Lecture 24

VE 311 Analog Circuits

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2023 Summer

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Recap of Last Lecture

- General Time constant approach

Topics to Be Covered

- General Time constant approach
- Biasing Circuit: Review and Insight

Further discussion on P and Z

- $H(s) = \frac{H^0 + H^1 \tau s}{1+\tau s} = H^\circ \frac{1+\frac{H^1}{H^0}\tau s}{1+\tau s}$
- $P = -\frac{1}{\tau}$
- $Z = -\frac{1}{H^1} = \frac{H^0}{H^0}\tau\left(-\frac{1}{\tau}\right)$
- $Z = \frac{H^0}{H^1}P$
- This tells us that we can look at the sign of H0 and H1 to infer the location of zeros and poles
- The ratio also tells whether poles or zeros come earlier.

Further discussion on P and Z

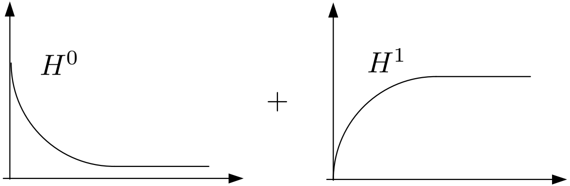
	$\left\|\frac{H_0}{H_1}\right\| < 1$	$\left\|\frac{H_0}{H_1}\right\| > 1$
$\frac{H_0}{H_1} > 0$	P and Z in same half plane $\left\|\frac{Z}{P}\right\| < 1$	P and Z in same half plane $\left\|\frac{Z}{P}\right\| > 1$
$\frac{H_0}{H_1} < 0$	P and Z in opposite half planes $\left\|\frac{Z}{P}\right\| < 1$	P and Z in opposite half planes $\left\|\frac{Z}{P}\right\| > 1$

Further discussion on P and Z

$$H(s) = \frac{H^0 + H^1 \tau S}{1+\tau S} = H^0 \frac{1}{1+\tau S} + H^1 \frac{\tau S}{1+\tau S} \tag{1}$$

Low path filter High path filter

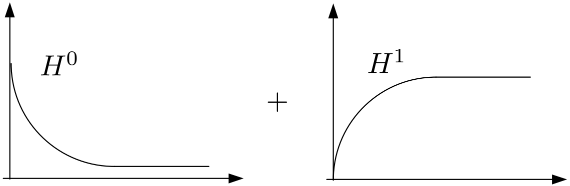
$$s(t) = H^0 \left(1 - e^{-t/\tau}\right) u(t) + H^1 e^{-t/\tau} u(t) \tag{2}$$



The final waveform also tells you the existence of poles and zeros.

Further discussion on P and Z

Transient Simulation



	$\lvert\frac{H_0}{H_1}\rvert < 1$	$\lvert\frac{H_0}{H_1}\rvert > 1$
$\frac{H_0}{H_1} < 0$	P and Z in opposite HP $\lvert Z\rvert < \lvert P\rvert$	P and Z in opposite HP $\lvert Z\rvert > \lvert P\rvert$

Further discussion on P and Z

	$\lvert\frac{H_0}{H_1}\rvert < 1$	$\lvert\frac{H_0}{H_1}\rvert > 1$
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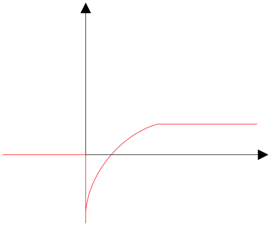
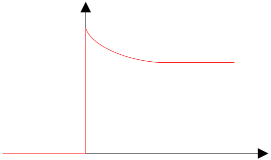


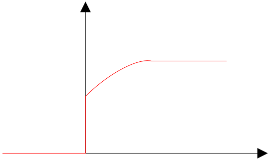
Opposite sign resulting in undershoot, left half plane pole and right half-plane zero.

