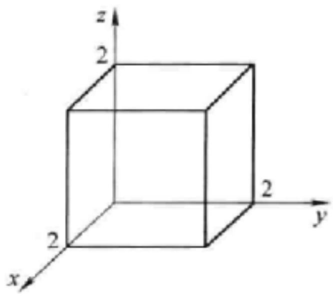
RC3

1 Quiz 1 Recap

Question 1

(a) Use the cube of side length 2 in the following picture and function $\boldsymbol{v} = (xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3xz)\hat{\boldsymbol{z}}$ to verify the divergence theorem.



① $\nabla \cdot \vec{v} = y + 2z + 3x$

② $\int (\nabla \cdot \vec{v})\, dv' = \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x)\, dx\, dy\, dz$

$= 48$

③ i) front: $d\vec{s} = \vec{a}_x\, dy\, dz$, $x = 2$

$\int \vec{v}\, d\vec{s} = \int_0^2 \int_0^2 2y\, dy\, dz = 8$

ii) back: $d\vec{s} = -\vec{a}_x\, dy\, dz$, $x = 0$

$\int \vec{v}\, d\vec{s} = 0$

iii) left: $d\vec{s} = -\vec{a}_y\, dx\, dz$, $y = 0$

$\int \vec{v}\, d\vec{s} = 0$

iv) right: $d\vec{s} = \vec{a}_y\, dx\, dz$, $y = 2$

$\int \vec{v}\, d\vec{s} = \int_0^2 \int_0^2 4z\, dx\, dz = 16$

v) top: $d\vec{s} = \vec{a}_z\, dx\, dy$, $z = 2$

$\int \vec{v}\, d\vec{s} = \int_0^2 \int_0^2 6x\, dx\, dy = 24$

vi) bottom: $d\vec{s} = -\vec{a}_z\, dx\, dy$, $z = 0$

$\int \vec{v}\, d\vec{s} = 0$.

Σ surfaces $= 8 + 16 + 24 = 48$

Verified.

Question 2

Assume the vector function $\mathbf{A} = \mathbf{a}_x 3x^2y^3 - \mathbf{a}_y x^3y^2$.

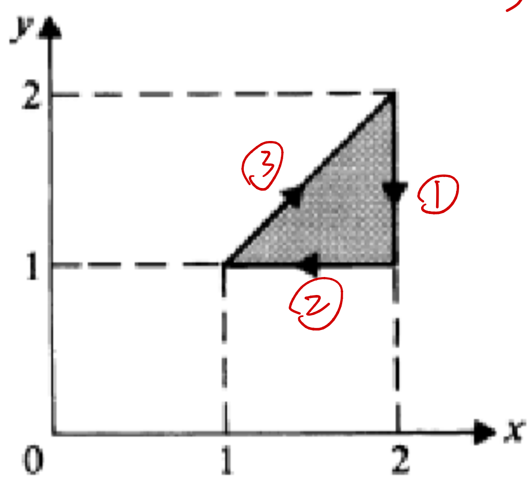
(a) Find $\oint \mathbf{A} \cdot d\ell$ around the triangular contour shown in the following figure.

(b) Evaluate $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over the triangular area.

(c) Can $\mathbf{A}$ be expressed as the gradient of a scalar? Explain.

(c) ✗.

$\nabla \times \vec{A} \neq 0$



(a). ① $x=2,\ dx=0$

$$\int \vec{A}\, d\vec{l} = \int_2^1 -8y^2\, dy = \frac{56}{3}$$

② $y=1,\ dy=0$

$$\int \vec{A}\, d\vec{l} = \int_2^1 3x^2\, dx = -7$$

③ $y=x$

$$\int \vec{A}\, d\vec{l} = \int (\vec{a}_x 3x^5 - \vec{a}_y x^5)(\vec{a}_x dx + \vec{a}_y dy) = \int_1^2 2x^5\, dx = 21$$

$$\oint \vec{A} \cdot d\vec{l} = 21 + \frac{56}{3} - 7 = \frac{98}{3}$$

(b) $$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^3 & -x^3y^2 & 0 \end{vmatrix} = -\vec{a}_z 12x^2y^2$$

$$d\vec{s} = -\vec{a}_z\, dx\, dy$$

$$\int (\nabla \times \vec{A})\, d\vec{s} = \int_1^2 \int_1^x 12x^2y^2\, dy\, dx = \frac{98}{3}$$

