VE216 SU23 Final RC

Chapter 9

Laplace Transforms

The Laplace Transform (LT) is the generalization of the Fourier Transform to include general complex exponential signals of the form

$$e^{st} = e^{(\sigma + jw)t} = e^{\sigma t} e^{jwt}$$

Laplace Transforms come in two flavors:

- Bilateral Laplace Transform (two-sided)
- Unilateral Laplace Transform (one-sided)

In this course, we only concern $\boldsymbol{Bilateral\ Laplace\ Transform}$

$$X(s) := \int_{-\infty}^{\infty} x(t) e^{-st}$$

In this way, the Inverse Laplace Transform is given

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}$$

don't use time wasting

and LT pairs can be denoted as

$$x(t) \stackrel{L}{\longleftrightarrow} X(s)$$

ROC: Region of Convergence

ROC is the set of values of s on the complex plane for which the bilateral Laplace transform is guaranteed to exist. Since the imaginary part of s does not contribute to the magnitude, we further conclude that

$$ROC = \left\{ s : \int_{-\infty}^{\infty} |x(t) e^{-real(s)t} dt < \infty \right\}$$

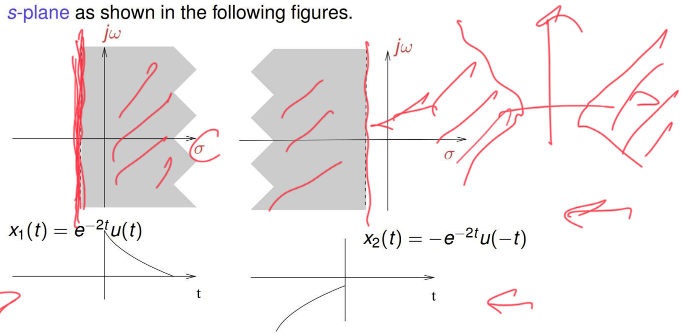
$$\int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t}$$

$$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t} e^{-j\omega t}| dt$$

$$|e^{-j\omega t}| = 1$$

Display of ROC

Often we display the ROC of a signal using the complex s-plane as shown in the following figures.



- The horizontal axis is usually called the σ axis, and the vertical axis is usually called the $j\omega$ axis.
- The shaded region indicates the set of points in the s-plane where the bilateral Laplace transform exists.
- Dotted lines for boundaries if ROC does not include its edges.
- If the shaded region includes the $j\omega$ axis, then the FT of the signal exists.

Rational LT

If the Laplace transform of a signal x(t) has the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M(s)^M + b_{M-1}(s)^{M-1} + \cdots + b_0}{a_N(s)^N + a_{N-1}(s)^{N-1} + \cdots + a_0} = \frac{b_M}{a_N} \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)}$$

where N(s) and D(s) are polynomials in s, then we say that the Laplace transform X(s) is rational.

$X(s) = s \to -\infty$

$X(s) \to 0$

$X(s) = +\infty$

$X(s) = \frac{1}{s}$ $+\infty$

- zeros: $z_1, z_2 \cdots z_M$, sometimes infinity
- poles: $p_1, p_2 \cdots p_N$, sometimes infinity
- gain: G = $\frac{b_M}{a_N}$
- A rational function is proper if $M \leq N$, otherwise improper.

$s \to +\infty$

$\frac{1}{s} \to 0$

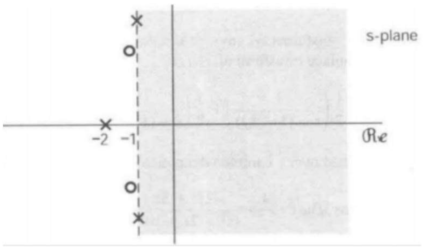
Pole-zero plot

$\frac{s}{(s+1)(s+2)}$

$\frac{(s+1)(s+2)}{s+3}$

im

Using 'o' to represent zeros, using 'X' to represent poles.



$X(s) = \frac{1}{s+2}$

$X(j\omega) = \frac{1}{j\omega+2}$

$-\omega$

$|X(j\omega)|$

- A rational LT can be completely described by its pole-zero plot, along with a gain G.
- The corresponding signal $x(t)$ is completely specified provided we know 3 things: the pole-zero plot, the gain G, and the ROC.
- The ROC of a rational Laplace transform $X(s)$ is bounded by its poles (or by infinity).

Properties of ROC

- The ROC of $X(s)$ consists of stripe-shaped regions parallel to the jω axis in the s plane.

The ROC of rational FT does not contain any poles.

