Ve 216: Introduction to Signals and Systems

Yong Long

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

May 8, 2023

Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

Outline

Outline

Overview

Signals

- definition
- classes
- notation
- transformations (operations)
- important signals

(Skip: 1.3.2, 1.3.3, 1.4.1)

Systems

- definition
- block diagrams
- system interconnection
- classes
- linearity, time-invariance

Goal: eventually system design; must first learn to analyze!

Outline

Signal Definition

Definition

A **signal** is any "physical" quantity that varies with time or space (or any other independent variable or variables).

Often when we discuss signals we refer to mathematical representation of the physical quantity.

Example

An approaching ambulance siren produces a time-varying change in acoustic pressure that our ears perceive as sound.

$$s(t) = (1 + t)\sin\left(2\pi[1000t + 10t^2 + 300\sin(2\pi t/2)]\right)$$

Example: Ambulance Siren

$$s(t) = (1+t)\sin\Big(2\pi[1000t + 10t^2 + 300\sin(2\pi t/2)]\Big)$$

- The $(1+t)$ amplitude term represents increasing loudness as the ambulance approaches.
- The 1000t term represents the 1kHz siren oscillation.
- The $10t^2$ term represents increasing pitch due to the Doppler effect as the ambulance approaches.
- The $300\sin(2\pi 2t)$ term represents the eeh-ooh-eeh-ooh periodic variation in pitch.

System Definition

Definition

A **system** is a physical "device" that performs an operation on a signal.

Example

The human ear converts acoustic signals into electrical nerve synapses (another signal) that are processed by the brain. The input and output signals are different physical quantities.

Signal processing

One of the main roles of electrical engineers is to design and analyze systems that take some input signal and produce some related (but almost always different) output signal.

We refer such operations as **signal processing**.

Signals and Systems

Example

For an audio amplifier, ideally the output signal is "simply" an amplified version of the input signal. (On paper it is easy:

$$s_{\mathrm{out}}(t) = a s_{\mathrm{in}}(t).$$

But implementing this in analog hardware with minimal distortion is nontrivial.)

This course will emphasize **continuous-time** or **analog** signals, and briefly introduce **discrete-time** or **digital** signals at the end.
(Portions of Chapters 1-10)

Outline

A signal is a function

Mathematically, a signal is a function of one or more independent variables.

Question

What is a function?

Classification of Signals: Dimensionality (1)

One way to classify signals is by the dimension of the domain of the function, *i.e.* how many arguments the function has.

Definition

A **one-dimensional** signal is a function of a single variable, *e.g.* time.

Classification of Signals: Dimensionality (2)

Question

*What is a **M-dimensional** signal?*

Classification of Signals: Dimensionality (3)

Example

A sequence of BW TV pictures $I(x, y, t)$ is a scalar valued function of two spatial coordinates x and y and time t, so it is a 3D signal.

We will focus on one-dimensional signals in this course, generally considering the independent variable to be time t.

Classification of Signals: Dimensionality (3)

Example

A sequence of BW TV pictures $I(x, y, t)$ is a scalar valued function of two spatial coordinates x and y and time t, so it is a 3D signal.

We will focus on one-dimensional signals in this course, generally considering the independent variable to be time t.

Classification of Signals: Dimensionality (4)

Another way to classify signals is by the dimension of the range of the function, *i.e.*, the space of values the function can take.

Definition

A **scalar** or **single-channel** signal is a function of a real-valued scalar or complex-valued scalar.

Question

*What is a **multichannel** signal?*

Classification of Signals: Dimensionality (5)

Example

A color TV picture can be described by a red, blue and green signal, whereas a BW TV picture is scalar valued.

We will focus on scalar signals in this course, both real and complex.

Most of the design/analysis techniques generalize to multichannel and multidimensional signals.

Continuous-time signals

Definition

A **continuous-time signal** or **analog signal** is a function defined for all times $t \in (-\infty, \infty)$, or at least over some continuous interval (a, b).

Example

$x(t) = e^{-t^2}, \ -\infty < t < \infty$. ***(Picture)***

Discrete-time signals

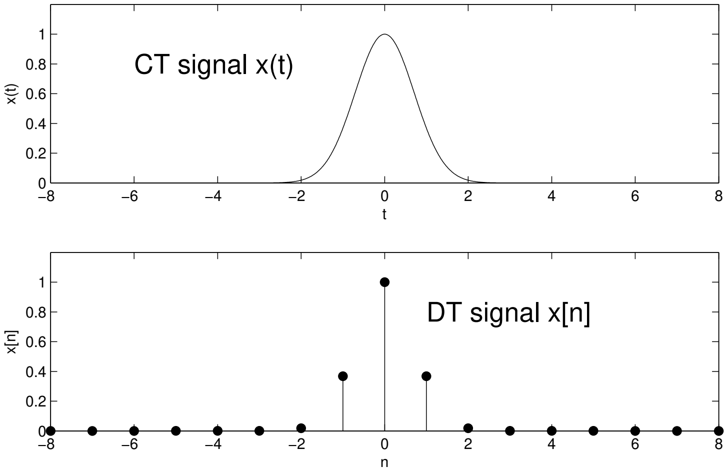
Definition

A **Discrete-time signal** is a function defined only at certain specific values of time.

Example

$x[n] = x(t_n)$. ***(Picture)***

Continuous-time signals vs. discrete-time signals



Classification of Signals: time characteristics

Classify signals by time characteristics

1. Continuous-time signals or analog signals
2. Discrete-time signals

Discrete-time signals arise from

- Sampling a continuous signal at discrete time instants
- Accumulating a quantity over a period of time

Example

When counting number of heart attacks per month, n would index the month, and $x[n]$ would be the number.

Classification: Value characteristics

Definition

A **continuous-valued signal** or **continuous-amplitude signal** can take any value in some continuous interval.

Example

Voltage between 0 and 5 volts.

Definition

A **discrete-valued signal** or **discrete-amplitude signal** only takes values from a discrete set of possible values.

Example

In heart attack example, $x[n]$ could be $0, 1, 2, \ldots,$ population of world.

Determinstic vs Random signals

1. **Deterministic signals** can be described by an explicit mathematical representation.
2. **Random signals** evolve over time in an unpredictable manner.

Example

"Hiss" or "noise" in an audio system.

We will focus on deterministic signals, although reducing noise (eliminating a random component) is often a goal in designing signal processing systems.

Classification: Our focus

We will focus on

- single-channel, one-dimensional, continuous-valued, continuous-time signals.
- $x(t)$ is a scalar valued function of a real independent variable.
- Mathematically

$$x : \mathbb{R} \to \mathbb{R} \text{ or } x : \mathbb{R} \to \mathbb{C}$$

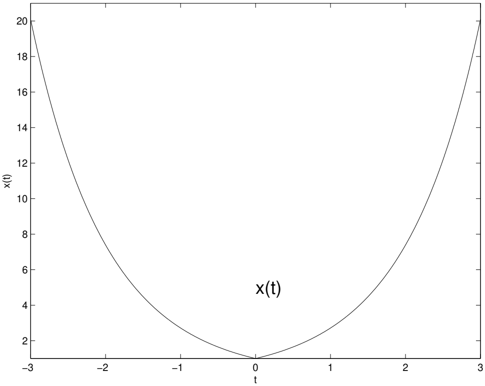
Outline

Signal notation (mathematical representation)(1)

1. Graphically ***(Picture)***:
2. Braces or piecewise notation: $x(t) = \begin{cases} e^{-t}, & t > 0 \\ e^{t}, & t \leq 0. \end{cases}$
3. Formula: $x(t) = e^{-|t|}$.
4. In terms of other functions: $x(t) = s(t) + s(-t)$ where
$$s(t) = \begin{cases} e^{-t}, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0. \end{cases}$$
5. Fourier representation:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t}\, d\omega$$

Skill: ***Convert between different signal representations.***
Skill: ***Choose representation most appropriate for a given problem.***

Signal notation (mathematical representation)(2)



Outline

Eventual goal

- Eventual goal: analyze interesting signals and to analyze and design useful systems.
- Such signals and systems consist of combinations of simpler (less interesting?) signals and systems.
- So we walk before we run...

