

Homework 4 Solution

Problems:

1. (a) $x(t) = 2\text{rect}\left(\frac{t-2}{4}\right)$ so $X(\omega) = 8e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$ by time scaling and shifting property.

(b) $x(t) = e^{-3t}\text{rect}\left(\frac{t-2}{4}\right) = e^{-3t}u(t) - e^{-3(t-4)}u(t-4)e^{-12}$ so $X(\omega) = \frac{1}{3+j\omega} [1 - e^{-12}e^{-4j\omega}]$

(c) $x(t) = t\text{rect}\left(\frac{t-2}{4}\right) = ty(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

Thus $X(\omega) = \frac{d}{d\omega}jY(\omega) = \frac{d}{d\omega}j2e^{-2j\omega}\frac{\sin(2\omega)}{\omega} = \frac{e^{-4j\omega}(1+4j\omega) - 1}{\omega^2}$

(d) $x(t) = \cos(4\pi t)y(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

Hence $X(\omega) = \frac{1}{2}(Y(\omega - 4\pi) + Y(\omega + 4\pi)) = 2e^{-2j\omega} \left(\text{sinc}\left(\frac{2\omega - 8\pi}{\pi}\right) + \text{sinc}\left(\frac{2\omega + 8\pi}{\pi}\right) \right)$

2. If $f(t)$ is odd, $f(t) = -f(-t)$.

$$\left. \begin{aligned} f(t) &\xleftrightarrow{\mathcal{F}} F(\omega) \\ f(-t) &\xleftrightarrow{\mathcal{F}} F(-\omega) \end{aligned} \right\} \Rightarrow F(\omega) = -F(-\omega) \Rightarrow F(\omega) \text{ is odd.}$$

If $f(t)$ is real, $f(t) = f^*(t)$.

$$\left. \begin{aligned} f(t) &\xleftrightarrow{\mathcal{F}} F(\omega) \\ f^*(t) &\xleftrightarrow{\mathcal{F}} F^*(-\omega) \end{aligned} \right\} \Rightarrow F(\omega) = -F(-\omega) = F^*(-\omega) \Rightarrow F(\omega) = -F^*(\omega) \Rightarrow F(\omega) \text{ is purely imaginary.}$$

3.

$$x(t) = t\text{sinc}^2(t) \xleftrightarrow{\mathcal{F}} X(\omega) = j\frac{d}{d\omega}\text{tri}\left(\frac{\omega}{2\pi}\right) = \pm j\frac{1}{2\pi}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

So $E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \frac{1}{4\pi^2} d\omega = \frac{1}{2\pi^2}$

4. (a)

$$H(s) = \frac{s^2 + s + 1}{(s^2 + 6s + 25)(s^2 + 2)} = \frac{s^2 + s + 1}{(s+2)(s+3-4j)(s+3+4j)} = \frac{r_1}{s+2} + \frac{r_2}{s+3-4j} + \frac{r_2^*}{s+3+4j}$$

with $r_1 = \frac{s^2 + s + 1}{s^2 + 6s + 25} \Big|_{s=-2} = \frac{3}{17}$ and $r_2 = \frac{s^2 + s + 1}{(s+3+4j)(s+2)} \Big|_{s=-3+4j} = \frac{7}{17} + \frac{71}{136}j = 0.665e^{j0.903}$.

Thus

$$H(j\omega) = \frac{3/17}{j\omega + 2} + \frac{0.665e^{j0.903}}{j\omega + 3 - 4j} + \frac{0.665e^{-j0.903}}{j\omega + 3 + 4j}$$

Taking the inverse FT yields

$$h(t) = \frac{3}{17}e^{-2t}u(t) + 1.33e^{-3t}\cos(4t + 0.903)u(t)$$

(b) Expanding the denominator polynomial:

$$\frac{Y(\omega)}{X(\omega)} = \frac{(j\omega)^2 + j\omega + 1}{(j\omega)^3 + 8(j\omega)^2 + 37j\omega + 50}$$

so by cross multiplying

$$50y(t) + 37\frac{d}{dt}y(t) + 8\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = x(t) + \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$$

5. [(a)]

$$[e^{-at} \cos \omega_0 t]u(t) = \frac{1}{2}e^{-at}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-at}e^{-j\omega_0 t}u(t)$$

$$X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} + \frac{1}{2(\alpha + j\omega_0 + j\omega)} = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}$$

