Homework2 Solutions

Problems:

1. (a) This system is $\boxed{\text{time-invariant}}$ as $y_1(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{s} x_1(\tau-5)d\tau\right]ds = \int_{-\infty}^{t}\left[\int_{-\infty}^{s} x(\tau-d-5)d\tau\right]ds$
$= \int_{-\infty}^{t}\left[\int_{-\infty}^{s-d} x(\tau-d-5)d(\tau-d)\right]ds = \int_{-\infty}^{t}\left[\int_{-\infty}^{s-d} x(\tau^*-5)d\tau^*\right]ds = \int_{-\infty}^{t-d}\left[\int_{-\infty}^{s} x(\tau-5)d\tau\right]ds$
$= y(t-d)$.
We can get its impulse response by substituting $x(t)$ with $\delta(t)$: $h(t) = \int_{-\infty}^{t}\left[\int_{-\infty}^{s}\delta(\tau-5)d\tau\right]ds = \int_{-\infty}^{t} u(s-5)ds = \boxed{(t\text{-}5)u(t\text{-}5)}$.

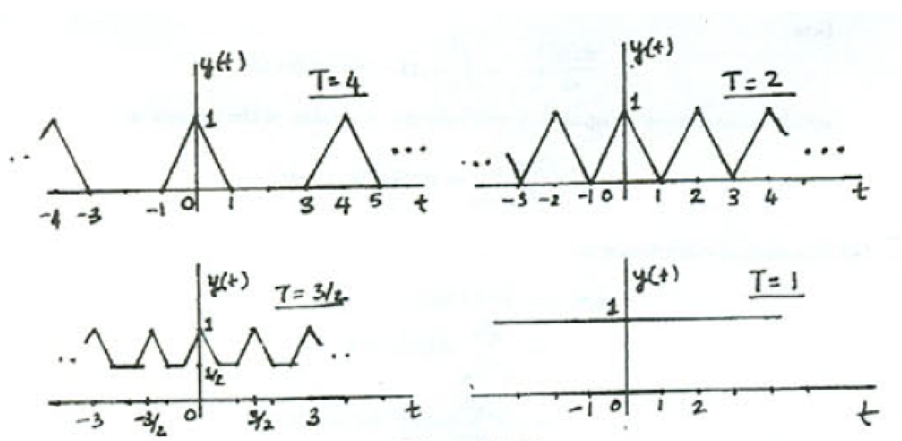
   (b) This system is $\boxed{\text{TI}}$ because we could write it in the exact form of convolution:
$y(t) = \int_{-\infty}^{\infty}(\tau^2 rect(\tau/6))x(t-\tau)d\tau + \int_{-\infty}^{\infty}((t-\tau+3)^{-2}u(t+1-\tau))x(\tau)d\tau$.
The impulse response is given by $\boxed{h(t) = t^2 rect(t/6) + (t+3)^{-2}u(t+1)}$.

2. The answer is:

(a) $h(t-\tau)$ is nonzero over $t-a<\tau<t-b$ and $x(\tau)$ is nonzero over $c<\tau<d$. The integral of their product is nonzero when $t-a>c$ and $t-b<d$, so that the intervals overlap. Thus $\boxed{a+c<t<b+d}$ is the range of values of t for which $y(t)$ is possibly nonzero.
(b) $a=1, b=3, c=-5, d=-1$ so $\boxed{-4<t<+2.}$
For $t-1>-5$ but $t-3<-5$, $y(t) = \int_{-5}^{t-1} 1\,dt = t+4$. For $t-3>-5$ but $t-1<-1$, $y(t) = \int_{t-3}^{t-1} 1\,dt = 2$.
For $t-3<-1$ but $t-1>-1$, $y(t) = \int_{t-3}^{-1} 1\,dt = 2-t$. So $\boxed{y(t) = \begin{cases} t+4, & -4<t<-2 \\ 2, & -2<t<0 \\ 2-t, & 0<t<2 \\ 0, & \text{otherwise.} \end{cases}}$

3. (a) $\boxed{\text{Not time-invariant}}$. A simple counterexample is $x(t) = \sin(t)$, $x_d(t) = \sin(t-1)$ (when $T \neq 1/m$, $m \in Z \setminus \{0\}$).

