# Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

# Outline

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# Overview

**Signals**

- definition
- classes
- notation
- transformations (operations)
- important signals

(Skip: 1.3.2, 1.3.3, 1.4.1)

**Systems**

- definition
- block diagrams
- system interconnection
- classes
- linearity, time-invariance

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Goal: eventually system design; must first learn to analyze!

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# Signal Definition

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A **signal** is any "physical" quantity that varies with time or space (or any other independent variable or variables).

Often when we discuss signals we refer to mathematical representation of the physical quantity.

## Example

An approaching ambulance siren produces a time-varying change in acoustic pressure that our ears perceive as sound.

$$s(t) = (1 + t)\sin\left(2\pi[1000t + 10t^2 + 300\sin(2\pi t/2)]\right)$$

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# Example: Ambulance Siren

$$s(t) = (1+t)\sin\Big(2\pi[1000t + 10t^2 + 300\sin(2\pi t/2)]\Big)$$

- The $(1+t)$ amplitude term represents increasing loudness as the ambulance approaches.
- The 1000t term represents the 1kHz siren oscillation.
- The $10t^2$ term represents increasing pitch due to the Doppler effect as the ambulance approaches.
- The $300\sin(2\pi 2t)$ term represents the eeh-ooh-eeh-ooh periodic variation in pitch.

# System Definition

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A **system** is a physical “device” that performs an operation on a signal.

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One of the main roles of electrical engineers is to design and analyze systems that take some input signal and produce some related (but almost always different) output signal.

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# Signals and Systems

**Example**

For an audio amplifier, ideally the output signal is "simply" an amplified version of the input signal. (On paper it is easy:

$$s_{\text{out}}(t) = a s_{\text{in}}(t).$$

But implementing this in analog hardware with minimal distortion is nontrivial.)

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This course will emphasize **continuous-time** or **analog** signals, and briefly introduce **discrete-time** or **digital** signals at the end.
(Portions of Chapters 1-10)

# Outline

# A signal is a function

Mathematically, a signal is a function of one or more independent variables.

**Question**

*What is a function?*

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## Definition

A **function** is a mapping from one set of values, called the **domain**, to another set called the **range**.

*It is a rule that assigns to each value in the range set at least one of the values in the domain set.*

# Classification of Signals: Dimensionality (1)

One way to classify signals is by the dimension of the domain of the function, *i.e.* how many arguments the function has.

**Definition**

A **one-dimensional** signal is a function of a single variable, *e.g.* time.

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*What is a **M-dimensional** signal?*

*A **M-dimensional** signal is a function of $M$ independent variables.*

# Classification of Signals: Dimensionality (3)

## Example

A sequence of BW TV pictures $I(x, y, t)$ is a scalar valued function of two spatial coordinates $x$ and $y$ and time $t$, so it is a 3D signal.

We will focus on one-dimensional signals in this course, generally considering the independent variable to be time $t$.

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Another way to classify signals is by the dimension of the range of the function, *i.e.*, the space of values the function can take.

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A **scalar** or **single-channel** signal is a function of a real-valued scalar or complex-valued scalar.

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*A* ***multichannel*** *signal is a function of a real vector or complex vector.*

# Classification of Signals: Dimensionality (5)

**Example**

A color TV picture can be described by a red, blue and green signal, whereas a BW TV picture is scalar valued.

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A **continuous-time signal** or **analog signal** is a function defined for all times $t \in (-\infty, \infty)$, or at least over some continuous interval $(a, b)$.

**Example**

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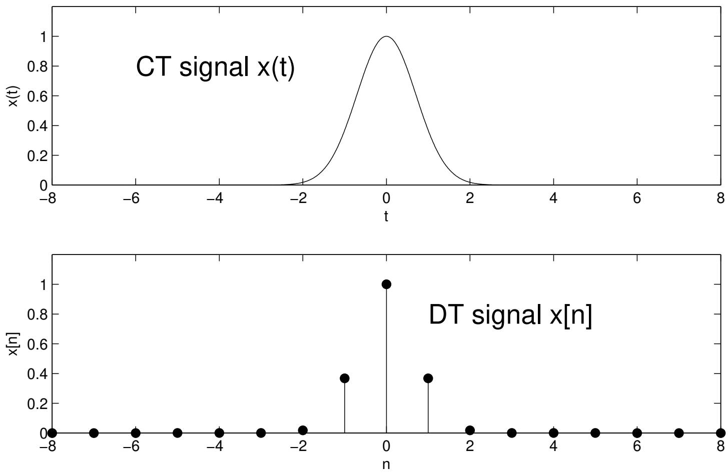
# Continuous-time signals vs. discrete-time signals



# Classification of Signals: time characteristics

Classify signals by time characteristics

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2. Discrete-time signals

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Discrete-time signals arise from

- Sampling a continuous signal at discrete time instants
- Accumulating a quantity over a period of time

**Example**

When counting number of heart attacks per month, $n$ would index the month, and $x[n]$ would be the number.

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A **continuous-valued signal** or **continuous-amplitude signal** can take any value in some continuous interval.

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Voltage between 0 and 5 volts.

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"Hiss" or "noise" in an audio system.

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- single-channel, one-dimensional, continuous-valued, continuous-time signals.
- $x(t)$ is a scalar valued function of a real independent variable.
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$$x : \mathbb{R} \to \mathbb{R} \text{ or } x : \mathbb{R} \to \mathbb{C}$$

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# Outline

# Signal notation (mathematical representation)(1)

1. Graphically ***(Picture)***:
2. Braces or piecewise notation: $x(t) = \begin{cases} e^{-t}, & t > 0 \\ e^{t}, & t \leq 0. \end{cases}$
3. Formula: $x(t) = e^{-|t|}$.
4. In terms of other functions: $x(t) = s(t) + s(-t)$ where
$$s(t) = \begin{cases} e^{-t}, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0. \end{cases}$$
5. Fourier representation:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t}\, d\omega$$

**Skill:** ***Convert between different signal representations.***
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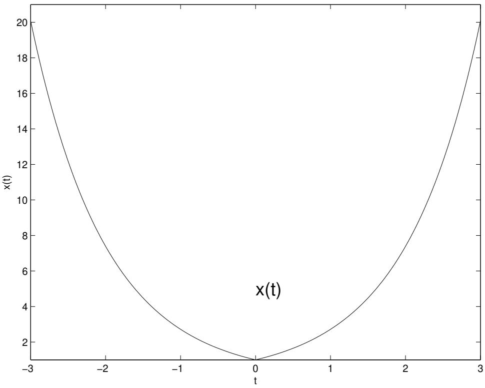

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# Transformations of CT signals

- Time transformations
  - Folding/reflection/time-reversal
  - Time-scaling
  - Time-shifting/time-delay
  - General time transformation
- Amplitude transformations
  - Amplitude reversal
  - Amplitude scaling
  - Amplitude shifting
- More signal operations
  - Differentiator
  - Integrator
- Operations with two signals

# Change of variables

If $x(t) = e^{-(t-2)}$ then $y(t) = x\left(\frac{t-1}{3}\right)$ is another function;

$$y(t) = e^{-[(t-1)/3-2]} = e^{-\left(\frac{t-7}{3}\right)}.$$

In calculus, this type of transformation is called a **change of variables**.

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# Time transformations

Here we give some new names to such transformations to reflect the physical meaning of the mathematics.

**Example**

$$x(t) = \begin{cases} e^{-(t-2)}, & t \geq 2, \\ 0, & \text{otherwise}. \end{cases} \quad \textbf{\textit{(Picture)}}.$$

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One can apply time transformations both graphically and mathematically. Both approaches are useful.

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**Folding/reflection/time-reversal**

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**Example**

- Backwards play a movie/audio tape.
- A mirror is an optical system that does “space reversal”.

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Find $y(t) = x(-t)$ for $x(t)$ above.

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# Folding/reflection/time-reversal: $y(t) = x(-t)$ (2)

Mathematical method:

- Replace all $t$'s with $-t$,
- Simplify where possible.

$$y(t) = x(-t)$$
$$= \begin{cases} e^{-(-t-2)}, & -t \geq 2, \\ 0, & \text{otherwise} \end{cases}$$
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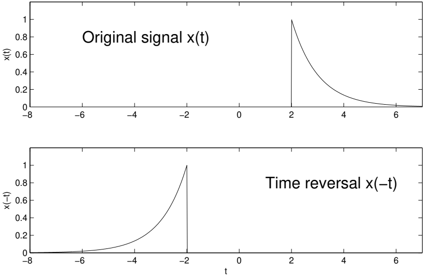


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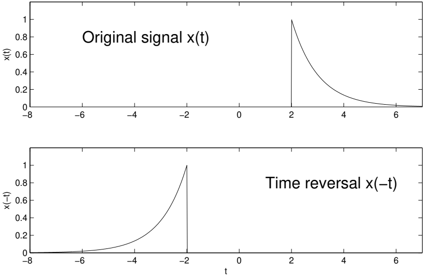


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# Time-scaling: $y(t) = x(at), a > 1$

**Time-scaling**

$$y(t) = x(at)$$

$a > 1$ will shrink or compress the signal

**Example**

Playing a recording at 3 times the normal speed.

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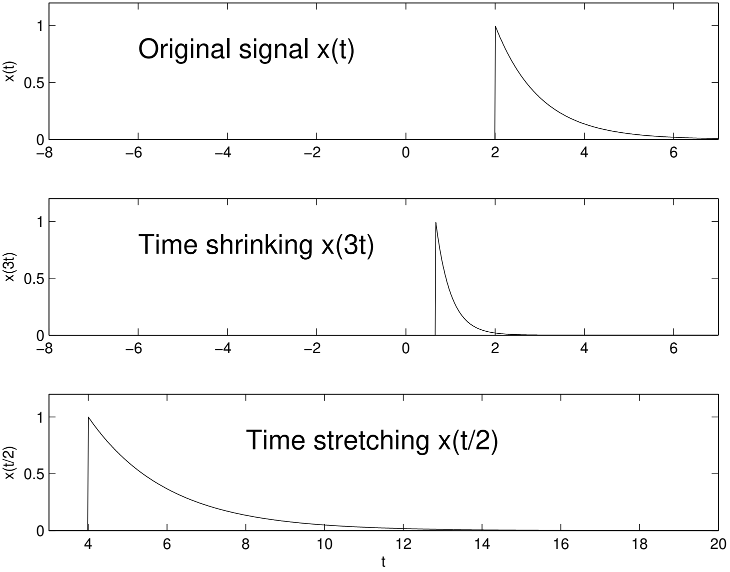
Find $y(t) = x(t/2)$ for $x(t)$ above.

$$y(t) = x(t/2) = \begin{cases} e^{-(t/2-2)}, & t/2 \geq 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-(t-4)/2}, & t \geq 4 \\ 0, & \text{otherwise} \end{cases}$$

# Time-scaling: $y(t) = x(at)$



# Time shifting: $y(t) = x(t - t_0)$

$$\boxed{\text{Time shifting: } y(t) = x(t - t_0)}$$

- $t_0$ can be positive (delayed signal) or negative (advanced signal).
- Physical systems can only delay, not advance, in time.

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"Park distance control" propagation delay

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Find $y(t) = x(t - 1)$ for $x(t)$ above.

