Homework 2

HW Notes:

- $\boxed{\text{Box}}$ your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

Problems:

1. [14!]Here are input-output relationships for a few systems, all of which are linear. Some of them are time-invariant, some are not. Determine which are which. Find the impulse response of the time-invariant systems.

   (a) $y(t) = \int_{-\infty}^{t} \left[\int_{-\infty}^{s} x(\tau - 5)d\tau\right] ds$,

   (b) $y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} x(\tau)d\tau$.

   Hint: Note that if you can transform the above relationships into the exact form of convolution $y(t) = g(t) * x(t)$, then the system is immediately time-invariant with $g(t)$ being the impulse response $h(t)$. That is because different from algebraic operators like multiplication, the convolution operator implies time-invariance itself.

2. [16!] Often we will be convolving two signals that are zero everywhere except over some finite range (called **finite support** signals). Suppose $x_1(t)$ is non-zero over the range $a \le t \le b$ and that $x_2(t)$ is non-zero over the range $c \le t \le d$. Suppose $y(t) = x_1(t) * x_2(t)$.

   (a) Find the range of values of t for which $y(t)$ is possibly non-zero.

   (b) Compute $rect((t-2)/2) * rect((t+3)/4)$ (express answer with braces and carefully sketch). Check your result with part (a).

3. [18!]

   (a) Consider a linear system with input $x(t)$ and output $y(t)$ given by

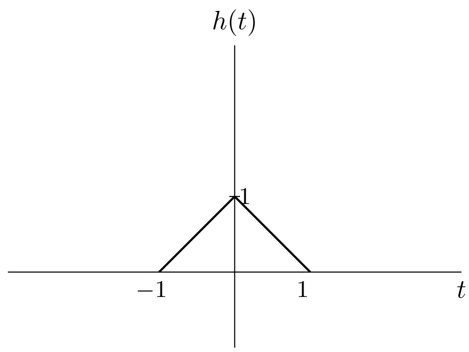
   $$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT).$$

   Is this system time-invariant?

   (b) Consider another LTI system. Let its impulse response $h(t)$ be the triangular pulse shown below, and $x(t)$ be the **impulse train**

   $$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

   SKETCH $y(t) = x(t) * h(t)$ for $T = 4, 2, 1.5$ and 1.(No formulae are needed though you still want to label your graphs clearly.)



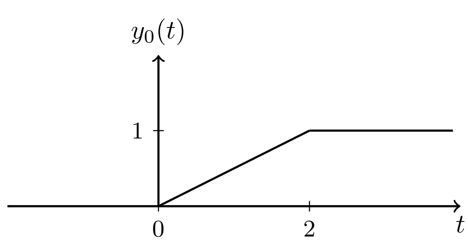
4. [12!] Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.
   (a) $y(t) = \int_{-\infty}^{t} (t-\tau)e^{-(t-\tau)}x(\tau)d\tau$
   (b) $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}x(\tau)d\tau$

5. [16!] Consider an LTI system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If
$$x(t) \to y(t)$$
and
$$\frac{dx(t)}{dt} \to -3y(t) + e^{-2t}u(t)$$
determine the impulse response $h(t)$ of S.

6. [12!] We are given a certain LTI system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched below. We are then given the following set of inputs to LTI systems with the indicated impulse responses.

	Input $x(t)$	Impulse response $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t+1)$
(c)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(e)	$x(t) = x_0'(t)$	$h(t) = h_0(t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If so, provide an accurate sketch of it with numerical values clearly indicated on the graph.



7. [12!] Suppose you have a "black box" LTI system and you discover that for a unit-step input signal the response of the system is the function $(3-t)rect(\frac{t-1}{2})$. Determine the impulse response of the system. (*Hint: See Optional Problem 2.*)

Optional Problems:

1. Assume a system has the input-output relationship $y(t) = f(t)x(t)$, where $x(t)$ is the input and $y(t)$ is the output. $f(t)$ is not constant, i.e., there exists t_0, t_1 that $f(t_0) \neq f(t_1)$. Show that this system is time-variant. That is, a system with time-variant gain cannot be time-invariant. (*Hint: find a counterexample.*)

