Ve 216: Introduction to Signals and Systems

Yong Long

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

May 8, 2023

Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

Outline

Overview

- impulse response
- convolution (graphical, properties, LTI interconnection)
- differential equation (diffeq) systems (important class of LTI systems)
- Skip: 2.1, 2.4.2, 2.5.3 (the doublets part, but do read about unit ramp signal)

Outline

Linear time-invariant (LTI) systems

Our primary focus hereafter will be on CT **linear time-invariant (LTI)** systems. This chapter is about analyzing such systems.

Why analysis? So far we only have input-output relationships. For a given system could compute $y(t)$ for a given $x(t)$, but it would be very difficult to design filters etc. by such trial-and-error.

Linear and time-invariant system

- Recall, for a **linear** system:

$$a_1 x_1(t) + a_2 x_2(t) \xrightarrow{\mathcal{T}} y(t) = a_1 y_1(t) + a_2 y_2(t)$$

  where $x_1(t) \xrightarrow{\mathcal{T}} y_1(t)$ and $x_2(t) \xrightarrow{\mathcal{T}} y_2(t)$.
- Recall, for a **time-invariant** system,

$$\text{if} \quad x(t) \xrightarrow{\mathcal{T}} y(t), \quad \text{then} \quad x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0).$$

These two properties greatly simplify analysis of systems!

LTI overview (1)

Overview:

$$x(t) \to \boxed{\text{LTI with impulse response } h(t)} \to y(t) = x(t) * h(t),$$

where $\delta(t) \xrightarrow{\mathcal{T}} h(t)$.

$$\boxed{\text{Input-output relationship: } y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

Remarkably, the input-output relationship for any LTI system is given by the above **convolution integral**.

LTI overview (2)

$$\boxed{\text{Input-output relationship: } y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

- Conversely, any system whose input-output relationship can be expressed in the above form is an LTI system (easy to verify).
- An LTI system is completely described by its impulse response $h(t)$.
- We can determine the response $y(t)$ due to any input signal $x(t)$ by performing the convolution of the input signal with the impulse response.

Outline

Techniques for the analysis of linear systems

General strategy:

1. Decompose input signal $x(t)$ into a weighted sum of elementary functions $x_k(t)$, *i.e.* $x(t) = \sum_k c_k x_k(t)$
   Sometimes this decomposition is itself of interest in terms of studying the signal properties (*e.g.* Fourier analysis).
2. Determine response of system to each elementary function (this should be easy from input-output relationship):
   $$x_k(t) \xrightarrow{\mathcal{T}} y_k(t)$$
3. Apply superposition property:
   $$x(t) = \sum_k c_k x_k(t) \xrightarrow{\mathcal{T}} y(t) = \sum_k c_k y_k(t).$$

Elementary functions

Two particularly good choices for the elementary functions $x_k(t)$:

1. impulse functions $\delta(t-\tau)$
2. complex exponentials $e^{j\omega t}$ (later).

- Linearity leads to the superposition property, which simplifies the analysis.
- Time-invariance then further simplifies.

Outline

Impulse response

Definition

Impulse response $h(t)$ of an LTI system is defined as the response of the system to an input signal that is unit impulse.

$$\delta(t) \xrightarrow{\mathcal{T}} h(t).$$

The key ingredient to our analysis is the impulse response.

Impulse response: example

Example

Find the impulse response of the moving average system $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$. (Verify that it is an LTI system.)

A toy car example (1)

Example

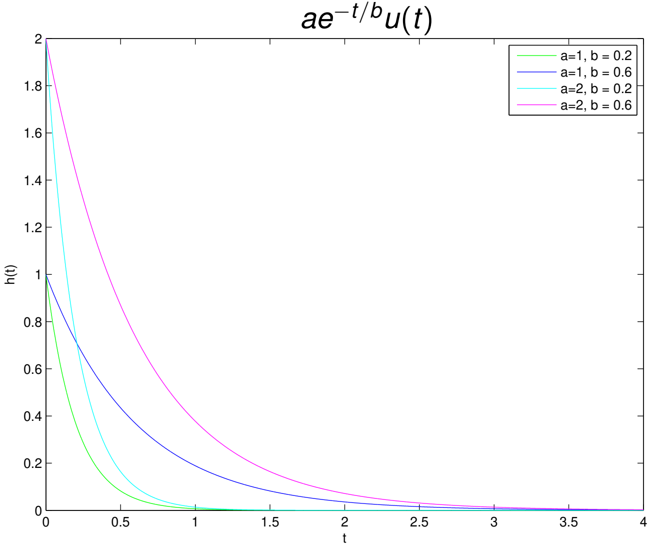
A toy car on a carpet. The input signal $x(t)$ is the applied force, and the output signal $y(t)$ is the velocity of the car.