Transformations of CT signals

- Time transformations
  - Folding/reflection/time-reversal
  - Time-scaling
  - Time-shifting/time-delay
  - General time transformation
- Amplitude transformations
  - Amplitude reversal
  - Amplitude scaling
  - Amplitude shifting
- More signal operations
  - Differentiator
  - Integrator
- Operations with two signals

Change of variables

If $x(t) = e^{-(t-2)}$ then $y(t) = x\left(\frac{t-1}{3}\right)$ is another function;

$$y(t) = e^{-[(t-1)/3-2]} = e^{-\left(\frac{t-7}{3}\right)}.$$

In calculus, this type of transformation is called a **change of variables**.

Time transformations

Here we give some new names to such transformations to reflect the physical meaning of the mathematics.

Example

$$x(t) = \begin{cases} e^{-(t-2)}, & t \geq 2, \\ 0, & \text{otherwise.} \end{cases} \quad \textit{\textbf{(Picture)}}.$$

(used throughout)

One can apply time transformations both graphically and mathematically. Both approaches are useful.

Folding/reflection/time-reversal: $y(t) = x(-t)$ (1)

Folding/reflection/time-reversal

$$y(t) = x(-t)$$

Example

- Backwards play a movie/audio tape.
- A mirror is an optical system that does "space reversal".

Example

Find $y(t) = x(-t)$ for $x(t)$ above.

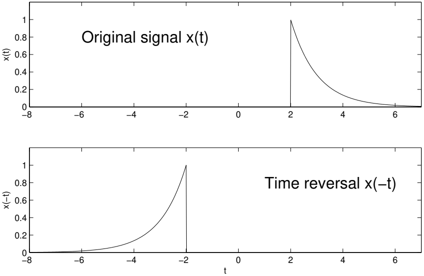
Folding/reflection/time-reversal: $y(t) = x(-t)$ (2)

Mathematical method:

- Replace all t's with $-t$,
- Simplify where possible.

$$y(t) = x(-t)$$
$$= \begin{cases} e^{-(-t-2)}, & -t \geq 2, \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} e^{t+2}, & t \leq -2, \\ 0, & \text{otherwise} \end{cases}$$

Graphical method:



Note that $y(t)$ becomes a "mirror image" of $x(t)$ around $t = 0$.

Time-scaling: $y(t) = x(at), a > 1$

Time-scaling

$$y(t) = x(at)$$

$a > 1$ will shrink or compress the signal

Example

Playing a recording at 3 times the normal speed.

Example

Find $y(t) = x(3t)$ for $x(t)$ above.

Time-scaling: $y(t) = x(at), a < 1$

Time-scaling

$$y(t) = x(at)$$

$a < 1$ will stretch or expand the signal

Example

slow-motion part of a movie

Example

Find $y(t) = x(t/2)$ for $x(t)$ above.

Time shifting: $y(t) = x(t - t_0)$

$$\boxed{\text{Time shifting: } y(t) = x(t - t_0)}$$

- t_0 can be positive (delayed signal) or negative (advanced signal).
- Physical systems can only delay, not advance, in time.

Example

"Park distance control" propagation delay

Example

Find $y(t) = x(t - 1)$ for $x(t)$ above.

General time transformation

General time transformation involves all three of the above time transformations (time reversal, time scaling, and time shifting).

Two distinct (but related) forms:

$$y(t) = x(at - b) = x\left(\frac{t - t_0}{w}\right)$$

where $t_0 = b/a$ and $w = 1/a$ or equivalently $a = 1/w$ and $b = t_0/w$.

Mathematical time transformation

mathematical recipe:

1. Replace all occurrences of t in the definition of $x(t)$ with $at - b$ or with $\frac{t-t_0}{w}$.
2. Manipulate algebraically to simplify.

Example

Find $y(t) = x(-t/2 + 5) = x\left(\frac{t-10}{-2}\right)$ for $x(t)$ above.

Graphical time transformations

Question

How to perform a general time transformation graphically? Should you "shift first" or "scale first?"

Form 1: $y(t) = x(at - b)$

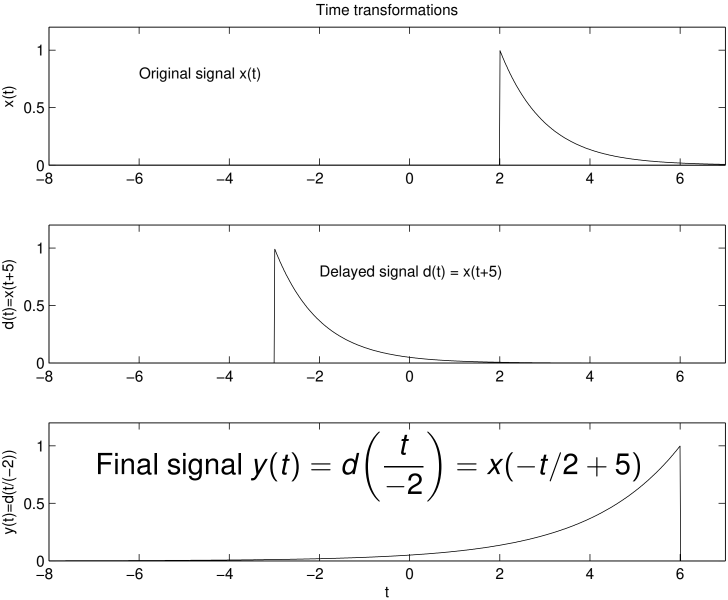
$$\boxed{\text{Form 1}: y(t) = x(at - b)}$$

- Introduce an intermediate signal $d(t) = x(t - b)$.
- Clearly $d(t)$ is just a delayed version of $x(t)$ by amount b.
- But $y(t) = d(at)$, which is just a scaled version of the signal $d(t)$.

To find $y(t) = x(at - b)$ graphically we must

1. time-delay the signal $x(t)$ by b.
2. time-scale that delayed signal by a.

Form 1: $y(t) = x(at - b)$ (Cont.)



Form 2: $y(t) = x\left(\frac{t-t_0}{w}\right)$

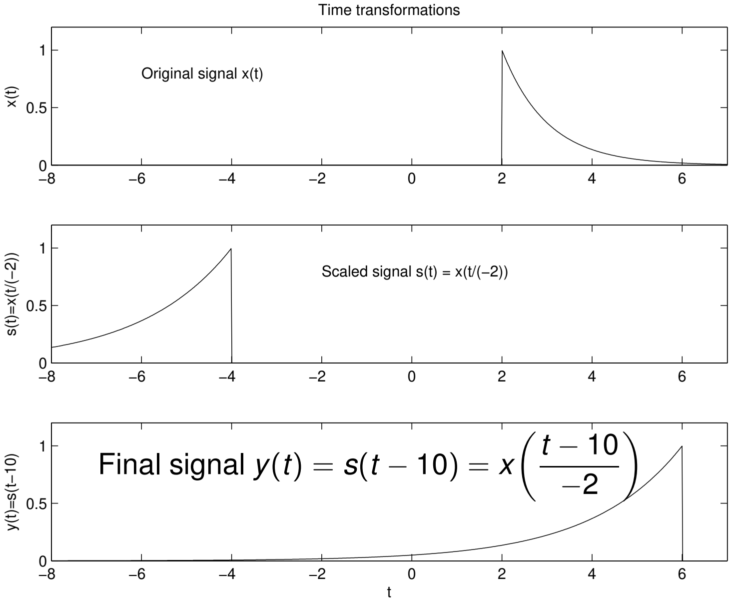
$$\boxed{\text{Form 2}: y(t) = x\left(\frac{t-t_0}{w}\right)}$$

- Introduce an intermediate signal $s(t) = x(t/w)$.
- Clearly $s(t)$ is a time-scaled version of $x(t)$ by the factor $1/w$.
- But $y(t) = s(t - t_0)$, which is just time delay of the signal $s(t)$.

