Solution for Homework 1

Problems:

1. [10!]

   (a) $\boxed{x(t) = |\sin(t)|}$

   (b)

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |\sin(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |\frac{1-\cos(2t)}{2}| dt \\ &= \boxed{\infty} \end{aligned}$$

   $x(t)$ is not an energy signal, but is a power signal.

2. [12!]

   (a)

$$E = \int_{-\infty}^{\infty} |x_1(t)|^2 \,\mathrm{d}t = \int_0^{\infty} e^{-4t} \mathrm{d}t = \boxed{\frac{1}{4}}$$

$$P = \boxed{0} \quad \text{since } E < \infty$$

   (b)

$$E = \int_{-\infty}^{\infty} |x_3(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{\infty} \cos^2(t) \mathrm{d}t = \boxed{\infty}$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x_3(t)|^2 \,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos^2(t) \mathrm{d}t = \boxed{\frac{1}{2}}$$

3. [14!]

   (a) If T_1/T_2 is rational, then there exists some integers n_1 and n_2 such that $T_1/T_2 = n_2/n_1$. Let $T = n_1 T_1 = n_2 T_2$. Then

$$x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t+n_1T_1) + x_2(t+n_2T_2) = x_1(t) + x_2(t) = x(t)$$

   Thus $x(t)$ is periodic with period T.

   (b) i. The periods of $\sin(\pi t/3)$, $cos(\pi t/4)$, $\sin(\pi t/5)$, and $\sin(\pi t/2)$ are 6, 8, 10, and 4, respectively. Then least common multiple is 120, so one period of $x(t)$ is $\boxed{120}$.

   ii. The ratio of two periods is not rational, so the sum of the two signals is not periodic.

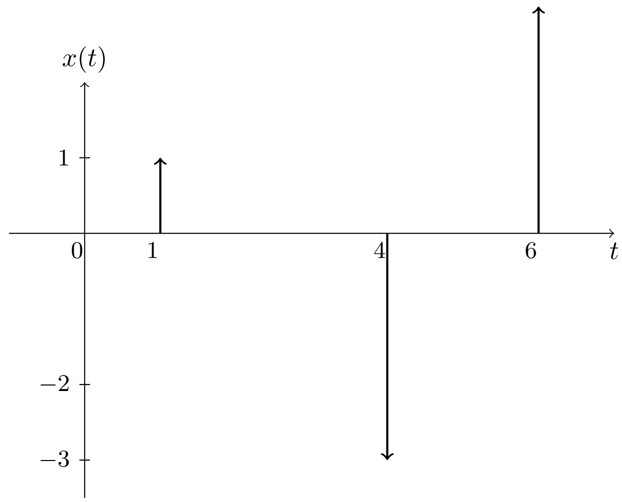
4. [12!]

   (a)

$$x(t) = \boxed{\text{u(t-1) - 3u(t-4)u(6-t)}}$$

   (b)

$$\frac{dx(t)}{dt} = \boxed{\delta(t-1) - 3\delta(t-4) + 3\delta(t-6)}$$



5. [15!]

(a) The system is $\boxed{\text{linear}}$ and $\boxed{\text{stable}}$.

$$y_1(t) = x_1(t-2) + x_1(2-t)$$

$$y_2(t) = x_2(t-2) + x_2(2-t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = x(t-2) + x(2-t)$$

$$y(t) = \alpha x_1(t-2) + \beta x_2(t-2) + \alpha x_1(2-t) + \beta x_2(2-t)$$

$$y(t) = \alpha(x_1(t-2) + x_1(2-t)) + \beta(x_2(t-2) + x_2(2-t)) = \alpha y_1(t) + \beta y_2(t)$$

