VE216 Recitation Class 3

Fourier Series

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Why FS?

For a LTI system: When

$$x(t) = e^{st}$$

Then

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{st-s\tau}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= x(t) \cdot H(s) \end{aligned}$$

Frequency Response

When we take s to be purely imaginary, i.e. s=jω

$$
\begin{aligned}
H(s)\mid_{s=j\omega} &= H(j\omega) = H(\omega) \\
&= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\
&= \mid H(\omega) \mid e^{j\angle H(\omega)}
\end{aligned}
$$

Then

$$y(t) = x(t) \cdot H(\omega) = e^{j\omega t} \mid H(\omega) \mid e^{j\angle H(\omega)}$$

Summary

A LTI system only changes the magnitude and phase of the input.

FS Expressions

Linearity tells us that, if

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{(Synthesis Equation)}$$

where

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \\ c_0 &= \frac{1}{T_0}\int_{T_0} x(t)dt \end{aligned} \qquad \text{(Analysis Equation)}$$

Then

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k e^{jk\omega_0 t}$$

Real Forms of FS

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} \mid c_k \mid cos(k\omega_0 t + \angle c_k)$$

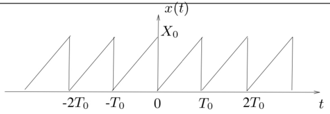
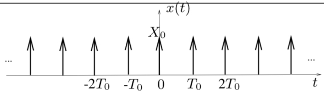
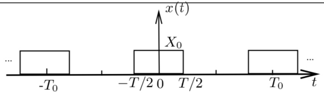
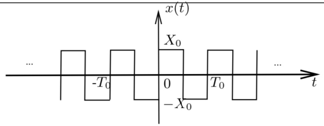
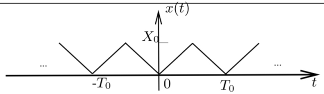
(Combined Trigonometric Form)

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} Re(c_k) \cdot cos(k\omega_0 t) - Im(c_k) \cdot sin(k\omega_0 t)$$

(Trigonometric Form)

Exercises

Table of Fourier Series for Common Signals

Name	Waveform	c_0	$c_k, k \neq 0$	Comments
Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	
Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\left(\frac{Tk\omega_0}{2\pi}\right)$	$\frac{Tk\omega_0}{2\pi} = \frac{Tk}{T_0}$
Square wave		0	$-j\frac{2X_0}{\pi k}$	$c_k = 0$, k even
Triangular wave sine		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$c_k = 0$, k even

Gibbs Phenomenon

Overshoot/Undershoot near discontinuity

Exercise

Sketch the following code in MATLAB;

```
a0=1;
t=-10:0.01:10;
x=0*t;
x=x+a0;
for k=1:50
x=x+cos(k.*pi.*t)*(sin(pi.*k./2))./(pi.*k./2);
end
plot(t,x);
axis([-5 5 -0.5 1.5]);
xlabel('time t [Hz]');
ylabel('inout signal x(t)');
title('plot 1');
```

FS Properties

- Hermitian Symmetry: x(t) Real $\rightarrow c_{-k} = c_k^*$ (Also hold for systems)
  - x(t) Real & Even: c_k Real & Even
  - x(t) Real & Odd: c_k Purely imaginary & Odd
- Amplitude: y(t)=ax(t) $\rightarrow \omega' = \omega_0$, $c_k' = ac_k$
- Time: y(t)=x(at+b) $\rightarrow \omega' = a\omega_0$, $c_k' = c_k \cdot e^{jk\omega_0 b}$
- Conjugation: y(t)=[x(t)]* $\rightarrow \omega' = \omega_0$, $c_k' = c_{-k}^*$
- Differentiation: y(t)=$\frac{d}{dt}$x(t) $\rightarrow \omega' = \omega_0$, $c_k' = jk\omega_0 \cdot c_k$
- Parseval's Relation: $P = \frac{1}{T_0}\int_{T_0} \mid x(t) \mid^2 dt = \sum_{-\infty}^{\infty} \mid c_k \mid^2$

Spectra

- Power Density Spectrum: $\mid c_k \mid^2$ vs. $k\omega$
- Magnitude Spectrum: $\mid c_k \mid$ vs. $k\omega$
- Phase Spectrum: $\angle c_k$ vs. $k\omega$

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Filters

Filters are used to allow certain frequencies to pass while blocking others

- Lowpass Filter
- Highpass Filter
- Bandpass Filter

Differential Equations

$$H(s) = \frac{\sum_0^M a_k s^k}{\sum_0^N b_k s^k}$$

Where:
a_k's are coefficients in front of x(t),
b_k's are coefficients in front of y(t)

Exercise

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) = x(t)$$

- H(s)?
- When input is sin(2t)+sin(3t)+sin(4t), what is the output?

Total Harmonic Distortion

$$
\begin{aligned}
THD &= 1 - \frac{\text{Average Power in First Fundamental}}{\text{Average Signal Power}} \\
&= 1 - \frac{\mid c_1 \mid^2}{\sum_{-\infty}^{\infty} \mid c_k \mid^2}
\end{aligned}
$$

Exercise

$y(t) = 3[x(t) - 2cos(\omega_0 t) \cdot x(t)]$

- x(t)=cos(ω_0t)
- x(t)=sin(ω_0t)