2. Let $y(t) = (x * h)(t)$. Show the following properties of convolution.

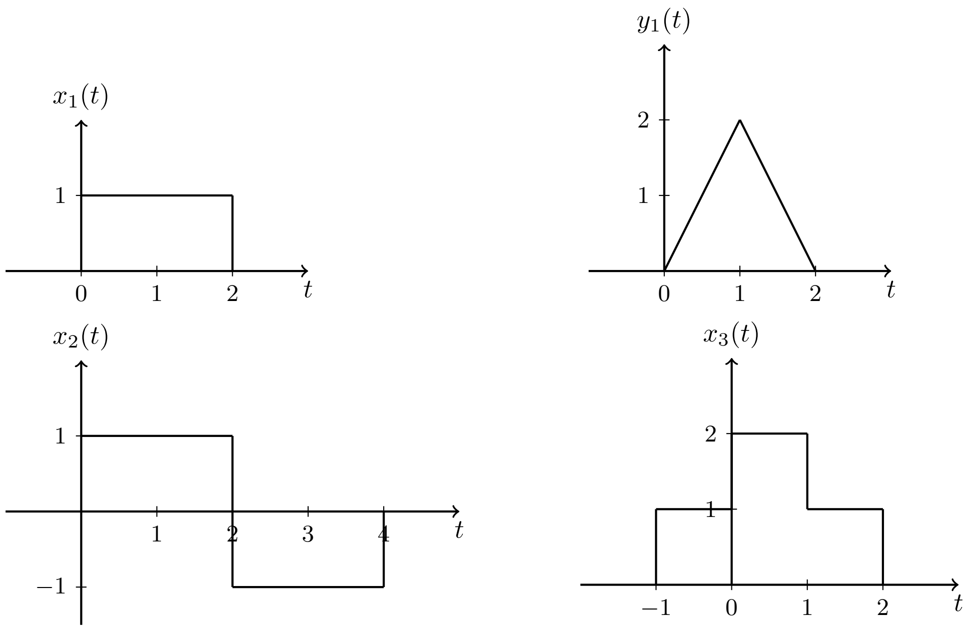
   (a) $\int_{-\infty}^{\infty} y(t) = \left[\int_{-\infty}^{\infty} x(t)dt\right]\left[\int_{-\infty}^{\infty} h(t)dt\right]$,

   (b) $\frac{d}{dt}y(t) = \left[\frac{d}{dt}x(t)\right] * h(t) = x(t) * \left[\frac{d}{dt}h(t)\right]$,

3. Consider an LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ which are illustrated below.

   (a) Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted below.

   (b) Determine and sketch carefully the response of the system to the input $x_3(t)$ depicted below.



4. Determine if each of the following statements concerning LTI systems is true or false. Justify your answers.

   (a) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and non-zero, the system is unstable.

   (b) The inverse of a causal LTI system is always causal.

   (c) If an LTI system is causal, it is stable.

   (d) The cascade of a noncausal LTI system with a causal one is necessarily noncausal.

5. Consider an LTI system described by the following differential equation and an input signal $x(t)$.

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t), \qquad x(t) = e^{-t}u(t) \tag{1}$$

   (a) Determine the family of signals y(t) that satisfies the associated homogeneous equation.

   (b) Assume that for t >0, one solution of eq. 1, with $x(t)$ as specified, is of the form

$$y_1(t) = Ae^{-t}, \qquad t>0$$

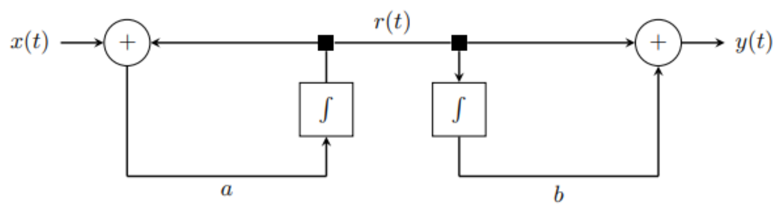
   Determine the value of A.

(c) By substituting into eq. 1, show that

$$y_1(t) = [2e^{-t/2} - 2e^{-t}]u(t)$$

is one solution for all t.

6. Consider the system shown in the figure below.



(a) Find the differential equation relating $x(t)$ and $y(t)$.

(b) Suppose the system is in initial rest. If $a = 2, b = 1, x(t) = e^t cos(t)u(t)$, find the full response of this system.