(b)

$$x(t) = x_1(t) + x_2(t)$$

where,

$$x_1(t) = e^{-3t} \sin(2t)u(t) \xrightarrow{\mathcal{F}} X_1(j\omega) = \frac{1/(2j)}{3 - j2 + j\omega} - \frac{1/(2j)}{3 + j2 + j\omega} = \frac{2}{(3 + j\omega)^2 + 4}$$

$$x_2(t) = e^{3t} \sin(2t)u(-t) \xrightarrow{\mathcal{F}} X_2(j\omega) = -X_1(-j\omega) = -\frac{1/(2j)}{3 - j2 - j\omega} + \frac{1/(2j)}{3 + j2 - j\omega} = \frac{-2}{(3 - j\omega)^2 + 4}$$

Thus,

$$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}$$

6. [(a)]

$$X(j\omega) = \frac{e^{(4\omega + \pi/3)j} + e^{-(4\omega + \pi/3)j}}{2}$$

$$x(t) = \frac{1}{2}e^{-j\pi t/12}\delta(t + 4) + \frac{1}{2}e^{j\pi t/12}\delta(t - 4)$$

(b) From the given figure we can get

$$X(j\omega) = \begin{cases} \omega e^{-3j\omega} & , |\omega| < 1 \\ 0 & , |\omega| > 1 \end{cases}$$

$$x(t) = \frac{\cos(t - 3) - 1 - (t - 3)\sin(t - 3)}{\pi(t - 3)^2}$$

7. From the figure we can get the frequency response of the lowpass differentiator is,

$$H(j\omega) = \begin{cases} \frac{j\omega}{3\pi} & , -3\pi \leq \omega \leq 3\pi \\ 0 & , otherwise \end{cases}$$

- (a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/2\pi}\pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$. It is zero outside the region $-3\pi \leq \omega \leq 3\pi$. Thus,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{j\omega}{3\pi}X(j\omega)$$

$$y(t) = \frac{1}{3\pi} \frac{dx(t)}{dt} = -\frac{2}{3} \sin(2\pi t + \theta)$$

- (b) Since $x(t) = \cos(4\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/4\pi}\pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$. The nonzero portions of $X(j\omega)$ lie outside the region $-3\pi \leq \omega \leq 3\pi$. This implies,

$$Y(j\omega) = X(j\omega)H(j\omega) = 0$$

$$y(t) = 0$$

- (c) The Fourier series coefficients of the signal $x(t)$ are given by

$$c_k = \frac{1}{1} \int_0^{0.5} \sin(2\pi t) e^{-jk2\pi t} dt$$

Also we have

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Where $\omega_0 = 2\pi$. Thus, we can see that only the portions when $k = 0, \pm 1$ lie in the region $-3\pi \leq \omega \leq 3\pi$.

$$c_0 = \frac{1}{\pi}, c_1 = c_{-1}^* = \frac{-1}{4j}$$

The portion that can pass the filter is $x_{lp}(t) = \frac{1}{\pi} + \frac{1}{2} \sin(2\pi t)$. Finally, $y(t) = \frac{1}{3\pi} \frac{dx_{lp}(t)}{dt} = \frac{1}{3} \cos(2\pi t)$

Optional Problems:

- From the FT table, $\frac{1}{\pi} \text{rect}\left(\frac{t}{\pi}\right) \xleftrightarrow{\mathcal{F}} \text{sinc}\left(\frac{\omega}{2}\right)$ and $\frac{1}{\pi} \text{tri}\left(\frac{t}{\pi}\right) \xleftrightarrow{\mathcal{F}} \text{sinc}^2\left(\frac{\omega}{2}\right)$ Hence $F(\omega) = \text{sinc}^3(\omega/2) \xleftrightarrow{\mathcal{F}} \frac{1}{\pi} \text{rect}\left(\frac{t}{\pi}\right) * \frac{1}{\pi} \text{tri}\left(\frac{t}{\pi}\right)$. By using graphical convolution,

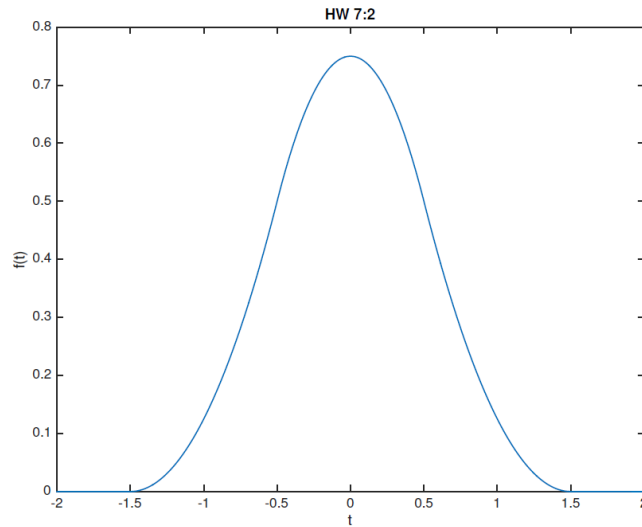
$$\text{rect}\left(\frac{t}{\pi}\right) * \text{tri}\left(\frac{t}{\pi}\right) = \begin{cases} \frac{1}{2\pi} \left(\frac{3}{2}\pi - t\right)^2 & \frac{1}{2}\pi < t < \frac{3}{2}\pi \\ \frac{1}{2\pi} \left(\frac{3}{2}\pi + t\right)^2 & -\frac{3}{2}\pi < t < -\frac{1}{2}\pi \\ \frac{3}{4}\pi - \frac{t^2}{\pi} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$$

