Time Frequency Analysis

The sixth lesson

Time-frequency analysis is a method of analyzing **time-varying** signals. It is used to analyze **non-stationary** signals, which are signals whose frequency content changes over time.

The Fourier Transform is not suitable for analyzing non-stationary signals because it assumes that the signal is stationary. The Short-Time Fourier Transform (STFT) is a method of analyzing non-stationary signals by applying the Fourier Transform to small windows of the signal.

Knowledge that must be mastered:

1. ***Fourier Transform***
   For a continuous-time function $x(t)$, the Fourier transform of $x(t)$ can be defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

   And the inverse Fourier transform is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

2. ***Parseval's Theorem of Fourier Transform***
   Parseval's theorem states that the energy of signal $x(t)$ [if $x(t)$ is aperiodic] or power of signal $x(t)$ [if $x(t)$ is periodic] in the time domain is equal to the energy or power in the frequency domain.
   Therefore,if

$$x_1(t) \leftrightarrow X_1(\omega)$$

$$x_2(t) \leftrightarrow X_2(\omega)$$

   Then, Parseval's theorem of Fourier transform states that:

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega$$

Where, $x_1(t)$ and $x_2(t)$ are complex functions.
Proof:

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)e^{j\omega t}d\omega\right) x_2^*(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left(\int_{-\infty}^{\infty} x_2^*(t)e^{j\omega t}dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left(\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt\right)^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega$$

3. ***Parseval's Identity of Fourier Transform***

The Parseval's identity of Fourier transform states that the energy content of the signal $x(t)$ is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- The Parseval's identity is also called **energy theorem** or **Rayleigh's energy theorem**
- The quantity $|X(\omega)|^2$ is called the energy density spectrum of the signal $x(t)$

Proof:
If $x1(t)$ = $x2(t)$ = $x(t)$, then the energy of the signal is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

4. ***Possion's Summation Formula***

$g(t)$ is a periodic function with time period T.

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT)$$

Then, the Fourier series of $g(t)$ is given by:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

Where, c_n is the Fourier series coefficient of $g(t)$ and is given by:

$$c_n = \frac{1}{T}\int_T g(t)e^{-jn\omega_0 t}dt$$

Where, $\omega_0 = \frac{2\pi}{T}$

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = x(t) + x(t+T) + x(t+2T) + x(t+3T) + \cdots$$

Extract a period from the periodic pulse sequence, then perform a Fourier transform on it.

$$X(\omega) = \int_T x(t)e^{-j\omega t}dt$$

$$c_n = \frac{1}{T}\int_T g(t)e^{-jn\omega_0 t}dt$$

$$c_n = \frac{1}{T}\int_T (\sum_{n=-\infty}^{\infty} x(t+nT))e^{-jn\omega_0 t}dt$$

$$c_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

$$c_n = \frac{1}{T} X(\omega)|_{n\omega_0}$$

$$c_n = \frac{1}{T} X(n\omega_0)$$

Where, $X(\omega)$ is the Fourier transform of $x(t)$
Therefore, Possion's Summation Formula is given by:

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}$$

5. ***Sample for*** $y(t)$

$$y_1(n) = y(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Fourier Transform of $y(k)$ is given by:

$$Y_1(\omega) = \frac{1}{2\pi} Y(\omega) * (w_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0))$$

$$Y_1(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} Y(\omega - n\omega_0)$$

where, $\omega_0 = \frac{2\pi}{T}$

6. ***Notation T、M***

i. $$T_x(f(t)) = f(t-x)$$

ii. $$M_\omega(f(t)) = f(t)e^{j\omega t}$$

iii. $$T_x M_\omega(f(t)) = T_x(f(t)e^{j\omega t}) = f(t-x)e^{j\omega(t-x)}$$

iv. $$M_\omega T_x(f(t)) = M_\omega(f(t-x)) = f(t-x)e^{j\omega(t)}$$

v. $$\mathcal{F}\left[T_x M_{\omega 0}(f(t))\right] = e^{-j\omega x} F(\omega - \omega_0) = M_{-x} T_{\omega 0} F(\omega)$$