$x(t) = e^{-2t}u(t+2)$ $s=0$ $s = 1$ or > 1

$e^{-(2+s)t}u(t+2)$

- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is right sided, and if the line Re{s} = σ_0 is in the ROC, then all values of s whose Re{s} > σ_0 will also be in the ROC.
- If $x(t)$ is left sided, and if the line Re{s} = σ_0 is in the ROC, then all values of s whose Re{s} < σ_0 will also be in the ROC.
- If $x(t)$ is two sided, and if the line Re{s} = σ_0 is in the ROC, then the ROC is composed of a stripe-shaped region including the line Re{s} = σ_0 on the s plane.

$e^{-2t}u(t)$ $e^{-2t-st}u(t)$ $-(2+s)$ $s > -2$

- If $X(s)$ is rational, its ROC is constrained by the poles or extents to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightest pole. If $X(s)$ is rational and left sided, its ROC is on the left of the leftest pole.

ROC RHP

Table of Laplace transform pairs

causal.

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Inverse Laplace transforms

→ causal LTI $H(s) = \frac{2s+12}{(s+5)(s+7)} = \frac{1}{s+5} + \frac{1}{s+7}$

$e^{-5t}u(t) + e^{-7t}u(t)$

Requirement: rational X(s) and ROC

Steps:

- Decompose $X(s)$ into the sum of atomic expressions $X(s) = \Sigma E_i(s)$
- Look up the Laplace transforms pairs table, find the correct $e_i(t)$ for each $E_i(s)$ following the given ROC

CAUTION: If the ROC is not given explicitly, discuss all the situations

ROC and System Properties

$h(t)$

→ $y(t) = h(t) * x(t)$ ← bounded

The Laplace transform H(s) of a system's impulse response h(t) is called its system function.

$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$ ←

$\int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau$

$x(t) = 1 \quad t \in (-\infty, \infty)$

$\int_{-\infty}^{\infty} h(\tau)d\tau$ ←

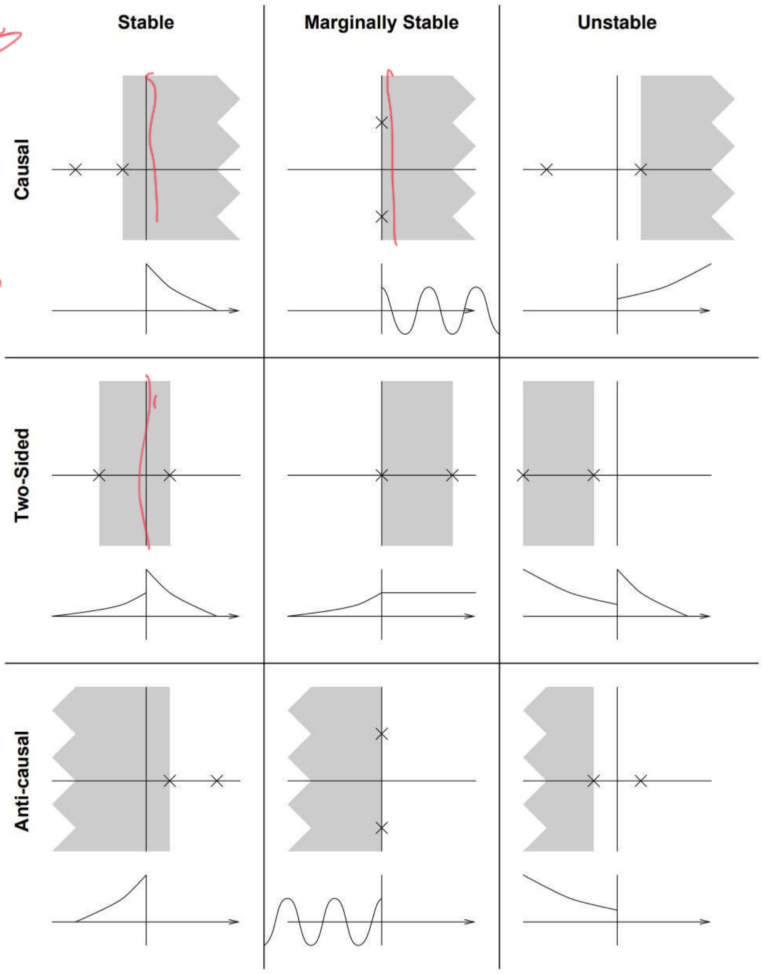
$\int_{-\infty}^{\infty} h(\tau)e^{-\sigma\tau}d\tau$

→ $s = 0$

- Stability $\longleftrightarrow$ jw axis in ROC $\longleftrightarrow$ $h(t)$ absolutely integrable
- Causality $\longleftrightarrow$ $h(t)$ is a right-sided signal $\longleftrightarrow$ ROC of the system function $H(s)$ is RHP
- Stability and Causality $\longleftrightarrow$ All of its poles lie strictly within the LHP.

CAUTION: Consider poles at $\pm\infty$! e.g. $H(s)$ = s has a pole at s = ∞.

- All improper systems are unstable!
- A diffeq system is stable iff all roots of its characteristic polynomial are in the left half plane.



