Lecture 11: Transformation of Variables

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2023-06-06

Recap: Factor Analysis (ANOVA)

- Partition of the response-variable sum of squares into "explained" and "unexplained"
- Procedures for fitting and testing linear models in which the explanatory variables are categorical.
- Suppose that there are no quantitative explanatory variables—only a single factor in your model:

$$Y_i = \beta_0 + \gamma_2 E_{i2} + \gamma_3 E_{i3} + \epsilon_i$$

- $E(\hat{Y}_i)$ in each category (group, level of factor) is the population group mean, can be denoted by μ_j

Recap: One-way ANOVA

Education(E)	Indicator	E(Y)
1 (HS)	$E_2 = E_3 = 0$	$\mu_1 = \beta_0$
2 (B.S.)	$E_2 = 1, E_3 = 0$	$\mu_2 = \beta_0 + \gamma_2$
3 (Advanced)	$E_2 = 0, E_3 = 1$	$\mu_3 = \beta_0 + \gamma_3$

- One-way ANOVA focuses on testing for differences among group means.
- One-way ANOVA examines the relationship between a quantitative response variable and a factor.
- H_0: $\gamma_2 = \gamma_3 = 0$, which implies $\mu_1 = \mu_2 = \mu_3$
- F-statistic for the regression of the response variable on $0/1$ dummy regressors constructed from the factor tests for differences in the response means across levels of the factor.

Recap: One-way ANOVA Table

Source	df	Sum of Squares	Mean Squares	F
Treatment	c-1	SSR	MSR	$F = \frac{MSR}{MSE}$
Error	n-c	SSE	MSE	
Total	n-1	SST		

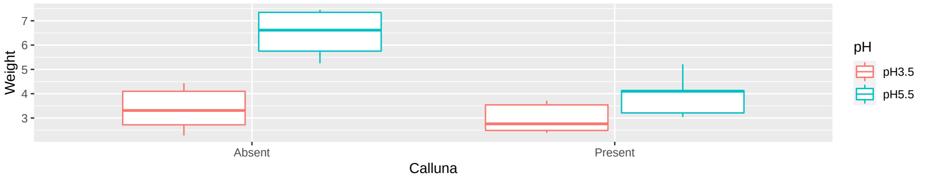
- $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$
- $SSR = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 = \sum_{j=1}^{c} n_j(\hat{\mu}_j - \bar{y})^2$
- $SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{j=1}^{c}(n_j - 1)s_j^2$

Where sample variance: $s_j^2 = \dfrac{\sum_{i \in \text{group}_j}(y_i - \mu_j)^2}{n_j - 1}$

Recap: Two-way ANOVA

- Two-way ANOVA allows us to determine whether there are significant differences between the effects of two categorical variables.
- Change of response variable may depend on
  - the level of the categorical variable (additive model)
  - the level of the interaction model.

```
festuca <- read.table(file = "festuca.txt",header = T)
ggplot(data = festuca, aes(x = Calluna, y = Weight,
                           colour = pH)) +
  geom_boxplot()
```



Recap: ANOVA and Linear Regression

Model	Terms	Regression Sum of Squares
1	pH, Calluna, pH*Calluna	SSR_1
2	pH, Calluna	SSR_2
3	pH	SSR_3
4	Calluna	SSR_4

The four models will produce the two-way ANOVA table

Source	Model Contrasted	df	Sum of Squares	Mean Squares	F
pH	2-4	1	$SSR_2 - SSR_4$	MSR	$F = \frac{MSR}{MSE}$
Calluna	2-3	1	$SSR_2 - SSR_3$	MSR	$F = \frac{MSR}{MSE}$
Interaction	1-2	1	$SSR_1 - SSR_2$	MSR	$F = \frac{MSR}{MSE}$
Error	SSE from Model 1	n-4	SSE	MSE	
Total		n-1	SST		

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- When and Why?

  Original variables violates one or more of the standard regression assumptions.

  - Linearity of the model
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Our primary focus:

- Polynomial Models
- Ordinal Categorical Predictors

Data: Smoking and FEV (Lung Capacity)

Sample of 654 youths, aged 3 to 19, in the area of East Boston during middle to late 1970's. The variables are