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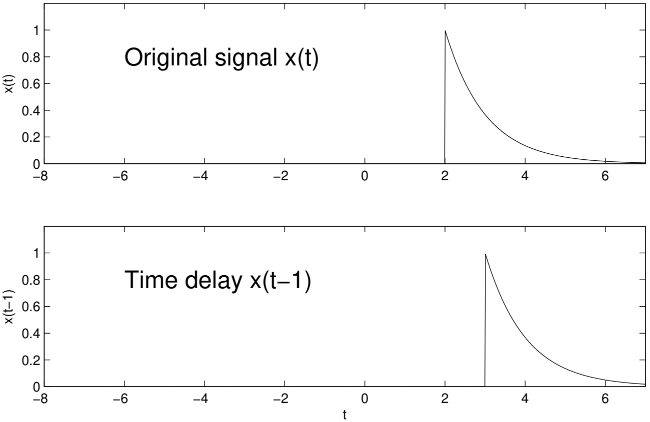
$$y(t) = x(t-1) = \begin{cases} e^{-(t-1-2)}, & t-1 \geq 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-(t-3)}, & t \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

# Time shifting: $y(t) = x(t - t_0)$ (Cont.)




*Note that $x(t-1)$ delays the signal, which means it shifts to the right (starts later in time).*

# General time transformation

General time transformation involves all three of the above time transformations (time reversal, time scaling, and time shifting).

Two distinct (but related) forms:

$$y(t) = x(at - b) = x\left(\frac{t - t_0}{w}\right)$$

where $t_0 = b/a$ and $w = 1/a$ or equivalently $a = 1/w$ and $b = t_0/w$.

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mathematical recipe:

1. Replace all occurrences of $t$ in the definition of $x(t)$ with $at - b$ or with $\frac{t-t_0}{w}$.
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$$x(t) = \begin{cases} e^{-(t-2)}, & t \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} y(t) &= x(-t/2+5) \\ &= \begin{cases} e^{-(-t/2+5-2)}, & -t/2+5 \geq 2 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} e^{(t-6)/2}, & t \leq 6 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

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*How to perform a general time transformation graphically? Should you "shift first" or "scale first?"*

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*The answer is: it depends on which of the two above forms you use to write the transformation!*

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- Introduce an intermediate signal $d(t) = x(t - b)$.
- Clearly $d(t)$ is just a delayed version of $x(t)$ by amount $b$.
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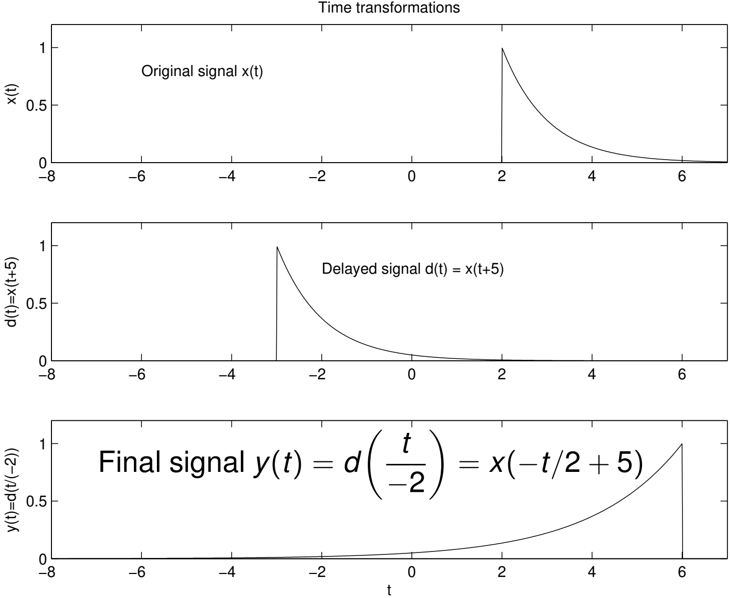

# Form 2: $y(t) = x\left(\frac{t-t_0}{w}\right)$

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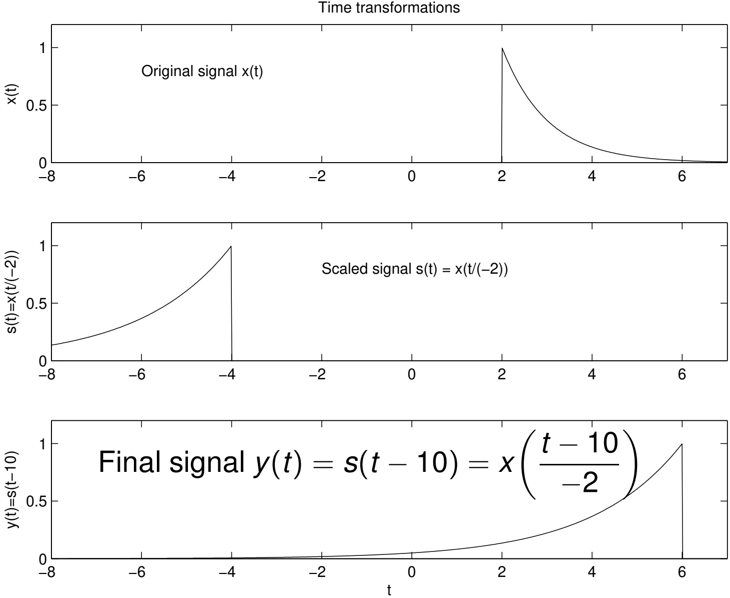

# Amplitude transformations

1. **amplitude reversal** $y(t) = -x(t)$
2. **amplitude scaling** $y(t) = ax(t)$
3. **amplitude shifting** $y(t) = x(t) + b$

**Example**

(Using all three.) Find $y(t) = -3x(t) + 2$ for the $x(t)$ above.

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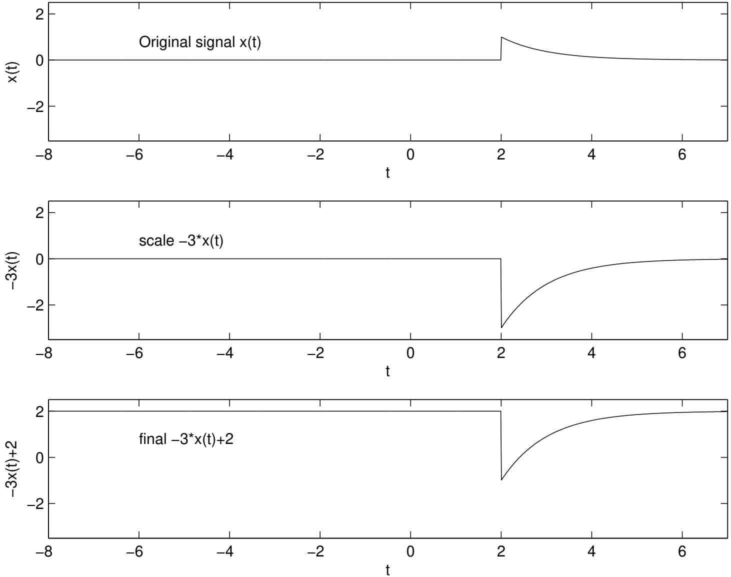
(Using all three.) Find $y(t) = -3x(t) + 2$ for the $x(t)$ above.

$$y(t) = -3x(t) + 2 = \begin{cases} -3e^{-(t-2)} + 2, & t \geq 2 \\ -3 \cdot 0 + 2, & o.w. \end{cases}$$

# Amplitude transformations

*Graphically: scale amplitude by -3 and then shift up by 2.*



# Outline

# Differentiator $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$

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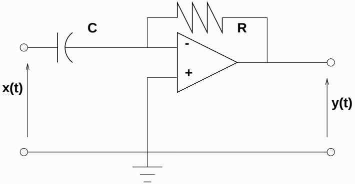
$$y(t) = -RC\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

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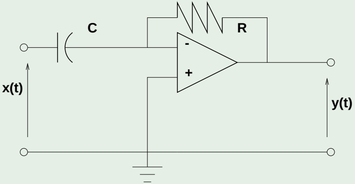


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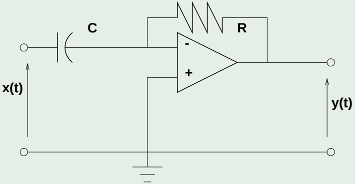


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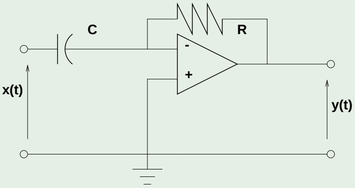


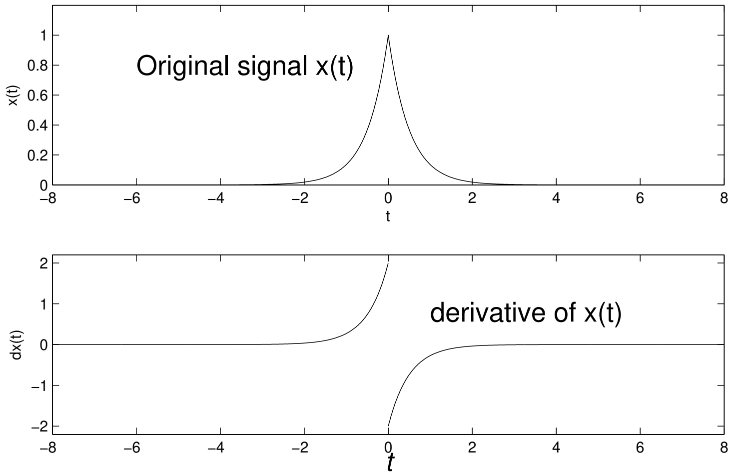

## Example

Find the differentiated signal of $x(t) = e^{-2|t|}$.

$$\begin{aligned} y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t) \\ &= \begin{cases} -2e^{-2t}, & t > 0 \\ 2e^{2t}, & t < 0 \\ ?, & t = 0. \end{cases} \end{aligned}$$

*The derivative of $x(t)$ is undefined at $t = 0$.*

# Differentiator of $x(t) = e^{-2|t|}$



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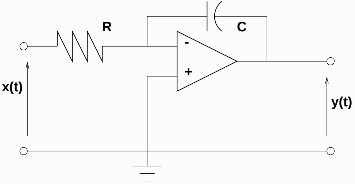


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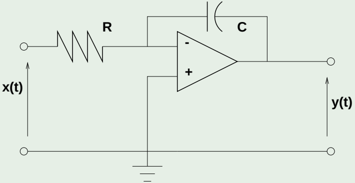


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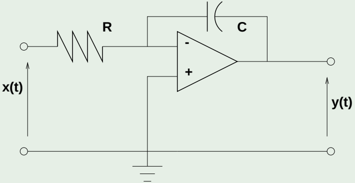


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# Integrator: example

## Solution

1. *rewrite $x(\cdot)$ in terms of $\tau$*

$$x(\tau) = \begin{cases} e^{-\tau}, & \tau > 0 \\ e^{\tau}, & \tau \leq 0. \end{cases}$$

2. *For $t \leq 0$:*

$$y(t) = \int_{-\infty}^{t} e^{\tau}\, d\tau = e^{t}$$

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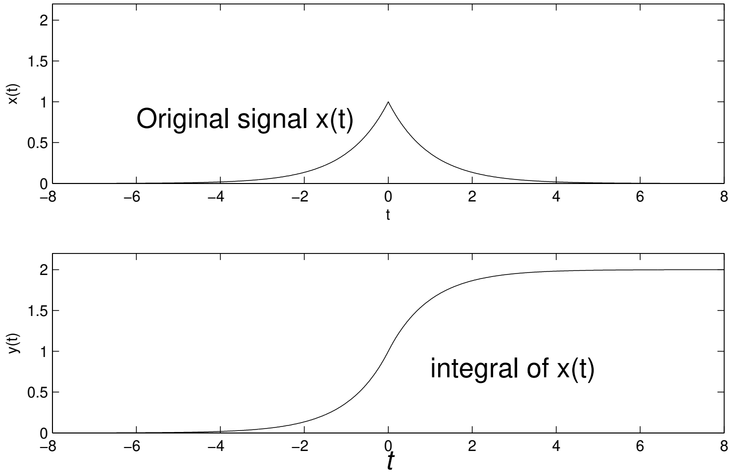

# Integrator vs. Integration in calculus

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- *When you calculate "area under a curve" you compute a single number from the curve, e.g.* $\int_0^1 t^2\,dt = 1/3$.
- *Integrator system in signals and systems transforms one function* $x(t)$ *into another function* $y(t)$.
  *It is a "running integral" whose upper limit is* $t$.

