Chapter 5:
Steady Electric Currents

Lecturer: Nana Liu

Summer 2023

JOINT INSTITUTE
交大密西根学院

Many slides courtesy of Sung-Liang Chen

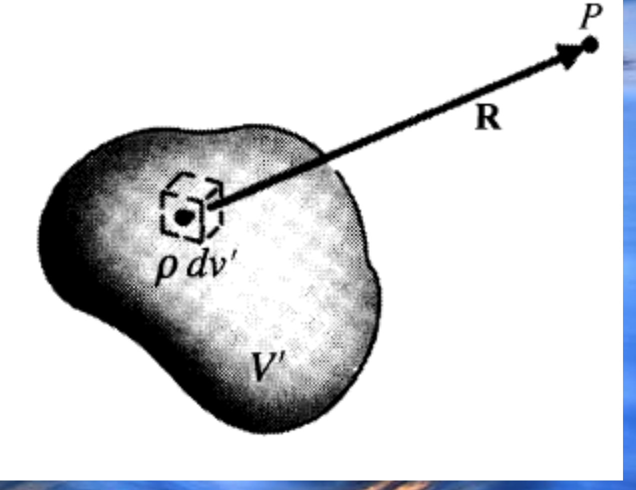
Everything you need to know about steady electric currents in this course…


Static electric charge density
Electric field

Static electric charge density

Electric field

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},$$



Static electric charge density
Electric field

Inside a Conductor *(Under Static Conditions)*
$\rho = 0$
$\mathbf{E} = 0$

Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$
$E_n = \dfrac{\rho_s}{\epsilon_0}$


Electric current density
Electric field density ?

Electric current density

Electric field density ?

Deriving Ohm's Law:

Relating speed, current, resistivity/conductivity

Electric current density

Electric field density

Deriving Ohm's Law:

$$V = IR$$

$$J = \sigma E$$


Currents in closed loops
Conservation laws

Currents in closed loops
Kirchoff's voltage law
(loop analysis)

$$\text{EMF} \sum_j V_j = \sum_k R_k I_k$$

Currents in closed loops
Kirchoff's circuit law
(node analysis)

$$\sum_j I_j = 0$$

Ohm's law versus Kirchoff's circuit law

Ohm's Law describes the relationship between voltage and current across a resistive element.

Ohm's Law states that voltage across a conductor is proportional to the current flows through it.

Ohm's Law is applicable to a single resistive element or set of resistive circuits as a whole.

Kirchhoff's Law describes the behaviour of current and voltage respectively in a circuit branch.

KCL states that the sum of current flows to a node is equal to zero while KVL states that the sum of voltages in a closed loop is zero.

KCL and KVL are applicable to a series of resistive elements in a circuit.

Steady currents governing equations

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$

Steady currents governing equations

Normal component

$$\nabla \cdot \mathbf{J} = 0$$

$$\boxed{J_{1n} = J_{2n} \qquad (\mathrm{A/m^2}).}$$

Tangential component

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$$

Analogy of physics at the boundary

Dielectric

D ϵ

Conductor

J σ

Main parts in steady electric currents:

(I) Ohm's law

(II) Kirchoff's law (voltage)

(III) Kirchoff's law (current)

(IV) Joule's law

(V) Boundary conditions for current density

5-1 Introduction

- Charges at rest (Ch3 and Ch4); Charges in motion (Ch5)
- Different types of currents
  - Conduction currents:
    - in conductors and semiconductors
    - electrons and/or holes
  - Electrolytic currents:
    - essentially in a liquid medium
    - Ions (e.g., Li-ion batteries)
  - Convection currents:
    - in vacuum or rarefied gas
    - electrons and/or ions

Average drift velocity of electrons for good conductors

Option	Speed in one direction	Number of votes
A	$10^3 - 10^4$ m/s	
B	$10^{-4} - 10^{-3}$ m/s	
C	$10^{-14} - 10^{-13}$ m/s	

Average drift velocity of electrons for good conductors

Option	Speed in one direction	Number of votes
A	$10^3 - 10^4\ \mathrm{m/s}$	
B	$10^{-4} - 10^{-3}\ \mathrm{m/s}$	
C	$10^{-14} - 10^{-13}\ \mathrm{m/s}$	

Topics for Conduction Currents

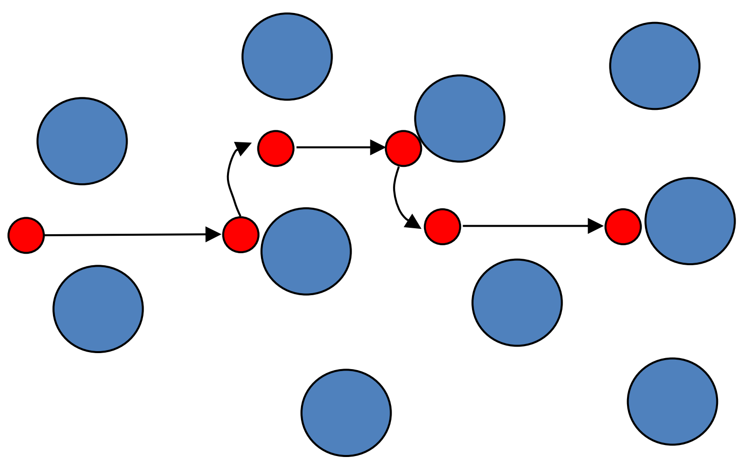
- Point form of Ohm’s law
- Kirchhoff’s voltage law
- Kirchhoff’s current law
  - Conservation of charge
  - Equation of continuity
- Boundary conditions for current density

Conduction Currents

- For a conductor, atoms consist of positively charged nuclei surrounded by electrons
  - Inner shell: tightly bound charges
  - Outermost shell: loosely bound charges (valance or conduction electrons)
- Without external **E**: conduction electrons wander randomly ➔ **no net drift motion** of conduction electrons

Conduction Currents

- With external **E**: **organized motion** of conduction electrons
  - Very low drift velocity due to collision with atoms
  - Conductor remains electrically neutral
    (electric forces prevent excess electrons from accumulating at any point of a conductor)



5-2 Current Density and Ohm's Law

Charge q across surface Δs with a velocity **u**
N: #/volume

The amount of charge passing Δs $\quad \Delta Q = Nq\underline{\mathbf{u}\cdot\mathbf{a}_n\,\Delta s\,\Delta t} \qquad \text{(C)}.$

differential volume with **Δs** along **a$_n$**

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u}\cdot\mathbf{a}_n\,\Delta s = \underline{Nq\mathbf{u}}\cdot\Delta\mathbf{s} \qquad \text{(A)}.$$

$$\boxed{\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2),} \qquad \boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),}$$

