Homework 1

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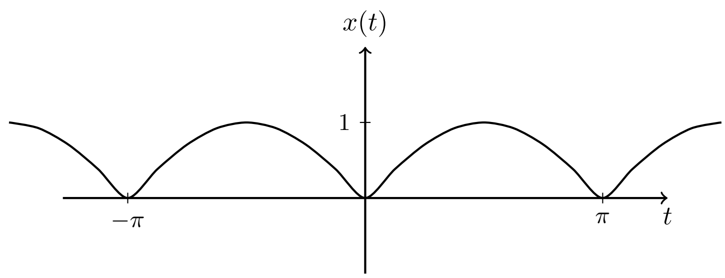
HW Notes:

- $\boxed{\text{Box}}$ your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

Problems:

1. [10!] Consider the periodic sinusoidal signal illustrated below.



   (a) Find the mathematical representation for this signal.
   (b) Find the energy of this signal. Is it an energy signal, power signal, or neither?

2. [12!] Determine the values of average power and energy for each of the following signals:
   (a) $x_1(t) = e^{-2t}u(t)$
   (b) $x_3(t) = cos(t)$

3. [14!] Suppose $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental periods $T_1 > 0$ and $T_2 > 0$ respectively.
   (a) Show that if T_1/T_2 is rational, then $x(t) = x_1(t) + x_2(t)$ is periodic.
   (b) Determine whether the following signals are periodic. If so, find a period. Otherwise, specify the reason.

$$x(t) = \sin(\pi t/3)\cos(\pi t/4) + \sin(\pi t/5)\sin(\pi t/2)$$
$$x(t) = \sin\left(\sqrt{3}\pi t/3\right) + \sin(\pi t/5)$$

4. [12!] Consider the signal illustrated below.

1. (a)

$x(t) = |\sin t|$

(b)

energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \sin^2 t \, dt = (\frac{t}{2} - \frac{\sin 2t}{4}) \Big|_{-\infty}^{\infty} = \infty \Rightarrow$ not energy signal

power: $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T \to \infty} (\frac{1}{2} - \frac{\sin 2T}{4T}) = \frac{1}{2} \in (0, \infty) \Rightarrow$ power signal

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2. (a)

$x_1(t) = e^{-2t} u(t)$

$\Rightarrow P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{-2t} u(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} (0 + \int_{0}^{T} |e^{-2t} \cdot 1|^2 dt)$

$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} e^{-4t} dt = 0$

$\Rightarrow E = \int_{-\infty}^{\infty} |e^{-2t} u(t)|^2 dt = \int_{-\infty}^{0} |e^{-2t} \cdot 0|^2 dt + \int_{0}^{\infty} |e^{-2t} \cdot 1|^2 dt = \int_{0}^{\infty} e^{-4t} dt = \frac{1}{4}$

(c)

$x_3(t) = \cos t$

$\Rightarrow P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\cos^2 t| \, dt = \lim_{T \to \infty} (\frac{1}{2T} + \frac{\sin 2T}{4T}) = \frac{1}{2}$

$\Rightarrow E = \int_{-\infty}^{\infty} |\cos t|^2 dt = (\frac{t}{2} + \frac{\sin 2t}{4}) \Big|_{-\infty}^{\infty} = \infty$

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3. (a)

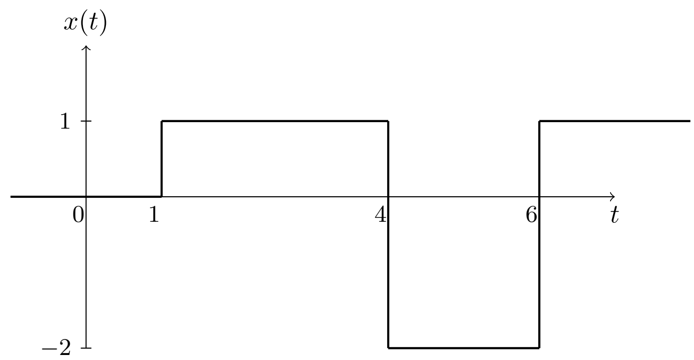
Assume $T = n_1 T_1 + n_2 T_2$ where $\frac{T_1}{T_2} = \frac{n_2}{n_1}$

