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Bachelor MAA107

Trinomial model for option pricing

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This paper presents a lattice-based computational model used in financial mathematics to price options. It is an extension of the binomial options pricing model which was popular for some time but in the last 18 years has become significantly outdated and is of little practical use, and is conceptually similar. After the derivation of our model, we will introduce Black-Scholes model which allows us to compute the price of call option explicitly and without implementing a trinomial tree. We will calculate the "greeks" which are a necessary tool for any trader or structurer used to eliminate risk and they give instructions as to how the trader or structurer can disperse the risk . Finally, we will do numerical analysis of what we derived.

1 Introduction

Options have been traded for centuries, but they remained relatively obscure financial instruments until the introduction of a listed options exchange in 1973. Option pricing theory has a long and illustrious history, but it also underwent a revolutionary change in 1973. At that time, Fischer Black and Myron holes presented the first completely satisfactory equilibrium option pricing model. Indeed, the theory applies to a very general class of economic problems - the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event.

The trinomial model is considered to produce more accurate results than the binomial model when fewer time steps are modelled, and is therefore used when computational speed or resources may be an issue. It improves upon the binomial model by allowing a stock price to move up, down or stay the same with certain probabilities. For vanilla options, as the number of steps increases, the results rapidly converge, and the binomial model is then preferred due to its simpler implementation. For exotic options the trinomial model (or adaptations) is sometimes more stable and accurate, regardless of step-size. Option pricing and hedging are two of the most important topics in option theory. The price of an option is the initial premium that the buyer pays to the seller in order to create a binding contract. How to find a fair price which does not trigger arbitrage is not trivial, and that is the first problem that we tackle within this chapter.

2 Derivation

In this problem, we aim at computing the non-arbitrage price of a general risky financial product together with the hedging strategy n_S used by the seller to cover the associated payoff.

We consider a one period model for an arbitrage- free market AoA which evolves between an initial time t = 0 and some maturity $T > 0$.
Since the pricing of put option and call option is similar, we take call options as an example to describe trinomial Markov [1] tree method. Over a small time interval this distribution is approximated by a three-point jump process in such a way that the expected return on the asset is the riskless rate.

We consider 2 types of assets on it: A risky asset S and a non-risky asset.
The portfolio P is given by the triplet $(-1, n_S, n_M)$ with n_S and n_M the number of risky asset S.

The parameters of our model are as follows:

- **T** is defined as time to option maturity meaning the period of the time the option can be exercised
- **K** is the Strike price, the predetermined price, at which the holder of the option can buy the underlying asset (in the case of call options)
- **r** is the risk free interest rate over the maturity/ period of time.
- **n** the number of time steps which the time interval is divided by.
- $\Delta t = T/n$, length of the individual time steps
- S_t is the asset underlying price at time t
- $S_{0,0}$ the price of the non-risky asset

Let $R = S_{0,0}(1 + r)$ be the return of the non-risky asset.
For simplicity let $S_{0,0} = 1$. This gives that $R := (1 + r)$.

[1] A Markov model is a stochastic model which describes a sequence of possible events (states) in which the probability of each event depends on a subset of previous events This report will focus on First-Order Markov Chains, in which the probability of a future state depends only on the current state.

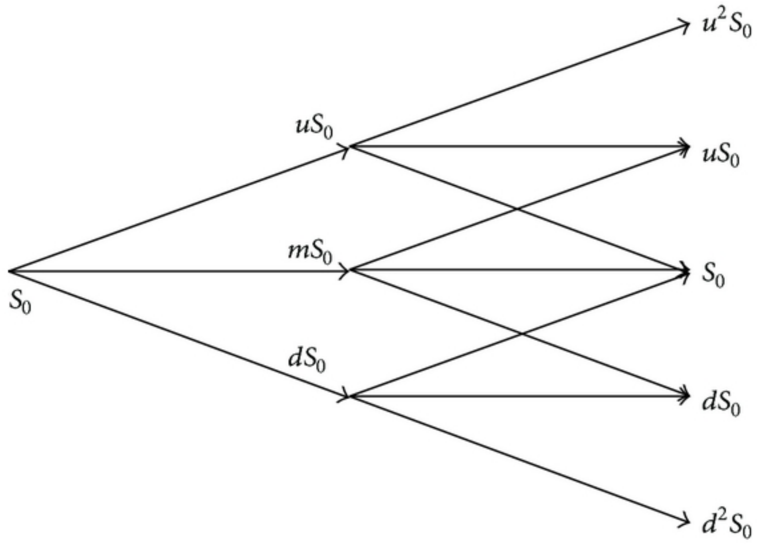
Consider a finite probability space $(\Omega, \mathcal{P}(\omega), \mathbb{P})$. Let Ω be the set of all possible moves of the price of the asset : down, middle and up respectively. $\Omega := \{\omega_d, \omega_m, \omega_u\}$. Those moves occur with respect to the following probabilities (given in the same order):

