Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

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Review FS and FT

We have seen that Fourier methods are very useful in the study of many problems involving signals and LTI systems.

- We can represent a broad class of signals using linear combinations of complex exponential signals ($e^{j\omega t}$).

$$\text{FS} \quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt, k = 0, \pm 1, \ldots$$

$$\text{FT} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega, \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- Those complex exponential signals are eigenfunctions of LTI systems.
- Those complex exponential signals are solutions to linear constant coefficient differential equations too.

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General complex exponential signal

- **Fourier series** and **Fourier transforms** use signals of the form e^{st} where $s = j\omega$, because signals of the form $e^{j\omega t}$ are eigenfunctions of LTI systems, which led to the concept of frequency response, filtering, etc.
- Many of the properties of e^{st} also apply when s is a general complex number $s = \sigma + j\omega$, rather than a pure imaginary number $s = j\omega$.

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Laplace transform

In particular, the eigenfunction property holds, as shown previously:

$$e^{st} \to \boxed{\text{LTI } h(t)} \to H(s)e^{st}, \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

Often there are advantages in reformulating some of the previously discussed ideas in the more general context of $s = \sigma + j\omega$.

The **Laplace transform** (LT) is the generalization of the **Fourier transform** to include general complex exponential signals of the form

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t}.$$

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Bilateral Laplace transform

Laplace transforms come in two flavors

- Bilateral Laplace transform (two-sided)
- Unilateral Laplace transform (one-sided)

- We will focus on the bilateral form of the LT, and will probably not have time to discuss the unilateral form.
- For causal systems and signals, the two forms are identical, so for many problems of interest there is no need to make a distinction. (Most properties are the same or very similar.)
- For brevity, I will speak simply of the "Laplace transform" and specify "bilateral" only occasionally.
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Overview

- bilateral
- ROC
- rational Laplace transforms
- pole-zero plots
- inverse LT
- ROC, causality, stability
- Magnitude response from pole-zero plot
- Properties
- application to LTI systems / filtering
- Feedback control

Outline

Bilateral Laplace transform

Definition

The **bilateral Laplace transform** is defined as

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}\,dt,$$

where $s = \sigma + j\omega$ is a complex variable with real part σ and imaginary part ω.

The following notation denotes LT pairs:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s).$$

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Unilateral Laplace transform

Definition

The **unilateral Laplace transform** is defined as

$$X_+(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt,$$

where the 0^- is included to handle an impulse function at 0.

Laplace transform: example (1)

Example

Find LT of $x(t) = e^{-at}u(t)$.

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{1}{-(s+a)} e^{-(s+a)t} \Big|_{0}^{\infty} = \frac{1}{s+a}[1 - e^{-(s+a)\infty}] = ?$$

Question

When is this integral finite?

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Laplace transform: example (2)

When

$$\text{real}\{s+a\} > 0 \Longrightarrow \sigma = \text{real}\{s\} > \text{real}\{-a\}\,.$$

So we write

$$\boxed{e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \text{real}\{s\} > \text{real}\{-a\}\,.}$$

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Region of convergence (ROC)

In general, the bilateral Laplace transform will exist for some values of $\text{real}\{s\}$ and not for others.

Definition

The set of values of s for which the bilateral Laplace transform is guaranteed to exist ($x(t)e^{-\text{real}\{s\}\,t}$ is absolutely integrable) is given by

$$\text{ROC} \triangleq \left\{ s : \int_{-\infty}^{\infty} |x(t)| e^{-\text{real}\{s\}\,t}\, dt < \infty \right\},$$

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The ROC depends on the signal $x(t)$.

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The ROC is

$$\{s : \text{real}\{s\} > \text{real}\{-a\}\}$$

or just

$$\text{real}\{s\} > \text{real}\{-a\} \textit{ for short}$$

Laplace transform: example (2)

Example

Find LT and ROC of $x(t) = -e^{-at}u(-t)$.

Laplace transform: solution

$$X(s) = \int_{-\infty}^{\infty} [-e^{-at} u(-t)] e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= \frac{1}{s+a} e^{-(s+a)t} \Big|_{-\infty}^{0} = \frac{1}{s+a}[1 - e^{-(s+a)(-\infty)}] = \frac{1}{s+a},$$

this time the integral only exists if $\text{real}\{s+a\} < 0$*, i.e.,* $\text{real}\{s\} < \text{real}\{-a\}$.

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The algebraic component of the two preceding signals is the same: $1/(s+a)$ in both cases. But their ROCs are different. To completely specify the Laplace transform of a signal, one must provide both the “formula” and the ROC.

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Why an ROC?

The LT for any given s is the signed "area" within the product of $x(t)$ with e^{-st} and t-axis.

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

Example

$$x(t) = e^{-2t}u(t)$$

- If $\text{real}\{s\} > -2$, then the product of $x(t) = e^{-2t}u(t)$ with e^{-st} is a decaying exponential, so the area is finite.

$$e^{-2t}u(t)e^{-st} = e^{-(2+s)t}, \text{ for } t \geq 0 \Longrightarrow \text{real}\{2+s\} > 0$$

- If $\text{real}\{s\} \leq -2$, then the product of the two functions has "infinite area" (more precisely the LT is undefined since things like $\infty - \infty$ would come into play).

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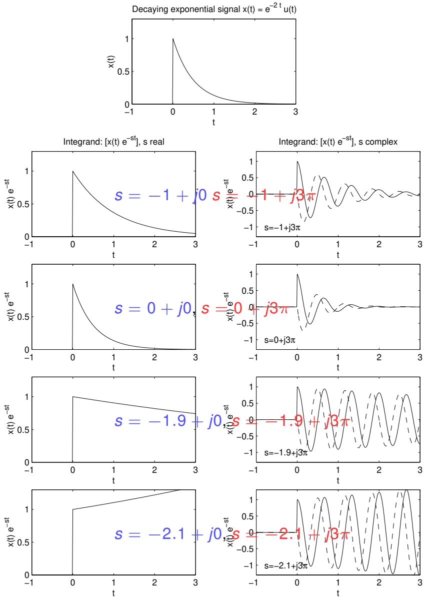
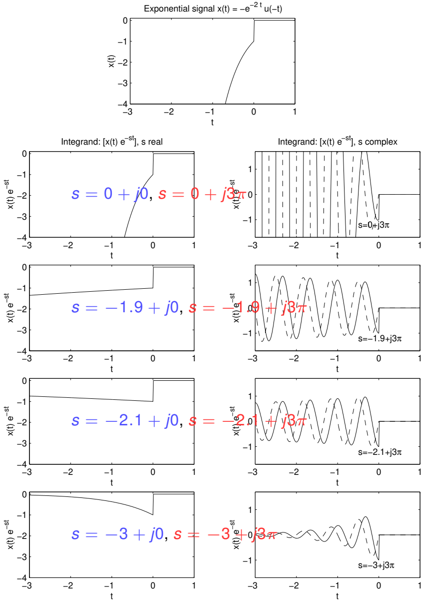
Decaying exponential