Then $x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t + n_1 T_1) + x_2(t + n_2 T_2) = x_1(t) + x_2(t) = x(t)$

Therefore, $x(t)$ is periodic with period T

(b) for $x(t) = \sin(\pi t/3)\cos(\pi t/4) + \sin(\pi t/5)\sin(\pi t/2)$: least_common_multiple $(6, 8, 10, 4) = 120$

for $x(t) = \sin\left(\sqrt{3}\pi t/3\right) + \sin(\pi t/5)$: $\frac{T_1}{T_2} = \frac{2\sqrt{3}}{10} = \frac{\sqrt{3}}{5}$, not rational. so $x(t)$ isn't periodic



(a) Express the signal $x(t)$ using a sum of step functions.

(b) Find the derivative of the signal and carefully sketch it.

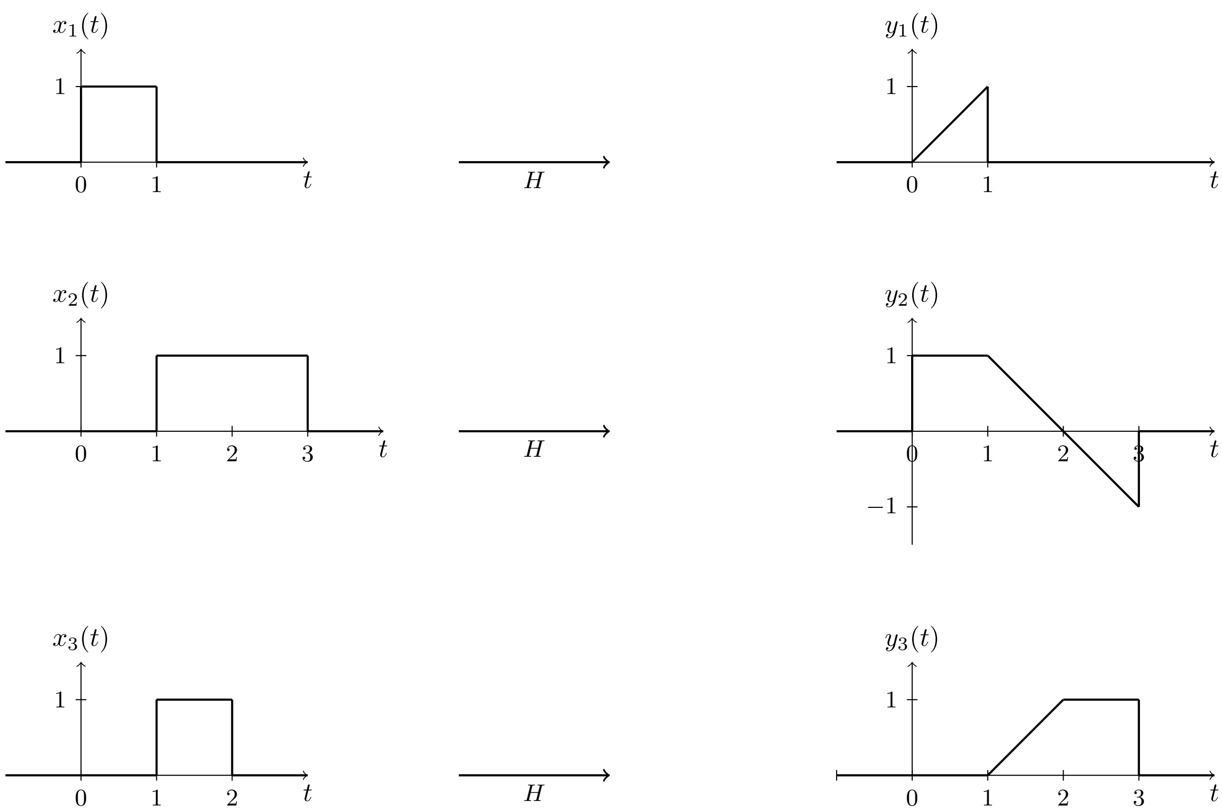
5. [15!] Indicate whether the following systems are Memoryless, Time Invariant, Linear, Causal, Stable. Justify your answers. (3! for each)

   (a) $y(t) = x(t-2) + x(2-t)$

   (b) $y(t) = cos(x(t))$

   (c) $y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau$

6. [12!] A linear system H has following input-output pairs. Answer the following question, and justify your answers.