# Outline

# Operations with two signals

**Operations with two signals**

1. sum of two signals $y(t) = x_1(t) + x_2(t)$
2. product of two signals $y(t) = x_1(t)\, x_2(t)$.

   Add or multiply two signals at every time point.

**Example**

If $x_1(t) = e^{-(t-3)^2}$, $x_2(t) = \sin(13t)$, then
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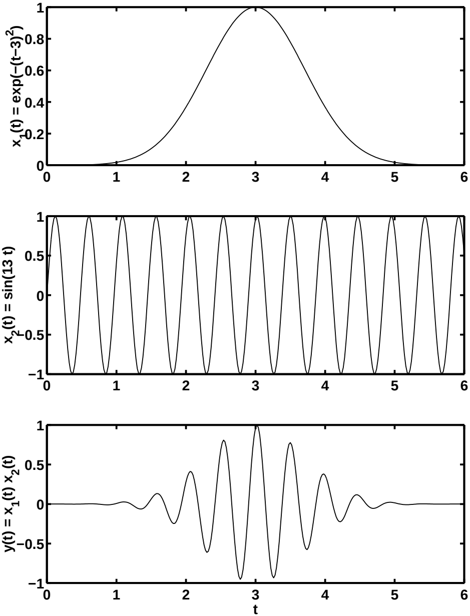

# Outline

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Why study periodic signals?

- important for analysis
- solution to ideal LC electrical circuits
- periodic physical phenomena: frictionless pendulums, earth rotation, heart rhythms, etc.

**Definition**

$x(t)$ is **periodic** with a **period** $T > 0$ iff

$$x(t+T) = x(t) \ \forall t \tag{1}$$

e,p

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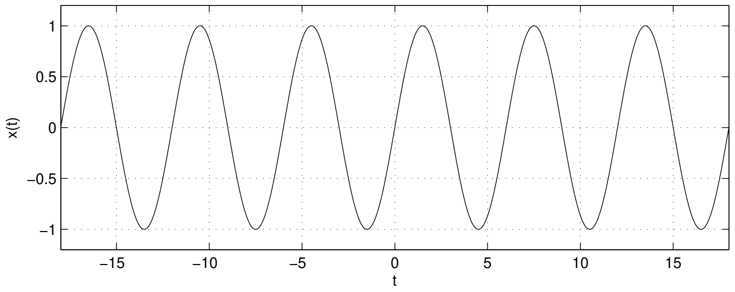

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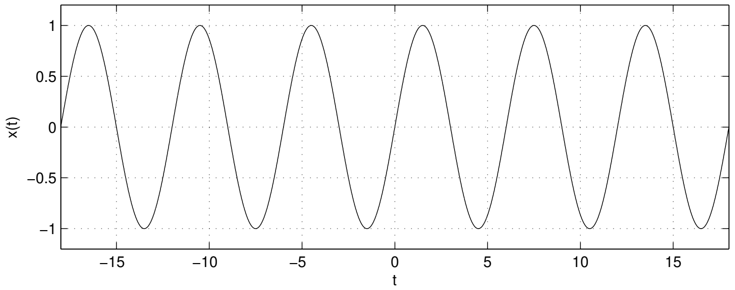


$T_0 = 6 = \frac{2\pi}{\pi/3}$.

# Sums of two periodic signals

## Question

*Suppose $x_1(t)$ is periodic with period $T_1$ and $x_2(t)$ is periodic with period $T_2$ and $x(t) = x_1(t) + x_2(t)$.*

- *Is $x(t)$ periodic?*
- *If so, what is a period $T$ of $x(t)$?*

# Sums of two periodic signals: solution (1)

## Solution

1. *Easy case: if $T_1 = T_2$ then, $x(t)$ is periodic, and $T = T_1 = T_2$.*
2. *General case. We know $x_1(t) = x_1(t + T_1)$ and $x_2(t) = x_2(t + T_2)$ and $x(t) = x_1(t) + x_2(t)$.*
   *We want to determine if there is any value of $T > 0$ such that $x(t) = x(t + T)$.*

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# Sums of two periodic signals: solution (2)

Suppose there is a value of $T > 0$ that satisfies $T = n_1 T_1$ and $T = n_2 T_2$, for some nonzero integers $n_1$ and $n_2$. Then

$$\begin{aligned} x(t+T) &= x_1(t+T) + x_2(t+T) = x_1(t + n_1 T_1) + x_2(t + n_2 T_2) \\ &= x_1(t) + x_2(t) = x(t), \end{aligned}$$

so $x(t)$ is periodic with period $T$.

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The conditions $T = n_1 T_1$ and $T = n_2 T_2$ are equivalent to requiring that

$$n_1 T_1 = n_2 T_2 \text{ so } T_1/T_2 = n_2/n_1,$$

which means that $T_1/T_2$ is rational, a ratio of two integers.

**Theorem**

*A sum of two periodic signals is periodic iff the ratio of their periods is rational.*

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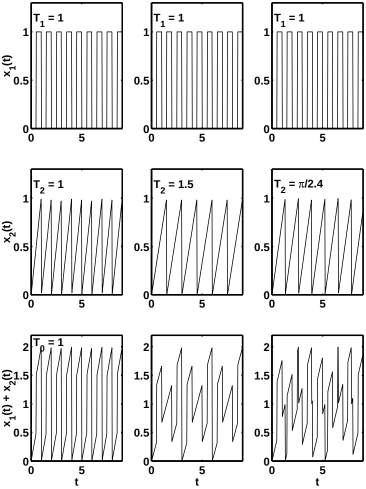
# Sums of two periodic signals: example

Are the signals in the third row periodic?



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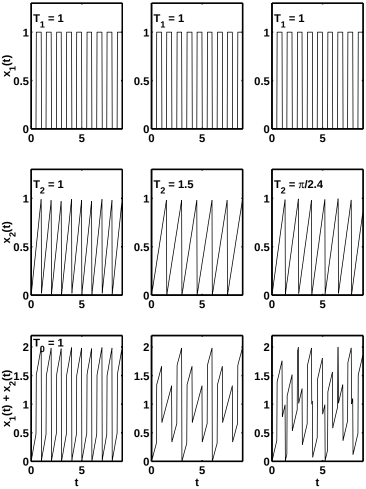


*Yes. Yes. No.*

# Sums of two periodic signals: fundamental period

**Question**

*The least common multiple of the fundamental periods of the two signals is a period of the sum. Is it the fundamental period?*

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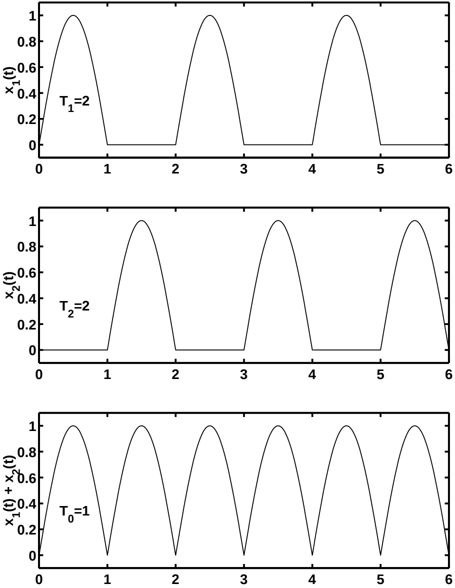
*In general it may not be the fundamental period.*

*To find a least common multiple, you find the smallest values of $n_1$ and $n_2$ such that $n_1 T_1 = n_2 T_2$.*

# Sums of two periodic signals: fundamental period



# Outline

# Even and odd symmetry

**Definition**

$x(t)$ has **even symmetry** iff $x(-t) = x(t)\ \forall t$

**Definition**

$x(t)$ has **odd symmetry** iff $x(-t) = -x(t)\ \forall t$

Note that if $x(t)$ has odd symmetry, then $x(0) = -x(0)$ so $x(0) = 0$.

**Question**

*If $x(0) = 0$, does $x(t)$ have odd symmetry?*

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**Question**

*If $x(0) = 0$, does $x(t)$ have odd symmetry?*

*No! The property $x(0) = 0$ is a necessary condition for $x(t)$ to have odd symmetry, but it is not a sufficient condition.*

# Even and odd components

We can decompose any signal into even and odd components:

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) \triangleq \frac{1}{2}\left[x(t) + x(-t)\right], \quad x_o(t) \triangleq \frac{1}{2}\left[x(t) - x(-t)\right]$$

## Question

*Is the following $x(t)$ even or odd?*

$$x(t) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{otherwise}, \end{cases}$$

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## Question

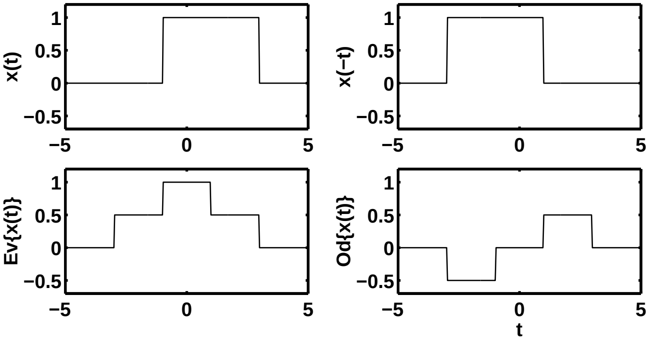
*Is the following $x(t)$ even or odd?*

$$x(t) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{otherwise}, \end{cases}$$

*Neither. Find even and odd components.*

# Even and odd signals decomposition

$$x_e(t) = \begin{cases} 1/2, & -3 < t \leq -1 \\ 1, & -1 < t < 1 \\ 1/2 & 1 \leq t < 3 \\ 0, & \text{otherwise}, \end{cases} \qquad x_o(t) = \begin{cases} -1/2, & -3 < t < -1 \\ 1/2 & 1 < t < 3 \\ 0, & \text{otherwise}. \end{cases}$$



# Outline

# Average value and energy

**Definition**

The **average value** of a signal is defined as

$$A \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt.$$

**Example**

The average value of an odd signal is zero.

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The **energy** of a signal $x(t)$ is defined as

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## Definition

If $E$ is finite ($E < \infty$) then $x(t)$ is called an **energy signal** and $P = 0$.

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# Energy and power signals

**Definition**

If $E$ is infinite, then $P$ can be either finite or infinite. If $P$ is finite and nonzero, then $x(t)$ is called a **power signal**.

Some signals are neither energy signals nor power signals, such as $x(t) = t^2$, for which $E = \infty$ and $P = \infty$. Such signals are generally of little practical engineering importance.

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# Energy and power signals: example

**Example**

consider $x(t) = 5 + a\cos t$ where $0 < a < \infty$.

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- Is $x(t)$ a power signal, an energy signal, or neither?

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## Solution

*The average value of this signal is:*

$$\begin{aligned} A &\triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt = \lim_{T\to\infty} \frac{1}{2T} \left(5t + a\sin t\right)\Big|_{-T}^{T} \\ &= \lim_{T\to\infty} \frac{1}{2T}(10T + 2a\sin T) = \lim_{T\to\infty} \left(5 - \frac{a\sin T}{2T}\right) = 5. \end{aligned}$$

# Energy and power signals: solution

## Solution

*The average power of this signal is (by a similar integral):*

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (5 + a\cos t)^2 \, dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 25 + 10a\cos t + a^2 (\tfrac{1}{2} + \tfrac{1}{2}\cos 2t) \, dt \\
&= \lim_{T\to\infty} \frac{25(2T) + 10a(2\sin T) + \frac{a^2}{2}(2T + \sin(2T))}{2T} \\
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*Since* $0 < P < \infty$, $x(t)$ *is a power signal.*
*Since* $P$ *is nonzero and* $E$ *is infinite,* $x(t)$ *is not an energy signal.*

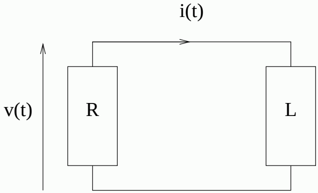
# Outline

# Exponential signals (1)

Sinusoidal signals, exponential signals, and complex exponentials signals are particularly important because they arise from the solutions of linear constant-coefficient differential equations.

