1. [20] Use the table of FT pairs and the table of properties to find the FT of each of the following signals (*DO NOT USE INTEGRATION*):

   (a) [5!] $x(t) = 2\text{rect}\left(\frac{t-2}{4}\right)$
   (b) [5!] $x(t) = e^{-3t}\text{rect}\left(\frac{t-2}{4}\right)$
   (c) [5!] $x(t) = t\text{rect}\left(\frac{t-2}{4}\right)$
   (d) [5!] $x(t) = \cos(4\pi t)\text{rect}\left(\frac{t-2}{4}\right)$

(a)

$$X(w) = 8e^{-2jw}\,\text{sinc}\left(\frac{2W}{\pi}\right)$$

by time – scaling and shifting property

(b)

$$x(t) = e^{-3t}u(t) - e^{-3(t-4)}u(t-4)e^{-12}$$

$$\Rightarrow X(w) = \frac{1}{3+jw}\left[1-e^{-12}e^{-4jw}\right]$$

(c)

since $x(t) = t\,y(t) \Rightarrow Y(w) = 4e^{-2jw}\,\text{sinc}\left(\frac{2W}{\pi}\right)$

$$\Rightarrow X(w) = \frac{dj\,Y(w)}{dw} = \frac{d}{dw}j\,2e^{-2jw}\frac{\sin(2w)}{w} = \frac{e^{-4jw}(1+4jw)-1}{w^2}$$

(d)

since $x(t) = \cos(4\pi t)\,y(t) \Rightarrow Y(w) = 4e^{-2jw}\,\text{sinc}\left(\frac{2W}{\pi}\right)$

$$\Rightarrow X(w) = \frac{1}{2}\left[Y(w-4\pi) + Y(w+4\pi)\right]$$

$$= 2e^{-2jw}\left[\text{sinc}\left(\frac{2W-8\pi}{\pi}\right) + \text{sinc}\left(\frac{2W+8\pi}{\pi}\right)\right]$$

2. [10] Show that if $f(t)$ is real and odd, then $F(\omega)$ is purely imaginary and odd.

since $f(t)$ is odd, then $f(t) = -f(-t)$

from $f(t) \xleftrightarrow{F} F(w)$ and $f(-t) \xleftrightarrow{F} F(-w)$

we can know $F(w) = -F(-w)$

so $F(w)$ is also odd

since $f(t)$ is real. $f(t) = f^*(t)$

from $f(t) \xleftrightarrow{F} F(w)$ and $f^*(t) \xleftrightarrow{F} F^*(-w)$

we can know $F(w) = -F^*(w)$

so F is purely imaginary

3. [10] Find the energy of the signal $x(t) = t\text{sinc}^2(t)$ by Fourier methods.

$$\Rightarrow X(w) = j\,\frac{d\,\text{tri}\left(\frac{W}{2\pi}\right)}{dw} = \pm j\frac{1}{2\pi}\,\text{rect}\left(\frac{W}{4\pi}\right)$$

$$\text{So } E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(w)|^2 dw = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\frac{1}{4\pi^2}dw = \frac{1}{2\pi^2}$$

4. [20] A LTI system has the following frequency response:

$$H(j\omega) = \frac{-\omega^2 + j\omega + 1}{(-\omega^2 + 6j\omega + 25)(j\omega + 2)}.$$

(a) [10] Find the impulse response of the LTI system.
Hint: first find the partial differential equation.

(b) [10] Find the differential equation corresponding to the LTI system.
Hint: write $H(\omega) = Y(\omega)/X(\omega)$ and cross multiply.

(a)

$$H(s) = \frac{s^2+s+1}{(s^2+6s+25)(s+2)} = \frac{s^2+s+1}{(s+2)(s+3-4j)(s+3+4j)} = \frac{r_1}{s+2} + \frac{r_2}{s+3-4j} + \frac{r_2^*}{s+3+4j}$$

with $r_1 = \left.\frac{s^2+s+1}{s^2+6s+25}\right|_{s=-2} = \frac{3}{17}$ and $r_2 = \left.\frac{s^2+s+1}{(s+3+4j)(s+2)}\right|_{s=-3+4j} = \frac{7}{17} + \frac{71}{136}j = 0.665e^{j0.903}$

Thus

$$H(j\omega) = \frac{\frac{3}{17}}{j\omega+2} + \frac{0.665e^{j0.903}}{j\omega+3-4j} + \frac{0.665e^{-j0.903}}{j\omega+3+4j}$$

$$\Rightarrow \quad h(t) = \frac{3}{17}e^{-2t}u(t) + 1.33e^{-3t}\cos(4t+0.903)u(t)$$

(b) Expanding the denominator polynomial:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(j\omega)^2 + j\omega + 1}{(j\omega)^3 + 8(j\omega)^2 + 37j\omega + 50}$$

$$\Rightarrow 50y(t) + 37y'(t) + 8y''(t) + y'''(t) = x(t) + x'(t) + x''(t)$$

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5. [10] Compute the Fourier transform of each of the following signals