$$\underset{x(t)}{\text{force}} \rightarrow \boxed{\text{car system}} \rightarrow \underset{y(t)}{\text{velocity}}$$

- If at time $t = 0$ someone hits the (previously stationary) car with a hard stick, then the car begins moving with some velocity.
- But friction gradually decreases the velocity towards zero. So the impulse response is something like:

$$h(t) = ae^{-t/b}u(t), \ \textbf{\textit{(Picture)}}.$$

A toy car example (2)



A toy car example (3)

$$h(t) = ae^{-t/b}u(t), .$$

Question

1. *What is the constant a related to?*
2. *What is the constant b related to?*

A toy car example (4)

This toy car system is time-invariant. If we hit the car with the stick at time $t = t_0$ instead, then the velocity would look like

$$h(t) = ae^{-(t-t_0)/b}u(t - t_0).$$

A toy car example (4)

Question

What if we hit the car twice with the same force, i.e. $x(t) = \delta(t) + \delta(t-3)$*?*

A toy car example (5)

The key question that is answered in this chapter is the following.

- Suppose that, instead of hitting the car (like an impulse), we give it a gentle push for a couple of seconds. What does the output signal (velocity vs time) look like?
- Now we are considering $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ for example, and we want to find $y(t)$.

We know the response of the system to any impulse. We want to find the response of the system to a rect signal (or a general input $x(t)$ for that matter). If only we could somehow express $x(t)$ as a "sum of delta functions", then we could use the LTI properties to say that the output is the "sum of the responses" due to all those delta functions.

Outline

Superposition of impulse function

Question

How to represent $x(t)$ as a "superposition" of impulse functions?

One way is to use the sifting property of the unit impulse function:

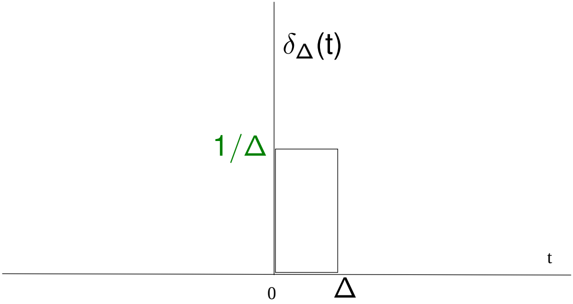
$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.}$$

$$\boxed{\text{sifting property } \int_{-\infty}^{\infty} x(t)\delta(t-t_0)\, dt = x(t_0).}$$

Superposition of impulse function

The following representation may be more intuitive. Practical impulse function, defined for any $\Delta > 0$:

$$\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise.} \end{cases} = \frac{1}{\Delta} \operatorname{rect}\left(\frac{t}{\Delta} - \frac{1}{2}\right)$$



Impulse representation of CT signals (2)

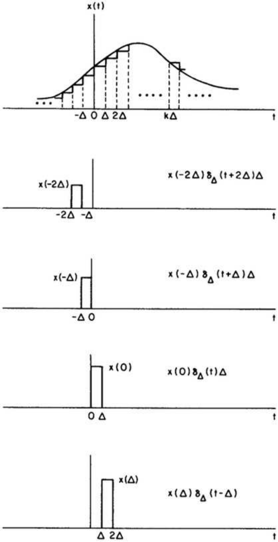
We can approximate any signal $x(t)$ by a stairstep signal formed from these pulse functions as follows

$$x(t) \approx \hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\Delta)\, \Delta\delta_{\Delta}(t - k\Delta).$$

Formally: $x(t) = \lim_{\Delta \to 0} \hat{x}(t)$.

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.}$$

Impulse representation of CT signals (3)



(MIT Lecture Notes 4-5)

Outline

Response of practical impulse function

Question

Let $h_\Delta(t)$ *denote the response of the system to the practical impulse function* $\delta_\Delta(t)$*, i.e.*

$$\delta_\Delta(t) \xrightarrow{\mathcal{T}} h_\Delta(t).$$

- *What happens to* δ_Δ *as* $\Delta \to 0$*?*
- *What happens to* h_Δ *as* $\Delta \to 0$*?*

Convolution for CT LTI systems (1)

If the system $\mathcal{T}$ is LTI, then shifting the input signal causes a shifted output signal

$$\delta_\Delta(t-k\Delta) \xrightarrow{\mathcal{T}} h_\Delta(t-k\Delta).$$

superposition property of a LTI system

$$\hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\Delta)\,\Delta\delta_\Delta(t-k\Delta) \xrightarrow{\mathcal{T}} \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_\Delta(t-k\Delta).$$

This holds for any Δ, so we can take the limit as $\Delta \to 0$.