- $x(t) = e^{-2t}u(t)$
- Solid line: real
- Dashed line: imag
- ROC: $\text{real}\{s\} > \text{real}\{-2\}$

Exponential



- $x(t) = -e^{-2t}u(-t)$
- Solid line: real
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- ROC: $\text{real}\{s\} < \text{real}\{-2\}$

Plotting a Laplace transform

- Since $s = \sigma + j\omega$ varies with both its real part σ and its imaginary part ω, in a sense the Laplace transform is a 2D function.
- To “plot” a Laplace transform one would use a `mesh` or `surface` plot in MATLAB, showing the value of $X(s)$ as a function of (σ,ω) over the complex plane.
- This is rarely done, since we will see later that a **pole-zero plot** provides the same information more easily.

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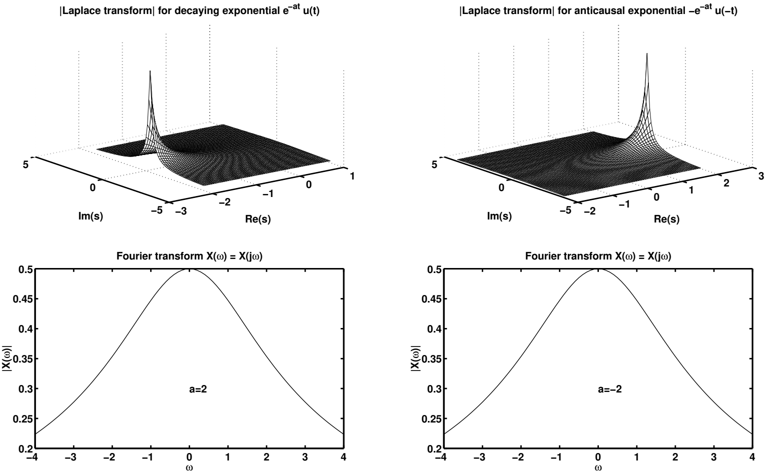
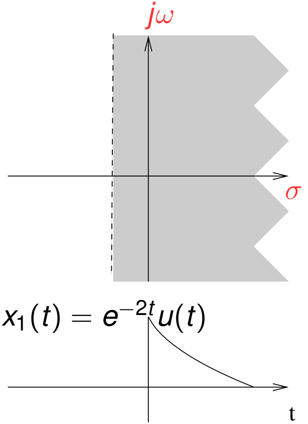
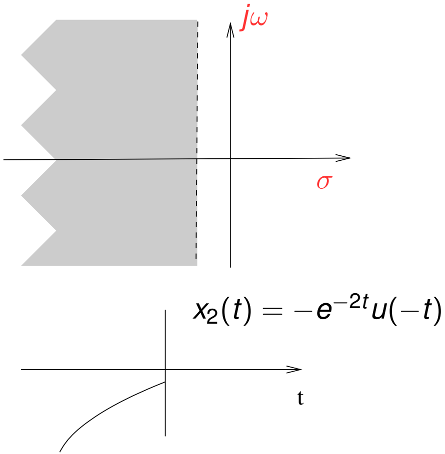


Display of the ROC (1)

Often we display the ROC of a signal using the complex s-plane as shown in the following figures.





Display of the ROC (2)

- The horizontal axis is usually called the σ **axis**, and the vertical axis is usually called the $j\omega$**axis**.
- The shaded region indicates the set of points in the *s*-plane where the bilateral Laplace transform exists, *i.e.*, the ROC.
- Dotted lines for boundaries if ROC does not include its edges.
- If the shaded region includes the $j\omega$axis, then the FT of the signal exists.

Display of the ROC: Example

Example

Find the LT of $x(t) = 3e^{-2t}u(t) + 4e^{t}u(-t) + \delta(t)$ and sketch its ROC.

Solution

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_0^{\infty} 3e^{-2t}e^{-st}\,dt + \int_{-\infty}^{0} 4e^{t}e^{-st}\,dt + \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt$$

$$= 3\int_0^{\infty} e^{-(s+2)t}\,dt + 4\int_{-\infty}^{0} e^{-(s-1)t}\,dt + 1$$

$$\boxed{= \frac{3}{s+2} - \frac{4}{s-1} + 1}$$

provided $\text{real}\{s+2\} > 0$ *and* $\text{real}\{s-1\} < 0$. *Both conditions must be satisfied, so the ROC is*

$$\boxed{-2 < \text{real}\{s\} < 1}$$

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Relation to Fourier transform

- When $\sigma = 0$, the Laplace transform integral is the same as the Fourier transform integral.
- Thus, the value of the bilateral Laplace transform along the $j\omega$axis is the FT of the signal. Mathematically

$$X(\omega) = X(s)|_{s=j\omega} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

if ROC includes $j\omega$axis.

- Note that $X(\omega)$ and $X(j\omega)$ are just different notations for the same thing (the integral above); this reuse of notation is common in most books and papers.
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$$X(\omega) = X(s)|_{s=j\omega} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

if ROC includes $j\omega$axis.

- Note that $X(\omega)$ and $X(j\omega)$ are just different notations for the same thing (the integral above); this reuse of notation is common in most books and papers.
- The Fourier transform of a signal $x(t)$ exists if and only if the ROC of $X(s)$ includes the imaginary axis.

Relation to Fourier transform

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When does the FT of the preceding signal exist?

