1. [5]Find the Fourier series coefficients $a_k's$ of $x(t) = \sin(3\pi t) + \cos(4\pi t)$

$T_1 = \frac{2}{3}$, $T_2 = \frac{1}{2}$, $T = lcm(T_1, T_2) = 2$, $W_0 = \pi$

$$x(t) = \sin(3\pi t) + \cos(4\pi t) = \frac{1}{2j}\left(e^{j(3)\pi t} - e^{j(-3)\pi t}\right) + \frac{1}{2}\left(e^{j(4)\pi t} + e^{j(-4)\pi t}\right)$$

Therefore, we can get that

$$a_k = \begin{cases} \frac{1}{2}, & k = \pm 4 \\ \frac{1}{2j}, & k = 3 \\ \frac{-1}{2j}, & k = -3 \\ 0, & \text{otherwise} \end{cases}$$

2. [9]Find the Fourier series representations of the following signals. Express your answer in a real form.

(a) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - 4n)$

(b) $x(t) = \sum_{n=-\infty}^{\infty} rect(\frac{t - 5n - 3}{6})$

(c) The signal illustrated below,

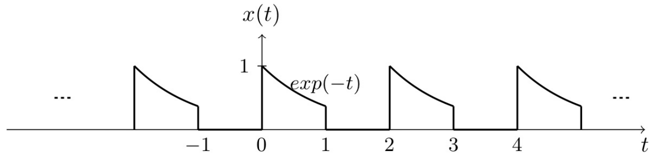


Figure 1: HW3-4(c)

(a)

$\Rightarrow$ It's an impulse train with a period of $T=4$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\frac{2\pi}{T}t} = \frac{1}{4}$$

Therefore, a_k is a constant

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{4} e^{jk\omega_0 t} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{2}\cos(k \cdot \frac{\pi}{2} t)$$

(b)

$$x(t) = 1 + \sum_{n=-\infty}^{\infty} rect(t - 5n - \tfrac{5}{2}) = 1 + \sum_{k=-\infty}^{\infty} \frac{1}{5}\,\mathrm{sinc}(\tfrac{k}{5})\, e^{jk(\frac{2\pi}{5})\cdot(\frac{5}{2})}\, e^{jk(\frac{2\pi}{5})t} = \frac{6}{5} + \sum_{k=1}^{\infty} \frac{1}{5}\,\mathrm{sinc}(\tfrac{k}{5})\cos\left(\frac{2\pi k(t-\frac{5}{2})}{5}\right)$$

(c)

We can know that $T=2$, $W_0 = \pi$, so $a_k = \frac{1}{2}\int_0^1 e^{-t} e^{-jk\omega_0 t}\, dt = \frac{1}{2}\int_0^1 e^{-(1+jk\pi)t}\, dt = \frac{1 - e^{-(1+jk\pi)}}{2(1+jk\pi)}$

$$x(t) = \frac{1 - e^{-1}}{2} + \sum_{k=1}^{\infty}\left[\frac{1 - e^{-1}\cos(k\pi)}{1+\pi^2k^2}\cos(k\pi t) + \frac{k\pi(1 - e^{-1}\cos(k\pi))}{1+k^2\pi^2}\sin(k\pi t)\right]$$

3. [9] Let $x(t)$ be a period signal whose Fourier series coefficient are

$$a_k = \begin{cases} 2 & , k = 0 \\ j(\frac{1}{2})^{|k|} & , otherwise \end{cases}$$

Use Fourier series properties to answer the following questions:

(a) Is $x(t)$ real?

(b) Is $x(t)$ even?

(c) Is $dx(t)/dt$ even?

(a) If $x(t)$ is real, then $x(t) = x^*(t)$ $\Rightarrow a_k = a^*_{-k}$ which is false $\Rightarrow x(t)$ is not real

(b) If $x(t)$ is even. then $x(t) = x(-t)$ and $a_k = a_{-k}$ which is true. So $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk\frac{2\pi}{T_0} a_k$$

Therefore.