To find $y(t) = x\left(\frac{t-t_0}{w}\right)$ graphically, we must

1. time-scale the signal $x(t)$ by $1/w$.
2. time-delay that scaled signal by t_0.

Form 2: $y(t) = x\left(\frac{t-t_0}{w}\right)$ (Cont.)



Final signal $y(t) = s(t-10) = x\left(\frac{t-10}{-2}\right)$

Amplitude transformations

1. **amplitude reversal** $y(t) = -x(t)$
2. **amplitude scaling** $y(t) = ax(t)$
3. **amplitude shifting** $y(t) = x(t) + b$

Example

(Using all three.) Find $y(t) = -3x(t) + 2$ for the $x(t)$ above.

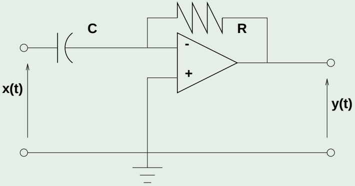
Outline

Differentiator $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$

Differentiator $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$

Example

$$y(t) = -RC\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$



Example

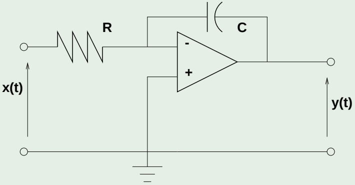
Find the differentiated signal of $x(t) = e^{-2|t|}$.

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Example

$$y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\tau)\, d\tau$$



Example

Find the integrated signal of $x(t) = e^{-|t|}$.

Integrator: example

Solution

1. *rewrite* $x(\cdot)$ *in terms of* τ

$$x(\tau) = \begin{cases} e^{-\tau}, & \tau > 0 \\ e^{\tau}, & \tau \leq 0. \end{cases}$$

2. *For* $t \leq 0$*:*

$$y(t) = \int_{-\infty}^{t} e^{\tau}\, d\tau = e^{t}$$

3. *For* $t \geq 0$*:*

$$\begin{aligned} y(t) &= \int_{-\infty}^{0} e^{\tau}\, d\tau + \int_{0}^{t} e^{-\tau}\, d\tau \\ &= e^{0} + (-e^{-\tau})\big|_{0}^{t} = 1 + (1 - e^{-t}) \\ &= 2 - e^{-t}. \end{aligned}$$

Integrator vs. Integration in calculus

Question

What is the distinction between simple integration of the kind learned in calculus (computing area under a curve) and the integrator system described here.

Outline

Operations with two signals

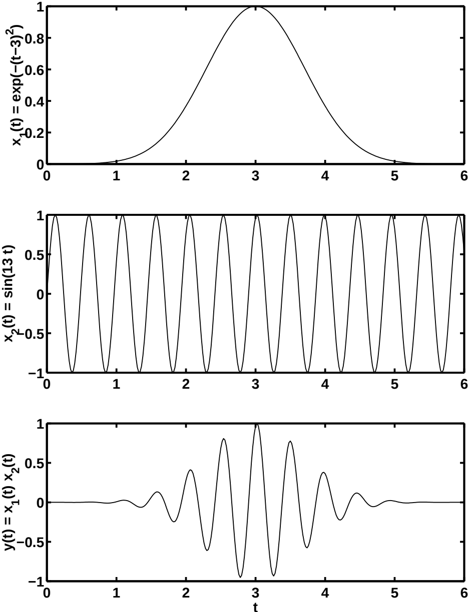
Operations with two signals

1. sum of two signals $y(t) = x_1(t) + x_2(t)$
2. product of two signals $y(t) = x_1(t)\, x_2(t)$.
   Add or multiply two signals at every time point.

Example

If $x_1(t) = e^{-(t-3)^2}$, $x_2(t) = \sin(13t)$, then
$y(t) = x_1(t)x_2(t) = e^{-(t-3)^2}\sin(13t)$.
This is called **amplitude modulation**.

amplitude modulation



Outline

Periodic signals

Why study periodic signals?

- important for analysis
- solution to ideal LC electrical circuits
- periodic physical phenomena: frictionless pendulums, earth rotation, heart rhythms, etc.

Definition

$x(t)$ is **periodic** with a **period** $T > 0$ iff

$$x(t+T) = x(t) \; \forall t \tag{1}$$

Definition

If no such $T > 0$ exists, $x(t)$ is called **aperiodic**

Fundamental period

Definition

The **fundamental period** T_0 of a signal is the smallest value of T satisfying (1).

Theorem

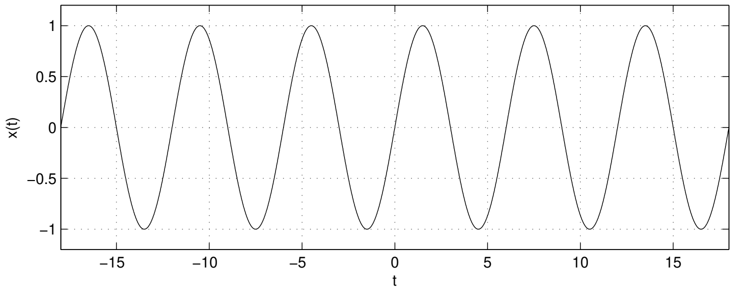
A signal that is periodic with period $T > 0$, is also periodic with period nT for any integer $n \neq 0$, i.e.

$$x(t + nT) = x(t).$$

Periodic signals: example

Example

$x(t) = \sin\left(\frac{\pi}{3}t\right) = \sin\left(\frac{2\pi}{T_0}t\right)$ What is fundamental period?



Sums of two periodic signals

Question

Suppose $x_1(t)$ is periodic with period T_1 and $x_2(t)$ is periodic with period T_2 and $x(t) = x_1(t) + x_2(t)$.

- *Is $x(t)$ periodic?*
- *If so, what is a period T of $x(t)$?*

Sums of two periodic signals: solution (1)

Solution

1. *Easy case: if $T_1 = T_2$ then, $x(t)$ is periodic, and $T = T_1 = T_2$.*
2. *General case. We know $x_1(t) = x_1(t + T_1)$ and $x_2(t) = x_2(t + T_2)$ and $x(t) = x_1(t) + x_2(t)$.*
   We want to determine if there is any value of $T > 0$ such that $x(t) = x(t + T)$.

Sums of two periodic signals: solution (2)

Suppose there is a value of $T > 0$ that satisfies $T = n_1 T_1$ and $T = n_2 T_2$, for some nonzero integers n_1 and n_2. Then

$$\begin{aligned} x(t+T) &= x_1(t+T) + x_2(t+T) = x_1(t+n_1 T_1) + x_2(t+n_2 T_2) \\ &= x_1(t) + x_2(t) = x(t), \end{aligned}$$

so $x(t)$ is periodic with period T.

Sums of two periodic signals: solution (3)

The conditions $T = n_1 T_1$ and $T = n_2 T_2$ are equivalent to requiring that

$$n_1 T_1 = n_2 T_2 \;\text{ so }\; T_1/T_2 = n_2/n_1,$$

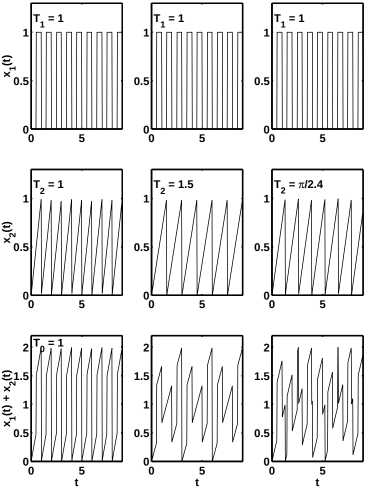
which means that T_1/T_2 is rational, a ratio of two integers.