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When does the FT of the preceding signal exist?

Previously, we found the LT of $x(t)$ is

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \text{real}\{s\} < \text{real}\{-a\}$$

Iff $\boxed{\text{real}\{a\} < 0}$ *so that the ROC includes the imaginary axis.*

Why both LT and FT?

So the Laplace transform generalizes the FT in the sense that some signals have a LT that do not have a FT.

Question

Why both LT and FT are needed? Are there signals that have FT but no LT?

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Why both LT and FT are needed? Are there signals that have FT but no LT?

Example

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1. Find LT of the causal cosinusoidal signal, $x(t) = \cos(\omega_0 t)\, u(t)$.
2. Find LT of the anti-causal cosinusoidal signal, $x(t) = \cos(\omega_0 t)\, u(-t)$.
3. Find LT of the cosinusoidal signal, $x(t) = \cos(\omega_0 t)$.

Solution (1)

Find LT of the causal cosinusoidal signal, $x(t) = \cos(\omega_0 t)\, u(t)$.

$$\boxed{e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a},\ \text{real}\{s\} > \text{real}\{-a\}}$$

$$X(s) = \int_0^\infty \cos(\omega_0 t)\, e^{-st}\, dt = \int_0^\infty \frac{1}{2} e^{j\omega_0 t} e^{-st}\, dt + \int_0^\infty \frac{1}{2} e^{-j\omega_0 t} e^{-st}\, dt$$

$$= \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2},\ \mathrm{ROC} = \text{real}\{s\} > 0,$$

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The ROC is the empty set and we say that the Laplace transform of $x(t) = \cos(\omega_0 t)$ does not exist.

Outline

Signals with rational Laplace transforms

Definition

If the Laplace transform of a signal $x(t)$ has the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where $N(s)$ and $D(s)$ are polynomials in s, then we say that the Laplace transform $X(s)$ is **rational**.

Poles and Zeros

Definition

Rational Laplace transform

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- The roots of the denominator are called **poles**, since if s_0 is a pole, then $X(s_0) = N(s_0)/0 = \infty$.
- The roots of the numerator are called **zeros**, since if s_0 is a zero, then $X(s_0) = 0/D(s_0) = 0$.
- The factor $G = b_m/a_n$ is called the **gain**.

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Outline

Pole-zero plot

- A rational LT can be completely described by its **pole-zero plot**, along with a gain G.
  (Picture MIT Lecture 20.2-4)
- The corresponding signal $x(t)$ is completely specified provided we know 3 things: the pole-zero plot, the gain G, and the ROC.
- The ROC of a rational Laplace transform $X(s)$ is bounded by its poles (or by infinity).

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Properties of any ROC (1)

Property

1. *The ROC consists of "strips" (which may be empty or include the entire s-plane) parallel to the $j\omega$ axis in the s-plane.* ***(Picture)****(MIT Lecture 20.2-4)*
2. *The ROC of $X(s)$ does not contain any poles of $X(s)$, if $X(s)$ is rational.*

For some signals, such as $e^{|t|}$ or $\cos(\omega_0 t)$, the ROC is the empty set and we say that the Laplace transform of $x(t)$ does not exist.

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Properties of ROC (2)

$$\mathrm{ROC} \triangleq \left\{ s : \int_{-\infty}^{\infty} |x(t)| e^{-\,\mathrm{real}\{s\}\,t}\, dt < \infty \right\},$$

absolutely integrable of $x(t)e^{-\,\mathrm{real}\{s\}\,t}$

For signals that have Laplace transforms that exist, the ROC always falls into one of the following categories, depending on the signal characteristics.

1. finite duration signal $\Longrightarrow$ the entire s-plane
2. right-sided signal $\Longrightarrow$ right half plane
3. left-sided signal $\Longrightarrow$ left half plane
4. two-sided signal $\Longrightarrow$ vertical strip

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Finite duration signals

Property

If $x(t)$ is finite duration and absolutely integrable, then $\mathrm{ROC} = \mathbb{C}$ (the entire s-plane).

The intuition behind this result is suggested in

- $x(t)$multiplied by a decaying exponential (MIT Lecture 20.5)
- $x(t)$multiplied by a growing exponential (MIT Lecture 20.6)

Since the interval over which $x(t)$ is nonzero is finite, the exponential weighting is never unbounded, and consequently, it is reasonable that the integrability of $x(t)$ is not destroyed by this exponential weighting.
(formal verification, textbook, p.664)

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Right-sided signals (1)

Property

If the signal $x(t)$ is a right-sided signal, i.e.,

$$x(t) = 0, \text{ for } t < T_1, \text{ where } T_1 \text{ is some constant,}$$

then the ROC of $x(t)$ will be a right half plane (RHP) of $\text{real}\{s\} > \sigma_0$*, for some σ_0.*

Suppose that the Laplace transform converges for some σ_0, then if $\sigma_1 > \sigma_0$, it must also be true that $x(t)e^{-\sigma_1 t}$ is absolutely integrable, since $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \to \infty$. ***(Picture)***(MIT Lecture 20.7)

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Property

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The ROC of $X(s)$ does not contain any poles of $X(s)$, if $X(s)$ is rational. ***(Picture)****(MIT Lecture 20.2)*

Left-sided signals

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then the ROC of $x(t)$ will be a left half plane (LHP) of $\text{real}\{s\} < \sigma_0$*, for some σ_0.*

Property

If $X(s)$ has a rational form, then if $x(t)$ is left-sided, the LHP of ROC *will be everything to the left of the leftmost pole.*

(Picture)(MIT Lecture 20.4)

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(Picture)(MIT Lecture 20.4)

Two-sided signals

Property

If the signal $x(t)$ is a two-sided signal, then the ROC will be a vertical strip.

(Explanation, textbook, p. 666)

Property

If $X(s)$ is rational, then the ROC will have the form $\sigma_1 < \text{real}\{s\} < \sigma_2$, for some $\sigma_1 < \sigma_2$, and in fact the ROC will be a strip between a pair of adjacent poles (not including any other poles).

(Picture)(MIT Lecture 20.3)

Two-sided signals

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If the signal $x(t)$ is a two-sided signal, then the ROC will be a vertical strip.