It is obviously $\boxed{\text{not Memoryless or Causal}}$.
It is $\boxed{\text{not Time-invariant}}$,

$$y_d(t) = x(t-2-d) + x(2-t-d) = x((t-d)-2) + x(2-(t+d) \neq y(t-d)$$

(b) The system is $\boxed{\text{Memoryless}}$ thus $\boxed{\text{Causal}}$. It is also $\boxed{\text{Stable}}$ because the absolute value of the output of *cos* function is bounded at 1.
It is $\boxed{\text{Time invariant}}$,

$$y_d(t) = cos(x(t-d)) = y(t-d)$$

It is $\boxed{\text{not linear}}$,

$$y_1(t) = cos(x_1(t))$$

$$y_2(t) = cos(x_2(t))$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = cos(x(t)) = cos(\alpha x_1(t))cos(\beta x_2(t)) - sin(\alpha x_1(t))sin(\beta x_2(t)) \neq \alpha y_1(t) + \beta y_2(t)$$

(c) it is $\boxed{\text{not Memoryless, Causal, Stable or Time invariant}}$. But it is $\boxed{\text{Linear}}$. It is $\boxed{\text{not Time invariant}}$ because,

$$y_d(t) = \int_{-\infty}^{t/2} x(\tau-d)d\tau = \int_{-\infty}^{t/2-d} x(\tau-d)d(\tau-d) = \int_{-\infty}^{(t-2d)/2} x(s)ds = y(t-2d)$$

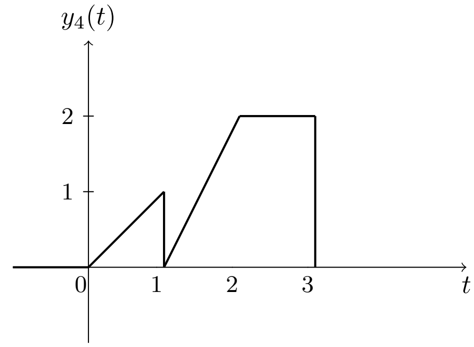
6. [12!]

   (a) $\boxed{\text{No}}$. Causal system's output depends only on values of the input at present and in the past. Look at the second pair, during the period (0, 1), if the system is causal, the output during this period should remain 0. Because the input left to 1 are all the same(all 0).

   (b) $\boxed{\text{No}}$. Look at the first pair and the third pair. $x_2(t) = x_1(t-1)$, but $y_2(t) \neq y_1(t-1)$, so it is not time invariant.

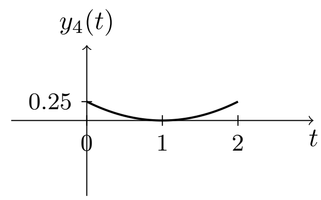
   (c) $\boxed{\text{No}}$. We already proved that the system is not causal in (a), so it is not memoryless, either.

   (d) Use the linearity of the system, cascade the input $x_1(t)$ and $x_3(t)$ to form $x_4(t)$. Then we can get the output $y_4(t)$ based on $y_1(t), y_3(t)$.



7. [10!]

   (a) The sketch is shown below.



   (b) By the scaling property of $\delta(t)$, $x(t) = \delta(t - \frac{1}{2}) + \delta(t-2) - \frac{1}{3}\delta(t - 4/3)$. $s(1/2) = \frac{1}{16}$, $s(2) = \frac{1}{4}$, $s(4/3) = \frac{1}{36}$. By Shifting property, $\int_{-\infty}^{\infty} s(t)x(t)dt = s(1/2) + s(2) - \frac{1}{3}s(4/3) = \boxed{\frac{131}{432} \text{ or } \frac{23}{432}}$

8. [15!]

