Ve 216: Introduction to Signals and Systems

Yong Long

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

May 23, 2023

Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

Outline

Useful mathematical formula

`http://web.eecs.umich.edu/~aey/eecs216/webstuff/lecture.html`

- Complex number
- Useful formula
- Phasors

Outline

Introduction

Skip: 3.6, 3.7, 3.11

- In previous chapter we focused on CT systems, specifically LTI systems, and analyzed them leading to the convolution relationship and properties. LTI systems are used to process signals.
- To get further insight, we need analysis methods for signals.
- The superposition property of linear systems suggests that decomposing signals into a sum of simpler signals will be a particularly convenient form for analysis.

Introduction

Skip: 3.6, 3.7, 3.11

- In previous chapter we focused on CT systems, specifically LTI systems, and analyzed them leading to the convolution relationship and properties. LTI systems are used to process signals.
- To get further insight, we need analysis methods for signals.
- The superposition property of linear systems suggests that decomposing signals into a sum of simpler signals will be a particularly convenient form for analysis.

Introduction

Skip: 3.6, 3.7, 3.11

- In previous chapter we focused on CT systems, specifically LTI systems, and analyzed them leading to the convolution relationship and properties. LTI systems are used to process signals.
- To get further insight, we need analysis methods for signals.
- The superposition property of linear systems suggests that decomposing signals into a sum of simpler signals will be a particularly convenient form for analysis.

Introduction

Skip: 3.6, 3.7, 3.11

- In previous chapter we focused on CT systems, specifically LTI systems, and analyzed them leading to the convolution relationship and properties. LTI systems are used to process signals.
- To get further insight, we need analysis methods for signals.
- The superposition property of linear systems suggests that decomposing signals into a sum of simpler signals will be a particularly convenient form for analysis.

Roadmap

	Signal	
Transform	Continuous Time	Discrete Time
Continuous Frequency	Fourier Transform	DTFT (periodic in frequency)
Discrete Frequency	Fourier Series (periodic in time)	DTFS or DFT (periodic in time and frequency) FFT

Overview

- LTI systems and complex-exponential signals
- Fourier series
- Convergence of Fourier series
- Properties of Fourier series
- Power density spectrum
- Fourier series and LTI systems
- Filtering and applications!

Outline

1. 3. Fourier Series
   - Introduction
   - History (3.1) (skim!)
   - LTI system response for complex-exponential input signals (3.2)
   - Preview
   - Fourier Series (3.3)
   - Convergence of Fourier series (3.4)
   - Properties of CT Fourier series (3.5)
     - One-signal properties(Fourier series transformations)
     - Two-signal properties
     - Parseval's Relation for CT Periodic Signals(3.5.7)
   - Power density spectrum
   - Fourier Series and LTI Systems (3.8)
   - Filtering (3.9)
   - Filters described by diffeqs (3.10)
   - summary

Outline

Complex-exponential signals

We have seen repeatedly that signals of the form e^{st} are particularly important.

Question

What happens if we pass such a signal through an LTI system?

$$x(t) = e^{st} \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t)$$

Complex-exponential signals

We have seen repeatedly that signals of the form e^{st} are particularly important.

Question

What happens if we pass such a signal through an LTI system?

$$x(t) = e^{st} \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t)$$

LTI system response for exponential input signals

By the convolution integral:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \end{aligned}$$

Transfer function

$$x(t) = e^{st} \to \boxed{\text{LTI } h(t)} \to y(t) = e^{st} H(s)$$

where

$$\boxed{H(s) \triangleq \int_{-\infty}^{\infty} h(t) e^{-st}\, dt}$$

is the system **transfer function**.

We will see later that $H(s)$ is the **Laplace transform** of $h(t)$.

So an exponential signal passed through an LTI system produces the same exponential signal, but scaled by $H(s)$.

Transfer function

$$x(t) = e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = e^{st} H(s)$$

where

$$\boxed{H(s) \triangleq \int_{-\infty}^{\infty} h(t) e^{-st} \, dt}$$

is the system **transfer function**.
We will see later that $H(s)$ is the **Laplace transform** of $h(t)$.

So an exponential signal passed through an LTI system produces the same exponential signal, but scaled by $H(s)$.

Transfer function

$$x(t) = e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = e^{st} H(s)$$

where

$$\boxed{H(s) \triangleq \int_{-\infty}^{\infty} h(t) e^{-st} \, dt}$$

is the system **transfer function**.
We will see later that $H(s)$ is the **Laplace transform** of $h(t)$.

So an exponential signal passed through an LTI system produces the same exponential signal, but scaled by $H(s)$.

LTI system response for complex-exponential signals

An important special case is when $s = j\omega$ so that the input is a complex-exponential signal:

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t)$$

$$\boxed{y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]}}$$

- The quantity $H(j\omega)$ is called the **frequency response** of the system, and is often just written $H(\omega)$.
- In general $H(\omega)$ is complex, so both the magnitude and phase of the complex-exponential signal are affected, as shown above.

LTI system response for complex-exponential signals

An important special case is when $s = j\omega$ so that the input is a complex-exponential signal:

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t)$$

$$\boxed{y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]}}$$

- The quantity $H(j\omega)$ is called the **frequency response** of the system, and is often just written $H(\omega)$.
- In general $H(\omega)$ is complex, so both the magnitude and phase of the complex-exponential signal are affected, as shown above.

LTI system response for complex-exponential signals

An important special case is when $s = j\omega$ so that the input is a complex-exponential signal:

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t)$$

$$\boxed{y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]}}$$

- The quantity $H(j\omega)$ is called the **frequency response** of the system, and is often just written $H(\omega)$.
- In general $H(\omega)$ is complex, so both the magnitude and phase of the complex-exponential signal are affected, as shown above.

LTI system response for complex-exponential signals

An important special case is when $s = j\omega$ so that the input is a complex-exponential signal:

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t)$$

$$\boxed{y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]}}$$

- The quantity $H(j\omega)$ is called the **frequency response** of the system, and is often just written $H(\omega)$.
- In general $H(\omega)$ is complex, so both the magnitude and phase of the complex-exponential signal are affected, as shown above.

Eigenfunction and eigenvalue (1)

Definition

When a signal has the property that when passed through a system it yields the same signal scaled by a (perhaps complex) constant, the signal is called an **eigenfunction** and the scaling factor is called the **eigenvalue**.

Example

eigenfunction: e^{st}
eigenvalue: $H(s)$

Eigenfunction and eigenvalue (1)

Definition

When a signal has the property that when passed through a system it yields the same signal scaled by a (perhaps complex) constant, the signal is called an **eigenfunction** and the scaling factor is called the **eigenvalue**.

Example

eigenfunction: e^{st}
eigenvalue: $H(s)$

Eigenfunction and eigenvalue

Question

Are the signals e^{st} the only eigenfunctions of LTI systems?

Solution

For most LTI systems, the signals e^{st} are the only eigenfunctions. *Consider the system with impulse response*

$$h(t) = \lambda\delta(t).$$

If $x(t)$ is any input signal, the output signal will be simply

$$y(t) = \lambda x(t)$$

So for this particular system (an idea amplifier), all signals are eigenfunctions.
But the idea amplifier is an unusual case.

Solution

For most LTI systems, the signals e^{st} are the only eigenfunctions. Consider the system with impulse response

$$h(t) = \lambda\delta(t).$$

If $x(t)$ is any input signal, the output signal will be simply

$$y(t) = \lambda x(t)$$

So for this particular system (an idea amplifier), all signals are eigenfunctions.
But the idea amplifier is an unusual case.

Solution

For most LTI systems, the signals e^{st} are the only eigenfunctions. Consider the system with impulse response

$$h(t) = \lambda\delta(t).$$

If $x(t)$ is any input signal, the output signal will be simply

$$y(t) = \lambda x(t)$$

So for this particular system (an idea amplifier), all signals are eigenfunctions.
But the idea amplifier is an unusual case.

Example (1)

Skill: ***Finding $H(s)$ from $h(t)$ (and vice versa - later)***

Example

Consider a RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$. Find the transfer function.

Example (1)

Skill: ***Finding*** $H(s)$ ***from*** $h(t)$ ***(and vice versa - later)***

Example

Consider a RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$. Find the transfer function.

Example (1)

Skill: ***Finding*** $H(s)$ ***from*** $h(t)$ ***(and vice versa - later)***

Example

Consider a RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$. Find the transfer function.

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st}\, dt = \int_0^{\infty} \alpha e^{-\alpha t} e^{-st}\, dt \\ &= \alpha \int_0^{\infty} e^{-(\alpha+s)t}\, dt = \frac{-\alpha}{\alpha+s} e^{-(\alpha+s)t} \Big|_0^{\infty} \\ &= \frac{\alpha}{\alpha+s} = \boxed{\frac{1}{1+(RC)s}}. \end{aligned}$$

Example (2)

Example

For RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$. What is the response to a DC (direct current) input signal?

Example (2)

Example

For RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$. What is the response to a DC (direct current) input signal?

The DC signal is

$$x(t) = a, \quad \text{or} \quad x(t) = ae^{0t}$$

so

$$s = 0 \Longrightarrow H(0) = 1$$

so

$$y(t) = H(0)ae^{0t} = a$$

Example (3)

Example

For RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$ and $RC = \frac{0.1}{2\pi}$sec. what happens to a 20Hz cosinusoidal input signal?

Example (3)

Example

For RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$ and $RC = \frac{0.1}{2\pi}$sec. what happens to a 20Hz cosinusoidal input signal?

$$x(t) = \cos(\omega_0 t) = \cos\left(\frac{2\pi}{T_0} t\right)$$

Example (3)

Example

For RC circuit (we showed previously) with $h(t) = \alpha e^{-\alpha t} u(t)$ where $\alpha = 1/RC$ and $RC = \frac{0.1}{2\pi}$sec. what happens to a 20Hz cosinusoidal input signal?

$$x(t) = \cos(\omega_0 t) = \cos\left(\frac{2\pi}{T_0} t\right)$$

$$= \cos(2\pi 20 t) = \frac{1}{2} e^{j40\pi t} + \frac{1}{2} e^{-j40\pi t}$$

Solution

$$y(t) = \frac{1}{2}H(j40\pi)e^{j40\pi t} + \frac{1}{2}H(-j40\pi)e^{-j40\pi t}$$

$$= \frac{1}{2}\frac{1}{1+\frac{0.1}{2\pi}\cdot j40\pi}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-\frac{0.1}{2\pi}\cdot j40\pi}e^{-j40\pi t} \quad \left(H(s) = \frac{1}{1+(RC)s}\right)$$

$$= \frac{1}{2}\frac{1}{1+j2}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-j2}e^{-j40\pi t}$$

$$\approx \frac{1}{2}0.45e^{-j1.1}e^{j40\pi t} + \frac{1}{2}0.45e^{+j1.1}e^{-j40\pi t}$$

$$= 0.45\frac{1}{2}\left[e^{j(40\pi t-1.1)} + e^{-j(40\pi t-1.1)}\right] = \boxed{0.45\cos(40\pi t - 1.1)}$$

Solution

$$y(t) = \frac{1}{2}H(j40\pi)e^{j40\pi t} + \frac{1}{2}H(-j40\pi)e^{-j40\pi t}$$

$$= \frac{1}{2}\frac{1}{1+\frac{0.1}{2\pi}\cdot j40\pi}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-\frac{0.1}{2\pi}\cdot j40\pi}e^{-j40\pi t} \quad (H(s) = \frac{1}{1+(RC)s})$$

$$= \frac{1}{2}\frac{1}{1+j2}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-j2}e^{-j40\pi t}$$

$$\approx \frac{1}{2}0.45e^{-j1.1}e^{j40\pi t} + \frac{1}{2}0.45e^{+j1.1}e^{-j40\pi t}$$

$$= 0.45\frac{1}{2}\left[e^{j(40\pi t-1.1)} + e^{-j(40\pi t-1.1)}\right] = \boxed{0.45\cos(40\pi t - 1.1)}$$

Solution

$$y(t) = \frac{1}{2}H(j40\pi)e^{j40\pi t} + \frac{1}{2}H(-j40\pi)e^{-j40\pi t}$$

$$= \frac{1}{2}\frac{1}{1+\frac{0.1}{2\pi}\cdot j40\pi}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-\frac{0.1}{2\pi}\cdot j40\pi}e^{-j40\pi t} \quad (H(s) = \frac{1}{1+(RC)s})$$

$$= \frac{1}{2}\frac{1}{1+j2}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-j2}e^{-j40\pi t}$$

$$\approx \frac{1}{2}0.45e^{-j1.1}e^{j40\pi t} + \frac{1}{2}0.45e^{+j1.1}e^{-j40\pi t}$$

$$= 0.45\frac{1}{2}\left[e^{j(40\pi t-1.1)} + e^{-j(40\pi t-1.1)}\right] = \boxed{0.45\cos(40\pi t - 1.1)}$$

Solution

$$y(t) = \frac{1}{2}H(j40\pi)e^{j40\pi t} + \frac{1}{2}H(-j40\pi)e^{-j40\pi t}$$

$$= \frac{1}{2}\frac{1}{1+\frac{0.1}{2\pi}\cdot j40\pi}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-\frac{0.1}{2\pi}\cdot j40\pi}e^{-j40\pi t} \quad \left(H(s) = \frac{1}{1+(RC)s}\right)$$

$$= \frac{1}{2}\frac{1}{1+j2}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-j2}e^{-j40\pi t}$$

$$\approx \frac{1}{2}0.45e^{-j1.1}e^{j40\pi t} + \frac{1}{2}0.45e^{+j1.1}e^{-j40\pi t}$$

$$= 0.45\frac{1}{2}\left[e^{j(40\pi t-1.1)} + e^{-j(40\pi t-1.1)}\right] = \boxed{0.45\cos(40\pi t - 1.1)}$$

Solution

$$y(t) = \frac{1}{2}H(j40\pi)e^{j40\pi t} + \frac{1}{2}H(-j40\pi)e^{-j40\pi t}$$

$$= \frac{1}{2}\frac{1}{1+\frac{0.1}{2\pi}\cdot j40\pi}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-\frac{0.1}{2\pi}\cdot j40\pi}e^{-j40\pi t} \quad \left(H(s) = \frac{1}{1+(RC)s}\right)$$

$$= \frac{1}{2}\frac{1}{1+j2}e^{j40\pi t} + \frac{1}{2}\frac{1}{1-j2}e^{-j40\pi t}$$

$$\approx \frac{1}{2}0.45e^{-j1.1}e^{j40\pi t} + \frac{1}{2}0.45e^{+j1.1}e^{-j40\pi t}$$

$$= 0.45\frac{1}{2}\left[e^{j(40\pi t-1.1)} + e^{-j(40\pi t-1.1)}\right] = \boxed{0.45\cos(40\pi t - 1.1)}$$

Lowpass filter

Question

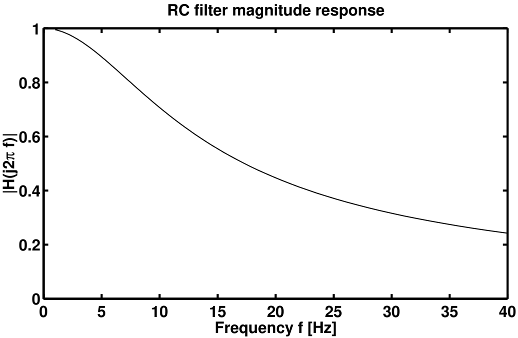
Why did the cosinusoidal signal come out attenuated?

Lowpass filter

Question

Why did the cosinusoidal signal come out attenuated?

*Because the RC circuit is a **lowpass filter**, which (roughly speaking) passes frequency components less than about 1/RC = 10Hz, but attenuates frequency components that are higher than that **cutoff frequency**.*



Outline

Transfer function

Question

The transfer function $H(s)$ is very important and useful. Why?

- We have seen how to determine the response of an LTI system (such as our RC circuit) to a sinusoidal input signal.
- Fine, but what if we wish to determine the response to a more interesting signal like a square wave? Convolution would be painful!
- We can decompose the square wave into a sum of sinusoidal signals.
  Video(MIT, Lecture 7, 39.26min)

Transfer function

Question

The transfer function $H(s)$ is very important and useful. Why?

- We have seen how to determine the response of an LTI system (such as our RC circuit) to a sinusoidal input signal.
- Fine, but what if we wish to determine the response to a more interesting signal like a square wave? Convolution would be painful!
- We can decompose the square wave into a sum of sinusoidal signals.

