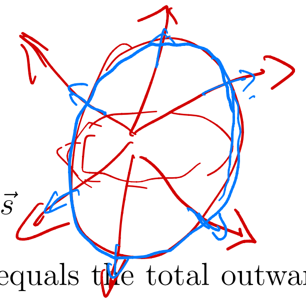
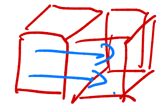
RC2

1 Vector Calculus

1.1 Divergence Theorem

3D → 2D.





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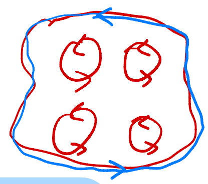
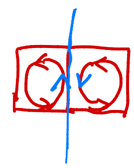
$$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

1.2 Stokes's Theorem

2D → 1D





$$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

1.3 Null Identities

$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla V$

$$\nabla \times (\nabla V) \equiv 0$$

- The curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.

$\nabla \cdot \vec{F} = 0 \quad \vec{F} = \nabla \times \vec{A}$.

$$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

- The divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field.

1.4 Exercise

- (HW1-4) Find the divergence of the following radial vector fields:

  a) $f_1(\mathbf{R}) = \mathbf{a_R} R^n$,

b) $f_2(\mathbf{R}) = \mathbf{a_R}\frac{k}{R^2}$.

$\cancel{\phi}$, $\cancel{0}$

In spherical coordinates, $\nabla\cdot\vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R)$ if $\vec{A} = \vec{a}_R A_R$

a) $\vec{A} = f_1(\vec{R}) = \vec{a}_R R^n$, $A_R = R^n$

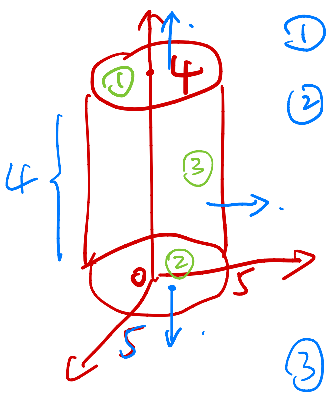
$$\nabla\cdot\vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}$$

b) $\vec{A} = f_2(\vec{R}) = \vec{a}_R \frac{k}{R^2}$, $A_R = \frac{k}{R^2}$

$$\nabla\cdot\vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0$$

$h_1 = 1$, $h_2 = r$, $h_3 = 1$.

- (HW1-5) For vector function $\mathbf{A} = \mathbf{a_r}r^2 + \mathbf{a_z}2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r = 5$, $z = 0$, and $z = 4$.



① $\nabla\cdot\vec{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}r^3 + \frac{\partial}{\partial z}2rz\right) = 3r+2 \Rightarrow \int_V \nabla\cdot\vec{A}\,dv = \int_0^4\int_0^{2\pi}\int_0^5 (3r+2)\,r\,dr\,d\phi\,dz = 1200\pi$.

② top face ($z=4$): $\vec{A} = \vec{a}_r r^2 + \vec{a}_z 8$, $d\vec{s} = \vec{a}_z ds$

$\int \vec{A}\cdot d\vec{s} = \int 8\,ds = 8\cdot(\pi 5^2) = 200\pi$

div thm verified

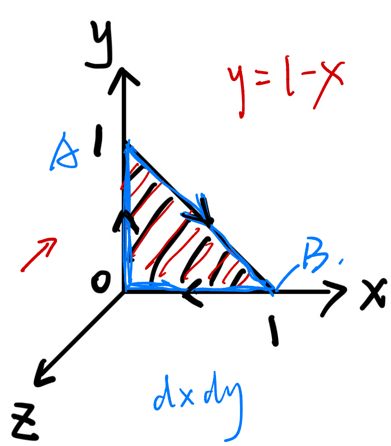
bottom face ($z=0$): $\vec{A} = \vec{a}_r r^2$, $d\vec{s} = -\vec{a}_z ds \Rightarrow \int \vec{A}\cdot d\vec{s} = 0$

wall ($r=5$): $\vec{A} = \vec{a}_r 25 + \vec{a}_z 2z$, $d\vec{s} = \vec{a}_r ds$

$\int \vec{A}\cdot d\vec{s} = 25\int ds = 25(2\pi\times5\times4) = 1000\pi$

③ $\boxed{\oint_S \vec{A}\cdot d\vec{s}} = 1000\pi + 0 + 200\pi = 1200\pi = \int_V \nabla\cdot\vec{A}\,dv$

- Given $\mathbf{F} = \mathbf{a_x}xy - \mathbf{a_y}2x$, verify Stokes's theorem using the triangle in the following picture.



