Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

Outline

Fourier series

The Fourier series analysis described previously provides several useful tools.

1. It allows us to analyze the frequency content of periodic signals by decomposing them into a linear combination of complex exponential signals (or sinusoids).
2. It also helps us understand conceptually what happens to periodic signals when passed through LTI systems (each frequency component gets a new amplitude and phase depending on frequency response of the system).
3. It gives us a simple mathematical expression for the response of an LTI system to a periodic input signal without performing convolution.

We would like to have similar tools for aperiodic signals as well. (Like speech or music.)

Roadmap

	Signal	
Transform	Continuous Time	Discrete Time
Continuous Frequency	Fourier Transform	DTFT (periodic in frequency)
Discrete Frequency	Fourier Series (periodic in time)	DTFS or DFT (periodic in time and frequency) FFT

Fourier transform

Fourier himself recognized the utility of representing aperiodic signals in the frequency domain, and to a large extent our development follows his original approach of treating an aperiodic signal as the limiting case of a set of periodic signals whose periods increase to infinity.

The primary focus of this chapter is on the "signals" part (frequency content of signals). The "systems" part will be emphasized further in the next chapter in the context of filtering.

Overview

- Definition
- Existence
- Examples
- Properties
- Convolution / filtering
- Multiplication / modulation (app: all electronic communication systems)
- Application to diffeq systems (app: RLC circuits)
- Partial fraction expansion (PFE)
- Finally: easy answer to $\cos(\omega t)\, u(t) \xrightarrow{\text{LTI}} y(t) = ?$ and related problems

Outline

Aperiodic signal

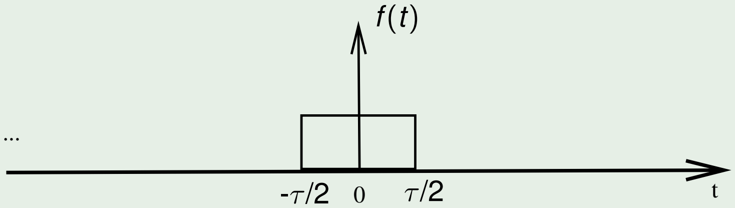
- Suppose we have an aperiodic, time-limited signal $f(t)$, and we would like to analyze its frequency content, either to better understand the signal itself, or to analyze what will happen to the signal when it passes through some type of filter, or both.
- As in most math and engineering fields, we develop such an analysis by building on what we already know.
- We know how to analyze the frequency content of periodic signals, so let us construct a periodic signal from $f(t)$, and then examine what happens to the frequency content of the periodic signal as the period increases.

Example: rectangular function

Example

consider the rectangular signal

$$f(t) = \mathrm{rect}\left(\frac{t}{\tau}\right).$$



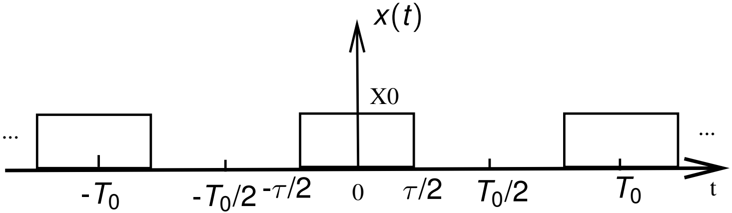
Question

Is this an energy or power signal?

Constructed periodic signal

Define a periodic signal

$$x_{T_0}(t) \triangleq \sum_{n=-\infty}^{\infty} f(t - nT_0) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_0}{\tau}\right)$$



Question

1. *What is this signal called?*
2. *Is it an energy or power signal?*
3. *What is the name of the special function that we defined to describe the c_k's of $x_{T_0}(t)$?*

Increasing the period

We have previously shown that this signal has a Fourier series representation with coefficients

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\tau\operatorname{sinc}\left(\tau\frac{k\omega_0}{2\pi}\right).$$

(Chap. 3, p.215)

- In the time domain, as T_0 increases, $x_{T_0}(t)$ approaches $f(t)$ for any given finite t.
- let us examine what happens in the frequency domain as T_0 increases.

Definition of FT (4)

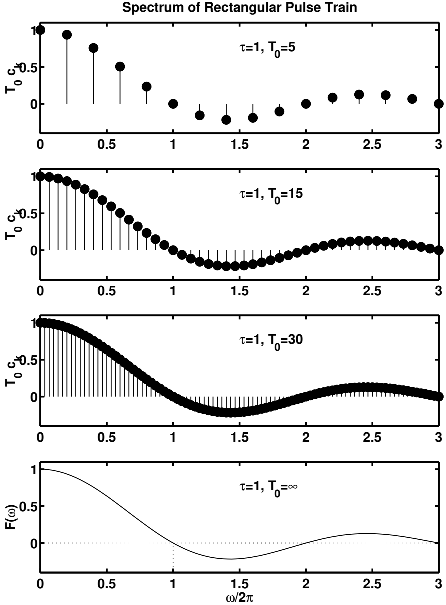
$$\boxed{c_k = \frac{1}{T_0}\tau \operatorname{sinc}\left(\tau \frac{k\omega_0}{2\pi}\right)}$$

When increasing T_0,

- The first thing we see is that $c_k \to 0$.
- This is due to the $1/T_0$ term, and reflects the fact that $x_{T_0}(t)$ is a power signal, whereas $f(t)$ is an energy signal (and hence has 0 power).
- So we normalize out the $1/T_0$ and instead look at what happens $T_0 c_k$ as the period T_0 increases.

$$\boxed{T_0 c_k = \tau \operatorname{sinc}\left(\tau \frac{k\omega_0}{2\pi}\right)}$$

Spectrum


Spectrum of Rectangular Pulse Train
τ=1, T0=5
τ=1, T0=15
τ=1, T0=30
τ=1, T0=∞
T0 ck
F(ω)
ω/2π
0
0.5
1
1.5
2
2.5
3

The envelop

The spectral lines of $x(t)$ become closer and closer, and in the limit as $T_0 \to \infty$, become a continuum described by the envelope.

Question

What is the envelope?

Definition of FT (1)

Question

Where did this $\text{sinc}(\cdot)$ *formula originate?*

Returning to the c_k formula:

$$
\begin{aligned}
T_0 c_k &= \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t}\, dt \\
&= \int_{-T_0/2}^{T_0/2} f(t) e^{-jk\omega_0 t}\, dt \\
&= \int_{-\infty}^{\infty} f(t) e^{-jk\omega_0 t}\, dt \\
&= \left. \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right|_{\omega = k\omega_0}
\end{aligned}
$$

Definition of FT (2)

So if we define

$$F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

then

$$T_0 c_k = F(\omega)|_{\omega=k\omega_0} \Longrightarrow c_k = \frac{1}{T_0} F(\omega)|_{\omega=k\omega_0}$$

where the c_k's are the FS coefficients of the periodic signal $x_{T_0}(t)$, but the $F(\omega)$ is solely related to the aperiodic signal $f(t)$.

General treatment (1)

- We have seen that we can represent periodic function $x(t)$ with period T_0 by the complex Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{where} \quad c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t}\, dt,$$

 where $\omega_0 = 2\pi / T_0$.
- The coefficients c_k define the spectrum of $x(t)$, and since the only frequency components present are at the harmonics $k\omega_0$, the spectrum is a discrete or line spectrum consisting of lines of height $|c_k|$ (with corresponding phase $\angle c_k$) at the frequencies $k\omega_0 = k\frac{2\pi}{T_0}$.