   (b) The sketches are shown below:



4. (a) $y(t) = \int_{-\infty}^{t}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty}(t-\tau)u(t-\tau)e^{-(t-\tau)}x(\tau)d\tau \Rightarrow \boxed{h(t) = te^{-t}u(t)}$
$\boxed{\text{Causal}}$: $h(t) = 0$ for $t<0$.
$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty}|h(t)|dt = \int_{0}^{\infty} te^{-t}dt = -(t+1)e^{-t}\big|_0^{\infty} = 1$.
$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t>0$.

(b) $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)} x(\tau) d\tau = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t-\tau}{2}\right) e^{-2(t-\tau)} x(\tau) d\tau \Rightarrow \boxed{h(t) = \text{rect}(t/2) e^{-2t}}$

$\boxed{\text{Non-causal}}$: $h(t) \neq 0$ for $t < 0$.

$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-1}^{1} e^{-2t} dt = \frac{e^2 - e^{-2}}{2}$.

$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t > 0$.

5.

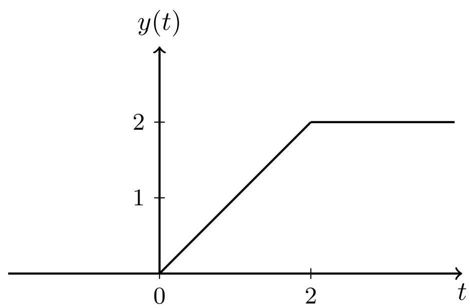
$$\frac{dx(t)}{dt} = -6e^{-3t}u(t-1) + 2e^{-3t}\delta(t-1) = -3x(t) + 2e^{-3}\delta(t-1) \to -3y(t) + e^{-2t}u(t)$$

Thus we know

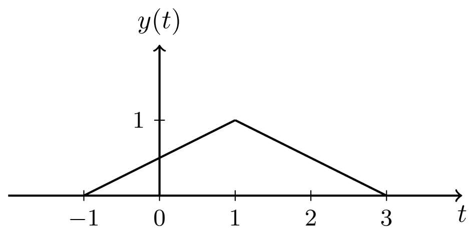
$$2e^{-3}\delta(t-1) \to e^{-2t}u(t) \quad \Rightarrow \quad \delta(t) \to \frac{1}{2}e^{-2t+1}u(t+1)$$

and it follows $\boxed{h(t) = \frac{1}{2}e^{-2t+1}u(t+1)}$.