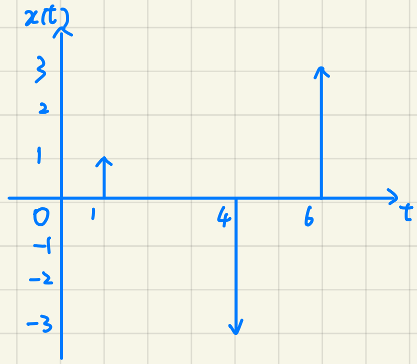


4. (a)

$x(t) = u(t-1) - 3u(t-4) + 3u(t-6)$

(b)

$$\frac{dx(t)}{dt} = \delta(t-1) - 3\delta(t-4) + 3\delta(t-6)$$



5. (a) $y(t) = x(t-2) + x(2-t)$

① Since the output $y(t)$ partly depends on $x(t-2)$ which is a previous input signal, it's $\boxed{\text{not Memory less}}$

② $y(t-t_0) = x(t-t_0-2) + x(2-t-t_0)$
since $y_d(t) = x(t-2-t_0) + x(2-t+t_0)$ $\Big\}$ not equal $\Rightarrow$ $\boxed{\text{not Time - invariant}}$

③ $y_1(t) = x_1(t-2) + x_1(2-t)$, $y_2(t) = x_2(t-2) + x_2(2-t)$

when $x(t) = a_1 x_1(t) + a_2 x_2(t)$, $y(t) = a_1 x_1(t-2) + a_2 x_2(t-2) + a_1 x_1(2-t) + a_2 x_2(2-t)$

$= a_1 y_1(t) + a_2 y_2(t)$ $\Rightarrow$ $\boxed{\text{Linear}}$

④ when $t<1$, $2-t>t$ $\Rightarrow$ $y(t)$ depends on the future input $x(2-t)$ $\Rightarrow$ $\boxed{\text{non causal}}$

⑤ If $|x(t)| < M < \infty$, then $|x(t-2)| < M$ and $|x(2-t)| < M$ $\Rightarrow |y(t)| < 2M < \infty$ $\Rightarrow$ $\boxed{\text{stable}}$

(b) $y(t) = cos(x(t))$

① just depend on current value of $x(t)$, so $y(t)$ is $\boxed{\text{Memoryless}}$

② $y(t-t_0) = \cos(x(t-t_0))$
$y_d(t) = \cos(x(t-t_0))$ $\Big\}$ equal $\Rightarrow$ $\boxed{\text{Time - invariant}}$

③ $y_1(t) = \cos(x_1(t))$, $y_2(t) = \cos(x_2(t))$

when $x(t) = a_1 x_1(t) + a_2 x_2(t)$, $y(t) = \cos(a_1 x_1(t) + a_2 x_2(t)) \neq a_1 y_1(t) + a_2 y_2(t)$ $\Rightarrow$ $\boxed{\text{not Linear}}$

④ Memoryless $\Rightarrow$ must $\boxed{\text{Causal}}$

⑤ If $|x(t)| \leq M$, $|y(t)| \leq 1$ $\Rightarrow$ $\boxed{\text{Stable}}$

(c) $y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau$

① $y(t)$ depends on the entire history of the input signal up to that point $\Rightarrow$ $\boxed{\text{not Memoryless}}$

② $y(t-t_0) = \int_{-\infty}^{\frac{t-t_0}{2}} x(\tau)\,d\tau$

$y_d(t) = \int_{-\infty}^{\frac{t}{2}} x(\tau - t_0)\,d\tau = \int_{-\infty}^{\frac{t}{2}-t_0} x(\tau - t_0)\,d(\tau - t_0)$

