1. [14!]Here are input-output relationships for a few systems, all of which are linear. Some of them are time-invariant, some are not. Determine which are which. Find the impulse response of the time-invariant systems.

(a) $y(t) = \int_{-\infty}^{t} \left[\int_{-\infty}^{s} x(\tau - 5)d\tau\right] ds$,

(b) $y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} x(\tau)d\tau$.

Hint: Note that if you can transform the above relationships into the exact form of convolution $y(t) = g(t) * x(t)$, then the system is immediately time-invariant with $g(t)$ being the impulse response $h(t)$. That is because different from algebraic operators like multiplication, the convolution operator implies time-invariance itself.

(a) $y(t) = \int_{-\infty}^{t} [\int_{-\infty}^{s} x(\tau-5)d\tau] ds$

Since $y_1(t) = \int_{-\infty}^{t} [\int_{-\infty}^{s} x_1(\tau-5)d\tau] ds = \int_{-\infty}^{t} [\int_{-\infty}^{s} x(\tau-d-5)d\tau] ds$

$= \int_{-\infty}^{t-d} [\int_{-\infty}^{s} x(\tau-5)d\tau] ds = y(t-d)$

$\Rightarrow$ this system is time-invariant

Then substituting $x(t)$ with $\delta(t)$, we can get its impulse response:

$h(t) = \int_{-\infty}^{t} [\int_{-\infty}^{s} \delta(\tau-5)d\tau] ds = (t-5)\,u(t-5)$

(b) $y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} x(\tau)d\tau$

$\Leftrightarrow y(t) = \int_{-\infty}^{\infty} (\tau^2 \mathrm{rect}(\frac{\tau}{6}))\, x(t-\tau)d\tau + \int_{-\infty}^{\infty} (\frac{1}{(t-\tau+3)^2} u(t+1-\tau))\, x(\tau)d\tau$

$\Rightarrow$ this system is time-invariant

The impulse response is given by $h(t) = t^2 \mathrm{rect}(\frac{t}{6}) + \frac{1}{(t+3)^2} u(t+1)$

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2. [16!] Often we will be convolving two signals that are zero everywhere except over some finite range (called **finite support** signals). Suppose $x_1(t)$ is non-zero over the range $a \le t \le b$ and that $x_2(t)$ is non-zero over the range $c \le t \le d$. Suppose $y(t) = x_1(t) * x_2(t)$.

(a) Find the range of values of t for which $y(t)$ is possibly non-zero.

(b) Compute $rect((t-2)/2) * rect((t+3)/4)$ (express answer with braces and carefully sketch). Check your result with part (a).

(a)

$x_1(t-\tau) \neq 0$ if $\tau \in (t-a, t-b)$

$x_2(\tau) \neq 0$ if $\tau \in (c, d)$

$\int x_1(t-\tau)x_2(\tau)d\tau \neq 0$ if $t \in (a+c, b+d)$

$\Rightarrow t \in (a+c, b+d)$ is the range of values of t for which $y(t)$ is possibly non-zero

(b) We now know that $a=1, b=3, c=-5, d=-1$

$\Rightarrow a+c=-4,\ b+d=2$

① $t-1>-5$ & $t-3<-5 \Rightarrow t\in(-4,-2)$

$y(t) = \int_{-5}^{t-1} dt = t+4$

② $t-3>-5$ & $t-1<1 \Rightarrow t\in(-2,0)$

$y(t) = \int_{t-3}^{t-1} dt = 2$

③ $t-3<1$ & $t-1>-1 \Rightarrow t\in(0,2)$

$y(t) = \int_{t-3}^{-1} dt = 2-t$

④ otherwise, $y(t)=0$

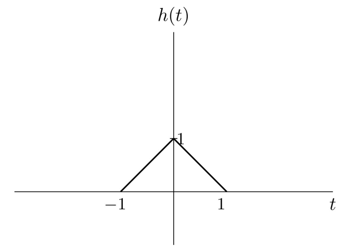
$$\Rightarrow y(t) = \begin{cases} t+4 & t\in(-4,-2) \\ 2 & t\in(-2,0) \\ 2-t & t\in(0,2) \\ 0 & \text{otherwise} \end{cases}$$

3. [18!]

(a) Consider a linear system with input $x(t)$ and output $y(t)$ given by

$$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT).$$

Is this system time-invariant?



(b) Consider another LTI system. Let its impulse response $h(t)$ be the triangular pulse shown below, and $x(t)$ be the **impulse train**

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

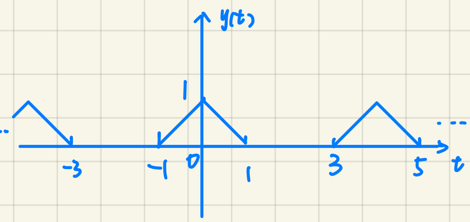
SKETCH $y(t) = x(t) * h(t)$ for $T = 4, 2, 1.5$ and 1.(No formulae are needed though you still want to label your graphs clearly.)