Theorem

A sum of two periodic signals is periodic iff the ratio of their periods is rational.

Sums of two periodic signals: example

Are the signals in the third row periodic?



Sums of two periodic signals: fundamental period

Question

The least common multiple of the fundamental periods of the two signals is a period of the sum. Is it the fundamental period?

Outline

Even and odd symmetry

Definition

$x(t)$ has **even symmetry** iff $x(-t) = x(t)\ \forall t$

Definition

$x(t)$ has **odd symmetry** iff $x(-t) = -x(t)\ \forall t$

Note that if $x(t)$ has odd symmetry, then $x(0) = -x(0)$ so $x(0) = 0$.

Question

If $x(0) = 0$, does $x(t)$ have odd symmetry?

Even and odd components

We can decompose any signal into even and odd components:

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) \triangleq \frac{1}{2}\left[x(t) + x(-t)\right], \quad x_o(t) \triangleq \frac{1}{2}\left[x(t) - x(-t)\right]$$

Question

Is the following $x(t)$ even or odd?

$$x(t) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{otherwise}, \end{cases}$$

Outline

Average value and energy

Definition

The **average value** of a signal is defined as

$$A \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt.$$

Example

The average value of an odd signal is zero.

Definition

The **energy** of a signal $x(t)$ is defined as

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2\, dt.$$

Average power and energy signal

Definition

The **average power** of a signal is defined as

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

Definition

If E is finite ($E < \infty$) then $x(t)$ is called an **energy signal** and $P = 0$.

Energy and power signals

Definition

If E is infinite, then P can be either finite or infinite. If P is finite and nonzero, then $x(t)$ is called a **power signal**.

Some signals are neither energy signals nor power signals, such as $x(t) = t^2$, for which $E = \infty$ and $P = \infty$. Such signals are generally of little practical engineering importance.

Energy and power signals: example

Example

consider $x(t) = 5 + a\cos t$ where $0 < a < \infty$.

- Find the average value of $x(t)$?
- Is $x(t)$ a power signal, an energy signal, or neither?

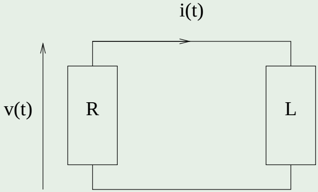
Outline

Exponential signals (1)

Sinusoidal signals, exponential signals, and complex exponentials signals are particularly important because they arise from the solutions of linear constant-coefficient differential equations.

Example

an RL circuit



$$v(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i(t) = L\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{v(t)}{R}\right)$$

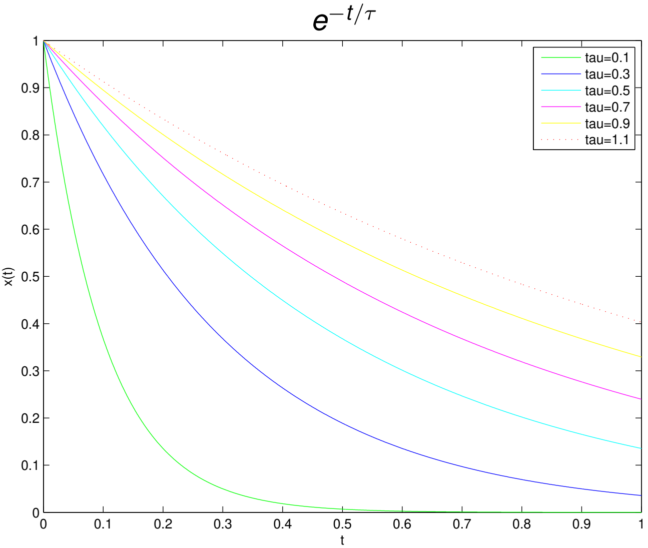
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \left(-\frac{R}{L}\right)v(t) = av(t)$$

where $a = -R/L$. Solution for $t > 0$ is

$$v(t) = v(0)e^{at} = v(0)e^{-t/\tau}$$

where $\tau = -1/a = L/R$ is called the time constant of the circuit. ***(Picture)***

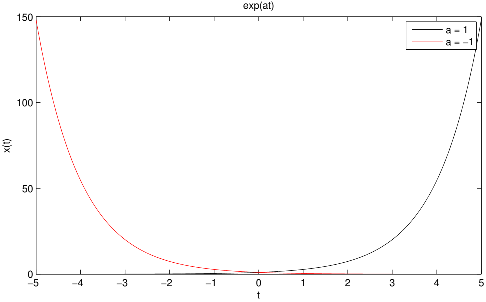
Exponential signals (2)



Exponential signals (3)

Signals of the form $x(t) = ce^{at}$ are very important, for both real and complex c and a.

- $a > 0$ real *e.g.*, population growth
- $a < 0$ real *e.g.*, radioactive decay,



Exponential signals (4)

- If a is purely imaginary, we get $x(t) = ce^{j\omega_0 t}$, a **complex exponential** signal.
- If $c = Ae^{j\phi}$, then $x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$, a **sinusoid** signal.
- If $x(t) = e^{st}$ where $s = a + j\omega_0$ and $a < 0$, then $x(t) = e^{at}(\cos\omega_0 t + j\sin\omega_0 t)$ which is called a **damped sinusoid** signal. (See textbook p21.)

Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Outline

Outline

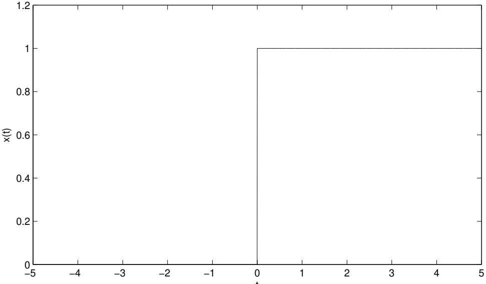
Unit step function/Signal

Definition

A **unit step function(signal)** is defined as

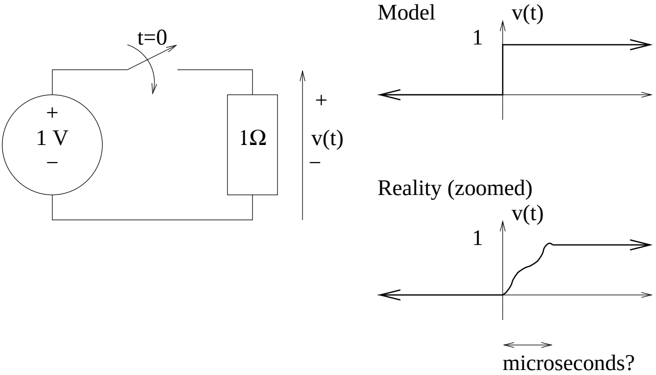
$$u(t) \triangleq \begin{cases} 1, & t > 0 \text{ or } t \geq 0 \\ 0, & t < 0 \end{cases}$$

The value at $t = 0$ is arbitrary and unimportant! Reasonable choices are 0, 1, and $\frac{1}{2}$; any will do.



Modeling a switch

The unit step is a useful model for a switch.



Question

- *For a real switch in above is the voltage exactly a step function?*
- *Does the final voltage exactly equal 1 Volt?*

Simplifying notation

The step function is useful for simplifying notation.

Example

The following step function "switches on" the signal from a guitar string plucked at time $t = 2$.

$$x(t) = \begin{cases} e^{-t}\sin(5t), & t > 2 \\ 0, & \text{otherwise} \end{cases} = e^{-t}\sin(5t)\,u(t-2).$$

The first notation is messy, the second way is neat, and hides the braces within the definition of $u(t-2)$.