Convolution for CT LTI systems (2)

On the input side, when we take the limit

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

On the output side, when we take the limit get:

$$\begin{aligned} y(t) &= \lim_{\Delta\to 0} \hat{y}(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_\Delta(t-k\Delta) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \end{aligned}$$

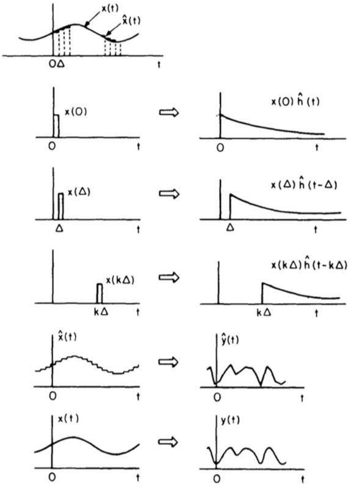
$$\boxed{\textbf{convolution integral: } x(t) \xrightarrow{\mathcal{T}} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

Convolution for CT LTI systems (3)

$$\boxed{\textbf{convolution integral: } x(t) \xrightarrow{\mathcal{T}} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

- Every value of the input signal that enters the system causes the system to respond.
- For a linear system, the overall response is the superposition of the contributions due to each input signal value.

Convolution for CT LTI systems (4)



(MIT, Lecture 4.7)

Convolution for CT LTI systems (5)

The response of an LTI system with impulse response $h(t)$ to an arbitrary input signal $x(t)$ is given by the convolution integral, which we also denote

$$\boxed{y(t) = x(t) * h(t) = (x * h)(t).}$$

This input-output relationship shows that with the impulse response $h(t)$, you can compute the output $y(t)$ for any input signal $x(t)$.

An LTI system is completely characterized by its impulse response $h(t)$.

Convolution for CT LTI systems (6)

Let us now check that the term “impulse response” is appropriately named.

Example

Suppose the input signal is $\delta(t-5)$. Find the output signal in terms of the impulse response $h(t)$.
By the sifting property of $\delta(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau-5)h(t-\tau)\,d\tau = h(t-5)$$

Thus, the convolution integral confirms that the response of the system to a unit impulse at time $t=5$ is the impulse response delayed to “start” at $t=5$.

Computing convolution (1)

Skill: *convolving*.

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

Recipe:

1. Fold: fold $h(\tau)$ about $\tau = 0$ to get $h(-\tau)$
2. Shift: shift $h(-\tau)$ by t to get $h(t-\tau)$
3. Multiply: $x(\tau)$ by $h(t-\tau)$ for every τ
4. Integrate: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

Repeat for all possible t; generally breaks in to a few intervals.

Mathematically, replace t with $t-\tau$ to complete step 1 and 2.

Computing convolution (2)

Question

Can we fold and shift $x(t)$ instead of $h(t)$?

Toy can example (1)

Example

Continue the toy car problem, but simplify by letting $b = 1$ so that $h(t) = e^{-at}u(t)$.
Find the response of the input $x(t) = u(t)$ (a steady push beginning at time 0).

Toy car example (2)

Example

Now return to the problem determining response of the toy car to a short push: $x(t) = \operatorname{rect}\left(\frac{t-1}{2}\right)$ and $h(t) = e^{-t}u(t)$.

Toy car example (3)

Example

A toy car on a carpet. The input signal $x(t)$ is the applied force, and the output signal $y(t)$ is the velocity of the car.

force $\qquad\qquad\qquad\qquad$ velocity

$$x(t) \rightarrow \boxed{\text{car system}} \rightarrow y(t)$$

- A gentle push for a couple of seconds. $x(t) = \mathrm{rect}\left(\frac{t-1}{2}\right)$.
- Impulse response $h(t) = e^{-t}u(t)$.
- Output velocity:

$$y(t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{-t}, & 0 < t < 2 \\ e^{-(t-2)}(1 - e^{-2}), & t \geq 2. \end{cases}$$

Video (MIT, Lecture 4, 36.38min)

Toy car example (4)

Question

- *What affects the rise portion?*
- *What affects the decay portion?*

Outline

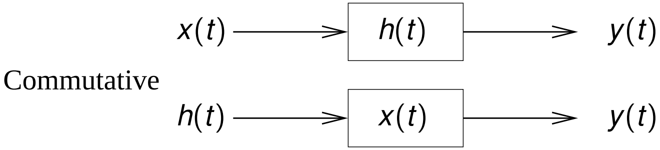
Commutative property (2.3.1)

Skill: *Use properties to simplify LTI systems.*

Property

commutative property

$$x(t) * h(t) = h(t) * x(t).$$



Commutative property: proof

Question

Show the commutative property?