$$\mathbb{P}\left[\{\omega_d\}\right] = p_d, \mathbb{P}\left[\{\omega_m\}\right] = p_m, \mathbb{P}\left[\{\omega_u\}\right] = 1 - p_d - p_m$$

We then define the multipliers of each step $0 < d < m < u$ which correspond to non-negative down step, middle step and up step respectively, which when multiplied by our underlying price S_t will give us the S_{t+1} value.
For the first period we have that :

$$S_1\left(\omega^1\right) = dS_0, S_1\left(\omega^2\right) = mS_0, S_1\left(\omega^3\right) = uS_0, \text{ where } 0 < d < m < u$$

,as shown in the trinomial tree below



Assuming that there is no arbitrage strategy is equivalent to $d < R < u$, as we will prove below.

If the market is viable, there exists a probability $\mathbb{P}^\star$ equivalent to $\mathbb{P}$, under which $\left(\tilde{S}_n\right)$ is a martingale.[2] Thus,

$$\mathbb{E}^*\left(\frac{S_{t+1}}{S_t}\right) = 1 + r$$

Let $T_{n+1} = \frac{S_{t+1}}{S_t}$ and therefore $\mathbb{E}^*\left(T_{n+1}\right) = 1 + r$. Since T_{n+1} is equal to either of d , u or 1 with non-zero probability, we necessarily have $(1 + r) \in (d, u)$.

[2]In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values.

Using the **First theorem of finance**[3], the risk neutral measure $\mathbb{Q}$ must be equivalent to $\mathbb{P}$, and will therefore be characterised by $(p_d, p_m) \in (0,1)^2$, with $p_d + q_m < 1$ and

$$\mathbb{Q}\left[\{\omega_d\}\right] = p_d, \mathbb{Q}\left[\{\omega_m\}\right] = p_m, \mathbb{Q}\left[\{\omega^u\}\right] = 1 - p_d - p_m$$

Let $\widetilde{S}_1 = \frac{S_1}{1+r}$ denotes the future value of S_1 discounted at the present given the risk free interest rate.
Since we have a one step-Martingale in the absence of arbitrage.

$$\mathbb{E}^{\mathbb{Q}}\left[\widetilde{S}_1\right] = S_0$$

So we have

$$\begin{aligned}\big(dp_d + mp_m + (1 - p_m - p_d)u\big)\frac{S_0}{R} &= S_0 \\ \iff p_d\,(d-u) + p_m\,(m-u) + u &= R \\ \iff p_m &= \frac{u-R}{u-m} - \frac{u-d}{u-m}p_d\end{aligned}$$

Then

$$p_u = 1 - p_d - p_m = \frac{m-d}{u-m}p_d + \frac{R-m}{u-m}$$

The conditions $(p_d, p_m) \in (0,1)^2$ and $p_d + p_m < 1$ when we substitute our variables into the inequalities we get:

$$\begin{cases} 0 < p_d < 1, \\ 0 < \frac{u-R}{u-m} - \frac{u-d}{u-m}p_d < 1, \\ 0 < \frac{m-d}{u-m}p_d + \frac{R-m}{u-m} < 1, \end{cases} \Leftrightarrow \begin{cases} 0 < p_d < 1, \\ \frac{m-R}{u-d} < p_d < \frac{u-R}{u-d} \\ \frac{m-R}{m-d} < p_d < \frac{u-R}{m-d} \end{cases}$$

This system is itself implies the condition:

$$\frac{(m-R)^+}{m-d} < p_d < \min\left\{1, \frac{u-R}{u-d}, \frac{u-R}{m-d}\right\}$$

For the upper-bound condition, we can easily discard $\frac{u-R}{m-d}$, since when comparing with the second inequality $u - d > m - d$ by definition of the step multipliers. Further more we can discard the possibility of 1 with the condition: $u > R > d$ (due to AoA) leaving us with the upper bound of $\frac{u-R}{u-d}$.