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If $X(s)$ is rational, then the ROC will have the form $\sigma_1 < \text{real}\{s\} < \sigma_2$, for some $\sigma_1 < \sigma_2$, and in fact the ROC will be a strip between a pair of adjacent poles (not including any other poles).

(Picture)(MIT Lecture 20.3)

Example

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Where do causal signals fall into the above categories?

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A causal signal is a special case of a right-sided signal, so the ROC *of a causal signal will be a RHP.*

Outline

Table of Laplace transform pairs (1)

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

Table of Laplace transform pairs (2)

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\, u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Outline

Inverse Laplace transform (1)

Question

Given a LT $X(s)$, how do we invert it to find the signal $x(t)$?

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \Longrightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$\Longrightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\Longrightarrow x(t)e^{-\sigma t} \overset{\mathcal{F}}{\longleftrightarrow} X(\sigma + j\omega)$$

Inverse FT $\quad x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$

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where

$$s = \sigma + j\omega, \quad ds = j\, d\omega$$

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Solution

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$$x(t) = \frac{1}{2j}\left[-e^{-(1+j)t}u(t) + e^{-(1-j)t}u(t)\right]$$

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Example of inverse LT given pole-zero plot

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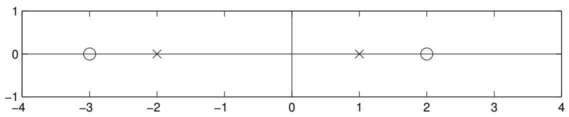

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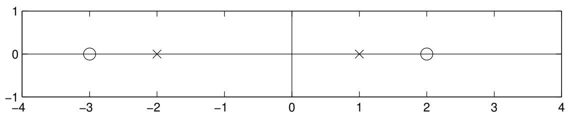

From the pole-zero plot:

$$X(s) = G\frac{(s+3)(s-2)}{(s+2)(s-1)} = 4\frac{s^2+s-6}{(s+2)(s-1)}.$$

Example of inverse LT given pole-zero plot (2)

The next step is to do the PFE.

$$X(s) = 4\frac{s^2+s-6}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

Question

Can we do PFE directly to the above rational function?

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Can we do PFE directly to the above rational function?

Proper and improper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- *A rational function is call* ***proper*** *if $m < n$.*
- *We can write any* ***improper*** *rational function ($m \geq n$) as the sum of a polynomial and a proper rational function.*

Example

$$\frac{s+5}{s+2} = \frac{s+2+3}{s+2} = 1 + \frac{3}{s+2}.$$

In general this is always possible using long division.

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Now we want to find $x(t)$ by taking the inverse LT by Table lookup. But recall that $e^{-at}u(t)$ and $-e^{-at}u(-t)$ both LT to the same algebraic form $1/(s+a)$. So for this problem there are $N+1=3$ possibilities for $x(t)$, depending on what the ROC is!

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Example of inverse LT given pole-zero plot (4)

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Outline

Stability

- Recall: An LTI system is BIBO stable if its impulse response is absolutely integrable, *i.e.*. $\int_{-\infty}^{\infty} |h(t)| < \infty$.
- If $h(t)$ is absolutely integrable, its FT $H(\omega)$ converges (exists).
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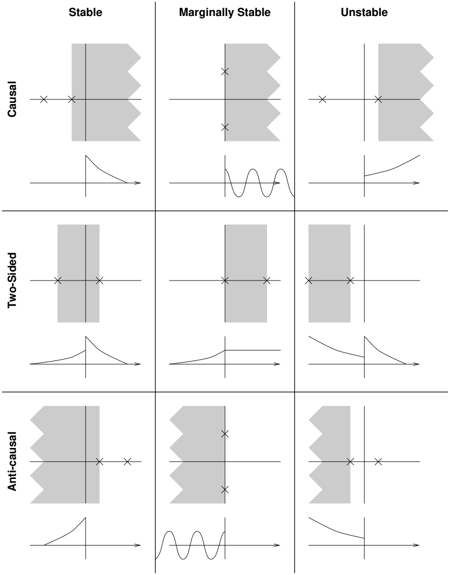
If it is known to be causal, then it is also stable iff all of its poles lie (strictly) within the LHP.

Causality and Stability (2)



Causality and Stability (3)

- The above figure summarizes the various cases.
- The zeros can be anywhere, without affecting causality or stability.
- The upper left plot is the case of greatest interest in practice: a causal system whose poles lie in the left half plane.

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This system appears to have no poles, so one might conclude it is stable.

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However, in fact this system is BIBO unstable because it is a differentiator system

$$\frac{d}{dt}\delta(t) \xleftrightarrow{\mathcal{L}} s, \forall s,$$

which has unbounded output for the bounded input signal $x(t) = \cos\left(t^2\right)$.

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- *These remarks apply to any system with rational but non-proper transfer function ($m > n$).*
- *Thus, all such systems are unstable!*

Solution (2)

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Stability of diffeq systems

A diffeq system (initially at rest) is stable iff all roots of its characteristic polynomial are in the left half plane ($\sigma < 0$).

Example

Is the following system stable?

$$6y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

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Solution

The characteristic polynomal is:

$$s^3 + 3s^2 + 2s + 6 = 0$$

In MATLAB*:* `roots([1 3 2 6])` *returns* $-3, \pm j\sqrt{2}$*.*
$\boxed{\textit{Not stable}}$*, since two roots on the imaginary axis - not in LHP.*

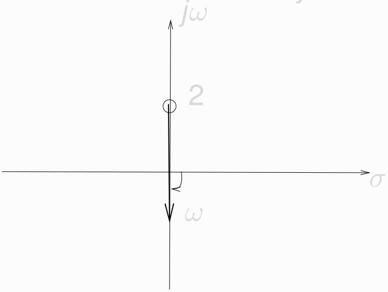
Outline

Geometric properties of FT from pole-zero plot

Given the pole-zero plot corresponding the the transfer function $H(s)$ of an LTI system, one can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$ of the system! This is very useful for understanding general system properties.

Example

$H(s) = s - j2$ which has a zero at $s = j2$.



