Foundations of Computation

Assignment 2

November 8, 2021

Q1: Write regular expressions which describe the following languages

1. $\{w \in \{0,1\}^* |\ w$ ends with 101 $\}$

   $(0 \cup 1)^*101$

2. $\{w \in \{0,1\}^* |\ w$ starts with 10 or ends with 1 $\}$

   $(10(0 \cup 1)^*) \cup ((0 \cup 1)^*1)$

3. $\{w \in \{0,1\}^* |\ w$ contains the substring 01 $\}$

   $(0 \cup 1)^*01(0 \cup 1)^*$

4. $\{w \in \{0,1\}^* |\ w$ does not contain the substring 01 $\}$

   1^*0^*

5. $\{w \in \{0,1\}^* |\ w$ contains exactly one occurrence of 01$\}$

   $1^*00^*11^*0^*$

6. $\{w \in \{0,1\}^* |\ w$ has even length $\}$

   $((0 \cup 1)(0 \cup 1))^*$

7. $\{w \in \{0,1\}^* |$ the number of 0s in w is odd $\}$

   $1^*0(1 \cup 01^*0)^*$

Q2: Convert the following regular expressions into nondeterministic finite automata using the method shown.

$(aba \cup bb \cup ba)^*$

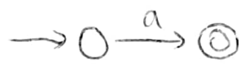
$ab(a \cup b)^*a(a \cup b)^*$

$b^*a^*(ba \cup b^*)$

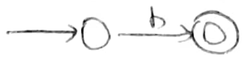
The solutions are in the diagrams below

Q2 a) $(aba \cup bb \cup ba)^*$

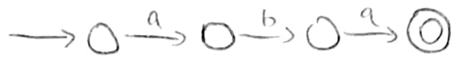
a