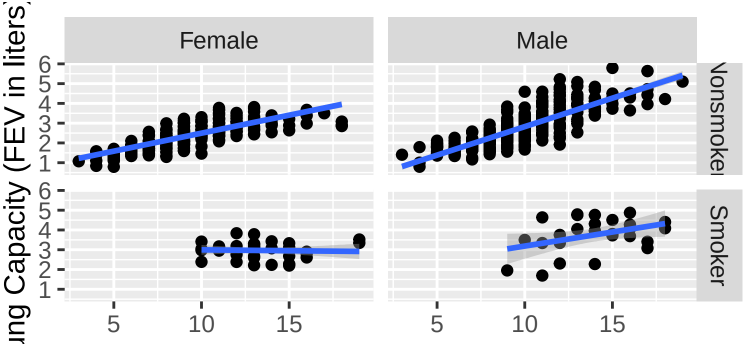
- age: Subject's age in years
- fev: Lung capacity of subject, measured by forced expiratory volume (abbreviated as FEV), the amount of air an individual can exhale in the first second of forceful breath in liters
- ht: Subject's height in inches
- sex: Gender of the subject coded as: 0 = Female, 1 = Male
- smoke:Smoking status coded as: 0 = Nonsmoker, 1 = Smoker

```
fevdata = read.table("fevdata.txt", header=TRUE)
fevdata$sex = factor(fevdata$sex, labels=c("Female","Male"))
fevdata$smoke = factor(fevdata$smoke, labels=c("Nonsmoker","Sm
```

Lung Capacity Dataset

```
ggplot(fevdata, aes(x = age, y = fev)) +
geom_point() + facet_grid(smoke~sex) +
geom_smooth(method='lm') + xlab("Age (years)") +
ylab("Lung Capacity (FEV in liters)")
```

```
## `geom_smooth()` using formula 'y ~ x'
```



Test Non-linearity: Female Nonsmokers

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- To test non-linearity, one can add some nonlinear function of age, e.g. age^2 and see if the nonlinear term is significant.

```
f.nonsmokers = subset(fevdata, sex == "Female" & smoke == "Nonsmoker")
summary(lm(fev ~ age + I(age^2), data=f.nonsmokers))$coef
```

```
##                Estimate  Std. Error   t value     Pr(>|t|)
## (Intercept) -0.50745967 0.210390388 -2.411991 1.651843e-02
## age          0.43979072 0.042969068 10.235054 4.714677e-21
## I(age^2)    -0.01297867 0.002120258 -6.121267 3.176433e-09
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- There is significant evidence of non-linearity.

Polynomial Models

- Fitting the polynomial model:

$$\text{fev} = \beta_0 + \beta_1 \text{age} + \beta_2 \text{age}^2 + \epsilon$$

doesn't mean we believe it is correct. It is just a decent approximation to the true underlying nonlinear model:

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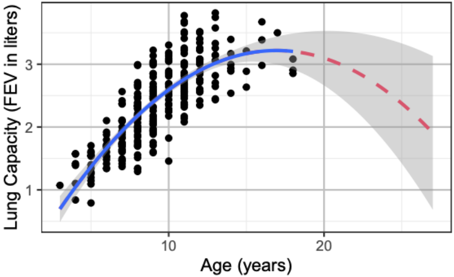
$$\text{fev} = f(\text{age}) + \epsilon$$

- One can try higher-order polynomials

$$\text{fev} = \beta_0 + \beta_1 \text{age} + \beta_2 \text{age}^2 + \cdots + \beta_k (\text{age})^k + \epsilon$$

if lower-order ones don't capture the nonlinear pattern well.

Caution: Don't trust extrapolation!



Does lung capacity decrease after children turn adults?

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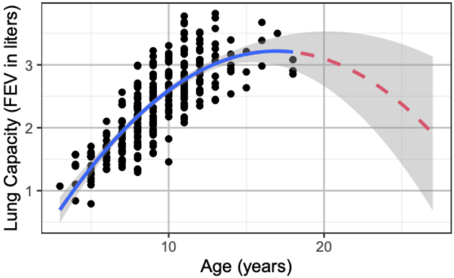

Does lung capacity decrease after children turn adults?

- We are not sure whether the nonlinear relations is a polynomial (it's just an approximation!).
- Extrapolating the model beyond the range of data is dangerous.

Test of Non-linearity: Male Nonsmokers

```
m.nonsmokers = subset(fevdata, sex == "Male" & smoke == "Nonsmoker")
summary(lm(fev ~ age + I(age^2), data=m.nonsmokers))$coef
```

```
##                Estimate  Std. Error   t value     Pr(>|t|)
## (Intercept) 0.143622429 0.298683403 0.4808517 6.309644e-01
## age         0.245874713 0.059137388 4.1576864 4.175341e-05
## I(age^2)    0.002057928 0.002820682 0.7295854 4.662000e-01
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- The large P-value 0.466 for the age^2 means little evidence of non-linearity
- This just means fev is approximaltely linear in age in the range of data for male smokers. Extrapolating the line beyond of the range of data remain dangerous
- The discrepancy in the significance of age^2 between boys and girls is an evidence of age:sex interaction — lung capacities of girls stop growing earlier than boys.