General treatment (2)

What happens as the period T_0 increases? The spacing of the lines decreases, and in the limit as $T_0 \to \infty$ we can think of the spectra as continuous curves (one for magnitude, one for phase), rather than discrete lines.

Now we formalize this idea mathematically to derive the Fourier transform of an aperiodic signal.

General treatment (3)

- Consider a "pulse" train

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} f(t - nT_0),$$

for some "pulse like" (energy) signal $f(t)$.

- As T_0 increases, the gap between the center pulse and the next pulse widens, and in the limit as $T_0 \to \infty$, eventually all that is left is central pulse. Formally:

$$\lim_{T_0 \to \infty} x_{T_0}(t) = f(t).$$

Since $f(t)$ is the limit of the $x_{T_0}(t)$ signals, it is natural to think that we should be able to define some type of spectrum for $f(t)$ by taking some type of limit of the FS expressions above.

General treatment (4)

- Since $x_{T_0}(t)$ is periodic, it is a power signal, whereas $f(t)$ is aperiodic and (at least in this typical example) is an energy signal.
- We need to scale the FS coefficients by a factor of T_0, since there is such a difference in the definitions of energy and power.

Energy and power

Recall

- The **energy** of a signal $x(t)$ is defined as

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

  If E is finite ($E < \infty$) then $x(t)$ is called an **energy signal** and $P = 0$.

- The **average power** of a signal is defined as

$$P \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

  If P is finite and nonzero, then $x(t)$ is called a **power signal**.

General treatment (5)

Define:

$$F_{T_0}(k\omega_0) \triangleq T_0 c_k = \int_{-T_0/2}^{T_0/2} f(t) e^{-jk\omega_0 t}\, dt,$$

then

$$F_{T_0}(\omega) = \int_{-T_0/2}^{T_0/2} f(t) e^{-j\omega t}\, dt.$$

Although $F_{T_0}(\cdot)$ is only valid for the values $\omega = k\omega_0$, as T_0 increases these values become ever closer together, so there are “more and more” valid values. In the limit we have the following expression, valid for all ω:

$$\boxed{\lim_{T_0 \to \infty} F_{T_0}(\omega) \triangleq F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt.}$$

General treatment (6)

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt}$$

This integral relationship, which defines a function $F(\omega)$ given a signal $f(t)$, is called the **Fourier transform** of $f(t)$.

In EE the convention is to use capital letters to denote the Fourier transform of a signal denoted with lower case letters, *e.g.* $Y(\omega)$ would be the FT of $y(t)$, defined of course by

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\, dt$$

General treatment (7)

So we see how to compute a FT $F(\omega)$ from an aperiodic signal $f(t)$. But this would be of limited utility if we could not also recover $f(t)$ from $F(\omega)$. Fortunately, we can!

For a periodic signal, such as our $x_{T_0}(t)$, we can recover it from its coefficients by summing:

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{F_{T_0}(k\omega_0)}{T_0} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F_{T_0}(k\omega_0) e^{jk\omega_0 t} \omega_0$$

General treatment (7)

In the limit as $T_0 = 2\pi/\omega_0 \to \infty$, this approaches the following integral:

$$f(t) = \lim_{T_0\to\infty} x_{T_0}(t)$$

$$= \lim_{T_0\to\infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F_{T_0}(k\omega_0) e^{jk\omega_0 t} \omega_0$$

$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega}$$

This integral is called the **inverse Fourier transform** of $F(\omega)$.

General treatment (8)

To summarize, for an aperiodic signal $f(t)$, we have derived the following relationships:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega, \quad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt}$$

The functions $f(t)$ and $F(\omega)$ are called **Fourier transform pairs** and we write

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega).$$

(we now switch from $x(t)$ to $f(t)$ to represent a generic signal.)

General treatment (9)

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega, \quad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt}$$

The annoying asymmetry (extra 2π) is due to our choice to use ω in radians/unit time as the frequency variable. If instead we had used cycles/unit time (*e.g.* Hz), then the 2π out front disappears.

Systems perspective for FT formula

$$x(t) = e^{j\omega t} \rightarrow \boxed{\text{LTI}} \rightarrow H(\omega)e^{j\omega t}$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt.$$

So

$$h(t) \xleftrightarrow{\mathcal{F}} H(\omega)$$

So the same formula is central to both signals and systems perspectives.

FT: Example (1)

Example

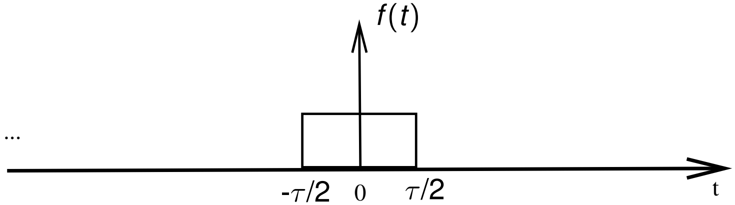
Find the FT of a rectangular signal $f(t) = \text{rect}(t/\tau)$.

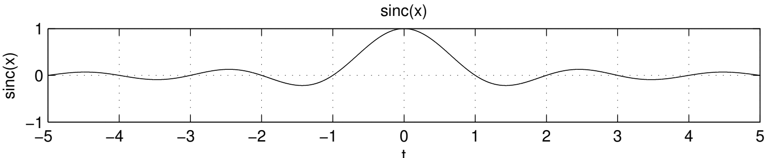
FT: Example (2)

$$\mathrm{sinc}(x) \triangleq \begin{cases} 1, & x = 0 \\ \frac{\sin \pi x}{\pi x}, & x \neq 0. \end{cases}$$

Thus we have derived our first FT pair

$$\boxed{\mathrm{rect}(t/\tau) \xleftrightarrow{\mathcal{F}} \tau\, \mathrm{sinc}\left(\tau \frac{\omega}{2\pi}\right).}$$





FT: Example (3)

Question

Where do $F(\omega) = \tau\,\mathrm{sinc}\left(\tau\frac{\omega}{2\pi}\right)$ *have its peak of* τ *and zeros?*

Outline

Conditions for existence of the CT FT

In the rect signal example above, we could easily perform the integral. But any time we see infinite sums or integrals, we must consider existence of the sum or integral.

Example

$\sum_{k=0}^{n}(-1)^k$ is well defined for any finite integer n. But $\sum_{k=0}^{\infty}(-1)^k$ is undefined!

Question

When in general will the FT exist?

Square integrable signals

If $f(t)$ is an energy signal, also known as **square integrable**, *i.e.* if $\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$, then

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$ exists and is finite, since by the triangle inequality: $|F(\omega)|^2 \leq \int_{-\infty}^{\infty} |f(t) e^{-j\omega t}|^2 \, dt < \infty$.
- If we "reconstruct" $\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$, then the error signal will have zero energy, *i.e.* $\int_{-\infty}^{\infty} |\tilde{f}(t) - f(t)|^2 \, dt = 0$.

This is completely adequate for engineering purposes, so we write $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$ throughout the rest of the course, even though strictly speaking the "equality" in that expression only holds in an L_2 sense rather than in the strict mathematical sense.

Dirichlet conditions (1)

skip

- Unfortunately, square integrable is a little bit too restrictive of a condition for many engineering problems.
- The **Dirichlet conditions** are a set of sufficient conditions on $f(t)$ that have been shown to ensure that the FT exists.
- There are various versions of these conditions that appear in different books. Here is one set of sufficient conditions.
  - $f(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$
  - $f(t)$ has a finite number of maxima and minima on any finite interval.
  - $f(t)$ has a finite number of finite discontinuities on any finite interval.

