Time Frequency Analysis

The sixth lesson

Time-frequency analysis is a method of analyzing **time-varying** signals. It is used to analyze **non-stationary** signals, which are signals whose frequency content changes over time.

The Fourier Transform is not suitable for analyzing non-stationary signals because it assumes that the signal is stationary. The Short-Time Fourier Transform (STFT) is a method of analyzing non-stationary signals by applying the Fourier Transform to small windows of the signal.

Knowledge that must be mastered:

1. ***Fourier Transform***
   For a continuous-time function $x(t)$, the Fourier transform of $x(t)$ can be defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

   And the inverse Fourier transform of $X(\omega)$ is given by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

2. ***Parseval's Theorem of Fourier Transform***
   Parseval's theorem states that **the energy of signal $x(t)$ [if $x(t)$ is aperiodic]** or **power of signal $x(t)$ [if $x(t)$ is periodic]** in the time domain is equal to the energy or power in the frequency domain.
   Therefore,if

$$x_1(t) \leftrightarrow X_1(\omega)$$

$$x_2(t) \leftrightarrow X_2(\omega)$$

   Then, Parseval's theorem of Fourier transform states that:

$$< x_1(t), x_2(t) >= \frac{1}{2\pi} < X_1(\omega), X_2(\omega) >$$

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega$$

Where, $x_1(t)$ and $x_2(t)$ are complex functions.
Proof one:

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)e^{j\omega t}d\omega\right)x_2^*(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left(\int_{-\infty}^{\infty} x_2^*(t)e^{j\omega t}dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left(\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt\right)^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega$$

Proof two:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt\right)X_2^*(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt\right)\left(\int_{-\infty}^{\infty} x_2(s)e^{-j\omega s}ds\right)^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)x_2^*(s)e^{-j\omega t}e^{j\omega s}dtdsd\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)x_2^*(s)\left(\int_{-\infty}^{\infty} e^{-j\omega(t-s)}d\omega\right)dtds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)x_2^*(s)2\pi\delta(t-s)dtds$$

$$= \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

3. ***Parseval's Identity of Fourier Transform***

   The Parseval's identity of Fourier transform states that the energy content of the signal $x(t)$ is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

   - The Parseval's identity is also called **energy theorem** or **Rayleigh's energy theorem**
   - The quantity $|X(\omega)|^2$ is called the energy density spectrum of the signal $x(t)$

   Proof:

   If $x_1(t)$ = $x_2(t)$ = $x(t)$, then the energy of the signal is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

4. ***Possion's Summation Formula***

   $g(t)$ is a periodic function with time period T.

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT)$$

   Then, the Fourier series of $g(t)$ is given by:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

   Where, c_n is the Fourier series coefficient of $g(t)$ and is given by:

$$c_n = \frac{1}{T}\int_T g(t)e^{-jn\omega_0 t}dt$$

Where, $\omega_0 = \frac{2\pi}{T}$

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = x(t) + x(t+T) + x(t+2T) + x(t+3T) + \cdots$$

Extract a period from the periodic pulse sequence, then perform a Fourier transform on it.

$$X(\omega) = \int_T x(t)e^{-j\omega t}dt$$

$$c_n = \frac{1}{T}\int_T g(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\int_T (\sum_{n=-\infty}^{\infty} x(t+nT))e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}X(\omega)|_{n\omega_0}$$

$$= \frac{1}{T}X(n\omega_0)$$

Where, $X(\omega)$ is the Fourier transform of $x(t)$

Therefore, Possion's Summation Formula is given by:

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(n\omega_0)e^{jn\omega_0 t}$$

Also, the other way to prove the Possion's Summation Formula:

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_T g(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_T (\sum_{n=-\infty}^{\infty} x(t+nT))e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_0^T x(t+nT)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{nT}^{(n+1)T} x(t^{'})e^{-jn\omega_0 (t^{'}-nT)}dt^{'}$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t^{'})e^{-jn\omega_0 t^{'}}dt^{'}$$

$$= \frac{1}{T}X(n\omega_0)$$

$$g(t) = \sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(n\omega_0)e^{jn\omega_0 t}$$

