sin n\phi+B_n\cos n\phi\right)+r^{-n}\left(A'_n\sin n\phi+B'_n\cos n\phi\right),\quad n\neq 0$$

(7) Special case: if V is independent of ϕ, k=0. Then we get

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=0$$

$$V(r)=C_1\ln r+C_2$$

Example An infinity long, thin, conducting circular tube of radius b is split in two halves. The upper half is kept at a potential $V = V_0$ and the lower half at $V = -V_0$.Determine the potential distribution both inside and outside the tube.



potential independent of z.

1 $V(b,\phi) = \begin{cases} V_0 & 0<\phi<\pi \\ -V_0 & \pi<\phi<2\pi \end{cases}$ 2 $\Rightarrow$ odd function.

1° inside $r<b$. $\Rightarrow$ no r^{-n} term.

$V_n(r,\phi) = A_n r^n \sin n\phi$

3 $V(r,\phi) = \sum_{n=1}^{\infty} V_n(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \sin n\phi.$

$\sum_{n=1}^{\infty} A_n b^n \sin n\phi = \begin{cases} V_0 & 0<\phi<\pi \\ -V_0 & \pi<\phi<2\pi. \end{cases}$

Fourier Transfor...

$A_n = \begin{cases} \frac{4V_0}{n\pi b^n} & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$

4 $V(r,\phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{r}{b}\right)^n \sin n\phi.$ $r<b$

2° $r>b$.

$V \to 0$ as $r \to \infty$.

r^n term doesn't exist.

$V(r,\phi) = \sum_{n=1}^{\infty} B_n' r^{-n} \sin n\phi.$

at $r=b$

$V(b,\phi) = \sum_{n=1}^{\infty} B_n' b^{-n} \sin n\phi.$

$= \begin{cases} V_0 & 0<\phi<\pi \\ -V_0 & \pi<\phi<2\pi \end{cases}$

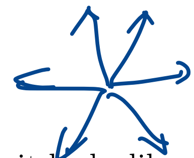
$\Rightarrow B_n' = \begin{cases} \frac{4V_0 b^n}{n\pi} & \text{if } n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$

$V(r,\phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{b}{r}\right)^n \sin n\phi.$ $r>b.$

5 Boundary-value Problem in Spherical Coordinates

(1) Since we only consider the situation that V is independent of ϕ, the Laplace Equation in Spherical coordinates is simplified to

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$



Symmetric.

(2) By using separation of variables, we assign $V(R,\theta)=\Gamma(R)\Theta(\theta)$. Then it looks like

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right]+\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]=0$$

(3) General solutions for $\Gamma(R)$. Firstly,we assume the part for $\Gamma(R)$ equals to k^2:

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right]=k^2$$

It is actually second differential Equation:

$$R^2\frac{d^2\Gamma(R)}{dR^2}+2R\frac{d\Gamma(R)}{dR}-k^2\Gamma(R)=0$$

The solution form is

$$\Gamma_n(R)=A_nR^n+B_nR^{-(n+1)},\ where\ k=n(n+1),n>0$$

(4) General Solutions for θ. Similarly, we can get

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]=-k^2$$

Since we already know n(n+1)=k^2, we can get the second differential equation:

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]+n(n+1)\Theta(\theta)\sin\theta=0$$

. It is called Legendre's equation and for $theta \in [0,\pi]$, the solution has special forms called Legendre's polynomials:

$$\Theta_n(\theta)=P_n(\cos\theta)$$

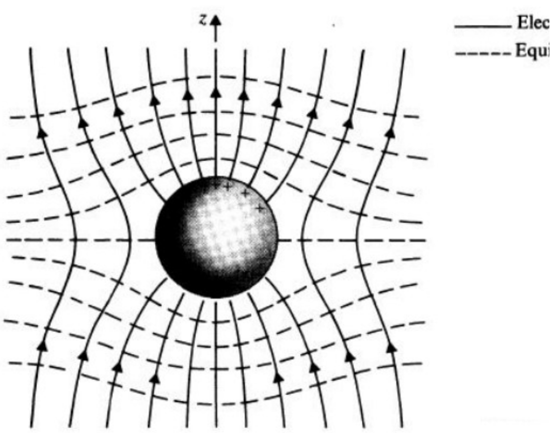
There are some solutions forms for usual n.