## Example

an RL circuit




$$v(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i(t) = L\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{v(t)}{R}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \left(-\frac{R}{L}\right)v(t) = av(t)$$

where $a = -R/L$. Solution for $t > 0$ is

$$v(t) = v(0)e^{at} = v(0)e^{-t/\tau}$$

where $\tau = -1/a = L/R$ is called the time constant of the circuit. ***(Picture)***

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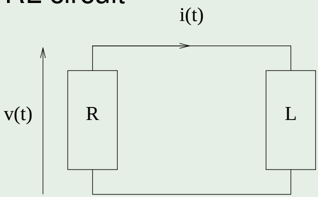


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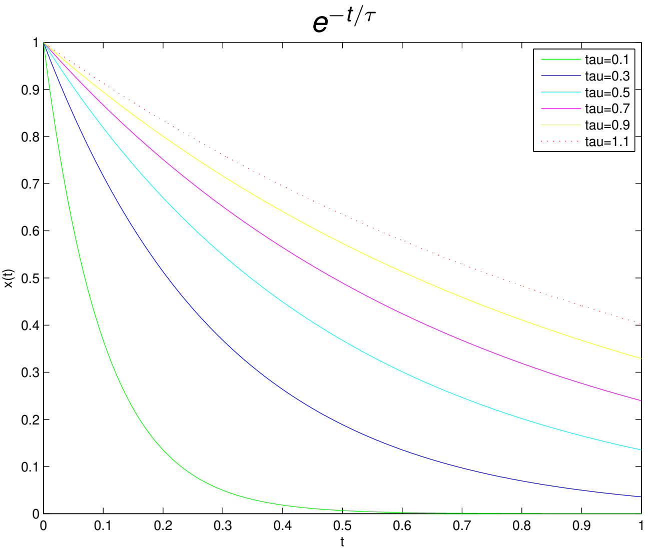
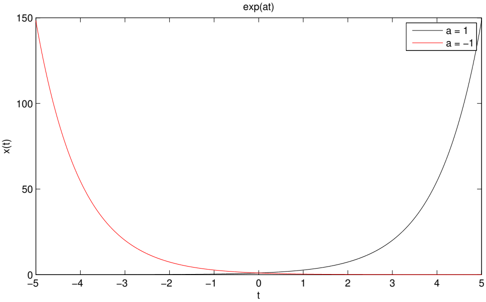


# Exponential signals (3)

Signals of the form $x(t) = ce^{at}$ are very important, for both real and complex $c$ and $a$.

- $a > 0$ real *e.g.*, population growth
- $a < 0$ real *e.g.*, radioactive decay,



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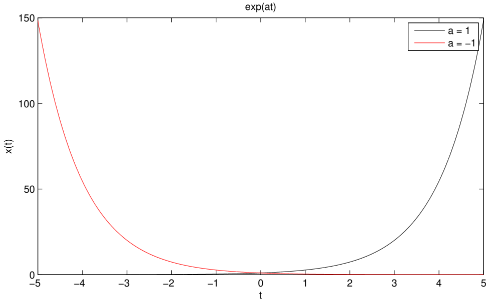

# Exponential signals (4)

- If $a$ is purely imaginary, we get $x(t) = ce^{j\omega_0 t}$, a **complex exponential** signal.
- If $c = Ae^{j\phi}$, then $x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$, a **sinusoid** signal.
- If $x(t) = e^{st}$ where $s = a + j\omega_0$ and $a < 0$, then $x(t) = e^{at}(\cos\omega_0 t + j\sin\omega_0 t)$ which is called a **damped sinusoid** signal. (See textbook p21.)

Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

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# Outline

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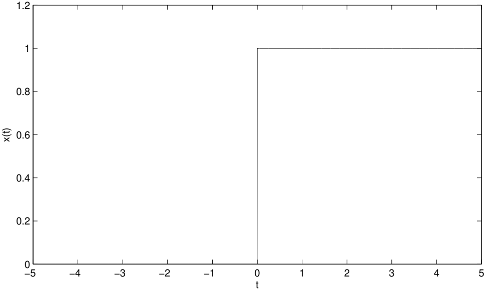
# Unit step function/Signal

## Definition

A **unit step function(signal)** is defined as

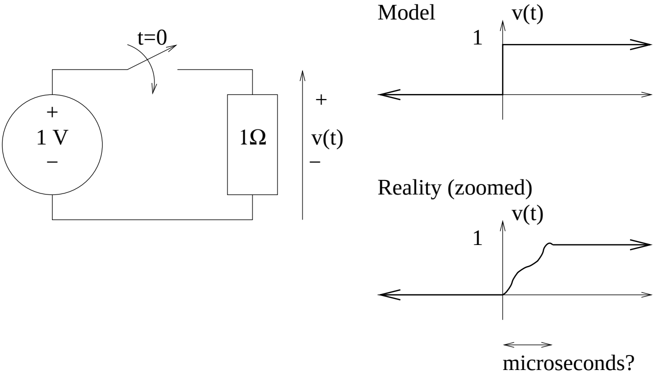
$$u(t) \triangleq \begin{cases} 1, & t > 0 \text{ or } t \geq 0 \\ 0, & t < 0 \end{cases}$$

The value at $t = 0$ is arbitrary and unimportant! Reasonable choices are 0, 1, and $\frac{1}{2}$; any will do.



# Modeling a switch

The unit step is a useful model for a switch.




## Question

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- *Does the final voltage exactly equal 1 Volt?*

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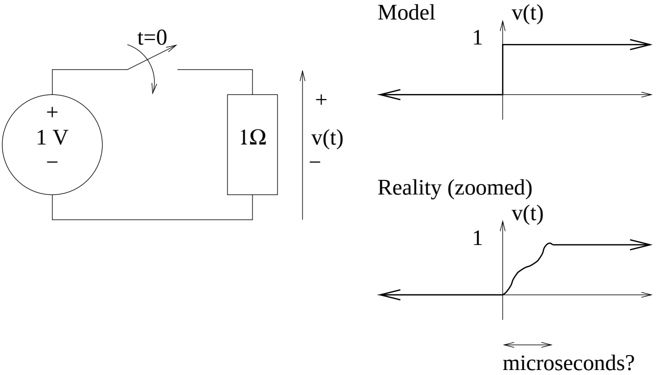


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# Modeling a switch (2)

## Solution

- *No, as the first atoms of the switch contacts begin to "touch", there will be a small current flow. As more atoms touch, more current will flow. Eventually the contacts will "touch completely". and the system will settle to steady state.*
- *No, because of internal resistance of source and resistances of switch itself and wires.*

# Modeling a switch (3)

- *But all of these effects are very small, so generally we can ignore them and the step function model $v(t) = u(t)$ is very reasonable for most purposes.*
- *In some high speed applications the switching time is important, so a more accurate model is necessary.*
- *Any real system can have no discontinuities of the type exhibited by the unit step function. There is always a small transition.*
- *It is OK to ignore this transition as long as the transition time is small relative to other time constants in the system being studied.*

# Simplifying notation

The step function is useful for simplifying notation.

**Example**

The following step function "switches on" the signal from a guitar string plucked at time $t = 2$.

$$x(t) = \begin{cases} e^{-t}\sin(5t), & t > 2 \\ 0, & \text{otherwise} \end{cases} = e^{-t}\sin(5t)\,u(t-2).$$

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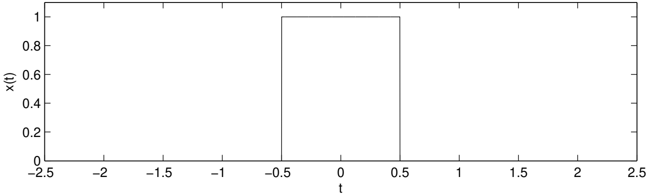
# Rect(angle) function

## Definition

A **rect(angle) function** is defined as

$$\mathrm{rect}(t) \triangleq \begin{cases} 1, & -1/2 < t < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

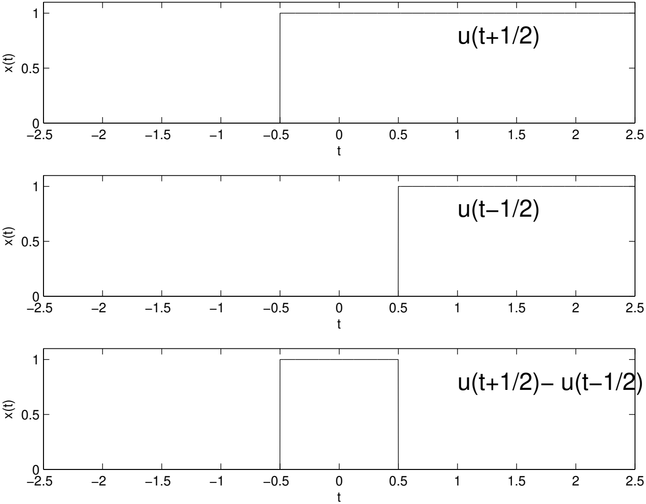
Centered at zero with unit width and unit height.



# Rect function and step functions (1)

Can be represented using step functions:

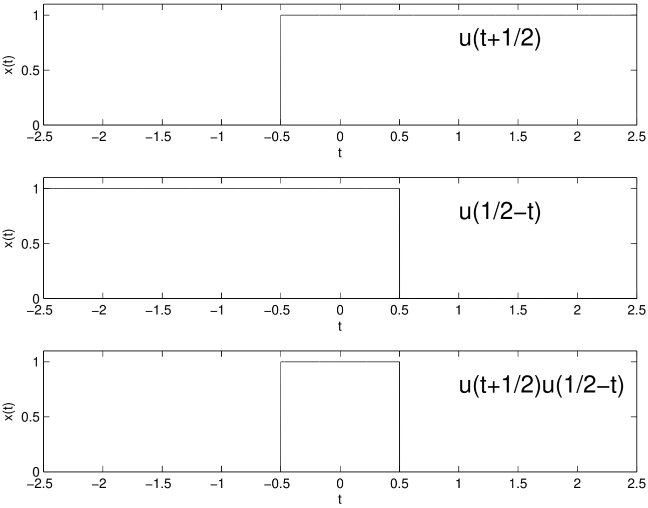
$$\mathrm{rect}(t) = u(t+1/2) - u(t-1/2)$$



# Rect function and step functions (2)

Can be represented using step functions:

$$\mathrm{rect}(t) = u(t + 1/2)u(1/2 - t)$$

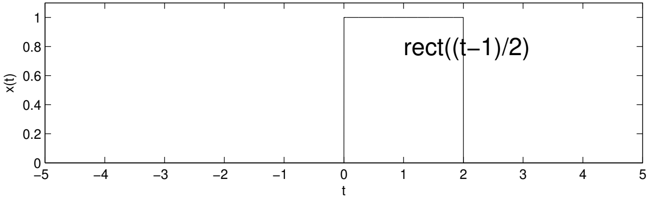


# Transformed rect functions

Time-scaled and time-shifted rect function

$$\mathrm{rect}\left(\frac{t-t_0}{T}\right) = \begin{cases} 1, & -1/2 < \frac{t-t_0}{T} < 1/2 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & t_0 - T/2 < t < t_0 + T/2 \\ 0, & \text{otherwise.} \end{cases}$$

Centered at $t_0$ with width $T$

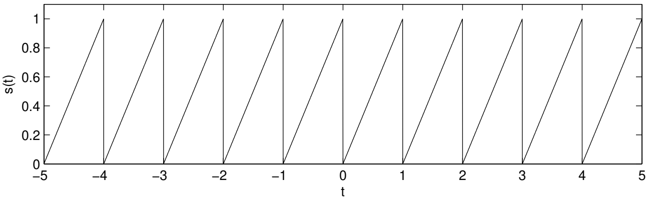


# Rect function: example

Useful for "switching on and off" other functions, or for "extracting" on part of a signal, such as one period of a periodic signal.