Interactions

```r
nonsmokers = subset(fevdata, smoke == "Nonsmoker")
summary(lm(fev ~ (age + I(age^2))*sex, data=nonsmokers))$coef
```

```
##                      Estimate  Std. Error   t value     Pr(>|t|)
## (Intercept)       -0.50745967 0.263273891 -1.927497 5.440320e-02
## age                0.43979072 0.053769726  8.179151 1.795959e-15
## I(age^2)          -0.01297867 0.002653204 -4.891696 1.295007e-06
## sexMale            0.65108210 0.369659312  1.761303 7.871126e-02
## age:sexMale       -0.19391600 0.074369400 -2.607470 9.354961e-03
## I(age^2):sexMale  0.01503659 0.003611741  4.163254 3.611388e-05
```

- For girls: $\hat{\text{fev}} = -0.507 + 0.44\text{age} - 0.013(\text{age})^2$
- For boys: $\hat{\text{fev}} = (-0.507 + 0.651) + (0.44 - 0.194)\text{age} + (0.015 - 0.013)\text{age}^2 = 0.144 + 0.246\text{age} + 0.002(\text{age})^2$

Interpretation of Coefficients in a Polynomial Model

- Recall in MLR, we said β_j is the mean change in the response Y when X_j is increased by one unit holding other X_i's constant.
- For a model that involves polynomial terms like:

$$Y = \beta_0 + \underbrace{\beta_1 X_1 + \beta_2 X_1^2}_{\text{Polynomial of } X_1} + \beta_3 X_3 + \cdots + \beta_p X_p + \epsilon$$

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- it makes no sense to interpret a single coefficient for a polynomial like β_1 or β_2 since it's impossible to change X_1 while holding X_1^2 constant
- Interpret the polynomials all together: the mean of Y change with X_1 following the curve $\beta_1 X_1 + \beta_2 X_1^2$ holding other X_i's constant.

Recap: Interpretation of Coefficients of Indicator Variables

$S = \beta_0 + \beta_1 X + \gamma_2 E_2 + \gamma_3 E_3 + \alpha M_1 + \epsilon$

- Interpret γ_2 as the mean difference in salary S between HS graduates and those with a Bachelor's degree if they were at the same management status and had the same years of experience.

Test of Non-linearity: Smokers

```r
m.smokers = subset(fevdata, sex == "Male" & smoke == "Smoker")
summary(lm(fev ~ age + I(age^2), data=m.smokers))$coef
```

```
##                Estimate Std. Error   t value   Pr(>|t|)
## (Intercept) -6.05764029 4.77931185 -1.267471 0.21766965
## age          1.31395132 0.70557307  1.862247 0.07539297
## I(age^2)    -0.04253231 0.02550048 -1.667902 0.10889460
```

```r
f.smokers = subset(fevdata, sex == "Female" & smoke == "Smoker")
summary(lm(fev ~ age + I(age^2), data=f.smokers))$coef
```

```
##                Estimate Std. Error   t value    Pr(>|t|)
## (Intercept)  5.98969790  1.9596750  3.056475 0.004204225
## age         -0.43746338  0.2843111 -1.538679 0.132627095
## I(age^2)     0.01536442  0.0101347  1.516022 0.138246407
```

- age^2 is insignificant for male smokers or female smokers, which might be just due to the small sample size that makes it difficult to detect the non-linearity.

Interactions

```
smokers = subset(fevdata, smoke == "Smoker")
summary(lm(fev ~ (age + I(age^2))*sex, data=smokers))$coef
```

```
##                        Estimate Std. Error   t value   Pr(>|t
## (Intercept)          5.98969790 2.79718654  2.141329 0.036388
## age                 -0.43746338 0.40581786 -1.077980 0.285430
## I(age^2)             0.01536442 0.01446599  1.062106 0.292515
## sexMale            -12.04733819 4.53161519 -2.658509 0.010086
## age:sexMale          1.75141470 0.66462640  2.635187 0.010726
## I(age^2):sexMale    -0.05789673 0.02389848 -2.422612 0.018498
```

Male smokers still have significantly larger lung capacities than female nonsmokers, though neither show significant non-linearity

Question: Can we remove the square term age^2 for smokers?