Dirichlet conditions (2)

Rule of thumb:
if you can draw a complete picture of $f(t)$, then its FT exists.

But there are signals for which we cannot draw exact pictures (such as $\delta(t)$), but for which the FT nevertheless is "defined" in a practical engineering sense.

Outline

Unit impulse (in time)(1)

Example

Find FT of unit impulse function. (not square integrable, not bounded, but FT still can be defined.)

Decaying exponential function (1)

Example

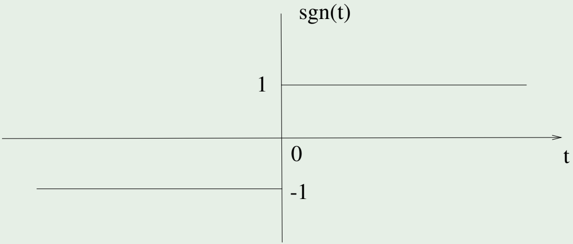
Find FT of decaying exponential function

$$f(t) = e^{-at}u(t), \quad \text{real}\{a\} > 0$$

Sign function (1)

Example

$$f(t) = \mathrm{sgn}(t) \triangleq \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$
$$= u(t) - u(-t) = 2u(t) - 1.$$



Is it absolutely integrable? Is it square integrable?

Unit step function

Example

Find the FT of $u(t)$.

Unit step function

Example

Find the FT of $u(t)$.

Hint: using $\operatorname{sgn}(t)$ and its FT.

Outline

FT of periodic signals (1)

- We already have a perfectly-sized tool for analyzing periodic signals: the Fourier series.
- So strictly speaking, analysis of periodic signals by FT methods is redundant.
- However, when considering signals that have mixed periodic and aperiodic components, such as AM (amplitude modulation) signals "with carrier" (Chap. 1, p.128), it is convenient to be able to use one tool to handle both the periodic and aperiodic component.
- Fortunately, the FT is sufficiently general to treat both periodic and aperiodic signals, provided we allow impulse functions in the spectrum.

FT of periodic signals (2)

- One cannot directly calculate the FT of a periodic signal because a periodic signal (which has infinite energy) is neither square integrable nor absolutely integrable.
- We use an alternate representation of the periodic signal as a Fourier series and then employ known properties of the Dirac delta function and Fourier transform to obtain the FT of periodic signals.
- Unlike Fourier transforms of finite-energy functions, the Fourier transforms of periodic functions are not ordinary functions but rather distributions which have a literature of their own.

FT from FS (1)

Suppose $x(t)$ is periodic with period T_0 and fundamental frequency $\omega_0 = 2\pi/T_0$. We saw earlier that we can represent $x(t)$ by its Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Question

What is the FT of $x(t)$?

FT from FS (2)

Solution

1. *FT of a complex exponential signal*
2. *Superposition property*

Unit impulse in spectrum

Question

What signal corresponds to a spectrum consisting of a single impulse?

Linearity of FT

Linearity of FT:

$$f(t) = a_1 f_1(t) + a_2 f_2(t) \xleftrightarrow{\mathcal{F}} F(\omega) = a_1 F_1(\omega) + a_2 F_2(\omega)$$

Question

Show the linearity property of FT.

Superposition of FT

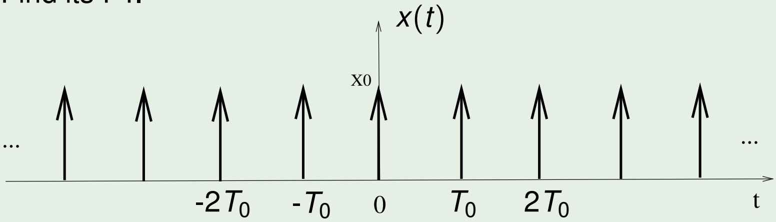
The linearity property is easily extended to the superposition property

$$\boxed{\sum_n x_n(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_n X_n(\omega)}$$

FT from FS: Example (1)

Example

What is the periodic signal $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ called? Find its FT.



FT from FS: Example (2)

Example

For the 0.5Hz pervious square wave

$$x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 1/2 - 2n) \text{ with } c_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{jk\pi}, & k \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

and $\omega_0 = \pi$, find its FT.

Outline

Motivation

Fourier transform pairs:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt, \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega}$$

We would like to be able to find $F(\omega)$ and $f(t)$ without recomputing everything. Another motivation is to avoid inverse FT via integration. Thus we study properties of the FT.

Linearity (1)

Property

Linearity property:

$$f(t) = a_1 f_1(t) + a_2 f_2(t) \xleftrightarrow{\mathcal{F}} F(\omega) = a_1 F_1(\omega) + a_2 F_2(\omega)$$

Example

Find FT of $f(t) = \cos \omega_0 t$.

Linearity (2)

Example

Find FT of $\cos(\omega_0 t + \phi)$.

Time-transformations

Property

Time transforms:

$$f(at+b) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} e^{j\omega b/a} F(\omega/a), \quad \textit{for} \;\; a \neq 0$$

Time-shift

Property

Time transforms:

$$f(at+b) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} e^{j\omega b/a} F(\omega/a), \quad \textit{for} \;\; a \neq 0$$

Property

Time-shift (use $a = 1$ and $b = -t_0$)

$$\boxed{f(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} F(\omega)} \textit{(phase shift)}$$

Time-scale

Property

Time transforms:

$$f(at+b) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} e^{j\omega b/a} F(\omega/a), \quad \textit{for} \;\; a \neq 0$$

Property

Time-scale (use $b = 0$)

$$\boxed{f(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \; a \neq 0}$$

Time-reversal

Property

Time transforms:

$$f(at+b) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} e^{j\omega b/a} F(\omega/a), \quad \textit{for} \;\; a \neq 0$$

Property

Time-reversal (use $a=-1$ and $b=0$)

$$f(-t) \overset{\mathcal{F}}{\longleftrightarrow} F(-\omega)$$

Even signals

Property

If $x(t)$ is an even signal, i.e.,

$$f(t) = f(-t)$$

then its FT is also even, i.e.,

$$F(\omega) = F(-\omega)$$

Conjugation

Property

Conjugation:

$$f^*(t) \overset{\mathcal{F}}{\longleftrightarrow} F^*(-\omega)$$

Hermitian symmetric

Property

If $f(t)$ is real, i.e.,

$$f(t) = f^*(t)$$

then

$$F(\omega) = F^*(-\omega)$$

so the spectrum of a real signal is ***Hermitian symmetric****.*
Furthermore

- $\angle F(\omega) = \angle F^*(-\omega) = -\angle F(-\omega)$
- $|F(\omega)| = |F^*(-\omega)| = |F(-\omega)|$

It can be easily proved using the conjugation property.

Hermitian symmetric: example

Example

Show the Hermitian symmetry property for $f(t) = e^{-t}u(t)$.

Real and even signals

Property

If $f(t)$ is real and even $F(\omega)$ is also real and even.