Where, $X(\omega)$ is the Fourier transform of $x(t)$

5. ***Sample for*** $y(t)$

$$y_1(n) = y(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Fourier Transform of $y(k)$ is given by:

$$Y_1(\omega) = \frac{1}{2\pi} Y(\omega) * (w_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0))$$

$$Y_1(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} Y(\omega - n\omega_0)$$

where, $\omega_0 = \frac{2\pi}{T}$

6. ***Notation T、M***

   i. $$T_x(f(t)) = f(t-x)$$

   ii. $$M_\omega(f(t)) = f(t)e^{j\omega t}$$

   iii. $$T_x M_\omega(f(t)) = T_x(f(t)e^{j\omega t}) = f(t-x)e^{j\omega(t-x)}$$

   iv. $$M_\omega T_x(f(t)) = M_\omega(f(t-x)) = f(t-x)e^{j\omega t}$$

   v. $\mathcal{F}\left[T_x M_{\omega 0}(f(t))\right] = \int_{-\infty}^{\infty} f(t-x)e^{j\omega_0(t-x)}e^{-j\omega t}dt = e^{-j\omega x}F(\omega - \omega_0) = M_{-x}T_{\omega 0}(I$

Short Time Fourier Transform (STFT)

1. ***FT***

$$F(\omega) = \int_{-\infty}^{\infty} f(t')e^{-j\omega t'}dt'$$

我们付出了整个时间域的代价，得到了某点ω的信息，将时间局部化后：

$$\int_{-\infty}^{\infty} f(t')g(t'-t)e^{-j\omega t'}dt'$$

其中，$g(t)$是一个窗函数，$g(t) = I_{[-a,a]}(t)$
因为

$$-a \le t' - t \le a$$

$$t - a \le t' \le t + a$$

所以

$$\int_{-\infty}^{\infty} f(t')g(t'-t)e^{-j\omega t'}dt' = \int_{t-a}^{t+a} f(t')e^{-j\omega t'}dt'$$

由此，我们就得到了STFT的定义：同时具有时间和频率二维信息的函数，$V_g f(t,\omega)$是以g为窗函数，f为对象的短时傅里叶变换。

$$V_g f(t,\omega) = \int_{-\infty}^{\infty} f(t')\overline{g(t'-t)}e^{-j\omega t'}dt'$$

从不同的角度去认识短时傅里叶变换：

$$V_g f(t,\omega) = \int_{-\infty}^{\infty} f(t')\overline{g(t'-t)}e^{-j\omega t'}dt'$$

$$V_G F(\omega,t) = \int_{-\infty}^{\infty} F(\omega')\overline{G(\omega'-\omega)}e^{-jt\omega'}d\omega'$$

$V_g f(t,\omega)$ 和 $V_G F(\omega,t)$的关系如何?
很明显，$V_g f(t,\omega)$可以间接看作$< f,h >$，$V_G F(\omega,t)$也可以间接看作$< F,H >$。我们知道，$< f,h >= \frac{1}{2\pi} < F,H >$，所以可以从此入手，得到$V_g f(t,\omega)$和$V_G F(\omega,t)$的关系。

$$V_G F(\omega,t) = \int_{-\infty}^{\infty} F(\omega')\overline{G(\omega'-\omega)}e^{-jt\omega'}d\omega'$$

$$= \int_{-\infty}^{\infty} F(\omega')\overline{G(\omega'-\omega)e^{jt\omega'}}d\omega'$$

所以对 $G(\omega'-\omega)e^{jt\omega'}$ 做傅里叶逆变换：

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega'-\omega)e^{jt\omega'}e^{j\omega' t'}d\omega'$$

令$\omega''=\omega'-\omega$，则$\omega'=\omega''+\omega$，所以原式等于：

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega'')e^{jt(\omega''+\omega)}e^{j(\omega''+\omega)t'}d\omega''$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\omega'')e^{j\omega''(t+t')}d\omega''e^{j\omega(t+t')}$$

$$=g(t+t')e^{j\omega(t+t')}$$

接下来，

$$V_GF(\omega,t)=\int_{-\infty}^{\infty}F(\omega')\overline{G(\omega'-\omega)e^{jt\omega'}}d\omega'$$

$$=2\pi\int_{-\infty}^{\infty}f(t')\overline{g(t+t')e^{j\omega(t+t')}}dt'$$

$$=2\pi\int_{-\infty}^{\infty}f(t')\overline{g(t'-(-t))}e^{-j\omega(t+t')}dt'$$

$$=2\pi\int_{-\infty}^{\infty}f(t')\overline{g(t'-(-t))}e^{-j\omega t'}dt'e^{-j\omega t}$$

