VE216 Recitation Class 4

Fourier Series & Fourier Transform

MA, Anlin

UM-SJTU Joint Institute

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FS Expressions

Linearity tells us that, if

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{(Synthesis Equation)}$$

where

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \\ c_0 &= \frac{1}{T_0} \int_{T_0} x(t) dt \end{aligned} \qquad \text{(Analysis Equation)}$$

Then

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k e^{jk\omega_0 t}$$

Real Forms of FS

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} \mid c_k \mid cos(k\omega_0 t + \angle c_k)$$

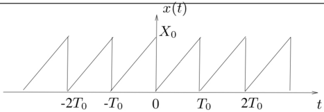
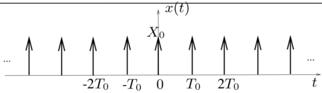
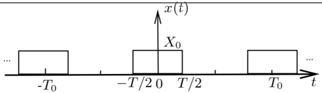
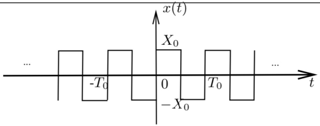
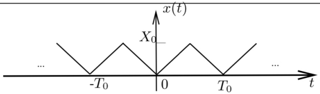
(Combined Trigonometric Form)

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} Re(c_k) \cdot cos(k\omega_0 t) - Im(c_k) \cdot sin(k\omega_0 t)$$

(Trigonometric Form)

Common Signals (Provided in Exam)

Table of Fourier Series for Common Signals

Name	Waveform	c_0	$c_k, k \neq 0$	Comments
Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	
Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\,\mathrm{sinc}\left(\frac{Tk\omega_0}{2\pi}\right)$	$\frac{Tk\omega_0}{2\pi} = \frac{Tk}{T_0}$
Square wave		0	$-j\frac{2X_0}{\pi k}$	$c_k = 0$, k even
Triangular wave sine		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$c_k = 0$, k even

Gibbs Phenomenon

Overshoot/Undershoot near discontinuity

FS Properties

- Hermitian Symmetry: x(t) Real $\rightarrow c_{-k} = c_k^*$ (Also hold for systems)
  - x(t) Real & Even: c_k Real & Even
  - x(t) Real & Odd: c_k Purely imaginary & Odd
- Amplitude: y(t)=ax(t) $\rightarrow \omega^{'} = \omega_0,\ c_k^{'} = ac_k$
- Time: y(t)=x(at+b) $\rightarrow \omega^{'} = a\omega_0,\ c_k^{'} = c_k \cdot e^{jk\omega_0 b}$
- Conjugation: y(t)=[x(t)]* $\rightarrow \omega^{'} = \omega_0,\ c_k^{'} = c_{-k}^*$
- Differentiation: y(t)=$\frac{d}{dt}$x(t) $\rightarrow \omega^{'} = \omega_0,\ c_k^{'} = jk\omega_0 \cdot c_k$
- Parseval's Relation: $P = \frac{1}{T_0}\int_{T_0} \mid x(t) \mid^2 dt = \sum_{-\infty}^{\infty} \mid c_k \mid^2$

Spectra

- Power Density Spectrum: $\mid c_k \mid^2$ vs. $k\omega$
- Magnitude Spectrum: $\mid c_k \mid$ vs. $k\omega$
- Phase Spectrum: $\angle c_k$ vs. $k\omega$

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FT Expressions

For aperiodic signals:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

For periodic signals:

$$x(t) = \sum_{n=-\infty}^{\infty} f(t - nT) = \sum_{n=-\infty}^{\infty} F(\omega) e^{-j\omega nT}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k 2\pi \delta(\omega - k\omega_0)$$

Exercises

- x(t)=δ(t)
- x(t)=$\sum_{-\infty}^{\infty} \delta(t-3n)$

Common FT Pairs (Provided in Exam)

Table of Fourier transform pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega) = \delta\left(\frac{\omega}{2\pi}\right)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
$\mathrm{e}^{\jmath\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$
$\sin\omega_0 t$	$\frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)$
e^{-bt^2}	$\sqrt{\pi/b}\,\mathrm{e}^{-\omega^2/(4b)}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\sum_{k=-\infty}^{\infty}\omega_0\,\delta(\omega-k\omega_0)$