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta-1)$
3	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$

(5) By Combing them together,

$$V_n(R,\theta) = \left[A_n R^n + B_n R^{-(n+1)}\right] P_n(\cos\theta)$$

Example An uncharged conducting sphere of radius b is placed in an initially uniform Electric Field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine (a) the potential distribution V(R,θ) and (b) the electric field intensity $\mathbf{E}(R,\theta)$ after the introduction of sphere.



Solution After the conducting sphere is introduced into the electric field, a separation and redistribution of charges will take place in such a way that the surface of the sphere is maintained equipotential. The electric field intensity within the sphere is zero. Outside the sphere the field lines will intersect the surface normally, and the field intensity at points very far away from the sphere will not be affected appreciably. The geometry of this problem is depicted in Fig. 4–21. The potential is, obviously, independent of the azimuthal angle ϕ, and the solution obtained in this section applies.

a) To determine the potential distribution $V(R, \theta)$ for $R \geq b$, we note the following boundary conditions:

$$V(b, \theta) = 0^{\dagger} \qquad (4\text{–}155\text{a})$$

$$V(R, \theta) = -E_0 z = -E_0 R \cos\theta, \qquad \text{for } R \gg b. \qquad (4\text{–}155\text{b})$$

Equation (4–155b) is a statement that the original $\mathbf{E}_0$ is not disturbed at points very far away from the sphere. By using Eq. (4–154) we write the general solution as

$$V(R, \theta) = \sum_{n=0}^{\infty} \left[A_n R^n + B_n R^{-(n+1)}\right] P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}156)$$

However, in view of Eq. (4–155b), all A_n except A_1 must vanish, and $A_1 = -E_0$. We have, from Eq. (4–156) and Table 4–2,

$$V(R, \theta) = -E_0 R P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta)$$

$$= B_0 R^{-1} + (B_1 R^{-2} - E_0 R)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}157)$$

Actually, the first term on the right side of Eq. (4–157) corresponds to the potential of a charged sphere. Since the sphere is uncharged, $B_0 = 0$, and Eq. (4–157) becomes

$$V(R, \theta) = \left(\frac{B_1}{R^2} - E_0 R\right)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}158)$$

Actually, the first term on the right side of Eq. (4–157) corresponds to the potential of a charged sphere. Since the sphere is uncharged, $B_0 = 0$, and Eq. (4–157) becomes

$$V(R, \theta) = \left(\frac{B_1}{R^2} - E_0 R\right)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \qquad (4\text{–}158)$$

Now applying boundary condition (4–155a) at $R = b$, we require

$$0 = \left(\frac{B_1}{b^2} - E_0 b\right)\cos\theta + \sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos\theta),$$

from which we obtain

$$B_1 = E_0 b^3$$

and

$$B_n = 0, \qquad n \geq 2.$$

We have, finally, from Eq. (4–158),

$$V(R, \theta) = -E_0\left[1 - \left(\frac{b}{R}\right)^3\right] R\cos\theta, \qquad R \geq b. \qquad (4\text{–}159)$$

b) The electric field intensity $\mathbf{E}(R, \theta)$ for $R \geq b$ can be easily determined from $-\nabla V(R, \theta)$:

$$\mathbf{E}(R, \theta) = \mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta, \qquad (4\text{–}160)$$

where

$$E_R = -\frac{\partial V}{\partial R} = E_0\left[1 + 2\left(\frac{b}{R}\right)^3\right]\cos\theta, \qquad R \geq b \qquad (4\text{–}160\text{a})$$

and

$$E_\theta = -\frac{\partial V}{R\,\partial\theta} = -E_0\left[1 - \left(\frac{b}{R}\right)^3\right]\sin\theta, \qquad R \geq b. \qquad (4\text{–}160\text{b})$$

The surface charge density on the sphere can be found by noting that

$$\rho_s(\theta) = \epsilon_0 E_R\Big|_{R=b} = 3\epsilon_0 E_0 \cos\theta, \qquad (4\text{–}161)$$