(a)

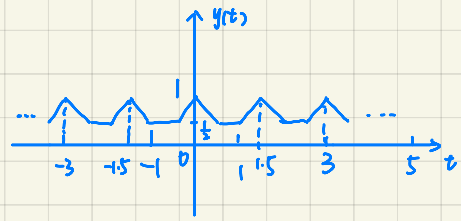
If we let $x(t) = \cos t$ and $x_d(t) = \cos(t-1)$, then we can find that it's not time-invariant

(b)

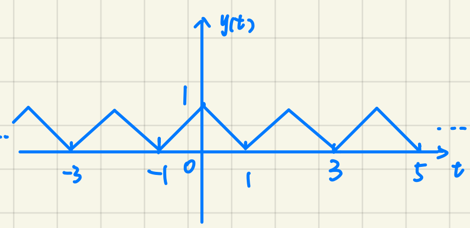
For $T=4$: ···



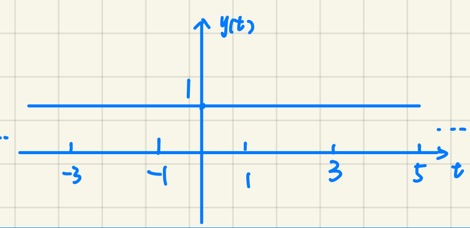
For $T=\frac{3}{2}$:



For $T=2$: ···



For $T=4$: ···



4. [12!] Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.

(a) $y(t) = \int_{-\infty}^{t} (t-\tau)e^{-(t-\tau)}x(\tau)d\tau$

(b) $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}x(\tau)d\tau$

(a)

$\Rightarrow y(t) = \int_{-\infty}^{\infty} (t-\tau)\, u(t-\tau)\, e^{-(t-\tau)} x(\tau)\, d\tau$

$\Rightarrow h(t) = t \cdot e^{-t}\, u(t)$

Since $h(t) = 0$ for $t<0$ $\Rightarrow$ causal

since $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{\infty} t e^{-t} dt = 1 \Rightarrow$ stable

since $h(t) \neq 0$ for $t>0$ $\Rightarrow$ dynamic

(b)

$\Rightarrow y(t) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t-\tau}{2}\right) e^{-2(t-\tau)} x(\tau)\, d\tau$

$\Rightarrow h(t) = \text{rect}\left(\frac{t}{2}\right) \cdot e^{-2t}$

since $h(t) \neq 0$ for $t<0$ $\Rightarrow$ non-causal

since $\int_{-\infty}^{\infty} |h(t)| = \frac{e^2 - e^{-2}}{2} \Rightarrow$ stable

since $h(t) \neq 0$ for $t>0$ $\Rightarrow$ dynamic

5. [16!] Consider an LTI system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If

$$x(t) \to y(t)$$

and

$$\frac{dx(t)}{dt} \to -3y(t) + e^{-2t}u(t)$$

determine the impulse response $h(t)$ of S.

$x(t) = 2e^{-3t}u(t-1) \Rightarrow \frac{d\,x(t)}{dt} = -6e^{-3t}u(t-1) + 2e^{-3t}\delta(t-1)$

Therefore we know: $2e^{-3}\delta(t-1) \longrightarrow e^{-2t}u(t) \Rightarrow \delta(t) \longrightarrow \frac{1}{2}e^{-2t+1}u(t+1)$

and it follows $h(t) = \frac{1}{2}e^{-2t+1}u(t+1)$

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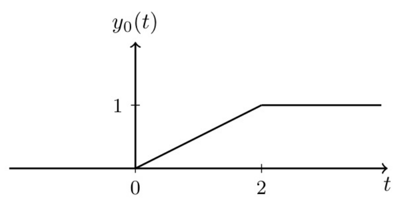
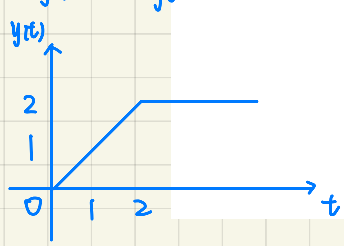
6. [12!] We are given a certain LTI system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched below. We are then given the following set of inputs to LTI systems with the indicated impulse responses.

	Input $x(t)$	Impulse response $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t+1)$
(c)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(e)	$x(t) = x_0'(t)$	$h(t) = h_0(t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

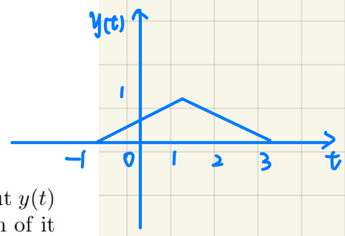
In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If so, provide an accurate sketch of it with numerical values clearly indicated on the graph.



(a) $y(t) = 2y_0(t)$

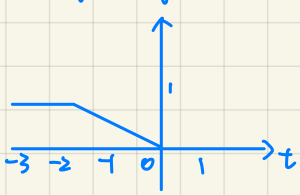


(b) $y(t) = y_0(t+1) - y_0(t-1)$

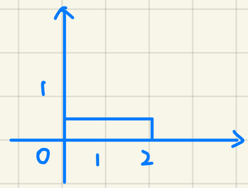


(c) $y(t)$ can't be determined

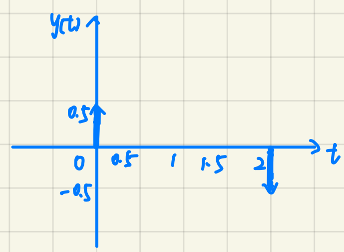
(d) $y(t) = y_0(-t)$



(e) $y(t) = y_0'(t)$



(f) $y(t) = y_0''(t)$



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7. [12!] Suppose you have a "black box" LTI system and you discover that for a unit-step input signal the response of the system is the function $(3-t)rect(\frac{t-1}{2})$. Determine the impulse response of the system. (*Hint: See Optional Problem 2.*)

$h(t) = \frac{d\,s(t)}{dt} \Rightarrow h(t) = 3\delta(t) - \delta(t-2) - rect\left(\frac{t-1}{2}\right)$