6. (a) $\boxed{y(t) = 2y_0(t)}$

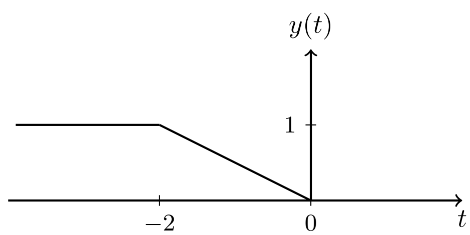


(b) $\boxed{y(t) = y_0(t+1) - y_0(t-1)}$

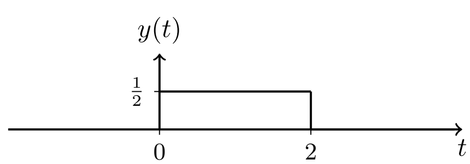


(c) $\boxed{y(t) \text{ cannot be determined.}}$

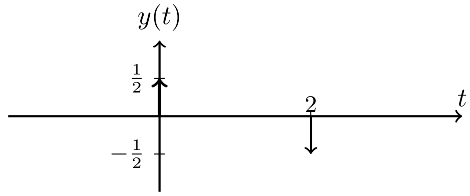
(d) $\boxed{y(t) = y_0(-t)}$



(e) $\boxed{y(t) = y_0'(t)}$



(f) $\boxed{y(t) = y_0''(t)}$



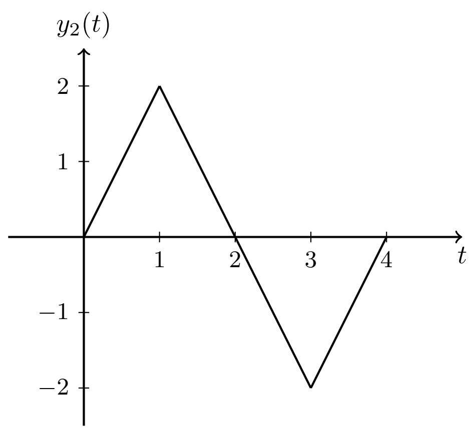
7. $h(t) = \frac{d}{dt}s(t)$. Sketch $s(t)$ and we easily get from graph that $h(t) = \boxed{3\delta(t) - \delta(t-2) - \text{rect}(\frac{t-1}{2})}$.

Optional Problems:

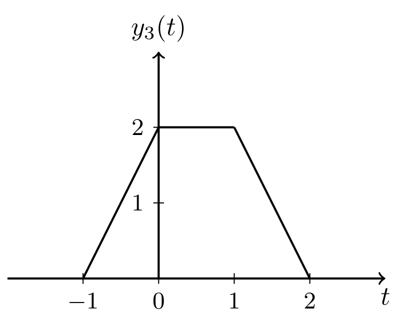
1. Let $d = t_1 - t_0$. When $x(t) = \delta(t - t_0)$, $y(t) = f(t)x(t) = f(t_0)\delta(t - t_0)$. So we have $y(t - d) = f(t_0)\delta(t - t_0 - d) = f(t_0)\delta(t - t_1)$. On the other hand, Let $x_1(t) = x(t - d) = \delta(t - t_0 - d) = \delta(t - t_1)$, the corresponding $y_1(t) = f(t)x_1(t) = f(t_1)\delta(t - t_1)$. Since we have $f(t_0) \neq f(t_1)$, thus $y_1(t) \neq y(t - d)$ at $t = t_1$. $\boxed{\text{The system is not time-invariant}}$.
(*A proof without using counterexamples is also OK.*)

2. (a) $\int_{-\infty}^{\infty} y(t)dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right] dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)dtd\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)dt\right] d\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t^*)dt^*\right] d\tau = \left[\int_{-\infty}^{\infty} x(t)dt\right]\left[\int_{-\infty}^{\infty} h(\tau)d\tau\right]$.
   (b) $\frac{d}{dt}y(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau = h(t) * (\frac{d}{dt}x(t))$.
   The other part of the statement was very similar.

3. (a) $x_2(t) = x_1(t) - x_1(t-2) \Rightarrow y_2(t) = y_1(t) - y_1(t-2)$



   (b) $x_3(t) = x_1(t) + x_1(t+1) \Rightarrow y_3(t) = y_1(t) + y_1(t+1)$



4. The answer is:

(a) $\boxed{\text{True}}$.

$$\int_{-\infty}^{\infty} |h(t)|\mathrm{d}t = \sum_{k=-\infty}^{\infty} \int_{0}^{T} |h(t)|\mathrm{d}t = \infty$$

(b) $\boxed{\text{False}}$. If $h(t) = \delta(t - t_0)$ for $t_0 > 0$, then the inverse system impulse response is $\delta(t + t_0)$, which is noncausal.

(c) $\boxed{\text{False}}$. $h(t) = u(t)$ implies causality, but $\int_{-\infty}^{\infty} u(t)\mathrm{d}t = \infty$ implies that the system is not stable.

(d) $\boxed{\text{False}}$.

$$\begin{aligned} &h_1(t) = \delta(t - t_1), \quad t_1 > 0 \quad \text{Causal} \\ &h_2(t) = \delta(t + t_2), \quad t_2 > 0 \quad \text{Noncausal} \\ &h(t) = h_1(t) * h_2(t) = \delta(t + t_2 - t_1), \quad t_2 \leq t_1 \quad \text{Causal} \end{aligned}$$

5. (a) The homogeneous equation is

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = 0$$

Plug $y(t) = Ce^{st}$ into the equation, and we get $s = -\frac{1}{2}$. Hence the family of signals $y(t)$ that satisfies the associated homogeneous equation is

$$\boxed{y(t) = Ce^{-\frac{1}{2}t}, \quad C \in \mathbb{R}}$$

(b) substituting $y_1(t) = Ae^{-t}, t > 0$ into the equation, we find

$$\frac{dy_1(t)}{dt} + \frac{1}{2}y_1(t) = -Ae^{-t} + \frac{1}{2}Ae^{-t} = e^{-t}, > 0$$