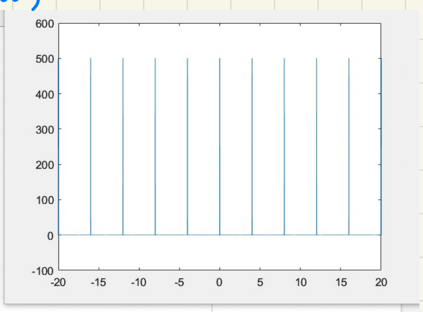
$$b_k = \begin{cases} 0 & k = 0 \\ -k(\frac{1}{2})^{|k|}\frac{2\pi}{T_0} & otherwise \end{cases} \Rightarrow b_k \text{ is not even} \Rightarrow \frac{dx(t)}{dt} \text{ is not even}$$

4. [10]Based on the results obtained in question 2(a)(b)(c), use MATLAB to plot the Fourier series representations. For the summation in the FS, since we can not go infinite, just choose a k that is large enough.

(a)

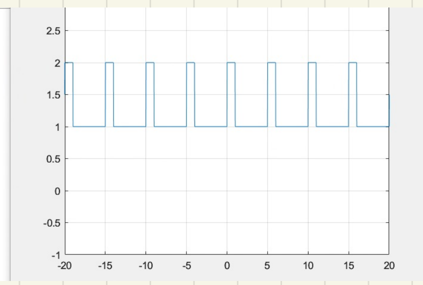
a2023_6_12_1.m

```
a_0 = 0.25;
t= -20: 0.01: 20;
x = 0*t;
x = x + a_0;
for i=1:1000
    x = x+0.5.*cos(pi.*i.*t./2);
end
plot(t,x)
```



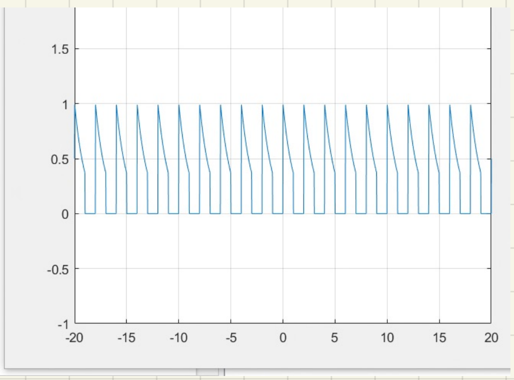
(b)

```
a_0 = 1.2;
t = -20:0.01:20;
x = zeros(size(t)) + a_0;

for i = 1:50000
    x = x + 0.4 * (sin(pi*i/5) / (pi*i/5)) * cos(2*pi*i*(t-0.5)/5);
end

plot(t, x)
ylim([-1 3])
grid on
```



(c)

```
e_val = exp(1);
a_0 = (1- (1/e_val))/2;
t = -20:0.01:20;
x = zeros(size(t)) + a_0;
for i = 1:50000
    kp=i*pi;
    coskpt=cos(kp*t);
    sinkpt=sin(kp*t);
    x = x + (1-(cos(kp)/e_val))*coskpt/(1+kp*kp)+ kp*(1-(cos(kp)/e_val))*sinkpt/(1+kp*kp);
end

plot(t, x)
ylim([-1 2])
grid on
```