## Example

Find mathematical expression for a sawtooth signal $s(t)$.



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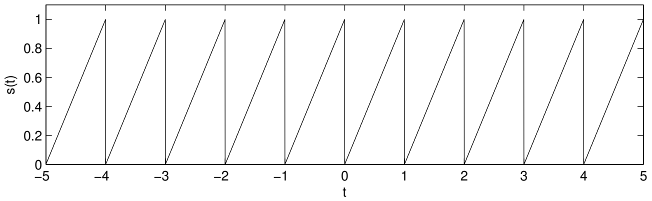
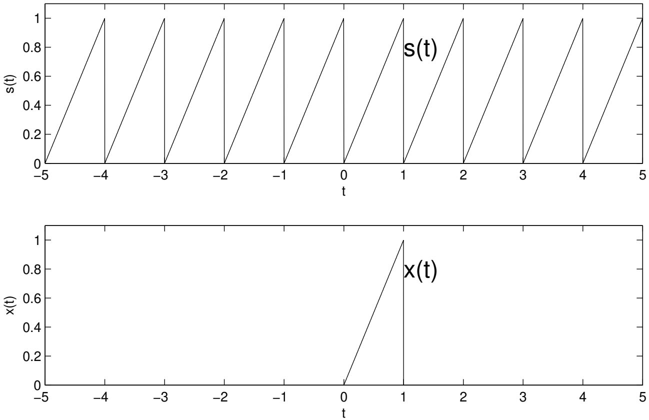


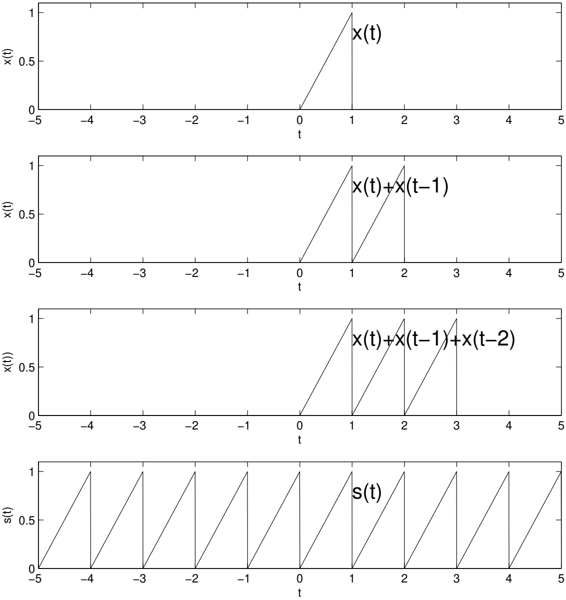
# Rect function: solution (1)

*Let* $x(t)$ *be the signal that is nonzero only over one period of* $s(t)$ *(one "tooth").*

$$x(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} = t\,\mathrm{rect}(t - 1/2)\,.$$



# Rect function: solution (2)



# Rect function: solution (3)

$$s(t) = x(t) + x(t-1) + x(t-2) + \cdots$$

$$s(t) = \sum_{k=-\infty}^{\infty} x(t-k) = \sum_{k=-\infty}^{\infty} (t-k)\,\mathrm{rect}(t-k-1/2), \quad k \in \mathbb{Z}.$$

*Very convenient closed-form representation. This form will be particularly useful for analyzing periodic signals using the Fourier series.*

# Outline

# Unit impulse function $\delta(t)$(1)

- **Unit impulse function** $\delta(t)$, aka **Dirac delta function** or just **delta function**.
- It is another mathematical idealization that cannot occur in nature (like the unit step function), but is nevertheless useful for modeling certain phenomena, just as the step function is a useful idealization of a switch.
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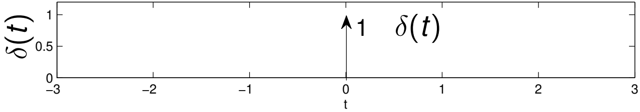
# Unit impulse function $\delta(t)$(2)

$\delta(t)$ is like a pulse of zero width but infinite height and unit area.




Graphical representation using upward arrow, labeled with area (called **weight**). (see text p.34 for scaled impulse.)

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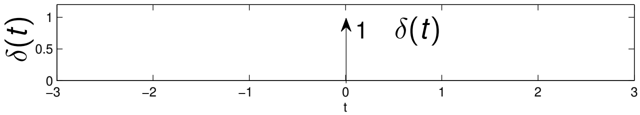


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# Minor Properties

## Property

1. *unit area property* $\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$ *for any* $t_0$
2. *scaling property* $\delta(at + b) = \frac{1}{|a|}\delta(t + b/a)$ *for* $a \neq 0$.
3. *symmetry property* $\delta(t) = \delta(-t)$
4. *support property* $\delta(t - t_0) = 0$ *for* $t \neq t_0$
5. *relationships with unit step function:* $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$, $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$

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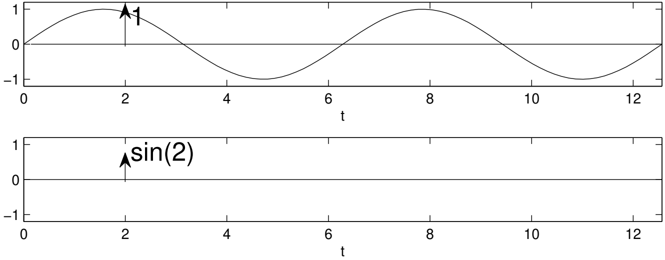
# Major properties: Sampling property

## Property

***Sampling property*** *holds when $x(t)$ is continuous at $t_0$:*

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$



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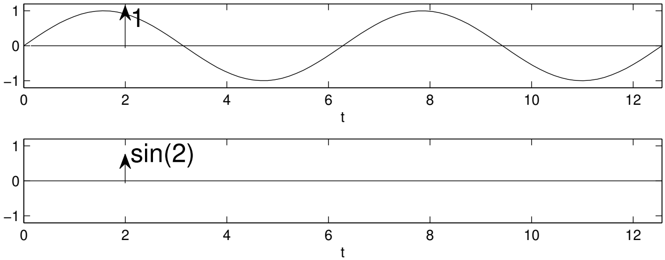

# Algebraic property

## Property

***Algebraic property***

$$t\delta(t) = 0$$

# Scaling property

**Example**

Show that $\delta(at) = \frac{1}{|a|}\delta(t)$ for $a \neq 0$.

**Property**

*Let $g_1(t)$ and $g_2(t)$ be generalized functions. Then the **equivalence property** states that $g_1(t) = g_2(t)$ iff*

$$\int_{-\infty}^{\infty} \phi(t)g_1(t)\,dt = \int_{-\infty}^{\infty} \phi(t)g_2(t)\,dt$$

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# Scaling property: proof (1)

*We prove this in two steps, for $a > 0$ and then for $a < 0$.*

- *When $a > 0$,*

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\,dt = \int_{-\infty}^{\infty} \phi(t)\delta(|a|\,t)\,dt$$

$$= \int_{-\infty}^{\infty} \phi(\frac{1}{|a|}y)\delta(y)\frac{1}{|a|}dy \quad (y = |a|\,t \Longrightarrow dy = |a|\,dt)$$

$$= \frac{1}{|a|}\,\phi(\frac{1}{|a|}y)\Big|_{y=0} = \boxed{\frac{1}{|a|}\phi(0)} \quad (\textit{sifting property})$$

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*Thus, for any a*

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# Ideal and practical impulse function

- *Mathematically, one can use the above properties to solve all the problems in this course.*
- *But physically we would like more insight, so we consider the "ideal" unit impulse function as a type of limit of "practical" impulse functions, just like the unit step function is a limit of practical almost-step functions.*

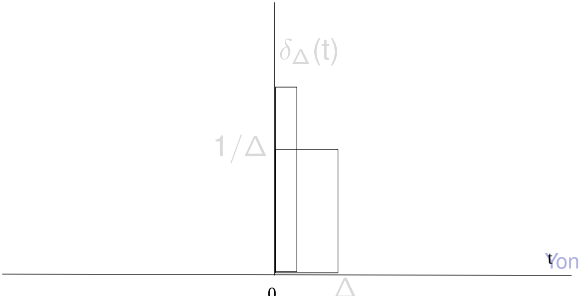
# Practical impulse function

## Definition

**Practical impulse function**, defined for any $\Delta > 0$:

$$\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise}. \end{cases}$$

Note that area is unity, width approaches zero as $\Delta \to 0$; height approaches infinity as $\Delta \to 0$.



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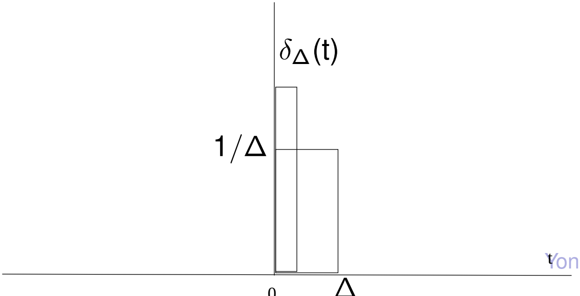

# Practical impulse function: example

**Example**

drumstick striking a drum (applied force vs time)

**Example**

metal hammer tapping a pendulum (applied force vs time)

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It is tempting to try to write

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

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Instead we "define" $\delta(t)$ in terms of its properties, making sure that the properties are consistent with the above "limit". Such objects are called generalized functions in mathematics.

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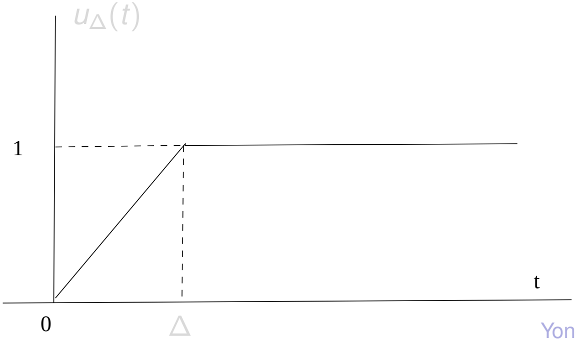
# Relationship to unit step function (1)

## Explanation of $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$ using limiting step function.

Define a practical almost-step function as

$$u_\Delta(t) \triangleq \begin{cases} 0, & t \le 0 \\ t/\Delta, & 0 < t < \Delta \\ 1, & t \ge \Delta \end{cases}$$



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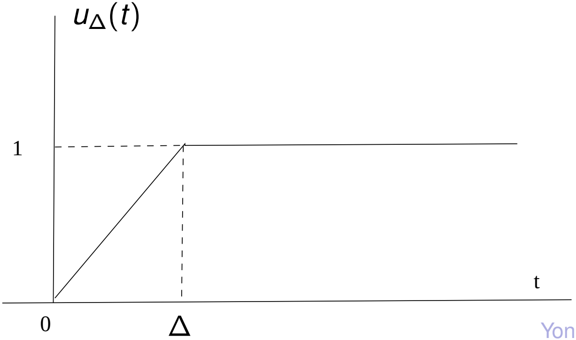

# Relationship to unit step function (2)

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\Delta(t) = \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases} = \delta_\Delta(t)$$

Then since $u(t) = \lim_{\Delta \to 0} u_\Delta(t)$, by taking the limit of both sides we have that

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# Relationship to unit step function (3)

Show graphically that $\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$.

