7.15.2016
Contest

Contest Paper - AMSP Cornell 2016 Mock AIME

Instructions

1. Do not open this booklet until your have set your timer to 3 hours.
2. FORMAT: This contest contains 15 problems ranging from relatively straighforward to extremely advanced. Problems are roughly in order of increasing difficulty (although there are definitely problems easier or harder than they should be). The answer to each problem is an integer ranging from 000 to 999, inclusive. Only one answer is correct. You will have 3 hours to work on these 15 problems.
3. SCORING: You may only give 1 answer for each problem. If you give more than 1 answer, all answers will be marked as incorrect. Each correct answer is worth 1 points; each unanswered problem or incorrect answer will earn 0 points.
4. Only scratch paper, graph paper, rulers, protractors, and compasses are allowed as an aid in this contest. Calculators, dictionaries, geometric objects, and other aids are not permitted.
5. Figures may not be drawn to scale.
6. Thanks for participating, and I hope you enjoy the contest!

Contributions (AoPS Usernames)
Problem 1 was proposed by magicarrow.
Problems 2, 4, 7-8, 10-11, 14-15 were proposed by joey8189681.
Problems 3, 13 were proposed by wu2481632.
Problems 5-6, 9 were proposed by Generic_Username.
Problem 12 was proposed by amplreneo.
LaTeX geek: azmath333.
Programmer: doodlemaster7
Helper: champion999
Problem Selection Committee: amplreneo, azmath333, Generic_Username, joey8189681, wu2481632

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1. There are 50 people who play Pokemon Go. Suppose that there is an RA such that for every person who passes him, he checks if they play Pokemon Go. If so, the person will need to do 10 push-ups. Given that 270 people go to AMSP, everyone is up 24/7, and every 3 hours someone passes the RA, then the expected number of days until 100 push-ups have been done due to Pokemon Go can be expressed as $\frac{m}{n}$, where m, n are relatively prime. Find $m+n$.

2. Joey is staring at a standard 12-hour clock. He waits until the next time the hour and minute hand form a right angle before he stops staring at the clock. Let E be the expected number of minutes he stares at the clock. Given E can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

3. Let ABC be a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. Let D be the foot of the altitude from A, and let I_A be the A-excenter (the center of the circle tangent to lines AB, AC, and side BC). Then DI_A can be expressed as $a\sqrt{b}$, where b is not divisible by the square of any prime. Find $a+b$.

4. A circle with center O has integer diameter. A point P is selected inside the circle such that $OP = 12$ and there are 5 chords of integer length passing through P with respect to the given circle. Then the sum of the radii of all such circles can be expressed as $\frac{m}{n}$, where m, n are relatively prime. Find the remainder when $m+n$ is divided by 1000.

5. Given a 5×5 chessboard, define S to be the set of all colorings of rows and columns of the chessboard in black and white, where a colored row or column contains exclusively black squares and a square that does not belong to a colored row or column is white.

   Define value(x) to be the number of black squares in an arbitrary coloring x. Find the remainder when
$$\sum_{x \in S} \text{value}(x)$$
   is divided by 1000.

6. Let a, b, c be constants such that
$$\begin{cases} \log_x a + \log_x b^2 + \log_x c^3 = 35 \\ \log_x b + \log_x c^2 + \log_x a^3 = 11 \\ \log_x c + \log_x a^2 + \log_x b^3 = 24 \end{cases}$$
   has a real solution. If $\log_x ab + \log_x bc + \log_x ca = \dfrac{m}{n}$, find $m+n$ where m, n are relatively prime positive integers.

7. Given that
$$N = \sum_{i=1}^{2017} \sum_{k=0}^{i} k \binom{i}{k},$$
   find the remainder when N is divided by 1000.

8. Let N be the smallest positive integer that cannot be expressed as the sum of 2017 (not necessarily distinct) or less integers of the form $n!$ where n is a positive integer. If N can be expressed as $a \cdot b! - c$ for integers a, b, c where b is as large as possible, find $a+b+c$.

9. Let P be the set of $\dfrac{m}{n}$ where every element of P is a root of $x^{97} - 1 = 0$ and

$$m = e^{2\pi i/p},\ n = e^{2\pi i/q}$$

for integers p, q. Find the number of ordered pairs (p, q).

10. In triangle BAC, Γ is its circumcircle, $CB = 14$, $AB = 13$, and $CA = 15$. The angle bisectors of $\angle CAB$ intersects Γ at a point $D \neq A$. Circles ω_1, ω_2 are distinct, internally tangent to Γ, tangent to CB and AD, and have overlapping area with triangle ABC. Given that ω_1 is tangent to AD at H_1 and ω_2 is tangent to AD at H_2, $DH_1 + DH_2$ can be expressed as $a\sqrt{b}$ where b is square-free. Find $a + b$.

11. Alvin is getting very tired and is sitting at the point $(0, 0)$ in the coordinate plane. He must get to Siva, who is sitting at the point $(8, 8)$, to get the caffeine necessary to pull off the all-nighter. Given that Alvin is at the point (x, y) he can do one on the three following moves:

$$1 : (x + 1, y)$$

$$2 : (x, y + 1)$$

$$3 : (x + 1, y + 1)$$

Let M be the number of ways Alvin can get to Siva. Find the remainder when M is divided by 1000.

12. Let $\mathcal{F}$ be the set of functions $f : \{1, 2, 3, ..., 42\}^3 \to \mathbb{N}_0$ that follow the following conditions:
   1. If $f(x, y, z) > 0$, then exactly one of $\{f(x-1, y, z), f(x, y-1, z), f(x, y, z-1)\}$ is greater than zero.
   2. $\sum_{1 \le x,y,z \le 42} f(x, y, z) = 2016$.
   3. $f(42, 42, 42) > 0$.

   Find the maximum integer value m such that $\frac{|\mathcal{F}|}{13^m}$ is an integer.

13. Let ABC be a triangle, and let Ω and ω be its circumcircle and incircle, respectively. Let M_B and M_C be the midpoints of arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BA}$ in Ω, not containing B and C, respectively. Denote by E and F the tangency points of ω with AC and AB, respectively, and let E, F, M_C, M_B all lie on a line. Let $M_B I : IB = 1$ and $M_C I : IC = \frac{1}{2}$, where I denotes the incenter of triangle ABC. If X is the intersection of BM_C and CM_B, then $\frac{[ABC]}{[XBC]}$ can be expressed as $\frac{m}{n}$, where m, n are relatively prime. Find $m + n$.

14. Let S be the set of all possible remainders when $\binom{n}{2}$ is divided by 2016 for all positive integers $n > 1$. Find the number of elements in S.

15. Quadrilateral $ABCD$ is inscribed in a circle of radius 7. Let $BC = 11$, let E be the intersection of rays $\overrightarrow{CD}$ and $\overrightarrow{BA}$, let F be the intersection of rays $\overrightarrow{BC}$ and $\overrightarrow{AD}$, and let G be the intersection of diagonals BD and AC. A point P lies on EF such that $GP \perp EF$, $\angle APE = 60^\circ$ and $\angle AED = 30^\circ$. Given that the area of triangle DEF can be expressed as $m\sqrt{n}$, where n is square-free, find $m + n$.