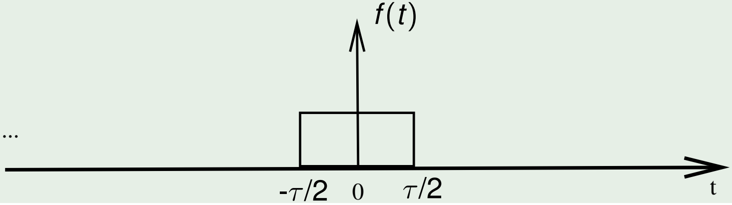
Property

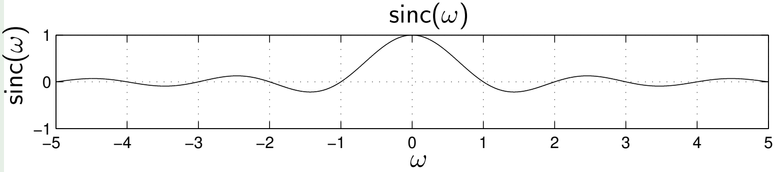
If $f(t)$ is real and odd, then $F(\omega)$ is purely imaginary and odd.

Real and even signals: example

Example

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$





Duality

Question

We have shown that

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right).$$

If we want to find the FT of $\mathrm{sinc}(t)$*, do we have to start from scratch?* *No!*

Property

The principle of duality says that FT pairs have the following dual relationship. If

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)$$

then

$$x(t) = F(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = 2\pi f(-\omega)$$

Example (1)

Example

Find the FT of $x(t) = \text{sinc}(t)$.

Example (1)

Example

Find the FT of $x(t) = \mathrm{sinc}(t)$.

Integrating sinc to compute the FT would be painful.

Example (1)

Example

Find the FT of $x(t) = \text{sinc}(t)$.

We learned the following:

- FT pair

$$\text{rect}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

- Duality property

$$f(t) \stackrel{\mathcal{F}}{\longleftrightarrow} F(\omega) \Longrightarrow x(t) = F(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) = 2\pi f(-\omega)$$

- Time-scale property

$$f(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \; a \neq 0$$

Solution (1)

Method 1:

Time scale:

$$\mathrm{rect}(t/\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} \tau\,\mathrm{sinc}\left(\tau\frac{\omega}{2\pi}\right)$$

so for $\tau = 2\pi$:

$$f(t) = \mathrm{rect}\left(\frac{t}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} F(\omega) = 2\pi\mathrm{sinc}(\omega).$$

Thus, by duality

$$2\pi\mathrm{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\mathrm{rect}\left(\frac{-\omega}{2\pi}\right) = 2\pi\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

or equivalently $\boxed{\mathrm{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

Solution (1)

Method 1:
Time scale:

$$\mathrm{rect}(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} \tau \,\mathrm{sinc}\left(\tau \frac{\omega}{2\pi}\right)$$

so for $\tau = 2\pi$:

$$f(t) = \mathrm{rect}\left(\frac{t}{2\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega) = 2\pi \mathrm{sinc}(\omega).$$

Thus, by duality

$$2\pi \mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \mathrm{rect}\left(\frac{-\omega}{2\pi}\right) = 2\pi \,\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

or equivalently $\boxed{\mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

Solution (1)

Method 1:

Time scale:

$$\mathrm{rect}(t/\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} \tau\,\mathrm{sinc}\left(\tau \frac{\omega}{2\pi}\right)$$

so for $\tau = 2\pi$:

$$f(t) = \mathrm{rect}\left(\frac{t}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} F(\omega) = 2\pi \mathrm{sinc}(\omega).$$

Thus, by duality

$$2\pi \mathrm{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi \mathrm{rect}\left(\frac{-\omega}{2\pi}\right) = 2\pi\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

or equivalently $\boxed{\mathrm{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

Solution (2)

Method 2:

Duality:

$$\mathrm{rect}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\Longrightarrow \quad \mathrm{sinc}\left(\frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} 2\pi\mathrm{rect}(-\omega) = 2\pi\,\mathrm{rect}(\omega)$$

Time scale:

$$f(t) = \mathrm{sinc}\left(2\pi\frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} F(\omega) = \frac{1}{2\pi}2\pi\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right).$$

or equivalently $\boxed{\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

This type of signal, whose frequency spectrum is nonzero only over a finite interval, is called **bandlimited**.

Solution (2)

Method 2:
Duality:

$$\mathrm{rect}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$
$$\Longrightarrow \quad \mathrm{sinc}\left(\frac{t}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\mathrm{rect}(-\omega) = 2\pi\,\mathrm{rect}(\omega)$$

Time scale:

$$f(t) = \mathrm{sinc}\left(2\pi\frac{t}{2\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} F(\omega) = \frac{1}{2\pi}2\pi\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right).$$

or equivalently $\boxed{\mathrm{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

This type of signal, whose frequency spectrum is nonzero only over a finite interval, is called **bandlimited**.

Solution (2)

Method 2:
Duality:

$$\mathrm{rect}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$
$$\Longrightarrow \quad \mathrm{sinc}\left(\frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} 2\pi\mathrm{rect}(-\omega) = 2\pi\,\mathrm{rect}(\omega)$$

Time scale:

$$f(t) = \mathrm{sinc}\left(2\pi\frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} F(\omega) = \frac{1}{2\pi}2\pi\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right).$$

or equivalently $\boxed{\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2\pi}\right).}$

This type of signal, whose frequency spectrum is nonzero only over a finite interval, is called **bandlimited**.

Time differentiation

Property

Time differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \xleftrightarrow{\mathcal{F}} j\omega F(\omega)$$

- DC component vanishes (derivative of a constant is zero).
- Higher frequencies are amplified! (Usually causes undesirable noise amplification (MIT Lecture 9-10).)

Filtering (ideal lowpass filter and differentiator), Video (MIT, Lecture 9, 28:40min)

Time differentiation: example (1)

Example

$f(t) = e^{-at}u(t)$, with $\mathrm{real}\{a\} > 0$. Find the FT of $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$.

Time differentiation: example (2)

Example

Find the FT of $f(t) = \mathrm{sgn}(t)$.

Time differentiation: example (2)

Example

Find the FT of $f(t) = \text{sgn}(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\text{sgn}(t) = 2\delta(t), \quad \text{and} \quad \delta(t) \xleftrightarrow{\mathcal{F}} 1 \Rightarrow j\omega F(\omega) = 2$$

Time differentiation: example (2)

Example

Find the FT of $f(t) = \text{sgn}(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\text{sgn}(t) = 2\delta(t), \quad \text{and} \quad \delta(t) \xleftrightarrow{\mathcal{F}} 1 \Rightarrow j\omega F(\omega) = 2$$

For this result, it is determined that

$$F(\omega) = \frac{2}{j\omega} + k\delta(\omega),$$

where the term $k\delta(\omega)$ is nonzero only at $\omega = 0$ and accounts for the time-averaged value of $f(t)$.

Time differentiation: example (3)

$$F(\omega) = \frac{2}{j\omega} + k\delta(\omega)$$

- In the general case, this $k\delta(\omega)$ term must be included; otherwise the time-derivative operation implied by the expression $j\omega F(\omega)$ would cause a loss of this information about the time-averaged value of $f(t)$.
- In this particular case, the time-averaged value of $\text{sgn}(t)$ is zero. Therefore, $k = 0$.

$$\boxed{\text{sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}}$$

Time differentiation: example (3)

$$F(\omega) = \frac{2}{j\omega} + k\delta(\omega)$$

- In the general case, this $k\delta(\omega)$ term must be included; otherwise the time-derivative operation implied by the expression $j\omega F(\omega)$ would cause a loss of this information about the time-averaged value of $f(t)$.
- In this particular case, the time-averaged value of $\mathrm{sgn}(t)$ is zero. Therefore, $k = 0$.

$$\boxed{\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega}}$$

Time differentiation: example (3)

$$F(\omega) = \frac{2}{j\omega} + k\delta(\omega)$$

- In the general case, this $k\delta(\omega)$ term must be included; otherwise the time-derivative operation implied by the expression $j\omega F(\omega)$ would cause a loss of this information about the time-averaged value of $f(t)$.
- In this particular case, the time-averaged value of $\mathrm{sgn}(t)$ is zero. Therefore, $k = 0$.

$$\boxed{\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega}}$$

Frequency differentiation

Property

Frequency differentiation:

$$(-jt)f(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega}F(\omega)$$

$$(-jt)^n f(t) \xleftrightarrow{\mathcal{F}} \frac{d^n}{d\omega^n}F(\omega)$$

Example

Example

Find FT of $y(t) = te^{-\alpha t}u(t)$ for $\text{real}\{\alpha\} > 0$.