Further discussion on P and Z



$$0 < \frac{H_0}{H_1} < 1$$

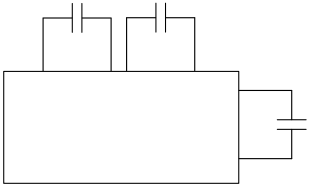
Left half plane low frequency zero that happens before the pole in a first order system.



$$1 < \frac{H_0}{H_1}$$

Zero happens after the pole freq.

Nth order system



$$H(s) = \frac{a_0 + a_1 S + a_2 S^2 + \ldots}{1 + b_1 S + b_2 S^2 + \ldots} \tag{3}$$

- Only the caps and inductors produces S.
- To get a_1 we have to have a cap (or inductor)

Nth order system

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- We can also infer that the s^2 term comes from two capacitors.
- $H(s) = \frac{a_0+\left(\sum_{i=1}^{N} \alpha_1^i C_i\right)s+\left(\sum_i^{1\leqslant i<j\leqslant N} \alpha_j^{ij} C_i C_j\right)s^2+\ldots}{1+\left(\sum_{i=1}^{N} \beta_1^i C_i\right)s+\left(\sum_i^{1\leqslant i<j\leqslant N} \sum_j^{ij} \beta_2^{ij} C_i C_j\right)s^2+\ldots}$
- If we set all c's except ci as zeros
- $H_i(s) = \frac{a_0+\alpha_1^i C_i s}{1+\beta_1^i C_i s}$
- $\tau_i^0 = R_i^0 C_i$
- $\beta_1^i = R_i^0$
- $b_1 = \sum_{i=1}^{N} \tau_i^0$

N th order system

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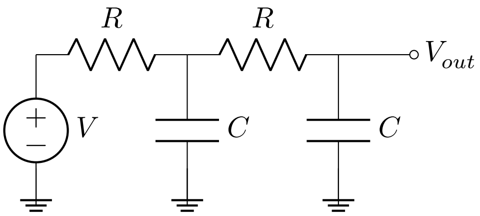
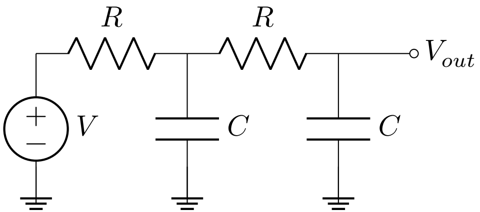
- $b_1 = \sum_{i=1}^{N} \tau_i^0$
- coefficient b_1 is the sum of all zero valued time constant.
- $a_1 = \sum_{i=1}^{N} \tau_i^0 H^i$
- This allows us to find out the dominant time constant.

N th order system



It is a two cap system, therefore the number of poles are two.
It is a system with an infinite response of zero if we short both capacitors, so there is no zeros.

Nth order system



- $H(S) = \frac{1}{1+3RCS+(RC)^2S^2}$
- $b_1 = \sum_{i=1}^{N} \tau_i^0 = \tau_1^0 + \tau_2^0$
- $\tau_1^0 = RC$
- $\tau_2^0 = 2RC$
- $b_1 = 3R_c$

Nth order system

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$$H(S) = \frac{1}{1 + 3RCS + (RC)^2 S^2} \tag{4}$$

- If the impedance seen by one cap does not change as we open or short the other cap, we say that the two-time constant are uncoupled to each other, and the expression can be written as

$$H(s) = \frac{H^0}{(1 + \tau_1 S)(1 + \tau_2 S)} \tag{5}$$

Nth order system

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$$H(S) = \frac{1}{1 + 3RCS + (RC)^2S^2} \tag{6}$$

- We now have the $3RC$ term. The question is whether it is true in this case to determine $(RC)^2S^2$

$$H(s) = \frac{1}{\left(1+\tau_1^0 s\right)\left(1+\tau_2^0 s\right)} = \frac{1}{1+\left(\tau_1^0+\tau_2^0\right)s+\tau_1^0\tau_2^0 s^2} \tag{7}$$

$$\tau_1^0\tau_2^0 = 2(RC)^2 \tag{8}$$

N th order system

We now know how to calculate a_1, b_1, and a_0.