Therefore,under the AoA condition, there are infinitely many risk-neutral measures for the one period trinomial model $\mathbb{Q}$ described by the system

$$\mathbb{Q}\left[\{\omega^1\}\right] = p_d, \mathbb{Q}\left[\{\omega^2\}\right] = \frac{u-R}{u-m} - \frac{u-d}{u-m}p_d, \mathbb{Q}\left[\{\omega^3\}\right] = \frac{m-d}{u-m}p_d + \frac{R-m}{u-m}$$

[3] **First theorem of finance** There exists a risk neutral probability if and only if the market is without any arbitrage opportunity.

with $p_d \in \left(\frac{(m-R)^+}{m-d}, \frac{u-R}{u-d}\right)$. For the sake of simplicity we define this interval as (α, β)

The main aim of our model was to the determine the price of an option knowing the desired pay-off ξ.Let $p(\xi)$ be this optimal price which must

The non-uniqueness of the probability which lies in the previously defined interval makes it hard to replicate the option.(see section 4) and imposes comparing the expectation of the payoff at step 1 in order to determine the price at the desired payoff which will the one which yields the maximal price since the seller wants to sell at the highest price possible to maximise seller profit or prevent possible buyer arbitrage.

$$p(\xi) = \sup_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}[\frac{\xi}{1+r}]$$

Moreover, if $p(\xi) < +\infty$, there exists $\Delta^\star$ such that $\mathbb{P}\left[V_T \geq \xi\right] = 1$ with V_T the value of the portfolio at time t=T
If in addition, we also have $(\frac{\xi}{1+r})^+$, the set of viable prices for ξ is the relative interior of $[-p(-\xi), p(\xi)]$, and $p(\xi) = -p(-\xi)$ if and only if ξ is replicable.

3 European options

There are two styles of option: European and American. In this section, we will first discuss about the European style option. European option is a contract that a buyer receives an opportunity to buy an asset with a strike price (K) and the buyer can only exercise on the date of expiration. A call option is when a buyer and a seller agrees on a contract that gives a right to a buyer to purchase an underlying asset with an expiration date and an exercise price (also known as strike price). When it is a call option, the buyer would expect the price to go higher than the exercise price so that they can gain profit and if it is a put option, it would be the opposite. However, if the price ends up lower than the exercise price, the buyer has a right to not purchase the asset. Therefore, for the European options, the stock price compared to the exercise price only on the expiration date matters. Since it is the buyer has an opportunity to buy the asset, the payoff would be $max(S_T - K, 0)$ which is also denoted as $(S_T - K)^+$. Thus, if $S_T > K$, the buyer would gain a profit and if not, they have the choice to not exercise the option.

4 Incomplete market - flaw of the model

4.1 Contingent claim

A contingent claim (ξ) is a type of a derivative that is influenced by unknown future events. Thus, it is also dependent on underlying assets. In this section, we are going to discuss particularly about European options.

4.2 Complete market

The market is defined to be complete when every contingent claim is attainable. Let us show that there are contingent claims that are not attainable which leads to an incomplete market. In other words, if any contingent claim cannot be generated by a strategy portfolio, the market is not complete.

Under the case of risk neutral probability, $\mathbb{E}^{\mathbb{Q}}[\frac{S_1}{1+r}] = S_0$. Under this assumption, we get a system of equation:

$$\begin{cases} \frac{1}{1+r}(p_u u + p_d d + p_m m)S_0 = S_0 \\ p_u + p_d + p_m = 1 \end{cases}$$

Let us demonstrate this with matrix. $\begin{pmatrix} u & m & d \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_u \\ p_d \\ p_m \end{pmatrix} = \begin{pmatrix} 1+r \\ 1 \end{pmatrix}$
in order to solve this system of equation to find a unique solution, we need three equations. Therefore, this proves that there exists multiple solution instead of unique and there is no perfect replication for every contingent claim. Thus, it proves that the market is not complete according to the definition of complete market. In the real markets, the incompleteness of market comes from limitation to trade and of short sells.