Associative property (2.3.3)

Property

Associative property

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

This property holds in general for any number of systems connected in series. So the following notation is acceptable:

$$h(t) = h_1(t) * h_2(t) * \cdots * h_k(t).$$

In particular:

$$\begin{aligned} (x * h_1) * h_2 &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1 \end{aligned}$$

Commutative and associative property

> The order of serial connection of LTI systems does not affect the overall impulse response.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Commutative

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h(t) = h_1(t) * h_2(t)} \longrightarrow y(t)$$

Commutative

$$x(t) \longrightarrow \boxed{h(t) = h_2(t) * h_1(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$$

Associative property: proof (1)

Question

Show the Associative property.

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

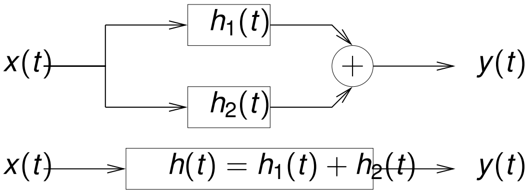
Distributive property

Property

Distributive property

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

Distributive



Question

Show the distributive property.

Example

Example

Given $u(t) * h(t) = (1 - e^{-t})u(t)$, find an easier approach to previous problem of finding $y(t) = x(t) * h(t)$ where $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ and $h(t) = e^{-t}u(t)$.

Properties of convolution and impulse functions (1)

Property

$x(t) * \delta(t) = x(t)$

impulse representation of a CT signal

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Properties of convolution and impulse functions (2)

Property

Delay property: $x(t) * \delta(t - t_0) = x(t - t_0)$

Question

Prove the above property.

Properties of convolution and impulse functions (3)

Property

$\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$

Property

If $y(t) = x(t) * h(t)$*, then* $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$.
(Due to time invariance of system.)

Question

Prove the above property.

Outline

LTI system properties via impulse response

Since an LTI system is completely characterized by its impulse response, we should be able to express the remaining four properties in terms of $h(t)$.

- T-1 causality
- T-2 memory
- A-2 stability
- A-3 invertibility

Outline

T-1 Causal LTI systems (1)

Recall a system is **causal** iff output $y(t)$ depends only on present and past values of input.

For an LTI system with impulse response $h(t)$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau + \int_{-\infty}^{0^-} h(\tau)x(t-\tau)\,d\tau. \end{aligned}$$

1. The first term depends on present and past input values $x(t), x(t-\tau)$ for $\tau \geq 0$.
2. The second term depends on future input values $x(t-\tau)$ for $\tau < 0$.

T-1 Causal LTI systems (2)

Thus the system is causal iff the impulse response terms corresponding to the second term are zero. These terms are $h(\tau)$ for $\tau < 0$.

Definition

An LTI system is **causal** iff its impulse response $h(t) = 0$ for all $t < 0$.

In the causal case the convolution integral simplifies slightly since we can drop the right term above:

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau')h(t-\tau')\,d\tau' \text{ (using } \tau' = t - \tau\text{)}.$$

T-1 Causal LTI systems (3)

Example

Is the LTI system with $h(t) = u(t - t_0 - 5)$ causal?

T-1 Causal LTI systems (4)

Definition

A **causal signal** is a signal $x(t)$ which is zero for all $t < 0$.

If the input to a causal LTI system is a causal signal, then the output is simply

$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t h(\tau)x(t-\tau)\,d\tau = \int_0^t x(\tau)h(t-\tau)\,d\tau, & t \geq 0. \end{cases}$$

MATLAB's `conv` function computes a discrete-time version of the above integral, for finite-duration $x(t)$ and $h(t)$.

Outline

T-2 Memory (1)

Recall a system is **static** or **memoryless** if the output $y(t)$ depends only on the current input $x(t)$, not on previous or future values of the input signal.