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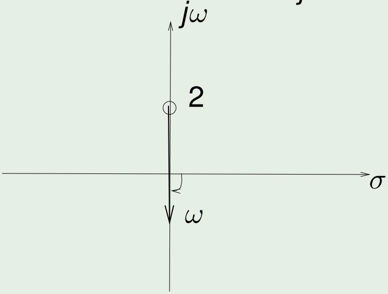

Example

- ***magnitude response*** $|H(\omega)|$

$$H(\omega) = j\omega - j2, \quad \textit{so} \quad |H(\omega)| = |j\omega - j2| = |\omega - 2|$$

which is the Euclidean distance between the point $(0, \omega)$ *and* $(0, 2)$ *in the s-plane.*

- ***phase response*** $\angle H(\omega)$

$$\angle H(\omega) = \angle(j\omega - j2)$$

which is the angle, from the real axis, of the vector pointing from j2 to the point $j\omega$*.*

  - *For* $\omega > 2$, $\angle H(\omega) = \pi/2$.
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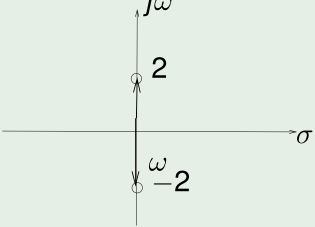
Example (2)

Example

Consider two zeros at $\pm 2j$. then $H(s) = (s - j2)(s + j2)$.



Multiply and divide complex numbers in polar form.

$$|z_1|e^{j\theta_1}|z_2|e^{j\theta_2} = |z_1||z_2|e^{j(\theta_1+\theta_2)}$$

$$\frac{|z_1|e^{j\theta_1}}{|z_2|e^{j\theta_2}} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$$

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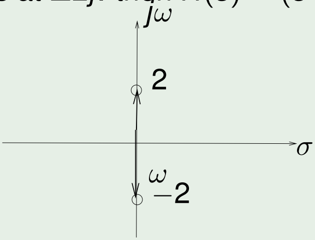

magnitude response $|H(\omega)|$

$$|H(\omega) = |(j\omega - j2)(j\omega + j2)| = |\omega - 2||\omega + 2|$$

which is the product of the Euclidean distances from the point $(0, \omega)$ *to* $(0, 2)$ *and from the point* $(0, \omega)$ *to* $(0, -2)$ *in the s-plane.*

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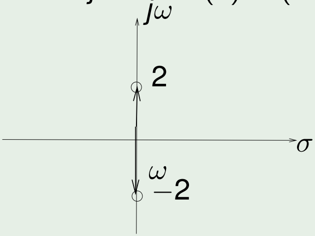

phase response $\angle H(\omega)$

$$\angle H(\omega) = \angle(j\omega - j2) + \angle(j\omega + j2)$$

General cases

If the LT of a signal $h(t)$ is rational, then it can be expressed

$$H(s) = G\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)},$$

where the z_k's and p_k's are the zeros and poles of the system, and G is a constant scale factor.

Mathematically, the frequency response of the system is given by

$$H(\omega) = H(s)|_{s=j\omega} = G\frac{(j\omega-z_1)\cdots(j\omega-z_m)}{(j\omega-p_1)\cdots(j\omega-p_n)}.$$

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The **magnitude response** is given by

$$|H(\omega)| = |G| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}.$$

- The magnitude response at that frequency ω is proportional to the product of the geometric distances in the complex plane from the point $(0,j\omega)$ to each of the zeros, divided by the product of the distances to each of the poles.
- As one "slides along the $j\omega$ axis" to make a plot of $|H(\omega)|$ vs ω, as the point $(0, j\omega)$ gets closer to a zero, the contribution of that term decreases, and as we get closer to a pole, the contribution of that term increases.

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Example

This property was used to design a 60Hz notch filter much earlier in the course.

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Phase response

The **phase response** is given by

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \cdots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \cdots - \angle(j\omega - p_n).$$

One can add and subtract the angles formed by the line segment between the point $(0, j\omega)$ and each zero or pole location in the s-plane to determine the overall phase response for each frequency ω.

Second order system

Example

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_n \frac{\mathrm{d}}{\mathrm{d}t}y(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 X(s)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(Picture MIT Lecture 21.1-4)

Video [MIT Lecture 21, 15:08-29:00min]

Second order system

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Outline

Important properties

Most important: linearity, differentiation, convolution. With these 3 we can solve most of the LTI systems problems of greatest interest to us.

Linearity

Property

Linearity

If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ *with* ROC_1 *and* $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ *with* ROC_2 *then*

$$x(t) = a_1 x_1(t) + a_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = a_1 X_1(s) + a_2 X_2(s),$$

and the ROC *of* $X(s)$ *is at least as large as the intersection of the* ROC_1 *and* ROC_2.

Question

Can the ROC *of* $X(s)$ *be larger?*

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Can the ROC *of* $X(s)$ *be larger?*

Yes. Because of term cancellation.

Linearity: Example

Example

$$x_1(t) = [e^{-2t} + e^{-t}]u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) = \frac{1}{s+2} + \frac{1}{s+1}, \ \mathrm{real}\{s\} > -1$$

$$x_2(t) = [e^{-3t} - e^{-t}]u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s) = \frac{1}{s+3} - \frac{1}{s+1}, \ \mathrm{real}\{s\} > -1$$

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which is a ROC that is larger than the intersection of the individual signal's ROC.

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*The **differentiation property** (in time)*

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s)$$

the ROC of $sX(s)$ will be at least as large (due to possible cancellation with a pole at $s = 0$) as before.

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Differentiation property: example

Example

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} 1/s, \quad \text{real}\{s\} > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1, \forall s$$

Differentiation property: example

Example

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Convolution property

Property

The ***convolution property***

$$y(t) = x(t) * h(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = H(s)X(s)$$

The ROC is at least as large as the intersection of the ROC's of $X(s)$ and $H(s)$.

Convolution property: example

Example

$$X_1(s) = \frac{s+1}{s+2},, \quad \text{real}\{s\} > -2$$

$$X_2(s) = \frac{s+2}{s+1},, \quad \text{real}\{s\} > -1$$

then

$$X_1(s)X_2(s) = 1, \quad \forall s$$

Time shift property

Property

time shift property

$$x(t - t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-t_0 s}X(s)$$

Same ROC as $X(s)$.