Geometric properties of FT from pole-zero plot

Given the pole-zero plot corresponding the the transfer function $H(s)$ of an LTI system, one can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$ of the system.

- $|H(\omega)|$

For rational transfer function $H(s)$, $|H(s)| = \left|\frac{b_M}{a_N} \cdot \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}\right| = \left|\frac{b_M}{a_N}\right| \cdot \frac{|(s-z_1)||(s-z_2)|\cdots|(s-z_M)|}{|(s-p_1)||(s-p_2)|\cdots|(s-p_N)|}$

With $s = jw$, We can focus on distances between jw and each zeros or poles on the pole-zero plot, then plug them into the $|H(w)|$

- $\angle H(\omega)$

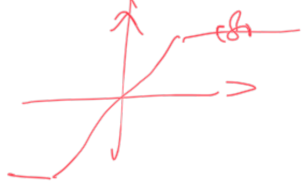
find the angles formed between $s - z_i \; or \; p_i$ and positive x axis. Then $\angle H(w) = \angle G + \angle(s-z_1) + \cdots + \angle(s-z_M) - \angle(s-p_1) - \cdots - \angle(s-p_N) \quad s = jw$

→ $s = jw$

$s \to +\infty$ $\quad$ $s \to -\infty$

Important properties of LT

- linearity



$$x_1(t) + x_2(t) \overset{L}{\longleftrightarrow} X_1(s) + X_2(s)$$

- differentiation

$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s)$$

- convolution

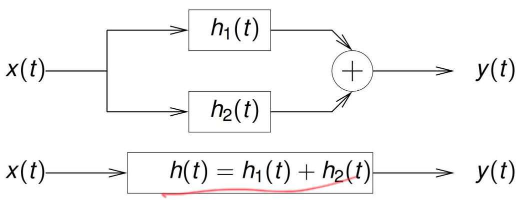
$$x(t) * y(t) \overset{L}{\longleftrightarrow} X(s)Y(s)$$

Those are similar to Fourier Transform because LT is an extension of FT.

System functions and block diagram representations

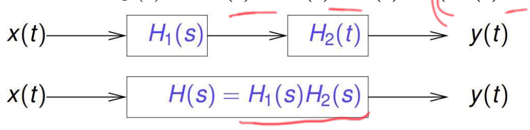
- parallel interconnection

For a system $y(t) = h_1(t) * x(t) + h_2(t) * x(t) = (h_1(t) + h_2(t)) * x(t)$



- series combination

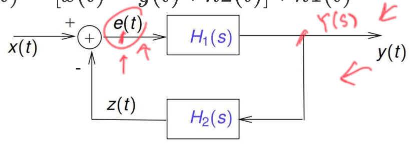
For a system $y(t) = h_2(t) * h_1(t) * x(t) = (h_2(t) * h_1(t)) * x(t)$



- feedback innerconnection

A feedback system uses the output of a system to control or modify the input.

i.e. $y(t) = [x(t) - y(t) * h2(t)] * h1(t)$



$E(s)H_1(s) = Y(s)$ ①

$E(s) = X(s) - H_2(s) \cdot Y(s)$ ②

$E(s) = \frac{Y(s)}{H_1(s)}$

$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \implies y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$\frac{Y(s)}{H_1(s)} = X(s) - H_2(s) \cdot Y(s)$

Block diagram representations for diffeq systems

The differentiator is both difficult to implement and extremely sensitive to noise. So we often use the integration block rather than differentiation block, $\frac{1}{s}$ instead of s.

To draw a block representation of $H(s) = \frac{2s^2+4s-6}{s^2+3s+2}$

s^2

s^2

- Change the form from "s" to "$\frac{1}{s}$"

$$H(s) = \frac{2 + \frac{4}{s} - \frac{6}{s^2}}{1 + \frac{3}{s} + \frac{2}{s^2}}$$

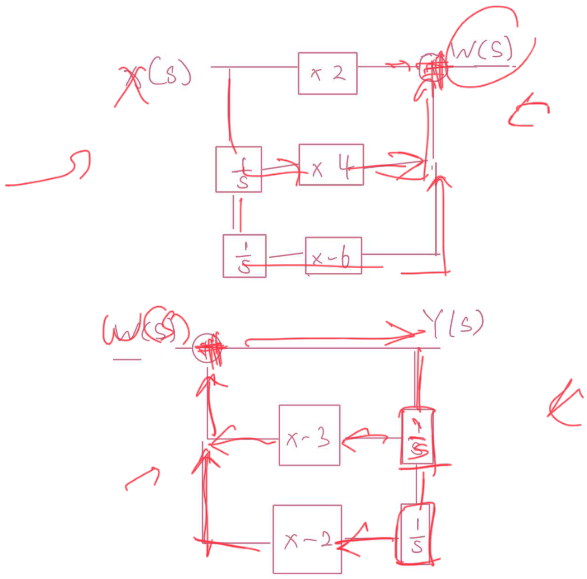
- Introduce $W(s)$ to split the system into two parts