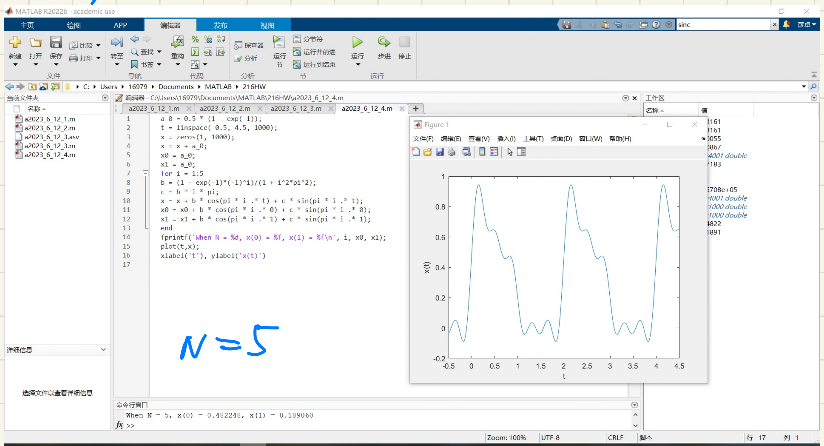


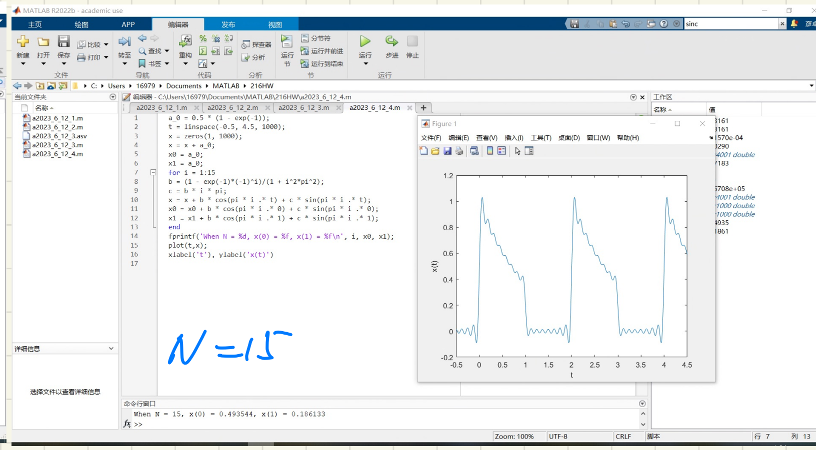
5. [8] In Problem 2(c), we see that the Fourier Series expansion of the "periodical exponential" signal is given by $S_N(t)$, where $x(t) = S_N(t)$ when $N \to \infty$.

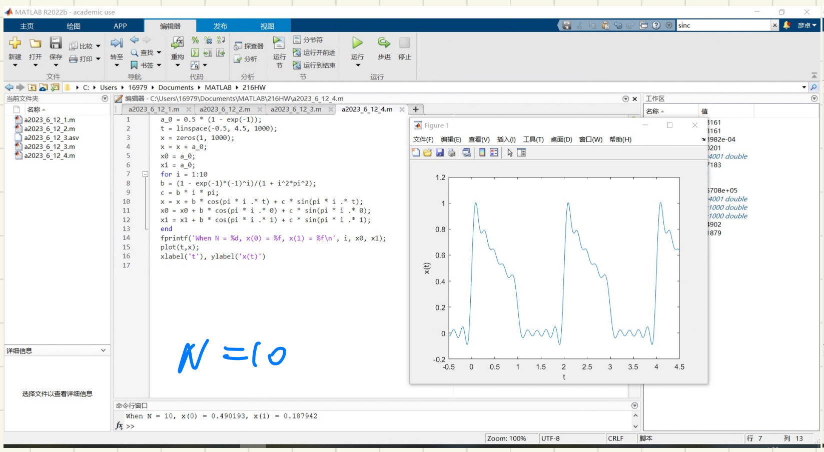
(a) Now, instead of plotting $x(t)$ using a N "large enough", we would like to plot the "intermediate steps". Specifically, please use Matlab to plot $S_N(t)$ with $N = 5, 10, 15, 19$ for $t \in [0.5, 4.5]$. Interpret what you see from the graphs.