Let us suggest another system of equation to prove that the market is incomplete. We will define x as initial wealth and n_S as a number of risky asset S. The probability space Ω has three elements, $\omega^u, \omega^m, \omega^d$ representing the three cases of change of price. $P = (n_s, n_m)$. We will define V_t as a value of the portfolio P at time t. In order to find a portfolio for ξ,

$$\mathbb{P}[V_1 = \xi] = 1$$

has to be satisfied. Let us define Since x is the initial wealth, we can write an equation for x:

$$V_0 = n_s S_0 + (x - n_s S_0) = x$$

After one step, we get

$$V_1 = n_s S_1 + (x - n_s S_0)(1+r)$$

This generates a system of equation:

$$\begin{cases} n_s u S_0 + (x - n_S S_0)(1+r) = \xi(\omega^u) \\ n_s d S_0 + (x - n_S S_0)(1+r) = \xi(\omega^d) \\ n_s m S_0 + (x - n_S S_0)(1+r) = \xi(\omega^m) \end{cases}$$

In this equation, $n_s S_0$ and $x - n_s S_0 = n_m$ indicate risky and non-risky part respectively. Let us solve this equation. In order to find x and n_S, ξ has to satisfy:

$$\xi(\omega^m) = \frac{u-m}{u-d}\xi(\omega^d) + \frac{m-d}{u-d}\xi(\omega^u)$$

$$\Leftrightarrow (mS_0 - K)^+ = \frac{m-d}{u-d}(dS_0 - K)^+ + \frac{u-m}{u-d}(uS_0 - K)^+$$

This shows that $\xi(\omega_m)$ is a linear combination of $\xi(\omega_d)$ and $\xi(\omega_u)$. If this is not satisfied, we can realize that the system of equation above don't have solution, in other words, we cannot generate a portfolio for ξ. Therefore, according to the definition of incomplete market, the trinomial model is incomplete since not every contingent claim is attainable which is a flaw of the model.

5 Black-Scholes model

5.1 Black-Scholes model

Black-Scholes model, which was first introduced and developed by Fischer Black, Myron Scholes and Robert C. Merton, allows us to estimate the European option price. This model only concerns the European options since American options may be exercised before they become expired. This model is available under a few assumptions:

- It only concerns European option.
- It follows stochastic process
- it is under the log-normal (Gaussian) distribution.
- It doesn't have any transaction cost.
- Volatility and interest rate of option are constant.

5.2 Black-Scholes equation and Black-Scholes model

It starts from geometric Brownian motion which is based on stochastic differential equation: (1)

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

, where μ, σ and W_t represent drift rate, volatility and Wiener process of Brownian motion respectively. The volatility is how unstable the price of the option is. This differential equation is out of our topic so we will present it briefly.

We will not go into the derivation of equation in this paper.

From the result of the derivation the Black-Scholes equation is:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\frac{\sigma^2 S^2 \delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0$$

where V is the option price depending on the stock price and time.

Now, we will present the Black-Scholes model:

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Let us first start with defining the variables. C, S and X indicates the unique arbitrage-free European call, strike price and exercise price respectively. Also T, σ, r represent the time remaining until the expiration date, volatility of the option price and risk-free interest rate respectively. The model suggests that, C, the call price is composed of $SN(d_1)$, what we earn and $Xe^{-rT}N(d_2)$, what we paid. Also $N(d_1), N(d_2)$ indicate the normal distribution of d_1, d_2 since it was previously assumed that the underlying asset follows log-normal distribution. As the volatility increases, C, the call increases because the more the price is unstable and oscillate more, the higher price it reaches.

5.3 Convergence of trinomial tree to Black-Scholes formula

The Black-Scholes model converges to the result of trinomial tree model. As it was mentioned above, the European call option price is defined as $C = max(0, S - K)$. The value obtained from Black-Scholes model converges to $max(0, S - K)$.

6 American options

The trinomial tree for American style options is almost identical for European style options, the only difference being that at each node the value is going to be the larger of the Binomial valuation and the Exercise value. For call options the exercise value is defined as the difference between the underlying asset price and the strike price of the option, while for a put it is defined as the difference between the strike price and the price of the underlying asset.

To derive the price of an American call (similar logic for a put) we use backwards induction, considering only two periods n and $n-1$ we have that if $S_n < S_{n-1}$ then it is advantageous to exercise the option at time $n-1$ and the price of the option should clearly be $S_{n-1} - K$.
Yet if we have that $S_{n-1} < S_n$ then it is advantageous for the holder of the option to wait until the next period to exercise. During the period $n-1$ if the option was priced at $S_{n-1} - K$ we could buy the option for $S_{n-1} - K$ and then exercise it during period n to earn a risk free profit of $S_n - S_{n-1}$. Hence we need to use the AoA condition to find the fair price of the option.
Let C_i denote the price of the option at time i. Although we are unable to prove this mathematically, with a few violations we can reason through the intuition.
To satisfy AoA, we must have that the return of the option over one period is the same as the return of a risk free asset of equal value, $\mathbb{E}[S_n - K] = (1+r)C_{n-1}$. Using

the fact that $\mathbb{E}[S_n] = S_{n-1}(1+r)$ we can solve for C_{n-1}.