Defined as **volume** current density

$$\Delta I = \mathbf{J}\cdot\Delta\mathbf{s}.$$

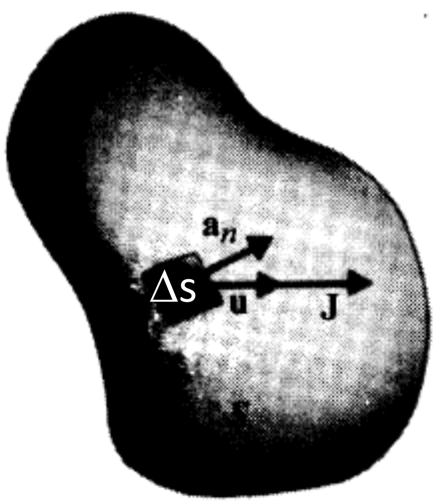


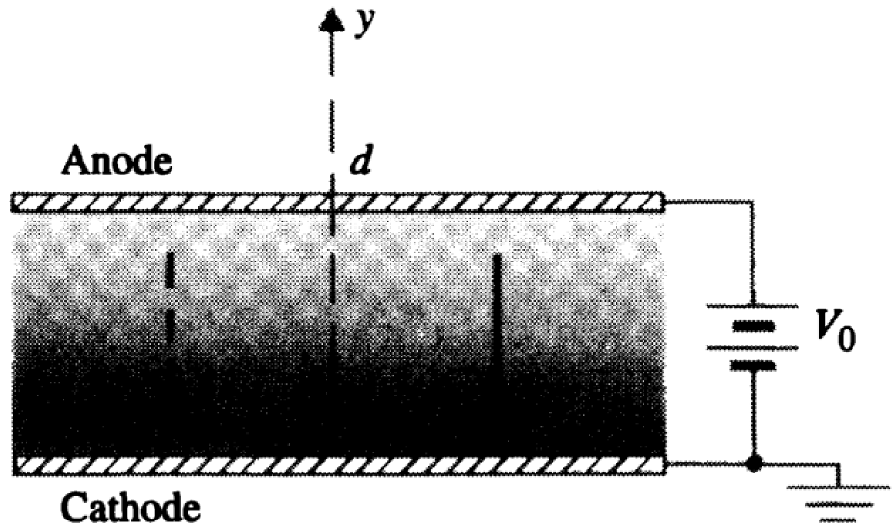
FIGURE 5–1
Conduction current due to drift motion of charge carriers across a surface.

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{s}.$$

Total current I flowing through a surface S

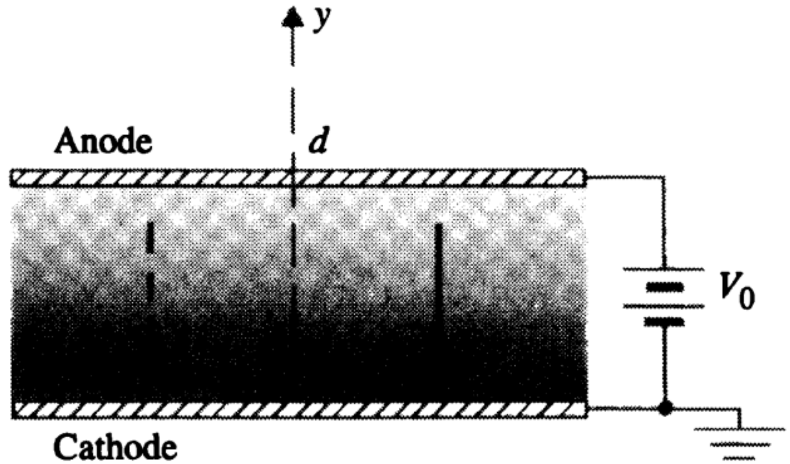
$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.}$$

Example



In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

Example



In other words, the net electric field at the cathode is zero. Neglecting fringing effects, we have

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left.\frac{dV(y)}{dy}\right|_{y=0} = 0.$$

In the steady state the current density is constant, independent of y:

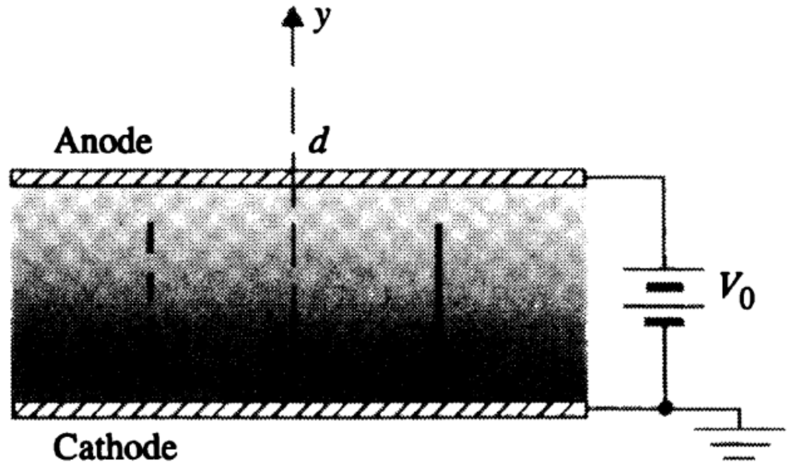
$$\mathbf{J} = -\mathbf{a}_y J = \mathbf{a}_y \rho(y) u(y),$$

where the charge density $\rho(y)$ is a negative quantity. The velocity $\mathbf{u} = \mathbf{a}_y u(y)$ is related to the electric field intensity $\mathbf{E}(y) = \mathbf{a}_y E(y)$ by Newton's law of motion:

$$m\frac{du(y)}{dt} = -eE(y) = e\frac{dV(y)}{dy},$$

where $m = 9.11 \times 10^{-31}$ (kg) and $-e = -1.60 \times 10^{-19}$ (C)

Example



$$m\frac{du}{dt} = m\frac{du}{dy}\frac{dy}{dt} = mu\frac{du}{dy}$$
$$= \frac{d}{dy}\left(\frac{1}{2}mu^2\right),$$