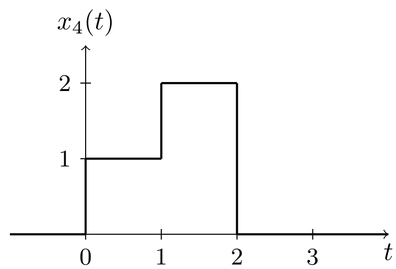
not equal $\Rightarrow$ $\boxed{\text{not Time invariant}}$

③ $y_1(t) = \int_{-\infty}^{\frac{t}{2}} x_1(\tau)\,d\tau$, $y_2(t) = \int_{-\infty}^{\frac{t}{2}} x_2(\tau)\,d\tau$

when $x(t) = a_1 x_1(t) + a_2 x_2(t)$, $y(t) = \int_{-\infty}^{\frac{t}{2}} a_1 x_1(\tau) + a_2 x_2(\tau)\,d\tau = a_1 y_1(t) + a_2 y_2(t)$ $\Rightarrow$ $\boxed{\text{Linear}}$

④ when $t < 0$, $t < \frac{t}{2}$ $\Rightarrow$ $y(t)$ depends on the future time $\frac{t}{2}$ $\Rightarrow$ $\boxed{\text{not Causal}}$

⑤ Assume $x(t) = \overset{\text{constant}}{M} > 0$ which is bounded . $y(t) = \int_{-\infty}^{\frac{t}{2}} M\,d\tau = +\infty$ $\Rightarrow$ $\boxed{\text{not Stable}}$



(a) Could this system be causal?

(b) Could this system be time invariant?

(c) Could this system be memoryless?

(d) What is the output for the input $x_4(t)$, sketch it.

7. [10!] Let $s(t) = (\dfrac{t-1}{2})^2 rect(\dfrac{t-1}{2})$

(a) Make a sketch of s(t).

(b) Evaluate $\int_{-\infty}^{\infty} s(t)x(t)dt$, where $x(t) = \delta(t - \frac{1}{2}) + \delta(t-2) - \delta(3t-4)$.

8. [15!] A system has the input and output relation given by

$$y(t) = tx(t).$$

Is the system

(a) linear?

(b) time invariant?

(c) bounded input bounded output (BIBO) stable?

(d) memoryless?

(e) causal?

6. (a) For a causal system, the output $y(t)$ at any time t depends only on the "present" and various "past" inputs.

should be same for $t \le 1$

However, according to $x_2(t)$, the input remains 0 for $t \le 1$. the output becomes different on $t \in (0,1)$ from $t \in (-\infty, 0)$.

$\Rightarrow$ $\boxed{\text{not Causal}}$

(b) A system T is time invariant iff

$$x(t) \xrightarrow{T} y(t) \quad \text{implies that} \quad x(t-t_0) \xrightarrow{T} y(t-t_0)$$

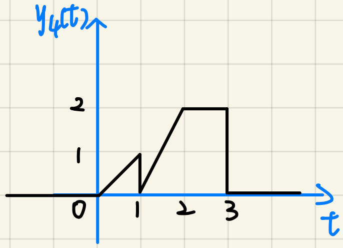
for every input signal $x(t)$ and time shift t_0.

Then we can see from $x_1(t)$ and $x_3(t)$ that $x_3(t) = x_1(t-1)$

but $y_3(t) \ne y_1(t-1)$ $\Rightarrow$ $\boxed{\text{not Time invariant}}$

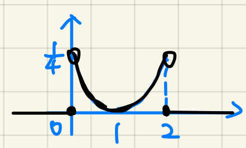
(c) not causal $\Rightarrow$ $\boxed{\text{not Memoryless}}$

(d) since H is Linear. and $x_4(t) = x_1(t) + 2x_3(t)$. we can plot the output:



7. (a)

$$s(t) = \begin{cases} \left(\frac{t-1}{2}\right)^2 & 0<t<2 \\ 0 & \text{otherwise} \end{cases}$$