$$C_{n-1} = \frac{(\mathbb{E}[S_n] - K)}{1+r} = \frac{S_{n-1}(1+r) - K}{1+r} = S_{n-1} - \frac{K}{1+r}$$

Let S_i^0 denote the price of a risk free asset at time i. Then using the fact that $E[S_n^0] = (1+r)S_{n-1}^0$, and assuming that the return of the risk free asset and the return of the stock are independent, we can rearrange rearrange this to get:

$$S_{n-1} - \frac{K}{1+r} = \mathbb{E}[S_n - K](1+r) = \mathbb{E}\left[\frac{S_n - K}{S_n^0}\right] S_{n-1}^0$$

Which is similar to the formula presented by lamberton lapeyre in their textbook "stochastic calculus". Hence the price for american options is given by the formula

$$C_n = max\left(S_{n-1} - K; S_{n-1}^0 \mathbb{E}\left[\frac{S_n - K}{S_n^0}\right]\right)$$

7 The Greeks

In finance the term "Greeks" is used to refer to the sensitivity of the price of a derivative to some parameter tied to the underlying asset. Here we will focus only on the first order Greeks. These are:

- Delta, $\Delta = \frac{\partial C}{\partial S}$, the sensitivity of the price of the option to the price of the underlying
- Vega, $V = \frac{\partial C}{\partial \sigma}$, the sensitivity of the price of the option to the volatility of the underlying.
- Theta, $\Theta = \frac{\partial C}{\partial T}$, the sensitivity of the price of the option to time remaining to expiration.
- Rho, $\rho = \frac{\partial C}{\partial r}$, the sensitivity of the price of the option to the risk free interest rate

Since the volatility of the stock is implicit in the model, or rather, individual to each investor, we have that Vega is individual to each investor. Furthermore the trinomial tree does not tell us anything interesting about Theta, since the price movement of the stock, and therefore the Greek is discretized. This leaves us with and Δ to analyse.

7.1 The Δ

First focusing on Δ, this is most likely the easiest one to calculate. Since it is simply the derivative of the price of the option with respect to the price of the underlying hence we can simply discretize it and calculate it numerically using the following formula:

$$\Delta = \frac{C(S(0) + \Delta S) - C(S(0) - \Delta S)}{2\Delta S}$$

To do this numerically we create two different price trees, one for $S(0) - \Delta S$ the other for $S(0) + \Delta S$ and then at each node we calculate the value difference between the nodes of the two trees (Clifford, Wang, Zaboronski).
If we actually implement this algorithm then we find that for a one period trinomial tree the Δ at each node is simply equal to either u,m or d. Depending on the node's position.

7.2 The ρ

Calculating the ρ should also be relatively straightforward by discretizing the derivative and using the formula:

$$\rho = \frac{C(r + \Delta r) - C(r)}{\Delta r}$$

and following the same procedure as for Δ by generating two distinct trees and finding the difference at each node.

8 Introducing our model

To use the file: "trinomial_tree.py" with custom inputs first go to the bottom and uncomment the statement with "if __name__ = "__main__":" Then comment out the test inputs which is everything below.

"get_params()" prompts the user in the terminal for useful parameters and calculates some other parameters. $S0, T, r, dt, K, u, d,$ and m. n is fixed at 1 since we're only modeling a one step trinomial tree.

"calc_formulas()" calculates the

$$p_u, p_d, p_m$$

based on the formulas, specifically it creates a p_d array of values restricted by the conditions. From that array of p_d values we calculate the rest of the probabilities with the linear equation we previously derived

"calc_underlying_price_tree()" takes the input and creates an underlying price matrix where the diagonals are where the underlying prices say the same, the row above are where the price goes up and the row below is where the price goes down. All of this by multiplying u, d, m to the previous nodes.

"options_tree()" takes the price_tree and looks at the last column of the price tree matrix and takes $max(S_n - K, 0)$ as our last column options tree matrix. It then calculates backwards the the value of the option price at the specific node using backwards induction:

$$C_0 = \sup_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}[\frac{\xi}{1+r}]$$

$$\max_{p \in (\alpha,\beta)} \left((uS_0 - K)^+ p_u + (mS_0 - K)^+ p_m + (dS_0 - K)^+ p_d\right) \frac{1}{1+r}$$

it goes all the way back to the matrix[0,0] which represents the initial option value.