   (a) Yes

   (b) No

   (c) No

   (d) Yes

   (e) Yes

Optional Problems:

1.

$$\begin{aligned} A &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} e^{-t}dt \\ &= \lim_{T\to\infty} \frac{1}{2T}\left(1-e^{-T}\right) \\ &= \boxed{0} \end{aligned}$$

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} e^{-2t}dt \\ &= \lim_{T\to\infty} \frac{1}{4T}\left(1-e^{-2T}\right) \\ &= \boxed{0} \end{aligned}$$

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{0}^{\infty} e^{-2t}dt \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

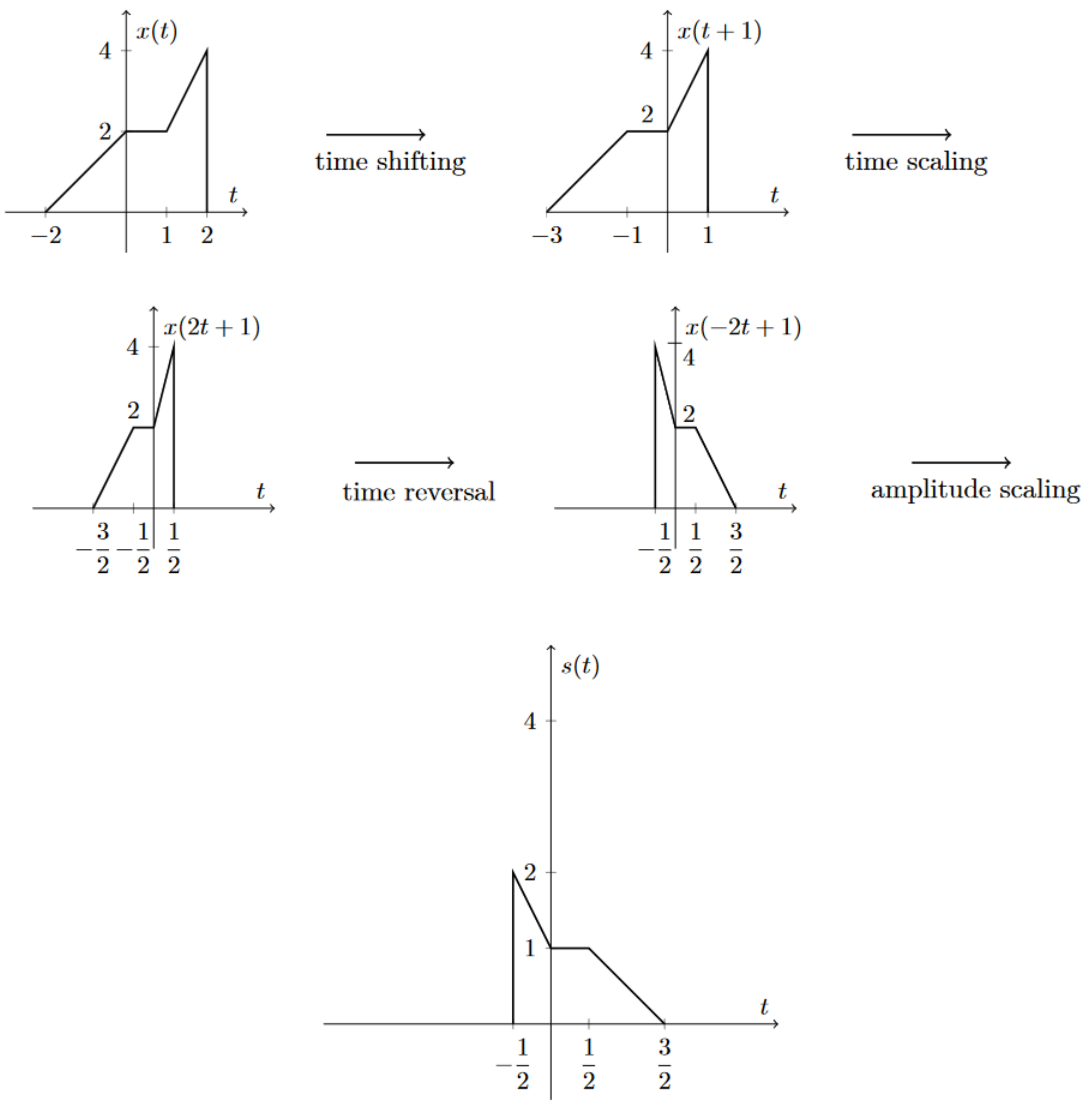
2. (a)

$$x(t) = (t+2)\mathrm{rect}\left(\frac{t+1}{2}\right) + 2\ \mathrm{rect}(t-1/2) + 2t\ \mathrm{rect}(t-3/2)$$