Since e^{-t} never equals zero, we can divide it out. This gives us an equation for A,

$$-A + \frac{A}{2} = 1 \quad as \boxed{\text{A=-2}}$$

(c) For $y_1(t) = [2e^{-t/2} - 2e^{-t}]u(t)$,

$$\frac{dy_1(t)}{t} = \begin{cases} [-e^{-t/2} + 2e^{-t}] & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$

$$\begin{aligned} \frac{dy_1(t)}{t} + \frac{1}{2}y_1(t) &= \begin{cases} (-e^{-t/2} + 2e^{-t}) + \frac{1}{2}(2e^{-t/2} - 2e^{-t}) = e^{-t} & , t > 0 \\ 0 & , t < 0 \end{cases} \\ &= x(t) \end{aligned}$$

6. The answer is:

(a) Relating $r(t)$ to $x(t)$ first, we have

$$\int a[x(t)+r(t)]\mathrm{d}t = r(t), \qquad \text{or} \qquad \frac{\mathrm{d}r(t)}{\mathrm{d}t} - ar(t) = ax(t), \tag{1}$$

and the signal $y(t)$ is related to $r(t)$ as follows:

$$r(t) + b\int r(t)\mathrm{d}t = y(t), \qquad \text{or} \qquad \frac{\mathrm{d}r(t)}{\mathrm{d}t} + br(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t} \tag{2}$$

Solving for $\frac{\mathrm{d}r(t)}{\mathrm{d}t}$ in eqs. 1 and 2 and equating, we obtain

$$ar(t) + ax(t) = -br(t) + \frac{\mathrm{d}y(t)}{\mathrm{d}(t)}$$

Therefore,

$$r(t) = \frac{-a}{a+b}x(t) + \frac{1}{a+b}\frac{\mathrm{d}y(t)}{\mathrm{d}t} \tag{3}$$

We now substitute eq. 3 into eq. 1 (or eq. 2), which, after simplification, yields,

$$\boxed{\frac{\mathrm{d}y^2(t)}{\mathrm{d}t^2} - a\frac{\mathrm{d}y(t)}{\mathrm{d}t} = a\frac{\mathrm{d}x(t)}{\mathrm{d}t} + abx(t)}$$

(b) Substitute $a = 2, b = 1, x(t) = e^t\cos(t)u(t)$ into the relationship we derived in (a), we obtain

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 2(2e^t\cos(t) - e^t\sin(t)), \quad t > 0 \tag{4}$$

It's obvious to see that the homogeneous solution of $y(t)$ is in the form of $\boxed{y_h(t) = C_1 + C_2e^{2t}}$.
Then we set the particular solution of $y(t)$ as $y_p(t) = e^t(a\sin(t) + b\cos(t))$ and,

$$\frac{\mathrm{d}y_p(t)}{\mathrm{d}t} = e^t[(a-b)\sin(t) + (a+b)\cos(t)]$$
$$\frac{\mathrm{d}^2y_p(t)}{\mathrm{d}t^2} = e^t(-2b\sin(t) + 2a\cos(t))$$

then substitute it into eq. 4, we get $\boxed{a = 1, b = -2}$. So $\boxed{y_p(t) = e^t(\sin(t) - 2\cos(t))}$.
Therefore, $\boxed{y(t) = C_1 + C_2e^{2t} + e^t(\sin(t) - 2\cos(t))}$.
Then we take the initial condition into consideration,

$$\begin{cases} y(0) = C_1 + C_2 - 2 = 0 \\ y'(0) = 2C_2 - 1 = 0 \end{cases} \implies \begin{cases} C_1 = \frac{3}{2} \\ C_2 = \frac{1}{2} \end{cases}$$

So the full response of this system is $\boxed{y(t) = \left[\frac{3}{2} + \frac{1}{2}e^{2t} + e^t(\sin(t) - 2\cos(t))\right]u(t)}$.