Example

Example

Find FT of $y(t) = te^{-\alpha t}u(t)$ for real$\{\alpha\} > 0$.

One approach would be to integrate by parts. Using properties greatly simplifies.

$\omega = 0$ & $t = 0$

Property

$\omega = 0$ (DC) value

$$F(0) = F(\omega)|_{\omega=0} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t)\,dt$$

Property

$t = 0$ value

$$f(0) = f(t)|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega\Big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,d\omega$$

Outline

Convolution

Property

Convolution (particularly useful for LTI systems)

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega)X(\omega)$$

Convolution: example

Example

eigenfunction revisited

$$x(t) = e^{j\omega_0 t} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t)$$

$$Y(\omega) = H(\omega)X(\omega) = H(\omega)2\pi\delta(\omega - \omega_0) = H(\omega_0)2\pi\delta(\omega - \omega_0),$$

by the sampling property of impulse functions. So

$$Y(\omega) = H(\omega_0)2\pi\delta(\omega - \omega_0) \xleftrightarrow{\mathcal{F}} \boxed{y(t) = H(\omega_0)e^{j\omega_0 t}}$$

as we have seen previously.

Practical use of the convolution property

The convolution property says

$$x(t) \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = h(t) * x(t)$$

so

$$y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega) = H(\omega)X(\omega)$$

$$e^{j\omega_0 t} \rightarrow \boxed{h(t)} \rightarrow y(t) = h(t) * e^{j\omega_0 t} = H(\omega_0)e^{j\omega_0 t}$$

Interchangeable notation

$H(\omega)$ and $H(j\omega)$ are interchangeable notation

$$H(\omega) = H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

- $H(\omega)$: **frequency response**
- $|H(\omega)|$: **magnitude response**
- $\angle H(\omega)$: **phase response**

Time integration

Property

time integration

$$\int_{-\infty}^{t} f(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

Finding LTI system response via FT methods

Skill: ***Finding LTI system response (output signal) via FT methods***

Recipe:

1. Find input spectrum $X(\omega)$ (often using FT table)
2. Find system frequency response $H(\omega)$ (often using FT table)
3. Multiply: $Y(\omega) = H(\omega)X(\omega)$
4. Take inverse FT to get $y(t)$ (often using PFE and FT table).

Basic Fourier Transform Pairs (Text TABLE 4.2, p.329)

Example (1)

Example

Suppose the aperiodic input signal $x(t) = \cos(t) + \cos(2\pi t)$ is applied to an LTI system with impulse response $h(t) = \operatorname{sinc}(t/2)$. Determine the output signal $y(t)$.

Example (2)

1. Find input spectrum

$$X(\omega) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1) + \pi\delta(\omega - 2\pi) + \pi\delta(\omega + 2\pi)$$

2. Find system frequency response

$$h(t) = \operatorname{sinc}(t/2) \stackrel{\mathcal{F}}{\longleftrightarrow} H(\omega) = 2\,\operatorname{rect}\left(\frac{\omega}{\pi}\right)$$

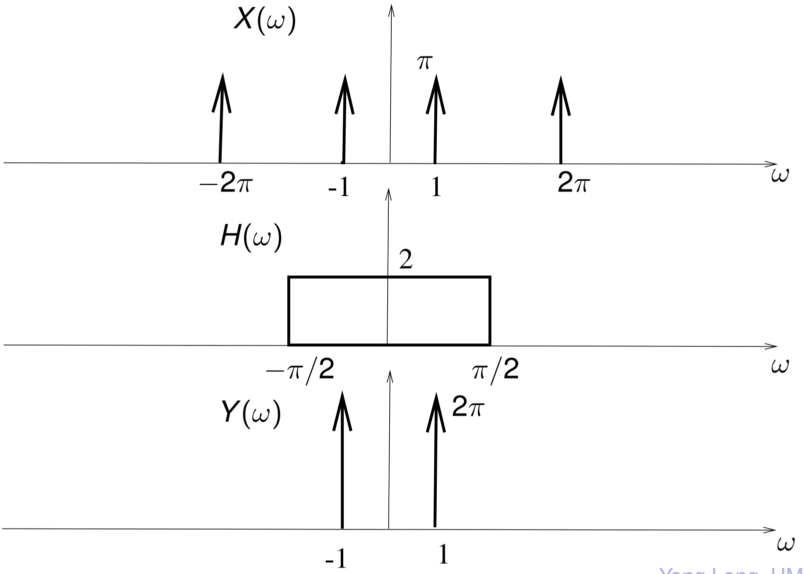
3. Multiply

$$Y(\omega) = H(\omega)X(\omega) = 2[\pi\delta(\omega - 1) + \pi\delta(\omega + 1)]$$

4. Take inverse FT to get $y(t)$

$$y(t) = \boxed{2\cos(t)}$$

Example (3)


$X(\omega)$
π
-2π
-1
1
2π
ω
$H(\omega)$
2
$-\pi/2$
$\pi/2$
ω
$Y(\omega)$
2π
-1
1
ω

Taking an inverse FT

The final step of this process involves taking an inverse FT.
There are several ways to do this:

- Table lookup
- Inverse FT formula (integration)
- Use of FT properties (along with table)
- PFE, followed by table lookup

Taking an inverse FT: example (1)

Example

Find $y(t) = [e^{-at}u(t)] * [e^{-at}u(t)]$.

Taking an inverse FT: example (2)

Example

Find $y(t) = [e^{-t}u(t)] * [e^{-2t}u(t)]$.

Taking an inverse FT: example (3)

Example

Find the FT of $y(t) = \text{tri}(t)$.

Convolution and LTI systems (1)

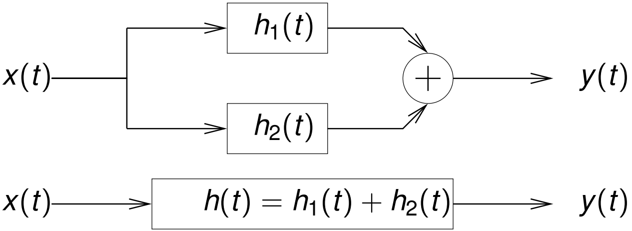
We have seen that when two LTI systems are connected in parallel, *i.e.*

$$y(t) = [h_1(t) * x(t)] + [h_2(t) * x(t)],$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where} \quad h(t) = h_1(t) + h_2(t).$$

Distributive



Convolution and LTI systems (2)

$$h(t) = h_1(t) + h_2(t)$$

Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

Convolution and LTI systems (3)

When two LTI systems are connected in series, *i.e.*

$$y(t) = h_2(t) * [h_1(t) * x(t)], ,$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where} \quad h(t) = h_1(t) * h_2(t).$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h(t) = h_1(t) * h_2(t)} \longrightarrow y(t)$$

Convolution and LTI systems (4)

$$h(t) = h_1(t) * h_2(t)$$

Thus the overall frequency response of two LTI systems connected in series is given by the product of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

Since multiplication is commutative, the order of serial interconnection of LTI subsystems has no effect on the overall frequency response of the system.

