1. $E[X] = \frac{a+b}{2} = \frac{1}{2}$

$E[Y] = E[kX^2] = kE[X^2]$

To find $E[X^2]$:

PDF: $f(x) = \begin{cases} \frac{1}{3} & \text{on } (-1,2) \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow E[X^2] = \int_{-1}^{2} x^2 \cdot \frac{1}{3}\, dx$

$= \frac{1}{9} x^3 \Big|_{x=-1}^{2} = 1$

$\Rightarrow E[Y] = k$

$\Rightarrow Cov[X,Y] = E[(x-\frac{1}{2})(kx^2-k)]$

$= kE[(x-\frac{1}{2})(x^2-1)]$

$= kE[x^3 - \frac{1}{2}x^2 - x + \frac{1}{2}]$

$= k(E[x^3] - \frac{1}{2}E[x^2] - E[x] + E[\frac{1}{2}])$

since $E[x^3] = \int_{-1}^{2} x^3 \cdot \frac{1}{3} dx$

$= \frac{1}{12} x^4 \Big|_{x=-1}^{2} = \frac{5}{4}$

$\Rightarrow Cov[X,Y] = k(\frac{5}{4} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2})$

$= \frac{3}{4}k \neq 0 \Rightarrow$ correlated

$E[XY] = kE[x^3] = \frac{5}{4}k \neq 0 \Rightarrow$ not orthogonal

since correlated $\Rightarrow$ not independent

2. $E[x] = \frac{1}{2}$

$E[Y] = kE[x^2] + c = k + c$

$\Rightarrow Cov[X,Y] = E[(x-\frac{1}{2})(kx^2+c-k+c)]$

$= \frac{3}{4}k \neq 0 \Rightarrow$ correlated and dependent

$E[X,Y] = E[x(kx^2+c)] = \frac{5}{4}k + \frac{1}{2}c$

if $c = -\frac{5}{2}k$, then orthogonal

otherwise, not orthogonal

3. $E[X] = \frac{a+b}{2} = 0$

$E[Y] = k \cdot \int_{-1}^{1} \frac{1}{2} x^2 dx$

$= \frac{1}{3}k$

$\Rightarrow Cov[X,Y] = E[X \cdot (kx^2 - \frac{1}{3}k)]$

$= kE[x^3 - \frac{1}{3}x]$

$= k(E[x^3] - \frac{1}{3}E[x])$

$= kE[x^3]$

since $E[x^3] = \int_{-1}^{1} x^3 \cdot \frac{1}{2} dx$

$= 0$

$\Rightarrow Cov[X,Y] = 0 \Rightarrow$ uncorrelated

$E[X \cdot Y] = E[kx^3] = 0 \Rightarrow$ orthogonal

$\Rightarrow$ X, Y are independent

4.

If two random variables X and Y are uncorrelated, it means that their covariance is zero: Cov(X, Y) = 0. This implies that there is no linear relationship between X and Y, and the fluctuations of one variable do not have a systematic effect on the fluctuations of the other variable. However, uncorrelated variables can still be dependent or related in a nonlinear or non-monotonic manner.
On the other hand, orthogonality is a specific concept used in the context of vectors in a vector space. In this context, two vectors are orthogonal if their dot product is zero. Orthogonality signifies a perpendicular or independent relationship between the vectors.
In the case of random variables, orthogonality is not directly applicable. While uncorrelated variables can be considered "uncorrelated in a linear sense," it doesn't necessarily imply orthogonality in the vector space sense.
In conclusion, uncorrelated random variables do not necessarily mean they are orthogonal. Uncorrelated variables imply a lack of linear relationship, whereas orthogonality refers to a specific geometric relationship between vectors.

2. (1)

$$SSR(B) = tr\left(\frac{1}{2}(yy^T - XBy^T - yB^TX^T + XBB^TX^T)\right)$$

(2)

$$\frac{\partial SSR(B)}{\partial B} = \frac{1}{2}\left(-X^Ty - X^Ty + X^TXB + X^TXB\right) = 0$$

(3)

$$B = (X^TX)^{-1}X^Ty$$

$$\Rightarrow \hat{\beta}_k = (X^TX)^{-1}X^Ty_k$$