Outline

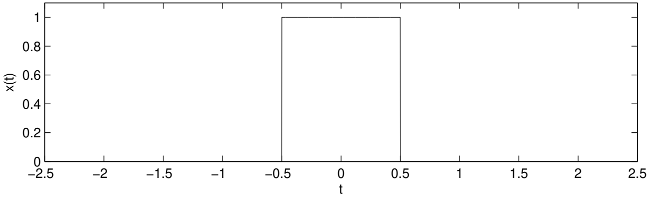
Rect(angle) function

Definition

A **rect(angle) function** is defined as

$$\mathrm{rect}(t) \triangleq \begin{cases} 1, & -1/2 < t < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

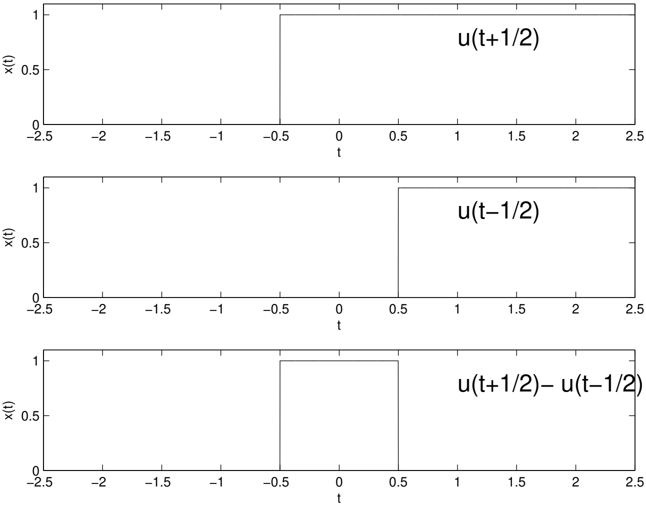
Centered at zero with unit width and unit height.



Rect function and step functions (1)

Can be represented using step functions:

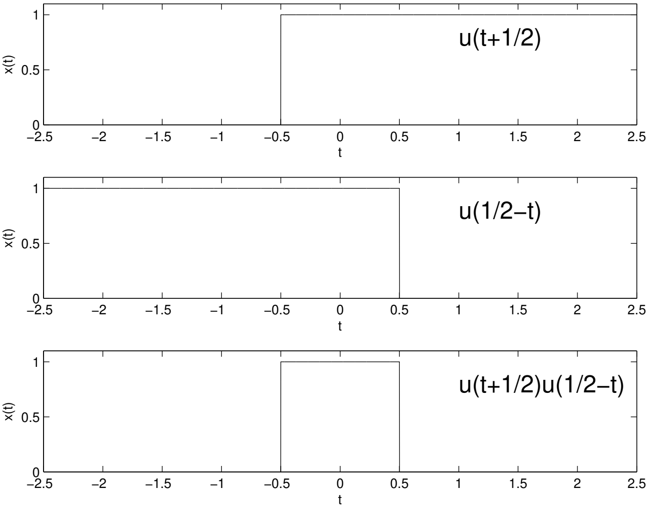
$$\mathrm{rect}(t) = u(t + 1/2) - u(t - 1/2)$$



Rect function and step functions (2)

Can be represented using step functions:

$$\text{rect}(t) = u(t + 1/2)u(1/2 - t)$$

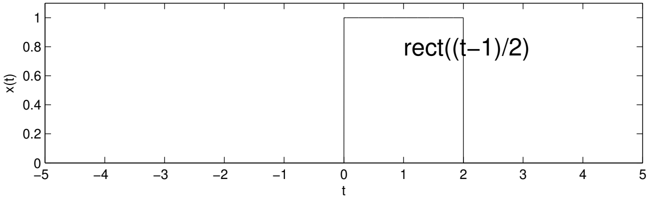


Transformed rect functions

Time-scaled and time-shifted rect function

$$\mathrm{rect}\left(\frac{t - t_0}{T}\right) = \begin{cases} 1, & -1/2 < \frac{t-t_0}{T} < 1/2 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & t_0 - T/2 < t < t_0 + T/2 \\ 0, & \text{otherwise.} \end{cases}$$

Centered at t_0 with width T

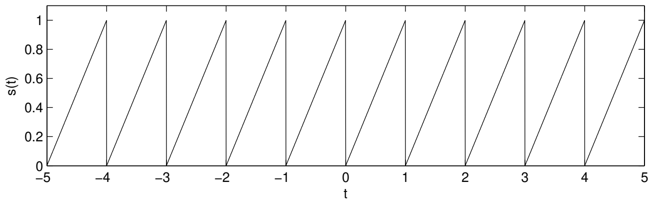


Rect function: example

Useful for “switching on and off” other functions, or for “extracting” on part of a signal, such as one period of a periodic signal.

Example

Find mathematical expression for a sawtooth signal $s(t)$.



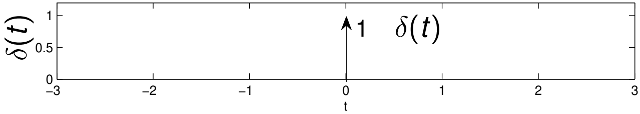
Outline

Unit impulse function $\delta(t)$(1)

- **Unit impulse function** $\delta(t)$, aka **Dirac delta function** or just **delta function**.
- It is another mathematical idealization that cannot occur in nature (like the unit step function), but is nevertheless useful for modeling certain phenomena, just as the step function is a useful idealization of a switch.
- More importantly, it will greatly simplify our analysis of LTI systems later.

Unit impulse function $\delta(t)$(2)

$\delta(t)$ is like a pulse of zero width but infinite height and unit area.



Graphical representation using upward arrow, labeled with area (called **weight**). (see text p.34 for scaled impulse.)

Such a thing is clearly not quite a “function” in the usual sense defined in calculus.
We can “define” an impulse function through its properties.

Minor Properties

Property

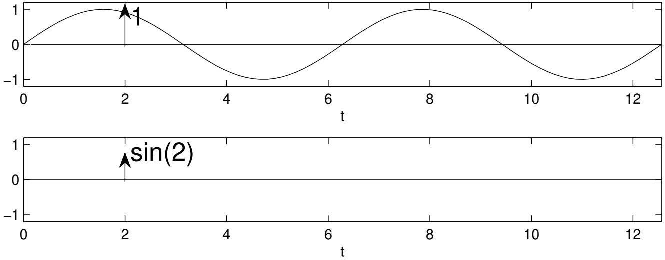
1. *unit area property* $\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$ *for any* t_0
2. *scaling property* $\delta(at + b) = \frac{1}{|a|}\delta(t + b/a)$ *for* $a \neq 0$.
3. *symmetry property* $\delta(t) = \delta(-t)$
4. *support property* $\delta(t - t_0) = 0$ *for* $t \neq t_0$
5. *relationships with unit step function:* $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$, $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$

Major properties: Sampling property

Property

Sampling property *holds when $x(t)$ is continuous at t_0:*

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

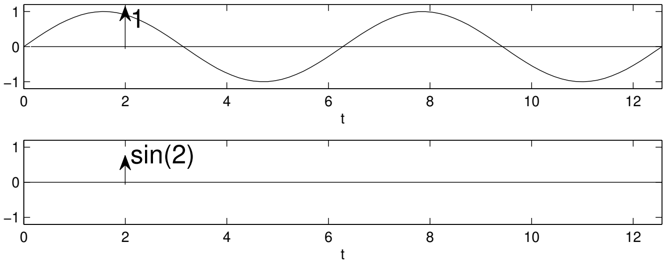


Major properties: Sifting property

Property

Sifting property *holds when $x(t)$ is continuous at t_0:*

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0).$$



Algebraic property

Property

Algebraic property

$$t\delta(t) = 0$$

Scaling property

Example

Show that $\delta(at) = \frac{1}{|a|}\delta(t)$ for $a \neq 0$.