A series of graphs are then plotted:

- p_d vs initial option price
- The option tree
- The underlying price tree

"predict_init_opt()" uses the analytical method as described below to predict the initial value based on u, d, m, K, S_0.

Alternatively, to skip the main function, the test inputs on the bottom can be left uncommented and the program will run test simulations with the variables defined there.

9 Numerical Analysis of the Equations

Recall we have the following equations to derive the initial call price:

- $C_0 = \sup_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}[\frac{\xi}{1+r}] = \max_{p \in (\alpha,\beta)} \left((uS_0 - K)^+ p_u + (mS_0 - K)^+ p_m + (dS_0 - K)^+ p_d\right) \frac{1}{1+r}$
- $p_d = p_d$
- $p_m = \frac{u-R}{u-m} - \frac{u-d}{u-m} p_d$
- $p_u = \frac{m-d}{u-m} p_d + \frac{R-m}{u-m}$

Since all of the equations are functions of $p_d \in (\alpha, \beta)$, more precisely linear ones, we can only expect a linear relationship between p_d and C_n in our plotted graphs.
The derivative of the call prices before taking the maximum as a function of p_d can be described as:

$$(uS_0 - K)^+ \frac{m-d}{u-m} - (mS_0 - K)^+ \frac{u-d}{u-m} + (dS_0 - K)^+$$

This means that if the derivative is positive, specifically:

$$(uS_0 - K)^+ \frac{m-d}{u-m} + (dS_0 - K)^+ > (mS_0 - K)^+ \frac{u-d}{u-m}$$

Then we'd use probability $p_d = \beta$ to find C_0; with equality, any p_d value will yield the same C_0, otherwise we use $p_d = \alpha$ since the function is strictly monotone on (α, β).

Now we analyse the strike price in the range $uS_0 > K > dS_0$. We will break them into cases.
If $uS_0 > K \geq mS_0$ since the last two terms will disappear will yield:

$$C_0 = (uS_0 - K)\left(\frac{m-d}{u-m}\beta + \frac{R-m}{u-m}\right)\frac{1}{1+r}$$

And if $mS_0 > K > dS_0$ and if the derivative is positive, meaning:

$$(uS_0 - K)(m-d) > (mS_0 - K)(u-d)$$

$$muS_0 - mK - duS_0 + Kd > muS_0 - Ku - dmS_0 + Kd$$

$$dS_0 > K$$

And so:

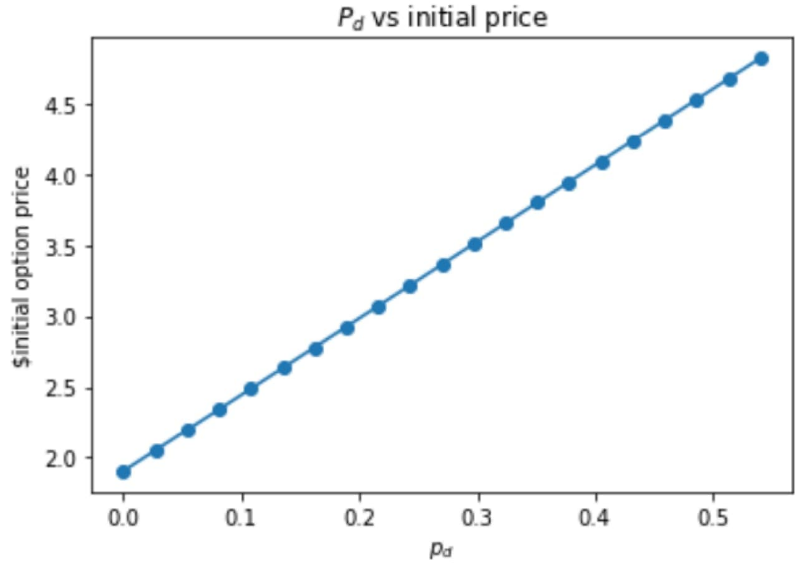
$$C_0 = \left((uS_0 - K)\left(\frac{m-d}{u-m}\beta + \frac{R-m}{u-m}\right) + (mS_0 - K)\left(\frac{u-R}{u-m} - \frac{u-d}{u-m}\beta\right)\right)\frac{1}{1+r}$$

Under no circumstances will we use the probability α as p_d value.