(b) $s(\frac{1}{2}) = \frac{1}{16}$, $s(2) = 0$, $s(\frac{4}{3}) = \frac{1}{36}$

$$\Rightarrow \int_{-\infty}^{\infty} s(t)\, x(t)\, dt = \frac{1}{16} + 0 - \frac{1}{36} = \boxed{\frac{23}{432}}$$

8. (a)

$y_1(t) = t x_1(t)$, $y_2(t) = t x_2(t)$

When $x(t) = a_1 x_1(t) + a_2 x_2(t)$, $y(t) = a_1 t x_1(t) + a_2 t x_2(t) = a_1 y_1(t) + a_2 y_2(t)$ equal $\Rightarrow$ $\boxed{\text{Linear}}$

(b)

$y(t-t_0) = (t-t_0)\, x(t-t_0)$

$y_d = t\, x(t-t_0)$

not equal $\Rightarrow$ $\boxed{\text{not Time invariant}}$

(c)

if $|x(t)| \le M$, $|y(t)| \le t \cdot M$ $\Rightarrow$ when $t \to \infty$, $t \cdot M \to +\infty$ $\Rightarrow$ $|y(t)|$ is not bounded

$\Rightarrow$ $\boxed{\text{not Stable}}$

(d)

only depends on current input $x(t)$ and t $\Rightarrow$ $\boxed{\text{Memoryless}}$

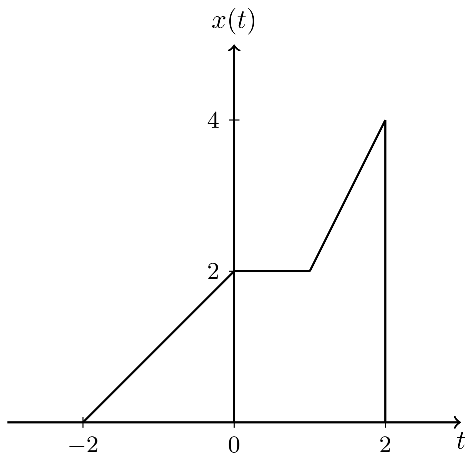
(e)

doesn't depend on future inputs $\Rightarrow$ $\boxed{\text{Causal}}$

Optional Problems:

1. Find the average value, power, and energy of signal $x(t) = \begin{cases} e^{-t} & t > 0, \\ 0 & \text{otherwise.} \end{cases}$

2. Consider the signal illustrated below.



   (a) Find a mathematical representation for $x(t)$.
   (b) Sketch $s(t) = x(-2t+1)/2$ by performing graphical time transformations. Sketch the intermediate signal each time you make a transformation, like time-shifting, or time-scaling.
   (c) Decompose $x(t)$ into its even and odd components. Carefully sketch the even and odd components of $x(t)$.

3. Considering the signals

$$x(t) = cos\frac{2\pi t}{3} + 2sin\frac{16\pi t}{3}$$
$$y(t) = sin(\pi t)$$

   Show that $z(t) = x(t)y(t)$ is periodic, and write $z(t)$ as a linear combination of harmonically related complex exponentials. That is, find a number T and complex numbers C_k such that

$$z(t) = \sum_k c_k e^{jk(2\pi/T)t}$$

4. Use MATLAB to plot the following three signals.
   (a) $y(t) = e^t$
   (b) $y(t) = e^{-0.1t} sin(\pi t)$

5. Prove that the product of two odd signals is an even signal.

6. Show that causality for a continuous-time linear system is equivalent to the following statement:

   For any time t_0 and any input $x(t)$ such that $x(t) = 0$ for $t < t_0$, the corresponding output $y(t)$ must also be zero for $t < t_0$.

7. Given a signal $x(t)$,
   (a) suppose it is an energy signal with energy $E[x(t)] = E_x$. Then what is the energy of the signal $x(-at + b)$, i.e. $E[x(-at + b)]$?
   (b) suppose it is a power signal with power $P[x(t)] = P_x$. Then what is the power of the signal $x(-at + b)$, i.e. $P[x(-at + b)]$?