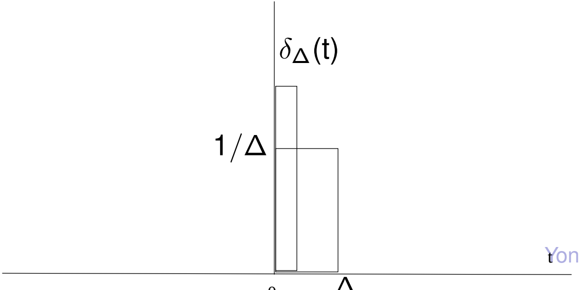
Practical impulse function

Definition

Practical impulse function, defined for any $\Delta > 0$:

$$\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Note that area is unity, width approaches zero as $\Delta \to 0$; height approaches infinity as $\Delta \to 0$.



Practical impulse function: example

Example

drumstick striking a drum (applied force vs time)

Example

metal hammer tapping a pendulum (applied force vs time)

Practical impulse function: example

It is tempting to try to write

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

but the limit is not well defined mathematically. Nevertheless, this is the intuition.

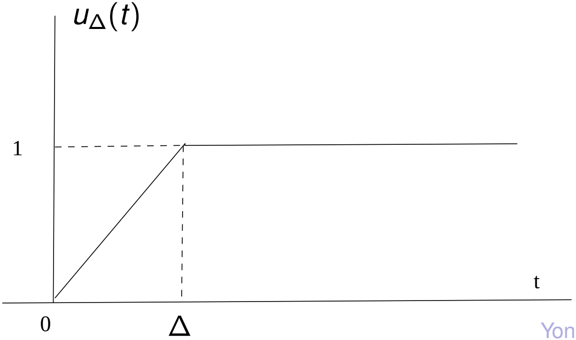
Instead we “define” $\delta(t)$ in terms of its properties, making sure that the properties are consistent with the above “limit”. Such objects are called generalized functions in mathematics.

Relationship to unit step function (1)

Explanation of $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$ using limiting step function.

Define a practical almost-step function as

$$u_\Delta(t) \triangleq \begin{cases} 0, & t \leq 0 \\ t/\Delta, & 0 < t < \Delta \\ 1, & t \geq \Delta \end{cases}$$



Relationship to unit step function (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\Delta(t) = \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases} = \delta_\Delta(t)$$

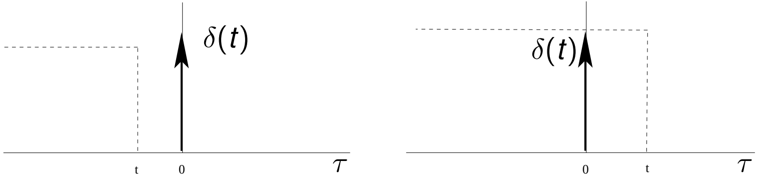
Then since $u(t) = \lim_{\Delta \to 0} u_\Delta(t)$, by taking the limit of both sides we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \lim_{\Delta \to 0} \frac{\mathrm{d}}{\mathrm{d}t}u_\Delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t).$$

Relationship to unit step function (3)

Show graphically that $\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$.

- For any $t < 0$, the range of integration over $(-\infty, t)$ will not cover zero, so the integral is simply zero.
- For any $t > 0$, integration over the range $(-\infty, t)$ covers zero, so the entire unit area of the impulse is included in the integral, so the integral evaluates to 1 for any $t > 0$.



Example (1)

Example

- By Newton's laws, velocity is the time-integral of acceleration.
- When a hammer taps a stationary pendulum, the pendulum (almost) instantaneously changes from being stationary to moving with some velocity that is related to how "hard" the hammer taps the pendulum.
- The acceleration is like a Dirac delta function, and the velocity is like a step function.

Example (2)

Example

Let $x(t) = 2\,\mathrm{rect}(t/2 - 3)$. Find $y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$.

Outline

Continuous-time systems

Definition

A **continuous-time system** is a device or process that, according to some well-defined rule, transforms one CT signal called the **input signal** or **excitation** into another CT signal called the **output signal** or **response**.

The input signal $x(t)$ is transformed by the system into a signal $y(t)$, which we express mathematically as

$$y(\cdot) = \mathcal{T}[x(\cdot)] \quad \text{or} \quad y(t) = \mathcal{T}[x(\cdot)](t) \quad \text{or} \quad x(\cdot) \xrightarrow{\mathcal{T}} y(\cdot).$$

Notation

Question

The notation $y(t) = \mathcal{T}[x(t)]$ *is mathematically vague. Why?*

Diagram

$$\begin{array}{c} \text{Input Signal} \\ x(t) \end{array} \rightarrow \boxed{\text{CT system}} \rightarrow \begin{array}{c} \text{Output Signal} \\ y(t) \end{array}$$

The arrows in this diagram are not necessarily wires! They represent whatever medium transports the signal from one part of the system to another part.

At the systems level, we are less interested in the details of the implementation than in the mathematical relationships and system properties.

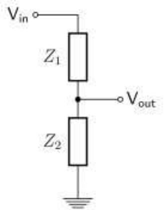
Example (1)

Example

voice (acoustic pressure) $\rightarrow$ $\boxed{\text{microphone}}$ $\rightarrow$ electrical current

Example

voltage divider
(https://en.wikipedia.org/wiki/Voltage_divider)



For identical resistors, the output is $y(t) = \frac{1}{2}x(t)$. Called a **static** system.

Example (2)

Example

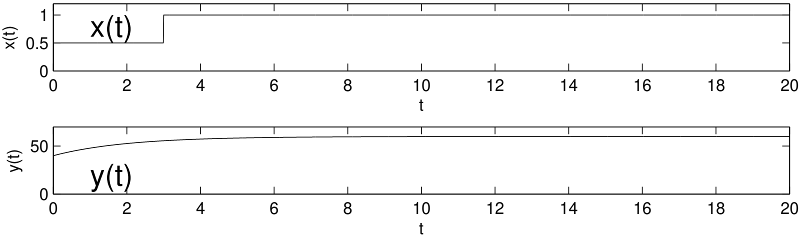
accelerator pedal position $\rightarrow$ $\boxed{\text{engine/car}}$ $\rightarrow$ car velocity

- Input signal induces a response of the system.
- $x(t) = \frac{1}{2} + \frac{1}{2}u(t-3)$ (one pushes the gas pedal to the floor)
- $y(t) = 40 + 20(1 - e^{-t/\tau})$ (rise time or transient response)
- $y(t)$ is not solely a function of $x(t)$ at time t, but also a function of previous input signal values and the present and past state of the system.
- Called a **dynamic** system.

Example (3)

$$x(t) = \tfrac{1}{2} + \tfrac{1}{2}u(t-3)$$

$$y(t) = 40 + 20(1 - e^{-t/\tau}), \quad \tau = 2$$



Outline

Input-output description of systems (1)

Pictures/diagrams are a starting point, but for quantitative analysis every system must have an **input-output relationship**.

Definition

Input-output relationship is a mathematical expression that precisely defines how the output signal is related to the input signal.

Example

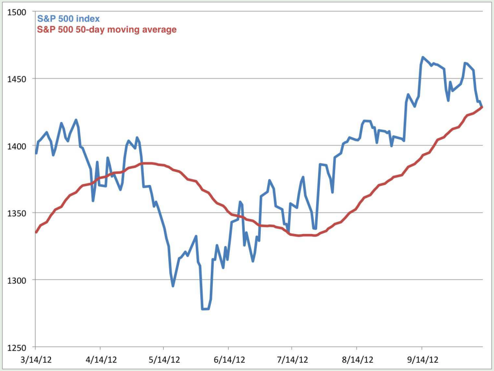
integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Moving average

Example

moving average filter

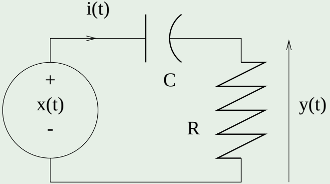
$$y(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)\, d\tau.$$



Input-output description of systems (2)

Example

RC circuit driven by voltage source (input signal).