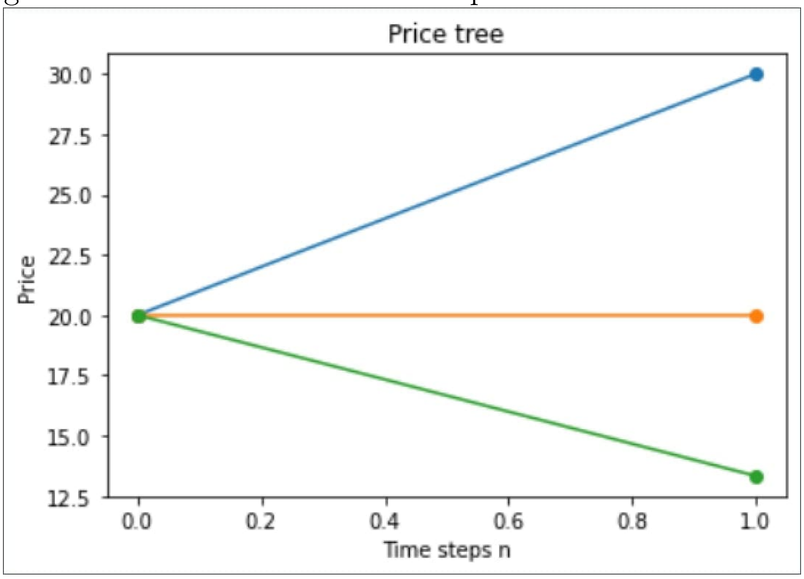
10 Simulations

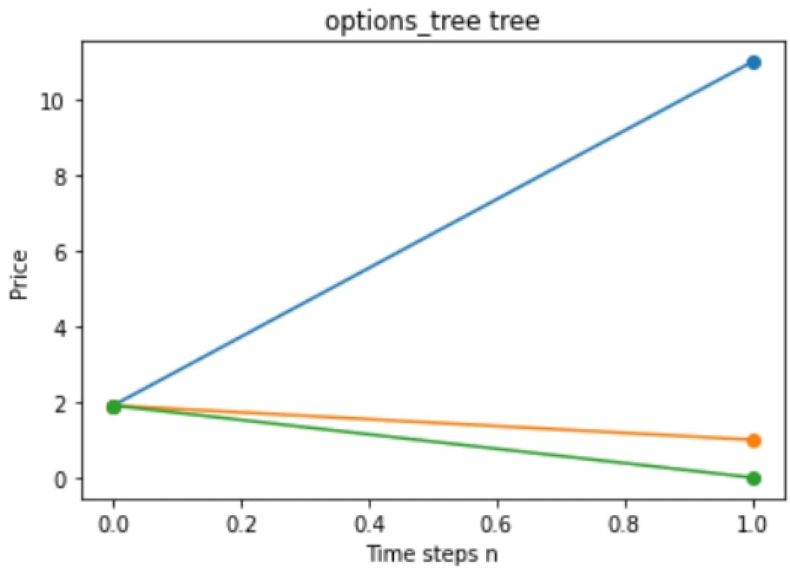
10.1 Analysis of equations

Trial 1: $uS_0 > K > mS_0$
We first set our variables: $S0 = 20, T = 3, r = 0.05, n = 1, K = 12, dt = T/n, u = 1.5, d = 1/u, m = 1$
Results: Initial option calculated by the backwards induction and predictions match at : 4.81904761904762 and p_d vs initial option price is increasing linear as predicted, and $(dS_0 - K)^+$ vanishes as predicted in the options tree graph