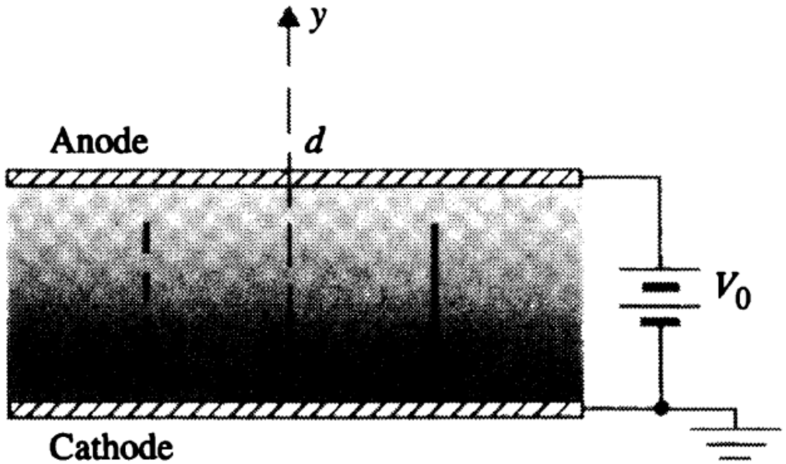
$$\frac{d}{dy}\left(\frac{1}{2}mu^2\right) = e\frac{dV}{dy}.$$

$$\tfrac{1}{2}mu^2 = eV,$$

$$u = \left(\frac{2e}{m}V\right)^{1/2}$$

where the constant of integration has been set to zero because at $y = 0$, $u(0) = V(0) = 0$.

Example



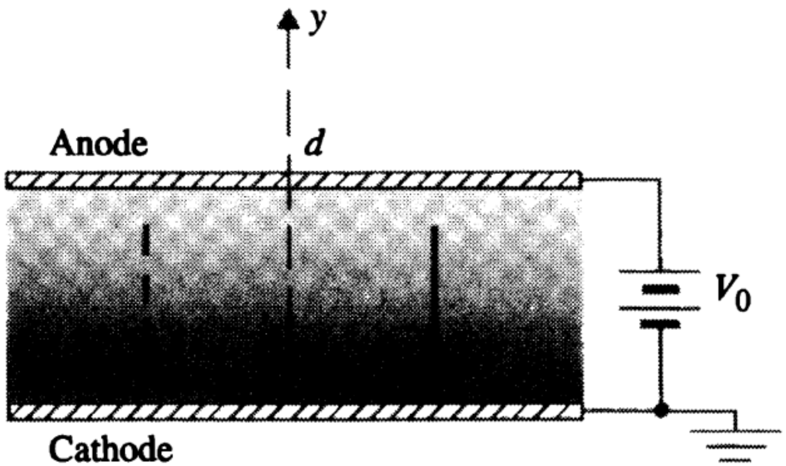
In order to find $V(y)$ in the interelectrode region we must solve Poisson's equation with ρ expressed in terms of $V(y)$

$$\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}\,V^{-1/2}.$$

Poisson's equation

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{-1/2}$$

Example



Poisson's equation

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{-1/2}$$

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{1/2} + c$$

At $y = 0$, $V = 0$, and $dV/dy = 0$

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left.\frac{dV(y)}{dy}\right|_{y=0} = 0$$

$$c = 0$$

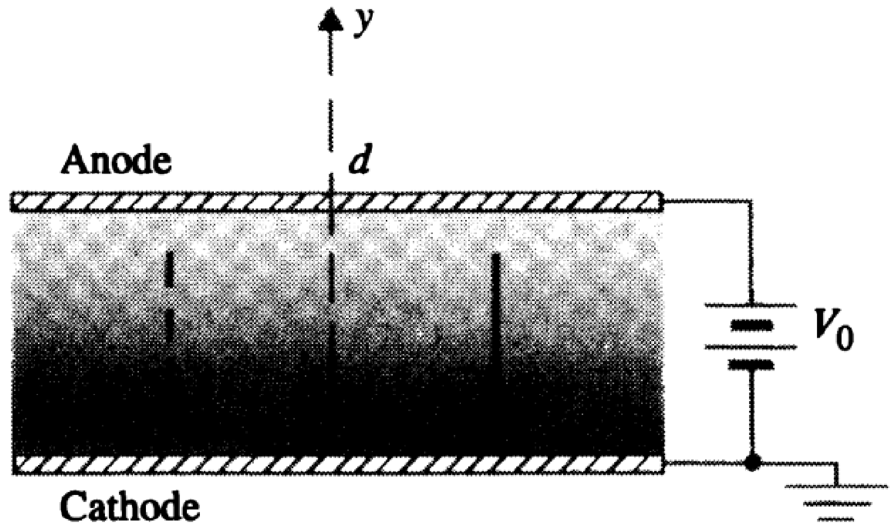
$$V^{-1/4}\,dV = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy$$

Integrate V from $V = 0$ to $V = V_0$

Integrate y from $y = 0$ to $y = d$

$$\frac{4}{3}V_0^{3/4} = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} d$$

Example



Space-charge-limited vacuum diode

$$J = \frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}\, V_0^{3/2} \qquad (\mathrm{A/m^2}).$$

Child-Langmuir Law

Conduction Currents

- For more than one kind of charge carriers (electrons, holes, and ions) drifting, current density:

$$\boxed{\mathbf{J} = Nq\mathbf{u} \qquad (\mathrm{A/m^2}),} \quad \Rightarrow \quad \mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).$$

- For most conducting materials,

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\mathrm{m/s}),$$

u: averaged drift velocity
$-\mu_e$: electron mobility ($\mathrm{m^2/V{\cdot}s}$)

Conduction Currents

- For most conducting materials,

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad \text{(m/s)},$$

u: averaged drift velocity
$-\mu_e$: electron mobility ($m^2/V{\cdot}s$)

copper	3.2×10^{-3} ($m^2/V{\cdot}s$)
aluminum	1.4×10^{-4} ($m^2/V{\cdot}s$)
silver	5.2×10^{-3} ($m^2/V{\cdot}s$)

$$\boxed{\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2),}$$

$$\boxed{\mathbf{u} = -\mu_e\mathbf{E} \qquad (\text{m/s}),}$$

$$\mathbf{J} = -\rho_e\mu_e\mathbf{E},$$

where $\rho_e = -Ne$

$$\boxed{\mathbf{J} = \sigma\mathbf{E} \qquad (\text{A/m}^2),}$$

Such materials are called ohmic media

where $\sigma = -\rho_e\mu_e$ $\boxed{\rho_e<0}$

σ: conductivity (A/V·m or S/m)

For conductors, $\sigma = -\rho_e\mu_e$

For semiconductors, $\sigma = -\rho_e\mu_e + \rho_h\mu_h,$

In general $\mu_e \neq \mu_h$

Conductivities

unit for σ is ampere per volt-meter (A/V·m) or siemens per meter (S/m).