For an LTI system with impulse response $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau,$$

the only way this can be true is if

$$h(t) = 0, \quad \text{for} \quad t \neq 0$$

T-2 Memory (2)

Definition

An LTI system is **memoryless** iff its impulse response is $h(t) = a\delta(t)$. Otherwise the system is **dynamic** (has memory).

In this case the response is
$y(t) = x(t) * h(t) = x(t) * a\delta(t) = ax(t)$.

T-2 Memory (3)

There are two classes of dynamic systems.

Definition

A **finite impulse response** or **FIR** system has $h(t)$ that is nonzero only over some finite interval $t_1 < t < t_2$.

Example

$h(t) = \mathrm{rect}(t)$

T-2 Memory (3)

Definition

An **infinite impulse response** or **IIR** system has $h(t)$ that persists indefinitely.

Example

$h(t) = e^{-t}(\cos t)u(t)$

Outline

A-2 Stability of LTI systems (1)

Recall $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$. Suppose $x(t)$ is a bounded input signal, *i.e.* $|x(t)| \le M_x < \infty \; \forall t$. Under what conditions on $h(t)$ is the response $y(t)$ a bounded signal?

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \right| \\ &\le \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau \quad \text{(triangle inequality)} \\ &= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\,d\tau \le M_x \int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \end{aligned}$$

Sufficient condition:

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$$

A-2 Stability of LTI systems (3)

Definition

An LTI system is **BIBO stable** iff its impulse response is absolutely integrable, *i.e.* $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

Example

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$.
What is impulse response? Is it stable?

A-2 Stability of LTI systems (3)

Example

Is the moving average $h(t) = \frac{1}{T}\,\mathrm{rect}\left(t/T - \frac{1}{2}\right)$ stable?

A-2 Stability of LTI systems (4)

Example

Show the stability of the decaying sinusoid system with impulse response of $h(t) = e^{-t}(\cos t)u(t)$.

Outline

A-3 Invertibility of LTI systems (1)

Recall if a system $\mathcal{T}$ is **invertible**, then there exists a system $\mathcal{T}^{-1}$ such that

$$x(t) \to \boxed{\mathcal{T}} \to y(t) \to \boxed{\mathcal{T}^{-1}} \to z(t) = x(t).$$

Fact: if a system is LTI, then if it is also invertible then the inverse system is also LTI (Problem 2.50, textbook). So

$$x(t) \to \boxed{\text{LTI } h(t)} \to y(t) \to \boxed{\text{LTI } h_i(t)} \to z(t) = x(t),$$

where $h_i(t)$ is the impulse response of the inverse system.

A-3 Invertibility of LTI systems (2)

- The cascade of two LTI systems is also LTI, so can be characterized by its impulse response.
- If the input is $x(t) = \delta(t)$, then the output is

$$x(t) * h(t) * h_i(t) = \delta(t) * h(t) * h_i(t)$$

- Thus, if the system is invertible, then

$$\delta(t) * h(t) * h_i(t) = \delta(t) \Longrightarrow \boxed{h(t) * h_i(t) = \delta(t).}$$

A-3 Invertibility of LTI systems (3)

Definition

An LTI system is **invertible** iff there exist an inverse system whose impulse response $h_i(t)$ satisfies the following relationship with $h(t)$

$$h(t) * h_i(t) = \delta(t).$$

Example

Consider the delay system with $h(t) = 2\delta(t-5)$. Is it invertible? If yes, find the impulse response of its inverse system.

A-3 Invertibility of LTI systems (3)

Example

The impulse response of an invertible LTI system is $h(t) = \delta(t) - e^{-t}u(t)$. Show a system with the impulse response of $h_i(t) = \delta(t) + u(t)$ is its inverse system.

Outline

Impulse response of an LTI system

Question

How can we find the impulse response $h(t)$ of an LTI system in practice?

1. One way is to put a “practical impulse” through the system and observe output. This approach can be difficult since a large input spike could drive some systems into nonlinear regime.
2. An alternative is to first find the **step response**, and then differentiate.