$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

Commutative property of convolution.

Outline

Energy signal

We have previously defined the energy of a CT signal $x(t)$ to be

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- If $E < \infty$ then we say $x(t)$ is an energy signal.
- If $x(t)$ has finite duration or if $x(t)$ decays to zero rapidly enough as $|t| \to \infty$, then $x(t)$ will be an energy signal.

The above definition is for the time domain. How can we measure energy in the frequency domain? Answered by Parseval!

Parseval's relation

Property

Parseval's relation

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

time-domain energy vs frequency domain!

For this reason, $|X(\omega)|^2$ is sometimes called the **energy density spectrum**.

Energy density spectrum

Generally, since $Y(\omega) = H(\omega)X(\omega)$, we have

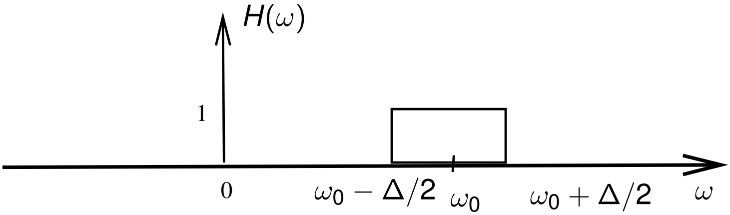
$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

This expression relates the energy density spectrum of the output of an LTI system to the energy density spectrum of its input.

Physical interpretation

Physical interpretation.
Imagine passing a signal $x(t)$ through a bandpass filter with a narrow passband centered at some ω_0, *i.e.*

$$H(\omega) = \text{rect}\left(\frac{\omega - \omega_0}{\Delta}\right)$$

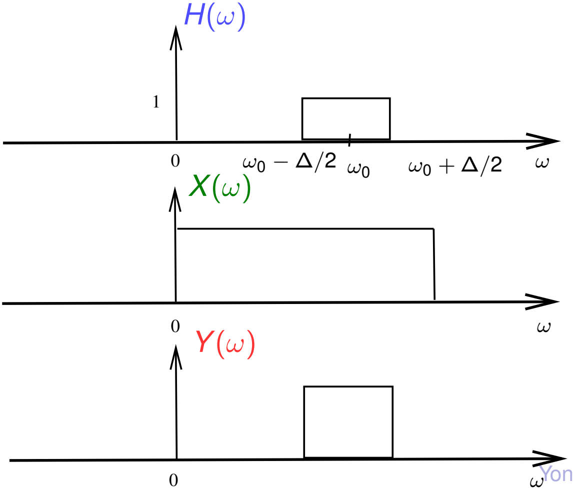


By convolution property, the output spectrum is

$$Y(\omega) = H(\omega)X(\omega) = X(\omega)\,\text{rect}\left(\frac{\omega - \omega_0}{\Delta}\right).$$

Example

$$Y(\omega) = H(\omega)X(\omega) = X(\omega)\mathrm{rect}\left(\frac{\omega-\omega_0}{\Delta}\right).$$



Total energy of the output

By the Parseval's relation, the total energy of the output signal is

$$\int_{-\infty}^{\infty} |y(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 |H(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \left|\mathrm{rect}\left(\frac{\omega-\omega_0}{\Delta}\right)\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{\omega_0-\Delta/2}^{\omega_0+\Delta/2} |X(\omega)|^2\, d\omega.$$

So the total energy of the output signal is the integral of the input signal's energy density spectrum over the filter passband.

Average power

skip We previously defined power as follows:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 \, dt$$

where

$$f_T(t) \triangleq x(t) \operatorname{rect}(t/T)$$

is a truncated signal.

Question

This is a time-domain expression. How do we express power in the frequency domain?

Power Density Spectra

skip Since $f_T(t)$ is finite duration and hence an energy signal, by Parseval's relation

$$\int_{-\infty}^{\infty} |f_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 \, d\omega.$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{1}{T} |F_T(\omega)|^2}_{} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_f(\omega) \, d\omega$$

Power spectral density

skip

Definition

$$P_f(\omega) \triangleq \lim_{T\to\infty} \frac{1}{T} |F_T(\omega)|^2$$

is called the **power spectral density** (when limit exists).

$$f_T(t) \triangleq x(t)\,\mathrm{rect}(t/T)$$

Periodic signals

skip Most useful case is when $x(t)$ is periodic with Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}.$$

We have shown previously that

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2$$

(the latter expression for real signals).

Periodic signals and LTI systems

skip By similar arguments for energy, if

$$x(t) \rightarrow \boxed{\text{H}} \rightarrow y(t)$$

then

$$\boxed{P_y(\omega) = |H(\omega)|^2 P_x(\omega).}$$

So $|H(\omega)|^2$ describes the transfer of signal power or energy from the input to the output of an LTI system, as a function of frequency.

Cross correlation

skip

Property

Cross Correlation

$$r_{xy}(t) = x(t) * y^*(-t) \overset{\mathcal{F}}{\longleftrightarrow} S_{xy}(\omega) = X(\omega)Y^*(\omega)$$

If $x(t)$ and $y(t)$ real, then

$$r_{xy}(t) = x(t) * y(-t) \overset{\mathcal{F}}{\longleftrightarrow} S_{xy}(\omega) = X(\omega)Y(-\omega).$$

Autocorrelation

skip

Property

Autocorrelation

$$r_{xx}(t) = x(t) * x^*(-t) \xleftrightarrow{\mathcal{F}} S_{xx}(\omega) = |X(\omega)|^2$$

Outline

Time-domain multiplication

Property

Time-domain multiplication

$$f_1(t)f_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

Time-domain multiplication: proof

skip

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f_1(t)f_2(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\lambda)e^{j\lambda t}\,d\lambda\right] f_2(t)e^{-j\omega t}\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\lambda)\left[\int_{-\infty}^{\infty} f_2(t)e^{-j(\omega-\lambda)t}\right]\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega-\lambda)\,d\lambda \\
&= \frac{1}{2\pi}F_1(\omega) * F_2(\omega).
\end{aligned}
$$

Frequency shift

Property

Frequency shift (complex modulation)

$$e^{j\omega_0 t} f(t) \xleftrightarrow{\mathcal{F}} F(\omega - \omega_0)$$

Modulation: example

Example

Find the FT of $f(t) \cos \omega_0 t$.

Summary

Summary

1. Convolution in time domain corresponds to multiplication in frequency domain.
2. Multiplication in time domain corresponds to convolution in frequency domain (with an extra $1/2\pi$).

Time-domain multiplication: example(1)

Example

1. Find FT of a causal cosine $x(t) = \cos(\omega_0 t)\, u(t)$.
2. Find the FT of a causal cosine $x(t) = \cos(\omega_0 t + \phi)\, u(t)$.

Time-domain multiplication: example(1)

Example

1. Find FT of a causal cosine $x(t) = \cos(\omega_0 t)\, u(t)$.
2. Find the FT of a causal cosine $x(t) = \cos(\omega_0 t + \phi)\, u(t)$.