Silicon

Copper

Germanium

5.80×10^{7} (S/m)

2.2 (S/m)

1.6×10^{-3} (S/m)

Votes	Right answer

Conductivities

Silicon

Copper

Germanium

5.80×10^{7} (S/m)

2.2 (S/m)

1.6×10^{-3} (S/m)

Votes	Right answer
	Copper
	Germanium
	Silicon

- Ohm's law $V_{12} = RI.$

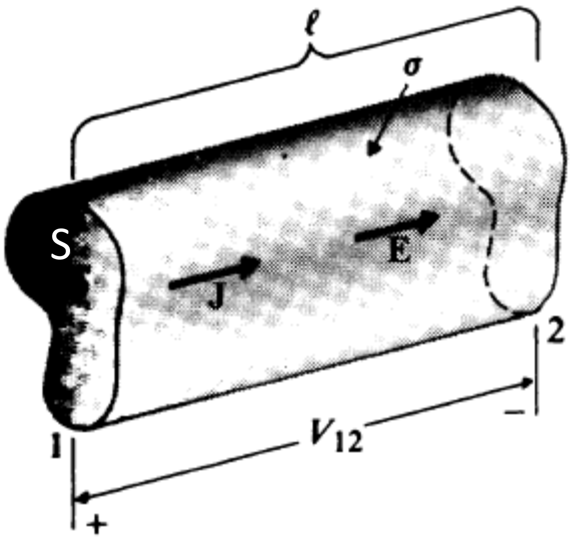
- Point form of Ohm's law $\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\mathrm{A/m^2}),}$
  - Holds at all points
  - σ can be a function of space

$V_{12} = RI.$

Integration?

$\mathbf{E} = (1/\sigma)\mathbf{J}$

Ohm's Law: Point Form to Circuit Form



$$V_{12} = E\ell \quad \text{or} \quad E = \frac{V_{12}}{\ell}.$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS \quad \text{or} \quad J = \frac{I}{S}.$$

FIGURE 5–3
Homogeneous conductor with a constant cross section.

$$\boxed{\mathbf{J} = \sigma \mathbf{E},}$$

$$J = \frac{I}{S}. \qquad E = \frac{V_{12}}{\ell}.$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI,$$

The resistance

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).}$$

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

Determine the d-c resistance of 1-(km) of wire having a 1-(mm) radius

(a) if the wire is made of copper, and (b) if the wire is made of aluminum

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

a) For copper wire, $\sigma_{cu} = 5.80 \times 10^7$ (S/m):

$$\ell = 10^3 \text{ (m)}, \qquad S = \pi(10^{-3})^2 = 10^{-6}\pi \quad (\text{m}^2).$$

We have

$$R_{cu} = \frac{\ell}{\sigma_{cu} S} = \frac{10^3}{5.80 \times 10^7 \times 10^{-6}\pi} = 5.49 \quad (\Omega).$$

b) For aluminum wire, $\sigma_{al} = 3.54 \times 10^7$ (S/m):

$$R_{al} = \frac{\ell}{\sigma_{al} S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49 = 8.99 \quad (\Omega).$$

Conductance

The ***conductance***, G, or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \qquad \text{(S)}.$$

From circuit theory we know the following:

a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

$$\boxed{R_{sr} = R_1 + R_2.}$$

b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\boxed{\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

or

$$\boxed{G_{||} = G_1 + G_2.}$$

Currents in closed loops → Kirchoff's voltage law (loop analysis)

$$\mathrm{EMF} \sum_j V_j = \sum_k R_k I_k$$

Currents in closed loops
Kirchoff's circuit law
(node analysis)

$$\sum_j I_j = 0$$

Ohm's law versus Kirchoff's circuit law

Ohm's Law describes the relationship between voltage and current across a resistive element.

Ohm's Law states that voltage across a conductor is proportional to the current flows through it.

Ohm's Law is applicable to a single resistive element or set of resistive circuits as a whole.

Kirchhoff's Law describes the behaviour of current and voltage respectively in a circuit branch.

KCL states that the sum of current flows to a node is equal to zero while KVL states that the sum of voltages in a closed loop is zero.

KCL and KVL are applicable to a series of resistive elements in a circuit.

5-3 Electromotive Force and Kirchhoff's Voltage Law

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

For an ohmic material

$$\mathbf{J} = \sigma \mathbf{E},$$

$$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0.$$

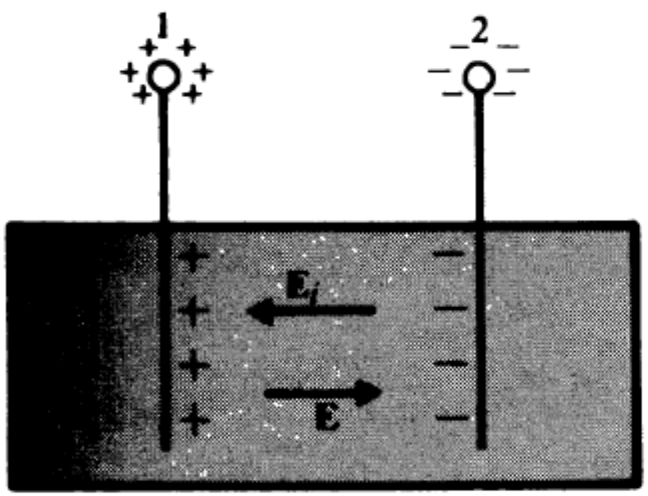
A steady current cannot be maintained in the same direction in a closed circuit by **an electrostatic field** (**conservative field**)

That is, to maintain a steady current in a closed circuit, there must be **non-conservative field** (e.g., electric batteries, etc.), which termed as **impressed electric field intensity $\mathbf{E}_i$**

Electromotive Force

- Chemical action ($\mathbf{E}_i$) ➔ cumulation of + and – charges on electrodes due to $\mathbf{E}_i$ ➔
  - Inside: $\mathbf{E}$ and $\mathbf{E}_i$
    - $\mathbf{E} = -\mathbf{E}_i$ due to I=0 for open circuit
  - Outside: $\mathbf{E}$ only

$\mathbf{E}$: electrostatic field
$\mathbf{E}_i$: nonconservative field



Electric battery

Open circuit

FIGURE 5–4
Electric fields inside an electric battery.