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```
anova(lm(fev ~ (age + I(age^2))*sex, data=smokers),
  lm(fev ~ age*sex, data=smokers))
```

```
## Analysis of Variance Table
##
## Model 1: fev ~ (age + I(age^2)) * sex
## Model 2: fev ~ age * sex
##   Res.Df    RSS Df Sum of Sq      F  Pr(>F)
## 1     59 21.279
## 2     61 23.489 -2   -2.2098 3.0635 0.05422 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

Ordinal Categorical Predictors; Salary Data

- An ordinal variable is a categorical variable with ordered categories.
  - e.g., E = education (HS only, BA or BS, advance degree) in the salary survey data is an ordinal variable

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Ordinal Categorical Predictors; Salary Data

- An ordinal variable is a categorical variable with ordered categories.
  - e.g., E = education (HS only, BA or BS, advance degree) in the salary survey data is an ordinal variable
- When we create indicators for E and include them in a model, we ignore the fact that the 3 education levels are ordered. Therefore, the estimated salary might not be ordered by education levels.
- We can incorporate the ordinal info of a ordinal predictor by assigning a score to each its category like

$$E = \begin{cases} 1 & \text{if HS only} \\ 2 & \text{if Bachelor's degree} \\ 3 & \text{if Advanced degree} \end{cases}$$

Ordinal Categorical Predictors: Salary Data

$S = \beta_0 + \beta_1 X + \beta_2 E + \alpha M_1$

This way, the fitted salary will always be ordered by education levels:

- a BS increases salary by β_1 from HS
- an advanced degree increases salary by another β_1 from BS

```
salary = read.table("salary.txt", header=TRUE)
salary$Man = factor(salary$M,
                    labels=c("No manager","Manager"))
lmsalary = lm(S ~ X+E+Man, salary)
summary(lmsalary)$coef
```

```
##                Estimate Std. Error   t value     Pr(>|t|)
## (Intercept) 6963.4777  665.69473 10.460467 2.876500e-13
## X            570.0874   38.55905 14.784789 3.000130e-18
## E           1578.7503  262.32162  6.018377 3.737405e-07
## ManManager  6688.1299  398.27563 16.792717 3.043277e-20
```

Flexibilty with Oridinal Predictors

- If one believe the salary gap between a Bachelor's deg. and HS diploma is greater than that between a Bachelor's deg. and an adv. deg., one may try a different scoring (1, 2.5, 3), i.e.,

$$E = \begin{cases} 1 & \text{if HS only} \\ 2.5 & \text{if Bachelor's degree} \\ 3 & \text{if Advanced degree} \end{cases}$$

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This way, the fitted salary will always be ordered by education levels:

- a BS increases salary by $1.5\beta_1$ from HS
- an advanced degree increases salary by another $0.5\beta_1$ from BS

R Example

```r
salary$E.score1 = ifelse(salary$E == 2, 2.5, salary$E)
summary(lm(S ~ M + E.score1 + X, data=salary))$coef
```

```
##              Estimate Std. Error   t value     Pr(>|t|)
## (Intercept) 6401.1019  561.45563 11.400904 1.943394e-14
## M           6741.1167  330.93922 20.369652 2.183795e-23
## E.score1    1703.2956  204.22077  8.340462 1.882827e-10
## X            562.2457   32.00685 17.566421 5.778756e-21
```

- A BS increases mean salary by $1.5 \times 1703 = 2554.5$ than HS
- An advanced degree increases mean salary by another $0.5 \times 1703 = 851.5$ than BS

Another scoring scheme

```r
salary$E.score2 = ifelse(salary$E >= 2, salary$E + 1, salary$E)
summary(lm(S ~ M + E.score2 + X, data=salary))$coef
```

```
##              Estimate Std. Error   t value     Pr(>|t|)
## (Intercept) 7072.138  535.55418 13.205270 1.516893e-16
## M           6711.605  349.58912 19.198552 2.074099e-22
## E.score2    1135.099  148.43586  7.647068 1.750709e-09
## X            565.975   33.81701 16.736401 3.442138e-20
```

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```

- Which model fits better? How to compare?

```r
summary(lm(S ~ M + E.score1 + X, data=salary))$r.squared
```

```
## [1] 0.9493052
```

```r
summary(lm(S ~ M + E.score2 + X, data=salary))$r.squared
```

```
## [1] 0.943712
```

Comparison of Models with Ordinal and Nominal Predictors

- Whatever scoring one uses for E, the ordinal model is always a nested model of that treats E as nominal

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```
anova(lm(S ~ M + E.score1 + X, data=salary),
lm(S ~ M + as.factor(E) + X, data=salary))
```

```
## Analysis of Variance Table
##
## Model 1: S ~ M + E.score1 + X
## Model 2: S ~ M + as.factor(E) + X
##   Res.Df      RSS Df Sum of Sq      F Pr(>F)
## 1     42 50750487
## 2     41 43280719  1   7469768 7.0761 0.0111 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

Pros and Cons of Using Ordinal Predictors

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Cons:

- The choice of scores seems arbitrary
- If the scores are not chosen properly, the model might not be reasonable.