Or,

$$x(t) = \boxed{\begin{cases} t+2 & -2 < t < 0, \\ 2 & 0 \leqslant t \leqslant 1, \\ 2t & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}}$$

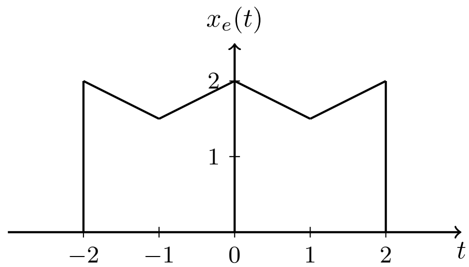
(b)

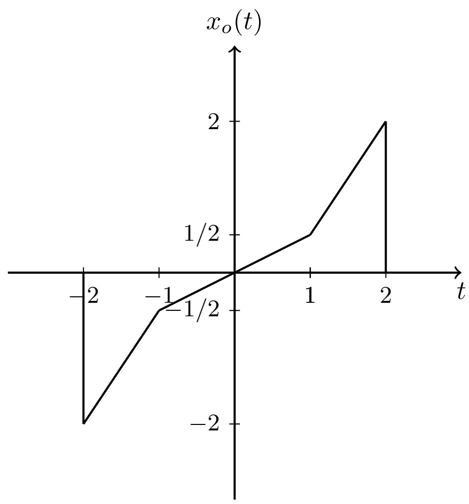

$x(t)$
4
2
t
-2
1
2
time shifting
$x(t+1)$
4
2
t
-3
-1
1
time scaling
$x(2t+1)$
4
2
t
$-\frac{3}{2}$
$-\frac{1}{2}$
$\frac{1}{2}$
time reversal
$x(-2t+1)$
4
2
t
$-\frac{1}{2}$
$\frac{1}{2}$
$\frac{3}{2}$
amplitude scaling
$s(t)$
4
2
1
t
$-\frac{1}{2}$
$\frac{1}{2}$
$\frac{3}{2}$

(c)

$$x_e(t) = \frac{x(t)+x(-t)}{2} = \begin{cases} -t/2+1 & -2 < t \leqslant -1, \\ t/2+2 & -1 < t \leqslant 0, \\ -t/2+2 & 0 < t \leqslant 1, \\ t/2+1 & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$x_o(t) = \frac{x(t)-x(-t)}{2} = \begin{cases} 3t/2+1 & -2 < t \leqslant -1, \\ t/2 & -1 < t \leqslant 1, \\ 3t/2-1 & 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$





3.

We first decompose $x(t)$ and $y(t)$ into sums of exponentials. Thus,

$$x(t) = \frac{1}{2}e^{j(2\pi t/3)} + \frac{1}{2}e^{-j(2\pi t/3)} + \frac{e^{j(16\pi t/3)}}{j} - \frac{e^{-j(16\pi t/3)}}{j},$$

$$y(t) = \frac{e^{j\pi t}}{2j} - \frac{e^{-j\pi t}}{2j}$$

Multiplying $x(t)$ and $y(t)$, we get

$$z(t) = \frac{1}{4j}e^{j(5\pi/3)t} - \frac{1}{4j}e^{-j(\pi/3)t} + \frac{1}{4j}e^{j(\pi/3)t} - \frac{1}{4j}e^{-j(5\pi/3)t}$$
$$- \frac{1}{2}e^{j(19\pi/3)t} + \frac{1}{2}e^{j(13\pi/3)t} + \frac{1}{2}e^{-j(13\pi/3)t} - \frac{1}{2}e^{-j(19\pi/3)t}$$

We see that all complex exponentials are powers of $e^{j(\pi/3)t}$. Thus, the fundamental period is $2\pi/(\pi/3) = 6$ s.