$$H(s) = \frac{a_0 + \left(\sum_{i=1}^{N} \alpha_1^i C_i\right) s + \left(\sum_i^{1 \leqslant i < j \leqslant N} \alpha_j^{ij} C_i C_j\right) s^2 + \ldots}{1 + \left(\sum_{i=1}^{N} \beta_1^i C_i\right) s + \left(\sum_i^{1 \leqslant i < j \leqslant N} \sum_j \beta_2^{ij} C_i C_j\right) s^2 + \ldots} \tag{9}$$

A_0 is the zero frequency response. B_1 is the summation of time constant A_1 can be obtained from infinite time response.

Nth order system

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$$b_2 = \sum_{i}^{i<j<N} \sum_{j}^{0} \tau_j^i \tag{10}$$

- That means you don't repeat $\tau 12$ and $\tau 21$
- τ_j^i means the time constant of element j when element I is infinite frequency

$$a_2 = \sum \sum_{j}^{i<j<N} \tau_i^0 \tau_j^i H^{ij} \tag{11}$$

- We can expect b_n is a multiple summation of the product of many time constants

Nth order system

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2} \tag{12}$$

$$H(s) = \frac{H^0 + (H^1 R_1^0 C_1 + H^2 R_2^0 C_2)s + (\alpha_2^{12} C_1 C_2)s^2}{1 + (R_1^0 C_1 + R_2^0 C_2)s + (\beta_2^{12} C_1 C_2)s^2} \tag{13}$$

- We notice that relabeling C_1 as C_2 and vice versa should not change the derived transfer function.
- So that $\alpha_2^{12} = \alpha_2^{21}$ and $\beta_2^{12} = \beta_2^{21}$
- R_2^1 is the resistance seen by C_2 (the subscript) when C_1 (the superscript) is infinite valued (shorted).

Nth order system

$$H(s) = \frac{H^0 + (H^1 R_1^0 C_1 + H^2 R_2^0 C_2)s + (\alpha_2^{12} C_1 C_2)s^2}{1 + (R_1^0 C_1 + R_2^0 C_2)s + (\beta_2^{12} C_1 C_2)s^2} \tag{14}$$

I want to determine the value of α_2^{12} and β_2^{12}.
It should be valid for $C_1 \to \infty$.
If I short C_1, $C_1 \to \infty$.

$$H(s)|_{C_1 \to \infty} = \frac{C_1 s \cdot (H^1 R_1^0 + \alpha_2^{12} C_2 s)}{C_1 s \cdot (R_1^0 + \beta_2^{12} C_2 s)} = H^1 \cdot \frac{1 + \frac{\alpha_2^{12}}{H^1 R_1^0} C_2 s}{1 + \frac{\beta_2^{12}}{R_1^0} C_2 s} \tag{15}$$

It should be the same as a 1 cap system.
Therefore, $\beta_2^{12} = R_1^0 R_2^1$ and $b_2 = R_1^0 C_1 R_2^1 C_2 = \tau_1^0 \tau_2^1$.

$\mathbf{N}^{th}$ order system

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$$H(s) = \frac{H^0 + (H^1 R_1^0 C_1 + H^2 R_2^0 C_2)s + (\alpha_2^{12} C_1 C_2)s^2}{1 + (R_1^0 C_1 + R_2^0 C_2)s + (\beta_2^{12} C_1 C_2)s^2} \tag{16}$$

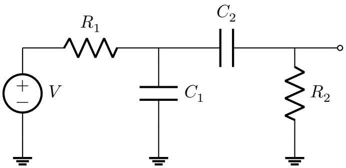
We determine a_2 in by setting both C_1 and C_2 to infinity

- $C_1 \to \infty$ and $C_2 \to \infty$, $H^{12} = \frac{\alpha_2^{12}}{\beta_2^{12}}$
- $\beta_2^{12} = R_1^0 R_2^1$
- Therefore, $\alpha_2^{12} = R_1^0 R_2^1 H^{12}$.
- $a_2 = R_1^0 C_1 R_2^1 C_2 H^{12} = \tau_1^0 \tau_2^1 H^{12}$
- H_{12} is the low-frequency input-output transfer constant with the reactive elements 1 and 2 at their infinite value (C_1 and C_2 shorted).