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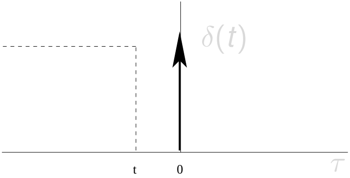
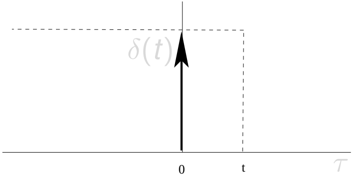
- For any $t < 0$, the range of integration over $(-\infty, t)$ will not cover zero, so the integral is simply zero.
- For any $t > 0$, integration over the range $(-\infty, t)$ covers zero, so the entire unit area of the impulse is included in the integral, so the integral evaluates to 1 for any $t > 0$.



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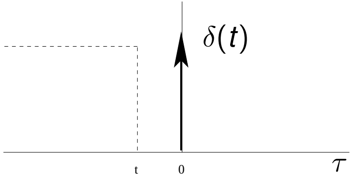
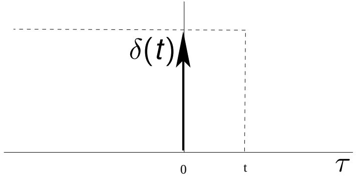


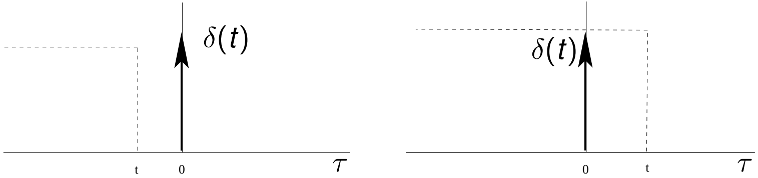



# Relationship to unit step function (3)

Show graphically that $\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$.

---

- For any $t < 0$, the range of integration over $(-\infty, t)$ will not cover zero, so the integral is simply zero.
- For any $t > 0$, integration over the range $(-\infty, t)$ covers zero, so the entire unit area of the impulse is included in the integral, so the integral evaluates to 1 for any $t > 0$.



# Example (1)

## Example

- By Newton's laws, velocity is the time-integral of acceleration.
- When a hammer taps a stationary pendulum, the pendulum (almost) instantaneously changes from being stationary to moving with some velocity that is related to how "hard" the hammer taps the pendulum.
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# Example (2)

**Example**

Let $x(t) = 2\,\mathrm{rect}(t/2 - 3)$. Find $y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$.

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## Solution

$x(t) = 2\,\mathrm{rect}(t/2 - 3) = 2\,\mathrm{rect}((t - 6)/2) = 2u(t - 5) - 2u(t - 7)$

*turns on at $t = 5$ and turns off at $t = 7$.*

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t) = 2\delta(t - 5) - 2\delta(t - 7)$$

# Example (3)

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\begin{aligned}
x(t) &= 2\,\mathrm{rect}(t/2-3) = 2\,\mathrm{rect}((t-6)/2) \\
&= 2u(\frac{t-6}{2}+\frac{1}{2}) - 2u(\frac{t-6}{2}-\frac{1}{2}) \\
&= 2u(\frac{t-5}{2}) - 2u(\frac{t-7}{2})
\end{aligned}
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*directly plug in* $\mathrm{rect}(t) = u(t+1/2) - u(t-1/2)$.

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y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \delta(\frac{t-5}{2}) - \delta(\frac{t-7}{2}) \\
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# Time-scaled unit step function

**Question**

$u(\frac{t-t_0}{w})? = u(t-t_0)$ *where* $w > 0$

# Time-scaled unit step function

## Question

$u(\frac{t-t_0}{w})? = u(t - t_0)$ *where* $w > 0$ *Yes.*

- $u(\frac{t-t_0}{w})$ *turns on at* $t_0$.

$$\frac{t-t_0}{w} \geq 0 \Longrightarrow t \geq t_0$$

- $u(t - t_0)$ *turns on at* $t_0$.

$$t - t_0 \geq 0 \Longrightarrow t \geq t_0$$

# Outline

# Continuous-time systems

**Definition**

A **continuous-time system** is a device or process that, according to some well-defined rule, transforms one CT signal called the **input signal** or **excitation** into another CT signal called the **output signal** or **response**.

The input signal $x(t)$ is transformed by the system into a signal $y(t)$, which we express mathematically as

$$y(\cdot) = \mathcal{T}[x(\cdot)] \quad \text{or} \quad y(t) = \mathcal{T}[x(\cdot)](t) \quad \text{or} \quad x(\cdot) \xrightarrow{\mathcal{T}} y(\cdot).$$

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**Question**

*The notation* $y(t) = \mathcal{T}[x(t)]$ *is mathematically vague. Why?*

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*The notation $y(t) = \mathcal{T}[x(t)]$ is mathematically vague. Why?*

*In general $y(t)$ is a function of the entire signal $x(\cdot)$ not just the single time point $x(t)$ as illustrated by auto example below.*

# Diagram

$$\begin{array}{c} \text{Input Signal} \\ x(t) \end{array} \rightarrow \boxed{\text{CT system}} \rightarrow \begin{array}{c} \text{Output Signal} \\ y(t) \end{array}$$

The arrows in this diagram are not necessarily wires! They represent whatever medium transports the signal from one part of the system to another part.

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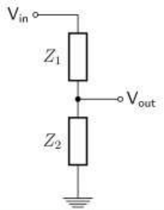
# Example (1)

## Example

voice (acoustic pressure) $\rightarrow$ $\boxed{\text{microphone}}$ $\rightarrow$ electrical current

## Example

voltage divider (https://en.wikipedia.org/wiki/Voltage_divider)




For identical resistors, the output is $y(t) = \frac{1}{2}x(t)$. Called a **static** system.

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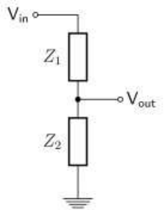


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# Example (2)

**Example**

accelerator pedal position $\rightarrow$ $\boxed{\text{engine/car}}$ $\rightarrow$ car velocity

- Input signal induces a response of the system.
- $x(t) = \frac{1}{2} + \frac{1}{2}u(t-3)$ (one pushes the gas pedal to the floor)
- $y(t) = 40 + 20(1 - e^{-t/\tau})$ (rise time or transient response)
- $y(t)$ is not solely a function of $x(t)$ at time $t$, but also a function of previous input signal values and the present and past state of the system.
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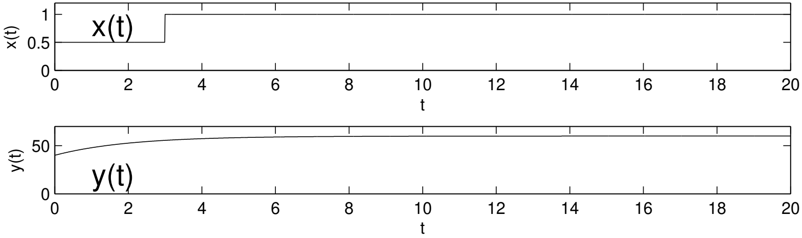
# Example (3)

$$x(t) = \tfrac{1}{2} + \tfrac{1}{2}u(t-3)$$

$$y(t) = 40 + 20(1 - e^{-t/\tau}), \quad \tau = 2$$



# Outline

# Input-output description of systems (1)

Pictures/diagrams are a starting point, but for quantitative analysis every system must have an **input-output relationship**.

**Definition**

**Input-output relationship** is a mathematical expression that precisely defines how the output signal is related to the input signal.

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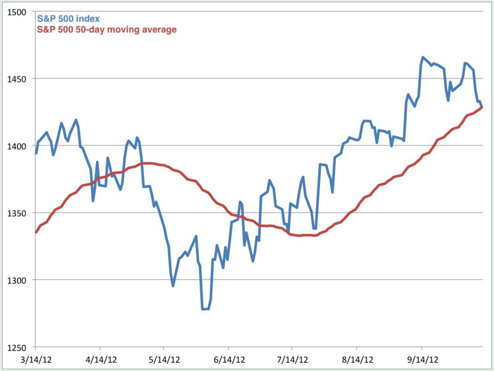
# Moving average

## Example

**moving average** filter

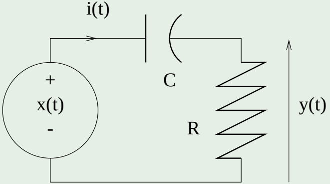
$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau.$$



# Input-output description of systems (2)

## Example

RC circuit driven by voltage source (input signal).



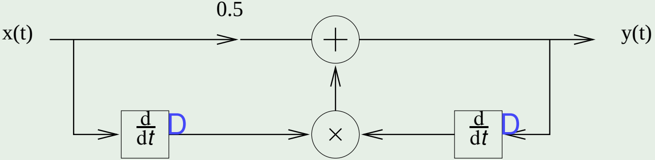

$$\frac{1}{CR}y(t) + \frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t).$$

- This input-output relation is not of the form $y(t) = \text{some\_function}[x(t)]$.
- When combined with an appropriate initial condition (such as 0 charge on the capacitor at time $t = 0$) one can solve the diffeq to determine $y(t)$ for any $x(t)$(**later lectures.**)

# Outline

# Block diagram representation of CT systems

## Example




The lower-right part is called a **feedback connection.**

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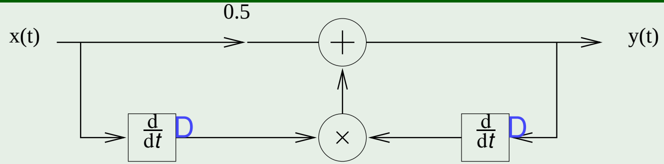


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Basic elements:

- adder (see text Fig. 2.29 on P. 126)
- constant multiplier (amplifier) (see text Fig. 2.29 on P. 126)
- signal multiplier
- differentiator (see text Fig. 2.29 on P. 126)
- integrator (see text Fig. 2.31 & 2.32 on P. 127)

# Block diagram representation of CT systems

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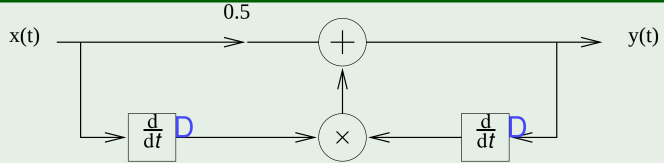


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Input-output relationship defined by the diagram:

$$y(t) = 0.5x(t) + \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right).$$

# Outline

# Interconnection of systems

1. Series connection

$$x(t) \rightarrow \boxed{\mathcal{T}_1} \rightarrow \boxed{\mathcal{T}_2} \rightarrow y(t)$$

   Mathematically: $y(t) = \mathcal{T}_2[\mathcal{T}_1[x(t)]]$.

2. Parallel connection
   (See text Fig. 1.42(b) on P. 42 )
   Mathematically: $y(t) = \mathcal{T}_1[x(t)] + \mathcal{T}_2[x(t)]$.

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In this example the order of interconnection is irrelevant. We will learn soon that this is because both subsystems are linear.

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# Classification of CT systems

Two general aspects to categorize:

- Amplitude properties
  - A-1 linearity (1.6.6)
  - A-2 stability (1.6.4)
  - A-3 invertibility (1.6.2)
- Time properties
  - T-1 causality (1.6.3)
  - T-2 memory (1.6.1)
  - T-3 time-invariance (1.6.5)

**Skill:** ***Determining classifications of a given CT system***

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# A-1 Linearity (1)

**Definition**

A system $\mathcal{T}$ is **linear** iff

$$\mathcal{T}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{T}[x_1(t)] + a_2 \mathcal{T}[x_2(t)]$$

for any signals $x_1(t), x_2(t)$ and any (even complex) constants $a_1$ and $a_2$. Otherwise the system is called **nonlinear**.