4. Matlab

(a)
```
t = 0:0.1:10;
y = exp(t);
plot(t, y)
```

(b)
```
t = 0:0.1:10;
y = exp(-0.1.*t).*sin(pi.*t);
plot(t, y)
```

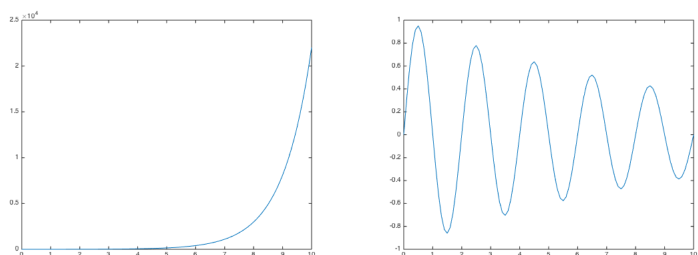


Figure 1: $y(t) = e^t$, $y(t) = e^{-0.1t}sin(\pi t)$.

5. Let $y(t) = x_1(t)x_2(t)$, where $x_1(-t) = -x_1(t)$ and $x_2(-t) = -x_2(t)$.
So $y(-t) = x_1(-t)x_2(-t) = (-x_1(t))(-x_2(t)) = x_1(t)x_2(t) = y(t)$. $y(t)$ is even.

6. Assumption: If $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$. To prove that: The system is causal.

Let us consider an arbitrary signal $x_1(t)$. Then, let us consider another signal $x_2(t)$ which is the same as $x_1(t)$ for $t < t_0$. But for $t > t_0$, $x_2(t) \neq x_1(t)$. Since the system is linear,

$$x_1(t) - x_2(t) \to y_1(t) - y_2(t).$$

Since $x_1(t) - x_2(t) = 0$ for $t < t_0$, by our assumption $y_1(t) - y_2(t) = 0$ for $t < t_0$. This implies that $y_1(t) = y_2(t) fort < t_0$. In other words, the output is not affected by input values for $t \geq t_0$. Therefore, the system is causal.

Assumption: The system is causal. To prove that: If $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$.

Let us assume that the signal $x(t) = 0$ for $t < t_0$. Then we may express $x(t)$ as $x(t) = x_1(t) - x_2(t)$, where $x_1(t) = x_2(t)$ for $t < t_0$. Since the system is linear, the output to $x(t)$ will be $y(t) = y_1(t) - y_2(t)$.

Now, since the system is causal, $x_1(t) = x_2(t)$ for $t < t_0$ implies that $y_1(t) = y_2(t)$ for $t < t_0$. Therefore, $y(t) = 0$ for $t < t_0$.

7. Time-shift transformation will not change the energy or power of a signal.

   (a)

$$\begin{aligned} E[x(-at+b)] = E[x(at)] &= \int_{-\infty}^{\infty} |x(at)|^2 dt \\ &= \int_{-\infty}^{\infty} |x(u)|^2 \frac{1}{|a|} du \\ &= \int_{-\infty}^{\infty} |x(u)|^2 \frac{1}{|a|} du \\ &= \boxed{\tfrac{1}{|a|} E_x} \end{aligned}$$

   (b)

$$\begin{aligned} P[x(-at+b)] = P[x(at)] &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(at)|^2 dt \\ &= \lim_{aT\to\infty} \frac{1}{2T} \int_{-aT}^{aT} |x(u)|^2 \frac{1}{a} du \\ &= \lim_{T'\to\infty} \frac{a}{2T'} \int_{-T'}^{T'} |x(u)|^2 \frac{1}{a} du \\ &= \boxed{P_x} \end{aligned}$$