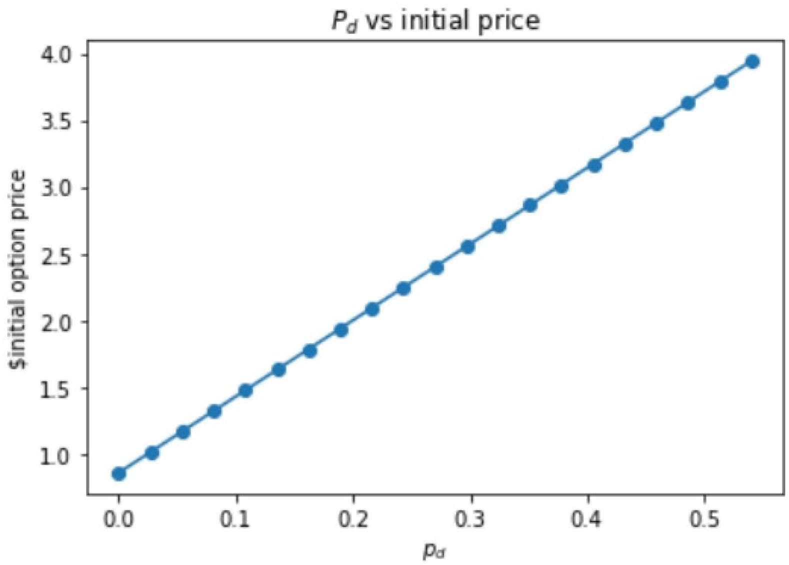


The following are trees of the solution initial option = 4.81904761904762

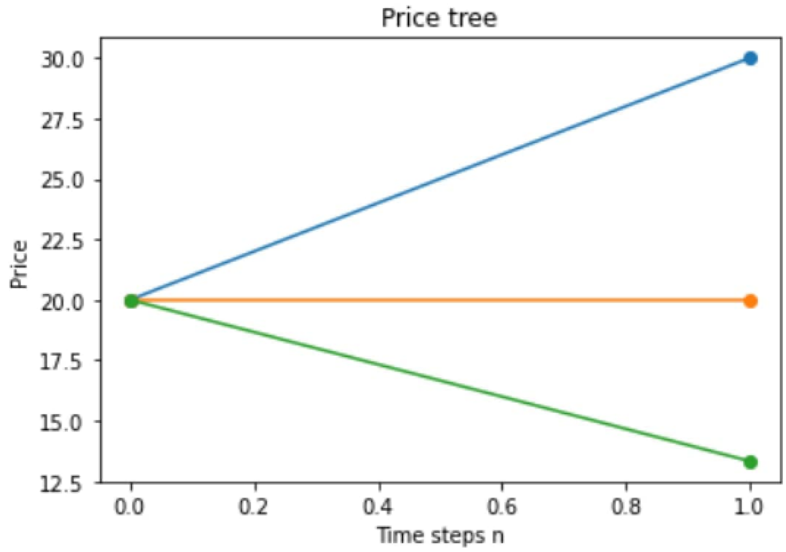


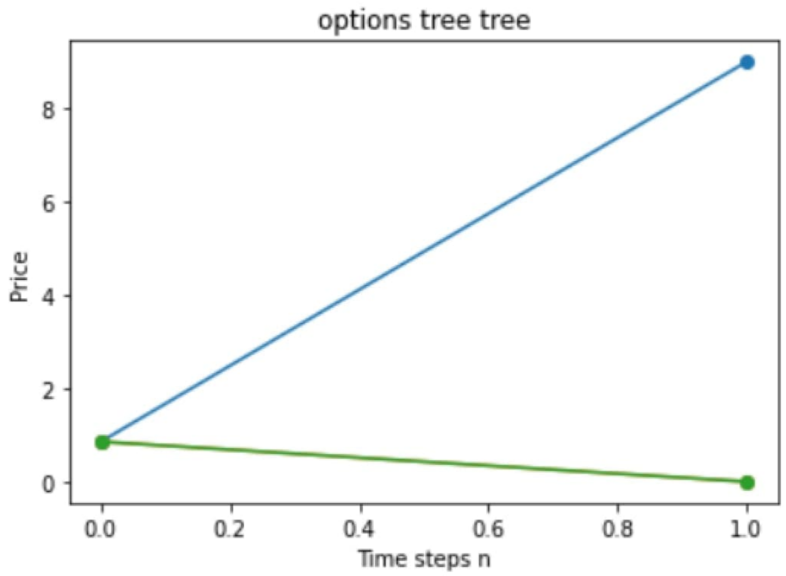


Trial 2: $uS_0 > K > mS_0$
We first set our variables: $S0 = 20, T = 3, r = 0.05, n = 1, K = 21, dt = T/n, u = 1.5, d = 1/u, m = 1$
Results: Initial option calculated by the backwards induction and predictions match at : 3.942857142857143 and p_d vs initial option price is increasing linear as predicted, and $(mS_0 - K)^+, (dS_0 - K)^+$ vanishes (two over-lapping lines connected to 0) as predicted in the options tree graph.



The following are trees of the solution initial option = 3.942857142857143. The price tree is the same since,S_0, u, d, m are held fixed.


Price tree
Price
Time steps n
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
0.0
0.2
0.4
0.6
0.8
1.0


options tree tree
Price
Time steps n
8
6
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0

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