  Video(MIT, Lecture 7, 39.26min)

Transfer function

Question

The transfer function $H(s)$ is very important and useful. Why?

- We have seen how to determine the response of an LTI system (such as our RC circuit) to a sinusoidal input signal.
- Fine, but what if we wish to determine the response to a more interesting signal like a square wave? Convolution would be painful!
- We can decompose the square wave into a sum of sinusoidal signals.
  Video(MIT, Lecture 7, 39.26min)

Transfer function

Question

The transfer function $H(s)$ is very important and useful. Why?

- We have seen how to determine the response of an LTI system (such as our RC circuit) to a sinusoidal input signal.
- Fine, but what if we wish to determine the response to a more interesting signal like a square wave? Convolution would be painful!
- We can decompose the square wave into a sum of sinusoidal signals.
  Video(MIT, Lecture 7, 39.26min)

Euler's identity

By Euler's identity, each sinusoidal signal can be expressed using complex exponential signals of the form $e^{j\omega t}$ for various ω (the harmonics).

$$\boxed{\begin{gathered} e^{j\theta} = \cos\theta + j\sin\theta \\ \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right), \quad \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{gathered}}$$

Multiplication in frequency domain

$$e^{j\omega t} \to \boxed{\text{LTI } h(t)} \to H(j\omega)e^{j\omega t}$$

Thus, by linearity (the superposition principle):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \to \boxed{\text{LTI } h(t)} \to y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}$$

Convolution in time domain becomes multiplication in frequency domain.

Multiplication in frequency domain

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

Thus, by linearity (the superposition principle):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}$$

Convolution in time domain becomes multiplication in frequency domain.

Multiplication in frequency domain

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

Thus, by linearity (the superposition principle):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}$$

Convolution in time domain becomes multiplication in frequency domain.

Periodic signal

When an periodic signal is passed through an LTI system

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}.$$

- the output signal is also periodic (with same period).
- the Fourier series of the output has coefficients $c_k H(j\omega_0 k)$, where the c_k's are the Fourier series coefficients of the input signal.

Periodic signal

When an periodic signal is passed through an LTI system

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \to \boxed{\text{LTI } h(t)} \to y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}.$$

- the output signal is also periodic (with same period).
- the Fourier series of the output has coefficients $c_k H(j\omega_0 k)$, where the c_k's are the Fourier series coefficients of the input signal.

Periodic signal

When an periodic signal is passed through an LTI system

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\omega_0 k) e^{j\omega_0 kt}.$$

- the output signal is also periodic (with same period).
- the Fourier series of the output has coefficients $c_k H(j\omega_0 k)$, where the c_k's are the Fourier series coefficients of the input signal.

Outline

Frequency analysis

Reasons why frequency analysis is important.

- Periodic physical phenomena lead to periodic signals, which can be decomposed into sinusoids parameterized by a frequency (phase and amplitude).
- Complex exponential signals (of any frequency) are eigenfunctions of LTI systems:

$$x(t) = e^{j\omega t} \xrightarrow{\text{LTI}} y(t) = H(j\omega)e^{j\omega t}$$

Frequency analysis

Reasons why frequency analysis is important.

- Periodic physical phenomena lead to periodic signals, which can be decomposed into sinusoids parameterized by a frequency (phase and amplitude).
- Complex exponential signals (of any frequency) are eigenfunctions of LTI systems:

$$x(t) = e^{j\omega t} \xrightarrow{\text{LTI}} y(t) = H(j\omega)e^{j\omega t}$$

Frequency analysis

Reasons why frequency analysis is important.

- Periodic physical phenomena lead to periodic signals, which can be decomposed into sinusoids parameterized by a frequency (phase and amplitude).
- Complex exponential signals (of any frequency) are eigenfunctions of LTI systems:

$$x(t) = e^{j\omega t} \xrightarrow{\text{LTI}} y(t) = H(j\omega)e^{j\omega t}$$

Frequency analysis

In previous chapter, we analyzed LTI systems using superposition.

- We decomposed the input signal $x(t)$ using delta functions:

$$x(t) = \int x(\tau)\delta(t-\tau)\,d\tau,$$

- determined the impulse response:

$$h(t) = \mathcal{T}[\delta(t)],$$

- and then used linearity and time-invariance to find the overall output signal:

$$y(t) = \int x(\tau)h(t-\tau)\,d\tau.$$

Frequency analysis

In previous chapter, we analyzed LTI systems using superposition.

- We decomposed the input signal $x(t)$ using delta functions:

$$x(t) = \int x(\tau)\delta(t-\tau)\, d\tau,$$

- determined the impulse response:

$$h(t) = \mathcal{T}[\delta(t)],$$

- and then used linearity and time-invariance to find the overall output signal:

$$y(t) = \int x(\tau)h(t-\tau)\, d\tau.$$

Frequency analysis

In previous chapter, we analyzed LTI systems using superposition.

- We decomposed the input signal $x(t)$ using delta functions:

$$x(t) = \int x(\tau)\delta(t-\tau)\,d\tau,$$

- determined the impulse response:

$$h(t) = \mathcal{T}[\delta(t)],$$

- and then used linearity and time-invariance to find the overall output signal:

$$y(t) = \int x(\tau)h(t-\tau)\,d\tau.$$

Frequency analysis

In previous chapter, we analyzed LTI systems using superposition.

- We decomposed the input signal $x(t)$ using delta functions:

$$x(t) = \int x(\tau)\delta(t - \tau)\, d\tau,$$

- determined the impulse response:

$$h(t) = \mathcal{T}[\delta(t)],$$

- and then used linearity and time-invariance to find the overall output signal:

$$y(t) = \int x(\tau)h(t - \tau)\, d\tau.$$

Elementary functions

- Decompose input signal $x(t)$into a weighted sum of elementary functions.
  1. delta function: $\delta(t-\tau)$.
  2. complex exponential signals: $\{e^{j\omega t}\}$ for various ω.
- We have already seen that the response of an LTI system to the input $e^{j\omega t}$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

  where $H(j\omega)$ is the **frequency response** of the system.
- So now we just need to figure out how to do the decomposition of $x(t)$ into a weighted combination of complex-exponential signals.
- We begin with the case where $x(t)$ is periodic.

Elementary functions

- Decompose input signal $x(t)$into a weighted sum of elementary functions.
  1. delta function: $\delta(t - \tau)$.
  2. complex exponential signals: $\{e^{j\omega t}\}$ for various ω.
- We have already seen that the response of an LTI system to the input $e^{j\omega t}$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

  where $H(j\omega)$ is the **frequency response** of the system.
- So now we just need to figure out how to do the decomposition of $x(t)$ into a weighted combination of complex-exponential signals.
- We begin with the case where $x(t)$ is periodic.

Elementary functions

- Decompose input signal $x(t)$into a weighted sum of elementary functions.
  1. delta function: $\delta(t-\tau)$.
  2. complex exponential signals: $\{e^{j\omega t}\}$ for various ω.
- We have already seen that the response of an LTI system to the input $e^{j\omega t}$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

  where $H(j\omega)$ is the **frequency response** of the system.
- So now we just need to figure out how to do the decomposition of $x(t)$ into a weighted combination of complex-exponential signals.
- We begin with the case where $x(t)$ is periodic.

Elementary functions

- Decompose input signal $x(t)$into a weighted sum of elementary functions.
  1. delta function: $\delta(t-\tau)$.
  2. complex exponential signals: $\{e^{j\omega t}\}$ for various ω.
- We have already seen that the response of an LTI system to the input $e^{j\omega t}$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(j\omega)e^{j\omega t}$$

  where $H(j\omega)$ is the **frequency response** of the system.
- So now we just need to figure out how to do the decomposition of $x(t)$ into a weighted combination of complex-exponential signals.
- We begin with the case where $x(t)$ is periodic.

Fourier Series: synthesis equation

Definition

A periodic signal $x(t)$ with fundamental period T_0 has the following **Fourier Series** representation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$ called **synthesis** equation ,

- T_0 is the **fundamental period** of the signal, $x(t + T_0) = x(t)\ \forall t$.
- ω_0 is the **fundamental frequency** of the signal: $\omega_0 = 2\pi/T_0$. Also called the **first harmonic.**
- The c_k's are called the **Fourier coefficients**.
- $k\omega_0$ is called the ***k*th harmonic**.

Fourier Series: synthesis equation

Definition

A periodic signal $x(t)$ with fundamental period T_0 has the following **Fourier Series** representation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{called } \textbf{synthesis} \text{ equation },$$

- T_0 is the **fundamental period** of the signal, $x(t + T_0) = x(t)\ \forall t$.
- ω_0 is the **fundamental frequency** of the signal: $\omega_0 = 2\pi/T_0$. Also called the **first harmonic.**
- The c_k's are called the **Fourier coefficients**.
- $k\omega_0$ is called the **kth harmonic**.

Fourier Series: synthesis equation

Definition

A periodic signal $x(t)$ with fundamental period T_0 has the following **Fourier Series** representation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{called } \textbf{synthesis} \text{ equation },$$

- T_0 is the **fundamental period** of the signal, $x(t + T_0) = x(t)\ \forall t$.
- ω_0 is the **fundamental frequency** of the signal: $\omega_0 = 2\pi / T_0$. Also called the **first harmonic.**
- The c_k's are called the **Fourier coefficients**.
- $k\omega_0$ is called the **kth harmonic**.

Fourier Series: synthesis equation

Definition

A periodic signal $x(t)$ with fundamental period T_0 has the following **Fourier Series** representation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{called } \textbf{synthesis} \text{ equation },$$

- T_0 is the **fundamental period** of the signal, $x(t + T_0) = x(t)\ \forall t$.
- ω_0 is the **fundamental frequency** of the signal: $\omega_0 = 2\pi/T_0$. Also called the **first harmonic.**
- The c_k's are called the **Fourier coefficients**.
- $k\omega_0$ is called the ***k*th harmonic**.

Fourier Series: synthesis equation

Definition

A periodic signal $x(t)$ with fundamental period T_0 has the following **Fourier Series** representation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{called } \textbf{synthesis} \text{ equation },$$

- T_0 is the **fundamental period** of the signal, $x(t + T_0) = x(t)\ \forall t$.
- ω_0 is the **fundamental frequency** of the signal: $\omega_0 = 2\pi / T_0$. Also called the **first harmonic.**
- The c_k's are called the **Fourier coefficients**.
- $k\omega_0$ is called the ***k*th harmonic**.

Analysis equation: analysis equation

We compute the Fourier coefficients by the following formula, called the **analysis equation**:

$$\boxed{c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt,\ k = 0, \pm 1, \ldots.}$$

$\int_{T_0}$ denotes integration over one period.

(See 3.3.2 for derivation of this formula.)

Note that for $k = 0$ we get the **average value** or **DC value** of the signal:

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt.$$

Analysis equation: analysis equation

We compute the Fourier coefficients by the following formula, called the **analysis equation**:

$$\boxed{c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt,\ k = 0, \pm 1, \dots.}$$

$\int_{T_0}$ denotes integration over one period.
(See 3.3.2 for derivation of this formula.)
Note that for $k = 0$ we get the **average value** or **DC value** of the signal:

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt.$$

Analysis equation: analysis equation

We compute the Fourier coefficients by the following formula, called the **analysis equation**:

$$\boxed{c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt, \; k = 0, \pm 1, \ldots.}$$

$\int_{T_0}$ denotes integration over one period.
(See 3.3.2 for derivation of this formula.)
Note that for $k = 0$ we get the **average value** or **DC value** of the signal:

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt.$$

Fourier series: exponential form

Definition

The synthesis and analysis equation defines the **Fourier Series** of a periodic CT signal:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \text{ synthesis equation} \\ c_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt,\ k = 0, \pm 1, \ldots, \text{ analysis equation} \end{aligned}$$

The above form is called the **exponential form** of the Fourier series, and is applicable even to complex-valued signals.

Fourier series: exponential form

Definition

The synthesis and analysis equation defines the **Fourier Series** of a periodic CT signal:

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \text{ synthesis equation} \\
c_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt,\ k = 0, \pm 1, \ldots, \text{ analysis equation}
\end{aligned}
$$

The above form is called the **exponential form** of the Fourier series, and is applicable even to complex-valued signals.

Example

Example

a 0.5Hz square wave $x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 1/2 - 2n)$. Find the Fourier series representation of $x(t)$.

Solution (1)

Since $T_0 = 2$, $\omega_0 = 2\pi/T_0 = \pi$.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_0^{1} 1e^{-jk\pi t}\,dt = \frac{1}{2}\begin{cases} 1, & k=0 \\ \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1, & k\neq 0 \end{cases}$$ **careful!**

$$= \frac{1}{2}\begin{cases} 1, & k=0 \\ \frac{1}{jk\pi}(1-e^{-jk\pi}), & k\neq 0 \end{cases} = \begin{cases} 1/2, & k=0 \\ \frac{1}{jk\pi}\frac{1-(-1)^k}{2}, & k\neq 0 \end{cases}$$

$$= \boxed{\begin{cases} 1/2, & k=0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Solution (1)

Since $T_0 = 2$, $\omega_0 = 2\pi/T_0 = \pi$.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_0^{1} 1\,e^{-jk\pi t}\,dt = \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1, & k \neq 0 \end{cases}$$

careful!

$$= \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}), & k \neq 0 \end{cases} = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}\frac{1-(-1)^k}{2}, & k \neq 0 \end{cases}$$

$$= \boxed{\begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Solution (1)

Since $T_0 = 2$, $\omega_0 = 2\pi/T_0 = \pi$.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_0^{1} 1\,e^{-jk\pi t}\,dt = \frac{1}{2}\begin{cases} 1, & k=0 \\ \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1, & k \neq 0 \end{cases}$$ **careful!**

$$= \frac{1}{2}\begin{cases} 1, & k=0 \\ \frac{1}{jk\pi}(1-e^{-jk\pi}), & k \neq 0 \end{cases} = \begin{cases} 1/2, & k=0 \\ \frac{1}{jk\pi}\frac{1-(-1)^k}{2}, & k \neq 0 \end{cases}$$

$$= \boxed{\begin{cases} 1/2, & k=0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Solution (1)

Since $T_0 = 2$, $\omega_0 = 2\pi/T_0 = \pi$.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_0^{1} 1\,e^{-jk\pi t}\,dt = \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1, & k \neq 0 \end{cases}$$ **careful!**

$$= \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}), & k \neq 0 \end{cases} = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}\frac{1-(-1)^k}{2}, & k \neq 0 \end{cases}$$

$$= \boxed{\begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Solution (1)

Since $T_0 = 2$, $\omega_0 = 2\pi/T_0 = \pi$.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^{2} x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\int_0^{1} 1\,e^{-jk\pi t}\,dt = \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1, & k \neq 0 \end{cases} \quad \textbf{\textit{careful!}}$$

$$= \frac{1}{2}\begin{cases} 1, & k = 0 \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}), & k \neq 0 \end{cases} = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}\frac{1-(-1)^k}{2}, & k \neq 0 \end{cases}$$

$$= \boxed{\begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Solution (2)

so the Fourier series representation of $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{k,\ \text{odd}} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=-\infty,\ \text{odd}}^{-1} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{-jk\pi} e^{-jk\pi t}$$

$$= \boxed{\frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)}$$

*So an infinite set of **harmonic** sinusoids must be summed to form this square wave signal.*

Solution (2)

so the Fourier series representation of $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{k,\ \text{odd}} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=-\infty,\ \text{odd}}^{-1} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{-jk\pi} e^{-jk\pi t}$$

$$= \boxed{\frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)}$$

*So an infinite set of **harmonic** sinusoids must be summed to form this square wave signal.*

Solution (2)

so the Fourier series representation of $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{k,\ \mathrm{odd}} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=-\infty,\ \mathrm{odd}}^{-1} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{1}{-jk\pi} e^{-jk\pi t}$$

$$= \boxed{\frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)}$$

*So an infinite set of **harmonic** sinusoids must be summed to form this square wave signal.*

Solution (2)

so the Fourier series representation of $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2} + \sum_{k,\ \text{odd}} \frac{1}{jk\pi} e^{jk\pi t}$$

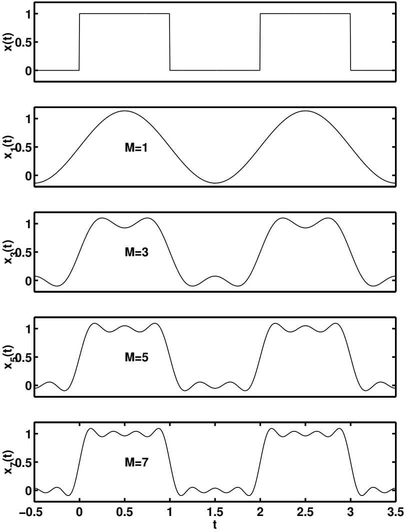
$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=-\infty,\ \text{odd}}^{-1} \frac{1}{jk\pi} e^{jk\pi t}$$

$$= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{jk\pi} e^{jk\pi t} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{1}{-jk\pi} e^{-jk\pi t}$$

$$= \boxed{\frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)}$$

*So an infinite set of **harmonic** sinusoids must be summed to form this square wave signal.*

Solution (3)



Note that only odd harmonics are present. *Video(MIT, Lecture 7, 39.26min)*

Hermitian symmetry

There are two other FS forms that are useful for real signals. To derive these forms, we first need the following fact:

Property

Hermitian symmetry:
If $x(t)$ is real, then $c_{-k} = c_k^*$.

