Solving constant coefficient linear systems

$$q_t + Aq_x = 0, \qquad A \in R^{m \times m}$$

We assume that A has a complete set of eigenvectors and real eigenvalues and so can be written as

$$A = R\Lambda R^{-1}$$

$$R = [r^1, r^2, \ldots r^m] \qquad \Lambda = \mathrm{diag}(\lambda^1, \lambda^2, \ldots \lambda^m)$$

Examples : Linearized shallow water wave equations, constant coefficient acoustics, ...

Solving a constant coefficient system

$$q_t + A\,q_x = 0 \quad \rightarrow \quad q_t + R\Lambda R^{-1}\,q_x = 0, \qquad A \in R^{m\times m}$$

Assume that A is diagonalizable

Define characteristic variables $\omega \in R^m$ as

$$\omega(x,t) = R^{-1}\,q(x,t), \qquad \omega(x,0) = R^{-1}\,q(x,0)$$

Characteristic equations decouple into m scalar equations :

$$\omega^p_t + \lambda^p\omega^p_x = 0, \qquad p = 1,2,\ldots,m$$

Solution to characteristic equations are given by

$$\omega^p(x,t) = \omega^p(x - \lambda^p t, 0)$$

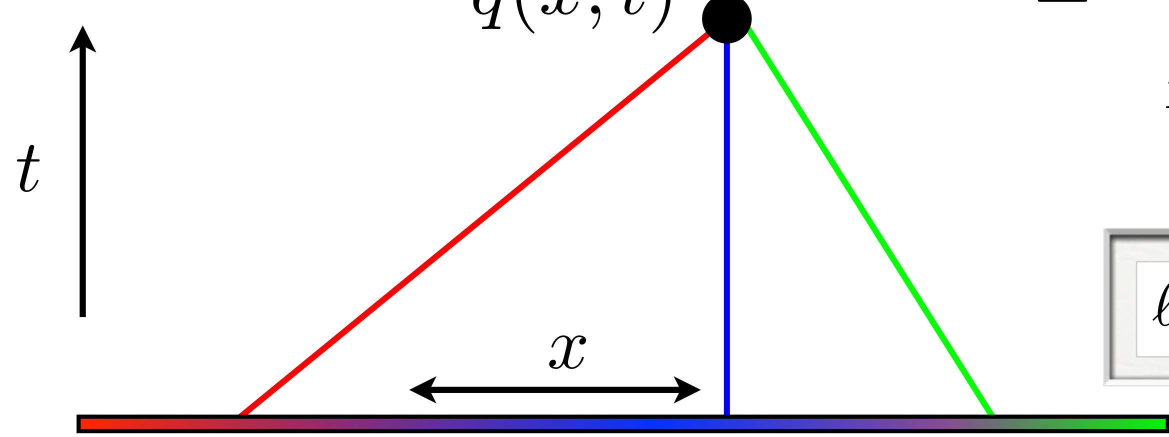
Solving a constant coefficient systems

$$q_t + A\,q_x = 0 \qquad \longrightarrow \qquad \omega^p_t + \lambda^p \omega^p_x = 0$$

Solution for general initial conditions $q(x,0)$:

$$q(x,t) = R\,\omega(x,t) \quad = \quad \sum_{p=1}^{m} \omega^p(x,t)\, r^p$$

$$\quad = \quad \sum_{p=1}^{m} \omega^p(x - \lambda^p t, 0)\, r^p$$



$$\boxed{\ell^p q(x - \lambda^p t, 0) = \omega^p(x - \lambda^p t, 0)}$$