Response to a weighted sum of input signals is the weighted sum of the individual responses. ***(Picture)***

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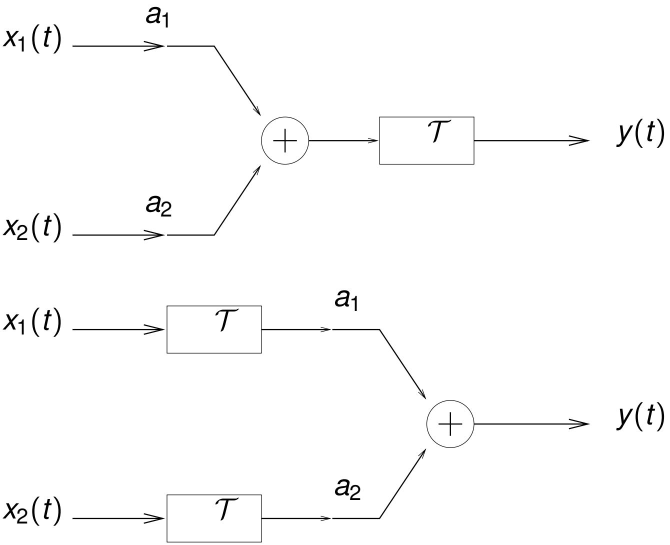

# A-1 Linearity (3)

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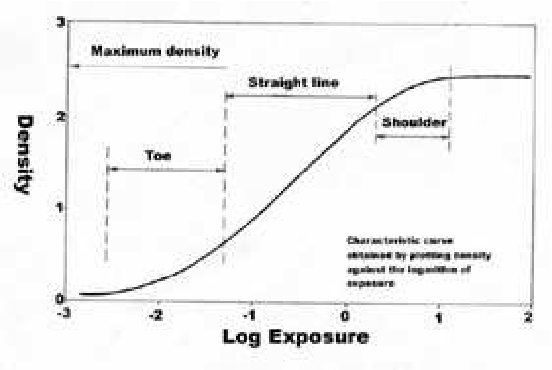
*We will focus on linear systems. Why?*

1. *The class of linear systems is easier to analyze.*
2. *Often linearity is desirable - avoids distortions (Example: amplifiers, audio mixers (superposition!)).*
3. *Many nonlinear systems are approximately linear (Example: the characteristic curve of a photographic film relating the optical density of the film to the logarithm of the incident exposure. **(Picture)**).*

# A-1 Linearity (4)




Real systems are never perfectly linear, but often they are approximately linear over an appropriate operating range.

# Two important special cases of linearity property (1)

## Property

*scaling property* or *homogeneity property*:

$$\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)]$$

Note that from $a = 0$ we see that zero input signal implies zero output signal for a linear system.

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Using proof-by-induction, one can easily extend this property to the general superposition property

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*general superposition property*

$$\mathcal{T}\left[\sum_{k=1}^{K} x_k(t)\right] = \sum_{k=1}^{K} \mathcal{T}[x_k(t)].$$

In words: the response of a linear system to the sum of several signals is the sum of the response to each of the signals.

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- In fact, we even assume that superposition holds for integrals:

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**Skill: *Determining a system is linear or nonlinear.***

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5. If it does not appear that $y(t) = a_1 y_1(t) + a_2 y_2(t)\ \forall t$, then find a specific counter-example.

# Determining a system is linear or nonlinear

**Skill: *Determining a system is linear or nonlinear.***

1. Find output signal $y_1(t)$ for a general input signal $x_1(t)$.
2. "Repeat" for input $x_2(t)$ and $y_2(t)$.
3. Find output signal $y(t)$ when input signal is $x(t) = a_1x_1(t) + a_2x_2(t)$.
4. If $y(t) = a_1y_1(t) + a_2y_2(t)\ \forall t$, then the system is linear.
5. If it does not appear that $y(t) = a_1y_1(t) + a_2y_2(t)\ \forall t$, then find a specific counter-example.

# Example (1)

**Example**

Prove that the integrator is a linear system, where
$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$.

# Example (1)

## Example

Prove that the integrator is a linear system, where $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$.

1. $y_1(t) = \int_{-\infty}^{t} x_1(\tau)\,d\tau$
2. $y_2(t) = \int_{-\infty}^{t} x_2(\tau)\,d\tau$.
3. *If the input is* $x(t) = a_1x_1(t) + a_2x_2(t)$, *then the output is*

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} [a_1x_1(\tau) + a_2x_2(\tau)]\,d\tau \\ &= a_1\int_{-\infty}^{t} x_1(\tau)\,d\tau + a_2\int_{-\infty}^{t} x_2(\tau)\,d\tau = a_1y_1(t) + a_2y_2(t). \end{aligned}$$

4. *Since this holds for all* $t$, *for all input signals* $x_1(t)$ *and* $x_2(t)$, *and for any constants* $a_1$ *and* $a_2$, *the integrator is linear.*

# Example (2)

**Example**

Determine whether linearity holds for $y(t) = \int_{-\infty}^{t} x^3(\tau)\, d\tau$.

# Example (2)

**Example**

Determine whether linearity holds for $y(t) = \int_{-\infty}^{t} x^3(\tau)\, d\tau$.

*Direct method:*

1. $y_1(t) = \int_{-\infty}^{t} x_1^3(\tau)\, d\tau$
2. $y_2(t) = \int_{-\infty}^{t} x_2^3(\tau)\, d\tau$.
3. *If the input is* $x(t) = a_1 x_1(t) + a_2 x_2(t)$, *then the output is*

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} [a_1 x_1(\tau) + a_2 x_2(\tau)]^3\, d\tau \\ &\neq a_1 y_1(t) + a_2 y_2(t). \end{aligned}$$

# Example (2)

**Example**

Determine whether linearity holds for $y(t) = \int_{-\infty}^{t} x^3(\tau)\, d\tau$.

*Counter-example:*

- *if $x(t) = u(t)$, then $y(t) = tu(t)$, which is called the **unit ramp signal**.*
- *But if $x(t) = 2u(t)$, then $y(t) = 8tu(t)$, so doubling the input did not double the output.*
- *Thus the scaling property is violated, so this system is nonlinear.*

# Example (3)

## Example

Are the following systems linear?

- $y(t) = x^3(t)$
- $y(t) = 2x(t) + 3$

# Example (3)

**Example**

Are the following systems linear?

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- $y(t) = 2x(t) + 3$

- *To show that a signal is nonlinear, all that is needed is a counter-example to the scaling and additivity properties.*
- *The scaling property will usually (but not always!) suffice.*

# Example (3)

## Example

Are the following systems linear?

- $y(t) = x^3(t)$
- $y(t) = 2x(t) + 3$

- *Let $x_1(t) = 1$, a constant signal. Then $y_1(t) = 1$.*
- *Now suppose the input is $x(t) = 2x_1(t) = 2$, then the output is $y(t) = 2^3 = 8 \neq 2y_1(t) = 2$*
- *So the system is nonlinear.*

# Example (3)

## Example

Are the following systems linear?

- $y(t) = x^3(t)$
- $y(t) = 2x(t) + 3$

*nonlinear, a zero input yields the output $y(t) = 3$, which is nonzero.*

# Example (4)

**Example**

Is $y(t) = \text{Real}[x(t)]$ linear?

# Example (4)

**Example**

Is $y(t) = \text{Real}[x(t)]$ linear?

*For a system to be called linear, we require the superposition property to hold even for complex signals and scaling constants.*

# Example (4)

**Example**

Is $y(t) = \text{Real}[x(t)]$ linear?

$y(t) = Real[x(t)]$ *satisfies additive property, but not scaling property for complex a. Thus it is not linear.*

# A-2 Stability (1)

**Definition**

A system is **bounded-input bounded-output (BIBO) stable** iff every bounded input produces a bounded output.

If $\exists M_x$ s.t. $|x(t)| \leq M_x < \infty \ \forall t$, then there must exist an $M_y$ s.t.
$$|y(t)| \leq M_y < \infty \ \forall t.$$

Usually $M_y$ will depend on $M_x$.

Otherwise the system is called **unstable**, and it is possible that a small input signal will make the output "blow up."

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# A-2 Stability: example (1)

**Example**

Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ BIBO stable?

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**Example**

Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ BIBO stable?

*Consider input signal $x(t) = u(t)$, which is bounded by $M_x = 1$. But $y(t) = tu(t)$ blows up, so the integrator is an unstable system.*

# Triangle inequality

The **triangle inequality** is sometimes useful for proving that a system is BIBO stable.

- $|a+b| \leq |a|+|b|$ (Easily proved by considering 4 cases where $a$ and $b$ are positive or negative.)
- $\left|\sum_n a_n\right| \leq \sum_n |a_n|$
- $\left|\int f(t)\,dt\right| \leq \int |f(t)|\,dt$

# A-2 Stability: example (2)

**Example**

Is the moving average $y(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)\, d\tau$ for $T > 0$ system BIBO stable?

# A-2 Stability: example (2)

**Example**

Is the moving average $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$ for $T > 0$ system BIBO stable?

*Suppose $|x(t)| \leq M_x < \infty\ \forall t$, so $x(t)$ is a bounded input. Then by the triangle inequality:*

$$|y(t)| \leq \frac{1}{T}\int_{t-T}^{t} |x(\tau)|\,d\tau \leq \frac{1}{T}\int_{t-T}^{t} M_x\,d\tau = M_x,$$

*so the output signal is also bounded for a bounded input. Thus the moving average system is BIBO stable.*

# A-2 Stability: example (3)

**Example**

Is $y(t) = x^5(t)$ BIBO stable?

# A-2 Stability: example (3)

**Example**

Is $y(t) = x^5(t)$ BIBO stable?

*Suppose* $|x(t)| \leq M_x < \infty$. *Then* $|y(t)| = |x^5(t)| \leq M_x^5 < \infty$. *So this system is BIBO stable.*

# A-3 Invertibility

**Definition**

A system $\mathcal{T}$ is called **invertible** iff each (possible) output signal is the response to only one input signal. Otherwise $\mathcal{T}$ is not **invertible**.

**Property**

*If a system $\mathcal{T}$ is invertible, then there exists a system $\mathcal{T}^{-1}$ such that*

$$x(t) \to \boxed{\mathcal{T}} \to y(t) \to \boxed{\mathcal{T}^{-1}} \to z(t) = x(t).$$

*Mathematically:*

$$\mathcal{T}^{-1}[\mathcal{T}[x(t)]] = x(t)$$

Design of $\mathcal{T}^{-1}$ is important in many signal processing applications.

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**Example**

encryption/decryption for secure communication. Needs to be invertible for no loss of information.

**Example**

digital speedometer

velocity $\rightarrow$ $\boxed{\text{speed sensor}}$ $\rightarrow$ voltage

$\rightarrow$ $\boxed{\text{mathematical inverse of sensor law}}$ $\rightarrow$ velocity display.

We display the velocity, not the voltage, so there should be a one-to-one relationship between the two.

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We display the velocity, not the voltage, so there should be a one-to-one relationship between the two.

# A-3 Invertibility: example (2)

**Example**

Is the full-wave rectifier: $y(t) = |x(t)|$ invertible?

# A-3 Invertibility: example (2)

**Example**

Is the full-wave rectifier: $y(t) = |x(t)|$ invertible?

*The distinct input signals $x(t) = \sin t$ and $x(t) = -\sin t$ yield the same output signal. So not invertible.*

# A-3 Invertibility: example (3)

**Example**

Is the exponential-law device: $y(t) = e^{x(t)}$ invertible?

# A-3 Invertibility: example (3)

**Example**

Is the exponential-law device: $y(t) = e^{x(t)}$ invertible?

*Invertible (for real input signals anyway). Inverse system is* $x(t) = \log y(t)$*, a log-law device.*

# A-3 Invertibility: example (4)

**Example**

Is the ideal amplifier: $y(t) = 2x(t)$ invertible?