$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{W(s)} \frac{W(s)}{X(s)}$

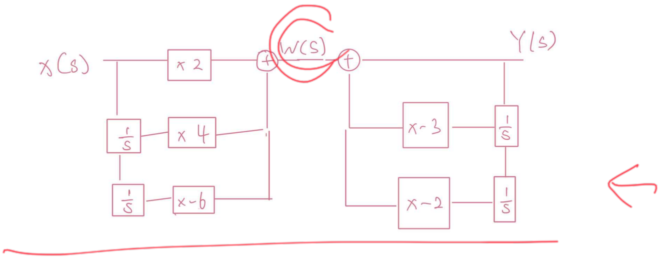
$$\frac{W(s)}{X(s)} = 2 + \frac{4}{s} - \frac{6}{s^2} \qquad \frac{Y(s)}{W(s)} = \frac{1}{1 + \frac{3}{s} + \frac{2}{s^2}}$$

$$W(s) = 2X(s) + \frac{4X(s)}{s} - \frac{6X(s)}{s^2} \qquad Y(s) = W(s) - \frac{3Y(s)}{s} - \frac{2Y(s)}{s^2}$$

- Draw each part of the system



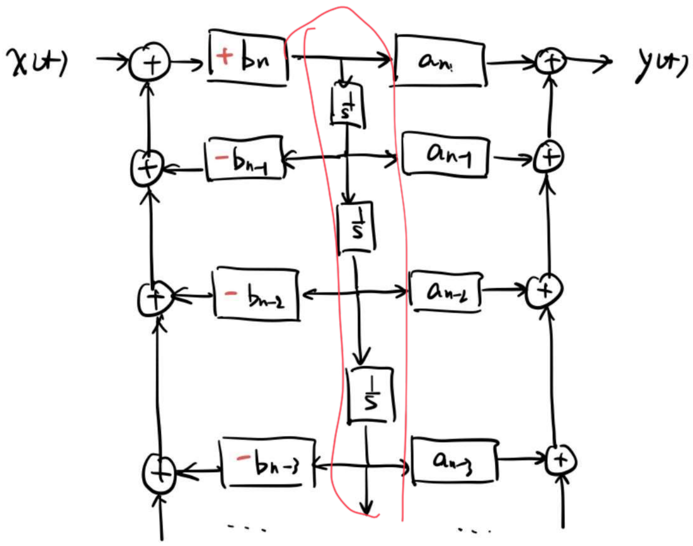
- Connect



Or you can choose a quicker method

more generalized.

$$H(s) = \frac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0}$$ (same order)



Chapter 6&7&8

Root Mean Square width

$$w_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} w^2|X(w)|^2dw}{\int_{-\infty}^{\infty} |X(w)|^2dw}} \qquad \tau_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} t^2|x(t)|^2dt}{\int_{-\infty}^{\infty} |x(t)|^2dt}}$$

$$w_{rms}\tau_{rms} \geq \frac{1}{2}$$

Sampling

- ideal sampling function

$$p(t) = \Sigma_{n=-\infty}^{\infty}\delta(t-nT_s) \overset{L}{\longleftrightarrow} P(w) = \Sigma_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(w-k\frac{2\pi}{T_s})$$

- sampled function

a series of points

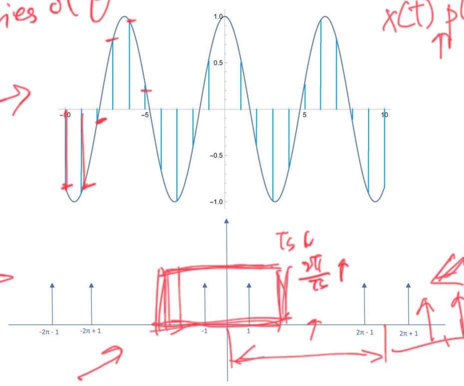
x(t) p(t)

p(t)x(t)

$\frac{1}{2\pi} P(s) * X(s)$

x[n]

