Q1 (a)

Gauss divergence theorem: $\int (\nabla V)\, d\ell = \int V\, ds$

Let $V = xy\,\hat{i} + 2yz\,\hat{j} + 3xz\,\hat{k}$

and $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ $\Rightarrow$ $\nabla \cdot V = y + 2z + 3x$

$\Rightarrow \int \nabla \cdot V\, dv = \int_0^2\int_0^2\int_0^2 (y+2z+3x)\, dx\,dy\,dz = 48$

For surface of the yz plane: $\int v\,ds = \int_0^2\int_0^2 xy\,dx\,dy = 8 \leftarrow x=2$

$\int v\,ds = \int_0^2\int_0^2 xy\,dx\,dy = 0 \leftarrow x=0$

For surface of the xz plane: $\int v\,ds = \int_0^2\int_0^2 2yz\,dx\,dz = 16 \leftarrow y=2$

$\int v\,ds = \int_0^2\int_0^2 2yz\,dx\,dz = 0 \leftarrow y=0$

For surface of the xy plane: $\int v\,ds = \int_0^2\int_0^2 3xz\,dx\,dy = 24 \leftarrow z=2$

$\int v\,ds = \int_0^2\int_0^2 3xz\,dx\,dy = 0 \leftarrow z=0$

since $8+16+24 = 48$. we have proved the divergence theorem

(b)

Stokes' Theorem

$$\int (\nabla \times \bar{A}) \cdot d\bar{s} = \oint_C \bar{A} \cdot dl$$

Assume $A = xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}$

$$\Rightarrow \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z}$$

$$\Rightarrow \int_S (\nabla \times A)\, ds = \int_0^2 \int_0^2 -2y \, dy\, dz = -\frac{8}{3}$$

$$\oint_C A\, dl = \int_0^2 2yz\, dy \quad (\text{since } z = -y+2)$$

$$= \int_0^2 2y(-y+2)\, dy = -\frac{8}{3}$$

$\rightarrow$ Hence Proved

Q2

$$A = 3x^2y^3\,\hat{a}_x - x^3y^2\,\hat{a}_y$$

(a) According to Stoke's theorem

$$\oint \vec{A}\cdot d\vec{l} = \int_s (\nabla\times\vec{A})\cdot d\vec{s}$$

$$\Rightarrow \nabla\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^3 & -x^3y^2 & 0 \end{vmatrix} = \frac{\partial 3x^2y^3}{\partial y}\,\hat{a}_z + 0 + 0 - 0 - \frac{\partial -x^3y^2}{\partial x}\,\hat{a}_z - 0$$

$$= 12x^2y^2\,\hat{a}_z$$

$$\Rightarrow \int (\nabla\times\vec{A})\, d\vec{s} = \int_1^2\int_1^2 12x^2y^2\, dy\, dx = 12\int_1^2 \left(\frac{x^5}{3} - \frac{x^2}{3}\right) dx = 32.66$$

(b) same as (a) $\Rightarrow$ 32.66

(c) we can't. Assume $\vec{E} = \nabla V$

$$\vec{E} = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z, \quad \nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

We need to let $V = x^3y^3$

$$\left.\begin{array}{l} \Rightarrow \vec{E} = 3x^2y^3\,\hat{a}_x + 3x^3y^2\,\hat{a}_y \\ \text{But } \vec{A} = 3x^2y^3\,\hat{a}_x - x^3y^3\,\hat{a}_y \end{array}\right\}\text{ not the same}$$

Or in other words. $\int 3x^2y^3\,dx \neq \int x^3y^3\,dy$ since

$$\left.\begin{array}{l} \text{LHS } y^3\cdot\square \\ \text{And RHS } y^4\cdot\square \end{array}\right\}\text{ can't be same}$$