Caution: after time shifting, a formerly rational LT becomes irrational due to $e^{-t_0 s}$.

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modulation property

Property

modulation property *(shifting in s-domain)*

$$e^{s_0 t}x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0),\ \mathrm{ROC_{new}} = \mathrm{ROC_{old}} + \text{real}\{s_0\}$$

The ROC *associated with* $X(s-s_0)$ *is that of* $X(s)$ *shifted by* real$\{s_0\}$ ***(Picture textbook, Figure 9.23)***

$$e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{L}} X(s-j\omega_0),\ \mathrm{ROC}\ \text{unchanged}$$

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Time scaling property

Property

time scaling $a \neq 0$

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|}X(\frac{s}{a}),\ \mathrm{ROC_{new}} = a\mathrm{ROC_{old}}$$

Differentiation in s-domain

Property

differentiation in s-domain

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\mathrm{d}}{\mathrm{d}s}X(s),\ \text{ROC}\ \textit{unchanged}$$

Running integration in time

Property

running integration in time

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s)$$

$$\mathrm{ROC}_{\mathrm{new}} \textit{ must contain } \mathrm{ROC}_{\mathrm{old}} \cap \{\text{real}\{s\} > 0\}$$

Initial and final value theorem

skip
initial value theorem

$$\lim_{t\to 0} x(t) = x(0+) = \lim_{s\to\infty} sX(s)$$

if $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, *i.e.* $M < N$ for rational $X(s)$ (no poles at infinity).

---

final value theorem

$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$ if $x(t)$ has a final value and $x(t) = 0$ for $t < 0$

Example

Example

Find the step response of the causal LTI system described by the following differential equation:

$$y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

Solution (1)

$$y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

Using linearity and differentiation properties, take LT of both sides:

$$Y(s) + 2sY(s) = X(s) + sX(s)$$

$$(1 + 2s)Y(s) = (1 + s)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1 + s}{1 + 2s} = \frac{1}{2}\frac{s + 1}{s + 1/2}.$$

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Solution (2)

$$x(t) = u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}, \text{real}\{s\} > 0$$

By the convolution property

$$Y(s) = H(s)X(s) = \frac{1}{2}\frac{s+1}{s+1/2}\frac{1}{s} = \frac{-1/2}{s+1/2} + \frac{1}{s}.$$

Since $x(t) = u(t)$ is causal and the system is causal, we know we are looking for the right-sided output signal. Thus our final answer is:

$$y(t) = -\frac{1}{2}e^{-0.5t}u(t) + u(t).$$

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Solution (3)

$$y(t) = -\frac{1}{2}e^{-0.5t}u(t) + u(t).$$

This answer consists of two parts.

- The $e^{-0.5t}u(t)$ part is called the **natural response**, and note that the decay rate is associated with the pole location $s = -1/2$. Since the natural response decays to zero, we also call it the **transient response**.
- The $u(t)$ part is called the **forced response** and is associated with the input signal. Since the forced response persists indefinitely, we also call it the **steady-state response**.

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Outline

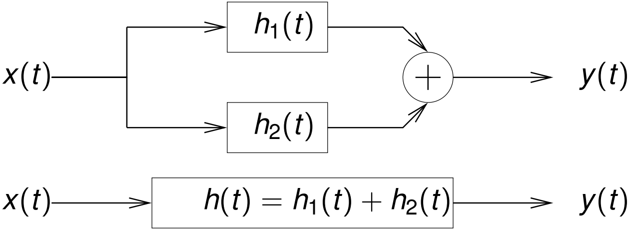
Parallel interconnection (1)

We have seen that when two LTI systems are connected in parallel, *i.e.*

$$y(t) = [h_1(t) * x(t)] + [h_2(t) * x(t)],$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where } h(t) = h_1(t) + h_2(t).$$



Distributive

Parallel interconnection (2)

Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$

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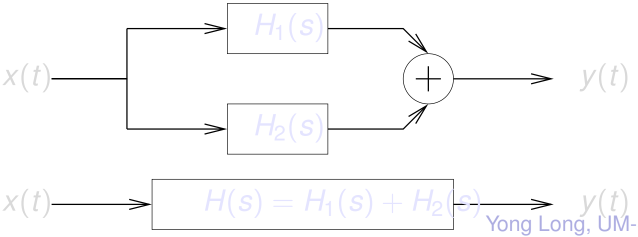

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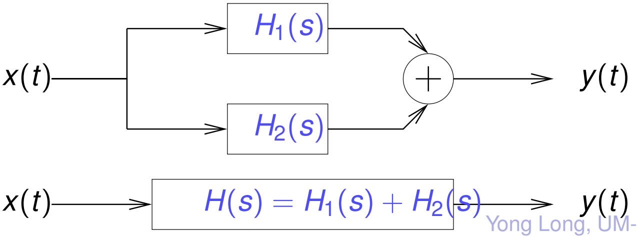

Distributive

Series combination (1)

When two LTI systems are connected in series, *i.e.*

$$y(t) = h_2(t) * [h_1(t) * x(t)], ,$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where} \quad h(t) = h_1(t) * h_2(t).$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h(t) = h_1(t) * h_2(t)} \longrightarrow y(t)$$

Series combination (2)

Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

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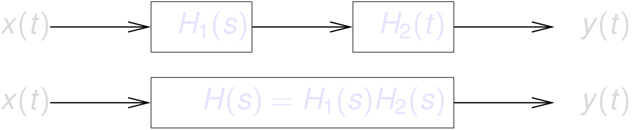
Associative



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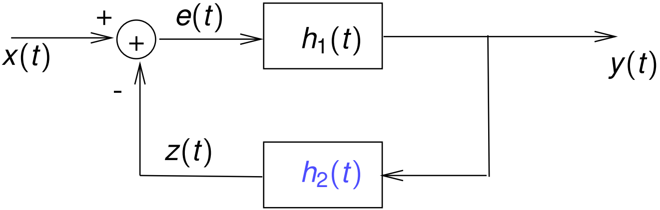
Feedback interconnection (1)

A **feedback system** uses the output of a system to control or modify the input.