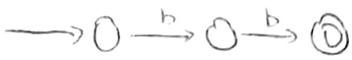
b



aba

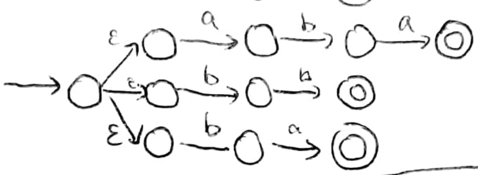


bb

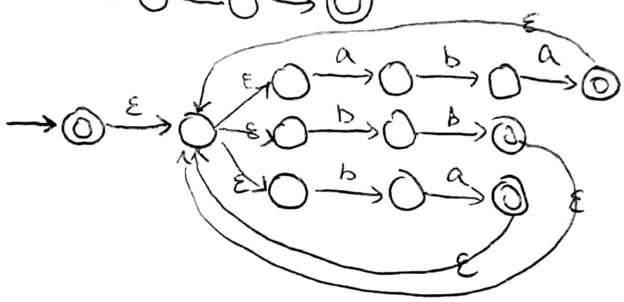


ba

$aba \cup bb \cup ba$

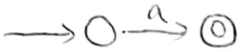


$(aba \cup bb \cup ba)^*$



Q 2 b) $ab(a \cup b)^* a (a \cup b)^*$

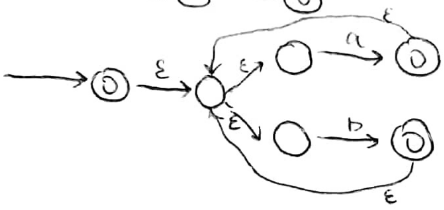
a



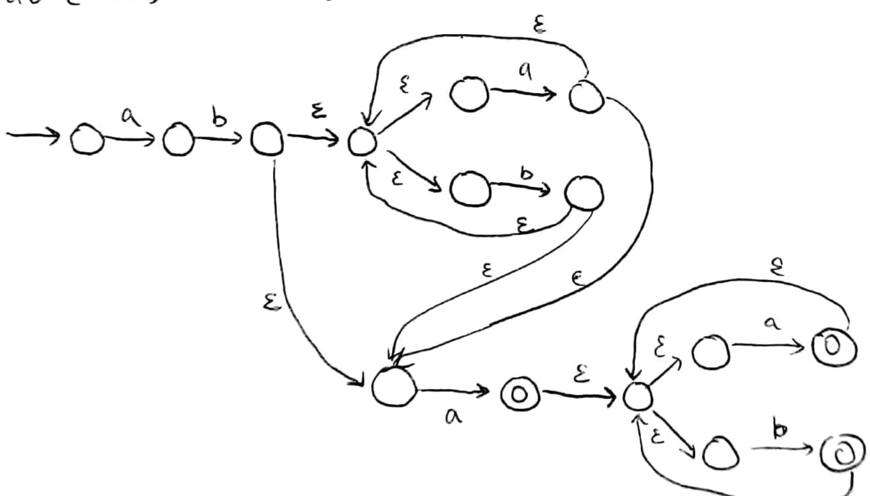
ab

$(a \cup b)$

$(a \cup b)^*$

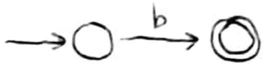


$ab(a \cup b)^* a (a \cup b)^*$

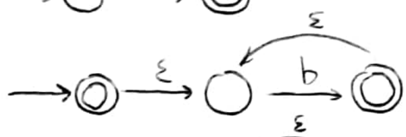


Q2C $b^*a^*(ba \cup b^*)$

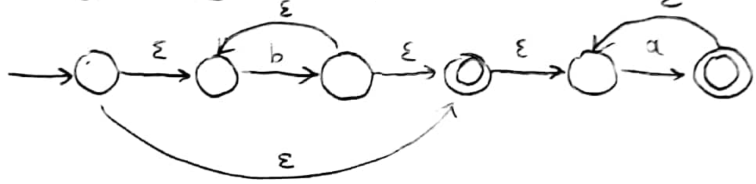
b



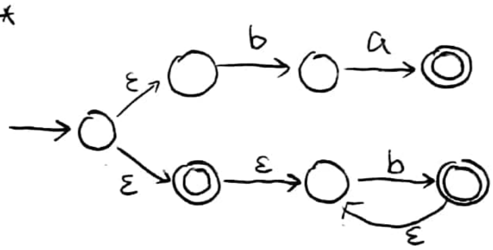
b^*



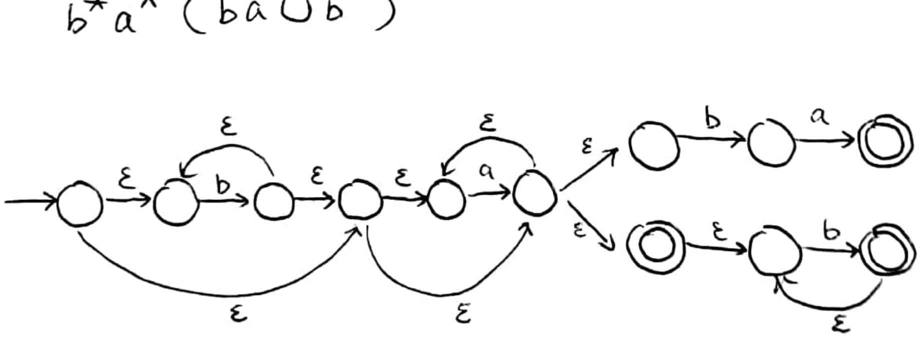
b^*a^*



$ba \cup b^*$



$b^*a^*(ba \cup b^*)$



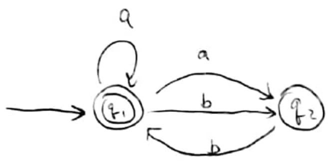
Q3: Convert these nondeterministic finite automata into regular expressions using the method shown in the lessons.

The final regular expression for the first NFA is $(a \cup ((a \cup b)b))^*$

The final regular expression for the second NFA is $(ab^*(a \cup b))(a(ab^*(a \cup b)))^*$

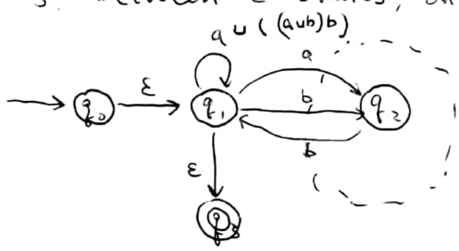
The steps are shown in the diagrams below.