2 Gauss's Law and Application

2.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

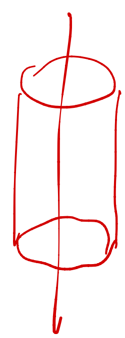
$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

2.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

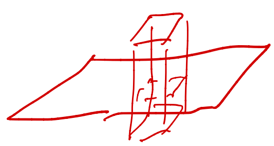
2.3 Example

- Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ.



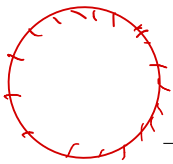
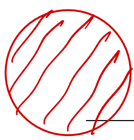
see RC

- Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.



2.4 Several Useful Models (paste on your ctpp!)

Note: The charge distribution should be **uniform**.

different models	E (magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r \epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
 uniform spherical surface charge with radius R	$\begin{cases} E = 0 \ \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ \ (r > R) \end{cases}$
 uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi\epsilon_0 R^3} \ \ (r < R) \\ E = \frac{Q}{4\pi\epsilon_0 r^2} \ \ (r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} \ \ (r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0} \ \ (r > R) \end{cases}$

3 Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

conservative

$\nabla \times \vec{E} = 0$

the reason for the negative sign: consistent with the convention that in going against the $\mathbf{E}$ field, the electric potential V increases.

- **Electric Potential Difference:**

$P_2 \to \infty \Rightarrow V_2 = 0$

$$\overset{0}{V_2} - V_1 = -\int_{P_1}^{\overset{\infty}{P_2}} \mathbf{E} \cdot d\ell$$

$V_1 = \int_{P_1}^{\infty} \vec{E} \cdot d\vec{\ell}$.

- **Electric Potential due to a Charge Distribution**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

i. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

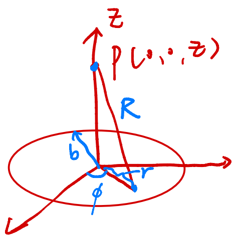
ii. **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{S}'} \frac{\rho_s}{R} ds' \quad (V)$$

iii. **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho}{R}dv' \quad (V)$$

- **Example:**
Obtain a formula for the electric field intensity and potential on the axis of a circular disk of radius b that carries a uniform surface charge ρ_s.



$d\vec{s} = r\,dr\,d\phi$

$R = \sqrt{r^2+z^2}$

$$V = \frac{1}{4\pi\epsilon_0}\int_{S'}\frac{\rho_s}{R}ds' \quad (V)$$

$$\vec{E} = -\nabla V$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s}{\sqrt{r^2+z^2}}\,d\vec{s}$$

$$= \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^b \frac{1}{\sqrt{r^2+z^2}}\,r\,dr\,d\phi$$

$$= \frac{\rho_s}{2\varepsilon_0}\left(\sqrt{b^2+z^2} - |z|\right)$$

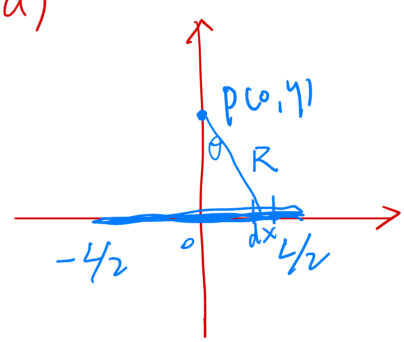
$$= \begin{cases} \vec{a}_z \dfrac{\rho_s}{2\varepsilon_0}\left(1 - \dfrac{z}{\sqrt{z^2+b^2}}\right), & z>0 \\ -\vec{a}_z \dfrac{\rho_s}{2\varepsilon_0}\left(1 + \dfrac{z}{\sqrt{z^2+b^2}}\right), & z<0 \end{cases}$$