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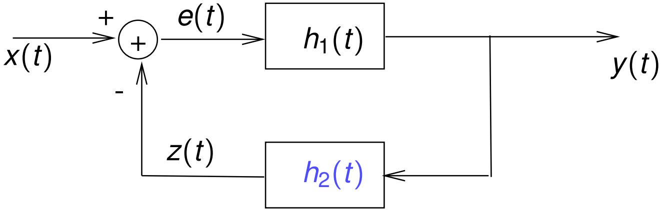

$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \Longrightarrow y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

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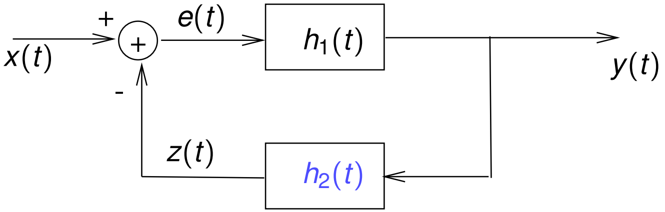

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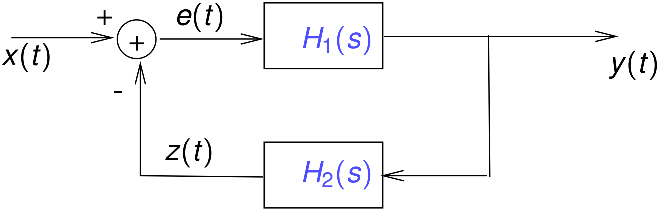

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Outline

Example # 1

Example

Draw the block diagram of the causal LTI system with system function

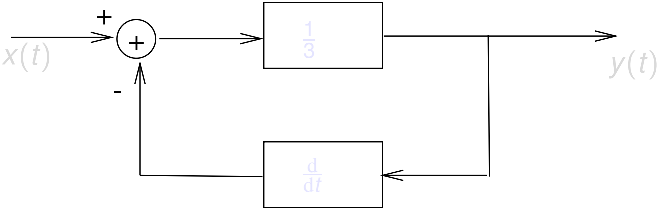
$$H(s) = \frac{1}{s+3}$$

Solution (1)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3} \Longrightarrow (s+3)\,Y(s) = X(s)$$

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t)$$

$$\Longrightarrow y(t) = \frac{1}{3}\left[x(t) - \frac{\mathrm{d}}{\mathrm{d}t}y(t)\right]$$



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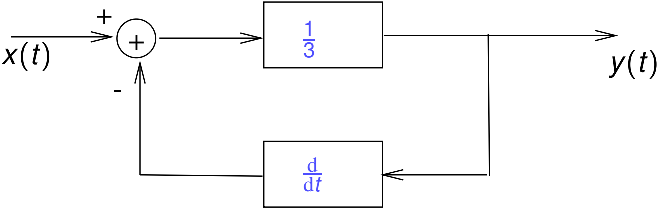
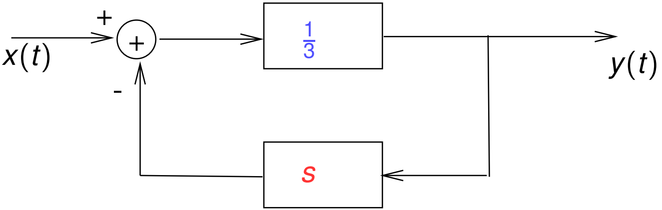


Solution (2)

$$y(t) = \frac{1}{3}\left[x(t) - \frac{\mathrm{d}}{\mathrm{d}t}y(t)\right] \Longrightarrow Y(s) = \frac{1}{3}[X(s) - sY(s)]$$

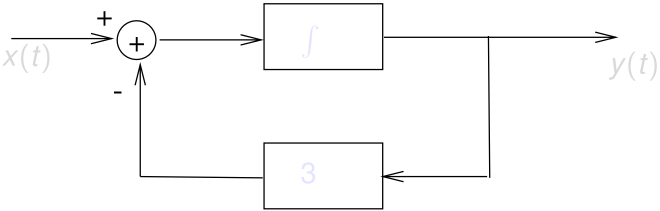


Solution (3)

- The previous diagram is a valid representation.
- But the differentiator is both difficult to implement and extremely sensitive to noise.

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t)$$

$$\Longrightarrow y(t) = \int_{-\infty}^{t} [x(\tau) - 3y(\tau)]\, d\tau$$



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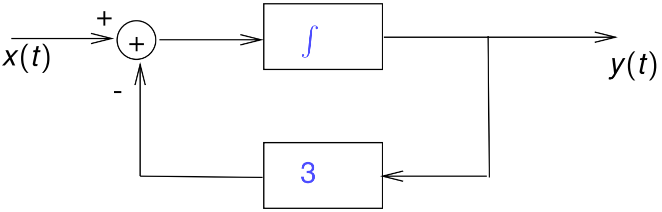

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$$y(t) = \int_{-\infty}^{t} [x(\tau) - 3y(\tau)] \, d\tau$$

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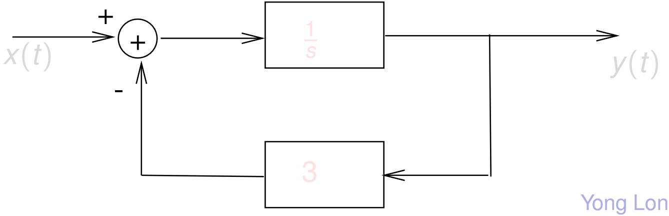
$$= [x(\tau) - 3y(\tau)] * u(t)$$

$$\Longrightarrow Y(s) = [X(s) - 3Y(s)] \frac{1}{s}$$



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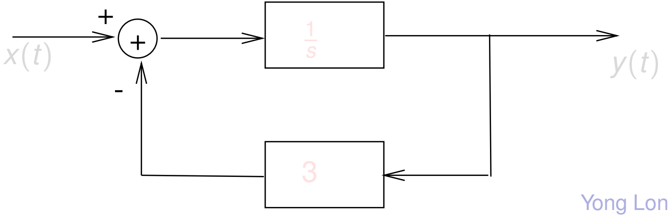