Electromotive Force

Against $\mathbf{E}_i$ from 1 to 2

- $\mathbf{E_i}$

$$\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside}\\ \text{the source}}}.$$

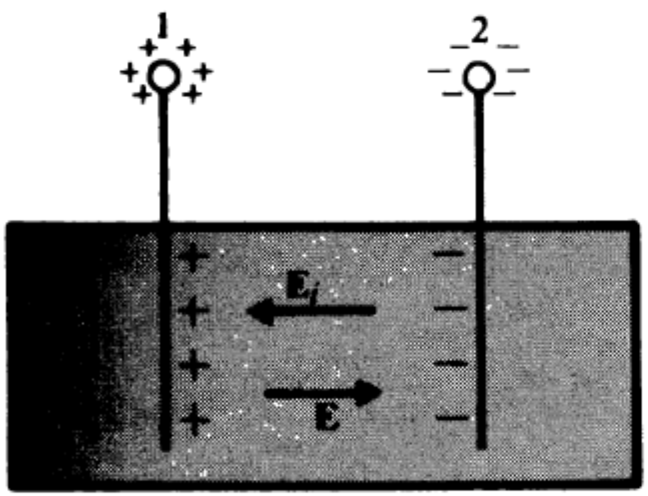
Open circuit: $\mathbf{E}_i = -\mathbf{E}$

The electromotive force is a measure of the strength of the nonconservative source, **not a force**

- **E**

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside}\\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside}\\ \text{the source}}} = 0.$$

Because of conservative field



Electric battery

Open circuit

FIGURE 5–4
Electric fields inside an electric battery.

$$\boxed{\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}}.}$$

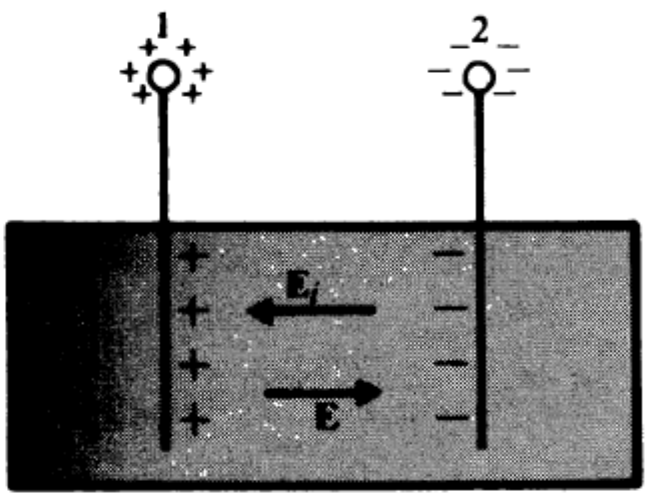
$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside} \\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}} = 0.}$$

$$\mathscr{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside} \\ \text{the source}}}$$

or

$$\mathscr{V} = V_{12} = V_1 - V_2.$$

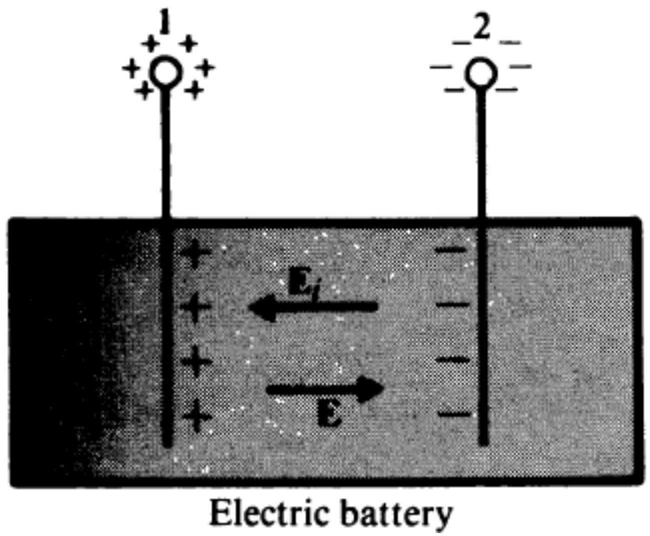
emf = voltage rise between + and – terminals



Electric battery

Open circuit

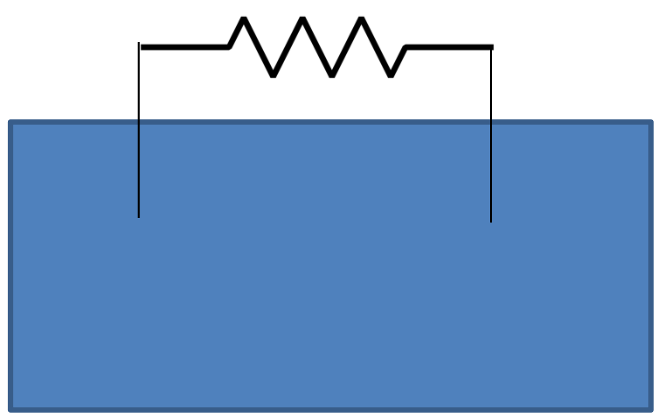
FIGURE 5–4
Electric fields inside an electric battery.



Open circuit ➔ No currents

FIGURE 5–4
Electric fields inside an electric battery.

If connected with a resistor
➔ Currents



Point form of Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i),$$

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.$$

$$\mathscr{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\boldsymbol{\ell} = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell}.$$

Integration due to 1st integrand = 0
Integration due to $\mathbf{E_i}$ outside the source = 0

$$\boxed{\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.}$$

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside} \\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}} = 0.}$$

$$\boxed{\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = \underbrace{-\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}}.}$$

For a resistor with uniform cross section:

$$J = I/S \qquad R = \frac{\ell}{\sigma S}$$

$$\int_1^2 \mathbf{E}_i \cdot dl = 0$$

$$\int_1^2 \mathbf{E} \cdot dl + \int_2^1 \mathbf{E} \cdot dl = 0$$

$$\mathscr{V} = RI.$$

$$\int_2^1 \mathbf{E}_i \cdot dl + 0 + 0 = \oint_c (\mathbf{E} + \mathbf{E}_i) \cdot dl$$

$$\mathcal{V} = RI.$$

For more-than-one emf and more-than-one resistor connected in series

$$\boxed{\sum_j \mathcal{V}_j = \sum_k R_k I_k \qquad \text{(V).}}$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances

Currents in closed loops
Kirchoff's circuit law
(node analysis)

$$\sum_j I_j = 0$$

5-4 Equation of Continuity and Kirchhoff's Current Law

- Principle of conservation of charge: electric charges may not be created or destroyed

$$\underbrace{I = \oint_S \mathbf{J} \cdot d\mathbf{s}}_{\text{Current leaving a volume}} = \underbrace{-\frac{dQ}{dt}}_{\text{Rate of charge decrease in the volume}} = -\frac{d}{dt}\int_V \rho \, dv.$$

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv.$$

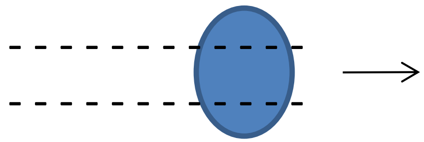
$$\int_V \nabla \cdot \mathbf{J}\, dv = -\int_V \frac{\partial \rho}{\partial t}\, dv.$$

Holds for arbitrary volume *V*

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

- For **steady current (I=constant)**, charge density does not vary with time (or charge in a volume is a constant over time although charge is moving): $\boxed{\partial\rho/\partial t = 0.}$



$\nabla \cdot \mathbf{J} = 0.$ (divergenceless: streamlines of steady currents close upon themselves)

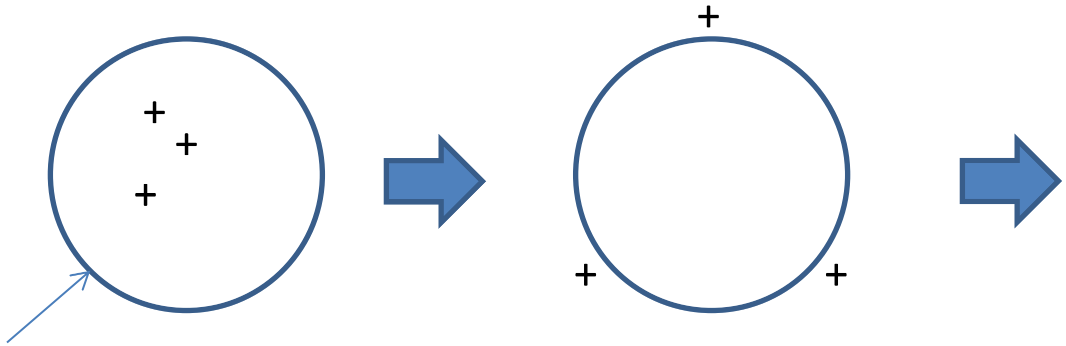
$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0,$ (integral form)

closed *S* → 0;
That is, volume (enclosed by *S)* → 0

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}}$$

Kirchhoff's current law: the algebraic sum of all the currents flowing **out of a junction** (a small volume) in an electric circuit is zero.

E and ρ inside a Conductor: time to equilibrium



Conductor surface

Charge distribution due to electrostatic force

ρ_{inside}=0

Inside a Conductor (*Under Static Conditions*)
$\rho = 0$
$\mathbf{E} = 0$

By Gauss's law

Time to Reach Equilibrium in a Conductor

- Inside a conductor, ρ=0, **E**=0 under equilibrium conditions
- Time to reach equilibrium?

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

Assume a constant σ

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

In a simple medium

$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

Solution: $$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\text{C/m}^3),}$$

Charge density inside a conductor will decrease with time exponentially.

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad (\text{s}).$$

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} s$	
B	$1.5 \times 10^{-8} s$	
C	$1.5 \times 10^{-19} s$	

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} s$	
	1/2 time for a housefly's wing flap Sound to travel 30 cm	
	Typical response time in high-end LCD computer monitors	

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

Option	Speed in one direction	Number of votes
	Half-life of Lithium-12	
B	$1.5 \times 10^{-8} s$	
	Half the time in fusion reaction in hydrogen bomb Light to travel 15 m in vacuum	

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

Option	Speed in one direction	Number of votes
C	$1.5 \times 10^{-19} s$	

Time it takes light to travel diameter of hydrogen atom

More than 200 times shorter than the shortest pulse of laser light ever created

10 attosecs to 1 sec is 1 sec to 32 billion years

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} s$	
B	$1.5 \times 10^{-8} s$	
C	$1.5 \times 10^{-19} s$	

Relaxation times

Relaxation time: time for ρ_0 to decay to 1/*e*

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}.$$

CHARGE RELAXATION TIMES OF TYPICAL MATERIALS

	Conductivity	ϵ/ϵ_0	Relaxation time s
Water, distilled	2×10^{-4}	81	3.6×10^{-6}
Corn oil	5×10^{-11}	3.1	0.55
Mica	10^{-11} - 10^{-15}	5.8	5.1 - 5.1×10^{4}

Main parts in steady electric currents:

(I) Ohm's law

(II) Kirchoff's law (voltage)

(III) Kirchoff's law (current)

(IV) Joule's law

(V) Boundary conditions for current density

5-5 Power Dissipation and Joule's Law

- Power dissipation:

  External **E** ➔ drift motion of electrons, which collide with atoms on lattice sites ➔ thermal energy

- Power by **E** to move a charge q

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \underline{q}\mathbf{E} \cdot \mathbf{u},$$

u: drift velocity

Total power in a volume dv $$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i \underline{N_i q_i} \mathbf{u}_i \right) \underline{dv},$$

$\boxed{\text{Total Q in a volume dv}}$

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv,$$

$$\boxed{\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).}$$

$$dP = \mathbf{E} \cdot \mathbf{J}\, dv$$

$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} \qquad (\mathrm{W/m^3}).$$

Power density

Joule's law

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad (\mathrm{W}).}$$

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad \text{(W)}.}$$

In a conductor with constant cross section

$$dv = ds\, d\ell$$

$$P = \int_L E\, d\ell \int_S J\, ds = VI,$$

We get the familiar expression $P = I^2 R$

Main parts in steady electric currents:

(I) Ohm’s law

(II) Kirchoff’s law (voltage)

(III) Kirchoff’s law (current)

(IV) Joule’s law

(V) Boundary conditions for current density

5-6 Boundary Conditions for Current Density

- Steady current density **J** on boundaries without nonconservative energy source

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\ell = 0$

Chapter 3

Here

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),}$$

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{J} = 0$$

$$\boxed{J_{1n} = J_{2n} \qquad (\mathrm{A/m^2}).}$$

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad (\mathrm{V/m}),}$$

The **tangential** component of a **curl-free** vector field is continuous across an interface