3.1 Exercise

- (HW2-5) A finite line charge of length L carrying uniform line charge density ρ_l is coincident with the x-axis.
  a) Determine V in the plane bisecting the line charge.
  b) Determine $\mathbf{E}$ from ρ_l directly by applying Coulomb's law.
  c) Check the answer in part (b) with $-\nabla V$.

(c). $\vec{E} = -\nabla(\quad)$.

a)



$$V = 2\,\frac{1}{4\pi\varepsilon_0}\int_0^{L/2}\frac{\rho_l}{R}\,dx$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\int_0^{L/2}\frac{dx}{\sqrt{x^2+y^2}}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\left(\ln\left[\sqrt{\left(\tfrac{L}{2}\right)^2+y^2} - \tfrac{L}{2}\right] - \ln y\right)$$

$\cos\theta = \frac{y}{R}$

b) $$\vec{E} = \vec{a}_y E_y = 2\,\frac{1}{4\pi\varepsilon_0}\int_0^{L/2}\vec{a}_y\,\frac{\rho_l}{R^2}\cdot\frac{y}{R}\,dx = \frac{\rho_l}{2\pi\varepsilon_0}\int_0^{L/2}\vec{a}_y\,\frac{y}{(x^2+y^2)^{3/2}}\,dx$$

$$= \vec{a}_y\,\frac{\rho_l}{2\pi\varepsilon_0 y}\,\frac{L/2}{\sqrt{(L/2)^2+y^2}}.$$

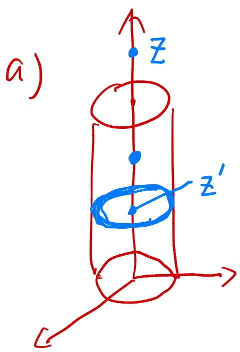
- (HW2-6) A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\mathbf{E}$ on its axis

  a) at a point outside the tube, then

  b) at a point inside the tube.

b) same expression.

a)



$\rho_s = \frac{Q}{2\pi b h}$

$dV = \frac{\rho_s b\, dz'}{2\epsilon_0 \sqrt{b^2+(z-z')^2}}$

$V = \int_0^h \frac{\rho_s b\, dz'}{2\epsilon_0\sqrt{b^2+(z-z')^2}} = \frac{b\rho_s}{2\epsilon_0} \ln \frac{z+\sqrt{b^2+z^2}}{(z-h)+\sqrt{b^2+(z-h)^2}}$

$\vec{E} = -\nabla V = \vec{a}_z \frac{b\rho_s}{2\epsilon_0}\left(\frac{1}{\sqrt{b^2+(z-h)^2}} - \frac{1}{\sqrt{b^2+z^2}}\right)$

ρ_l

$V = \oint \frac{\rho_l\, dl}{4\pi\epsilon_0 R} = \int_0^{2\pi} \frac{\rho_l \overbrace{b\, d\phi}^{dl}}{4\pi\epsilon_0 \underbrace{\sqrt{b^2+(z-z')^2}}_{R}} = \frac{\rho_l b}{2\epsilon_0\sqrt{b^2+(z-z')^2}}$