① $\nabla\times\vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -\vec{a}_z(x+2)$

② $\oint_S (\nabla\times\vec{F})\cdot d\vec{s} = \int_0^1\int_0^{1-x} -\vec{a}_z(x+2)\cdot(-\vec{a}_z\,dxdy)$.

$= \int_0^1\int_0^{1-x}(x+2)\,dxdy = \frac{7}{6}$

③ i) $O\to A$: $x=0$, $\vec{F} = 0$

$\oint_C \vec{F}\cdot d\vec{l} = 0$

ii) $B\to O$: $y=0$, $\vec{F} = -\vec{a}_y 2x$. $\oint_C \vec{F}\cdot d\vec{l} = 0$.

$\vec{F}\cdot d\vec{l} = (-\vec{a}_y 2x)(\vec{a}_x dx + \vec{a}_y \underbrace{dy}_{\to 0}) = 0$.

$y = 1-x$.

iii) $A\to B$. $\vec{F}\cdot d\vec{l} = (\vec{a}_x xy - \vec{a}_y 2x)(\vec{a}_x dx + \vec{a}_y dy)$.

$= xy\,dx - 2x\,dy$

$= x(1-x)\,dx + 2x\,dx = (-x^2+3x)\,dx$

$\oint_C \vec{F}\cdot d\vec{l} = \int_0^1 (-x^2+3x)\,dx = \frac{7}{6}$

2 Electrostatics in Free Space

Static electric charges (source) in free space → electric field

2.1 Electric field intensity

$\frac{\vec{F}}{q}$ $\quad q \bullet \longrightarrow \vec{F}$.

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathrm{V/m})$$

2.2 Fundamental Postulates of Electrostatics

- Differential form:

$$① \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (divergence)$$

$\rho = \varepsilon_0 \cdot (\nabla \cdot \vec{E})$.

$$② \quad \nabla \times \mathbf{E} = 0 \quad (curl)$$

where ρ is the volume charge density of free charges (C/m^3), ϵ_0 is the permittivity of free space, a universal constant.

- Integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

— Gauss's Theorem

$$\oint_C \mathbf{E} \cdot d\ell = 0$$

(stokes's theorem).

where Q is the total charge contained in volume V bounded by surface S. Also, the scalar line integral of the static electric field intensity around any closed path vanishes.

E is **not solenoidal** (unless $\rho = 0$), but **irrotational (conservative)**

3 Coulomb's Law

3.1 Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

$$\mathbf{E} = \mathbf{a_R} E_R = \mathbf{a_R} \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathrm{V/m})$$

$|E| \propto \frac{1}{R^2}$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\,(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 \,|\mathbf{R} - \mathbf{R}'|^3}. \quad (\mathrm{V/m})$$

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force $\mathbf{F_{12}}$ is experienced by q_2 due to the electric field intensity $\mathbf{E_{12}}$ of q_1 at q_2. Then we have:

$$\mathbf{F_{12}} = q_2 \mathbf{E_{12}} = \mathbf{a_R} \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

F_{12} q_2

q_1

$q_1 > 0, \; q_2 > 0$

- **several point charges:**

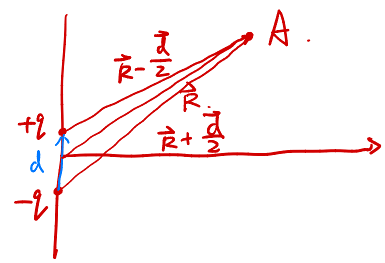
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k \left(\mathbf{R} - \mathbf{R}'_k\right)}{\left|\mathbf{R} - \mathbf{R}'_k\right|^3}$$

vectors

$\vec{E}$

3.2 Electric Dipole

- **Electric Field** general expression:

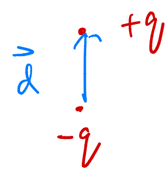


$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^3} \right\}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3} \left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d} \right]$$

- **Electric Dipole Moment Definition:**



$$\mathbf{p} = q\mathbf{d}$$

,where q is the charge, vector $\mathbf{d}$ goes from $-q$ to $+q$.