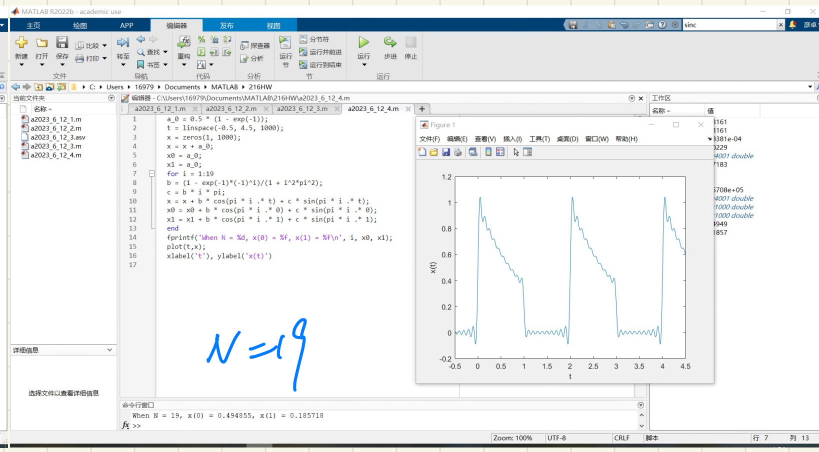
(b) We notice that the $x(t) = S_N(t)$ above is not "strict" in the sense that $x(t)$ has some discontinuities where its value is undefined, while $S_N(t)$ is always continuous even when $N \to \infty$. Use Matlab to calculate $S_N(0)$ and $S_N(1)$ for the N values in (a). Then make an educated guess for the values of $S_N(0)$ and $S_N(1)$ when $N \to \infty$.

(a)









We can see that as N increases, $S_N(t)$ becomes more like $x(t)$

(b)

```
>> a2023_6_12_4
When N = 5, x(0) = 0.482248, x(1) = 0.189060
>> a2023_6_12_4
When N = 10, x(0) = 0.490193, x(1) = 0.187942
>> a2023_6_12_4
When N = 15, x(0) = 0.493544, x(1) = 0.186133
>> a2023_6_12_4
When N = 19, x(0) = 0.494855, x(1) = 0.185718
```

$\Rightarrow$ When $N \to \infty$,

$S_N(0) \to 0.5$

and $S_N(1) \to \frac{1}{2e}$

6. [27] Consider a casual LTI system realized by the RLC circuit shown below. $x(t)$ is the voltage input (V) and y(t) is the voltage (V) across the capacitor.

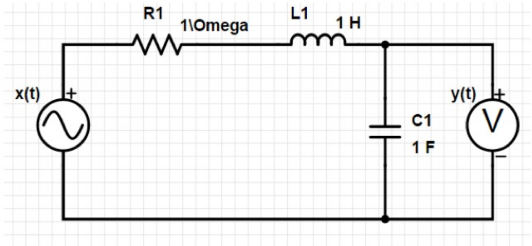


Figure 2: HW3-6

(a) Find the differential equation relating $x(t)$ and $y(t)$.

(b) Find the system's response to $x(t) = e^{j\omega t}$, where ω is arbitrary.

(c) Write out the system's transfer function $H(s)$.

(d) Calculate the magnitude of the system's frequency response $|H(j\omega)|$ and plot it as a function of ω.

(e) Use Matlab **freqs** to generate the exact same plot as part (d). Note that you want to **plot** the output argument of **freqs** because directly calling **freqs** with no output arguments will 1. give two figures; 2. plot both the magnitude and phase response in its default log scale. Besides, be sure to specify an appropriate range for the third input argument of **freqs** using **linspace**.

(f) Find the Fourier series expansion (in complex exponential form) for $x(t) = 1 + \sin(t) + \sin(4t)$ and plot its power density spectrum (by using **stem**).

(g) Use $x(t)$ as an example to verify Parseval's relation. You may want to use Matlab/Mathematica to calculate (integrate) the power of $x(t)$.

(h) Find the systems output $y(t)$ due to $x(t)$. Plot the power density spectrum of $y(t)$.

(i) Compare $x(t)$ and $y(t)$ and their PDS's. What frequency component of the $x(t)$ is attenuated? Is this RLC circuit a lowpass/highpass/bandpass/bandstop filter?