4 Conductors and Dielectrics in Static Electric Field

- Conductors:
  - electrons migrate easily.
  - charges reach the surface and conductor redistribute themselves in a way that both the charge and the field vanish.
  - **static state conditions**:
    * inside the conductor:

$$\rho = 0, \ \mathbf{E} = 0$$

      where $\rho = 0$ represents no charge in the interior
    * on the conductor surface (boundary conditions)

$$E_t = 0, \ E_n = \frac{\rho_s}{\epsilon_0} \ \vec{a}_n$$

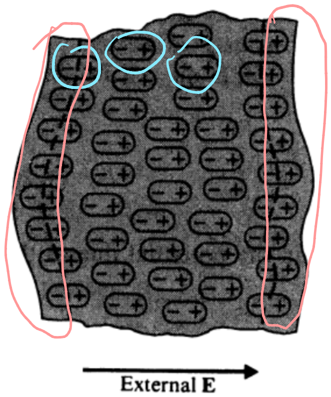
  - electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature
- Dielectrics (Insulators):
  - electrons are confined to their orbits.
  - external electric field polarizes a dielectric material and create electric dipoles. The induced electric dipoles will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

$-q$ $+q$

conductor

$E_{ind} = E_{ext}$

E.



E_{ind} not powerful enough to cancel the external field.

Figure 1: A cross section of a polarized dielectric medium

– **polarization charge densities/ bound-charge densities:**

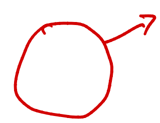
∗ **polarization vector, P**:

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p_k}}{\Delta v}$$

where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv.

∗ charge distribution on surface density:

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a_n}$$



∗ volume charge distribution density:

$$\rho_p = -\nabla \cdot \mathbf{P}$$

.

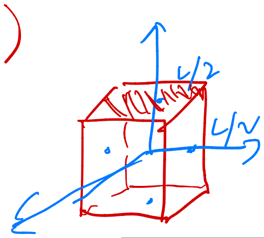
4.1 Exercise

- (HW3-1) The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_0(\mathbf{a_x}x + \mathbf{a_y}y + \mathbf{a_z}z)$.

  a) Determine the surface and volume bound-charge densities.

  b) Show that the total bound charge is zero.

a)



$\rho_{ps}(x, y, z = L/2) = \vec{P} \cdot \vec{a}_n = P_0(\vec{a}_x x + \vec{a}_y y + \vec{a}_z z) \cdot \vec{a}_z = P_0 \frac{L}{2}$

all six surfaces, $P_0 \frac{L}{2}$

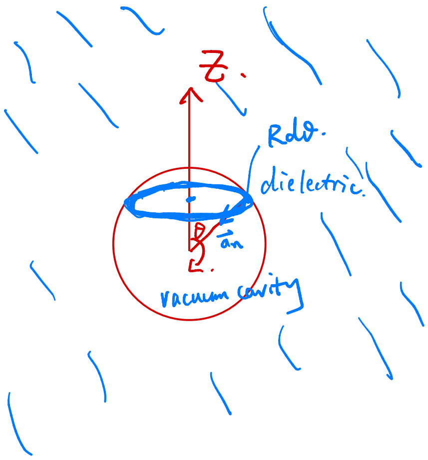
b) $Q_s = 6L^2 \cdot P_0 \frac{L}{2} = 3 P_0 L^3$

$Q_v = L^3 \cdot (-3P_0) = -3P_0 L^3$

$\rho_p = -\nabla \cdot \vec{P} = -3P_0$

$Q_s + Q_v = 0$

- (HW3-2) Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization **P** exists.



$$\vec{P} = P_0 \vec{a}_z$$

$$\rho_{ps} = \vec{P} \cdot \vec{a}_n = (\vec{a}_z P_0) \cdot \vec{a}_n = \underline{-P_0 \cos\theta}$$

$$\underline{dE_z = \frac{P_0 \cos\theta}{4\pi\varepsilon_0 R^2} \cdot (2\pi R \sin\theta)(R\,d\theta) \cos\theta}$$

$$= \frac{P_0}{2\varepsilon_0} \cos^2\theta \sin\theta \, d\theta$$

$$\vec{E} = \vec{a}_z E_z = \vec{a}_z \frac{P_0}{2\varepsilon_0} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \frac{P_0}{3\varepsilon_0} \vec{a}_z = \frac{\vec{P}}{3\varepsilon_0}.$$