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

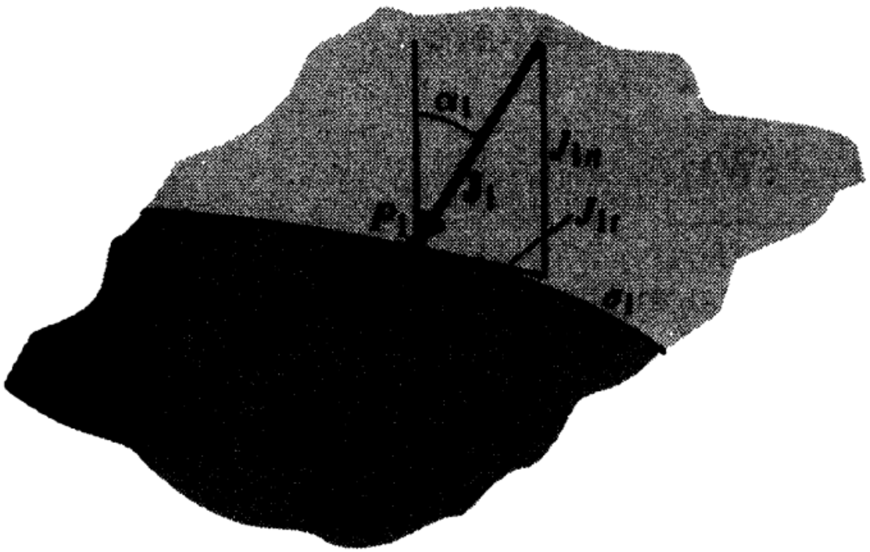
$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$$

The ratio of J_t at two sides of an interface is equal to **the ratio of the conductivities**

Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. The steady current density in medium 1 at point P_1 has a magnitude J_1 and makes an angle α_1 with the normal.

Determine the magnitude and direction of the current density at point P_2 in medium 2.

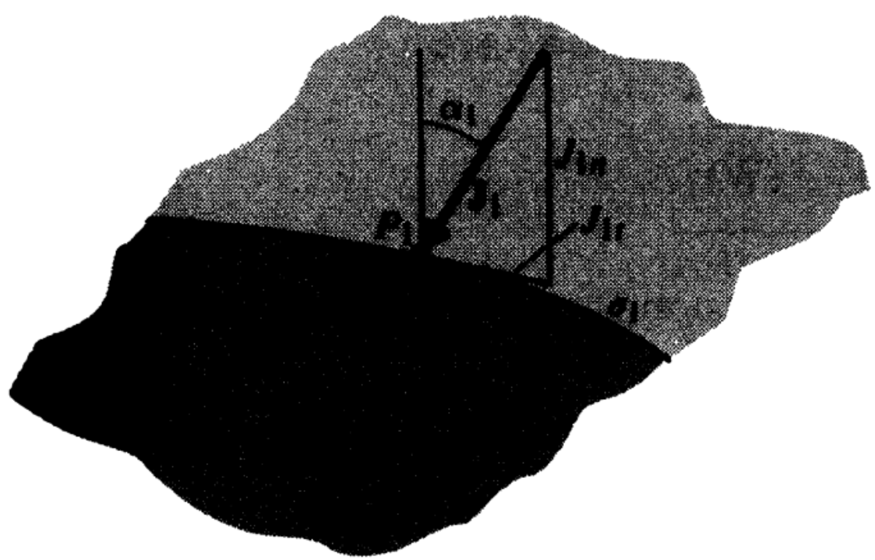


Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$



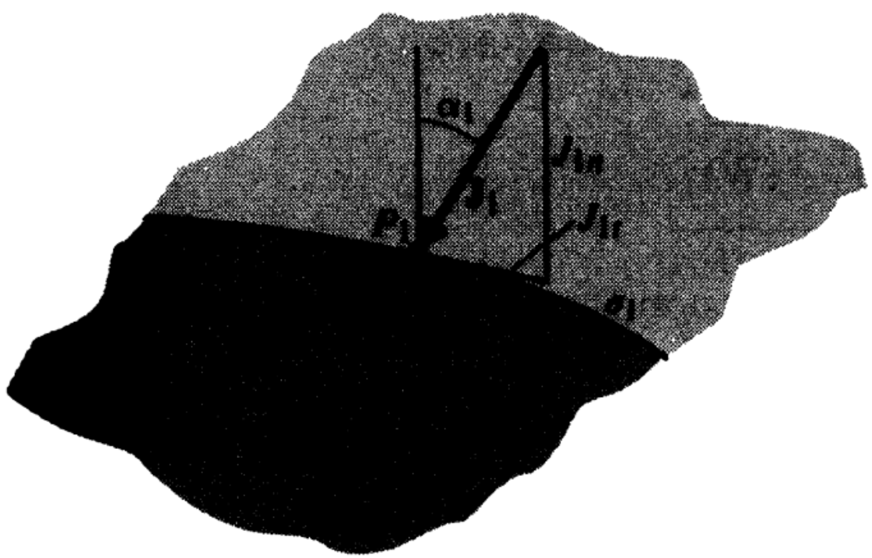
Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.}$$



Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.}$$

If medium 1 is a much better conductor than medium 2 ($\sigma_1 \gg \sigma_2$ or $\sigma_2/\sigma_1 \to 0$), α_2 approaches zero, and $\mathbf{J}_2$ emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of $\mathbf{J}_2$ is

$$J_2 = \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2 \sin \alpha_2)^2 + (J_2 \cos \alpha_2)^2}$$
$$= \left[\left(\frac{\sigma_2}{\sigma_1} J_1 \sin \alpha_1\right)^2 + (J_1 \cos \alpha_1)^2\right]^{1/2}$$

Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.}$$

If medium 1 is a much better conductor than medium 2 ($\sigma_1 \gg \sigma_2$ or $\sigma_2/\sigma_1 \to 0$), α_2 approaches zero, and $\mathbf{J}_2$ emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of $\mathbf{J}_2$ is

$$J_2 = \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2 \sin \alpha_2)^2 + (J_2 \cos \alpha_2)^2}$$
$$= \left[\left(\frac{\sigma_2}{\sigma_1} J_1 \sin \alpha_1\right)^2 + (J_1 \cos \alpha_1)^2\right]^{1/2}$$

$$\boxed{J_2 = J_1\left[\left(\frac{\sigma_2}{\sigma_1} \sin \alpha_1\right)^2 + \cos^2 \alpha_1\right]^{1/2}.}$$

Chapter 3

Here

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),}$$

$$\nabla \cdot \mathbf{J} = 0$$

$$\boxed{J_{1n} = J_{2n} \qquad (\text{A/m}^2).}$$

The **tangential** component of a **curl-free** vector field is continuous across an interface