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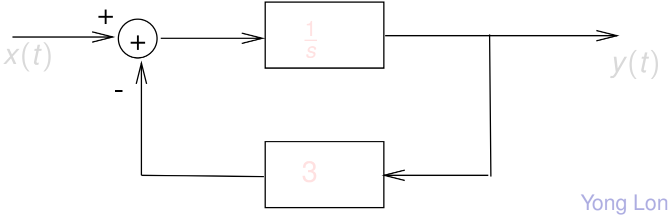

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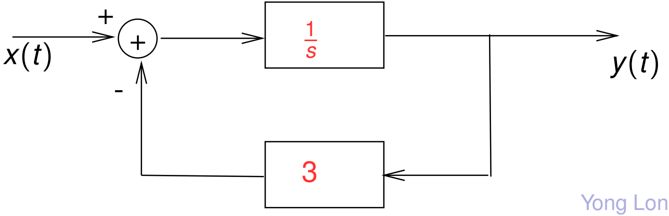

Example # 2

Example

Draw the block diagram of the causal LTI system with transfer function

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

Solution (1)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2 + 4s^{-1} - 6s^{-2}}{1 + 3s^{-1} + 2s^{-2}}$$

$$H(s) = \frac{2 + 4s^{-1} - 6s^{-2}}{1 + 3s^{-1} + 2s^{-2}} = \frac{Y(s)}{W(s)} \frac{W(s)}{X(s)}$$

$$\frac{Y(s)}{W(s)} = 2 + 4s^{-1} - 6s^{-2} \Longrightarrow Y(s) = 2W(s) + 4s^{-1}W(s) - 6s^{-2}W(s)$$

$$\frac{W(s)}{X(s)} = \frac{1}{1 + 3s^{-1} + 2s^{-2}} \Longrightarrow W(s) = X(s) - 3s^{-1}W(S) - 2s^{-2}W(s)$$

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Example # 2

(Picture Block diagram from spring semester)
Block diagram representation in direct form, cascade form and parallel form. (Textbook, Example 9.30)

Outline

Cruise control system

Example

The cruise control system in a car.

- The “input” to a car is the applied forces
  - external: wind, gravity (hills), road friction, etc.
  - internal: engine (controlled by gas pedal)
- The output is the car’s velocity, which a cruise control system should hold approximately constant (by adjusting gas pedal) even as road conditions/hills vary.

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Transfer function

System diagram:
force $f(t) \to \boxed{\text{Car}} \to v(t)$ velocity.

Newton's laws say

$$f(t) = ma(t)$$

where $a(t)$ is the acceleration and m is the mass of the car. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = a(t) = \frac{f(t)}{m}$$

is the input-output relationship for this system, so in the Laplace domain

$$sV(s) = F(s)/m$$

so the transfer function of this system is:

$$H(s) = V(s)/F(s) = \frac{1}{sm}.$$

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Stability

Question

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- *A cruise control system cannot work in "open loop."*

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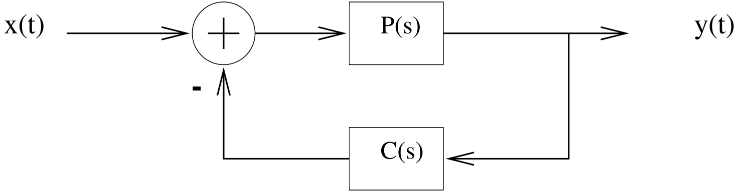
- *A cruise control system cannot work in "open loop."*

There must be feedback: some way of measuring the car's velocity and that information is "fed back" to the system input by adjusting the gas intake to maintain the desired velocity.

Negative feedback

A **negative feedback** control system looks like the following:



- $P(s)$ is the transfer function of the "plant" to be controlled (in this case the car).
- $C(s)$ is the transfer function of the controller system.
  - The simplest form of feedback is proportional control, where $C(s) = c$, simply some constant.
  - Intuition: if the car velocity is too high, then decrease the force (acceleration) a little to compensate. (And vice versa).

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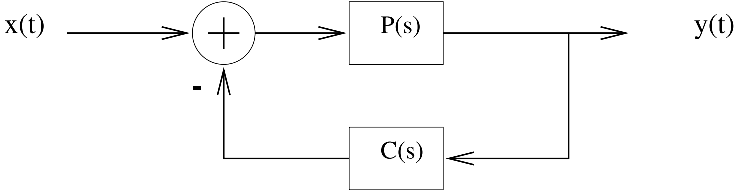

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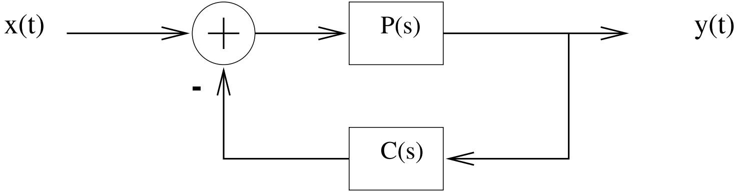

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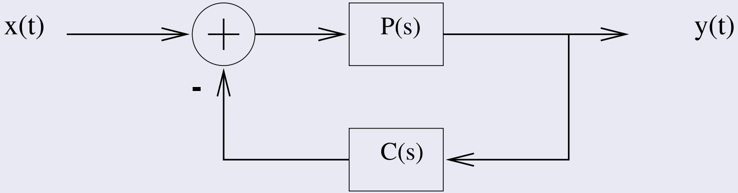
Overall transfer function

Question

For the car system, $P(s) = 1/(ms)$. Find overall transfer function of system with controller in place, when $C(s) = c$. Is this system stable?



Solution

By inspection:

$$Y(s) = P(s)[X(s) - C(s)Y(s)]$$

$$[1 + C(s)P(s)] = P(s)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{1 + C(s)P(s)} = \frac{1/(ms)}{1 + c/(ms)} = \frac{1/m}{s + c/m}.$$

This system is stable if $c > 0$ since then the pole is at $-c/m$ which is in the LHP.

So adding a simple proportional negative feedback stabilizes this system.

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Outline

Summary

- Laplace transform definition / computation by integration
- ROC of Laplace transform / properties
- relation to Fourier transform
- rational Laplace transforms / pole-zero plot
- inverse Laplace transform by PFE
- FT magnitude from pole-zero plot
- properties of LT
- application of LT to LTI systems