Step response and impulse response

Question

Find the ***step response****, and then differentiate to find the* ***impulse response*** $h(t)$*. Why does this work?*

$$u(t) \to \boxed{d/dt} \to \delta(t) \to \boxed{\text{LTI}} \to h(t)$$

by associative/commutative property:

$$u(t) \to \boxed{\text{LTI}} \to s(t) \to \boxed{d/dt} \to h(t)$$

Step response

Definition

The **(unit) step response** of an LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$\delta(t) \xrightarrow{\mathcal{T}} h(t) \text{ called } \textbf{impulse response}$$
$$u(t) \xrightarrow{\mathcal{T}} s(t) \text{ called } \textbf{step response}$$

Step response

If the system is causal, then $h(t) = 0$ for $t < 0$ so

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \left[\int_{0}^{t} h(\tau)\, d\tau\right] u(t)}$$

Relations:

$$\begin{aligned} s(t) &= \left[\int_{0}^{t} h(\tau)\, d\tau\right] u(t) \\ h(t) &= \frac{\mathrm{d}}{\mathrm{d}t} s(t) \end{aligned}$$

Example

Example

Earlier for car problem we showed that step response was $s(t) = (1 - e^{-t})u(t)$. Find the impulse response.

Outline

CT systems described by differential equation models (1)

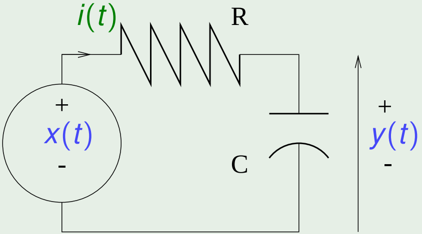
- The convolution integral $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$ looks fine on paper, but what about hardware (*e.g.* circuit) implementation?
- We might use mathematical analysis to design the "perfect" impulse response $h(t)$, but if there is no physical system that has that impulse response, then our design is of limited use.

Let us focus on RLC circuits for the moment, and see what types of input-output relationships (and hence what types of impulse response functions) can be realized.

CT systems described by differential equation models (2)

Example

The input signal $x(t)$ is the voltage source, and the output signal $y(t)$ is the voltage across the capacitor.



The current is $i(t) = (x(t) - y(t))/R$, and the current is also C times the derivative of the voltage across the capacitor: $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}y(t)$. Thus

$$\boxed{\frac{x(t) - y(t)}{R} = C\frac{\mathrm{d}}{\mathrm{d}t}y(t) \quad \text{or} \quad y(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)}$$

CT systems described by differential equation models (3)

Had we used a larger number of circuit elements, we would have found a differential equation (diffeq) with more terms.

Definition

The input-output relationship for an RLC circuit (with a finite number of components) is a diffeq of the following form, called a **linear constant-coefficient differential equation**:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Fortunately this class of systems is sufficiently large to be interesting and useful!

CT systems described by differential equation models (4)

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).}$$

- This is an implicit input-output relationship.
- We assume hereafter that that a_k's and b_k's are real.
- N is called the order of the system

Example

What is order of above RC circuit?

CT systems described by differential equation models (4)

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).}$$

In general there is not a simple time-domain method for finding the impulse response of such systems. We will do it systematically later using frequency response and the partial fraction expansion (PFE) method.

Outline

Solution of linear constant-coefficient diffeqs (1)

The general solution to a diffeq can be decomposed into the sum of two parts:

$$y(t) = y_h(t) + y_p(t).$$

- $y_h(t)$ is called the **homogeneous solution** or **zero-input solution** in the diffeq world, or the **natural response** in the systems world.
- The general form of $y_h(t)$ only depends on the system itself, although the coefficients in the form depend on the input signal.

Solution of linear constant-coefficient diffeqs (2)

$$y(t) = y_h(t) + y_p(t).$$

- $y_p(t)$ is called the **particular solution** in the diffeq world, or the **forced response** in the systems world. It is also called the **zero-state** response.
- $y_p(t)$ is independent of any initial conditions of the system.

Method of undetermined coefficients (1)

The **natural response** is found by solving the zero-input equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = 0. \tag{1}$$

It was discovered long ago that the solutions to the above equation have components of the form

$$y(t) = Ce^{st}$$

for values $C \neq 0$ and s to be determined. Plugging into (1) yields

$$\sum_{k=0}^{N} a_k C s^k e^{st} = Ce^{st} \sum_{k=0}^{N} a_k s^k = 0.$$

Method of undetermined coefficients (2)

Clearly this is only zero for all t if s satisfies

$$\sum_{k=0}^{N} a_k s^k = 0,$$

i.e. if s is a root of the above **characteristic equation** or **characteristic polynomial**.
The characteristic equation has N roots, let us label them $s_1, \ldots, s_N$. (They may be distinct or not, real or complex.)

$$\sum_{k=0}^{N} a_k s^k = a_k(s - s_1)\ldots(s - s_N) = 0$$

Method of undetermined coefficients (3)