Ts

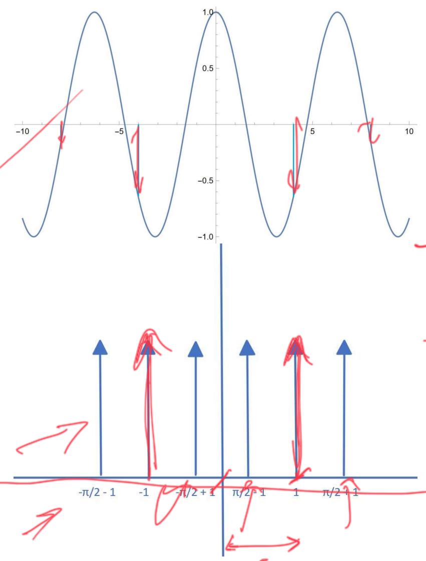


$w(jw)$

$\frac{2\pi}{T_s}$

- aliasing

$* \delta(t)$



Ts

$\frac{2\pi}{T_s}$

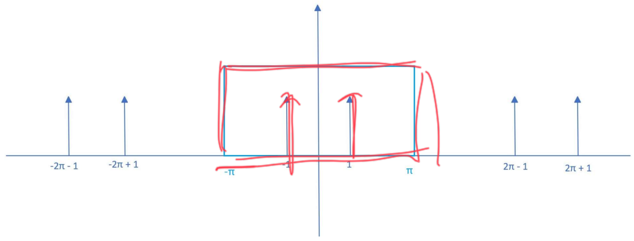
- Sinc Interplation (Ideal Lowpass Filter)

$w_{max} < w_c < w_s - w_{max}$ and usually $w_c = \frac{w_s}{2}$

$$h(t) = \frac{\omega_c T_s}{\pi}\,\text{sinc}\left(\frac{\omega_c t}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega) = T_s\,\text{rect}\left(\frac{\omega}{2\omega_c}\right)$$

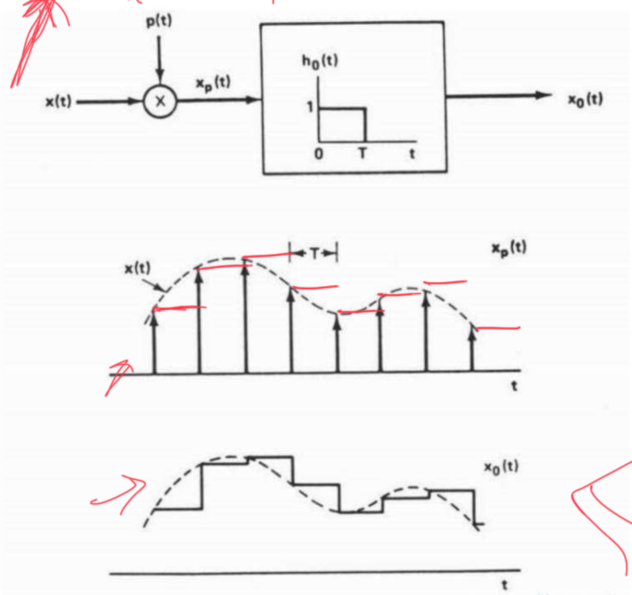
$h(t) * x(t)$

$H(w) \cdot X(w)$



- Linear Interplation (First-order hold filter)

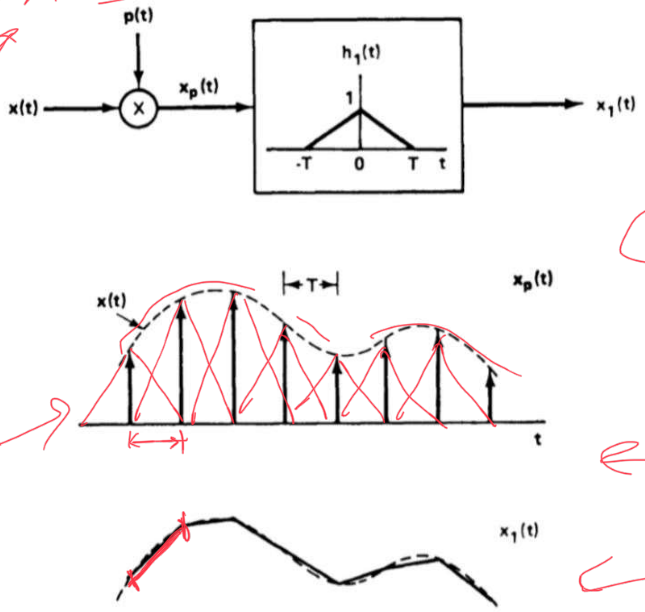
$$h_1(t) = \mathrm{tri}(t/T_s) \overset{\mathcal{F}}{\longleftrightarrow} H_1(\omega) = T_s\,\mathrm{sinc}^2\left(\frac{\omega}{\omega_s}\right)$$