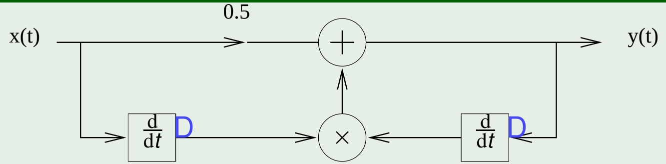
$$\frac{1}{CR}y(t) + \frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t).$$

- This input-output relation is not of the form $y(t) =$ some_function$[x(t)]$.
- When combined with an appropriate initial condition (such as 0 charge on the capacitor at time $t = 0$) one can solve the diffeq to determine $y(t)$ for any $x(t)$(**later lectures.**)

Outline

Block diagram representation of CT systems

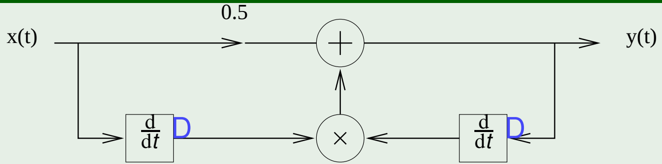
Example



The lower-right part is called a **feedback connection.**

Block diagram representation of CT systems

Example



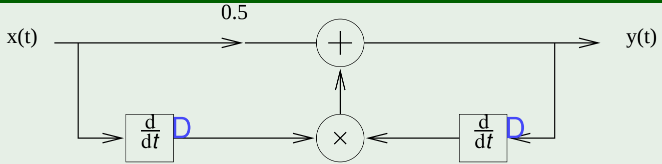
The lower-right part is called a **feedback connection.**

Basic elements:

- adder (see text Fig. 2.29 on P. 126)
- constant multiplier (amplifier) (see text Fig. 2.29 on P. 126)
- signal multiplier
- differentiator (see text Fig. 2.29 on P. 126)
- integrator (see text Fig. 2.31 & 2.32 on P. 127)

Block diagram representation of CT systems

Example



The lower-right part is called a **feedback connection.**

Input-output relationship defined by the diagram:

$$y(t) = 0.5x(t) + \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right).$$

Outline

Interconnection of systems

1. Series connection

$$x(t) \rightarrow \boxed{\mathcal{T}_1} \rightarrow \boxed{\mathcal{T}_2} \rightarrow y(t)$$

   Mathematically: $y(t) = \mathcal{T}_2[\mathcal{T}_1[x(t)]]$.

2. Parallel connection
   (See text Fig. 1.42(b) on P. 42)
   Mathematically: $y(t) = \mathcal{T}_1[x(t)] + \mathcal{T}_2[x(t)]$.

Example

Example

$x(t) \rightarrow \boxed{\text{amplifier, gain=5}} \rightarrow \boxed{\text{differentiator}} \rightarrow y(t)$

$$y(t) = 5\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

In this example the order of interconnection is irrelevant. We will learn soon that this is because both subsystems are linear.

Outline

Classification of CT systems

Two general aspects to categorize:

- Amplitude properties
  - A-1 linearity (1.6.6)
  - A-2 stability (1.6.4)
  - A-3 invertibility (1.6.2)
- Time properties
  - T-1 causality (1.6.3)
  - T-2 memory (1.6.1)
  - T-3 time-invariance (1.6.5)

Skill: *Determining classifications of a given CT system*

A-1 Linearity (1)

Definition

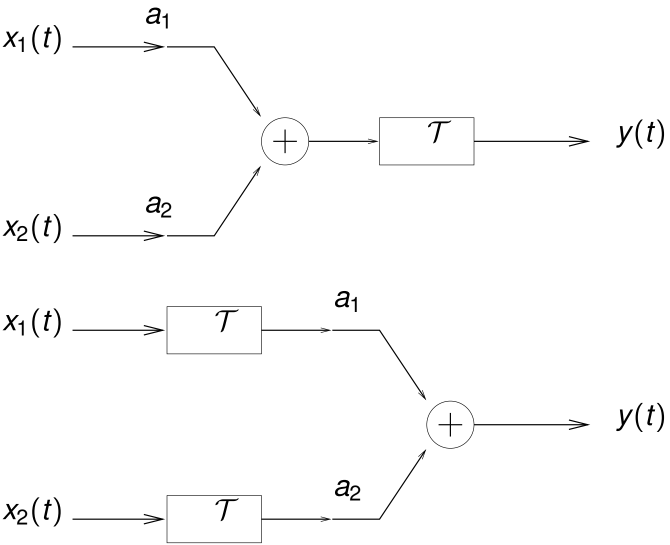
A system $\mathcal{T}$ is **linear** iff

$$\mathcal{T}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{T}[x_1(t)] + a_2 \mathcal{T}[x_2(t)]$$

for any signals $x_1(t), x_2(t)$ and any (even complex) constants a_1 and a_2. Otherwise the system is called **nonlinear**.

Response to a weighted sum of input signals is the weighted sum of the individual responses. ***(Picture)***

A-1 Linearity (2)

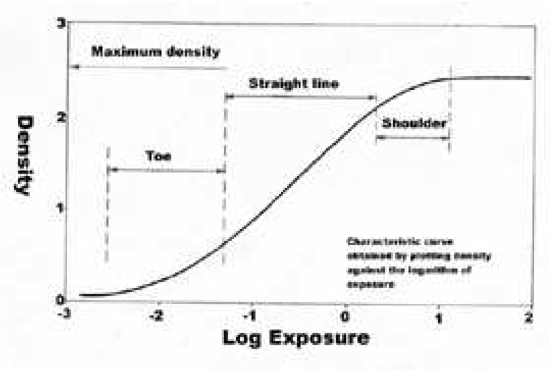


A-1 Linearity (3)

Question

We will focus on linear systems. Why?

A-1 Linearity (4)



Real systems are never perfectly linear, but often they are approximately linear over an appropriate operating range.

Two important special cases of linearity property (1)

Property

scaling property or *homogeneity property*:

$$\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)]$$

Note that from $a = 0$ we see that zero input signal implies zero output signal for a linear system.

Two important special cases of linearity property (2)

Property

additivity property:

$$\mathcal{T}[x_1(t) + x_2(t)] = \mathcal{T}[x_1(t)] + \mathcal{T}[x_2(t)]$$

Using proof-by-induction, one can easily extend this property to the general superposition property

Property

general superposition property

$$\mathcal{T}\left[\sum_{k=1}^{K} x_k(t)\right] = \sum_{k=1}^{K} \mathcal{T}[x_k(t)].$$

In words: the response of a linear system to the sum of several

General superposition property

- In general superposition need not hold for infinite sums; additional continuity assumptions are required.
- We assume the superposition summation holds even for infinite sums without further comment in this course.
- In fact, we even assume that superposition holds for integrals:

$$\boxed{\mathcal{T}\left[\int x(t;\nu)\,d\nu\right] = \int \mathcal{T}\left[x(t;\nu)\right]\,d\nu}$$

Determining a system is linear or nonlinear

Skill: *Determining a system is linear or nonlinear.*

1. Find output signal $y_1(t)$ for a general input signal $x_1(t)$.
2. "Repeat" for input $x_2(t)$ and $y_2(t)$.
3. Find output signal $y(t)$ when input signal is $x(t) = a_1 x_1(t) + a_2 x_2(t)$.
4. If $y(t) = a_1 y_1(t) + a_2 y_2(t)\ \forall t$, then the system is linear.
5. If it does not appear that $y(t) = a_1 y_1(t) + a_2 y_2(t)\ \forall t$, then find a specific counter-example.