$$=2\pi V_gf(-t,\omega)e^{-j\omega t}$$

$\tilde{g}(t)=g(-t)$，所以$V_gf(t,\omega)=\int_{-\infty}^{\infty}f(t')\overline{\tilde{g}(t-t')}e^{-j\omega t'}dt'$ 所以可以从卷积的角度来看待短时傅里叶变换。

$$V_gf(t,\omega)=\int_{-\infty}^{\infty}f(t')\overline{\tilde{g}(t-t')}e^{-j\omega t'}dt'$$

$$=\int_{-\infty}^{\infty}f(t')\overline{\tilde{g}(t-t')}e^{-j\omega(t-t')}dt'e^{-j\omega t}$$

$$=(f(t)*(\overline{\tilde{g}(t)}e^{j\omega t}))e^{-j\omega t}$$

这个卷积核一方面在时域上面移动，一方面在频域上面移动，所以STFT本质是拿一个滤波器在时域和频域上面移动，然后对信号进行滤波。是一个在时频平面的滤波器。
时间分辨率和频率分辨率不可兼得，时间分辨率和频率分辨率是互相矛盾的，时间分辨率越高，频率分辨率越低，时间分辨率越低，频率分辨率越高。

Uncertainty Principle

对于一个能量有限的信号，

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d(\omega) < \infty$$

我们可以得到均值，也就是平均位置：

$$m_t = \int_{-\infty}^{\infty} t|f(t)|^2 dt$$

$$m_\omega = \int_{-\infty}^{\infty} \omega|F(\omega)|^2 d(\omega)$$

于是，我们可以得到方差，也就是有效长度(宽胖程度)，与之对应也会是时间分辨率和频率分辨率：

$$\sigma_t^2 = \int_{-\infty}^{\infty} (t - m_t)^2 |f(t)|^2 dt$$

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - m_\omega)^2 |F(\omega)|^2 d(\omega)$$

我们可以得到不确定性原理：(两者不能同时达到最小值):

$$\sigma_t^2 \sigma_\omega^2 \geq C$$

proof:

假设有AB两个算子，A算子作用在f上，与g做内积:

$$< Af, g >= \int_{-\infty}^{\infty} (Af)(t) g^*(t) dt =< f, A^H g >$$

从矩阵上看：

$$< Af, g >= (Af)^H g = f^H A^H g =< f, A^H g >$$

A^H是A的伴随，伴随就理解成矩阵里面的共轭转置。

可交换性：

已知$AB - BA = [A, B]$，$< x, y >= \overline{< y, x >}$

$$< [A, B]f, f >=< (AB - BA)f, f >$$

$$=< ABf, f > - < BAf, f >$$

$$=< Bf, Af > - < Af, Bf >$$

$$= 2jIm < Bf, Af >$$

所以：

$$| < [A, B]f, f > | = 2|Im < Bf, Af > |$$

$$\leq 2| < Bf, Af > |$$

$$\leq 2||Bf|| \, ||Af||$$

假设$(Af)(t) = tf(t)$，$(Bf)(t) = \frac{1}{2\pi j} \frac{d}{dt} f(t)$

$$(AB - BA)f = (AB)f - (BA)f = \frac{1}{2\pi j} t \frac{d}{dt} f(t) - \frac{1}{2\pi j} \frac{d}{dt} (tf(t))$$

$$= -\frac{1}{2\pi j} f(t)$$

所以，$AB - BA = -\frac{1}{2\pi j} I$

$$| < [A,B]f, f > | = \frac{1}{2\pi} | < f, f > | = \frac{1}{2\pi} ||f||^2 \leq ||Af|| \, ||Bf||$$

令$< f, f >= ||f||^2 = 1$

$$||Af|| \, ||Bf|| \geq \frac{1}{2\pi}$$

$$||Af|| = \sqrt{< Af, Af >} = (\int_{-\infty}^{\infty} tf(t)\overline{tf(t)}dt)^{1/2}$$

$$= (\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt)^{1/2} = \sigma_t$$

$$||Bf|| = \sqrt{< Bf, Bf >} = (\int_{-\infty}^{\infty} (\frac{1}{2\pi j} \frac{d}{dt} f(t)) \overline{(\frac{1}{2\pi j} \frac{d}{dt} f(t))} dt)^{1/2}$$

$$= (\frac{1}{4\pi^2} (\int_{-\infty}^{\infty} |\frac{d}{dt} f(t)|^2 dt))^{1/2}$$

$$= (\frac{1}{4\pi^2} (\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega))^{1/2}$$