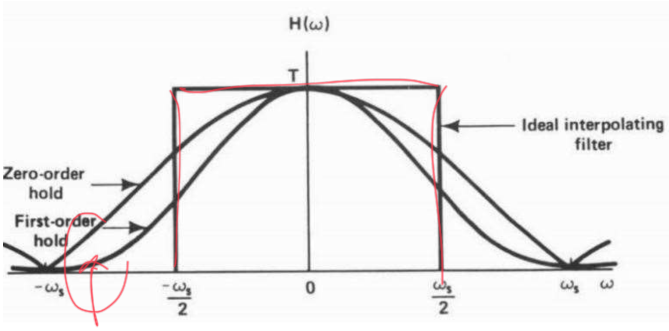
- Nearest Neighbor Interplation (Zero-order Hold Filter)

$$h_2(t) = \mathrm{rect}\left(\frac{t}{T_s} - \frac{1}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} H_2(\omega) = T_s\,\mathrm{sinc}\left(\frac{\omega}{\omega_s}\right) e^{-j\omega T_s/2}$$

sorry, wrong picture
swap these two



- Comparasion of interpolation filter on frequency domain

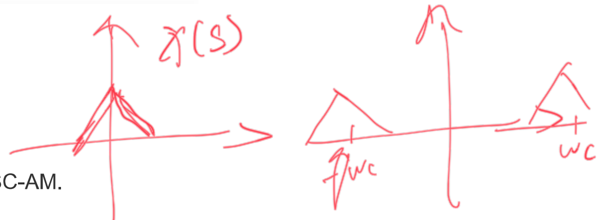


Modulation

● Synchronous

We modulate the signal with a carrier $c(t) = cos(w_c t + \theta_c)$. K $\theta_c = 0$

This method is called doublesideband, suppressed carrier, amplitude modulation or DSB/SC-AM.

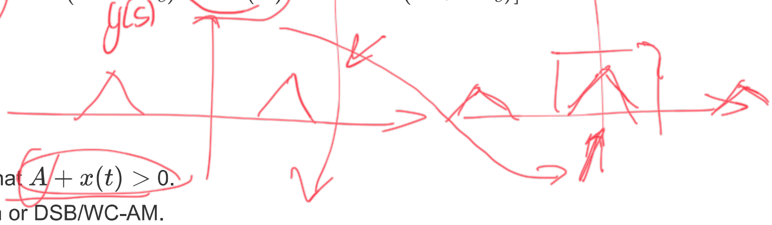


$$y(t) = x(t)c(t) = x(t)cos(w_c t + \theta_c) \overset{F}{\longleftrightarrow} Y(w) = \frac{1}{2}[e^{j\theta_c}X(w - w_c) + e^{-j\theta_c}X(w + w_c)]$$

When we do synchronous demoulation, we mutiply $y(t)$ with another $c(t) = cos(w_c t + \theta_c)$

$$w(t) = y(t)cos(w_c t + \theta_c) \overset{F}{\longleftrightarrow} W(w) = \frac{1}{4}[e^{2j\theta_c}X(w - 2w_c) + 2X(w) + e^{-2j\theta_c}X(w + 2w_c)]$$

y(s)



Then we use a lowpass filter to extract $X(w)$

● Asynchronous

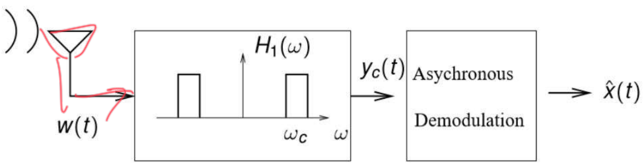
We modulate the signal with $y(t) = (A + x(t))cos(w_c t)$ and ensure that $A + x(t) > 0$.

This method is called double sideband, with carrier, amplitude modulation or DSB/WC-AM.

$$Y(w) = A\pi[\delta(w - w_c) + \delta(w + w_c)] + \frac{1}{2}[X(w - w_c) + X(w + w_c)]$$

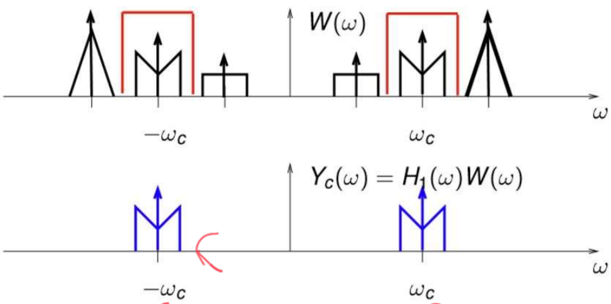
● Tunable bandpass filter

Tuning (Design #1)



narrow

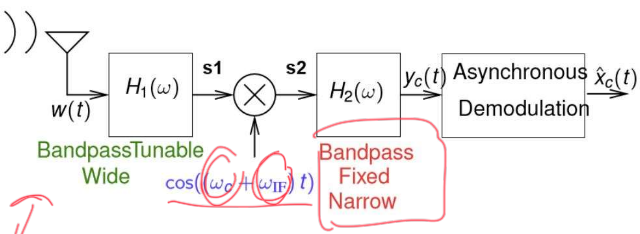
(Frequency Division Multiplexing)