(a)

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

$$\Rightarrow \text{diffeq}: \frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b)

output form should be: $H(jw)\,e^{jwt}$

$$\Rightarrow \frac{d^2}{dt^2}\left(H(jw)e^{jwt}\right) + \frac{d}{dt}\left(H(jw)e^{jwt}\right) + H(jw)e^{jwt} = e^{jwt} \Rightarrow H(jw) = \frac{1}{-w^2+jw+1}$$

(c)

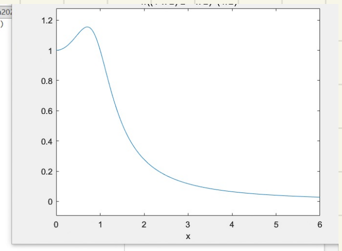
$$H(s) = \frac{1}{s^2+s+1}$$

(d)

$$|H(jw)| = \sqrt{\frac{1}{(1-w^2)^2+w^2}}$$

plot:

```
ezplot('1/((1-x^ 2)^ 2 + x^ 2) ^ (1/2)', [0 6])
```

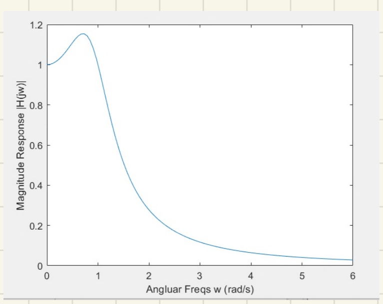


(e)

code:

```
n = [1];
den = [1 1 1];
x = linspace(0, 6);
H = freqs(n, den, x);
plot(x, abs(H));
xlabel('Angluar Freqs w (rad/s)'),
ylabel('Magnitude Response |H(jw)|'),
```

plot:



(f)

$T_0 = 2\pi$ and $w_0 = 1$

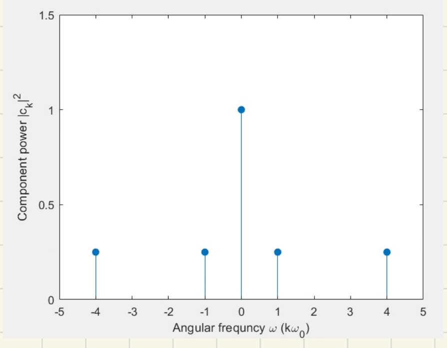
$$\Rightarrow x(t) = 1 + \frac{1}{2j}\left(e^{jt} - e^{-jt}\right) + \frac{1}{2j}\left(e^{4jt} - e^{-4jt}\right)$$

$$\Rightarrow C_1 = C_4 = \frac{1}{2j},\ C_{-1} = C_{-4} = \frac{-1}{2j},\ C_0 = 1$$

code:

```
x = [-4 -1 0 1 4];
y = [0.25 0.25 1 0.25 0.25];
stem(x,y,'filled');
axis([-5 5 0 1.5]);
xlabel('Angular frequncy \omega (k\omega_0)'),
ylabel('Component power |c_k|^2')
```

plot:



(g) since $\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^2 = 2$

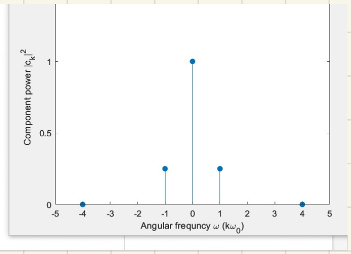
$$\Rightarrow \sum |C_k|^2 = 1 + 0.25 + 0.25 + 0.25 + 0.25 = 2$$

Hence the Parseval relation is verified

(h)

$$y(t) = 1 + \frac{1}{2j}\left(\frac{1}{j}e^{jt} - \frac{-1}{j}e^{-jt} + \frac{e^{j4t}}{-15+j4} - \frac{e^{-j4t}}{-15-j4}\right) = 1 - \cos t - \frac{4}{241}\cos(4t) - \frac{15}{241}\sin(4t)$$

```
x = [-4 -1 0 1 4];
y = [1/964 0.25 1 0.25 1/964];
stem(x,y,'filled');
axis([-5 5 0 1.5]);
xlabel('Angular frequncy \omega (k\omega_0)'),
ylabel('Component power |c_k|^2')
```



(i)

The high frequency component is attenuated.