Hints: Apply the delay property to the cosine part:

$$x(t) = \cos(\omega_0 t + \phi)\, u(t) = \cos(\omega_0 (t + \phi/\omega_0))u(t)$$

$$f(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} F(\omega)$$

Time-domain multiplication

Property

Time-domain multiplication

$$f_1(t)f_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

Pulsed cosine (1)

Example

pulsed cosine.
Find FT of

$$f(t) = \mathrm{rect}(t/T)\cos(\omega_0 t)\,.$$

Plot its signal spectrum and energy density spectrum.

Pulsed cosine (2)

$$f(t) = \text{rect}(t/T)\cos(\omega_0 t) = f_1(t/T)f_2(t)$$

$$f_1(t) \triangleq \text{rect}(t), \quad f_2(t) \triangleq \cos(\omega_0 t)$$

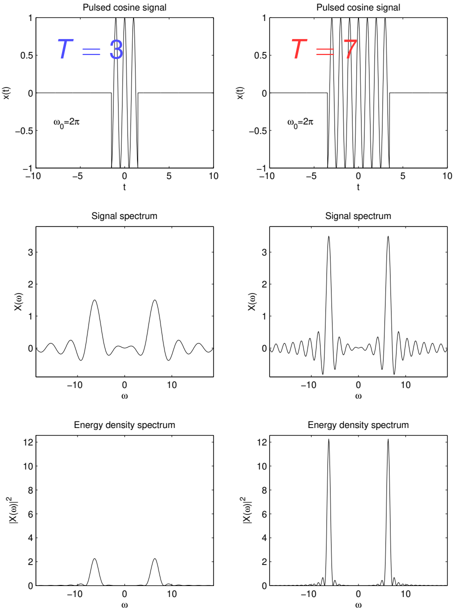
Using time-scaling and time-domain multiplication properties

$$F(\omega) = \frac{1}{2\pi} T F_1(\omega T) * F_2(\omega)$$

$$= \frac{1}{2\pi} T \,\text{sinc}\left(T\frac{\omega}{2\pi}\right) * \{\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\}$$

$$\boxed{= \frac{1}{2}\left[T\,\text{sinc}\left(T\frac{\omega - \omega_0}{2\pi}\right) + T\,\text{sinc}\left(T\frac{\omega + \omega_0}{2\pi}\right)\right]}$$

Pulsed cosine (3)



Pulsed cosine (4)

- As T increases, the spectrum becomes more concentrated at the center frequency ω_0.
- Recall that a pure periodic signal only has frequency components at multiples of the fundamental.
- Even thought the $f(t)$ above is not periodic, its spectrum is “similar” to that of a periodic signal in that most of its energy is near the frequency component ω_0.

Pulsed cosine (4)

This type of signal is used in digital communications.
The following practical tradeoff is unavoidable:

> increasing T will narrow the spectrum (use less bandwidth), but the corresponding signal is then longer in the time domain.

Outline

Application of the FT to RLC circuits

Using properties of the FT, we can solve many problems associated with diffeq systems in general and RLC circuits in particular.

- Find frequency response $H(\omega)$.
- Find impulse response $h(t)$.
- Determine response $y(t)$ to a given input signal $x(t)$

The key properties of the FT are:

- convolution property,
- linearity,
- (time-domain) differentiation property.

Outline

Finding response $y(t)$ of RLC circuit (1)

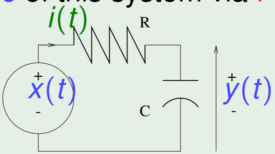
Finding response $y(t)$ of RLC circuit to a simple input.

Example

We showed that for the following RC circuit we have

$$h(t) = (1/RC)e^{-t/RC}u(t), \quad H(\omega) = \frac{1}{1 + j\omega RC}.$$

Find the step response of this system via FT methods.



Finding response $y(t)$ of RLC circuit (2)

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{1+j\omega RC}[\pi\delta(\omega) + 1/j\omega]$$

$$= \underbrace{\frac{1}{1+j\omega RC}\pi\delta(\omega)}_{\text{sampling property}} + \underbrace{\frac{1}{j\omega}\frac{1}{1+j\omega RC}}_{\text{PFE for simple inverse FT}}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}\frac{1+j\omega RC - j\omega RC}{1+j\omega RC}$$

$$= \pi\delta(\omega) + \frac{1+j\omega RC}{j\omega(1+j\omega RC)} - \frac{j\omega RC}{j\omega(1+j\omega RC)}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{1/RC + j\omega}$$

Finding response $y(t)$ of RLC circuit (3)

$$Y(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{1/RC + j\omega}$$

Taking the inverse FT by table lookup, we get the following system step response:

$$\boxed{y(t) = u(t) - e^{-t/RC}u(t) = (1 - e^{-t/RC})u(t)}$$

This example is simple enough that both the time-domain and frequency-domain approaches were comparable effort. But for more complicated systems, the frequency-domain method is usually easier than solving diffeqs and/or convolution!

Outline

Basic idea

Prior to this point, to find $H(\omega)$ for a diffeq system or RLC circuit, we had to first find the diffeq for the circuit (time domain). Now we can work in the frequency domain.

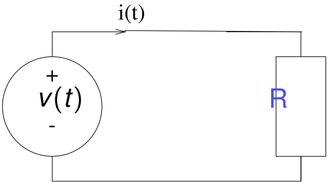
Basic idea:

$$X(\omega) \rightarrow \boxed{\text{LTI } H(\omega)} \rightarrow Y(\omega) = H(\omega)X(\omega)$$

we can rearrange above formula to get

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Resister



Resistor:

$$v(t) = i(t)R$$

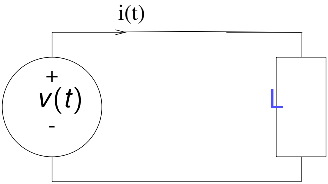
so

$$V(\omega) = I(\omega)R$$

or

$$\boxed{\frac{V(\omega)}{I(\omega)} = R}$$

Inductor



Inductor:

$$v(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i(t)$$

So by the differentiation property

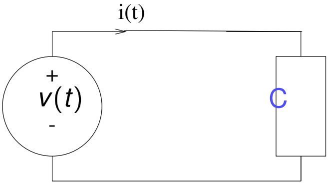
$$V(\omega) = Lj\omega I(\omega)$$

Thus

$$\boxed{\frac{V(\omega)}{I(\omega)} = j\omega L}$$

This is the **complex impedance** of an inductor derived by FT methods!

Capacitor



Capacitor:

$$i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t)$$

so by the differentiation property.

$$I(\omega) = Cj\omega V(\omega)$$

Thus

$$\boxed{\frac{V(\omega)}{I(\omega)} = \frac{1}{j\omega C}}$$

Impedance

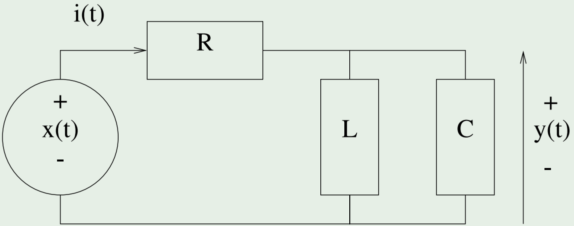
$$
\begin{aligned}
\text{Resistor} &: \quad \frac{V(\omega)}{I(\omega)} = R \\
\text{Inductor} &: \quad \frac{V(\omega)}{I(\omega)} = j\omega L \\
\text{Capacitor} &: \quad \frac{V(\omega)}{I(\omega)} = \frac{1}{j\omega C}
\end{aligned}
$$

- In the frequency domain, diffeq's become simply ratios!
- Usual rules for combining resistances in series and parallel apply to impedances.
- Impedance is an inherently frequency-domain concept due to ω.