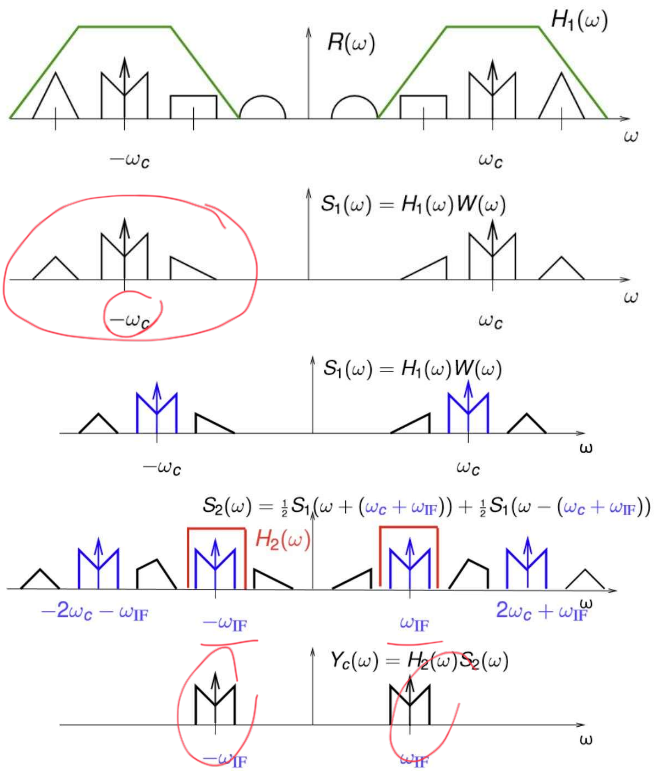


● Superheterodyning receiver

demodulate cos(wt)

Superheterodyning Tuner





$\cos(\omega_{IF})$

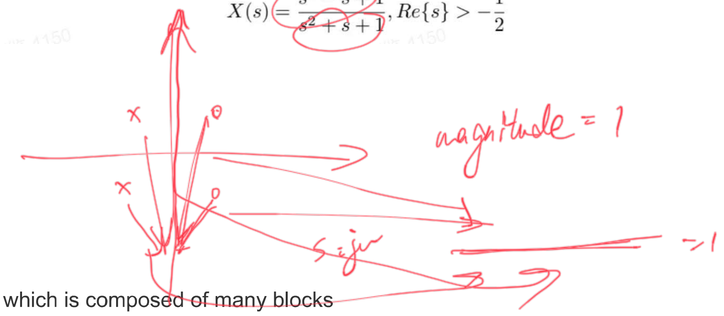
Exercise

1,

Use geometric evaluation from the pole-zero plot to determine the magnitude of the Fourier transform of the signal whose Laplace transform is specified as

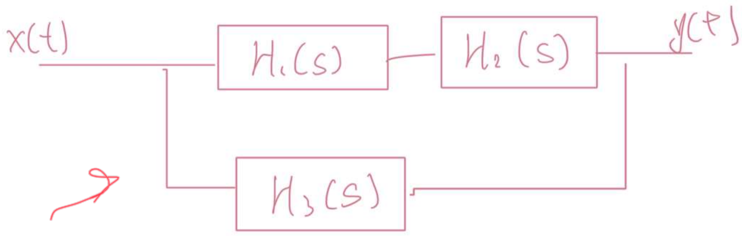
zeros $\frac{1 \pm \sqrt{3}i}{2}$ $\frac{-1 \pm \sqrt{3}i}{2}$ poles

$$X(s) = \frac{s^2 - s + 1}{s^2 + s + 1}, Re\{s\} > -\frac{1}{2}$$

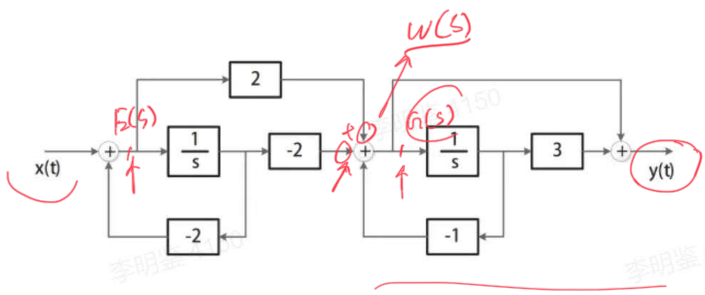


2,

There is an causal LTI system $H(s)$ which is composed of many blocks



For $H_1(s)$, its block diagram is given as



For $H_2(s)$, it is second order system which has its poles at -5 and -7, zeros at -6 and $+\infty$, and gain $= 2$

s^2

$$H_2(s) = 2\frac{(s+6)}{(s+5)(s+7)}$$

$$2\cdot 10^6 Y(s) + 10^5 sY(s) + 60 s^2 Y(s) + s^3 Y(s) = 8\cdot 10^6 X(s) - 10^4 sX(s)$$