Q3 a)

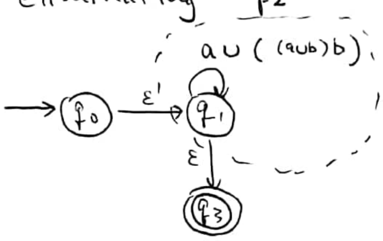


preprocessing stage

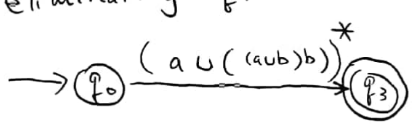
1. initial states shouldn't have any incoming arrows
2. accept states shouldn't have any outgoing arrows
3. between 2 states, only 1 arrow



eliminating q_2 $\qquad$ $q_1 - q_2 - q_3$

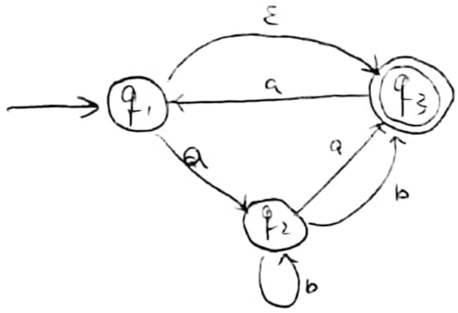


eliminating q_1 $\qquad$ $q_0 - q_1 - q_3$

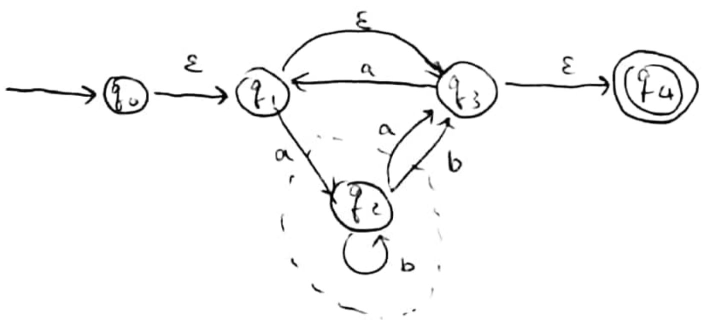


The final regular expression is $(a \cup ((a \cup b)b))^*$

Q3 b)

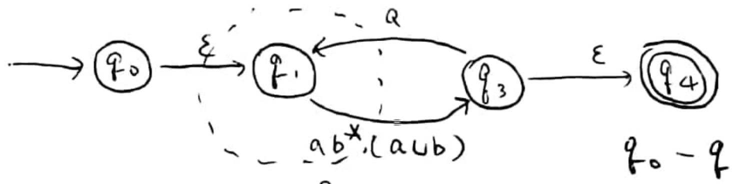


Following the same steps in Q3 a), we reach the preprocessing stage below:



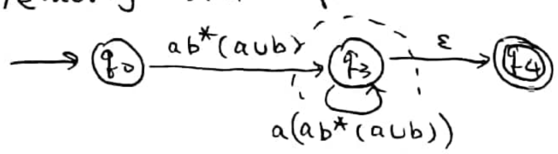
removing state q_2,

$q_1 - q_2 - q_3$



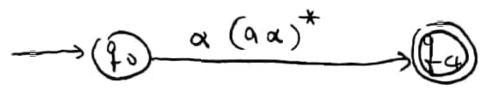
$q_0 - q_1 - q_3$
$q_3 - q_1 - q_3$

removing state q_1



$q_0 - q_3 - q_4$

removing state q_3, let $\alpha = ab^*(a \cup b)$



The final RE is $(ab^*(a \cup b))(a(ab^*(a \cup b)))^*$

Q4: Use the pumping lemma to prove that the following languages are not regular.

1. $\{b^{n^3} \in \{b\}^* | n \geq 0\}$

   Step 1:
   Suppose that L is regular. By the pumping lemma, $\exists$ some $m > 0$ (pumping length) s.t. $\forall$ w with $|w| \geq m$ can be pumped.