(a) $[e^{-\alpha t}\cos\omega_0 t]u(t), \alpha > 0$

(b) $e^{-3|t|}\sin 2t$

(a)

$$[e^{-\alpha t}\cos\omega_0 t]\cdot u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t) \Rightarrow X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} + \frac{1}{2(\alpha + j\omega_0 + j\omega)} = \frac{\alpha + j\omega}{(\alpha+j\omega)^2 + \omega_0^2}$$

(b)

$$x_1(t) = e^{-3t}\sin(2t)u(t) \xrightarrow{F} X_1(j\omega) = \frac{\frac{1}{2j}}{3-j2+j\omega} - \frac{\frac{1}{2j}}{3+j2+j\omega} = \frac{2}{(3+j\omega)^2+4}$$

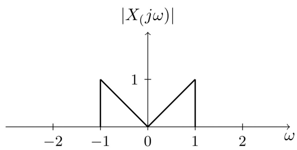
$$x_2(t) = e^{-3t}\sin(2t)u(-t) \xrightarrow{F} X_2(j\omega) = -X_1(-j\omega) = -\frac{\frac{1}{2j}}{3-j2-j\omega} + \frac{\frac{1}{2j}}{3+j2-j\omega} = \frac{-2}{(3-j\omega)^2+4}$$

$$\Rightarrow X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9+(\omega+2)^2} - \frac{3j}{9+(\omega-2)^2}$$

6. [10] Determine the continuous-time signal corresponding to the following transform.

(a) $X(j\omega) = cos(4\omega + \pi/3)$

(b) $X(j\omega)$ as given by magnitude and phase plots.



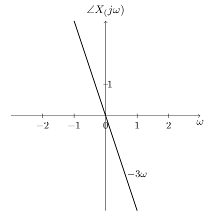


Figure: 0402

(a)

$$x(jw) = \frac{e^{j(4w+\frac{\pi}{3})} + e^{-j(4w+\frac{\pi}{3})}}{2} \Rightarrow x(t) = \frac{1}{2} e^{-\frac{j\pi t}{12}} \delta(t+4) + \frac{1}{2} e^{\frac{j\pi t}{12}} \delta(t-4)$$

(b)

From the figure we can know that $X(jw) = \begin{cases} we^{-3jw} & |w| < 1 \\ 0 & |w| > 1 \end{cases}$

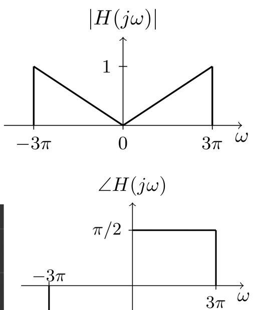
$$\Rightarrow x(t) = \frac{\cos(t-3) - 1 - (t-3)\sin(t-3)}{\pi(t-3)^2}$$

7. [10] Shown in the figure 0403 is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals x(t) below, determine the filter output signal $y(t)$.

(a) $x(t) = \cos(2\pi t + \theta)$

(b) $x(t) = \cos(4\pi t + \theta)$

(c) $x(t)$ is a half-wave rectified sine wave of period 1, as sketched in figure 0404.



From the figure we can know that $H(jw) = \begin{cases} \frac{jw}{3\pi}, & -3\pi \le w \le 3\pi \\ 0, & \text{otherwise} \end{cases}$

(a) since $x(t) = \cos(2\pi t + \theta)$, $X(jw) = e^{-\frac{jw\theta}{2\pi}} \pi [\delta(w-2\pi) + \delta(w+2\pi)]$

$$\Rightarrow Y(jw) = X(jw) H(jw) = \frac{jw}{3\pi} X(jw)$$

$$y(t) = \frac{1}{3\pi} \frac{dx(t)}{dt} = -\frac{2}{3} \sin(2\pi t + \theta)$$

(b) since $x(t) = \cos(4\pi t + \theta)$, $X(jw) = e^{-\frac{jw\theta}{4\pi}} \pi [\delta(w-4\pi) + \delta(w+4\pi)]$

$$\Rightarrow Y(jw) = X(jw) H(jw) = 0 \Rightarrow y(t) = 0$$

(c) The Fourier series coefficients of $x(t)$ are $C_k = \int_0^{0.5} \sin(2\pi t) e^{-jk2\pi t} dt$

Also we have $X(jw) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(w - kw_0)$ with $w_0 = 2\pi$

$$\Rightarrow C_0 = \frac{1}{\pi}, \quad C_1 = C_{-1}^* = \frac{-1}{4j} \Rightarrow x_{lp}(t) = \frac{1}{\pi} + \frac{1}{2}\sin(2\pi t)$$

$$\Rightarrow y(t) = \frac{1}{3\pi} x_{lp}'(t) = \frac{1}{3}\cos(2\pi t)$$