Nth order system

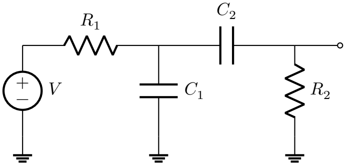
- $b_2 = \tau_1^0 \tau_2^1 = \tau_2^0 \tau_1^2$
- $a_2 = \tau_1^0 \tau_2^1 H^{12} = \tau_2^0 \tau_1^2 H^{12}$
- $b_1 = \tau_1^0 + \tau_2^0$
- $a_1 = \tau_1^0 H^1 + \tau_2^0 H^2$

Nth order system



- Two pole and one zero
- How do we test this?
- $H^0 = 0$
- $\tau_1^0 = R_1 C_1$
- $\tau_2^0 = (R_1 + R_2)\, C_2$

Nth order system

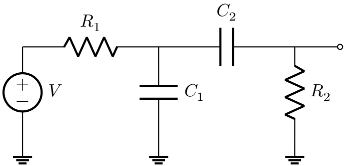


- $\tau_2^1 = R_2 C_2$
- $\tau_1^2 = (R_1 \parallel R_2)\, C_1$
- $\tau_1^0 \tau_2^1 = R_1 C_1 R_2 C_2 = \tau_2^0 \tau_1^2$
- $\tau_2^0 \tau_1^2 = (R_1 + R_2)\, C_2 \frac{R_1 R_2}{R_1+R_2} C_1$
- $\tau_2^0 = (R_1 + R_2)\, C_2$

- $H^1 = 0$
- $H^2 = \frac{R_2}{R_2+R_1}$
- $H^{12} = 0$

$$H(s) = \frac{R_2 C_2 S}{1 + (R_1 C_1 + R_1 C_2 + R_2 C_2)\, S + R_1 C_1 R_2 C_2 S^2} \tag{17}$$

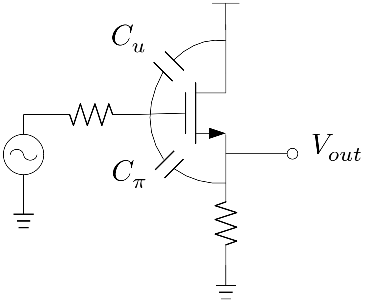
Nth order system



- $H(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2}$
- $b_2 = \tau_1^0 \tau_2^1 = \tau_2^0 \tau_1^2$
- $a_2 = \tau_1^0 \tau_2^1 H^{12} = \tau_2^0 \tau_1^2 H^{12}$
- $b_1 = \tau_1^0 + \tau_2^0$
- $a_1 = \tau_1^0 H^1 + \tau_2^0 H^2$

Nth order system

- Two poles and one zero
- $H^0 = \frac{R_2}{R_2 + r_m}$
- $\tau_\pi^0 = C_\pi \frac{R_1 + R_2}{1 + g_m R_2}$
- $\tau_\mu^0 = C_\mu R_1$
- $\tau_\mu^\pi = C_\mu \left(R_1 \parallel R_2\right)$
- $H^\mu = 0$
- $H^\pi = \frac{R_2}{R_1 + R_2}$
- $H^{\mu\pi} = 0$