Example (1)

Example

Prove that the integrator is a linear system, where
$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$.

Example (2)

Example

Determine whether linearity holds for $y(t) = \int_{-\infty}^{t} x^3(\tau)\, d\tau$.

Example (3)

Example

Are the following systems linear?

- $y(t) = x^3(t)$
- $y(t) = 2x(t) + 3$

Example (4)

Example

Is $y(t) = \text{Real}[x(t)]$ linear?

A-2 Stability (1)

Definition

A system is **bounded-input bounded-output (BIBO) stable** iff every bounded input produces a bounded output.

If $\exists M_x$ s.t. $|x(t)| \leq M_x < \infty \; \forall t$, then there must exist an M_y s.t.

$$|y(t)| \leq M_y < \infty \; \forall t.$$

Usually M_y will depend on M_x.

Otherwise the system is called **unstable**, and it is possible that a small input signal will make the output "blow up."

A-2 Stability: example (1)

Example

Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ BIBO stable?

Triangle inequality

The **triangle inequality** is sometimes useful for proving that a system is BIBO stable.

- $|a + b| \leq |a| + |b|$ (Easily proved by considering 4 cases where a and b are positive or negative.)
- $\left|\sum_n a_n\right| \leq \sum_n |a_n|$
- $\left|\int f(t)\, dt\right| \leq \int |f(t)|\, dt$

A-2 Stability: example (2)

Example

Is the moving average $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau$ for $T > 0$ system BIBO stable?

A-2 Stability: example (3)

Example

Is $y(t) = x^5(t)$ BIBO stable?

A-3 Invertibility

Definition

A system $\mathcal{T}$ is called **invertible** iff each (possible) output signal is the response to only one input signal. Otherwise $\mathcal{T}$ is not **invertible**.

Property

If a system $\mathcal{T}$ is invertible, then there exists a system $\mathcal{T}^{-1}$ such that

$$x(t) \to \boxed{\mathcal{T}} \to y(t) \to \boxed{\mathcal{T}^{-1}} \to z(t) = x(t).$$

Mathematically:

$$\mathcal{T}^{-1}[\mathcal{T}[x(t)]] = x(t)$$

Design of $\mathcal{T}^{-1}$ is important in many signal processing applications.

A-3 Invertibility: example (1)

Example

encryption/decryption for secure communication. Needs to be invertible for no loss of information.

Example

digital speedometer

velocity → $\boxed{\text{speed sensor}}$ →voltage

→ $\boxed{\text{mathematical inverse of sensor law}}$ →velocity display.

We display the velocity, not the voltage, so there should be a one-to-one relationship between the two.

A-3 Invertibility: example (2)

Example

Is the full-wave rectifier: $y(t) = |x(t)|$ invertible?

A-3 Invertibility: example (3)

Example

Is the exponential-law device: $y(t) = e^{x(t)}$ invertible?

A-3 Invertibility: example (4)

Example

Is the ideal amplifier: $y(t) = 2x(t)$ invertible?

T-1 Causal systems

Definition

For a **causal** system, the output $y(t)$ at any time t depends only on the "present" and (possibly) "past" inputs *i.e.* on $x(t)$ and on various $x(t_0)$ for $t_0 \leq t$ only, but not on future inputs.
Otherwise **noncausal** system.

Causality is necessary for real-time implementation. Noncausal systems arise primarily when t is some other variable than time, such as space.

T-1 Causal systems: example (1)

Example

Is the integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ causal?

T-1 Causal systems: example (2)

Example

Is the symmetric moving average: $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\tau)\, d\tau$ causal? (Useful for image processing.)

T-2 Memory

Definition

For a **static system** or **memoryless** system, the output $y(t)$ depends only on the current input $x(t)$, not on previous or future values of the input signal.
Otherwise it is a **dynamic system** and must have memory.

Example

- $y(t) = e^{x(t)}/\sqrt{|t+3|}$.
- moving average $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$, $T > 0$

Dynamic systems are the interesting ones and will be our focus. (This time we take the more complicated choice!)

Memory vs. causality

Question

- *Is a memoryless system necessarily causal?*
- *Is a dynamic system necessarily noncausal?*

T-3 Time-invariance (1)

Systems whose input-output behavior does not change with time are called **time-invariant** will be our focus.

- "Easier" to analyze.
- Time-invariance is a desired property of many systems.

We will focus primarily, but not exclusively, on time-invariant systems.

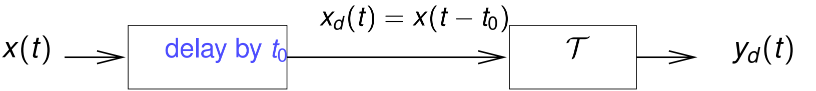
T-3 Time-invariance (2)

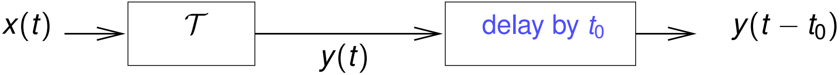
Definition

A system $\mathcal{T}$ is called **time invariant** or **shift invariant** iff

$$x(t) \xrightarrow{\mathcal{T}} y(t) \quad \text{implies that} \quad x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0)$$

for every input signal $x(t)$ and time shift t_0. Otherwise the system is called **time variant** or **shift variant**.





Recipe for showing time-invariance

Recipe for showing time-invariance

1. Determine output signal $y(t)$ due to a generic input signal $x(t)$.
2. Determine the delayed output signal $y(t-t_0)$, by replacing t with $t-t_0$ in $y(t)$ expression.
3. Determine output signal $y_d(t)$ due to a delayed input signal $x_d(t) = x(t-t_0)$.
4. If $y_d(t) = y(t-t_0)$, then system is time-invariant.

Time-invariance: example (1)

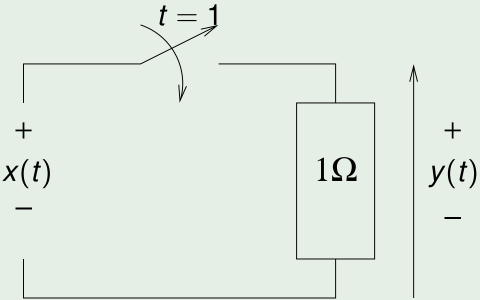
Example

Is the symmetric moving average filter
$y(t) = \frac{1}{3}[x(t-1) + x(t) + x(t+1)]$ time-invariant?

Time-invariance: example (2)

Example

A switch that closes at $t = 1$.



1. How to represent input-output relationship mathematically?
2. Is it Time invariant? If no, find a counter-example.

Time-invariance: example (3)

Example

Is the modulator $y(t) = \cos(\pi t)\, x(t)$ time-invariant?

Time-invariance: example (4)

Example

Is the amplified time reversal $y(t) = 3x(-t)$ time-invariant?

Time-invariance: example (5)

Example

Is the integrator $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ time-invariant?

Outline

Summary (1)

- signal notation
- signal transformations
  - time transformations
  - amplitude transformations
  - differentiator / integrator systems
  - two-signal operations
- signal classes
  - even/odd signals
  - energy/power signals
  - periodic/aperiodic signals
  - exponential signals

Summary (2)

- singularity functions
  - unit step / rect signals
  - unit impulse function
  - impulse function properties (sifting, sampling, scaling)
- CT systems
- block diagrams
- system classes
  - amplitude properties: linearity, stability, invertibility
  - time properties: causality, memory, time-invariance

Key concepts/skills to study

Key concepts/skills to study

- time transformations
- braces/plots to rects/steps
- running integral operation
- properties of $\delta(t)$
- identifying signal properties
- identifying (all six) system properties