Element-Based Galerkin method

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Galerkin methods are numerical methods used to solve PDEs. This short note on Galerkin methods presents the Continous Galerkin (CG), where a free flux is assumed across the boundary. We described the CG method using the following diffusion equation

$$\frac{\partial q(x,t)}{\partial t} = \nu \nabla^2 q(x,t), \tag{1}$$

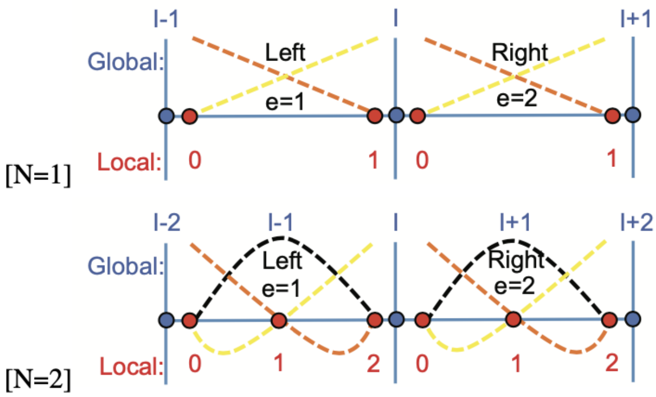
with either Neumann or Robin boundary conditions. Let's divide the problem domain $\Omega \in \mathbb{R}$ into N_e elements

$$\Omega = \bigcup_{i=1}^{Ne} \Omega_e,$$

We create an approximation $q_N^{(e)}(x,t)$ to $q(x,t)$ within each element Ω_e using

$$q_N^{(e)}(x,t) = \sum_{j=1}^{M_N} \psi_j q_j^{(e)}(t), \tag{2}$$

where M_N is the number of nodes in the element, ψ_j is a basis function (e.g. Lagrange polynomials), $q_j^{(e)}$ is the expansion coefficient and the superscript $^{(e)}$ denotes the element index.



Continuous Galerkin elements for polynomial order $N = 1$ and $N = 2$.

We expand both sides of equation (1) using the approximation (2), multiplying with the test function ψ_i and integrate within each element

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} d\Omega_e = \nu \int_{\Omega_e} \psi_i \nabla^2 q_N^{(e)} d\Omega_e \tag{3}$$

Let us use now the product rule

$$\nabla \cdot (\psi_i \nabla q_N) = \nabla \psi_i \cdot \nabla q_N + \psi_i \nabla^2 q_N$$

After integrating and using divergence theorem we obtain

$$\int_{\Omega_e} \psi_i d\Omega_e \frac{dq_N^{(e)}}{dt} = \nu \int_{\Gamma_e} \psi_i \mathbf{n} \cdot \nabla q_N^{(e)} d\Gamma_e - \nu \int_{\Omega_e} \nabla \psi_i \cdot \nabla q_N^{(e)} d\Omega_e. \tag{4}$$

Γ_e represent the boundary of the element Ω_e. Using the expansion (2), we get

$$\sum_{j=1}^{M_N} \int_{\Omega_e} \psi_i \psi_j d\Omega_e \frac{dq_j^{(e)}}{dt} = \nu \int_{\Gamma_e} \psi_i \mathbf{n} \cdot \nabla q_N^{(e)} d\Gamma_e - \nu \sum_{j=1}^{M_N} \int_{\Omega_e} \nabla \psi_i \cdot \nabla \psi_j d\Omega_e q_j^{(e)}. \tag{5}$$

The first term on the right-hand side is used to apply the boundary conditions. Due to the continuity across element interfaces, the inter-element edges vanish when the direct stiffness summation is applied to the first term on the right-hand side.

1 Neumann boundary conditions

The Neumann boundary condition is generally described as follows

$$\mathbf{n} \cdot \nabla q(x,t) = h(x), \quad x \in \partial\Omega. \tag{6}$$

Applying the Neumann condition (6), equation (5) becomes

$$\sum_{j=1}^{M_N} \int_{\Omega_e} \psi_i \psi_j d\Omega_e \frac{dq_j^{(e)}}{dt} = \nu \int_{\Gamma_e} \psi_i h(x) d\Gamma_e - \nu \sum_{j=1}^{M_N} \int_{\Omega_e} \nabla \psi_i \cdot \nabla \psi_j d\Omega_e q_j^{(e)}. \tag{7}$$

In the matrix form, we get

$$M^{(e)} \frac{dq^{(e)}}{dt} = \nu B^{(e)} - \nu L^{(e)} q^{(e)}. \tag{8}$$

where

$$M^{(e)} = \sum_{j=1}^{M_N} \int_{\Omega_e} \psi_i \psi_j d\Omega_e,$$

$$L^{(e)} = \sum_{j=1}^{M_N} \int_{\Omega_e} \nabla \psi_i \cdot \nabla \psi_j d\Omega_e,$$

$$B^{(e)} = \int_{\Gamma_e} \psi_i h(x) d\Gamma_e,$$

$q^{(e)}$ is a vector that contains the q_j^e, and $i, j = 1, 2, \ldots, M_N$.

2 Robin boundary condition

The general description of Robin boundary condition is given in below equation

$$\mathbf{n} \cdot \nabla q(x,t) + kq(x,t) = g(x), \quad x \in \partial\Omega.$$

Applying the above condition, equation (5) becomes

$$\sum_{j=1}^{M_N} \int_{\Omega_e} \psi_i \psi_j d\Omega_e \frac{dq_j^{(e)}}{dt} = \nu \int_{\Gamma_e} \psi_i g(x) - k\psi_i q(x,t) d\Gamma_e - \nu \sum_{j=1}^{M_N} \int_{\Omega_e} \nabla\psi_i \nabla\psi_j d\Omega_e q_j^{(e)}$$

$$\sum_{j=1}^{M_N} \int_{\Omega_e} \psi_i \psi_j d\Omega_e \frac{dq_j^{(e)}}{dt} = \nu \int_{\Gamma_e} \psi_i g(x) d\Gamma_e - k\nu \sum_{j=1}^{M_N} \int_{\Gamma_e} \psi_i \psi_j d\Gamma_e q_j^{(e)} - \nu \sum_{j=1}^{M_N} \int_{\Omega_e} \nabla\psi_i \nabla\psi_j d\Omega_e q_j^{(e)}$$

In the matrix form, we get

$$M^{(e)} \frac{dq^{(e)}}{dt} = \nu B^{(e)} - \nu \left(kF^{(e)} + L^{(e)} \right) q^{(e)}, \tag{9}$$

where the new term $F^{(e)}$ is

$$F^{(e)} = \sum_{j=1}^{M_N} \int_{\Gamma_e} \psi_i \psi_j d\Gamma_e.$$

The direct stiffness summation is applied to move from the element reference equation to the global equation. For more detail the reader is referred to Giraldo (2020).

References

Francis X Giraldo. *An Introduction to Element-Based Galerkin Methods on Tensor-Product Bases: Analysis, Algorithms, and Applications*, volume 24. Springer International Publishing, 2020.