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

- **Electric Field:** (spherical coordinate)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right) \quad (\mathrm{V/m})$$

3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

dq $\rho dv' = dq$

, where dv' is the differential volume element.

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2} d\ell' \quad (\mathbf{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'} \mathbf{a}_R \frac{\rho_s}{R^2} ds' \quad (\mathrm{V/m})$$

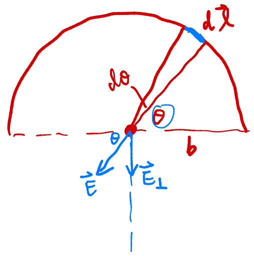
- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv' \quad (\mathbf{V/m})$$

3.4 Exercise

- (HW2-1) A line charge of uniform density ρ_l in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

$-\vec{a}_y$



$d\vec{l} = b\,d\theta$

$$d\vec{E} = \frac{\rho_l b\,d\theta}{4\pi\epsilon_0 b^2} = \frac{\rho_l\,d\theta}{4\pi\epsilon_0 b}$$

$$d\vec{E}_\perp = \frac{\rho_l\,d\theta}{4\pi\epsilon_0 b}\sin\theta$$

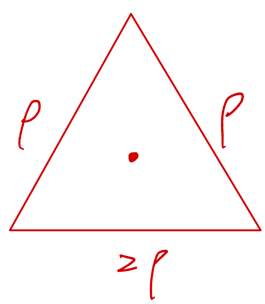
$$\int_0^\pi \frac{\rho_l\,d\theta}{4\pi\epsilon_0 b}\sin\theta = \frac{\rho_l}{4\pi\epsilon_0 b}\int_0^\pi \sin\theta\,d\theta = \boxed{\frac{\rho_l}{2\pi\epsilon_0 b}}$$ magnitude.

$$\vec{E} = -\vec{a}_y \frac{\rho_l}{2\pi\epsilon_0 b}$$

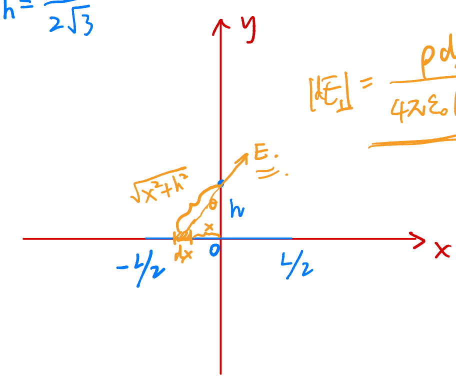
dir. mag.

- (HW2-2) Three uniform line charges——ρ_{l1}, ρ_{l2}, and ρ_{l3}, each of length L——form an equilateral triangle. Assuming that $\rho_{l1} = 2\rho_{l2} = 2\rho_{l3}$, determine the electric field intensity at the center of the triangle.

$2\rho \quad \rho \quad \rho$



$h = \frac{L}{2\sqrt{3}}$



$$\cos\theta = \frac{h}{\sqrt{x^2+h^2}}$$

$$|d\vec{E}_\perp| = \frac{\rho\,dx}{4\pi\epsilon_0(x^2+h^2)}\cdot\frac{h}{\sqrt{x^2+h^2}}$$

$$E_\perp = \int_{-L/2}^{L/2} \frac{\rho h\,dx}{4\pi\epsilon_0(x^2+h^2)^{\frac{3}{2}}} = \frac{\rho h}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{1}{(x^2+h^2)^{\frac{3}{2}}}\,dx.$$

$$\vec{E} = \vec{a}_y \frac{3\rho}{2\pi\epsilon_0 h} \qquad h = \frac{L}{2\sqrt{3}}$$

4 Gauss's Law and Application

4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

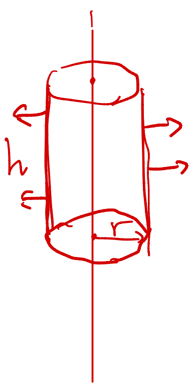
$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

4.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

4.3 Example

- Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ.