$f(t)$	$F(\omega)$
$\frac{1}{b^2+t^2}$	$\frac{\pi}{b}\,\mathrm{e}^{-b\lvert\omega\rvert}$
$\mathrm{e}^{-b\lvert t\rvert}$	$\frac{2b}{b^2+\omega^2}$
$\operatorname{rect}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}\left(T\frac{\omega}{2\pi}\right)$
$\operatorname{tri}(t)$	$\operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\frac{\omega_0}{2\pi}\operatorname{sinc}\left(\frac{\omega_0}{2\pi}t\right)$	$\operatorname{rect}\left(\frac{\omega}{\omega_0}\right)$
$\operatorname{sinc}^2(t)$	$\operatorname{tri}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{e}^{-at}\,u(t)$	$\frac{1}{j\omega+a}$
$\frac{t^{n-1}}{(n-1)!}\,\mathrm{e}^{-at}\,u(t)$	$\frac{1}{(j\omega+a)^n}$
$\frac{j}{\pi t}$	$\operatorname{sgn}(\omega)$

b is a real positive number throughout. a is a real or complex number throughout, with positive real part.

Important FT Properties

- Time Shift: f(t-t_0) $\longleftrightarrow$ $e^{-j\omega t_0}F(\omega)$
- Time Differentiation: $\frac{d}{dt}f(t) \longleftrightarrow j\omega F(\omega)$
- Time Convolution: $f_1(t) * f_2(t) \longleftrightarrow F_1(\omega) \cdot F_2(\omega)$
- Time Multiplication: $f_1(t) \cdot f_2(t) \longleftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
  - $f(t) \cdot \delta(t - t_0) = f(t_0) \cdot \delta(t - t_0)$
  - $f(t) * \delta(t - t_0) = f(t - t_0)$
- Time Integral: $\int_{-\infty}^{t} f(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
- Parseval's Relation E=$\int_{-\infty}^{\infty} \mid x(t) \mid^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mid X(\omega) \mid^2 d\omega$

FT Properties (Provided in Exam)

Properties of the Continuous-Time Fourier Transform

	Time	Fourier
Synthesis, Analysis	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
Eigenfunction	$h(t) * e^{j\omega_0 t} = H(\omega_0)e^{j\omega_0 t}$	$H(\omega)2\pi\delta(\omega-\omega_0)$ $= H(\omega_0)2\pi\delta(\omega-\omega_0)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Time transformation	$f(at+b),\ a \neq 0$	$\frac{1}{\lvert a\rvert}e^{j\omega b/a}F(\omega/a)$
Time shift	$f(t-\tau)$	$F(\omega)e^{-j\omega\tau}$
Time reversal	$f(-t)$	$F(-\omega)$
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Convolution	$f_1(t) * f_2(t)$	$F_1(\omega)\cdot F_2(\omega)$
Time-domain Multiplication	$f_1(t)\cdot f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Modulation (cosine)	$f(t)\cos(\omega_0 t)$	$\frac{F(\omega-\omega_0)+F(\omega+\omega_0)}{2}$
Time. Differentiation	$\frac{d^n}{dt^n}f(t)$	$(j\omega)^n F(\omega)$
Freq. Differentiation	$(-jt)^n f(t)$	$\frac{d^n}{d\omega^n}F(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)\,d\tau = f(t) * u(t)$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
Conjugation	$f^*(t)$	$F^*(-\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Relation to Laplace	$F(\omega) = F(s)\rvert_{s=j\omega}$, if ROC includes $j\omega$ axis	
Parseval's Theorem	$\int_{-\infty}^{\infty} f_1(t)f_2^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2^*(\omega)\,d\omega$	
Parseval/Rayleigh Theorem	$E = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2\,d\omega$	
DC Value	$\int_{-\infty}^{\infty} f(t)\,dt = F(0)$	

A function that satisfies $f(t) = f^*(-t)$ is said to have **Hermitian symmetry**.

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Filters & THD

Lowpass/ Highpass/ Bandpass Filter

$$
\begin{aligned}
THD &= 1 - \frac{\text{Average Power in First Hoarmonic/ Fundamental}}{\text{Average Signal Power}} \\
&= 1 - \frac{2 \mid c_1 \mid^2}{\sum_{-\infty}^{\infty} \mid c_k \mid^2}
\end{aligned}
$$

Common LTI Systems

- RLC Circuit
- $\cos(\omega t)$
- $\mathrm{rect}(\omega)$ (lowpass filter)

Skill

Partial Fraction:

$$\frac{(s+1)(s-3)}{(s-2)^2(s-1)} =$$