and it follows that

$$f(t) = \begin{cases} \frac{1}{2\pi^3} \left(\frac{3}{2}\pi - |t| \right)^2 & \frac{1}{2}\pi < |t| < \frac{3}{2}\pi \\ \frac{3}{4\pi} - \frac{t^2}{\pi^3} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$$

The matlab code is given by:

```
t = linspace(-2, 2, 301);
rect = inline('abs(t) <= 1/2');
f = (3/4 - t.^2) .* rect(t) + (t.^2/2 - 3 * abs(t)/2 + 9/8) .* rect(abs(t) - 1);
plot(t, f)
xlabel('t'), ylabel('f(t)'), title('HW 7:2')
print('h0702', '-deps')
```



2. According to the FT table,

$$e^{-t^2/4} \xleftrightarrow{\mathcal{F}} 2\sqrt{\pi}e^{-\omega^2} = F_1(\omega)$$

and also

$$(-jt)^2 e^{-t^2/4} \xleftrightarrow{\mathcal{F}} \frac{d^2}{d\omega^2} F_1(\omega) = -4\sqrt{\pi}e^{-\omega^2}(1 - 2\omega^2)$$

Thus $F(\omega) = -\frac{d^2}{d\omega^2} F_1(\omega) = \boxed{4\sqrt{\pi}e^{-\omega^2}(1 - 2\omega^2)}$

3.

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)] \xleftrightarrow{\mathcal{F}} F_e(\omega) = \frac{1}{2}[F(\omega) + F(-\omega)]$$

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)] \xleftrightarrow{\mathcal{F}} F_o(\omega) = \frac{1}{2}[F(\omega) - F(-\omega)]$$

Since $f(t)$ is real, $F(-\omega) = F^*(-\omega)$, and it follows that

$$f_e(t) \xleftrightarrow{\mathcal{F}} F_e(\omega) = \frac{1}{2}[F(\omega) + F^*(\omega)] = \text{real}\{F(\omega)\}$$

$$f_o(t) \xleftrightarrow{\mathcal{F}} F_o(\omega) = \frac{1}{2}[F(\omega) - F^*(\omega)] = j\text{imag}\{F(\omega)\}$$

4. $f(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} F(\omega) = \frac{1}{j\omega + 1}$, so the fraction in the band $[-7, 7]$ is

$$\frac{\int_{-7}^7 |F(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega} = \frac{\int_{-7}^7 \frac{1}{1 + \omega^2} d\omega}{\int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} d\omega} = \frac{\tan^{-1} \omega \Big|_{-7}^7}{\tan^{-1} \omega \Big|_{-\infty}^{\infty}} = \frac{2 \tan^{-1} 7}{\pi} = \boxed{90.97\%}$$

5. $P_y(\omega) = P_f(\omega)|H(\omega)|^2$, from the diagram $P_f(\omega) = 2\delta(\omega - 150) + 2\delta(\omega + 150) + 0.5\delta(\omega - 200) + 0.5\delta(\omega + 200)$. Note $|H(150)| = 18$ and $|H(200)| = 20$, so

$$\boxed{P_y(\omega) = 648\delta(\omega - 150) + 648\delta(\omega + 150) + 200\delta(\omega - 200) + 200\delta(\omega + 200)}.$$

The sketch is shown below.

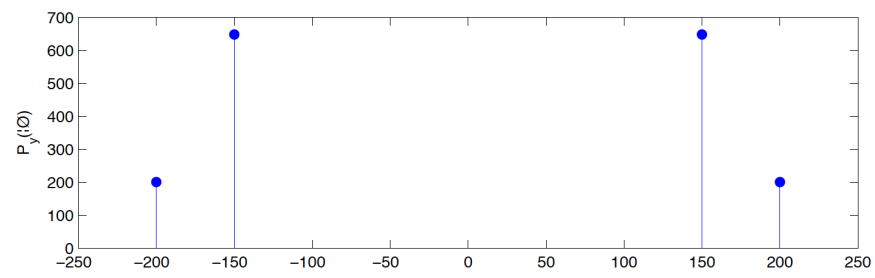


Figure 1: Problem 10