Question

Prove the above property.

Hermitian symmetry

There are two other FS forms that are useful for real signals. To derive these forms, we first need the following fact:

Property

Hermitian symmetry:
If $x(t)$ is real, then $c_{-k} = c_k^*$.

Question

Prove the above property.

Proof

If $x(t)$ is real, then

$$c_k{}^* = \left[\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt\right]^*$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{+jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(-k)\omega_0 t}\,dt = c_{-k}$$

$$\boxed{c_k^* = c_{-k}}$$

Proof

If $x(t)$ is real, then

$$c_k{}^{*} = \left[\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt\right]^{*}$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{+jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(-k)\omega_0 t}\,dt = c_{-k}$$

$$\boxed{c_k^{*} = c_{-k}}$$

Proof

If $x(t)$ is real, then

$$c_k{}^* = \left[\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt\right]^*$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{+jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(-k)\omega_0 t}\,dt = c_{-k}$$

$$\boxed{c_k^* = c_{-k}}$$

Proof

If $x(t)$ is real, then

$$c_k{}^* = \left[\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt\right]^*$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{+jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(-k)\omega_0 t}\,dt = c_{-k}$$

$$\boxed{c_k^* = c_{-k}}$$

Trigonometric forms of Fourier series (1)

Although the above exponential form is useful because it can represent complex signals, often we have real signals and it can be helpful to have an explicitly real representation, just as found in the preceding example.

Definition

Combined trigonometric form of the FS is

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k),$$

where

$$c_k = |c_k|\, e^{j\theta_k}.$$

Trigonometric forms of Fourier series (1)

Although the above exponential form is useful because it can represent complex signals, often we have real signals and it can be helpful to have an explicitly real representation, just as found in the preceding example.

Definition

Combined trigonometric form of the FS is

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k),$$

where

$$c_k = |c_k|\, e^{j\theta_k}.$$

Trigonometric forms of Fourier series (2)

Question

Derive the combined trigonometric form.

- *Recall if $x(t)$ is real, then $c_{-k} = c_k^*$.*
- *Thus if $c_k = |c_k| e^{j\theta_k}$ then $c_{-k} = c_k^* = |c_k| e^{-j\theta_k}$, where $\theta_k = \angle c_k$.*
- *Substituting into the exponential form of the FS.*

Trigonometric forms of Fourier series (2)

Question

Derive the combined trigonometric form.

- *Recall if $x(t)$ is real, then $c_{-k} = c_k^*$.*
- *Thus if $c_k = |c_k| e^{j\theta_k}$ then $c_{-k} = c_k^* = |c_k| e^{-j\theta_k}$, where $\theta_k = \angle c_k$.*
- *Substituting into the exponential form of the FS.*

Trigonometric forms of Fourier series (2)

Question

Derive the combined trigonometric form.

- *Recall if* $x(t)$ *is real, then* $c_{-k} = c_k^*$.
- *Thus if* $c_k = |c_k|\, e^{j\theta_k}$ *then* $c_{-k} = c_k^* = |c_k|\, e^{-j\theta_k}$, *where* $\theta_k = \angle c_k$.
- *Substituting into the exponential form of the FS.*

Trigonometric forms of Fourier series (2)

Question

Derive the combined trigonometric form.

- *Recall if* $x(t)$ *is real, then* $c_{-k} = c_k^*$.
- *Thus if* $c_k = |c_k| e^{j\theta_k}$ *then* $c_{-k} = c_k^* = |c_k| e^{-j\theta_k}$, *where* $\theta_k = \angle c_k$.
- *Substituting into the exponential form of the FS.*

Trigonometric forms of Fourier series (3)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(|c_k| \, e^{j\theta_k} e^{jk\omega_0 t} + |c_k| \, e^{-j\theta_k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \, \frac{e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$$= \boxed{c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \cos(k\omega_0 t + \theta_k)}$$

Trigonometric forms of Fourier series (3)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(|c_k| \, e^{j\theta_k} e^{jk\omega_0 t} + |c_k| \, e^{-j\theta_k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} 2 |c_k| \frac{e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$$= \boxed{c_0 + \sum_{k=1}^{\infty} 2 |c_k| \cos(k\omega_0 t + \theta_k)}$$

Trigonometric forms of Fourier series (3)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(|c_k| \, e^{j\theta_k} e^{jk\omega_0 t} + |c_k| \, e^{-j\theta_k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} 2 |c_k| \frac{e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$$= \boxed{c_0 + \sum_{k=1}^{\infty} 2 |c_k| \cos(k\omega_0 t + \theta_k)}$$

Trigonometric forms of Fourier series (3)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(|c_k| \, e^{j\theta_k} e^{jk\omega_0 t} + |c_k| \, e^{-j\theta_k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \, \frac{e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$$= \boxed{c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \cos(k\omega_0 t + \theta_k)}$$

Trigonometric forms of Fourier series (3)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_k e^{jk\omega_0 t}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(|c_k| \, e^{j\theta_k} e^{jk\omega_0 t} + |c_k| \, e^{-j\theta_k} e^{-jk\omega_0 t} \right)$$

$$= c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \, \frac{e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$$= \boxed{c_0 + \sum_{k=1}^{\infty} 2 \, |c_k| \cos(k\omega_0 t + \theta_k)}$$

Trigonometric forms of Fourier series (4)

Definition

Trigonometric form of the FS:

$$\boxed{x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t),}$$

where $A_k = \mathrm{Real}(c_k)$ and $B_k = \mathrm{Imag}(c_k)$.

Proof. Skip. (textbook, p. 189)
Writing c_k in rectangular form as

$$c_k = A_k + jB_k.$$

Trigonometric forms of Fourier series (4)

Definition

Trigonometric form of the FS:

$$\boxed{x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t),}$$

where $A_k = \mathrm{Real}(c_k)$ and $B_k = \mathrm{Imag}(c_k)$.

Proof. Skip. (textbook, p. 189)
Writing c_k in rectangular form as

$$c_k = A_k + jB_k.$$

Example

Example

For the 0.5Hz square wave example in previous lecture:
$x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 1/2 - 2n)$. We already found

$$c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

Find the alternate trigonometric forms of the Fourier series representation of x(t).

Solution (1)

1. *Combined trigonometric form of FS*
   For $k > 0$, $|c_k| = \frac{1}{\pi k}$ and $\angle c_k = -\pi/2$.

$$
\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k) \\
&= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \cos(k\pi t - \pi/2)
\end{aligned}
$$

2. *Trigonometric form of the FS:*
   For $k > 0$, $\mathrm{Real} = 0, \mathrm{Imag}(c_k) = -1/\pi k$.

$$
\begin{aligned}
x(t) &= c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t) \\
&= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)
\end{aligned}
$$

Solution (1)

1. *Combined trigonometric form of FS*
   For $k > 0$, $|c_k| = \frac{1}{\pi k}$ *and* $\angle c_k = -\pi/2$.

$$
\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k) \\
&= \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \cos(k\pi t - \pi/2)
\end{aligned}
$$

2. *Trigonometric form of the FS:*
   For $k > 0$, $\mathrm{Real} = 0, \mathrm{Imag}(c_k) = -1/\pi k$.

$$
\begin{aligned}
x(t) &= c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t) \\
&= \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)
\end{aligned}
$$

Solution (1)

1. *Combined trigonometric form of FS*
   For $k > 0$, $|c_k| = \frac{1}{\pi k}$ and $\angle c_k = -\pi/2$.

$$
\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k) \\
&= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \cos(k\pi t - \pi/2)
\end{aligned}
$$

2. *Trigonometric form of the FS:*
   For $k > 0$, $\text{Real} = 0, \text{Imag}(c_k) = -1/\pi k$.

$$
\begin{aligned}
x(t) &= c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t) \\
&= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)
\end{aligned}
$$

Solution (1)

1. *Combined trigonometric form of FS*
   For $k > 0$, $|c_k| = \frac{1}{\pi k}$ *and* $\angle c_k = -\pi/2$.

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k) \\ &= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \cos(k\pi t - \pi/2) \end{aligned}$$

2. *Trigonometric form of the FS:*
   For $k > 0$, $\mathrm{Real} = 0, \mathrm{Imag}(c_k) = -1/\pi k$.

$$\begin{aligned} x(t) &= c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t) \\ &= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) \end{aligned}$$

Solution (2)

Thus we have the alternate trigonometric forms of the Fourier series representation of $x(t)$:

$$\boxed{x(t) = \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi}\sin(k\pi t) = \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi}\cos(k\pi t - \pi/2).}$$

$$\cos(\omega t \pm \pi/2) = \mp \sin(\omega t)$$

Solution (2)

Thus we have the alternate trigonometric forms of the Fourier series representation of $x(t)$:

$$\boxed{x(t) = \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi}\sin(k\pi t) = \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi}\cos(k\pi t - \pi/2).}$$

$$\cos(\omega t \pm \pi/2) = \mp \sin(\omega t)$$

Three forms of Fourier series (1)

1. Complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where c_k are complex numbers

2. Trigonometric form (cosines minus Sines)

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t),$$

where $A_k = \mathrm{Real}(c_k)$ and $B_k = \mathrm{Imag}(c_k)$.

3. Combined trigonometric form (Phase-shifted sinusoids)

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k)$$

where $\theta_k = \angle c_k$.

Three forms of Fourier series (1)

1. Complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

   where c_k are complex numbers

2. Trigonometric form (cosines minus Sines)

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t),$$

   where $A_k = \mathrm{Real}(c_k)$ and $B_k = \mathrm{Imag}(c_k)$.

3. Combined trigonometric form (Phase-shifted sinusoids)

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k)$$

   where $\theta_k = \angle c_k$.

Three forms of Fourier series (1)

1. Complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

   where c_k are complex numbers

2. Trigonometric form (cosines minus Sines)

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t),$$

   where $A_k = \mathrm{Real}(c_k)$ and $B_k = \mathrm{Imag}(c_k)$.

3. Combined trigonometric form (Phase-shifted sinusoids)

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k|\cos(k\omega_0 t + \theta_k)$$

   where $\theta_k = \angle c_k$.

Three forms of Fourier series (2)

Question

So why do we need three different Fourier series?

Three forms of Fourier series (2)

Question

So why do we need three different Fourier series?

Each has a different ease of computation:

1. *It is easiest to compute, analogous to the Discrete Fourier Transform (DFT), and it will prove most useful later in the course. But it is the most abstract for you right now.*
2. *It is easier to compute, and represents the even and odd parts of $x(t)$ separately*
3. *It is the simplest to understand, but it requires the most work to compute it.*

Outline

Convergence of Fourier series (1)

In practice we often use just a finite series approximation:

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = x(t) \approx x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

The approximation improves as N increases.

Denote the error signal as

$$e_N(t) \triangleq x(t) - x_N(t).$$

Question

But does $e_N(t)$ decreases as N increases?

Convergence of Fourier series (1)

In practice we often use just a finite series approximation:

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = x(t) \approx x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

The approximation improves as N increases.

Denote the error signal as

$$e_N(t) \triangleq x(t) - x_N(t).$$

Question

But does $e_N(t)$ decreases as N increases?

Convergence of Fourier series (1)

In practice we often use just a finite series approximation:

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = x(t) \approx x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

The approximation improves as N increases.

Denote the error signal as

$$e_N(t) \triangleq x(t) - x_N(t).$$

Question

But does $e_N(t)$ decreases as N increases?

Convergence of Fourier series (1)

In practice we often use just a finite series approximation:

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = x(t) \approx x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

The approximation improves as N increases.

Denote the error signal as

$$e_N(t) \triangleq x(t) - x_N(t).$$

Question

But does $e_N(t)$ decreases as N increases?

Convergence of Fourier series (2)

- $x(t)$ square integrable ($\int_{T_0} |x(t)|^2 \, dt < \infty$)
  The error signal energy $E_N \to 0$ as $N \to \infty$

$$E_N \triangleq \int_{T_0} |e_N(t)|^2 \, dt.$$

- $x(t)$ satisfies the **Dirichlet conditions**
  if $\int_{T_0} |x(t)| \, dt < \infty$ and $x(t)$ "well behaved ",
  then the error signal $e_N(t) \to 0$ as $N \to \infty$ except at discontinuities.
  (The signal goes to the right value at every time instant except at the discontinuities.)

Convergence of Fourier series (2)

- $x(t)$ square integrable ($\int_{T_0} |x(t)|^2 \, dt < \infty$)
  The error signal energy $E_N \to 0$ as $N \to \infty$

$$E_N \triangleq \int_{T_0} |e_N(t)|^2 \, dt.$$

- $x(t)$ satisfies the **Dirichlet conditions**
  if $\int_{T_0} |x(t)| \, dt < \infty$ and $x(t)$ "well behaved ",
  then the error signal $e_N(t) \to 0$ as $N \to \infty$ except at discontinuities.
  (The signal goes to the right value at every time instant except at the discontinuities.)

Convergence of Fourier series (2)

- $x(t)$ square integrable ($\int_{T_0} |x(t)|^2\, dt < \infty$)
  The error signal energy $E_N \to 0$ as $N \to \infty$

$$E_N \triangleq \int_{T_0} |e_N(t)|^2\, dt.$$

- $x(t)$ satisfies the **Dirichlet conditions**
  if $\int_{T_0} |x(t)|\, dt < \infty$ and $x(t)$ "well behaved ",
  then the error signal $e_N(t) \to 0$ as $N \to \infty$ except at discontinuities.
  (The signal goes to the right value at every time instant except at the discontinuities.)

Dirichlet conditions

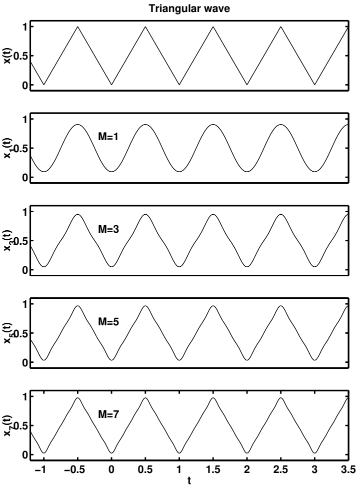
skip

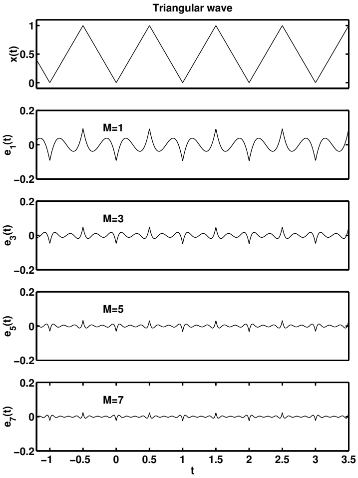
The following are the **Dirichlet conditions** that define rigorously what we mean by "well behaved" signals:

- $x(t)$ is bounded, or $x(t)$ is absolutely integrable (over each period): $\int_{T_0} |x(t)|\, dt < \infty$.
- $x(t)$ has a finite number of maxima and minima in each period. (bounded variation)
- $x(t)$ has at most a finite number of finite discontinuities over one period.