- A solution associated with one root

$$y_{hl}(t) = C_l e^{s_l t}$$

  satisfies the zero-input equation (1).
- Because the differential equation is linear, the sum of these solutions is also a soultion.
- For the case of no repeated roots, each such root can contribute a term to the natural response, yielding the general natural response

$$\boxed{y_h(t) = \sum_{l=1}^{N} C_l e^{s_l t}.}$$

Method of undetermined coefficients (3)

For the general case of an r-th order root s_i in the characteristic equation, the (almost) general natural response is

$$\boxed{y_h(t) = \left[\sum_{j=1}^{r} C_j t^{j-1}\right] e^{s_i t} + \sum_{l=r+1}^{N} C_l e^{s_l t}.}$$

Question

How to determine the coefficients $C_1, C_2, \ldots, C_N$?

Method of undetermined coefficients (5)

- Different choices for these auxiliary conditions result in different input-output relationships. (Textbook, Problem 2.34)
- Depending on how the auxiliary information is stated or whether the auxiliary information is available, the system may or may not correspond to a linear system, may or may not correspond to a linear time-invariant (LTI) system, may or may not correspond to a causal and LTI system.
- We focus on differential equations used to describe systems that are LTI and causal.

Method of undetermined coefficients (6)

Definition

Initial rest states that if $x(t) = 0$ for $t \leq t_0$, then $y(t) = 0$ for $t \leq t_0$.

In words the output must be zero up until the time when the input becomes nonzero.

$$\boxed{\text{Causal and LTI system} \Longleftrightarrow \text{initial rest}}$$

Initial rest (1)

- It is relatively straightforward to see

$$\text{Causal and LTI system} \Longrightarrow \text{initial rest}$$

  (textbook Problem 1.44)
- It is somewhat more difficulty to verify

$$\text{initial rest} \Longrightarrow \text{causal and LTI system}$$

  (textbook Problem 2.33)

Initial rest (2)

intuition as to why initial rest $\Longrightarrow$ time-invariant system.

If we perform identical experiments on two successive days, where the circuit starts from initial rest at noon on each day, then we would expect to see identical responses, *i.e.*, responses that are simply time-shifted by one day with respect to each other.

Initial conditions

initia/auxiliary conditions $= 0$

$$y(t_0) = \frac{\mathrm{d}}{\mathrm{d}t} y(t_0) = \ldots = \frac{d^{N-1}}{dt^{N-1}} y(t_0) = 0$$

- The initial conditions at $t = t_0^-$ must be translated to $t = t_0^+$ to reflect the effect of applying the input at $t = t_0$.
- A necessary/sufficient condition: the right-hand side of the differential equation contains no impulses or derivatives of impulses.
- If not, the initial conditions at $t = t_0^-$ are no longer equal to the initial conditions at $t = t_0^+$ (out of scope of this course).

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

The method of undetermined coefficients

- The **method of undetermined coefficients** assumes that the **forced response** or **particular solution** is the sum of functions of the mathematical form of the excitation $x(t)$ and all (distinct, nonzero) derivatives of $x(t)$ that differ in form from $x(t)$ *i.e.*,

$$\boxed{y_p(t) = P_0 x(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \cdots .}$$

- The method of undetermined coefficients applies only if the particular solution as described has a finite number of terms.

Forced response examples

Example

1.
$$x(t) = 5e^{-7t}u(t) \Longrightarrow y_p(t) = P_0 e^{-7t}u(t)$$

2.
$$x(t) = 170\cos(377t)\,u(t) \Longrightarrow$$
$$y_p(t) = [P_0\cos(377t) + P_1\sin(377t)]\,u(t)$$

3.
$$x(t) = 10u(t) \Longrightarrow y_p(t) = P_0 u(t)$$

Method of undetermined coefficients (7)

Combining the **natural response** and **forced response**, the overall solution has the form

$$\boxed{\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \sum_{l=1}^{N} C_l e^{s_l t} + P_0 x(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \cdots \end{aligned}}$$

which we plug into the differential equation and then solve for the undetermined coefficients (the C_l's and the P_k's).

Impulse response of LTI diffeq systems (1)

- In a diffeq class, the focus is on computing solutions to the diffeq for various input signals.
- In systems analysis, the diffeq indirectly describes the input-output relationship of an LTI system, and we are more interested in finding the impulse response $h(t)$ of that system (from which we can then look at stability and other system properties).