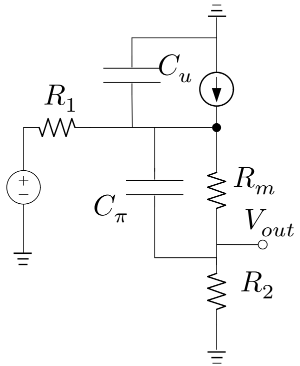
Nth order system

- Now we can write out the transfer function. If we assume that $R_2 >> R_1$
- And $R_2 >> R_m$

$$H^0 = \frac{R_2}{R_2 + r_m} \qquad (18)$$

$$\tau_\pi^0 = C_\pi \frac{R_1 + R_2}{1 + g_m R_2} = r_m C_\pi \qquad (19)$$

$$\tau_\mu^0 = C_\mu R_1 \qquad (20)$$



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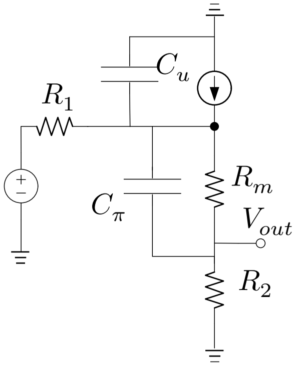
$$\tau_\mu^\pi = C_\mu \left(R_1 \parallel R_2\right) = C_\mu R_1 \tag{21}$$

$$H^\mu = 0 \tag{22}$$

$$H^\pi = \frac{R_2}{R_1 + R_2} \tag{23}$$



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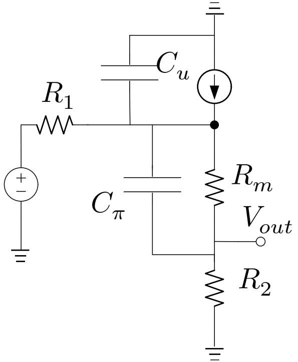
$$\tau_{\pi}^{0}\tau_{\mu}^{\pi} = r_m R_1 C_\pi C_\mu \quad (24)$$

$$H^{\pi}\tau_{\pi}^{0} = r_m C_\pi \frac{R_2}{r_m + R_2} = H^0 r_m C_\pi \quad (25)$$



$$H(S) = H^0 \frac{1 + r_m C_\pi S}{1 + (r_m C_\pi + R_1 C_\mu) S + r_m C_\pi R_1 C_\mu S^2} \quad (26)$$

Nth order system

$$H(S) = H^0 \frac{1 + r_m C_\pi S}{1 + \left(r_m C_\pi + R_1 C_\mu\right) S + r_m C_\pi R_1 C_\mu S^2} = \frac{H^0}{1 + R_1 C_\mu S} \tag{27}$$

- This essentially tells us the dominate pole is the C_μ, because C_π shares current between the capacitor and the resistor, so that it tells us to improve the bandwidth of operation, we need to use a inductor or some topology to cancel the effect of C_μ

Bandwidth Estimation

- The whole system can be expressed as the product of a high pass and a low pass transfer function.
- If I'm designing an analog circuit, I can use transfer function to estimate bandwidth, assuming I care about the lowpass one.
- For low frequency system, in many cases we can assume a zeroless system.
- For example, in common-source stage and the source-follower stage, the zero's frequencies are comparable to the cut-off frequency of the transistor itself

Bandwidth Estimation

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- $H(s) \approx \frac{a_0}{1+b_1 s+b_2 s^2+...+b_n s^n}$
- At dc $(\omega = 0)$, the only term in the denominator that matters is the leading 1.
- As the frequency goes up and starts approaching the wh, the first term that becomes non-negligible would be $b1$, so in the vicinity of the ωh
- The system is pretty much
- $H(s) \approx \frac{a_0}{1+b_1 s}$
- This tells us, in order to find out the cutoff frequency,
- We calculate b1

Bandwidth Estimation

- $\omega_h \approx \frac{1}{b_1} = \frac{1}{\sum_{i=1}^{N} \tau_i^0}$
- $\frac{a_1}{1+b_1 S+b_2 S^2} \Rightarrow H(j\omega) = \frac{a_0}{(1-b_2\omega^2)+j\omega b_1}$
- It is a conservative estimation of bandwidth