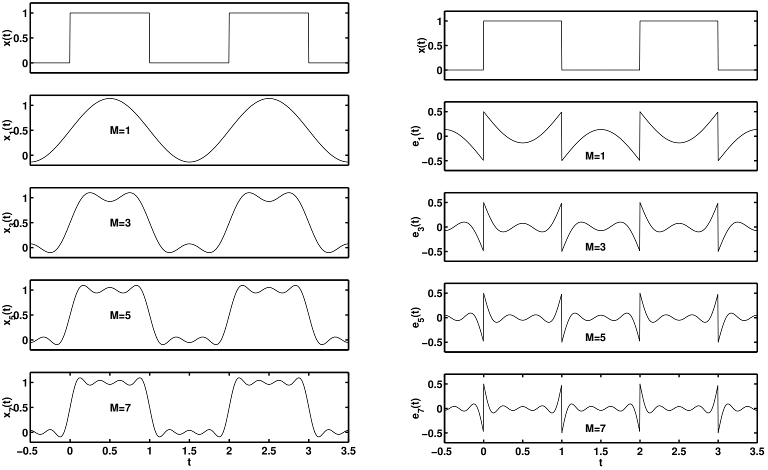
The signals $x(t)$ of interest in engineering always satisfy these conditions.

Example (1)


Triangular wave
x(t)
M=1
x_1(t)
M=3
x_3(t)
M=5
x_5(t)
M=7
x_7(t)
t


Triangular wave
x(t)
M=1
e_1(t)
M=3
e_3(t)
M=5
e_5(t)
M=7
e_7(t)
t

Example (2)



Gibbs phenomenon

Definition

Near the discontinuity there will usually be overshoot and/or undershoot that persists even as N increases, which is called **Gibbs phenomenon**.

(It is unsurprising since sinusoids have no jumps!)
Video (MIT, Lecture 7, 46.10min)

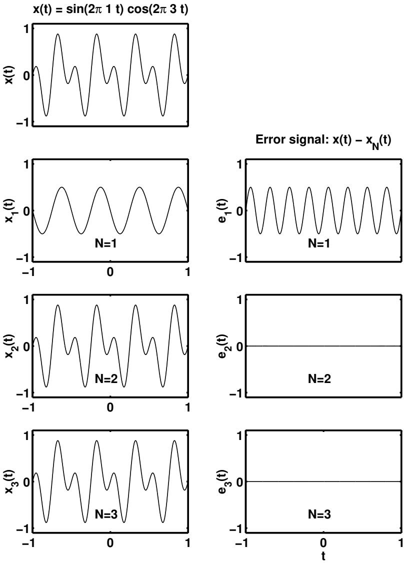
Gibbs phenomenon

Definition

Near the discontinuity there will usually be overshoot and/or undershoot that persists even as N increases, which is called **Gibbs phenomenon**.

(It is unsurprising since sinusoids have no jumps!)
Video (MIT, Lecture 7, 46.10min)

Example (3)


x(t) = sin(2π 1 t) cos(2π 3 t)
Error signal: x(t) − x_N(t)
x(t)
x_1(t)
x_2(t)
x_3(t)
e_1(t)
e_2(t)
e_3(t)
N=1
N=2
N=3
t
1
0
−1

Example (4)

$$
\begin{aligned}
x(t) &= \sin(2\pi t)\cos(2\pi 3t) = \frac{1}{2}\sin(2\pi 4t) - \frac{1}{2}\sin(2\pi 2t) \\
&= \frac{1}{2}\sin(2\omega_0 t) - \frac{1}{2}\sin(\omega_0 t) \quad (\omega_0 = 4\pi) \\
&= \frac{1}{2}\frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} - \frac{1}{2}\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \\
&= \frac{-1}{4j}e^{-j2\omega_0 t} + \frac{1}{4j}e^{-j\omega_0 t} + 0 + \frac{-1}{4j}e^{j\omega_0 t} + \frac{1}{4j}e^{j2\omega_0 t}
\end{aligned}
$$

$$
c_k = \begin{cases} 0, & k = 0 \\ \frac{-1}{4j} = c_{-k}^*, & k = 1 \\ \frac{1}{4j} = c_{-k}^*, & k = 2 \\ 0, & \text{otherwise} \end{cases}
$$

Example (4)

$$
\begin{aligned}
x(t) &= \sin(2\pi t)\cos(2\pi 3t) = \frac{1}{2}\sin(2\pi 4t) - \frac{1}{2}\sin(2\pi 2t) \\
&= \frac{1}{2}\sin(2\omega_0 t) - \frac{1}{2}\sin(\omega_0 t) \quad (\omega_0 = 4\pi) \\
&= \frac{1}{2}\frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} - \frac{1}{2}\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \\
&= \frac{-1}{4j}e^{-j2\omega_0 t} + \frac{1}{4j}e^{-j\omega_0 t} + 0 + \frac{-1}{4j}e^{j\omega_0 t} + \frac{1}{4j}e^{j2\omega_0 t}
\end{aligned}
$$

$$
c_k = \begin{cases} 0, & k = 0 \\ \frac{-1}{4j} = c_{-k}^*, & k = 1 \\ \frac{1}{4j} = c_{-k}^*, & k = 2 \\ 0, & \text{otherwise} \end{cases}
$$

Outline

Outline

Fourier series transformations (1)

Suppose we have already found the FS of a signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Now suppose we transform $x(t)$ (for example by a time or amplitude transformation) to form a new signal $y(t)$.
When $y(t)$ is also periodic (it will be for all of the transformations that follow) we can also express $y(t)$ by a FS, say:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

where ω_1 is the fundamental frequency of $y(t)$ (which may or may not equal ω_0, depending on the type of transformation).

Fourier series transformations (1)

Suppose we have already found the FS of a signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Now suppose we transform $x(t)$ (for example by a time or amplitude transformation) to form a new signal $y(t)$.
When $y(t)$ is also periodic (it will be for all of the transformations that follow) we can also express $y(t)$ by a FS, say:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

where ω_1 is the fundamental frequency of $y(t)$ (which may or may not equal ω_0, depending on the type of transformation).

Fourier series transformations (1)

Suppose we have already found the FS of a signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Now suppose we transform $x(t)$ (for example by a time or amplitude transformation) to form a new signal $y(t)$.
When $y(t)$ is also periodic (it will be for all of the transformations that follow) we can also express $y(t)$ by a FS, say:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

where ω_1 is the fundamental frequency of $y(t)$ (which may or may not equal ω_0, depending on the type of transformation).

Fourier series transformations (2)

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

We would like to be able to find the FS for $y(t)$ *i.e.*, find

1. FS coefficients d_k's
2. fundamental frequency ω_1

without recomputing everything. Thus we study properties of the FS.

First question to ask in each case: is it still periodic? If so, what is the fundamental period?

Fourier series transformations (2)

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

We would like to be able to find the FS for $y(t)$ *i.e.*, find

1. FS coefficients d_k's
2. fundamental frequency ω_1

without recomputing everything. Thus we study properties of the FS.

First question to ask in each case: is it still periodic? If so, what is the fundamental period?

Fourier series transformations (2)

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

We would like to be able to find the FS for $y(t)$ *i.e.*, find

1. FS coefficients d_k's
2. fundamental frequency ω_1

without recomputing everything. Thus we study properties of the FS.

First question to ask in each case: is it still periodic? If so, what is the fundamental period?

Fourier series transformations (2)

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},$$

We would like to be able to find the FS for $y(t)$ *i.e.*, find

1. FS coefficients d_k's
2. fundamental frequency ω_1

without recomputing everything. Thus we study properties of the FS.

First question to ask in each case: is it still periodic? If so, what is the fundamental period?

One-signal properties

- Amplitude transformations
- Time transformations (3.5.2, 3.5.3, 3.5.4)
- Conjugation (3.5.6)
- Complex modulation (frequency shift) (3.5.8)
- Differentiation (3.5.8)

Amplitude transformations

Recall amplitude transforms of signals

$$\begin{aligned} y(t) &= ax(t) + b = a\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] + b \\ &= b + \sum_{k=-\infty}^{\infty} a c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}, \end{aligned}$$

where ($\omega_1 = \omega_0$ is unchanged) and

$$\boxed{d_k = \begin{cases} b + a c_0, & k = 0 \\ a c_k, & k \neq 0. \end{cases}}$$

Amplitude transformations

Recall amplitude transforms of signals

$$
\begin{aligned}
y(t) &= ax(t) + b = a\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] + b \\
&= b + \sum_{k=-\infty}^{\infty} ac_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t},
\end{aligned}
$$

where ($\omega_1 = \omega_0$ is unchanged) and

$$
\boxed{d_k = \begin{cases} b + ac_0, & k = 0 \\ ac_k, & k \neq 0. \end{cases}}
$$

General time transformations

Recall general time transforms of signals

$$
\begin{aligned}
y(t) &= x(at+b) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(at+b)} \\
&= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 b} e^{jk(\omega_0 a)t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t},
\end{aligned}
$$

where $\omega_1 = a\omega_0$ and

$$\boxed{d_k = c_k e^{jk\omega_0 b}.}$$

The preceding formula makes the most sense if $a > 0$, because we usually think of fundamental frequencies as positive values.

General time transformations

Recall general time transforms of signals

$$\begin{aligned} y(t) &= x(at+b) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(at+b)} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 b} e^{jk(\omega_0 a)t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_1 t}, \end{aligned}$$

where $\omega_1 = a\omega_0$ and

$$\boxed{d_k = c_k e^{jk\omega_0 b}.}$$

The preceding formula makes the most sense if $a > 0$, because we usually think of fundamental frequencies as positive values.

Time reversal

To see how to handle negative values of a, consider the following time reversal property:

$$\begin{aligned} y(t) &= x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} c_k e^{j(-k)\omega_0 t} \\ &= \sum_{k'=-\infty}^{\infty} c_{-k'} e^{jk'\omega_0 t} \quad (k' = -k). \end{aligned}$$

Thus we can say that, for time reversal, the fundamental frequency of $y(t)$ remains the same as for $x(t)$, namely ω_0, but the relationship between the FS coefficients for $y(t)$ and $x(t)$ becomes $\boxed{d_k = c_{-k}.}$

Time reversal

To see how to handle negative values of a, consider the following time reversal property:

$$\begin{aligned} y(t) &= x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} c_k e^{j(-k)\omega_0 t} \\ &= \sum_{k'=-\infty}^{\infty} c_{-k'} e^{jk'\omega_0 t} \quad (k' = -k). \end{aligned}$$

Thus we can say that, for time reversal, the fundamental frequency of $y(t)$ remains the same as for $x(t)$, namely ω_0, but the relationship between the FS coefficients for $y(t)$ and $x(t)$ becomes $\boxed{d_k = c_{-k}.}$

Time shift

Consider the case where $a = 1$ and $b = -t_0$, so

$$y(t) = x(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} \left[c_k e^{-jk\omega_0 t_0}\right] e^{jk\omega_0 t}.$$

Then $\omega_1 = \omega_0$ again, but

$$\boxed{d_k = c_k e^{-jk\omega_0 t_0}.}$$

The effect of time delay is a **phase change** of Fourier coefficients. This property is expected because a time delay of a sinusoid only changes its phase.

Time shift

Consider the case where $a = 1$ and $b = -t_0$, so

$$y(t) = x(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} \left[c_k e^{-jk\omega_0 t_0} \right] e^{jk\omega_0 t}.$$

Then $\omega_1 = \omega_0$ again, but

$$\boxed{d_k = c_k e^{-jk\omega_0 t_0}.}$$

The effect of time delay is a **phase change** of Fourier coefficients. This property is expected because a time delay of a sinusoid only changes its phase.

Time shift

Consider the case where $a = 1$ and $b = -t_0$, so

$$y(t) = x(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} \left[c_k e^{-jk\omega_0 t_0}\right] e^{jk\omega_0 t}.$$

Then $\omega_1 = \omega_0$ again, but

$$\boxed{d_k = c_k e^{-jk\omega_0 t_0}.}$$

The effect of time delay is a **phase change** of Fourier coefficients. This property is expected because a time delay of a sinusoid only changes its phase.

Conjugation

Taking the complex conjugate of a period signal $x(t)$ has the effect of complex conjugation and time reversal of the corresponding Fourier series coefficients.

$$\begin{aligned} y(t) &= [x(t)]^* = \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]^* = \sum_{k=-\infty}^{\infty} {c_k}^* e^{-jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c^*_{-k} e^{jk\omega_0 t} \end{aligned}$$

$$\boxed{d_k = c^*_{-k}.}$$

Conjugation

Taking the complex conjugate of a period signal $x(t)$ has the effect of complex conjugation and time reversal of the corresponding Fourier series coefficients.

$$
\begin{aligned}
y(t) &= [x(t)]^* = \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]^* = \sum_{k=-\infty}^{\infty} {c_k}^* e^{-jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c^*_{-k} e^{jk\omega_0 t}
\end{aligned}
$$

$$\boxed{d_k = c^*_{-k}.}$$

Conjugation

Taking the complex conjugate of a period signal $x(t)$ has the effect of complex conjugation and time reversal of the corresponding Fourier series coefficients.

$$
\begin{aligned}
y(t) &= [x(t)]^* = \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]^* = \sum_{k=-\infty}^{\infty} c_k{}^* e^{-jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}
\end{aligned}
$$

$$\boxed{d_k = c_{-k}^*.}$$

Complex modulation (frequency shift)

Complex modulation: multiply $x(t)$ by a complex exponential signal whose frequency is a harmonic:

$$\begin{aligned} y(t) &= x(t)e^{j\omega_0 tN} = \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] e^{j\omega_0 tN} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k+N)\omega_0 t} \\ &= \sum_{k'=-\infty}^{\infty} c_{k'-N} e^{jk'\omega_0 t}, \quad (k' = k + N), \end{aligned}$$

so

$$\boxed{d_k = c_{k-N}}$$

which means the coefficients are all shifted by N.

Complex modulation (frequency shift)

Complex modulation: multiply $x(t)$ by a complex exponential signal whose frequency is a harmonic:

$$\begin{aligned} y(t) &= x(t)e^{j\omega_0 tN} = \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] e^{j\omega_0 tN} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k+N)\omega_0 t} \\ &= \sum_{k'=-\infty}^{\infty} c_{k'-N} e^{jk'\omega_0 t}, \quad (k' = k+N), \end{aligned}$$

so

$$\boxed{d_k = c_{k-N}}$$

which means the coefficients are all shifted by N.

Differentiation

Differentiation

$$
\begin{aligned}
y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c_k \frac{\mathrm{d}}{\mathrm{d}t} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{jk\omega_0 t}
\end{aligned}
$$

$$\boxed{d_k = jk\omega_0\, c_k, \quad k \neq 0.}$$

Question

Which frequency components are amplified more, high frequency or low frequency components?

Differentiation

Differentiation

$$\begin{aligned} y(t) &= \frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k \frac{\mathrm{d}}{\mathrm{d}t} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

$$\boxed{d_k = jk\omega_0 \, c_k, \quad k \neq 0.}$$

Question

Which frequency components are amplified more, high frequency or low frequency components?

Differentiation

Differentiation

$$\begin{aligned} y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k \frac{\mathrm{d}}{\mathrm{d}t} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

$$\boxed{d_k = jk\omega_0\, c_k, \quad k \neq 0.}$$

Question

Which frequency components are amplified more, high frequency or low frequency components?

Differentiation

Differentiation

$$
\begin{aligned}
y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} c_k \frac{\mathrm{d}}{\mathrm{d}t} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{jk\omega_0 t}
\end{aligned}
$$

$$\boxed{d_k = jk\omega_0\, c_k, \quad k \neq 0.}$$

Question

Which frequency components are amplified more, high frequency or low frequency components?

Coefficients of higher frequency terms are amplified more. This is why differentiators amplify noise.

Differentiation: example

Example

Find the FS of $x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-8n}{2}\right) + \text{rect}\left(\frac{t-4-8n}{4}\right)$ using the differentiation property.

Differentiation: example

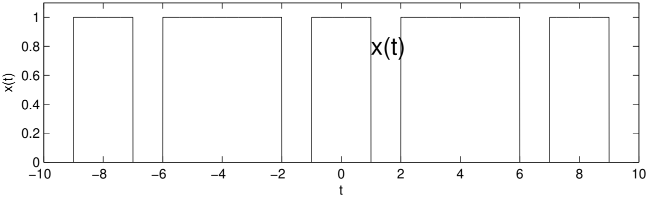
Example

Find the FS of $x(t) = \sum_{n=-\infty}^{\infty} \operatorname{rect}\left(\frac{t-8n}{2}\right) + \operatorname{rect}\left(\frac{t-4-8n}{4}\right)$ using the differentiation property.