   Step 2:
   Choose $w = b^{m^3}$. Here $|w| = m^3$. We want $m^3 \geq m$. $\Leftrightarrow m^2 \geq 1$ as $m > 0 \Leftrightarrow m \geq 1$
   Therefore, our choice of w is $w = b^{m^3}, m \geq 1$

   Step 3:
   By the pumping lemma, $\exists$ some division $w = xyz$ s.t.
   1. $y \neq e$
   2. $|xy| \leq m$
   3. $xy^iz \in L, i \geq 0$

   Step 4:
   Consider xyz, we have $|xyz| = |w| = m^3$. Consider xy^2z, we have $|xy^2z| = |xyz| + |y|$.
   By pumping lemma 2, $|xy| \leq m \Leftrightarrow |y| \leq m$.
   $\Rightarrow |xy^2z| = |xyz| + |y| \leq m^3 + m < m^3 + 3m^2 + 3m + 1 = (m+1)^3, m > 0.$
   By pumping lemma 1, $y \neq e \Rightarrow |y| > 0 \Rightarrow |xy^2z| = |xyz| + |y| > |xyz| = m^3$
   Therefore, we have $m^3 < |xy^2z| < (m+1)^3 \Rightarrow xy^2z \notin L$
   Contradiction!

2. $\{b^{3^n} \in \{b\}^* | n \geq 0\}$

   Step 1:
   Suppose that L is regular. By the pumping lemma, $\exists$ some $m > 0$ (pumping length) s.t. $\forall$ w with $|w| \geq m$ can be pumped.

   Step 2:
   Choose $w = b^{3^m}$. Here $|w| = 3^m$. Consider a function $y = 3^x - x, x > 0$, we have y is always positive either by plotting the graph of y or consider the derivative of y. Therefore $|w| = 3^m > m$

   Step 3:
   By the pumping lemma, $\exists$ some division $w = xyz$ s.t.
   1. $y \neq e$
   2. $|xy| \leq m$
   3. $xy^iz \in L, i \geq 0$

   Step 4:
   Consider xyz and xy^2z, we have $|xyz| = |w| = 3^m$ and $|xy^2z| = |xyz| + |y|$.
   By pumping lemma 1, $y \neq e \Rightarrow |y| > 0$
   $\Rightarrow |xy^2z| = |xyz| + |y| > |xyz| = 3^m$
   By pumping lemma 2, $|xy| \leq m \Rightarrow |y| \leq m$.
   $\Rightarrow |xy^2z| = |xyz| + |y| \leq 3^m + m < 3^{m+1}$
   the strictly less than sign is because $3^{m+1} = 3^m * 3 = 3^m + 2 * 3^m > 3^m + m$ as $3^m > m$ (Proved in Step 2)

   Therefore, we have $3^m < |xy^2z| < 3^{m+1} \Rightarrow xy^2z \notin L$
   Contradiction!

Q5: Let $\Sigma = \{a, b\}$.

1. Show that $\{a^k ba^k | k \geq 1\}$ is not regular

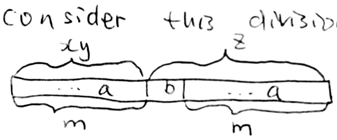
   Step 1:
   Suppose that L is regular. By the pumping lemma, $\exists$ some $m > 0$ (pumping length) s.t. $\forall$ w with $|w| \geq m$ can be pumped.

   Step 2:
   Choose $w = a^m ba^m, m \geq 1$
   Note that $|w| = 2m + 1 > m$ for $m \geq 1$

   Step 3:
   By the pumping lemma, $\exists$ some division $w = xyz$ s.t.
   1. $y \neq e$
   2. $|xy| \leq m$
   3. $xy^i z \in L, i \geq 0$

   Step 4:

   Consider this division $w = xyz$

   By pumping lemma 1 and 2, $y \neq e$ and $|xy| \leq m \Rightarrow |x| \leq m - 1$
   Therefore xz has less than m a's on the LHS of b but m a's on the RHS, thus breaking the symmetry.
   $\Rightarrow xz \notin L$
   Contradiction!

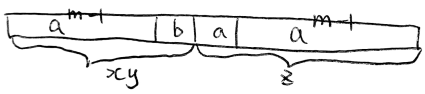
2. Show that $\{a^k bua^k | k \geq 1, u \in \Sigma^*\}$ is not regular

   Step 1:
   Suppose that L is regular. By the pumping lemma, $\exists$ some $m > 0$ (pumping length) s.t. $\forall$ w with $|w| \geq m$ can be pumped.