The circuit serves as a lowpass filter

7. [10] Consider a continues-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1 & , |\omega| \leq 100 \\ 0 & , |\omega| > 100 \end{cases}$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \longrightarrow y(t) = x(t)$$

For what value of k it is guaranteed that a_k must be zero?

$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0) e^{jkw_0 t}$ where $w_0 = 12$ in this question

$\Rightarrow$ $H(jw)$ must be 0 for $|w| > 100$. so $|k| w_0 > 100$.

$\Rightarrow$ For $|k| \geq 9$, a_k is guaranteed to be 0

8. [10] A distortion present in all amplifiers is in some form of nonlinearity. Nonlinearities introduce additional frequency components, transferring some of the signal power from the fundamental frequency component to higher harmonics. This is called **harmonic distortion**. The quality of an amplifier is (in part) judged by how small its **total harmonic distortion (THD)** is, defined by:

$$\text{THD} = \frac{\text{power in DC \& harmonics}}{\text{total power}} \cdot 100\% = \left[1 - \frac{\text{power in fundamental}}{\text{total power}}\right] \cdot 100\%.$$

Consider the following model for an amplifier:

$$y(t) = 7[x(t) + bx^5(t)],$$

where $b = 0.10$. (This is not a great amplifier.) Find the THD for this amplifier, when the input signal is $x(t) = \sin(3t)$.

$$\sin^5 x = \left(\frac{1}{2j}\right)^5 (e^{jx} - e^{-jx})^5 = \frac{1}{32j}(e^{j5x} - 5e^{j3x} + 10e^{jx} - 10e^{-jx} + 5e^{-j3x} - e^{-j5x})$$

$$= \frac{10\sin x - 5\sin 3x + \sin 5x}{16}$$

Thus $y(t) = 7[\sin 3t + \frac{10}{16} b \sin 3t - \frac{5}{16} b \sin 9t + \frac{1}{16} b \sin 15t] \Rightarrow C_1 = 7 \cdot \frac{1+10b}{16}$

$$\Rightarrow \text{THD} = \left(1 - \frac{\left(\frac{1+10b}{16}\right)^2}{\left(\frac{1+10b}{16}\right)^2 + \left(\frac{5b}{16}\right)^2 + \left(\frac{b}{16}\right)^2}\right) \cdot 100\% = 2.090\%$$

---

9. [12] Consider the following three continuous-time signals:

$$x(t) = \cos(4\pi t)$$
$$y(t) = \sin(4\pi t)$$
$$z(t) = x(t)y(t)$$

(a) Determine the Fourier series coefficients of $x(t)$.

(b) Determine the Fourier series coefficients of $y(t)$.

(c) Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.

(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part(c).

(a)

$x(t) = \cos(4\pi t) = \frac{1}{2}e^{4\pi t} + \frac{1}{2}e^{-4\pi t}$ $\Rightarrow$ nonzero FS coeffcients of $x(t)$ are $a_1 = a_{-1} = \frac{1}{2}$

(b)

$y(t) = \sin(4\pi t) = \frac{1}{2j}e^{4\pi t} - \frac{1}{2j}e^{-4\pi t}$ $\Rightarrow$ nonzero FS coeffcients of $x(t)$ are $b_1 = \frac{1}{2j}$, $b_{-1} = \frac{-1}{2j}$

(c)

$z(t) = x(t)y(t) \Rightarrow C_k = \sum_{t=-\infty}^{\infty} a_t b_{k-t} \Rightarrow C_k = a_k * b_k = \sum_{t=-\infty}^{\infty} a_t b_{k-t} = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]$

$\Rightarrow$ nonzero FS coeffcients of $x(t)$ are $C_2 = \frac{1}{4j}$, $C_{-2} = -\frac{1}{4j}$

(d)

$z(t) = \sin(4t)\cos(4t) = \frac{1}{2}\sin(8t) = \frac{1}{4j}e^{8\pi t} - \frac{1}{4j}e^{-8\pi t}$

$\Rightarrow$ nonzero FS coeffcients of $x(t)$ are $C_2 = \frac{1}{4j}$, $C_{-2} = \frac{-1}{4j}$