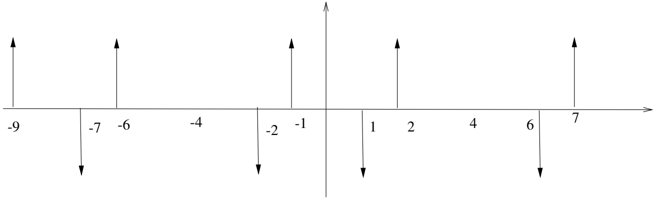
We could do this by integration, but the derivative of $x(t)$ is simply a sequence of impulses, and impulses are particularly easy to integrate using the sifting property.

- *Recall rectangle function can be represented using step functions.*
- *Recall $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$*

Solution (1)



$$x(t) = \sum_{n=-\infty}^{\infty} u(t+1-8n) - u(t-1-8n) + u(t-2-8n) - u(t-6-8n)$$



$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \sum_{n=-\infty}^{\infty} \delta(t+1-8n) - \delta(t-1-8n) + \delta(t-2-8n) - \delta(t-6-8n)$$

Solution (2)

The FS coefficients of $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ *are*

$k \neq 0$

$$d_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{8}\int_0^8 y(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{8}\int_{-4}^{4}[-\delta(t+2)+\delta(t+1)-\delta(t-1)+\delta(t-2)]e^{-jk(\pi/4)t}\,dt$$

$$= \frac{1}{8}\left[-e^{-jk(\pi/4)(-2)} + e^{-jk(\pi/4)(-1)} - e^{-jk(\pi/4)} + e^{-jk(\pi/4)2}\right]$$

$$= \frac{j}{4}\left[\sin(k\pi/4) - \sin(k\pi/2)\right]$$

Solution (2)

The FS coefficients of $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ *are*

$k \neq 0$

$$d_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{8}\int_0^8 y(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{8}\int_{-4}^{4}[-\delta(t+2)+\delta(t+1)-\delta(t-1)+\delta(t-2)]e^{-jk(\pi/4)t}\,dt$$

$$= \frac{1}{8}\left[-e^{-jk(\pi/4)(-2)}+e^{-jk(\pi/4)(-1)} - e^{-jk(\pi/4)} + e^{-jk(\pi/4)2}\right]$$

$$= \frac{j}{4}\left[\sin(k\pi/4)-\sin(k\pi/2)\right]$$

Solution (2)

The FS coefficients of $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ *are*

$k \neq 0$

$$d_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{8}\int_0^8 y(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{8}\int_{-4}^{4}[-\delta(t+2)+\delta(t+1)-\delta(t-1)+\delta(t-2)]e^{-jk(\pi/4)t}\,dt$$

$$= \frac{1}{8}\left[-e^{-jk(\pi/4)(-2)} + e^{-jk(\pi/4)(-1)} - e^{-jk(\pi/4)} + e^{-jk(\pi/4)2}\right]$$

$$= \frac{j}{4}\left[\sin(k\pi/4) - \sin(k\pi/2)\right]$$

Solution (2)

The FS coefficients of $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ *are*

$k \neq 0$

$$d_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{8}\int_0^8 y(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{8}\int_{-4}^{4}[-\delta(t+2)+\delta(t+1)-\delta(t-1)+\delta(t-2)]e^{-jk(\pi/4)t}\,dt$$

$$= \frac{1}{8}\left[-e^{-jk(\pi/4)(-2)} + e^{-jk(\pi/4)(-1)} - e^{-jk(\pi/4)} + e^{-jk(\pi/4)2}\right]$$

$$= \frac{j}{4}\left[\sin(k\pi/4) - \sin(k\pi/2)\right]$$

Solution (3)

$k = 0$

$$
\begin{aligned}
c_k &= \frac{1}{8}\int_0^8 x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{8}\left(\int_0^1 1\,dt + \int_2^6 1\,dt + \int_7^8 1\,dt\right) = 3/4
\end{aligned}
$$

Since $d_k = jk\omega_0 c_k$,

$$
\begin{aligned}
c_k &= \begin{cases} \frac{1}{jk\omega_0} d_k, k \neq 0 \\ 3/4, k = 0 \end{cases} \\
&= \boxed{\begin{cases} \frac{1}{k\pi}\left[\sin(k\pi/4) - \sin(k\pi/2)\right], k \neq 0 \\ 3/4, k = 0 \end{cases}}.
\end{aligned}
$$

Solution (3)

$k = 0$

$$
\begin{aligned}
c_k &= \frac{1}{8}\int_0^8 x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{8}\left(\int_0^1 1\,dt + \int_2^6 1\,dt + \int_7^8 1\,dt\right) = 3/4
\end{aligned}
$$

Since $d_k = jk\omega_0 c_k$,

$$
\begin{aligned}
c_k &= \begin{cases} \frac{1}{jk\omega_0} d_k, k \neq 0 \\ 3/4, k = 0 \end{cases} \\
&= \boxed{\begin{cases} \frac{1}{k\pi}\left[\sin(k\pi/4) - \sin(k\pi/2)\right], k \neq 0 \\ 3/4, k = 0 \end{cases}}.
\end{aligned}
$$

Outline

Two-signal properties

- Linearity (3.5.1)
- Multiplication (3.5.5)
- Circular convolution

Linearity

- If $x_1(t)$ and $x_2(t)$ are both periodic with the same period T_0, then the sum $x(t) = Ax_1(t) + Bx_2(t)$ is also periodic with period T_0.
- If $x_1(t)$ has FS coefficients a_k and $x_2(t)$ has FS coefficients b_k, then $x(t)$ has FS coefficients

$$\boxed{c_k = Aa_k + Bb_k}$$

The proof follow directly from

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt, k = 0, \pm 1, \pm 2, \ldots.$$

Linearity

- If $x_1(t)$ and $x_2(t)$ are both periodic with the same period T_0, then the sum $x(t) = Ax_1(t) + Bx_2(t)$ is also periodic with period T_0.
- If $x_1(t)$ has FS coefficients a_k and $x_2(t)$ has FS coefficients b_k, then $x(t)$ has FS coefficients

$$\boxed{c_k = Aa_k + Bb_k}$$

The proof follow directly from

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\, dt, k = 0, \pm 1, \pm 2, \ldots.$$

Linearity

- If $x_1(t)$ and $x_2(t)$ are both periodic with the same period T_0, then the sum $x(t) = Ax_1(t) + Bx_2(t)$ is also periodic with period T_0.
- If $x_1(t)$ has FS coefficients a_k and $x_2(t)$ has FS coefficients b_k, then $x(t)$ has FS coefficients

$$\boxed{c_k = Aa_k + Bb_k}$$

The proof follow directly from

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt, k = 0, \pm 1, \pm 2, \ldots.$$

Multiplication

- If $x(t)$ and $y(t)$ are both periodic with the same period T_0, then their product $x(t)y(t)$ is also periodic with period T_0.
- If $x(t)$ has FS coefficients a_k and $y(t)$ has FS coefficients b_k, then $x(t)y(t)$ has FS coefficients

$$\boxed{c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

- This is called **discrete convolution**. (Whether using this property is easier than integrating is debatable.)

Multiplication

- If $x(t)$ and $y(t)$ are both periodic with the same period T_0, then their product $x(t)y(t)$ is also periodic with period T_0.
- If $x(t)$ has FS coefficients a_k and $y(t)$ has FS coefficients b_k, then $x(t)y(t)$ has FS coefficients

$$\boxed{c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

- This is called **discrete convolution**. (Whether using this property is easier than integrating is debatable.)

Multiplication

- If $x(t)$ and $y(t)$ are both periodic with the same period T_0, then their product $x(t)y(t)$ is also periodic with period T_0.
- If $x(t)$ has FS coefficients a_k and $y(t)$ has FS coefficients b_k, then $x(t)y(t)$ has FS coefficients

$$\boxed{c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

- This is called **discrete convolution**. (Whether using this property is easier than integrating is debatable.)

Proof

$$z(t) = x(t)y(t) = \left[\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right] y(t) = \sum_{l=-\infty}^{\infty} a_l \left[y(t)e^{jl\omega_0 t}\right]$$

$$= \sum_{l=-\infty}^{\infty} a_l \left[\sum_{k=-\infty}^{\infty} b_{k-l} e^{jk\omega_0 t}\right] \quad \text{(complex modulation)}$$

$$= \sum_{k=-\infty}^{\infty} \left[\underbrace{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}_{c_k}\right] e^{jk\omega_0 t}$$

$$\boxed{z(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

Proof

$$z(t) = x(t)y(t) = \left[\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right] y(t) = \sum_{l=-\infty}^{\infty} a_l \left[y(t)e^{jl\omega_0 t}\right]$$

$$= \sum_{l=-\infty}^{\infty} a_l \left[\sum_{k=-\infty}^{\infty} b_{k-l} e^{jk\omega_0 t}\right] \quad \text{(complex modulation)}$$

$$= \sum_{k=-\infty}^{\infty} \left[\underbrace{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}_{c_k}\right] e^{jk\omega_0 t}$$

$$\boxed{z(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

Proof

$$z(t) = x(t)y(t) = \left[\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right] y(t) = \sum_{l=-\infty}^{\infty} a_l \left[y(t)e^{jl\omega_0 t}\right]$$

$$= \sum_{l=-\infty}^{\infty} a_l \left[\sum_{k=-\infty}^{\infty} b_{k-l} e^{jk\omega_0 t}\right] \qquad \text{(complex modulation)}$$

$$= \sum_{k=-\infty}^{\infty} \left[\underbrace{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}_{c_k}\right] e^{jk\omega_0 t}$$

$$\boxed{z(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

Proof

$$z(t) = x(t)y(t) = \left[\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right] y(t) = \sum_{l=-\infty}^{\infty} a_l \left[y(t)e^{jl\omega_0 t}\right]$$

$$= \sum_{l=-\infty}^{\infty} a_l \left[\sum_{k=-\infty}^{\infty} b_{k-l} e^{jk\omega_0 t}\right] \qquad \text{(complex modulation)}$$

$$= \sum_{k=-\infty}^{\infty} \left[\underbrace{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}_{c_k}\right] e^{jk\omega_0 t}$$

$$\boxed{z(t) = x(t)y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

Circular convolution

skip Suppose $x_1(t)$ and $x_2(t)$ are both periodic with fundamental frequency ω_0, and suppose $y(t)$ is defined as follows:

$$y(t) = \frac{1}{T_0} \int_{T_0} x_1(t-\tau) x_2(\tau)\, d\tau$$

which is called **periodic convolution** or **circular convolution**.

Let $\{a_k\}$, $\{b_k\}$, and $\{d_k\}$ denote the Fourier coefficients of $x_1(t)$, $x_2(t)$ and $y(t)$ respectively. Then

$$d_k = a_k b_k, \ \forall k.$$

So (periodic) convolution in the time domain yields multiplication in the frequency domain.

Circular convolution (2)

skip Proof: First, it is easy to verify that $y(t)$ is periodic with the same period.

$$
\begin{aligned}
d_k &= \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{T_0}\int_{T_0}\left[\frac{1}{T_0}\int_{T_0} x_1(t-\tau)x_2(\tau)\,d\tau\right]e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{T_0}\int_{T_0} x_2(\tau)\left[\frac{1}{T_0}\int_{T_0} x_1(t-\tau)e^{-jk\omega_0 t}\,dt\right]d\tau \\
&= \frac{1}{T_0}\int_{T_0} x_2(\tau)a_k e^{-jk\omega_0 \tau}\,d\tau = a_k\frac{1}{T_0}\int_{T_0} x_2(\tau)e^{-jk\omega_0 \tau}\,d\tau \\
&= a_k b_k.
\end{aligned}
$$

Consistent relationship

In each of the preceding 2 properties, we see the following consistent relationship.

> Convolution in one domain (time or frequency) corresponds to multiplication in the other domain.

This property is a significant part of the reason why the frequency domain is so important.

Outline

Parseval's Relation for CT Periodic Signals

Recall that periodic signal are **power signals**, and each such signal has a certain **average power** given by

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt.$$

Parseval's relation for periodic signals is:

$$\boxed{P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2 .}$$

The signal power is the sum of the power in each frequency component.

Parseval's Relation for CT Periodic Signals

Recall that periodic signal are **power signals**, and each such signal has a certain **average power** given by

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt.$$

Parseval's relation for periodic signals is:

$$\boxed{P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \,.}$$

The signal power is the sum of the power in each frequency component.

Proof

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}$$

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^*\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k^* \left[\underbrace{\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\, dt}_{c_k}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k^* c_k = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Proof

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}$$

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* \, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k^* \left[\underbrace{\frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} \, dt}_{c_k}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k^* c_k = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Proof

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}$$

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^*\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k^* \left[\underbrace{\frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt}_{c_k}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k^* c_k = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Proof

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}$$

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \frac{1}{T_0} \int_{T_0} x(t) x(t)^* \, dt$$

$$= \frac{1}{T_0} \int_{T_0} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} c_k^* \left[\underbrace{\frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \, dt}_{c_k} \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k^* c_k = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Example

Example

For the 0.5Hz square wave example in previous lecture: $x(t) = \sum_{n=-\infty}^{\infty} \text{rect}(t - 1/2 - 2n)$. Its FS coefficients are

$$c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases} .$$

Check the series table to verify that

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Solution

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \frac{1}{2}\int_0^1 1 \, dt = \boxed{\frac{1}{2}}$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |c_k|^2 &= \tfrac{1}{2}^2 + 2\sum_{k=1,\text{odd}}^{\infty} \frac{1}{jk\pi}\left(\frac{1}{-jk\pi}\right) \\
&= \frac{1}{4} + \frac{2}{\pi^2} \underbrace{\sum_{k=1,\text{odd}}^{\infty} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)}_{\pi^2/8} = \boxed{\frac{1}{2}}
\end{aligned}$$

This infinite series does in fact sum to $\frac{\pi^2}{8}$ (check series table); try computing its partial sums numerically.

Solution

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \frac{1}{2}\int_0^1 1 \, dt = \boxed{\frac{1}{2}}$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |c_k|^2 &= \tfrac{1}{2}^2 + 2 \sum_{k=1,\text{odd}}^{\infty} \frac{1}{jk\pi}\left(\frac{1}{-jk\pi}\right) \\
&= \frac{1}{4} + \frac{2}{\pi^2} \underbrace{\sum_{k=1,\text{odd}}^{\infty} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)}_{\pi^2/8} = \boxed{\frac{1}{2}}
\end{aligned}$$

This infinite series does in fact sum to $\frac{\pi^2}{8}$ (check series table); try computing its partial sums numerically.

Solution

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \frac{1}{2}\int_0^1 1 \, dt = \boxed{\frac{1}{2}}$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |c_k|^2 &= \tfrac{1}{2}^2 + 2\sum_{k=1,\text{odd}}^{\infty} \frac{1}{jk\pi}\left(\frac{1}{-jk\pi}\right) \\
&= \frac{1}{4} + \frac{2}{\pi^2} \underbrace{\sum_{k=1,\text{odd}}^{\infty} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)}_{\pi^2/8} = \boxed{\frac{1}{2}}
\end{aligned}$$

This infinite series does in fact sum to $\frac{\pi^2}{8}$ (check series table); try computing its partial sums numerically.

Outline

Power density spectrum

Definition

The **power density spectrum** of a periodic signal is a plot that shows how much power the signal has in each frequency component $k\omega_0$. It is a plot of component power $|c_k|^2$ vs frequency $k\omega_0$.

Since there is no power except at the frequencies $\{k\omega_0\}_{k=-\infty}^{\infty}$, we use MATLAB's `stem` command to make these **line spectra**, rather than "connecting the dots" like the `plot` command would.

Power density spectrum

Definition

The **power density spectrum** of a periodic signal is a plot that shows how much power the signal has in each frequency component $k\omega_0$. It is a plot of component power $|c_k|^2$ vs frequency $k\omega_0$.