For $H_3(s)$, it can be described by a diff equation

$$H_3(s) = \frac{8\cdot 10^6 - 10^4 s}{2\times 10^6 + 10^5 s + 60 s^2 + s^3}$$

$$2\cdot 10^6 y(t) + 10^5 \frac{d}{dt}y(t) + 60\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = 8\cdot 10^6 x(t) - 10^4 \frac{d}{dt}x(t)$$

- What is the $H(s)$ of this system? $H(s) = H_3(s) + H_1(s)\cdot H_2(s)$
- If a signal $e(t) = e^{-6|t|}$ is applied to $H_2(s)$, what's the result signal on time domain?

$$E(s) = X(s) + -2\times\frac{E(s)}{s} \qquad W(s) = -2\cdot\frac{E(s)}{s} + 2\times E(s)$$

$$E(s) = \frac{X(s)}{\frac{2}{s}+1} \qquad = \left(\frac{-2}{s}+2\right)E(s) = \left(\frac{-2}{s}+2\right)\frac{X(s)}{\frac{2}{s}+1} = \frac{-2+2s}{2+s}X(s)$$

$$-\frac{G(s)}{s} + W(s) = G(s) \qquad Y(s) = \frac{G(s)}{s}\times 3 + G(s)$$

$$G(s) = \frac{W(s)}{1+\frac{1}{s}} \qquad = G(s)\left(\frac{3}{s}+1\right) = \frac{W(s)}{1+\frac{1}{s}}\cdot\frac{3}{s}+1$$

$$Y(s) = \frac{3+s}{s+1}\cdot\frac{2(s-1)}{s+2}X(s) \qquad = \frac{3+s}{s+1}W(s)$$

$$H_1(s) = \frac{Y(s)}{X(s)}$$

$$2)\ E(s) = e^{-6t}u(t) + e^{6t}u(-t).$$

$$\frac{1}{s+6} \quad \frac{1}{6-s}$$

$-6<s<6$ ROC

$s>-5$

$-5<s<6$

$$-7 \quad -5$$

$$\left(\frac{1}{s+6}+\frac{1}{6-s}\right) 2\cdot\frac{(s+6)}{(s+7)(s+5)} \qquad \frac{24}{(6-s)(s+5)(s+7)}$$

$$= \frac{24}{143(6-s)} + \frac{12}{11(s+5)} - \frac{12}{13(7+s)} = \frac{24}{143}e^{6t}u(-t) + \frac{12}{11}e^{-5t}u(t) - \frac{12}{13}e^{-7t}u(t)$$

$u(t)$ or $u(-t)$

$e^{-7t}u(-t)$

3,

A violinist visits an AM radio station to make a “live” performance. The violinist plays a note with a fundamental frequency of 1200Hz. (A violin signal can be modeled as a sawtooth wave with a maximum amplitude of 1). To abide by FCC regulations, the radio station filters the signal to remove all components above 7kHz, The filtered signal is broadcasted using DSB/SC-AM at a carrier frequency of 1000kHz. Determine the interference signal that would be heard by someone who turned into the AM station centered at 1010kHz. phases of the carrier and demodulation signal are both 0. the receiver's lowpass filter(s) are ideal ($H(\omega)$ = rect($\frac{\omega}{2\omega_M}$) where f_M = 5kHz).

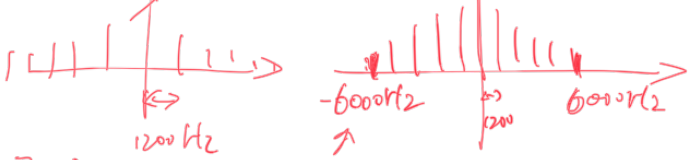
$k=0$ $\frac{X_0}{2}$ $\quad j\frac{X_0}{2\pi k}$ $k\neq 0$

$k=5$ or -5

$\omega = 1200$ Hz

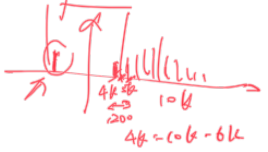
$1200k < 7000$

$k\leq 5$



$$\frac{1}{4}j\frac{X_0}{2\pi\cdot 5} \quad (4kHz)$$

$$\frac{1}{4}j\frac{X_0}{2\pi 5} \quad (-4kHz)$$



$$\frac{1}{4}j\frac{X_0}{2\pi\cdot 5}e^{4000\cdot 2\pi t} + \frac{1}{4}j\frac{X_0}{2\pi 5}e^{-4000\cdot 2\pi t}$$

$$\frac{X_0}{20\pi}\sin(4000\cdot 2\pi\cdot t)$$

Good luck!!!

always remember to check your answer!

Reference

[1] ve216.chap6.study.pdf
[2] ve216.chap7.study.pdf
[3] ve216.chap8.study.pdf
[3] ve216.chap9.study.pdf