Example (1)

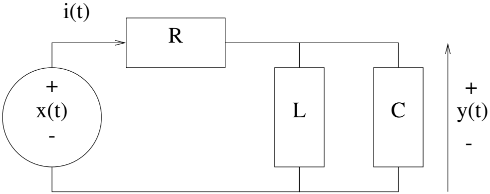
Example

Find frequency response $H(\omega)$, diffeq, and impulse response $h(t)$ for the following circuit.



Time domain approach (1)

Time domain approach using diffeq



1. $i(t)$ on R

$$i(t) = \frac{x(t) - y(t)}{R} \Longrightarrow I(\omega) = \frac{X(\omega) - Y(\omega)}{R}$$

2. $i(t)$ on L and C

$$i(t) = i_L(t) + i_C(t) \Longrightarrow I(\omega) = I_L(\omega) + I_C(\omega) = \frac{Y(\omega)}{j\omega L} + Y(\omega)(j\omega C)$$

Time domain approach (2)

Equating:

$$\frac{X(\omega) - Y(\omega)}{R} = \frac{Y(\omega)}{j\omega L} + Y(\omega)(j\omega C)$$

$$\implies \quad Y(\omega)\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right) = \frac{X(\omega)}{R}$$

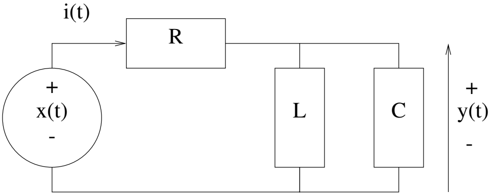
Thus

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/R}{1/R + 1/(j\omega L) + j\omega C}$$

$$= \boxed{\frac{j\omega}{R/L + j\omega + (j\omega)^2 RC}}$$

Frequency domain approach (1)

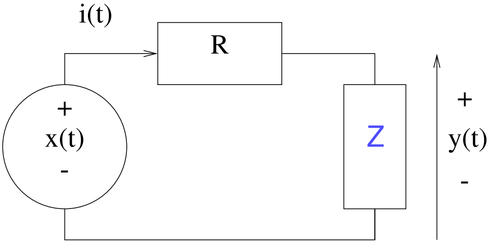
Frequency domain approach using complex impedances.



Equivalent impedance of parallel combination of inductor and capacitor:

$$Z(\omega) = \left[(j\omega L)^{-1} + j\omega C\right]^{-1}.$$

Frequency domain approach (2)



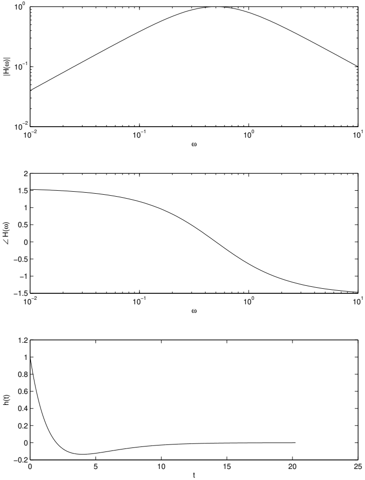
Considering equivalent circuit above as a (complex) voltage divider:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{Z(\omega)}{Z(\omega) + R} = \frac{1}{1 + R/Z(\omega)}$$

$$= \frac{1}{1 + R\left[(j\omega L)^{-1} + j\omega C\right]} = \frac{j\omega}{R/L + j\omega + (j\omega)^2 RC}$$

Frequency response from RLC circuits: Example (3)

Now it is trivial to plot magnitude and phase response using MATLAB's `freqs` command for given RLC values.



MATLAB code (1)

```
a = [1 1 1/4]; %RC = 1; R/L = 1/4
b = [1 0]; %start from higher-order coefficients
[H,o]= freqs(b,a);
sys = tf(b,a);
[h,t] = impulse(sys);
plot(h,t)

subplot(311)
loglog(o, abs(H))
xlabel('\omega'), ylabel('|H(\omega)|')
subplot(312)
semilogx(o, angle(H))
xlabel('\omega'), ylabel('\angle H(\omega)')
subplot(313)
plot(t, h)
xlabel('t'), ylabel('h(t)')
```

MATLAB code (2)

- [H, w] = freqs(b,a) evaluates the complex frequency response of the analog filter specified by coefficient vectors b and a at auto-generated angular frequencies (200 points by default) in rad/s specified in real vector w.
- sys = tf(b, a) creates a continuous-time transfer function with numerator(s) and denominator(s) specified by b and a.
- [y,t] = impulse(sys) returns the output response y and the time vector t used for simulation (if not supplied as an argument to impulse).
- loglog(X,Y) creates a plot using a logarithmic scale for both the x-axis and the y-axis.
- semilogx(X,Y) creates a plot with a logarithmic scale for the x-axis and a linear scale for the y-axis.

Find $H(\omega)$ experimentally

The analysis above is the mathematical approach.

Question

How would one find $H(\omega)$ experimentally?

Diffeq from $H(\omega)$(1)

Question

How to find the diffeq from $H(\omega)$?

$$H(\omega) = \frac{j\omega}{R/L + j\omega + (j\omega)^2 RC}.$$

Diffeq from $H(\omega)$(2)

We know that

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega}{R/L + j\omega + (j\omega)^2 RC}.$$

Cross multiplying yields

$$[R/L + j\omega + (j\omega)^2 RC]Y(\omega) = j\omega X(\omega).$$

Thus, by the time-domain differentiation property of the FT, the corresponding diffeq is

$$\boxed{\frac{R}{L}y(t) + \frac{\mathrm{d}}{\mathrm{d}t}y(t) + RC\frac{d^2}{dt^2}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)}$$

Impulse response from $H(\omega)$

- In principle, $h(t)$ is “simply” the inverse FT of $H(\omega)$.
- But you will not find this particular $H(\omega)$ in most FT tables, and trying to find the inverse FT by integration will be challenging!
- The solution is partial fraction expansions, which is discussed in an Appendix of the textbook.

Impulse response from $H(\omega)$: example (1)

General idea. First note that

$$H(\omega) = \frac{j\omega}{R/L + j\omega + (j\omega)^2 RC} = \left.\frac{s}{s^2 RC + s + R/L}\right|_{s=j\omega}.$$

Suppose $RC = 1$ and $R/L = 1/4$. Then

$$H(\omega) = \left.\frac{s}{s^2 + s + 1/4}\right|_{s=j\omega} = \left.\frac{s}{(s+1/2)^2}\right|_{s=j\omega} = \frac{j\omega}{(j\omega + 1/2)^2}.$$

We know that

$$te^{-t/2}u(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega + 1/2)^2$$

Impulse response from $H(\omega)$: example (2)

The extra $j\omega$ in $H(\omega)$ is equivalent to differentiating in the time domain. Thus

$$h(t) = \boxed{\frac{\mathrm{d}}{\mathrm{d}t} te^{-t/2}u(t) = (1 - t/2)e^{-t/2}u(t)}.$$

Question

- *How did we do this in this case?*
- *How do we do this in general?*

Outline

Summary

- Defined FT and inverse FT by limits of FS
- Existence of FT
- FT of many important signals
- FT properties (!)
- FT of periodic signals
- Parseval's relation (Energy density spectrum)
- convolution property and LTI systems
- Application of FT to RLC and diffeq systems