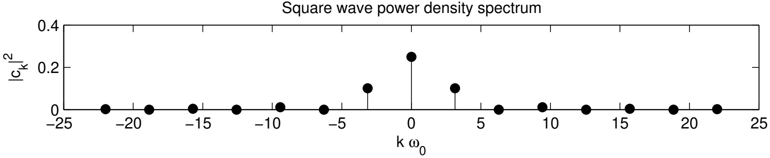
Since there is no power except at the frequencies $\{k\omega_0\}_{k=-\infty}^{\infty}$, we use MATLAB's `stem` command to make these **line spectra**, rather than "connecting the dots" like the `plot` command would.

Example

Example

Plot the power density spectrum of previous square wave.

$$\omega_0 = \pi,\ c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

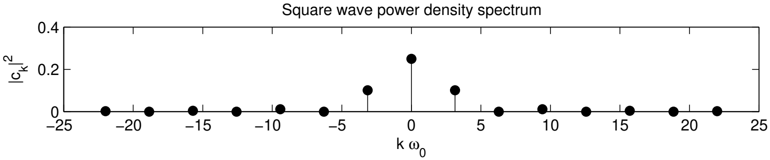


Example

Example

Plot the power density spectrum of previous square wave.

$$\omega_0 = \pi,\ c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$



Magnitude and phase spectrum

Sometimes we prefer to plot the amplitude and phase rather than the power.

- $|c_k|$ vs $k\omega_0$ is called the **magnitude spectrum**
- $\angle c_k$ vs $k\omega_0$ is called the **phase spectrum**

Note Hermitian symmetry of c_k's means phase spectrum is odd symmetric.

Magnitude and phase spectrum

Sometimes we prefer to plot the amplitude and phase rather than the power.

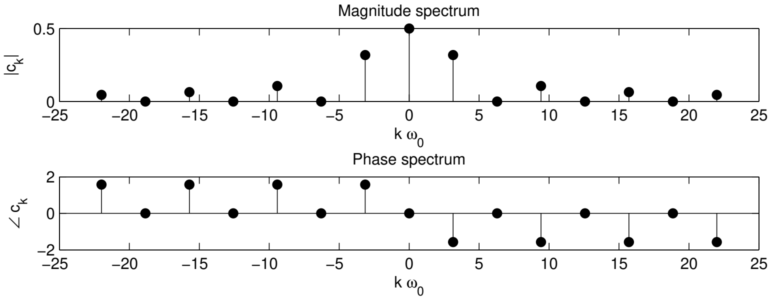
- $|c_k|$ vs $k\omega_0$ is called the **magnitude spectrum**
- $\angle c_k$ vs $k\omega_0$ is called the **phase spectrum**

Note Hermitian symmetry of c_k's means phase spectrum is odd symmetric.

Example

Example

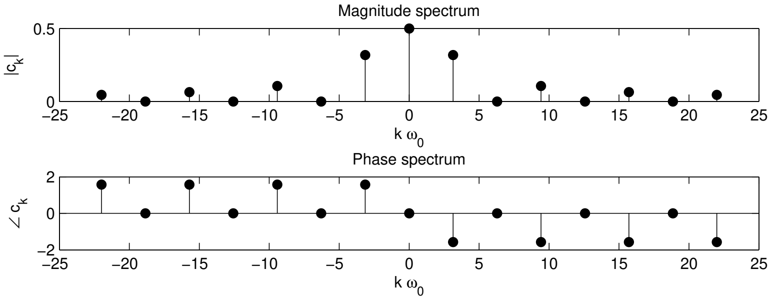
Plot the magnitude and phase density spectrum of previous square wave. $\omega_0 = \pi$, $c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases}$



Example

Example

Plot the magnitude and phase density spectrum of previous square wave. $\omega_0 = \pi$, $c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases}$



Equal power

Question

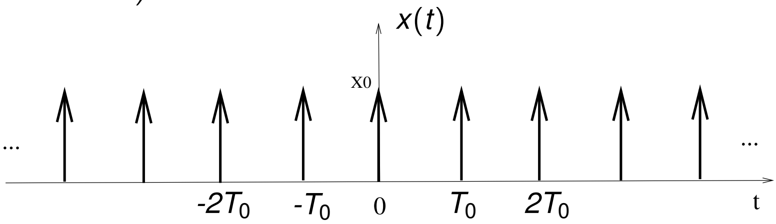
Is there a signal that has equal power at all frequencies?

Solution

*Yes. The **impulse train** signal is*

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)}$$

it is useful (for analyzing the sampling function of D/A converters).

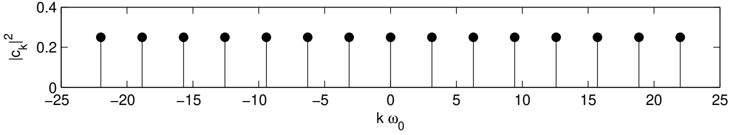


Impulse Train

Find its FS.

$$c_k = \frac{1}{T_0} \underbrace{\int_{T_0} \delta(t) e^{-jk\omega_0 t}\, dt}_{=1\,\textit{(sifting property)}} = \frac{1}{T_0}\ \forall k.$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}.$$



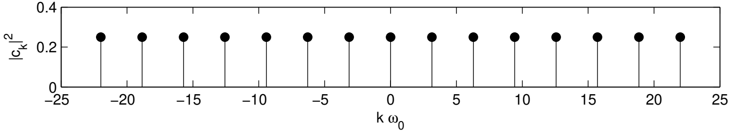
$T_0 = 2; \omega_0 = 2\pi/T_0 = \pi$

Impulse Train

Find its FS.

$$c_k = \frac{1}{T_0} \underbrace{\int_{T_0} \delta(t) e^{-jk\omega_0 t}\, dt}_{=1\,\textit{(sifting property)}} = \frac{1}{T_0}\ \forall k.$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}.$$



$T_0 = 2; \omega_0 = 2\pi / T_0 = \pi$

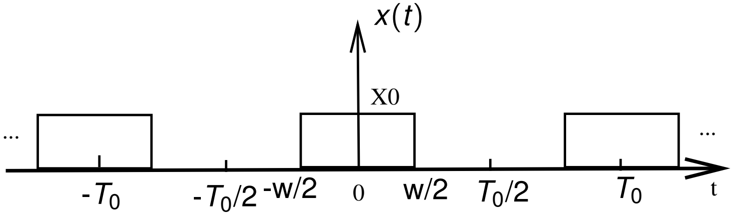
Rectangular Pulse Train

Example

rectangular pulse train (useful for clocking circuits etc.)

$$x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_0}{w}\right)$$

for $0 < w < T_0$.



FS of Rectangular Pulse Train

Find its FS.

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{-w/2}^{w/2} 1e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\,\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_{-w/2}^{w/2} \\
&= \frac{2}{T_0 k\omega_0}\frac{e^{jk\omega_0 w/2} - e^{-jk\omega_0 w/2}}{2j} = \frac{w}{T_0}\frac{\sin(k\omega_0 w/2)}{k\omega_0 w/2} \\
&= \frac{w}{T_0}\mathrm{sinc}\left(kw\frac{\omega_0}{2\pi}\right),
\end{aligned}
$$

where the **sine cardinal function** or just **sinc function** is:

$$
\boxed{\mathrm{sinc}(x) \triangleq \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}}
$$

FS of Rectangular Pulse Train

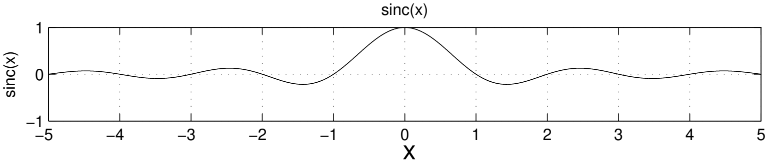
Find its FS.

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{-w/2}^{w/2} 1 e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\,\frac{1}{-jk\omega_0} e^{-jk\omega_0 t}\Big|_{-w/2}^{w/2} \\
&= \frac{2}{T_0 k\omega_0}\frac{e^{jk\omega_0 w/2} - e^{-jk\omega_0 w/2}}{2j} = \frac{w}{T_0}\frac{\sin(k\omega_0 w/2)}{k\omega_0 w/2} \\
&= \frac{w}{T_0}\mathrm{sinc}\left(kw\frac{\omega_0}{2\pi}\right),
\end{aligned}
$$

where the **sine cardinal function** or just **sinc function** is:

$$
\boxed{\mathrm{sinc}(x) \triangleq \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}}
$$

Sinc function



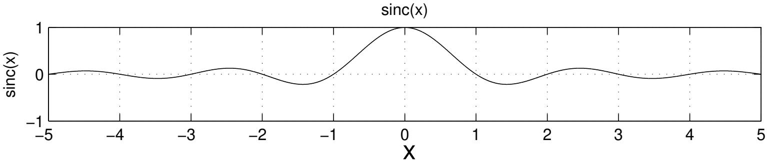
Caution! In MATLAB, `sinc(x)` = $\frac{\sin(\pi x)}{\pi x}$. This is consistent with the Oppenheim text.

But in some books,

$$\mathrm{sinc}(x) \triangleq \frac{\sin(x)}{x}$$

Whenever you see a sinc function in the future, make sure you check which version is meant!

Sinc function

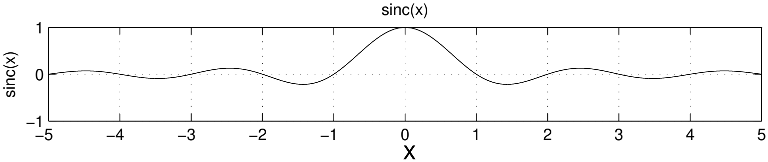


Caution! In MATLAB, `sinc(x)` = $\frac{\sin(\pi x)}{\pi x}$. This is consistent with the Oppenheim text.
But in some books,

$$\mathrm{sinc}(x) \triangleq \frac{\sin(x)}{x}$$

Whenever you see a sinc function in the future, make sure you check which version is meant!

Sinc function



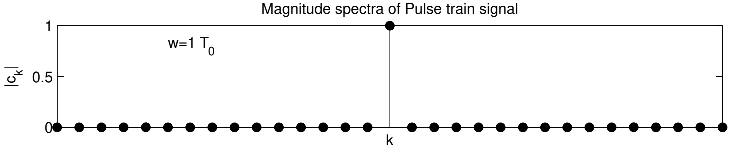
Caution! In MATLAB, `sinc(x)` = $\frac{\sin(\pi x)}{\pi x}$. This is consistent with the Oppenheim text.
But in some books,

$$\mathrm{sinc}(x) \triangleq \frac{\sin(x)}{x}$$

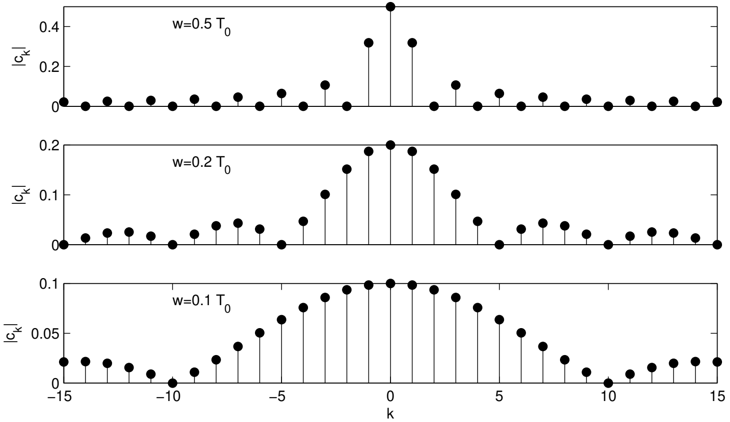
Whenever you see a sinc function in the future, make sure you check which version is meant!

Magnitude spectra of pulse train signal (1)

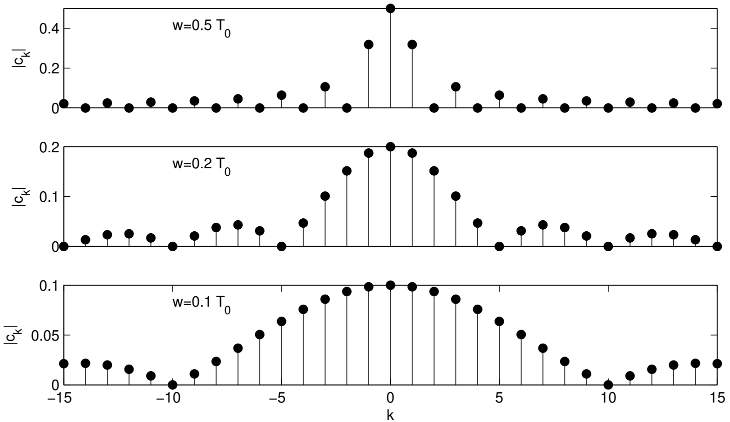
$$w = T_0, \quad c_k = \frac{w}{T_0}\operatorname{sinc}\left(kw\frac{\omega_0}{2\pi}\right) = \operatorname{sinc}(k) = \begin{cases} \frac{\sin \pi k}{\pi k} = 0, & k \neq 0 \\ 1, & k = 0 \end{cases}$$



Magnitude spectra of pulse train signal (1)



Magnitude spectra of pulse train signal (1)



With the decrease of w, the rectangular gets narrower and narrower, the magnitude spectra spreads out more to the high frequencies and the low frequency values are smaller.

Outline

Exponential signals

Exponential signals are eigenfunctions of LTI systems:

$$x(t) = e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = H(s)e^{st}$$

Proof (uses convolution formula derived earlier for LTI systems):

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \end{aligned}$$

Laplace transform of $h(t)$, **system function**

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt,$$

Exponential signals

Exponential signals are eigenfunctions of LTI systems:

$$x(t) = e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = H(s)e^{st}$$

Proof (uses convolution formula derived earlier for LTI systems):

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau \end{aligned}$$

Laplace transform of $h(t)$, **system function**

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt,$$

Complex exponential signals

Complex exponential signals are the most important special case ($s = j\omega$):

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}$$

Fourier transform of $h(t)$, **frequency response**:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt = H(s)|_{s=j\omega} = |H(j\omega)|e^{j\angle H(j\omega)},$$

Complex exponential signals

Complex exponential signals are the most important special case ($s = j\omega$):

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}$$

Fourier transform of $h(t)$, **frequency response**:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = H(s)|_{s=j\omega} = |H(j\omega)|e^{j\angle H(j\omega)},$$

Complex numbers

Mathematical review of complex numbers (Text p. 71)

- Cartesian or rectangular form:

$$z = x + jy, \quad x = \text{real}\{z\}, \quad y = \text{imag}\{z\}$$

- Polar form

$$z = |z|e^{j\theta}, \quad |z| = \sqrt{x^2 + y^2}, \quad \theta = \angle z = \operatorname{atan}(y/x)$$

Complex numbers

Mathematical review of complex numbers (Text p. 71)

- Cartesian or rectangular form:

$$z = x + jy, \quad x = \text{real}\{z\}, \quad y = \text{imag}\{z\}$$

- Polar form

$$z = |z|e^{j\theta}, \quad |z| = \sqrt{x^2 + y^2}, \quad \theta = \angle z = \operatorname{atan}(y/x)$$

Periodic signals

Periodic signals represented as sums of exponentials:

$$x(t) = \sum_k c_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_k c_k H(jk\omega_0) e^{jk\omega_0 t}$$

- $y(t)$ is also periodic with the same fundamental frequency ω_0.
- If $\{c_k\}$ is the set of FS coefficients for the input $x(t)$, then $\{c_k H(jk\omega_0)\}$ is the set of coefficients for the output $y(t)$.
- The effect of the LTI system is to modify individually each of the FS coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Periodic signals

Periodic signals represented as sums of exponentials:

$$x(t) = \sum_k c_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_k c_k H(jk\omega_0) e^{jk\omega_0 t}$$

- $y(t)$ is also periodic with the same fundamental frequency ω_0.
- If $\{c_k\}$ is the set of FS coefficients for the input $x(t)$, then $\{c_k H(jk\omega_0)\}$ is the set of coefficients for the output $y(t)$.
- The effect of the LTI system is to modify individually each of the FS coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Periodic signals

Periodic signals represented as sums of exponentials:

$$x(t) = \sum_k c_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_k c_k H(jk\omega_0) e^{jk\omega_0 t}$$

- $y(t)$ is also periodic with the same fundamental frequency ω_0.
- If $\{c_k\}$ is the set of FS coefficients for the input $x(t)$, then $\{c_k H(jk\omega_0)\}$ is the set of coefficients for the output $y(t)$.
- The effect of the LTI system is to modify individually each of the FS coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Periodic signals

Periodic signals represented as sums of exponentials:

$$x(t) = \sum_k c_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_k c_k H(jk\omega_0) e^{jk\omega_0 t}$$

- $y(t)$ is also periodic with the same fundamental frequency ω_0.
- If $\{c_k\}$ is the set of FS coefficients for the input $x(t)$, then $\{c_k H(jk\omega_0)\}$ is the set of coefficients for the output $y(t)$.
- The effect of the LTI system is to modify individually each of the FS coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Frequency response

Property

Hermitian symmetry

$\boxed{\textit{If } h(t) \textit{ is real, then } H^*(s) = H(s^*) \textit{ and } H(-j\omega) = H^*(j\omega).}$

Question

Show the above property.