# A-3 Invertibility: example (4)

**Example**

Is the ideal amplifier: $y(t) = 2x(t)$ invertible?

*Invertible. Inverse system is $x(t) = \frac{1}{2}y(t)$.*

# T-1 Causal systems

## Definition

For a **causal** system, the output $y(t)$ at any time $t$ depends only on the "present" and (possibly) "past" inputs *i.e.* on $x(t)$ and on various $x(t_0)$ for $t_0 \leq t$ only, but not on future inputs.
Otherwise **noncausal** system.

Causality is necessary for real-time implementation. Noncausal systems arise primarily when $t$ is some other variable than time, such as space.

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**Example**

Is the integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ causal?

# T-1 Causal systems: example (1)

**Example**

Is the integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ causal?

**Solution**

*It only depends on values of $x$ for $\tau \leq t$, so causal.*

# T-1 Causal systems: example (2)

**Example**

Is the symmetric moving average: $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\tau)\, d\tau$ causal? (Useful for image processing.)

# T-1 Causal systems: example (2)

**Example**

Is the symmetric moving average: $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\tau)\, d\tau$ causal? (Useful for image processing.)

**Solution**

*noncausal. It depends on* $t + T \geq \tau > t$

# T-2 Memory

## Definition

For a **static system** or **memoryless** system, the output $y(t)$ depends only on the current input $x(t)$, not on previous or future values of the input signal.
Otherwise it is a **dynamic system** and must have memory.

## Example

- $y(t) = e^{x(t)}/\sqrt{|t+3|}$.
- moving average $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$, $T > 0$

Dynamic systems are the interesting ones and will be our focus. (This time we take the more complicated choice!)

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## Example

- $y(t) = e^{x(t)}/\sqrt{|t+3|}$. *memoryless*
- moving average $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau$, $T > 0$ *dynamic*

Dynamic systems are the interesting ones and will be our focus. (This time we take the more complicated choice!)

# Memory vs. causality

**Question**

- *Is a memoryless system necessarily causal?*
- *Is a dynamic system necessarily noncausal?*

# Memory vs. causality

## Question

- *Is a memoryless system necessarily causal?* Yes
- *Is a dynamic system necessarily noncausal?* No. *Dynamic systems van be causal or noncausal.*

# Memory vs. causality

**Question**

- *Is a memoryless system necessarily causal?*
- *Is a dynamic system necessarily noncausal?*

*Memory is often associated with stored energy, such as the charge on a capacitor, or the kinetic energy of a moving object.*

# T-3 Time-invariance (1)

Systems whose input-output behavior does not change with time are called **time-invariant** will be our focus.

- "Easier" to analyze.
- Time-invariance is a desired property of many systems.

We will focus primarily, but not exclusively, on time-invariant systems.

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- Time-invariance is a desired property of many systems.

We will focus primarily, but not exclusively, on time-invariant systems.

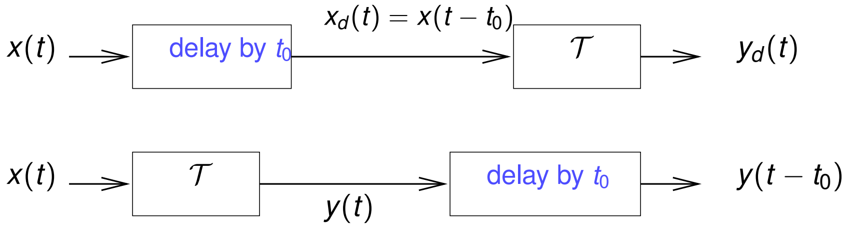
# T-3 Time-invariance (2)

## Definition

A system $\mathcal{T}$ is called **time invariant** or **shift invariant** iff

$$x(t) \xrightarrow{\mathcal{T}} y(t) \quad \text{implies that} \quad x(t - t_0) \xrightarrow{\mathcal{T}} y(t - t_0)$$

for every input signal $x(t)$ and time shift $t_0$. Otherwise the system is called **time variant** or **shift variant**.



# Recipe for showing time-invariance

Recipe for showing time-invariance

1. Determine output signal $y(t)$ due to a generic input signal $x(t)$.
2. Determine the delayed output signal $y(t - t_0)$, by replacing $t$ with $t - t_0$ in $y(t)$ expression.
3. Determine output signal $y_d(t)$ due to a delayed input signal $x_d(t) = x(t - t_0)$.
4. If $y_d(t) = y(t - t_0)$, then system is time-invariant.

# Time-invariance: example (1)

**Example**

Is the symmetric moving average filter
$y(t) = \frac{1}{3}[x(t-1) + x(t) + x(t+1)]$ time-invariant?

# Time-invariance: example (1)

## Example

Is the symmetric moving average filter
$y(t) = \frac{1}{3}[x(t-1) + x(t) + x(t+1)]$ time-invariant? *Yes*

1. *Output due to* $x(t)$ *is* $y(t) = \frac{1}{3}[x(t-1) + x(t) + x(t+1)]$
2. *Delayed output is*

$$y(t-t_0) = \frac{1}{3}[x(t-t_0-1) + x(t-t_0) + x(t-t_0+1)]$$

3. *Output due to delayed input* $x_d(t) = x(t-t_0)$ *is*

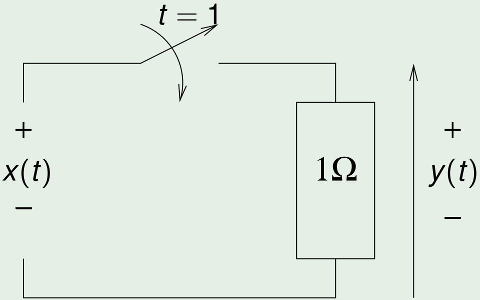
$$\begin{aligned} y_d(t) &= \frac{1}{3}[x_d(t-1) + x_d(t) + x_d(t+1)] \\ &= \frac{1}{3}[x(t-t_0-1) + x(t-t_0) + x(t-t_0+1)] \end{aligned}$$

4. $y_d(t) = y(t-t_0)$

# Time-invariance: example (2)

## Example

A switch that closes at $t = 1$.




1. How to represent input-output relationship mathematically?
2. Is it Time invariant? If no, find a counter-example.

# Time-invariance: solution (2)

## Solution

1. $y(t) = u(t-1)x(t)$.
2. No. Time-varying gain $u(t-1)$.
3. A counter-example.
   - If $x(t) = \delta(t)$, then $y(t) = 0$.
   - But if $x_d(t) = \delta(t-2)$, then $y_d(t) = \delta(t-2)u(t-1) = \delta(t-2) \neq y(t-2)$.

# Time-invariance: solution (2)

## Solution

1. $y(t) = u(t-1)x(t)$.
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   - *If* $x(t) = \delta(t)$, *then* $y(t) = 0$.
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# Time-invariance: solution (3)

**Example**

A switch that closes at $t = 1$: $y(t) = u(t-1)x(t)$.

*For certain $t_0$, $y_d(t) = y(t - t_0)$.*
*if $x_d(t) = \delta(t+2)$, then $y_d(t) = \delta(t+2)u(t-1) = 0 = y(t+2)$.*

---

*A time-varying gain results in a time-varying system while systems with constant gains are time-invariant (e.g., $y(t) = 2x(t)$).*

# Time-invariance: solution (3)

**Example**

A switch that closes at $t = 1$: $y(t) = u(t-1)x(t)$.

*For certain $t_0$, $y_d(t) = y(t - t_0)$.*
*if $x_d(t) = \delta(t+2)$, then $y_d(t) = \delta(t+2)u(t-1) = 0 = y(t+2)$.*

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*A time-varying gain results in a time-varying system while systems with constant gains are time-invariant (e.g., $y(t) = 2x(t)$).*

# Time-invariance: example (3)

**Example**

Is the modulator $y(t) = \cos(\pi t)\, x(t)$ time-invariant?

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**Example**

Is the modulator $y(t) = \cos(\pi t)\, x(t)$ time-invariant?

*No.*

# Time-invariance: example (3)

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Is the modulator $y(t) = \cos(\pi t)\, x(t)$ time-invariant?

*No. How do we show lack of a property? Find a counter-example.*

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**Example**

Is the modulator $y(t) = \cos(\pi t)\, x(t)$ time-invariant?

*No. How do we show lack of a property? Find a counter-example. A simple counter-example is all that is needed, and usually impulse functions will suffice .*

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$$\textit{If} \quad x(t) = \delta(t) \quad \textit{then} \quad y(t) = \delta(t)$$

$$\textit{If} \quad x_d(t) = \delta(t-1) \quad \textit{then} \quad y_d(t) = -\delta(t-1) \neq \delta(t-1) = y(t-1)$$

# Time-invariance: example (3)

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*No. How do we show lack of a property? Find a counter-example. A simple counter-example is all that is needed, and usually impulse functions will suffice .*

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$$\textit{If} \quad x_d(t) = \delta(t-1) \quad \textit{then} \quad y_d(t) = -\delta(t-1) \neq \delta(t-1) = y(t-1)$$

*The modulator is a useful system! (AM radio)*

# Time-invariance: example (4)

**Example**

Is the amplified time reversal $y(t) = 3x(-t)$ time-invariant?

# Time-invariance: example (4)

## Example

Is the amplified time reversal $y(t) = 3x(-t)$ time-invariant?

## Solution

*No. Any time shift in the input will be time-reversed.*

$$\begin{aligned} x(t) = \delta(t) \xrightarrow{\mathcal{T}} y(t) &= 3x(-t) = 3\delta(-t) \\ &= 3\delta(t) \textit{ (by symmetry property of unit impulse)} \end{aligned}$$

$$\begin{aligned} x_d(t) = \delta(t-1) \xrightarrow{\mathcal{T}} y_d(t) &= 3x_d(-t) = 3x_d(z) \quad (\textit{where} \quad z = -t) \\ &= 3\delta(z-1) = 3\delta(-t-1) = 3\delta(-(t+1)) \\ &= 3\delta(t+1) \neq 3\delta(t-1) = y(t-1), \end{aligned}$$

$y_d(t) \neq y(t-1)$ *for* $x_d(t) = \delta(t-1)$*, so system is time varying.*

# Time-invariance: example (4)

**Example**

Is the amplified time reversal $y(t) = 3x(-t)$ time-invariant?

*Any time shift in the input will be time-reversed. Similarly, time-scaled (compressed and expanded) systems are time-variant. (text Example 1.16)*

# Time-invariance: example (5)

## Example

Is the integrator $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ time-invariant?

# Time-invariance: example (5)

## Example

Is the integrator $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ time-invariant?

*Yes.*

1. $y(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$
2. $x_d(t) = x(t - t_0)$

$$
\begin{aligned}
y_d(t) &= \int_{-\infty}^{t} x_d(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau - t_0)\, d\tau \\
&= \int_{-\infty}^{t-t_0} x(\tau')\, d\tau' \quad (\tau' = \tau - t_0) \\
&= y(t - t_0)
\end{aligned}
$$

# Outline

# Summary (1)

- signal notation
- signal transformations
  - time transformations
  - amplitude transformations
  - differentiator / integrator systems
  - two-signal operations
- signal classes
  - even/odd signals
  - energy/power signals
  - periodic/aperiodic signals
  - exponential signals

# Summary (2)

- singularity functions
  - unit step / rect signals
  - unit impulse function
  - impulse function properties (sifting, sampling, scaling)
- CT systems
- block diagrams
- system classes
  - amplitude properties: linearity, stability, invertibility
  - time properties: causality, memory, time-invariance

# Key concepts/skills to study

Key concepts/skills to study

- time transformations
- braces/plots to rects/steps
- running integral operation
- properties of $\delta(t)$
- identifying signal properties
- identifying (all six) system properties