Impulse response of LTI diffeq systems (2)

- In principle we could find $h(t)$ by letting $x(t) = \delta(t)$ and solving the diffeq.
- Typically this time-domain approach requires more work and is less insightful than the frequency-domain / PFE approach described later.
- Nevertheless, for completeness we work one example here.
- In this example, we simplify things by first finding the step response, and then differentiating to find the impulse response, as in earlier discussion of step response.

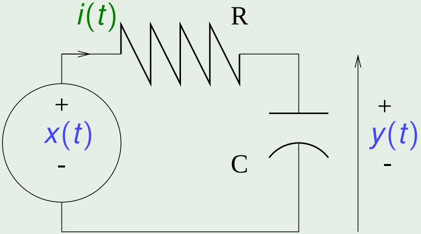
Example (1)

Example

Find step response and impulse response of the following RC circuit, which had diffeq

$$y(t) + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)$$

for $\alpha = 1/RC$.



Solution (1)

Find **natural response** $y_h(t) = Ce^{st}$.

$$y_h(t) + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y_h(t) = 0$$

$$\Longrightarrow Ce^{st} + \frac{1}{\alpha}sCe^{st} = 0$$

$$\Longrightarrow s + \alpha = 0, \quad (C \neq 0, e^{st} \neq 0)$$

$$\Longrightarrow s = -\alpha$$

$$\Longrightarrow \boxed{y_h(t) = Ce^{-\alpha t}}$$

Solution (2)

We analyze a unit-step input signal

$$x(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & otherwise \end{cases}.$$

Since $x(t) = 0$ for $t \leq 0$, the condition of initial rest implies the auxiliary condition $y(t) = 0$ for $t \leq 0$.

Since $x(t) = 1$, $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 0$ for $t \geq 0$, the forced response is

$$\boxed{y_p(t) = P \text{ for } \ t \geq 0}$$

Thus

$$\boxed{y(t) = y_h(t) + y_p(t) = Ce^{-\alpha t} + P \ \text{ for } \ t \geq 0}$$

Solution (3)

Plugging into the diffeq yields:

$$
\begin{aligned}
& y(t) + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t} y(t) = x(t) \\
\implies \quad & \left[Ce^{-\alpha t} + P\right] + \frac{1}{\alpha}\left[-\alpha Ce^{-\alpha t}\right] = 1 \quad (\text{for} \quad t \geq 0) \\
\implies \quad & P = 1.
\end{aligned}
$$

Furthermore, using the initial condition

$$
\begin{aligned}
& y(0) = 0 \\
\implies \quad & Ce^{-\alpha 0} + P = 0 \\
\implies \quad & C + P = 0 \\
\implies \quad & C = -P = -1.
\end{aligned}
$$

Solution (4)

$$y_h(t) = -e^{-\alpha t} \text{ and } y_p(t) = 1 \text{ for } t \geq 0.$$

Note that

$$y_h(t) \to 0 \text{ as } t \to \infty, \quad y(t) \to y_p(t) \text{ as } t \to \infty,$$

These effects are called the **transient response** and the **steady-state response**

Our final solution for the step response is

$$\boxed{y(t) = \begin{cases} [1 - e^{-\alpha t}], & t \geq 0 \\ 0, & t < 0 \end{cases} = [1 - e^{-\alpha t}]u(t).}$$

so taking the derivative yields the impulse response

$$\boxed{h(t) = \alpha e^{-\alpha t} u(t) = \frac{1}{RC} e^{-t/RC} u(t).}$$

Solution (5)

- We have the impulse response, we can determine the response of the system to any other input signal simply by performing convolution, rather than re-solving the differential equation!
- Find $h(t)$ by time-domain methods was not too painful in this case, but it gets laborious for higher-order systems. The Fourier and Laplace methods greatly simplify.

Summary of diffeqs

- Main point is hardware implementation with RLC circuits yields systems described by diffeqs. Thus this class of systems is extremely important.
- Analog filter design is partly about how to make efficient approximations to a desired impulse response! (n and m limited by cost/complexity)
- Just because the world is going digital does not obviate the need for analog. As we will see later (especially in 451), the first component of a DSP system is the sampler in an A/D converter, which requires a high-quality analog anti-alias filter!

Outline

Summary

- impulse response
- convolution integral for LTI systems
- graphical convolution
- properties of convolution
- convolution and LTI system interconnection
- impulse response vs step response
- LTI system properties characterized by $h(t)$ (causality, memory, stability, invertibility)
- diffeq systems
- solutions of diffeqs