Frequency response

Property

Hermitian symmetry

$\boxed{\textit{If } h(t) \textit{ is real, then } H^*(s) = H(s^*) \textit{ and } H(-j\omega) = H^*(j\omega).}$

Question

Show the above property.

Proof

$$H^*(s) = \left[\int_{-\infty}^{\infty} h(t)e^{-st}\,dt\right]^* = \int_{-\infty}^{\infty} h(t)e^{-s^*t}\,dt = H(s^*).$$

Let $s = j\omega$, then

$$H^*(j\omega) = H((jw)^*) = H(-j\omega)$$

Proof

$$H^*(s) = \left[\int_{-\infty}^{\infty} h(t)e^{-st}\,dt\right]^* = \int_{-\infty}^{\infty} h(t)e^{-s^*t}\,dt = H(s^*).$$

Let $s = j\omega$, then

$$H^*(j\omega) = H((jw)^*) = H(-j\omega)$$

Frequency response (1)

Example

Find response $y(t)$of a (real) LTI system to the sinusoidal signal $x(t) = \cos(\omega t + \phi)$.

Solution

$$x(t) = \cos(\omega t + \phi) = \frac{1}{2}[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}]$$

$$\begin{aligned}
y(t) &= \frac{1}{2}[H(j\omega)e^{j\omega t}e^{j\phi} + H(-j\omega)e^{-j\omega t}e^{-j\phi}] \\
&= \frac{1}{2}[|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}e^{j\phi} + |H(j\omega)|e^{-j\angle H(j\omega)}e^{-j\omega t}e^{-j\phi}] \\
&= |H(j\omega)|\frac{1}{2}[e^{j(\omega t+\phi+\angle H(j\omega))} + e^{-j(\omega t+\phi+\angle H(j\omega))}] \\
&= \boxed{|H(j\omega)|\cos(\omega t+\phi+\angle H(j\omega))}
\end{aligned}$$

Solution

$$x(t) = \cos(\omega t + \phi) = \frac{1}{2}[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}]$$

$$\begin{aligned}
y(t) &= \frac{1}{2}[H(j\omega)e^{j\omega t}e^{j\phi} + H(-j\omega)e^{-j\omega t}e^{-j\phi}] \\
&= \frac{1}{2}[|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}e^{j\phi} + |H(j\omega)|e^{-j\angle H(j\omega)}e^{-j\omega t}e^{-j\phi}] \\
&= |H(j\omega)|\frac{1}{2}[e^{j(\omega t+\phi+\angle H(j\omega))} + e^{-j(\omega t+\phi+\angle H(j\omega))}] \\
&= \boxed{|H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega))}
\end{aligned}$$

Frequency response (2)

Sums of sinusoidal signals:

$$x(t) = \sum_k A_k \cos(\omega_k t + \phi_k) \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t)$$

$$y(t) = \sum_k A_k |H(j\omega_k)| \cos(\omega_k t + \phi_k + \angle H(j\omega_k))$$

$$x(t) = \sum_k A_k \sin(\omega_k t + \phi_k) \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t)$$

$$y(t) = \sum_k A_k |H(j\omega_k)| \sin(\omega_k t + \phi_k + \angle H(j\omega_k))$$

Frequency response (2)

Sums of sinusoidal signals:

$$\begin{aligned} x(t) &= \sum_k A_k \cos(\omega_k t + \phi_k) \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t) \\ y(t) &= \sum_k A_k |H(j\omega_k)| \cos(\omega_k t + \phi_k + \angle H(j\omega_k)) \end{aligned}$$

$$\begin{aligned} x(t) &= \sum_k A_k \sin(\omega_k t + \phi_k) \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t) \\ y(t) &= \sum_k A_k |H(j\omega_k)| \sin(\omega_k t + \phi_k + \angle H(j\omega_k)) \end{aligned}$$

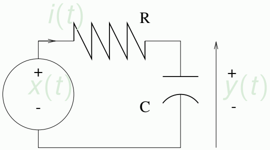
Outline

Filtering

We are now fully equipped to decompose periodic signals into sinusoidal components, so are finally in position to start looking carefully at what happens when such signals pass through LTI systems, aka **filters**.

Example

Return to RC example on p. 32 In the preceding example, we considered an input signal that was a single sinusoid. Now we consider what happens if a more complicated signal, such as a square wave (which is a sum of sinusoids) is applied to the RC circuit.

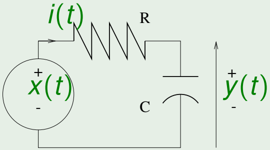


Filtering

We are now fully equipped to decompose periodic signals into sinusoidal components, so are finally in position to start looking carefully at what happens when such signals pass through LTI systems, aka **filters**.

Example

Return to RC example on p. 32 In the preceding example, we considered an input signal that was a single sinusoid. Now we consider what happens if a more complicated signal, such as a square wave (which is a sum of sinusoids) is applied to the RC circuit.



Filtering: example (1)

- Recall RC circuit has transfer function and hence frequency response (p. 32)

$$H(s) = \frac{1}{1 + sRC}, \quad H(j\omega) = \frac{1}{1 + j\omega RC}.$$

- So recalling earlier FS for our 1-0 square wave with $\omega_0 = \pi$ (p. 79):

$$x(t) = \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

Filtering: example (1)

- Recall RC circuit has transfer function and hence frequency response (p. 32)

$$H(s) = \frac{1}{1 + sRC}, \quad H(j\omega) = \frac{1}{1 + j\omega RC}.$$

- So recalling earlier FS for our 1-0 square wave with $\omega_0 = \pi$ (p. 79):

$$x(t) = \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

Filtering: example (2)

$$x(t) = \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) \rightarrow \boxed{\text{RC}} \rightarrow y(t) = ?$$

$$\begin{aligned} y(t) &= \frac{1}{2}H(0) + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} |H(jk\omega_0)| \sin(k\pi t + \angle H(jk\omega_0)) \\ &= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \left| \frac{1}{1 + jk\pi RC} \right| \sin\left(k\pi t + \angle \frac{1}{1 + jk\pi RC}\right) \end{aligned}$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

Filtering: example (2)

$$x(t) = \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) \rightarrow \boxed{\text{RC}} \rightarrow y(t) = ?$$

$$\begin{aligned} y(t) &= \frac{1}{2} H(0) + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} |H(jk\omega_0)| \sin(k\pi t + \angle H(jk\omega_0)) \\ &= \frac{1}{2} + \sum_{k=1,\ \text{odd}}^{\infty} \frac{2}{k\pi} \left| \frac{1}{1 + jk\pi RC} \right| \sin\left(k\pi t + \angle \frac{1}{1 + jk\pi RC} \right) \end{aligned}$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

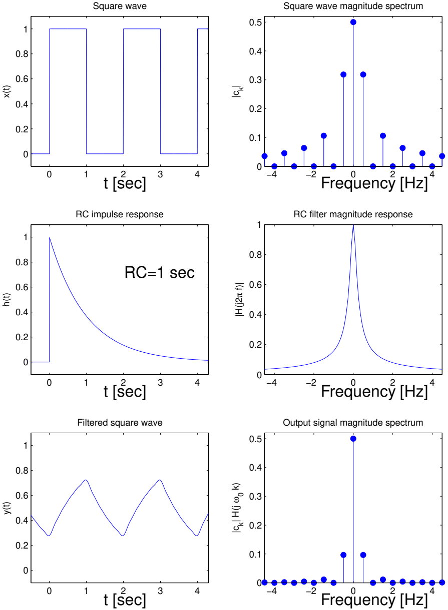
Filtering: example (2)

$$x(t) = \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t) \rightarrow \boxed{\text{RC}} \rightarrow y(t) = ?$$

$$\begin{aligned} y(t) &= \frac{1}{2} H(0) + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} |H(jk\omega_0)| \sin(k\pi t + \angle H(jk\omega_0)) \\ &= \frac{1}{2} + \sum_{k=1,\ \mathrm{odd}}^{\infty} \frac{2}{k\pi} \left| \frac{1}{1 + jk\pi RC} \right| \sin\left(k\pi t + \angle \frac{1}{1 + jk\pi RC}\right) \end{aligned}$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

Filtering: example (3)



Outline

Filters

In many applications, one must remove selected frequency components from signals, while preserving other frequency components.

Example

- AM radio tuning
- anti-aliasing filtering in A/D converters. (explained later in sampling)

Analog filters are often constructed from RLC circuits, and we have seen that the input-output relationship for such circuits is given by a diffeq.

Filters

In many applications, one must remove selected frequency components from signals, while preserving other frequency components.

Example

- AM radio tuning
- anti-aliasing filtering in A/D converters. (explained later in sampling)

Analog filters are often constructed from RLC circuits, and we have seen that the input-output relationship for such circuits is given by a diffeq.

Filters

In many applications, one must remove selected frequency components from signals, while preserving other frequency components.

Example

- AM radio tuning
- anti-aliasing filtering in A/D converters. (explained later in sampling)

Analog filters are often constructed from RLC circuits, and we have seen that the input-output relationship for such circuits is given by a diffeq.

Differential equation systems

When we want to find the frequency response $H(j\omega)$ of such a filter circuit,

1. one approach would be to first determine $h(t)$ by some method (such as time-domain method described previously), and then compute $H(s)$ by the integral transformation.
2. Fortunately, there is an easier way to find $H(s)$ directly for a diffeq system!

Differential equation systems

When we want to find the frequency response $H(j\omega)$ of such a filter circuit,

1. one approach would be to first determine $h(t)$ by some method (such as time-domain method described previously), and then compute $H(s)$ by the integral transformation.
2. Fortunately, there is an easier way to find $H(s)$ directly for a diffeq system!

Filters described by diffeqs (1)

- We have shown above that for any LTI system, the response to an input signal $x(t) = e^{st}$ is $y(t) = H(s)e^{st}$, for some value $H(s)$.

$$x(t) = e^{st} \to \boxed{\text{LTI } h(t)} \to y(t) = H(s)e^{st}$$

- Since diffeq systems (with initial rest) are LTI systems, the solution to the diffeq for such an input signal must also be $y(t) = H(s)e^{st}$, and we just need to find $H(s)$, which is essentially the "undetermined coefficient."

$$x(t) = e^{st} \to \boxed{\text{diffeq LTI } h(t)} \to y(t) = H(s)e^{st}$$

Filters described by diffeqs (1)

- We have shown above that for any LTI system, the response to an input signal $x(t) = e^{st}$ is $y(t) = H(s)e^{st}$, for some value $H(s)$.

$$x(t) = e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = H(s)e^{st}$$

- Since diffeq systems (with initial rest) are LTI systems, the solution to the diffeq for such an input signal must also be $y(t) = H(s)e^{st}$, and we just need to find $H(s)$, which is essentially the "undetermined coefficient."

$$x(t) = e^{st} \rightarrow \boxed{\text{diffeq LTI } h(t)} \rightarrow y(t) = H(s)e^{st}$$

Filters described by diffeqs (2)

Plugging $x(t) = e^{st}$ and $y(t) = H(s)e^{st}$ in to the general form for the diffeq

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$
$$\Longrightarrow \quad \sum_{k=0}^{N} a_k H(s) s^k e^{st} = \sum_{k=0}^{M} b_k s^k e^{st}.$$

So solving for $H(s)$ yields the general form for the transfer function of a diffeq system:

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.}$$

Filters described by diffeqs (2)

Plugging $x(t) = e^{st}$ and $y(t) = H(s)e^{st}$ in to the general form for the diffeq

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\Longrightarrow \quad \sum_{k=0}^{N} a_k H(s) s^k e^{st} = \sum_{k=0}^{M} b_k s^k e^{st}.$$

So solving for $H(s)$ yields the general form for the transfer function of a diffeq system:

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.}$$

Filters described by diffeqs (3)

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.}$$

- Since $H(s)$ is a ratio of two polynomials in s, it is called a **rational system function**.
- All diffeq systems have rational system functions. This is fortunate, since rational system functions are particularly easy to handle.

Filters described by diffeqs (3)

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.}$$

- Since $H(s)$ is a ratio of two polynomials in s, it is called a **rational system function**.
- All diffeq systems have rational system functions. This is fortunate, since rational system functions are particularly easy to handle.

Example

Example

For the previous RC circuit, we had $y(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)$.
Find its transfer function.

Example

Example

For the previous RC circuit, we had $y(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)$. Find its transfer function.

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Thus we can immediately write down that

$$\boxed{H(s) = \frac{1}{1 + RCs}}$$

Same result, less work!

Example: notch filter (1)

Example

A **notch filter** is a band-stop filter with a narrow stopband.
Consider a notch filter for removing 60 Hz noise from AC electrical lines based communication network.
Consider an LTI system described by the following diffeq:

$$(\omega_0^2 + \sigma^2)y(t) - 2\sigma\frac{\mathrm{d}}{\mathrm{d}t}y(t) + \frac{d^2}{dt^2}y(t) = \omega_0^2 x(t) + \frac{d^2}{dt^2}x(t),$$

where $\omega_0 = 2\pi 60$. Plot the magnitude response $|H(j\omega)|$ vs $\omega = 2\pi f$.

Example: notch filter (1)

Example

A **notch filter** is a band-stop filter with a narrow stopband.
Consider a notch filter for removing 60 Hz noise from AC electrical lines based communication network.
Consider an LTI system described by the following diffeq:

$$(\omega_0^2 + \sigma^2)y(t) - 2\sigma\frac{\mathrm{d}}{\mathrm{d}t}y(t) + \frac{d^2}{dt^2}y(t) = \omega_0^2 x(t) + \frac{d^2}{dt^2}x(t),$$

where $\omega_0 = 2\pi 60$. Plot the magnitude response $|H(j\omega)|$ vs $\omega = 2\pi f$.

Finding the impulse response first would be the hard way to approach this problem

Example: notch filter (2)

Since $n = m = 2$ in this problem:

$$\begin{aligned} H(s) &= \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{s^2 + \omega_0^2}{s^2 - 2\sigma s + (\omega_0^2 + \sigma^2)} \\ &= \frac{(s - j\omega_0)(s + j\omega_0)}{[s - (\sigma + j\omega_0)][s - (\sigma - j\omega_0)]} \end{aligned}$$

$$H(j\omega) = H(s)|_{s=j\omega} = \boxed{\frac{(j\omega - j\omega_0)(j\omega + j\omega_0)}{[j\omega - (\sigma + j\omega_0)][j\omega - (\sigma - j\omega_0)]}}.$$

Question

What happens when $\omega = \omega_0$?

Example: notch filter (2)

Since $n = m = 2$ in this problem:

$$\begin{aligned} H(s) &= \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{s^2 + \omega_0^2}{s^2 - 2\sigma s + (\omega_0^2 + \sigma^2)} \\ &= \frac{(s - j\omega_0)(s + j\omega_0)}{[s - (\sigma + j\omega_0)][s - (\sigma - j\omega_0)]} \end{aligned}$$

$$H(j\omega) = H(s)|_{s=j\omega} = \boxed{\frac{(j\omega - j\omega_0)(j\omega + j\omega_0)}{[j\omega - (\sigma + j\omega_0)][j\omega - (\sigma - j\omega_0)]}}.$$

Question

What happens when $\omega = \omega_0$?

Example: notch filter (2)

Since $n = m = 2$ in this problem:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{s^2 + \omega_0^2}{s^2 - 2\sigma s + (\omega_0^2 + \sigma^2)}$$

$$= \frac{(s - j\omega_0)(s + j\omega_0)}{[s - (\sigma + j\omega_0)][s - (\sigma - j\omega_0)]}$$

$$H(j\omega) = H(s)|_{s=j\omega} = \boxed{\frac{(j\omega - j\omega_0)(j\omega + j\omega_0)}{[j\omega - (\sigma + j\omega_0)][j\omega - (\sigma - j\omega_0)]}}.$$

Question

What happens when $\omega = \omega_0$?