   Step 2:
   Let $u = a$, choose $w = a^{m-1}baa^{m-1}, m \geq 2$
   Note that $|w| = 2m > m$ for $m \geq 2$

   Step 3:
   By the pumping lemma, $\exists$ some division $w = xyz$ s.t.
   1. $y \neq e$
   2. $|xy| \leq m$
   3. $xy^i z \in L, i \geq 0$

   Step 4:
   Consider this division
   $w' = xy^0 z = xz = a^{m-1}aa^{m-1} = a^{2m-1}$ which does not contain b
   Therefore $xz \notin L$ Contradiction!



3. Show that $\{a^k u a^k | k \geq 1, u \in \Sigma^*\}$ is regular

   To show that a language is regular, we need to show that this language can be described by finite automata. We have $u \in \Sigma^*$, u can be represented by $(a \cup b)^*$
   Because u can be any number of a's, it can be combined with a^k on the left, or a^k on the right to just mean a^*. Hence the above expression is equivalent to $\{a^i(a \cup b)^* a^j | i, j \geq 0\}$, which can be expressed by finite automata. Therefore $\{a^k u a^k | k \geq 1, u \in \Sigma^*\}$ is regular.

Q6: Consider the language $L = \{a^i b^j c^k | i, j, k \geq 0$, and if $i = 1$ then $j = k\}$.

1. Show that L is not regular, using the fact that $L' = \{ab^i c^i | i \geq 0\}$ is not regular. Hint: the regular languages are closed under the intersection operation.

   Because regular languages are closed under the intersection operation. We have the following: $L1$, $L2$ are regular $\Rightarrow L1 \cap L2$ is regular.

   Consider $M = \{ab^j c^k | j, k \geq 0\}$. We know that M is regular as we can easily draw its NFA.

   Note that $L' = M \cap L$. Suppose that L is regular. Since M is also regular, we have $M \cap L$ is regular (regular languages are closed under intersection), which means that L' must be regular. However, it is given fact that $L' = \{ab^i c^i | i \geq 0\}$ is not regular.

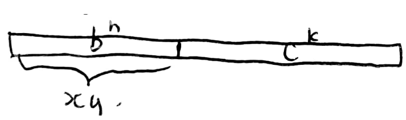
   Contradiction!

   Therefore, L is not regular.

2. Show that L meets the requirements of the regular language pumping lemma. In other words, find a pumping length n and show that the conditions hold for strings longer than n.

   $L = \{a^i b^j c^k | i, j, k \geq 0$, and if $i = 1$ then $j = k\}$
   $= \{ab^j c^j | j \geq 0\} \cup \{a^i b^j c^k | i, j, k \geq 0, i \neq 1\}$

   let $N = \{a^i b^j c^k | i, j, k \geq 0, i \neq 1\}$
   W.T.S $\exists$ a pumping length n and conditions hold for strings longer than n.
   Choose $w = a^0 b^n c^k = b^n c^k$ where $n, k \geq 0$
   Note that $|w| = n + k \geq n$ $\exists$ some division $w = xyz$ s.t.
   1. $y \neq e$
   2. $|xy| \leq m$
   3. $xy^i z \in L, i \geq 0$

   $w' = xy^i z = b^m c^k, m \geq n \geq 0, k \geq 0$
   $w' \in N \subset L$

3. Explain why the previous two parts do not contradict each other.

   Pumping lemma states that all regular languages have a special property.
   We have RE $\Rightarrow$ PL and $\neg$ PL $\Rightarrow$ $\neg$ RE
   If we can show that a language does not have this property, we can guarantee that it is not regular.
   However, this does not mean that if a language has this property for some n, this language is regular. In other words, PL $\Rightarrow$ RE is not true
   By proving that the conditions hold for some pumping length n does not prove the language is regular. The previous Q6b) does not prove the language is regular just by finding a pumping length n that meets the requirements of the pumping lemma.
   Therefore, the previous two parts do not contradict each other.