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m}),}$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$$

The ratio of J_t at two sides of an interface is equal to **the ratio of the conductivities**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$\nabla\times(\mathbf{D}/\varepsilon)=0$

Analogy

Interface of dielectric media	Interface of conducting media
D	**J**
ε	σ

A homogeneous conducting medium

Electrostatics analogy

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

If σ is a constant (homogeneous)

$$\nabla \times \mathbf{J} = 0.$$

$\nabla\times\mathbf{E}=0$

By null identity

$$\mathbf{J} = -\nabla\psi.$$

$\mathbf{E}=-\nabla V$

Laplace's eq.: $$\nabla^2\psi = 0.$$

$\nabla^2 V=0$

$\mathbf{J}=\sigma\mathbf{E}$

$$\psi = \sigma V.$$

Boundary Condition between Two Lossy Dielectrics (for a Steady Current)

- Two lossy dielectrics: ϵ_1 and ϵ_2 $\boxed{\sigma_1 \text{ and } \sigma_2}$

ϵ_1 and ϵ_2	σ_1 and σ_2
$E_{2t} = E_{1t}$	$J_{1t}/\sigma_1 = J_{2t}/\sigma_2$
$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s,$	$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}.$$

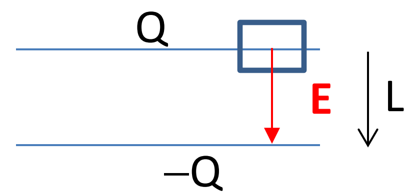
In most cases, a surface charge exists at the interface unless $\sigma_2/\sigma_1 = \varepsilon_2/\varepsilon_1$

What if medium 2 is a perfect conductor?

5-7 Resistance Calculations

- Capacitance between two conductors:

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}},$$

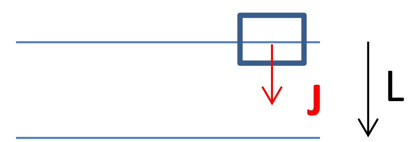


Numerator: surface integral over a surface enclosing the positive conductor

- Resistance between two conductors (medium between is lossy):

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$

$$\boxed{\mathbf{J} = \sigma \mathbf{E},}$$



Denominator: the same surface as in the numerator of the above equation

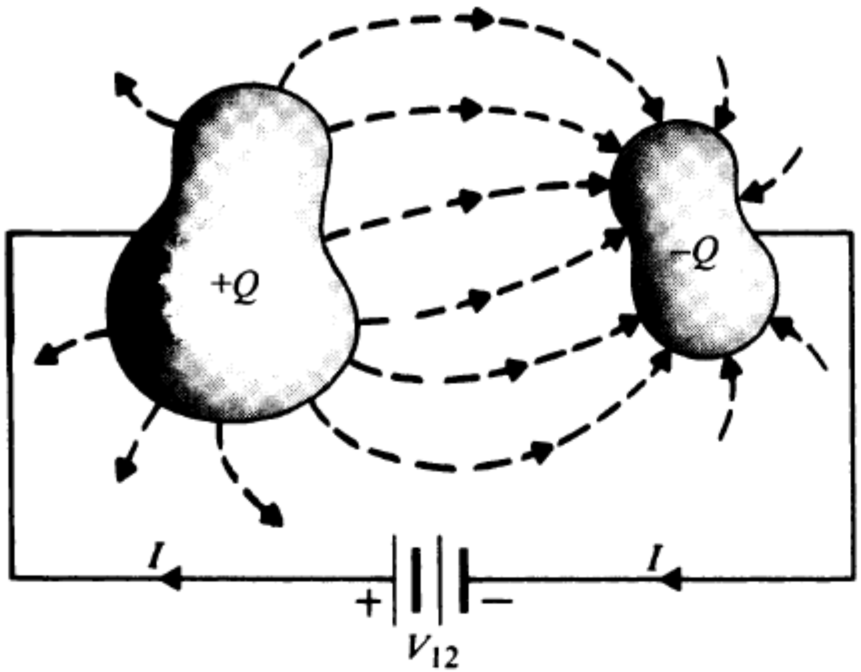


FIGURE 5–7
Two conductors in a lossy dielectric medium.

$$\boxed{C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon\mathbf{E} \cdot d\mathbf{s}}{-\int_L \cancel{\mathbf{E} \cdot d\boldsymbol{\ell}}},}$$

$$\times$$

$$\boxed{R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \cancel{\mathbf{E} \cdot d\boldsymbol{\ell}}}{\oint_S \sigma\mathbf{E} \cdot d\mathbf{s}},}$$

If ε and σ of the medium have the same space dependence or if the medium is homogeneous

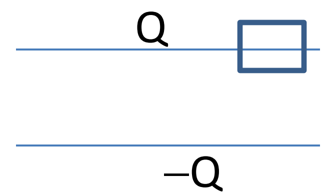
$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

C_ℓ and R_ℓ

- C_ℓ: capacitance per unit length
  $C_\ell = C/\ell$ (F/m)
  (ℓ longer → area S larger → C larger)

  $\boxed{C=\varepsilon S/d}$

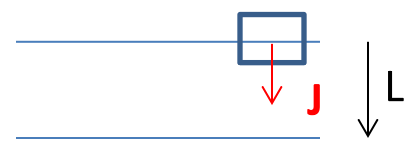
ℓ: in/out plane direction



- R_ℓ: Resistance per unit length
  $R_\ell = R\ell$ ($\Omega\cdot$m)
  (ℓ longer → area S larger → R smaller)

  $\boxed{R=\varepsilon L/S}$

ℓ: in/out plane direction



Note that ℓ and L are different!

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

→

$$\boxed{R_\ell C_\ell = \varepsilon/\sigma}$$

Procedure to Compute R between Specified Equi-potential Surfaces

- Procedure 1: **V_0 to I**
  - 1. Choose a coordinate
  - 2. Assume potential difference V_0 between conductors
  - 3. Find **E** between conductors
    - $\nabla^2 V = 0 \Rightarrow \mathbf{E} = -\nabla V$
  - 4. Find current $I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma \mathbf{E} \cdot d\mathbf{s},$
  - 5. Find $R = V_0 / I$
- Procedure 2: **I to V_0**
  - Assume $I \Rightarrow \mathbf{J} \Rightarrow \mathbf{E} \Rightarrow V_0$
  - $R = V_0 / I$

if **J** can be determined easily from I