Example: notch filter (2)

Since $n = m = 2$ in this problem:

$$\begin{aligned} H(s) &= \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{s^2 + \omega_0^2}{s^2 - 2\sigma s + (\omega_0^2 + \sigma^2)} \\ &= \frac{(s - j\omega_0)(s + j\omega_0)}{[s - (\sigma + j\omega_0)][s - (\sigma - j\omega_0)]} \end{aligned}$$

$$H(j\omega) = H(s)|_{s=j\omega} = \boxed{\frac{(j\omega - j\omega_0)(j\omega + j\omega_0)}{[j\omega - (\sigma + j\omega_0)][j\omega - (\sigma - j\omega_0)]}}.$$

Question

What happens when $\omega = \omega_0$?

Note that $H(j\omega) = 0$ when $\omega = \omega_0$! This is by design.

Example: notch filter (3)

Here is how to do it in MATLAB (Assume $\sigma = -1$)

```
f = linspace(0, 200, 201);
oo = 2*pi*60;
b = [1 0 oo^2];
a = [1 2 oo^2+1^2];
H = freqs(b, a, 2*pi*f);
subplot(211), plot(f, abs(H)), axis([0 200 0 1.1])
xlabel('frequency f [Hz]')
ylabel('Magnitude response |H(j 2\pi f)|')
title('Magnitude response of a notch filter')
```

H = freqs(b,a,w) evaluates the complex frequency response of the analog filter specified by coefficient vectors b and a at angular frequencies in rad/s specified in real vector w.

Example: notch filter (3)

Here is how to do it in MATLAB (Assume $\sigma = -1$)

```
f = linspace(0, 200, 201);
oo = 2*pi*60;
b = [1 0 oo^2];
a = [1 2 oo^2+1^2];
H = freqs(b, a, 2*pi*f);
subplot(211), plot(f, abs(H)), axis([0 200 0 1.1])
xlabel('frequency f [Hz]')
ylabel('Magnitude response |H(j 2\pi f)|')
title('Magnitude response of a notch filter')
```

H = freqs(b,a,w) evaluates the complex frequency response of the analog filter specified by coefficient vectors b and a at angular frequencies in rad/s specified in real vector w.

Example: notch filter (4)

And here is how to plot the impulse response:

```
sys = tf(b,a);
subplot(212), impulse(sys, 3)
print('fig,notch', '-deps')
```

- sys = tf(num,den) creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den.

$$sys = \frac{s^2 + 1.421e05}{s^2 + 2s + 1.421e05}$$

- impulse(sys,Tfinal) simulates the impulse response from t = 0 to the final time t = Tfinal.

Example: notch filter (4)

And here is how to plot the impulse response:

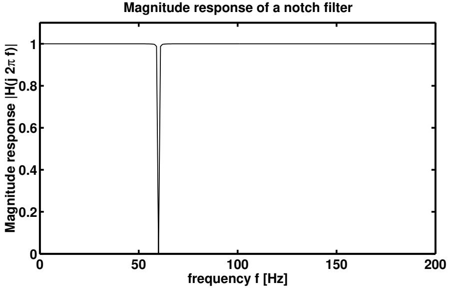
```
sys = tf(b,a);
subplot(212), impulse(sys, 3)
print('fig,notch', '-deps')
```

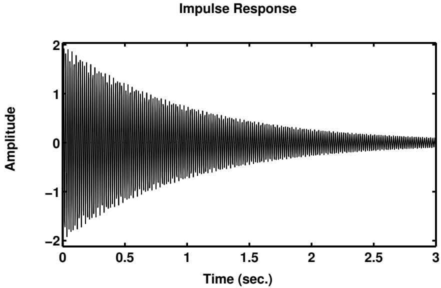
- sys = tf(num,den) creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den.

$$sys = \frac{s^2 + 1.421e05}{s^2 + 2s + 1.421e05}$$

- impulse(sys,Tfinal) simulates the impulse response from t = 0 to the final time t = Tfinal.

Example: notch filter (5)





Ideal amplifier

- An ideal amplifier with a gain of 5 would be described by the input-output relationship

$$x(t) \to \boxed{\text{ideal amplifier with a gian of 5}} \to y(t) = 5x(t)$$

- If the input to such an amplifier is $x(t) = \cos \omega t$, then

$$\cos \omega t \to \boxed{\text{ideal amplifier with a gian of 5}} \to y(t) = 5 \cos \omega t.$$

  All of the power of the output signal is in the fundamental frequency component, where it belongs.

Ideal amplifier

- An ideal amplifier with a gain of 5 would be described by the input-output relationship

$$x(t) \to \boxed{\text{ideal amplifier with a gian of 5}} \to y(t) = 5x(t)$$

- If the input to such an amplifier is $x(t) = \cos \omega t$, then

$$\cos \omega t \to \boxed{\text{ideal amplifier with a gian of 5}} \to y(t) = 5 \cos \omega t.$$

All of the power of the output signal is in the fundamental frequency component, where it belongs.

Harmonic distortion

Definition

A distortion present in all amplifiers is some form of nonlinearity, which will introduce additional frequency components, transferring some of the signal power from the fundamental frequency component to higher harmonics. This is called **harmonic distortion**.

Definition

The quality of an amplifier is (in part) judged by how small its **total harmonic distortion (THD)** is, defined by:

$$\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%$$

Harmonic distortion

Definition

A distortion present in all amplifiers is some form of nonlinearity, which will introduce additional frequency components, transferring some of the signal power from the fundamental frequency component to higher harmonics. This is called **harmonic distortion**.

Definition

The quality of an amplifier is (in part) judged by how small its **total harmonic distortion (THD)** is, defined by:

$$\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%$$

Total harmonic distortion of amplifiers

$$\boxed{\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%}$$

$$= \frac{c_0^2 + \sum_{k=2}^{\infty} 2|c_k|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\% \quad (x(t) \text{ real, } c_{-k} = c_k^* \text{ Hermitian symmetry})$$

$$= \frac{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2 - 2|c_1|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\%$$

$$= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% \quad (\text{Parseval's theorem})$$

$$\boxed{= \left[1 - \frac{\text{avg. power in fundamental}}{\text{avg. signal power}}\right] \cdot 100\%}$$

Total harmonic distortion of amplifiers

$$\boxed{\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%}$$

$$= \frac{c_0^2 + \sum_{k=2}^{\infty} 2|c_k|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\% \quad (x(t) \text{ real, } c_{-k} = c_k^* \text{ Hermitian symmetry})$$

$$= \frac{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2 - 2|c_1|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\%$$

$$= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% \quad (\text{Parseval's theorem})$$

$$\boxed{= \left[1 - \frac{\text{avg. power in fundamental}}{\text{avg. signal power}}\right] \cdot 100\%}$$

Total harmonic distortion of amplifiers

$$\boxed{\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%}$$

$$= \frac{c_0^2 + \sum_{k=2}^{\infty} 2|c_k|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\% \quad (x(t) \text{ real, } c_{-k} = c_k^* \text{ Hermitian symmetry})$$

$$= \frac{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2 - 2|c_1|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\%$$

$$= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% \quad (\text{Parseval's theorem})$$

$$\boxed{= \left[1 - \frac{\text{avg. power in fundamental}}{\text{avg. signal power}}\right] \cdot 100\%}$$

Total harmonic distortion of amplifiers

$$\boxed{\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%}$$

$$= \frac{c_0^2 + \sum_{k=2}^{\infty} 2|c_k|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\% \quad (x(t) \text{ real, } c_{-k} = c_k^* \text{ Hermitian symmetry})$$

$$= \frac{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2 - 2|c_1|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\%$$

$$= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% \quad (\text{Parseval's theorem})$$

$$\boxed{= \left[1 - \frac{\text{avg. power in fundamental}}{\text{avg. signal power}}\right] \cdot 100\%}$$

Total harmonic distortion of amplifiers

$$\boxed{\mathrm{THD} = \frac{\text{avg. power in DC \& harmonics}}{\text{avg. signal power}} \cdot 100\%}$$

$$= \frac{c_0^2 + \sum_{k=2}^{\infty} 2|c_k|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\% \quad (x(t) \text{ real, } c_{-k} = c_k^* \text{ Hermitian symmetry})$$

$$= \frac{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2 - 2|c_1|^2}{c_0^2 + \sum_{k=1}^{\infty} 2|c_k|^2} \cdot 100\%$$

$$= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% \quad (\text{Parseval's theorem})$$

$$\boxed{= \left[1 - \frac{\text{avg. power in fundamental}}{\text{avg. signal power}}\right] \cdot 100\%}$$

THD: example

Example

Consider the following model for an amplifier with a small 3rd-order nonlinearity:

$$y(t) = 5[x(t) + bx^3(t)].$$

Find the THD for this amplifier when the input signal is $x(t) = \cos \omega_0 t$.

$$y(t) = 5[\cos \omega_0 t + b \cos^3 \omega_0 t].$$

THD: example

Example

Consider the following model for an amplifier with a small 3rd-order nonlinearity:

$$y(t) = 5[x(t) + bx^3(t)].$$

Find the THD for this amplifier when the input signal is $x(t) = \cos \omega_0 t$.

$$y(t) = 5[\cos \omega_0 t + b \cos^3 \omega_0 t].$$

To compute THD, we must find the signal's average power P, and the average power in the fundamental $2|c_1|^2$.

Solution (1)

The binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k}$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

We have

$$\begin{aligned} \cos^3 x &= \left(\frac{e^{jx}+e^{-jx}}{2}\right)^3 = \frac{1}{8}\left[e^{j3x}+3e^{jx}+3e^{-jx}+e^{-j3x}\right] \\ &= \frac{1}{4}\left[3\cos x + \cos 3x\right] \end{aligned}$$

Solution (1)

The binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We have

$$\begin{aligned}\cos^3 x &= \left(\frac{e^{jx}+e^{-jx}}{2}\right)^3 = \frac{1}{8}\left[e^{j3x}+3e^{jx}+3e^{-jx}+e^{-j3x}\right] \\ &= \frac{1}{4}\left[3\cos x + \cos 3x\right]\end{aligned}$$

Solution (2)

$$
\begin{aligned}
y(t) &= 5[\cos\omega_0 t + b\cos^3\omega_0 t] \\
&= 5\left[\cos\omega_0 t + b\left(\frac{3}{4}\cos\omega_0 t + \frac{1}{4}\cos 3\omega_0 t\right)\right] \\
&= 5\left[1 + \frac{3b}{4}\right]\cos\omega_0 t + 5\frac{b}{4}\cos 3\omega_0 t.
\end{aligned}
$$

Question

Determine c_k and P from the above representation.

Solution (2)

$$
\begin{aligned}
y(t) &= 5[\cos\omega_0 t + b\cos^3\omega_0 t] \\
&= 5\left[\cos\omega_0 t + b\left(\frac{3}{4}\cos\omega_0 t + \frac{1}{4}\cos 3\omega_0 t\right)\right] \\
&= 5\left[1 + \frac{3b}{4}\right]\cos\omega_0 t + 5\frac{b}{4}\cos 3\omega_0 t.
\end{aligned}
$$

Question

Determine c_k and P from the above representation.

Solution (3)

$$y(t) = 5\left[1 + \frac{3b}{4}\right] \cos\omega_0 t + 5\frac{b}{4}\cos 3\omega_0 t$$

Thus

$$c_{\pm 1} = \frac{1}{2}\cdot 5\left[1+\frac{3b}{4}\right], \quad c_{\pm 3} = \frac{1}{2}\cdot 5\frac{b}{4}, \quad P = 2c_1^2 + 2c_3^2$$

Question

Why divide by 2?

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k)$$

where $\theta_k = \angle c_k$.

Solution (3)

$$y(t) = 5\left[1 + \frac{3b}{4}\right]\cos\omega_0 t + 5\frac{b}{4}\cos 3\omega_0 t$$

Thus

$$c_{\pm 1} = \frac{1}{2}\cdot 5\left[1 + \frac{3b}{4}\right], \quad c_{\pm 3} = \frac{1}{2}\cdot 5\frac{b}{4}, \quad P = 2c_1^2 + 2c_3^2$$

Question

Why divide by 2?

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k|\cos(k\omega_0 t + \theta_k)$$

where $\theta_k = \angle c_k$.

Solution (3)

$$y(t) = 5\left[1 + \frac{3b}{4}\right] \cos \omega_0 t + 5\frac{b}{4} \cos 3\omega_0 t$$

Thus

$$c_{\pm 1} = \frac{1}{2} \cdot 5\left[1 + \frac{3b}{4}\right], \quad c_{\pm 3} = \frac{1}{2} \cdot 5\frac{b}{4}, \quad P = 2c_1^2 + 2c_3^2$$

Question

Why divide by 2?

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \theta_k)$$

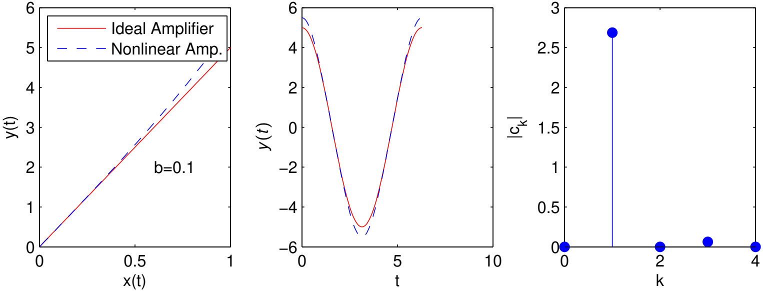
where $\theta_k = \angle c_k$.

Solution (4)

$$
\begin{aligned}
\text{THD} &= \left[1 - \frac{2|c_1|^2}{P}\right] \cdot 100\% = \left[1 - \frac{2c_1^2}{2c_1^2 + 2c_3^2}\right] \cdot 100\% \\
&= \frac{c_3^2}{c_1^2 + c_3^2} \cdot 100\% = \left[1 - \frac{(1 + 3b/4)^2}{(1 + 3b/4)^2 + (b/4)^2}\right] \cdot 100\% \\
&= \boxed{\frac{1}{(4/b + 3)^2 + 1} \cdot 100\%}.
\end{aligned}
$$

Total harmonic distortion of amplifiers (6)

If $b = 0.1$, then THD=0.05%.



Outline

Summary (1)

- 3.2 exponential signals through LTI systems
- 3.3 Fourier series
- Hermitian symmetry of Fourier coefficients
- trigonometric forms of FS
- 3.4 convergence of FS
- Gibbs phenomenon
- 3.5 properties of FS
  - time/amplitude transformation
  - differentiation/modulation properties

Summary (2)

- 3.5.7 Parseval's theorem
- power density spectrum
- magnitude/phase spectrum
- system transfer function (Laplace)
- frequency response (Fourier)
- Hermitian symmetry of frequency response
- sums of cosines through LTI
- 3.8 LTI system analysis
- 3.10 filters described by diffeq systems
- rational transfer functions for diffeq systems

Summary (3)

- With the tools developed in this chapter, we can finally do some interesting applications, such as the 60Hz notch filter described above.
- Specifically, cumbersome convolution in the time domain becomes replaced by simple multiplication in the frequency domain.
- Multiply each frequency component of the signal by the frequency response of the system at that frequency.

Summary (3)

- With the tools developed in this chapter, we can finally do some interesting applications, such as the 60Hz notch filter described above.
- Specifically, cumbersome convolution in the time domain becomes replaced by simple multiplication in the frequency domain.
- Multiply each frequency component of the signal by the frequency response of the system at that frequency.

Summary (3)

- With the tools developed in this chapter, we can finally do some interesting applications, such as the 60Hz notch filter described above.
- Specifically, cumbersome convolution in the time domain becomes replaced by simple multiplication in the frequency domain.
- Multiply each frequency component of the signal by the frequency response of the system at that frequency.

Summary (4)

- But our analysis is incomplete since essentially we have considered only periodic signals.
- The next chapter introduces the more general Fourier transform which will allow the same type of simplification even for aperiodic signals.

Summary (4)

- But our analysis is incomplete since essentially we have considered only periodic signals.
- The next chapter introduces the more general Fourier transform which will allow the same type of simplification even for aperiodic signals.