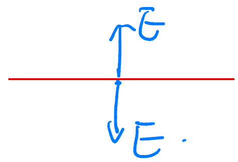


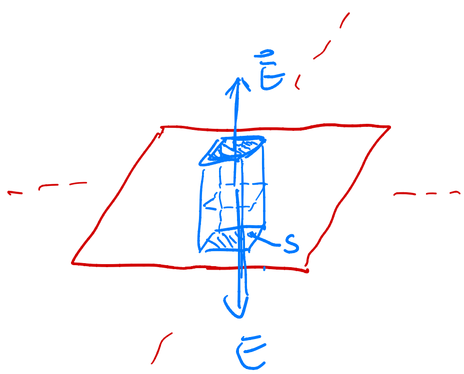
magnitude.

$$\oint_s \vec{E}\cdot d\vec{s} = \underset{\Delta}{E}\cdot 2\pi r \cancel{h} = \frac{\rho_\ell \cdot \cancel{h}}{\varepsilon_0}$$

$$E = \frac{\rho_\ell}{2\pi\varepsilon_0 r} \propto \frac{1}{r}.$$

- Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.





$$\oint_s \vec{E}\cdot d\vec{s} = E\cdot 2S = \frac{\rho_s \cdot S}{\varepsilon_0}$$

$$E = \frac{\rho_s}{2\varepsilon_0}.$$

4.4 Exercise

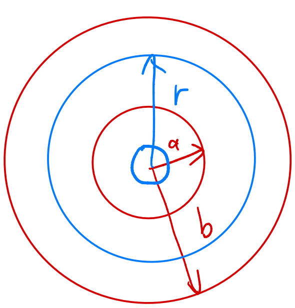
- (HW2-3) Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ $(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb}, respectively.

  a) Determine **E** everywhere.

  b) What must be the relation between a and b in order that **E** vanishes for $r > b$?

b). $a\rho_{sa} + b\rho_{sb} = 0$

$\Rightarrow \frac{b}{a} = -\frac{\rho_{sa}}{\rho_{sb}}$



a). i) $r < a$

no enclosed charge

$E = 0$

ii) $a < r < b$

$Q = \rho_{sa}\cdot 2\pi a l$

$\oint_s \vec{E}\cdot d\vec{s} = E\cdot 2\pi r l.$

$\frac{\rho_{sa}\cdot 2\pi a l}{\varepsilon_0} = E\cdot 2\pi r l \Rightarrow E = \frac{a\rho_{sa}}{\varepsilon_0 r}$

iii) $r > b$

$Q = \rho_{sa}\cdot 2\pi a l + \rho_{sb}\cdot 2\pi b l.$

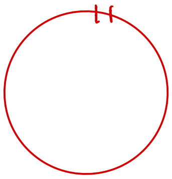
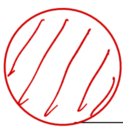
$\oint_s \vec{E}\cdot d\vec{s} = E\cdot 2\pi r l.$

$\frac{\rho_{sa}\cdot 2\pi a l + \rho_{sb}\cdot 2\pi b l}{\varepsilon_0} = E\cdot 2\pi r l$

$\Rightarrow E = \frac{a\rho_{sa} + b\rho_{sb}}{\varepsilon_0 r}.$

4.5 Several Useful Models (paste on your ctpp!)

Note: The charge distribution should be **uniform**.

different models	E (magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r\epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
 uniform spherical surface charge with radius R	$\begin{cases} E = 0 \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ (r > R) \end{cases}$
 uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi\epsilon_0 R^3} \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ (r > R) \end{cases}$
 infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} \ (r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0} \ (r > R) \end{cases}$

3.4 Exercise

(HW2-2) Caculation of $\int_{-L/2}^{L/2} \frac{1}{(x^2+h^2)^{\frac{3}{2}}} dx$

Let $x = h\tan\theta$, then $\int_{-L/2}^{L/2} \frac{1}{(x^2+h^2)^{\frac{3}{2}}} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{h^3\sec^3\theta} d\,h\tan\theta$

$$= \frac{1}{h^2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^3\theta \cdot \frac{1}{\cos^2\theta} d\theta = \frac{1}{h^2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\, d\theta = \frac{1}{h^2}\sin\theta\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{\sqrt{3}}{h^2}$$